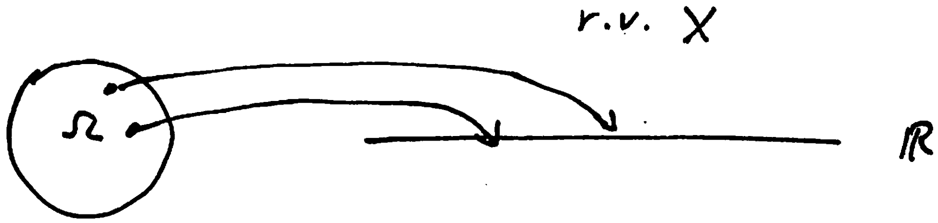
Random Variables

Given prob. model with sample space Ω, random variable (r.v.) assigns real number to each outcome $\omega \in \Omega$, i.e. r.v. is a function from Ω to $\mathbb{R}$.



For $\omega \in \Omega$,
$X(\omega) = x \in \mathbb{R}$
event $\{X = x\}$
$= \{\omega \in \Omega \mid X(\omega) = x\}$

Discrete r.v. − range of r.v. is finite or countable

$$S_X = \{x_1, x_2, \ldots\} \text{ or } S_X = \{x_1, \ldots, x_n\}$$

Continuous r.v. − range is uncountable

$$S_X = [-1, 1]$$

Discrete r.v.'s probability mass function (PMF)

$$P_X : S_X \to [0,1]$$

$$P_X(x) = P(X = x) = P(\{\omega \in \Omega \mid X(\omega) = x\})$$

(a) $P_X(x) \geq 0 \quad \forall x \in S_X$

(b) $\sum_{x \in S_X} P_X(x) = 1$

(c) For any $B \subset S_X$, $P(B) = \sum_{x \in B} P_X(x)$

e.g. 2 indep. tosses of fair coin. $\Omega = \{hh, ht, th, tt\}$ (16)

Let X = # heads, $S_X = \{0, 1, 2\}$

$$P_X(x) = \begin{cases} 1/4, & x = 0 \text{ or } x = 2 \\ 1/2, & x = 1 \\ 0, & \text{o/w} \end{cases}$$

Expectation or mean of $X \sim P_X$ is $\underset{\mu}{\underset{\parallel}{E[X]}} = \sum_{x \in S_X} x P_X(x)$

center of mass of PMF

Function of r.v.

$X : \Omega \to \mathbb{R}$

$g : \mathbb{R} \to \mathbb{R}$

Let $Y = g(X) : \Omega \to \mathbb{R} \to \mathbb{R}$ is another r.v.

e.g. X = temp in celsius

$Y = g(X) = 1.8X + 32$ = temp. in Fahrenheit

$$P_Y(y) = P(Y = y) = \sum_{\{x \mid g(x) = y\}} P_X(x)$$

$$E[g(X)] = E[Y] = \sum_{y \in S_Y} y P_Y(y) = \sum_{x \in S_X} g(x) P_X(x)$$

Moments: $X \sim P_X$, $Y = X^2$

$E[X^2]$ = 2nd moment of X

$E[X^n]$ = nth " "

- $Y = (X - E[X])^2$ is 2nd central moment of X

$E[Y] = 0, \quad Y \geq 0$

$E[Y] = E[(X - E[X])^2]$ = variance of X $= Var(X)$

$\sigma_X = \sqrt{Var(X)}$ = standard deviation of X

$$Var(X) = E[(X - E[X])^2]$$
$$= \sum_{x \in S_X} (x - E[X])^2 P_X(x)$$

- $E[aX + b] = a E[X] + b$

$Var[aX + b] = a^2 Var(X)$

$Var(X) = E[X^2] - E[X]^2$

Multiple r.v.'s (discrete)

Given r.v.'s X and Y assoc. w/ same experiment

Joint PMF of X, Y is

$\{X = x, Y = y\}$ is an event

$$P_{X,Y}(x,y) = P(X = x, Y = y)$$
$$= P(X = x \text{ and } Y = y)$$

Marginal PMF's

$$P_X(x) = \sum_y P_{X,Y}(x,y)$$

$$P_Y(y) = \sum_x P_{X,Y}(x,y)$$

Covariance of 2 r.v.'s ~~and~~ X and Y

$$\mathrm{Cov}(X,Y) = E[(X-E[X])(Y-E[Y])]$$

if $\mathrm{Cov}(X,Y)=0$, then say X, Y <u>uncorrelated</u>

$\mathrm{cov}(X,Y) > 0$ values of $X-E[X]$ and $Y-E[Y]$ tend to have same sign.

< 0 : opposite sign.

$$\mathrm{cov}(X,Y) = E[XY] - E[X]E[Y]$$

$$\mathrm{Cov}(X,X) = \mathrm{var}(X).$$

$$\mathrm{var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{cov}(X,Y)$$

<u>Conditional PMF</u> of X, <u>conditioned on event</u> B, $P(B)>0$,

$$P_{X|B}(x) = P(X=x\,|\,B) = \frac{P(\{X=x\}\cap B)}{P(B)}$$

$$\sum_x P_{X|B}(x) = 1$$

if $B_1, \ldots, B_m$ is an event space, then

$$P_X(x) = \sum_{i=1}^{m} P_{X|B_i}(x)\,P(B_i)$$

<u>Conditioning one r.v. on another</u>

Conditional PMF of X given Y

(specialize B to form $\{Y=y\}$, $P_Y(y)>0$)

$$P_{X|Y}(x|y) = P(X=x\,|\,Y=y)$$

$$= \frac{P(X=x, Y=y)}{P(Y=y)} = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

for fixed y, $P_{X|Y}(x|y)$ is a PMF for x.

Thus, $\sum_x P_{X|Y}(x|y) = 1$ for each y s.t. $P_Y(y) > 0$

Also, $P_{Y|X}(y|x) = \dfrac{P_{X,Y}(x,y)}{P_X(x)}$

$$P_{X,Y}(x,y) = P_Y(y)\, P_{X|Y}(x|y) = P_X(x)\, P_{Y|X}(y|x)$$

for $P_X(x) > 0$, $P_Y(y) > 0$

Conditional Expectation

$E[X|B] = \sum_x x P_{X|B}(x)$ conditional expectation of X given event B, $P(B) > 0$

$$E[g(X)|B] = \sum_x g(x) P_{X|B}(x).$$

For event space $B_1, .., B_m$, $P(B_i) > 0 \ \forall i$

$$E[X] = \sum_{i=1}^{m} E[X|B_i]\, P(B_i)$$

Specialize to $B = \{Y = y\}$

$E[X|Y=y] = \sum_x x P_{X|Y}(x|y)$ conditional expectation of X given $Y=y$

$$E[X] = \sum_y P_Y(y)\, E[X|Y=y]$$

Independence of r.v.'s

X, Y indep if

$$p_{X,Y}(x,y) = p_X(x)\,p_Y(y) \qquad \forall (x,y)$$

i.e. events $\{X=x\}$ and $\{Y=y\}$ are indep. $\forall (x,y)$.

$$\Leftrightarrow \qquad p_{X|Y}(x|y) = p_X(x) \qquad \forall y \text{ s.t. } p_Y(y)>0,\ \forall x$$

$$p_{Y|X}(y|x) = p_Y(y) \qquad \forall x \text{ s.t. } p_X(x)>0,\ \forall y$$

value of Y provides no info. on value of X and vice versa.

- If X, Y indep., then $E[XY] = E[X]\cdot E[Y]$.
- X, Y indep: $\operatorname{cov}(X,Y) = E[XY] - E[X]E[Y] = 0$

  $\Rightarrow$ X, Y uncorrelated, but reverse <u>not true</u>

  i.e. X, Y uncorrelated $\not\Rightarrow$ X, Y indep.

Sums of indep. r.v.'s

- If $X_1, \ldots, X_n$ are indep,

  $$\operatorname{var}(X_1 + \ldots + X_n) = \operatorname{var}(X_1) + \ldots + \operatorname{var}(X_n)$$

- Let $X_1, \ldots, X_n$ indep, ident, distr. (i.i.d)

  where $E[X_i] = \mu$, $\operatorname{var}(X_i) = \sigma^2$

  $$S_n = \frac{1}{n}(X_1 + \ldots + X_n).$$

$E[S_n] = \mu$, $\quad Var(S_n) = \sum_{i=1}^{n} \frac{1}{n^2} \sigma^2 = \frac{\sigma^2}{n} \to 0$ as $n \to \infty$

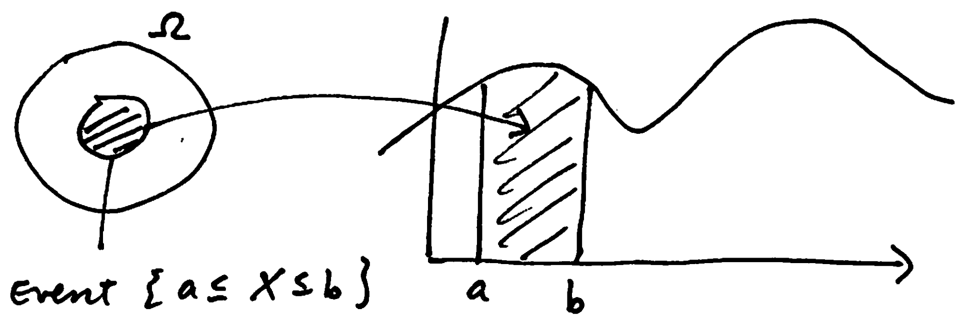
Continuous r.v.'s

- Main difference : cannot specify prob. of each outcome in Ω. In fact, prob. of each outcome $=0$!
- Instead, specify probs for intervals
- Probability density function (PDF) f_X

$$P(X \in B) = \int_B f_X(x)\,dx \qquad \text{(Riemann Integral)}$$

for every subset B of $\mathbb{R}$.

$$P(a \le X \le b) = \int_a^b f_X(x)\,dx$$



- $f_X(x) \geq 0 \quad \forall x \quad$ and $\quad \int_{-\infty}^{\infty} f_X(x)\,dx = 1$
- For δ small,

$$P(X \in [x, x+\delta]) = \int_{x}^{x+\delta} f_X(t)\,dt \sim f_X(x)\,\delta$$

- $f_X(x)$ as "prob. mass per unit length"
- $f_X(x)$ is not prob. of any event.

- Expectation $E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$

(need $\int_{-\infty}^{\infty} |x| f_X(x)\,dx < \infty$)

- $E[X]$ "center of gravity" of PDF
- $E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$
- $E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\,dx$ nth moment
- $Var(X) = E[(X - E[X])^2] = \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x)\,dx$

<u>Cumulative</u> <u>Distribution</u> <u>Function</u> (CDF)

- unify treatment for discrete and continuous r.v.'s
- $F_X(x) = P(X \le x) = \begin{cases} \sum_{k \le x} P_X(k) & X \text{ discrete} \\ \int_{-\infty}^{x} f_X(x)\,dx & X \text{ continuous} \end{cases}$

- By Fund. Th. of calculus

$$f_X(x) = \frac{d F_X(x)}{dx}$$

For continuous r.v.'s CDF is continuous.

- $F_X(x) \nearrow$ $F_X(x) \to 0$ as $x \to -\infty$

  $F_X(x) \to 1$ as $x \to +\infty$

Joint PDF

- say X, Y are jointly continuous with joint PDF $f_{X,Y}$ if $f_{X,Y}$ is non-neg. function s.t.

$$P((X,Y)\in B) = \iint_{(x,y)\in B} f_{X,Y}(x,y)\,dx\,dy$$

for every subset $B \subset \mathbb{R}^2$

$$P(a \le X \le b,\ c \le Y \le d) = \int_c^d \int_a^b f_{X,Y}(x,y)\,dx\,dy$$

- $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

- $$P(a \le X \le a+\delta,\ c \le Y \le c+\delta) \sim f_{X,Y}(a,c)\,\delta^2$$

$f_{X,Y}(a,c)$ is "prob. per unit area" near (a,c).

- $$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

marginal PDF's

- Joint CDF $F_{X,Y}(x,y) = P(X \le x, Y \le y)$

$$= \int_{\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t)\,dt\,ds$$

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}}{\partial x \partial y}(x,y)$$

- $(X, Y) \sim f_{X,Y}$

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\,dx\,dy$$

Conditional PDF of r.v. X, given event B, $P(B) > 0$ is non-neg. $f_{X|B}$ satisfying

$$P(X \in A \mid B) = \int_A f_{X|B}(x)\,dx$$

$$\int_{-\infty}^{\infty} f_{X|B}(x)\,dx = 1.$$

- $\{B_1, \ldots, B_n\}$ an event space. Then

$$f_X(x) = \sum_{i=1}^{n} f_{X|B_i}(x)\, P(B_i)$$

Conditioning one r.v. on another

$(X, Y) \sim f_{X,Y}$; conditional PDF of X given $\{Y = y\}$

is $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$ fn. of x for fixed y.

- $f_{X|Y}(x|y)$ has same shape as $f_{X,Y}(x,y)$ since $f_Y(y)$ does not depend on x.

- $\int_{-\infty}^{\infty} f_{X|Y}(x|y)\,dx = 1$

Conditional Expectation

- X, Y jointly continuous

$$E[X|B] = \int_{-\infty}^{\infty} x \, f_{X|B}(x)\,dx$$

$$E[g(X)|B] = \int_{-\infty}^{\infty} g(x) \, f_{X|B}(x)\,dx$$

e.g. $Var(X|B) = E[X^2|B] - (E[X|B])^2$

- $$E[X|Y=y] = \int_{-\infty}^{\infty} x \, f_{X|Y}(x|y)\,dx$$

- $\{B_1, \ldots, B_n\}$ event space $\quad P(B_i) > 0 \ \forall i$

$$E[X] = \sum_{i=1}^{n} P(B_i)\, E[X|B_i]$$

$$E[X] = \int_{-\infty}^{\infty} E[X|Y=y]\, f_Y(y)\,dy.$$

Independence

- X, Y indep. if

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) \quad \forall x, y$$

$$f_{X|Y}(x|y) = f_X(x) \quad \forall y \text{ with } f_Y(y) > 0$$
all x

$$f_{Y|X}(y|x) = f_Y(y) \quad \forall x \text{ with } f_X(x) > 0$$
all y.

- independence $\Rightarrow$ $F_{X,Y}(x,y) = F_X(x) F_Y(y)$

Continuous Bayes Rule

$X \sim f_X(x)$ unobserved phenomenon

Make noisy measurement $Y \sim f_{Y|X}$ cond. PDF

$$\xrightarrow[f_X(x)]{X} \boxed{\text{measurement}} \xrightarrow[f_{Y|X}(y|x)]{Y} \boxed{\text{Inference}} \longrightarrow f_{X|Y}(x|y)$$

$$f_{X|Y}(x|y) = \frac{f_X(x)\, f_{Y|X}(y|x)}{f_Y(y)}$$

$$= \frac{f_X(x)\, f_{Y|X}(y|x)}{\int_{-\infty}^{\infty} f_X(t)\, f_{Y|X}(y|t)\,dt}.$$

Inference about discrete r.v.

- Unobserved phenom is discrete (disease, no disease)

  $P(A)$, $f_{Y|A}(y)$, $f_{Y|A^c}(y)$

  event $\longrightarrow$

  want $P(A \mid Y=y)$

- Suppose A has form $\{N=n\}$, $N \sim p_N$ (PMF)

$$P(N=n \mid Y=y) = \frac{p_N(n)\, f_{Y|N}(y|n)}{\sum_i p_N(i)\, f_{Y|N}(y|i)}$$