Neyman - Pearson Test (Does not assume know a ~~pri~~ prior probs)

- Examine trade-off between $Pr(e|H=0)$ and $Pr(e|H=1)$ as a fn. of threshold η. See figure for 1-D Gaussian example.
- Let $\Lambda(\underline{y}) = \dfrac{f_{\underline{Y}|H}(\underline{y}|1)}{f_{\underline{Y}|H}(\underline{y}|0)}$, $L = \Lambda(\underline{Y})$ likelihood ratio r.v.

  Let $e(\eta)$ = error event as fn. of η.

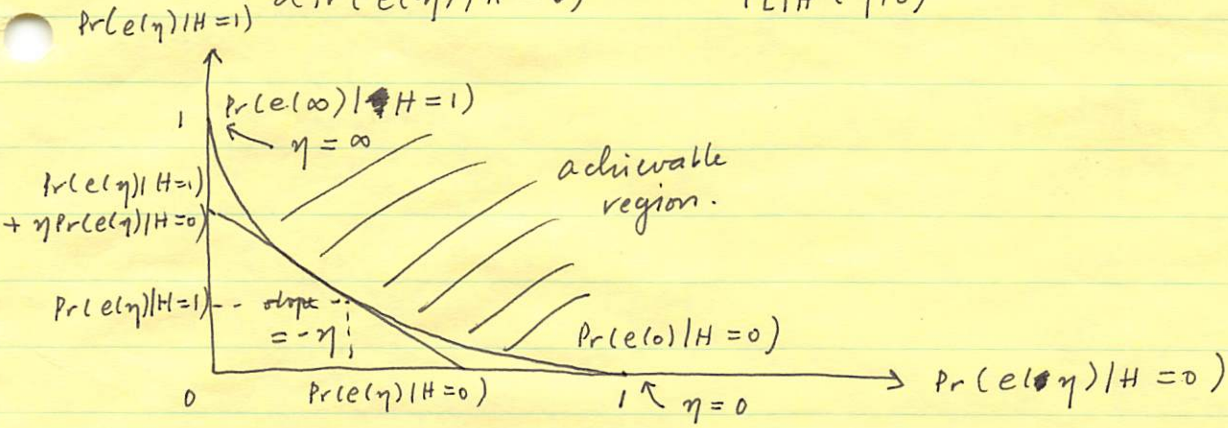
  $Pr(e(\eta)|H=0) = Pr(L \geq \eta | H=0)$, $Pr(e(\eta)|H=1) = Pr(L < \eta | H=1)$

  $$\frac{d\,Pr(e(\eta)|H=0)}{d\eta} = -f_{L|H}(\eta|0), \qquad \frac{d\,Pr(e(\eta)|H=1)}{d\eta} = f_{L|H}(\eta|1)$$

  assuming L has finite ~~non-zero~~ density under each hypothesis

- can show (see notes)

  $$\frac{d\,Pr(e(\eta)|H=1)}{d\,Pr(e(\eta)|H=0)} = -\frac{f_{L|H}(\eta|1)}{f_{L|H}(\eta|0)} = -\eta$$



- At $\eta = 0$, $Pr(e(0)|H=0) = 1$, $Pr(e(0)|H=1) = 0$

  $\eta = \infty$, $Pr(e(\infty)|H=0) = 0$, $Pr(e(\infty)|H=1) = 1$

- As $Pr(e(\eta)|H=0)$ ~~decreases~~ increases, η ~~increases~~ decreases, and slope $= -\eta$ ~~increases~~ ~~decreases~~ increases. Thus curve is convex.

- In radar, plot $1 - Pr(e(\eta)|H=1)$ (prob of detection) vs. $Pr(e(\eta)|H=0)$ (prob of false alarm) — Receiver Operating characteristic (ROC)

  flip error curve vertically around 1/2.

- Consider arbitrary test A

$$Pr(e(A)) = P_1 Pr(e(A) | H=1) + P_0 Pr(e(A) | H=0)$$
$$= P_1 [Pr(e(A) | H=1) + \eta Pr(e(A) | H=0)], \quad \eta = \frac{P_0}{P_1}$$

- Now the MAP test with threshold $\eta = \frac{P_0}{P_1}$ minimizes $Pr(e(A))$ over all tests.
  └ not assuming know a priori probs

So
$$P_1 [Pr(e(\eta) | H=1) + \eta Pr(e(\eta) | H=0)]$$
$$\le P_1 [Pr(e(A) | H=1) + \eta Pr(e(A) | H=0)] \quad (9)$$

- Now $\frac{LHS}{P_1}$ = vertical axis intercept of tangent line in Figure with slope $-\eta$.

  (9) says point $(Pr(e(A) | H=0), Pr(e(A) | H=1))$ lies above tangent line. True for any $P_0, P_1 > 0$ and so for all $0 < \eta < \infty$. So $(Pr(e(A) | H=0), Pr(e(A) | H=1))$ lies on or above all tagent lines, and thus the curve.

- So achievable region of trade-offs above curve and threshold tests

- Threshold tests are optimal: achieves best trade-off between $Pr(e(A) | H=1)$ and $Pr(e(A) | H=0)$. end of Wed.

- Neyman Pearson Test: test A that minimizes $Pr(e(A) | H=1)$ s.t. $Pr(e(A) | H=0) \le \alpha$
  (False alarm)

  This test is a threshold test: choose η s.t. $\alpha = Pr(e(\eta) | H=0)$

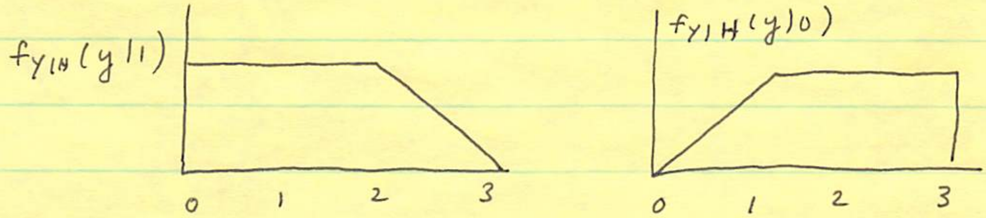
  On curve, find α on horizontal axis, value is value of error curve at α — min. $Pr(e | H=1)$ s.t. $Pr(e | H=0) \le \alpha$

  threshold = magnitude of slope at that pt.

- Derivation of $Pr(e(\eta) \mid H=1)$ as fn. of $Pr(e(\eta) \mid H=0)$ assumed likelihood ratio r.v. L has finite density everywhere.
- Suppose now distribution fn. of L, given $H=0$, contains discontinuities

This happens when $\underline{Y}$ is discrete and also when

$f_{\underline{Y}|H}(\underline{y}|1) = f_{\underline{Y}|H}(\underline{y}|0)$ for $\underline{y}$ in set of positive measure



- Suppose ~~so for~~ at some η^*, $Pr(L=\eta^* \mid H=0) > 0$

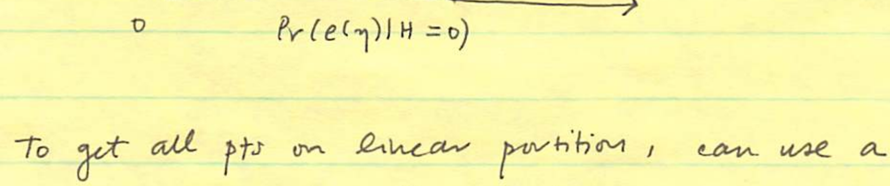
In this case, can verify that

$$Pr(L=\eta^* \mid H=1) = \eta^* \, Pr(L=\eta^* \mid H=0)$$ linear ~~as before~~

- As η goes from η^{*-} to η^{*+},

$Pr(e(\eta) \mid H=0)$ jumps down by $Pr(L=\eta^* \mid H=0)$

$Pr(e(\eta) \mid H=1)$ " up " $|\eta^*| \, Pr(L=\eta^* \mid H=0)$



- To get all pts on linear portion, can use a <u>randomized test</u>:

If $\Lambda(\underline{y}) = \eta^*$, $\hat{H} = \begin{cases} 1 & \text{w.p. } q \\ 0 & \text{w.p. } 1-q \end{cases}$

$Pr(e(\eta^*, q) \mid H=0) \nearrow$ with q
$Pr(e(\eta^*, q) \mid H=1) \searrow$ with q
} along straight line for η^* at which L is discontinuous.

- Randomized test can be applied to Neyman-Pearson setting.

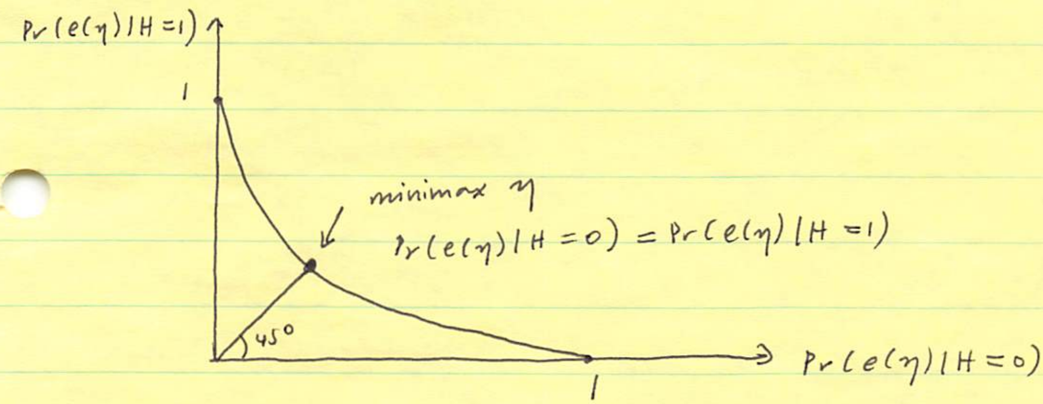
Minimax Tests

- If P_0, P_1 unknown, want to minimize max. prob of error

$$\hat{H}_{minimax} = \arg\min_{\hat{H}} \max_{i} Pr(e \mid H=i)$$

1) We choose test $\hat{H}$
2) Nature chooses $H=i$ to max. $Pr(e)$.

- adversarial setting.



- choose pt. on curve to min. $\max\{Pr(e(\eta) \mid H=0), Pr(e(\eta) \mid H=1\}$.
  Optimal pt = intersection of error curve with 45° line from origin.
- If error curve symmetric around 45°, threshold for minimax at $\eta = 1$.
  (ML Test)