# 18-661 Introduction to Machine Learning

## Logistic Regression

Spring 2023

ECE – Carnegie Mellon University

# Outline

1. Review of Naïve Bayes

2. Logistic Regression Model

3. Loss Function and Parameter Estimation

# Review of Naïve Bayes

# How to Tell Spam from Ham Emails?

FROM THE DESK OF MR. AMINU SALEH
DIRECTOR, FOREIGN OPERATIONS DEPARTMENT
AFRI BANK PLC
Afribank Plaza,
14th Floormoney344.jpg
51/55 Broad Street,
P.M.B 12021 Lagos-Nigeria

Attention: Honorable Beneficiary,

IMMEDIATE PAYMENT NOTIFICATION VALUED AT **US$10 MILLION**


Hi Virginia,

Can we meet today at 2pm?

thanks,

Carlee

# Bag of Words Model

| | |
|---|---|
| free | 100 |
| money | 2 |
| ⋮ | ⋮ |
| account | 2 |
| ⋮ | ⋮ |

$$\begin{pmatrix} 100 \times 0.2 \\ 2 \times 0.3 \\ \vdots \\ 2 \times 0.3 \\ \vdots \end{pmatrix}$$

$$\begin{pmatrix} 100 \times 0.01 \\ 2 \times 0.02 \\ \vdots \\ 2 \times 0.01 \\ \vdots \end{pmatrix}$$

**different weights for spam and ham: representing how compatible the word pattern is to each category**

**= 3.2**


**= 1.03**

# Naïve Bayes Model for Identifying Spam

- **Class label**: binary
  - $y = \{$ spam, ham $\}$
- **Features**: word counts in the document (bag-of-words)
  - $\mathbf{x} = \{$('free', 100), ('lottery', 5), ('money', 10)$\}$
  - Each pair is in the format of (word$_i$, #word$_i$), namely, a unique word in the dictionary, and the number of times it shows up
- **Model**

$$p(\mathbf{x}|spam) = p(\text{`}free'|spam)^{100} p(\text{`}lottery'|spam)^{5} p(\text{`}money'|spam)^{10} \cdots$$

  - Choose the "most likely" option: $p(\mathbf{x}|spam)p(spam)$ vs. $p(\mathbf{x}|ham)p(ham)$

These conditional probabilities are the parameters we need to estimate

# Why Is This Naïve?

- Strong assumption of conditional independence:

$$p(\text{word}_i, \text{word}_j|y) = p(\text{word}_i|y)p(\text{word}_j|y)$$

- Previous example:

$$p(\mathbf{x}|spam) = p(\text{`}free'|spam)^{100}p(\text{`}lottery'|spam)^{5}p(\text{`}money'|spam)^{10}\cdots$$

- Independence across different words as well as multiple occurrences of the same word
- This assumption makes estimation much easier (as we'll see)

# Bayes Classification Rule

**MAP rule:** For any document $\mathbf{x}$, we want to compare

$$p(\text{spam}|\mathbf{x}) \text{ versus } p(\text{ham}|\mathbf{x})$$

Recall that by Bayes rule we have:

$$p(\text{spam}|\mathbf{x}) = \frac{p(\mathbf{x}|\text{spam})p(\text{spam})}{p(\mathbf{x})}$$

$$p(\text{ham}|\mathbf{x}) = \frac{p(\mathbf{x}|\text{ham})p(\text{ham})}{p(\mathbf{x})}$$

Denominators are same, and easier to compute logarithms, so instead we compare:

$$\log[p(\mathbf{x}|\text{spam})p(\text{spam})] \quad \text{versus} \quad \log[p(\mathbf{x}|\text{ham})p(\text{ham})]$$

# Classifier in Linear Form

$$\log[p(\mathbf{x}|\text{spam})p(\text{spam})] = \log\left[\prod_i p(\text{word}_i|\text{spam})^{\#\text{word}_i} p(\text{spam})\right]$$
$$= \sum_i (\#\text{word}_i) \log p(\text{word}_i|\text{spam}) + \log p(\text{spam})$$

Similarly, we have

$$\log[p(\mathbf{x}|\text{ham})p(\text{ham})] = \sum_i (\#\text{word}_i) \log p(\text{word}_i|\text{ham}) + \log p(\text{ham})$$

We're back to the idea of comparing weighted sums of word occurrences!

$\log p(spam)$ *and* $\log p(ham)$ *are called "priors" (in our initial example we did not include them but they are important!)*

# Estimating the Conditional and Prior Probabilities

- Collect a lot of ham and spam emails as training examples
- Estimate the "prior"

$$p(\text{ham}) = \frac{\#\text{of ham emails}}{\#\text{of emails}}, \quad p(\text{spam}) = \frac{\#\text{of spam emails}}{\#\text{of emails}}$$

- Estimate the weights, e.g., $p(\text{funny\_word}|\text{ham})$

$$p(\text{funny\_word}|\text{ham}) = \frac{\#\text{of funny\_word in ham emails}}{\#\text{of words in ham emails}}$$

$$p(\text{funny\_word}|\text{spam}) = \frac{\#\text{of funny\_word in spam emails}}{\#\text{of words in spam emails}}$$

- Use Laplacian smoothing to avoid these probabilities being 0 for any word

# Missing Features: Some Words Never Occur in Ham Emails

| | free | bank | meet | time | y |
|---|---|---|---|---|---|
| Email 1 | 5 | 3 | 1 | 1 | Spam |
| Email 2 | 4 | 2 | 1 | 1 | Spam |
| Email 3 | 2 | 0 | 2 | 3 | Ham |
| Email 4 | 1 | 0 | 3 | 2 | Ham |

Find ML estimates of parameters $\pi_c$ and $\theta_{ck}$

In this training phase, we can just use all available values and ignore missing values

$$\theta_{spam,free} = Pr(free|spam) = 9/18 \qquad \theta_{ham,free} = Pr(free|ham) = 3/13$$
$$\theta_{spam,bank} = Pr(bank|spam) = 5/18 \qquad \theta_{ham,bank} = Pr(bank|ham) = 0/13$$
$$\theta_{spam,meet} = Pr(meet|spam) = 2/18 \qquad \theta_{ham,meet} = Pr(meet|ham) = 5/13$$
$$\theta_{spam,time} = Pr(time|spam) = 2/18 \qquad \theta_{ham,time} = Pr(time|ham) = 5/13$$
$$\pi_{spam} = Pr(spam) = 2/4 \qquad \pi_{ham} = Pr(ham) = 2/4$$

# Missing Features: Test Phase

$$\theta_{spam,free} = Pr(free|spam) = 9/18 \qquad \theta_{ham,free} = Pr(free|ham) = 3/13$$
$$\theta_{spam,bank} = Pr(bank|spam) = 5/18 \qquad \theta_{ham,bank} = Pr(bank|ham) = 0/13$$
$$\theta_{spam,meet} = Pr(meet|spam) = 2/18 \qquad \theta_{ham,meet} = Pr(meet|ham) = 5/13$$
$$\theta_{spam,time} = Pr(time|spam) = 2/18 \qquad \theta_{ham,time} = Pr(time|ham) = 5/13$$
$$\pi_{spam} = Pr(spam) = 2/4 \qquad \pi_{ham} = Pr(ham) = 2/4$$

New email with the word counts (free, bank, meet, time) = (1,3,4,2),

$$\begin{aligned}\log \Pr(ham|\mathbf{x}) &\propto \log \Pr(ham) \cdot \Pr(\mathbf{x}|ham) \\ &= \log\left(\frac{2}{4} \cdot \left(\frac{3}{13}\right)\left(\frac{0}{13}\right)^3\left(\frac{5}{13}\right)^4\left(\frac{5}{13}\right)^2\right) \\ &= -\infty\end{aligned}$$

Problem: The email is ALWAYS classified as spam. Just because an event has not happened yet, doesn't mean that it won't ever happen.

# Dealing with Missing Features: Solution 1

Remove the features that take zero values for one or more classes

| | free | bank | meet | time | y |
|---|---|---|---|---|---|
| Email 1 | 5 | 3 | 1 | 1 | Spam |
| Email 2 | 4 | 2 | 1 | 1 | Spam |
| Email 3 | 2 | 0 | 2 | 3 | Ham |
| Email 4 | 1 | 0 | 3 | 2 | Ham |

Remove the 'bank' column

We can then use the same procedure as before to learn the parameters, and then use these parameters to classify new emails

But then we are wasting a lot of useful data..

# Dealing with Missing Features: Solution 2

Use Laplacian smoothing: Pretend you've seen each word 1 extra time for each class (spam and ham). This is called a 'pseudo-count'.

| | free | bank | meet | time | y |
|---|---|---|---|---|---|
| Email 1 | 5 | 3 | 1 | 1 | Spam |
| Email 2 | 4 | 2 | 1 | 1 | Spam |
| Email 3 | 2 | 0 | 2 | 3 | Ham |
| Email 4 | 1 | 0 | 3 | 2 | Ham |

$$\theta_{ham,free} = Pr(free|ham) = (3+1)/(13+4)$$
$$\theta_{ham,bank} = Pr(bank|ham) = (0+1)/(13+4)$$
$$\theta_{ham,meet} = Pr(meet|ham) = (5+1)/(13+4)$$
$$\theta_{ham,time} = Pr(time|ham) = (5+1)/(13+4)$$
$$\pi_{ham} = Pr(ham) = 2/4$$

# Dealing with Missing Features: Solution 2

More generally, pretend you've seen each word $\alpha \geq 1$ extra times.

| | free | bank | meet | time | y |
|---|---|---|---|---|---|
| Email 1 | 5 | 3 | 1 | 1 | Spam |
| Email 2 | 4 | 2 | 1 | 1 | Spam |
| Email 3 | 2 | 0 | 2 | 3 | Ham |
| Email 4 | 1 | 0 | 3 | 2 | Ham |

$$p(\text{funny\_word}|\text{spam}) = \frac{\#\text{of funny\_word in spam emails} + \alpha}{\#\text{of words in spam emails} + \alpha \times \#\text{of unique words}}$$

# Laplacian Smoothing: History and Effect on ML Estimate

## History: What is the prob. that the sun will rise tomorrow?

- Given a large sample of days with the rising sun, we still can not be completely sure that the sun will still rise tomorrow

$$\Pr(\text{sun rising tomorrow} | \text{it rose } t \text{ times}) = \frac{t+1}{t+2}$$

## Effect on the ML estimate

- Laplace smoothing biases the ML estimate
- Equivalent to performing MAP estimation with a Dirichlet (multi-variate Beta) prior
- As training data size grows, the effect of Laplacian smoothing disappears

# Outline

1. Review of Naïve Bayes

2. Logistic Regression Model

3. Loss Function and Parameter Estimation

# Logistic Regression Model

# How Does Naïve Bayes Work?

Examine the classification rule for Naïve Bayes

$$y^* = \arg\max_c \left( \log \pi_c + \sum_k x_k \log \theta_{ck} \right)$$

For binary classification, we thus determine the label based on the sign of

$$\log \pi_1 + \sum_k x_k \log \theta_{1k} - \left( \log \pi_2 + \sum_k x_k \log \theta_{2k} \right)$$

This is just a linear function of the features (word-counts) $\{x_k\}$

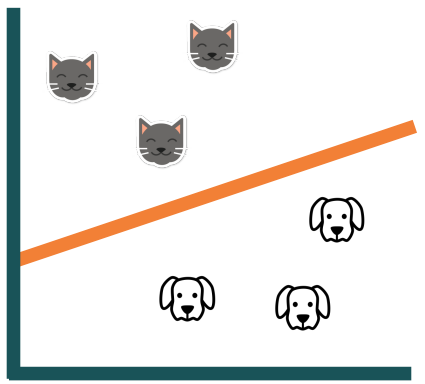
$$w_0 + \sum_k x_k w_k$$

where we "absorb" $w_0 = \log \pi_1 - \log \pi_2$ and $w_k = \log \theta_{1k} - \log \theta_{2k}$.

# Intuition: Logistic Regression

Learn the equation of the decision boundary $\mathbf{w}^\top \mathbf{x} = 0$ such that

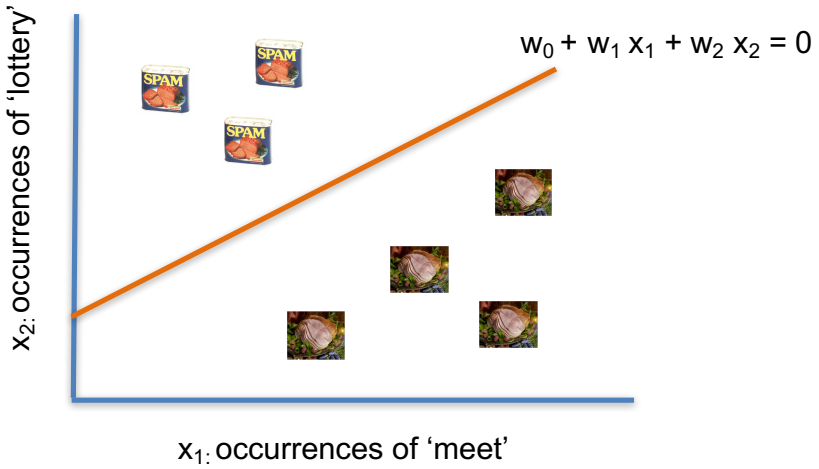
- If $\mathbf{w}^\top \mathbf{x} \geq 0$ declare $y = 1$ (cat)
- If $\mathbf{w}^\top \mathbf{x} < 0$ declare $y = 0$ (dog)



$y = 0$ for dog, $y = 1$ for cat

# Back to Spam vs. Ham Classification...

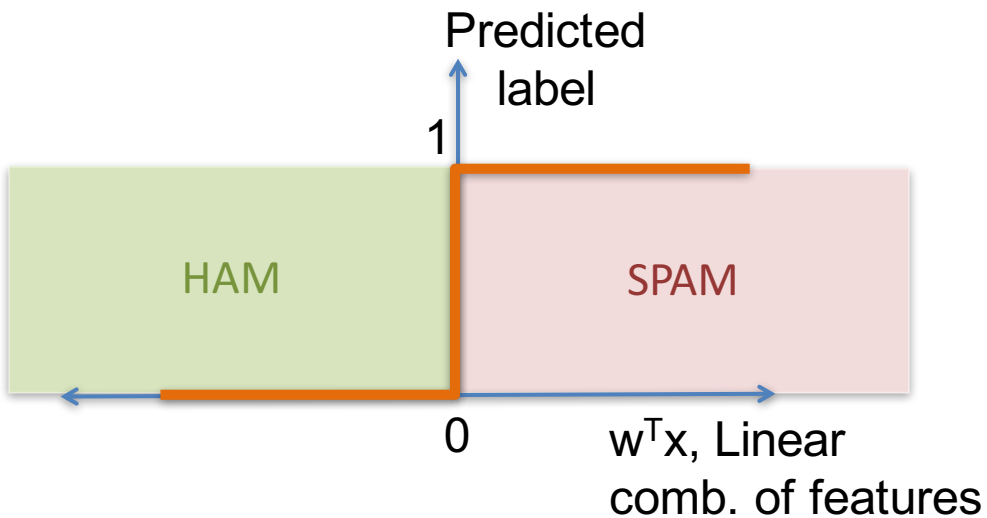
- $x_1 = \#$ of times 'meet' appears in an email
- $x_2 = \#$ of times 'lottery' appears in an email
- Define feature vector $\mathbf{x} = [1, x_1, x_2]$
- Learn the decision boundary $w_0 + w_1 x_1 + w_2 x_2 = 0$ such that
  - If $\mathbf{w}^\top \mathbf{x} \geq 0$ declare $y = 1$ (spam)
  - If $\mathbf{w}^\top \mathbf{x} < 0$ declare $y = 0$ (ham)




Key Idea: If 'meet' appears few times and 'lottery' appears many times than the email is spam

# Visualizing a Linear Classifier

- $x_1 = \#$ of times 'lottery' appears in an email
- $x_2 = \#$ of times 'meet' appears in an email
- Define feature vector $\mathbf{x} = [1, x_1, x_2]$
- Learn the decision boundary $w_0 + w_1 x_1 + w_2 x_2 = 0$ such that
  - If $\mathbf{w}^\top \mathbf{x} \geq 0$ declare $y = 1$ (spam)
  - If $\mathbf{w}^\top \mathbf{x} < 0$ declare $y = 0$ (ham)




$y = 1$ for spam, $y = 0$ for ham

# Your Turn

Suppose you see the following email:

```
CONGRATULATIONS!!  Your email address have won you the
lottery sum of US$2,500,000.00 USD to claim your prize,
   contact your office agent (Athur walter) via email
    claims2155@yahoo.com.hk or call +44 704 575 1113
```

Keywords are [lottery, prize, office, email]
The given weight vector is $\mathbf{w} = [0.3, 0.3, -0.1, -0.04]^\top$

Will we predict that the email is spam or ham?

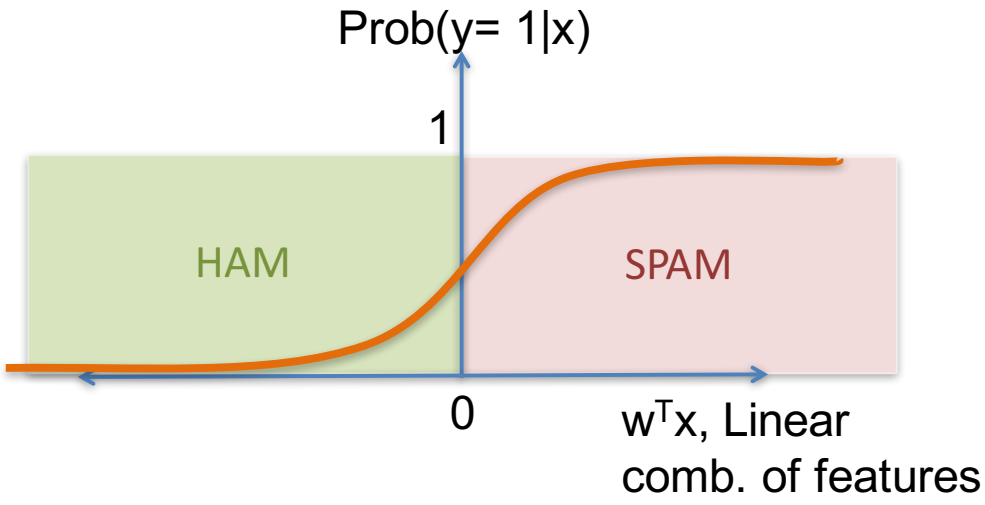
$$\mathbf{x} = [1, 1, 1, 2]^\top$$

$$\mathbf{w}^\top \mathbf{x} = 0.3 * 1 + 0.3 * 1 - 0.1 * 1 - 0.04 * 2 = 0.42 > 0$$

so we predict spam!

# Intuition: Logistic Regression

- Suppose we want to output the probability of an email being spam/ham instead of just 0 or 1
- This gives information about the confidence in the decision
- Use a function $\sigma(\mathbf{w}^\top \mathbf{x})$ that maps $\mathbf{w}^\top \mathbf{x}$ to a value between 0 and 1




Probability that predicted label is 1 (spam)

Key Problem: Finding optimal weights $\mathbf{w}$ that accurately predict this probability for a new email

# Formal Setup: Binary Logistic Classification

- Input/features: $\boldsymbol{x} = [1, x_1, x_2, \ldots x_D] \in \mathbb{R}^{D+1}$
- Output: $y \in \{0, 1\}$
- Training data: $\mathcal{D} = \{(\boldsymbol{x}_n, y_n), n = 1, 2, \ldots, N\}$
- Model:

$$p(y = 1|\boldsymbol{x}; \boldsymbol{w}) = \sigma[g(\boldsymbol{x})]$$

where

$$g(\boldsymbol{x}) = w_0 + \sum_d w_d x_d = \boldsymbol{w}^\top \boldsymbol{x}$$

and $\sigma[\cdot]$ stands for the sigmoid function

$$\sigma(a) = \frac{1}{1 + e^{-a}}$$

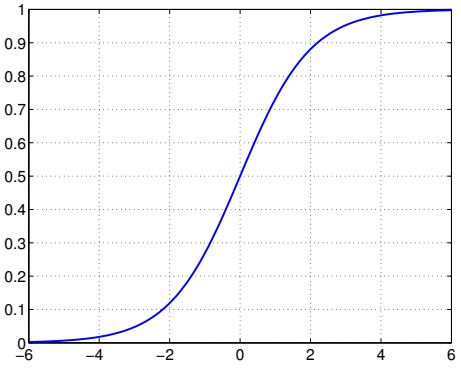
# Why the Sigmoid Function?

## What does it look like?

$$\sigma(a) = \frac{1}{1 + e^{-a}}$$

where

$$a = \boldsymbol{w}^\top \boldsymbol{x}$$

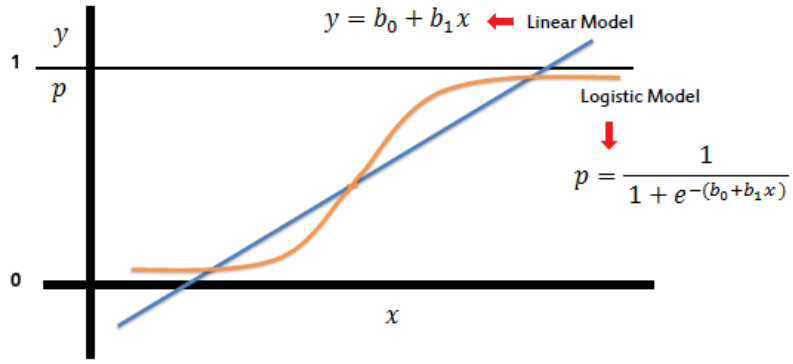



## Sigmoid properties

- Bounded between 0 and 1 ← thus, interpretable as probability
- Monotonically increasing ← thus, usable to derive classification rules
  - $\sigma(a) \geq 0.5$, positive (classify as '1')
  - $\sigma(a) < 0.5$, negative (classify as '0')
- Nice computational properties ← as we will see soon

# Comparison to Linear Regression

Sigmoid function returns values in [0,1]




Decision boundary is linear: linear classifier

# Your Turn

Suppose you see the following email:

```
CONGRATULATIONS!!  Your email address have won you the
lottery sum of US$2,500,000.00 USD to claim your prize,
   contact your office agent (Athur walter) via email
    claims2155@yahoo.com.hk or call +44 704 575 1113
```

Keywords are [lottery, prize, office, email]
The given weight vector is $\mathbf{w} = [0.3, 0.3, -0.1, -0.04]^\top$

What is the probability that the email is spam?

$$\mathbf{x} = [1, 1, 1, 2]^\top$$

$$\mathbf{w}^\top \mathbf{x} = 0.3 * 1 + 0.3 * 1 - 0.1 * 1 - 0.04 * 2 = 0.42 > 0$$

$$\Pr(y = 1|\mathbf{x}) = \sigma(\mathbf{w}^\top \mathbf{x}) = \frac{1}{1 + e^{-0.42}} = 0.603$$

# Loss Function and Parameter Estimation

# How Do We Optimize the Weight Vector w?

Learn from experience

- get a lot of spams
- get a lot of hams

But what to optimize?

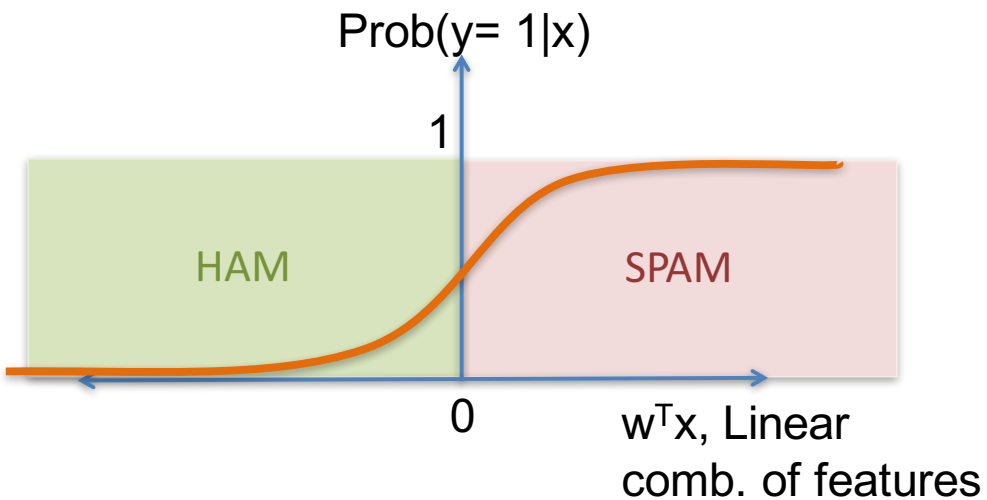

# Likelihood Function

Probability of a single training sample $(\boldsymbol{x}_n, y_n)$...

$$p(y_n|\boldsymbol{x}_n; \boldsymbol{w}) = \begin{cases} \sigma(\boldsymbol{w}^\top \boldsymbol{x}_n) & \text{if} \quad y_n = 1 \\ 1 - \sigma(\boldsymbol{w}^\top \boldsymbol{x}_n) & \text{otherwise} \end{cases}$$

Simplify, using the fact that $y_n$ is either 1 or 0

$$p(y_n|\boldsymbol{x}_n; \boldsymbol{w}) = \sigma(\boldsymbol{w}^\top \boldsymbol{x}_n)^{y_n}[1 - \sigma(\boldsymbol{w}^\top \boldsymbol{x}_n)]^{1-y_n}$$




Probability that predicted label is 1 (spam)

# Log Likelihood or Cross Entropy Error

**Log-likelihood of the whole training data $\mathcal{D}$**

$$P(\mathcal{D}) = \prod_{n=1}^{N} p(y_n|\boldsymbol{x}_n; \boldsymbol{w}) = \prod_{n=1}^{N} \left\{\sigma(\boldsymbol{w}^\top \boldsymbol{x}_n)^{y_n}[1-\sigma(\boldsymbol{w}^\top \boldsymbol{x}_n)]^{1-y_n}\right\}$$

$$\log P(\mathcal{D}) = \sum_n \{y_n \log \sigma(\boldsymbol{w}^\top \boldsymbol{x}_n) + (1-y_n)\log[1-\sigma(\boldsymbol{w}^\top \boldsymbol{x}_n)]\}$$

It is convenient to work with its negation, which is called the cross-entropy error function

$$\mathcal{E}(\boldsymbol{w}) = -\sum_n \{y_n \log \sigma(\boldsymbol{w}^\top \boldsymbol{x}_n) + (1-y_n)\log[1-\sigma(\boldsymbol{w}^\top \boldsymbol{x}_n)]\}$$

# Cross-Entropy as a Loss Function

**Supervised learning**

We aim to build a function $h(\boldsymbol{x})$ to predict the true value $y$ associated with $\boldsymbol{x}$. If we make a mistake, we incur a loss

$$\ell(h(\boldsymbol{x}), y)$$

Cross-entropy is also a sum over all data samples:

$$\mathcal{E}(\boldsymbol{w}) = -\sum_n \{y_n \log \sigma(\boldsymbol{w}^\top \boldsymbol{x}_n) + (1 - y_n) \log[1 - \sigma(\boldsymbol{w}^\top \boldsymbol{x}_n)]\}$$

where $h(\boldsymbol{x}) = \sigma\left(\boldsymbol{w}^T \boldsymbol{x}\right)$.

What is the loss function?

$$\ell(h(\boldsymbol{x}_n), y) = -\{y_n \log \sigma(\boldsymbol{w}^\top \boldsymbol{x}_n) + (1 - y_n) \log[1 - \sigma(\boldsymbol{w}^\top \boldsymbol{x}_n)]\}$$

# How to Find the Optimal Parameters for Logistic Regression?

**We will minimize the cross-entropy function**

$$\mathcal{E}(\boldsymbol{w}) = -\sum_n \{y_n \log \sigma(\boldsymbol{w}^\top \boldsymbol{x}_n) + (1 - y_n) \log[1 - \sigma(\boldsymbol{w}^\top \boldsymbol{x}_n)]\}$$

However, this function is complex and we cannot find the simple solution as we did in Naïve Bayes. So we need to use numerical methods.

- Numerical methods are messier, in contrast to cleaner closed-form solutions.
- In practice, we often have to tune a few optimization parameters — patience is necessary.
- A popular method: gradient descent and its variants.

# Gradient Descent Update for Logistic Regression

Finding the gradient of $\mathcal{E}(\boldsymbol{w})$ looks very hard, but it turns out to be simple and intuitive.

Let's start with the derivative of the sigmoid function $\sigma(a)$:

$$\begin{aligned}
\frac{d}{d\,a}\sigma(a) &= \frac{d}{d\,a}\left(1+e^{-a}\right)^{-1} \\
&= \frac{-1}{(1+e^{-a})^2}\frac{d}{d\,a}(1+e^{-a}) \\
&= \frac{e^{-a}}{(1+e^{-a})^2} \\
&= \frac{1}{1+e^{-a}}\frac{e^{-a}}{1+e^{-a}} \\
&= \frac{1}{1+e^{-a}}\frac{1+e^{-a}-1}{1+e^{-a}} \\
&= \sigma(a)[1-\sigma(a)]
\end{aligned}$$

# Gradients of the Cross-entropy Function

## Cross-entropy Error Function

$$\mathcal{E}(\boldsymbol{w}) = -\sum_n \{y_n \log \sigma(\boldsymbol{w}^\top \boldsymbol{x}_n) + (1 - y_n) \log[1 - \sigma(\boldsymbol{w}^\top \boldsymbol{x}_n)]\}$$

$$\frac{d}{d\,a}\sigma(a) = \sigma(a)[1 - \sigma(a)]$$

## Computing the gradient

$$\begin{aligned}
\frac{\partial \mathcal{E}(\boldsymbol{w})}{\partial \boldsymbol{w}} &= -\sum_n \left\{ y_n \frac{\sigma(\boldsymbol{w}^\top \boldsymbol{x}_n)[1 - \sigma(\boldsymbol{w}^\top \boldsymbol{x}_n)]}{\sigma(\boldsymbol{w}^\top \boldsymbol{x}_n)} \boldsymbol{x}_n \right. \\
&\qquad \left. - (1 - y_n) \frac{\sigma(\boldsymbol{w}^\top \boldsymbol{x}_n)[1 - \sigma(\boldsymbol{w}^\top \boldsymbol{x}_n)]}{1 - \sigma(\boldsymbol{w}^\top \boldsymbol{x}_n)} \boldsymbol{x}_n \right\} \\
&= -\sum_n \{y_n[1 - \sigma(\boldsymbol{w}^\top \boldsymbol{x}_n)]\boldsymbol{x}_n - (1 - y_n)\sigma(\boldsymbol{w}^\top \boldsymbol{x}_n)\boldsymbol{x}_n\} \\
&= \sum_n \underbrace{\{\sigma(\boldsymbol{w}^\top \boldsymbol{x}_n) - y_n\}}_{\text{Error of the } n\text{th training sample.}} \boldsymbol{x}_n
\end{aligned}$$

# Numerical Optimization

## Gradient descent for logistic regression

- Choose a proper step size $\eta > 0$
- Iteratively update the parameters following the negative gradient to minimize the error function

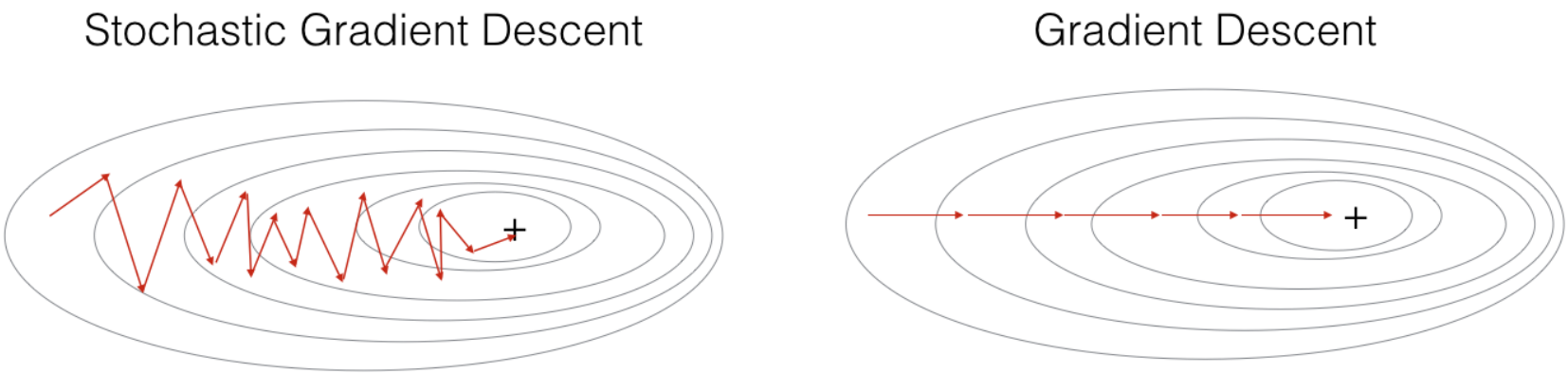
$$\boldsymbol{w}^{(t+1)} \leftarrow \boldsymbol{w}^{(t)} - \eta \sum_n \left\{ \sigma(\boldsymbol{w}^{(t)\top} \boldsymbol{x}_n) - y_n \right\} \boldsymbol{x}_n$$

## Stochastic gradient descent for logistic regression

- Choose a proper step size $\eta > 0$
- Draw a sample $n$ uniformly at random
- Iteratively update the parameters following the negative gradient to minimize the error function

$$\boldsymbol{w}^{(t+1)} \leftarrow \boldsymbol{w}^{(t)} - \eta \left\{ \sigma(\boldsymbol{w}^{(t)\top} \boldsymbol{x}_n) - y_n \right\} \boldsymbol{x}_n$$

# SGD versus Batch GD




- SGD reduces per-iteration complexity since it considers fewer samples.
- But it is noisier and can take longer to converge.

# Example: Spam Classification

| | free | bank | meet | time | y |
|---|---|---|---|---|---|
| Email 1 | 5 | 3 | 1 | 1 | Spam |
| Email 2 | 4 | 2 | 1 | 1 | Spam |
| Email 3 | 2 | 1 | 2 | 3 | Ham |
| Email 4 | 1 | 2 | 3 | 2 | Ham |

Perform gradient descent to learn weights $\mathbf{w}$

- Feature vector for email 1: $\mathbf{x}_1 = [1, 5, 3, 1, 1]^\top$
- Let $\boldsymbol{X} = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4]$, the matrix of all feature vectors.
- Initial weights $\mathbf{w} = [0.5, 0.5, 0.5, 0.5, 0.5]^\top$
- Prediction

  $$[\sigma(\mathbf{w}^\top \mathbf{x}_1), \sigma(\mathbf{w}^\top \mathbf{x}_2), \sigma(\mathbf{w}^\top \mathbf{x}_3), \sigma(\mathbf{w}^\top \mathbf{x}_4)]^\top = [0.996, 0.989, 0.989, 0.989]^\top$$

  which can be obtained by computing $\mathbf{w}^\top \mathbf{X}$ and then apply $\sigma(\cdot)$ entrywise, which we abuse the notation and write $\sigma(\boldsymbol{X}^\top \mathbf{w})$.

# Example: Spam Classification, Batch Gradient Descent

| | free | bank | meet | time | y |
|---|---|---|---|---|---|
| Email 1 | 5 | 3 | 1 | 1 | Spam |
| Email 2 | 4 | 2 | 1 | 1 | Spam |
| Email 3 | 2 | 1 | 2 | 3 | Ham |
| Email 4 | 1 | 2 | 3 | 2 | Ham |

Perform gradient descent to learn weights $\mathbf{w}$

- Prediction $\sigma(\boldsymbol{X}^\top \mathbf{w}) = [0.996, 0.989, 0.989, 0.989]^\top$
- Difference from labels $\boldsymbol{y} = [1, 1, 0, 0]^\top$ is

$$\sigma(\boldsymbol{X}^\top \mathbf{w}) - \boldsymbol{y} = [-0.004, -0.011, 0.989, 0.989]^\top$$

- Gradient of the first email,

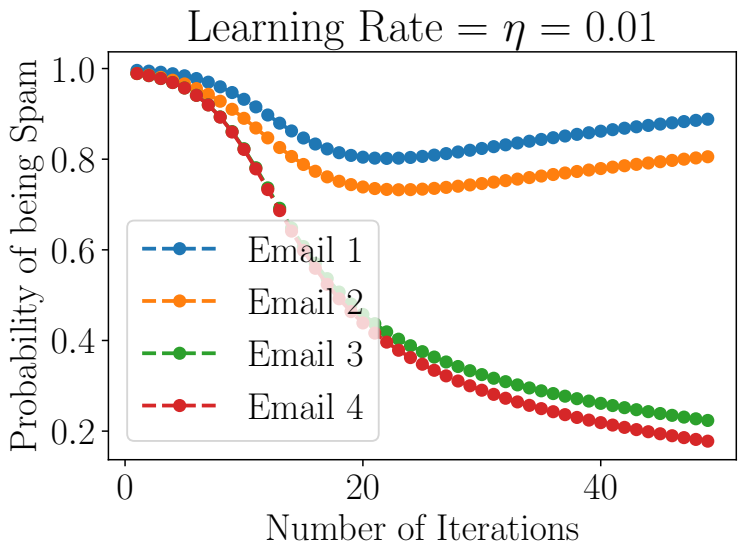
$$\boldsymbol{g}_1 = (\sigma(\mathbf{w}^\top \mathbf{x}_1) - y_1)\mathbf{x}_1 = -0.004[1, 5, 3, 1, 1]^\top$$

- $\mathbf{w} \leftarrow \mathbf{w} - \underbrace{0.01}_{\text{learning rate}} \sum_n \boldsymbol{g}_n = \mathbf{w} - \eta \boldsymbol{X}(\sigma(\boldsymbol{X}^\top \mathbf{w}) - \boldsymbol{y})$

notice the similarity with linear regression

# Example: Spam Classification, Batch Gradient Descent

| | free | bank | meet | time | y |
|---|---|---|---|---|---|
| Email 1 | 5 | 3 | 1 | 1 | Spam |
| Email 2 | 4 | 2 | 1 | 1 | Spam |
| Email 3 | 2 | 1 | 2 | 3 | Ham |
| Email 4 | 1 | 2 | 3 | 2 | Ham |




Predictions for Emails 3 and 4 are initially close to 1 (spam), but they converge towards the correct value 0 (ham)

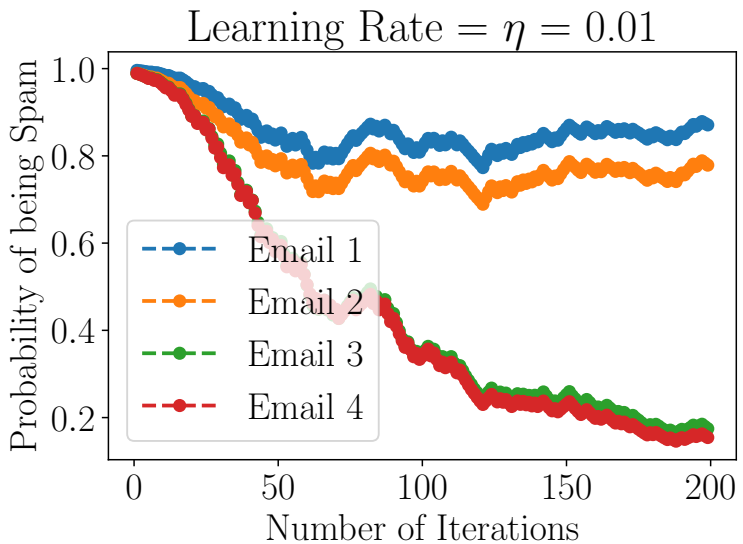
# Example: Spam Classification, Stochastic Gradient Descent

| | free | bank | meet | time | y |
|---|---|---|---|---|---|
| Email 1 | 5 | 3 | 1 | 1 | Spam |
| Email 2 | 4 | 2 | 1 | 1 | Spam |
| Email 3 | 2 | 1 | 2 | 3 | Ham |
| Email 4 | 1 | 2 | 3 | 2 | Ham |

- Prediction $\sigma(\mathbf{w}^\top \mathbf{x}_r) = 0.996$ for a randomly chosen email $r$
- Difference from label $y = 1$ is $-0.004$
- Gradient is $\boldsymbol{g}_r = (\sigma(\mathbf{w}^\top \mathbf{x}_n) - y)\mathbf{x}_r = -0.004\mathbf{x}_r$
- $\mathbf{w} \leftarrow \mathbf{w} - 0.01\boldsymbol{g}_r$

# Example: Spam Classification, Stochastic Gradient Descent

| | free | bank | meet | time | y |
|---|---|---|---|---|---|
| Email 1 | 5 | 3 | 1 | 1 | Spam |
| Email 2 | 4 | 2 | 1 | 1 | Spam |
| Email 3 | 2 | 1 | 2 | 3 | Ham |
| Email 4 | 1 | 2 | 3 | 2 | Ham |




Predictions for Emails 3 and 4 are initially close to 1 (spam), but they converge towards the correct value 0 (ham)

# Example: Spam Classification, Test Phase

| | free | bank | meet | time | y |
|---|---|---|---|---|---|
| Email 1 | 5 | 3 | 1 | 1 | Spam |
| Email 2 | 4 | 2 | 1 | 1 | Spam |
| Email 3 | 2 | 1 | 2 | 3 | Ham |
| Email 4 | 1 | 2 | 3 | 2 | Ham |

- Final $\mathbf{w} = [0.187, 0.482, 0.179, -0.512, -0.524]^\top$ after 50 batch gradient descent iterations.
- Given a new email with feature vector $\mathbf{x} = [1, 1, 3, 4, 2]$, the probability of the email being spam is estimated as $\sigma(\mathbf{w}^\top \mathbf{x}) = \sigma(-1.889) = 0.13$.
- Since this is less than 0.5 we predict ham.

# Contrast Naïve Bayes and Logistic Regression

Both classification models are linear functions of features

**Joint vs. conditional distribution**
Naive Bayes models the joint distribution: $P(X, Y) = P(Y)P(X|Y)$

Logistic regression models the conditional distribution: $P(Y|X)$

**Correlated vs. independent features**
Naive Bayes assumes independence of features and multiple occurences

Logistic Regression implicitly captures correlations when training weights

**Generative vs. Discriminative**
NB is a generative model, LR is a discriminative model

# Generative Model vs. Discriminative Model

$\{x : P(Y = 1|X = x) = P(Y = 0|X = x)\}$ is called the decision boundary of our data.

**Generative classifiers**
Model the class-conditional densities $P(Y|X = x)$ explicitly:

$$P(Y = 1|X = x) = \frac{P(X = x|Y = 1)P(Y = 1)}{P(X = x|Y = 1)P(Y = 1) + P(X = x|Y = 0)P(Y = 0)}$$

This means we need to separately estimate both $P(X|Y)$ and $P(Y)$.

**Discriminative classifier**
Directly model the decision boundary and avoid estimating the conditional probabilities.

# Logistic Regression vs. Linear Regression

| | **Logistic regression** | **Linear regression** |
|---|---|---|
| Training data | $(\boldsymbol{x}_n, y_n), y_n \in \{0, 1\}$ | $(\boldsymbol{x}_n, y_n), y_n \in \mathbb{R}$ |
| Loss function | cross-entropy | RSS |
| Interpretation of $y_n \vert \boldsymbol{x}_n, \boldsymbol{w}$ | $\sim \text{Ber}(\sigma(\boldsymbol{w}^\top \boldsymbol{x}_n))$ | $\sim \mathcal{N}(\boldsymbol{w}^\top \boldsymbol{x}_n, \sigma^2)$ |
| Gradient per sample | $\left(\sigma(\boldsymbol{x}_n^\top \boldsymbol{w}) - y_n\right) \boldsymbol{x}_n$ | $\left(\boldsymbol{x}_n^\top \boldsymbol{w} - y_n\right) \boldsymbol{x}_n$ |

**Cross-entropy loss function (logistic regression):**

$$\mathcal{E}(\boldsymbol{w}) = -\sum_n \{y_n \log \sigma(\boldsymbol{w}^\top \boldsymbol{x}_n) + (1 - y_n) \log[1 - \sigma(\boldsymbol{w}^\top \boldsymbol{x}_n)]\}$$

**RSS loss function (linear regression):**

$$RSS(\boldsymbol{w}) = \frac{1}{2} \sum_n (y_n - \boldsymbol{w}^\top \boldsymbol{x}_n)^2$$

# You Should Know

**Setup for binary classification**

- Logistic Regression models conditional distribution as: $p(y = 1|\boldsymbol{x}; \boldsymbol{w}) = \sigma[g(\boldsymbol{x})]$ where $g(\boldsymbol{x}) = \boldsymbol{w}^\top \boldsymbol{x}$
- Linear decision boundary: $g(\boldsymbol{x}) = \boldsymbol{w}^\top \boldsymbol{x} = 0$

**Minimizing cross-entropy error (negative log-likelihood)**

- $\mathcal{E}(b, \boldsymbol{w}) = -\sum_n \{y_n \log \sigma(b + \boldsymbol{w}^\top \boldsymbol{x}_n) + (1 - y_n) \log[1 - \sigma(b + \boldsymbol{w}^\top \boldsymbol{x}_n)]\}$
- No closed form solution; must rely on iterative solvers

**Numerical optimization**

- Gradient descent: simple, scalable to large-scale problems
  - Move in direction opposite of gradient!
  - Gradient of the cross-entropy error takes nice form

# What's Next?

What about when we want to predict multiple classes?

- Dog vs. cat. vs crocodile
- Movie genres (action, horror, comedy, . . . )
- Yelp ratings (1, 2, 3, 4, 5)
- Part of speech tagging (verb, noun, adjective, . . . )
- . . .