Series Search

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1 Introduction

This project is intended to study computational applications in analysis. Inspired by the elusive value of $\zeta(3)$ (Apery's constant), I wanted to see a potential computational approach to developing closed forms for infinite series. While Apery's constant is unknown, I intend to derive the famous result $\zeta(2) = \frac{\pi^2}{6}$ through the algorithm below.

1.1 Formal Development

Let S be a sequence whose sum converges absolutely to a value δ, with terms Δ_n. Further, let S' be a sequence that converges absolutely to a value δ', with terms Δ'_n.

1.2 Basic Definitions

i $Ord_R(S)$ is the sequence K, such that S is reordered by a relation R.

ii $Abs(S)$ is the sequence K, such that $K = (|\Delta_i|, |\Delta_{i+1}|, |\Delta_{i+2}|, |\Delta_{i+3}|, ...)$

iii S' is an undercompensate of S if there exists a term $\Delta_k \neq \Delta_i$, for all appropriate terms in S', Δ_i. We denote this with $S \sqsubset S'$

iv S' is an overcompensate of S if there exists a term $\Delta'_i \neq \Delta_k$, for all appropriate terms in S, Δ_k. We denote this with $S \sqsupset S'$

v S' is said to be **total** to S if $Ord(Abs(S')) = Ord(Abs(S))$ We denote this with $S \equiv S'$. Considering the sums of the respective series are absolutely convergent, one can see that the OrdAbs operator on S' and S is purely a formality, and not necessary. S' is said to be **equal** to S if and only if $S' \equiv S$ and $\delta' = \delta$

Note: one may rewrite the above definitions in the language of functions, that is through injections, surjections, and bijections on the set of terms $\{x|x = \Delta_n\}$ and $\{x|x = \Delta'_n\}$

-Abuse of Notation-

i Let A and B be sequences whose sum absolutely converges, with terms a_n and b_n. The sequence union of A and B, denoted $A \cup B$ is the sequence $(a_1, b_1, a_2, b_2, ...)$.

ii Let A and B be absolutely convergent sequences with terms a_n and b_n. The sequence intersection of A and B, denoted $A \cap B$, is the sequence $(c_i, c_{i+1}, c_{i+2}...)$, where $c_i = a_i$ where $a_i = b_i$.

iii Let A and B be absolutely convergent sequences with terms a_n and b_n. The sequence minus of A and B, denoted $A \setminus B$, is the sequence $(c_i, c_{i+1}, c_{i+2}...)$, where $c_i = a_i, b_i$, where $a_i \neq b_i$.

iv Let $A_1, A_2, A_3...A_n$ be sequences whose sum converges absolutely. Then, $\mathcal{A} = \{A_i\}$ is said to be partitions of S if A_i are undercompensates of S and $Ord_{\leq}(Abs(\bigcup_{i=1}^{n} A_i)) = Ord_{\leq}(Abs(S))$.

1.3 Series Encoding

We formalize the notion of encoding a series. The **encoding** of a series is a tuple $\mathcal{E} = (\rho_0, \mathcal{T}, N)$, where ρ_0 denotes the initial value of a series, $\mathcal{T}$ denotes a transform $\mathcal{T} : \mathbb{Q} \to \mathbb{Q}$ (for this purpose, a series is over the rationals), and N which denotes the number of transition states in the transform for the summation. The transform $\mathcal{T}$ should be defined such that $\mathcal{T}(\rho_i) = \rho_{i+1}$. The n'th iteration of $\mathcal{T}$ on ρ_i, (i.e. $\mathcal{T}(\mathcal{T}(\mathcal{T}...(\rho_i)...)))$ will be denoted as $\mathcal{T}^n(\rho_i)$.

1.4 Series Decomposition

We formalize the notion of decomposing a series. Given a series, S $(= \Delta_n)$, S may be decomposed into a partition $\mathcal{A}$ and encoded as $\mathcal{Z}$.

1.4.1 Example

Consider a geometric series, G, given by

$$G = \sum_{n=1}^{\infty} (\frac{1}{3})^{n-1}$$

A valid partition for G is given by:
$\mathcal{A} = \{A_1, A_2\}$, where

$$A_1 = \sum_{n=0}^{\infty} (\frac{1}{3})^{2n}, A_2 = \sum_{n=0}^{\infty} (\frac{1}{3})^{2n+1}$$

Encoded as: Enc($\mathcal{A}$) $= \mathcal{Z} = \{(1, \mathcal{T}_0(s) = \frac{1}{9}s, |\mathbb{N}|), (\frac{1}{3}, \mathcal{T}_1(s) = \frac{1}{9}s, |\mathbb{N}|)\}$

1.5 Series Searching

Theorem 1.1 Let S be an absolutely convergent series with term Δ_n.

Let $\mathcal{U}$ be the set of all absolutely converging series with term $u_{i_n} \in \mathcal{U}$. Let $\mathcal{P}$ be the set of all encodings of series in the partitions of $\mathcal{U}$. Construct a directed complete graph where $G = (U, V)$, where the set of vertices, $V = \{p \in P\}$. A set $P \subseteq \mathcal{P}$ is said to *complete* S if there exists