

# Series Search

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## 1 Introduction

This project is intended to study computational applications in analysis. Inspired by the elusive value of  $\zeta(3)$  (Apery's constant), I wanted to see a potential computational approach to developing closed forms for infinite series. While Apery's constant is unknown, I intend to derive the famous result  $\zeta(2) = \frac{\pi^2}{6}$  through the algorithm below.

### 1.1 Formal Developments

Let  $S$  be a sequence whose sum converges absolutely to a value  $\delta$ , with terms  $\Delta_n$ . Further, let  $S'$  be a sequence that converges absolutely to a value  $\delta'$ , with terms  $\Delta'_n$ .

### 1.2 Basic Definitions

- i  $Ord_R(S)$  is the sequence  $K$ , such that  $S$  is reordered by a relation  $R$ .
- ii  $Abs(S)$  is the sequence  $K$ , such that  $K = (|\Delta_i|, |\Delta_{i+1}|, |\Delta_{i+2}|, |\Delta_{i+3}|, \dots)$
- iii  $S'$  is an undercompensate of  $S$  if there exists a term  $\Delta_k \neq \Delta_i$ , for all appropriate terms in  $S'$ ,  $\Delta_i$ . We denote this with  $S \sqsubset S'$
- iv  $S'$  is an overcompensate of  $S$  if there exists a term  $\Delta'_i \neq \Delta_k$ , for all appropriate terms in  $S$ ,  $\Delta_k$ . We denote this with  $S \sqsupset S'$
- v  $S'$  is said to be **total** to  $S$  if  $Ord(Abs(S')) = Ord(Abs(S))$  We denote this with  $S \equiv S'$ . Considering the sums of the respective series are absolutely convergent, one can see that the OrdAbs operator on  $S'$  and  $S$  is purely a formality, and not necessary.  $S'$  is said to be **equal** to  $S$  if and only if  $S' \equiv S$  and  $\delta' = \delta$

**Note:** one may rewrite the above definitions in the language of functions, that is through injections, surjections, and bijections on the set of terms  $\{x|x = \Delta_n\}$  and  $\{x|x = \Delta'_n\}$

### -Abuse of Notation, Additional Definitions-

- i Let  $A$  and  $B$  be sequences whose sum absolutely converges, with terms  $a_n$  and  $b_n$ . The sequence union of  $A$  and  $B$ , denoted  $A \cup B$  is the sequence  $(a_1, b_1, a_2, b_2, \dots)$ .
- ii Let  $A$  and  $B$  be absolutely convergent sequences with terms  $a_n$  and  $b_n$ . The sequence intersection of  $A$  and  $B$ , denoted  $A \cap B$ , is the sequence  $(c_i, c_{i+1}, c_{i+2}, \dots)$ , where  $c_i = a_i$  where  $a_i = b_i$ .
- iii Let  $A$  and  $B$  be absolutely convergent sequences with terms  $a_n$  and  $b_n$ . The sequence minus of  $A$  and  $B$ , denoted  $A \setminus B$ , is the sequence  $(c_i, c_{i+1}, c_{i+2}, \dots)$ , where  $c_i = a_i, b_i$ , where  $a_i \neq b_i$ .
- iv Let  $A_1, A_2, A_3 \dots A_n$  be sequences whose sum converges absolutely. Then,  $\mathcal{A} = \{A_i\}$  is said to be partitions of  $S$  if  $A_i$  are undercompensates of  $S$  and  $Ord_{\leq}(Abs(\bigcup_{i=1}^n A_i)) = Ord_{\leq}(Abs(S))$ .

## 1.3 Series Encoding

We formalize the notion of encoding a series. The **encoding** of a series is a tuple  $\mathcal{E} = (\rho_0, \mathcal{T}, N)$ , where  $\rho_0$  denotes the initial value of a series,  $\mathcal{T}$  denotes a transform  $\mathcal{T} : \mathbb{Q} \rightarrow \mathbb{Q}$  (for this purpose, a series is over the rationals), and  $N$  which denotes the number of transition states in the transform for the summation. The transform  $\mathcal{T}$  should be defined such that  $\mathcal{T}(\rho_i) = \rho_{i+1}$ . The  $n$ 'th iteration of  $\mathcal{T}$  on  $\rho_i$ , (i.e.  $\mathcal{T}(\mathcal{T}(\mathcal{T} \dots (\rho_i) \dots))$ ) will be denoted as  $\mathcal{T}^n(\rho_i)$ .

## 1.4 Series Decomposition

We formalize the notion of decomposing a series. Given a series,  $S (= \Delta_n)$ ,  $S$  may be decomposed into a partition  $\mathcal{A}$  and encoded as  $\mathcal{Z}$ .

### 1.4.1 Example

Consider a geometric series,  $G$ , given by

$$G = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1}$$

A valid partition for  $G$  is given by:

$\mathcal{A} = \{A_1, A_2\}$ , where

$$A_1 = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{2n}, A_2 = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{2n+1}$$

Encoded as:  $\text{Enc}(\mathcal{A}) = \mathcal{Z} = \{(1, \mathcal{T}_0(s) = \frac{1}{9}s, |\mathbb{N}|), (\frac{1}{3}, \mathcal{T}_1(s) = \frac{1}{9}s, |\mathbb{N}|)\}$

## 1.5 Series Searching

Theorem 1.1 Let  $S$  be an absolutely convergent series with term  $\Delta_n$ . Let  $\mathcal{U}$  be the set of all absolutely converging series with term  $u_{i_n} \in \mathcal{U}$ . Let  $\mathcal{P}$  be the set of all encodings of series in the partitions of  $\mathcal{U}$ . Construct a directed complete graph where  $G = (U, V)$ , where the set of vertices,  $V = \{p \in \mathcal{P}\}$ . A set  $P \subseteq \mathcal{P}$  is said to *complete*  $S$  if there exists