MTH 304 - Differential Equations

Introductory Notes

Differential Equations is an applied field of mathematics

Because it is applied, modeling is important.

What is modeling?

It is the process of translating a real-life problem into a mathematical problem.

Quantities in any model fall into 3 categories

1. Independent variables (almost always time t).
2. Dependent variables
3. parameters.

Ex: If we are modeling the motion of a rocket, velocity, height, are variables that depend on time. the initial mass of the rocket is a parameter.

Ex: Population is another example that is modeled quite often. Parameters may be availability of food resources, the birth rate, the death rate,...

More precisely, one elementary ~~method~~ model is based on the assumption that "the rate of growth of the population is proportional to the size of the population."

Let us model this problem together.

* t = independent variable
* $p(t)$ = dependent variable (population size at time t).
* Rate of growth / Rate of change translates into a derivative $\Rightarrow \frac{dP}{dt} = p'(t)$
* proportional to $\Rightarrow$ $k.(\ )$; k = constant.

In conclusion, the model is:

$$\boxed{\frac{dP}{dt} = k.\,p(t)}$$

This is an example of a <u>differential equation</u>.

<u>Definition:</u> A differential equation is an equation that relates a dependent variable to its derivatives (of arbitrary order) and to the independent variables.

<u>Ex:</u> $\underbrace{\frac{dy}{dt} = ky}_{(a)}$; $\underbrace{\frac{dy}{dt} = 5e^{-t}}_{(b)}$; $\underbrace{\frac{d^2y}{dt^2} = 5y' - 6y}_{(c)}$.

those are examples of <u>ordinary differential</u> equations (ODE) because there is <u>one</u> independent variable in the equations (t).

Equations (a) and (b) are <u>first-order</u> ODE's

Equation (c) is <u>second-order</u> ODE.

<u>Ex</u>. Suppose $z = f(x,y)$. Here is an example of a differential equation:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

This equation is called a <u>partial</u> differential equation or <u>PDE</u>

Course Goal:

Given a differential equation, find the function (the dependent variable) that satisfies the equation.

In other words, in such equations the unknowns are the functions/dependent variables.

Solving an ODE or a PDE is finding the function or functions that satisfy the equation.

Ex: Solve $\frac{dy}{dt} = 5e^{-t}$. This is a simple exercise. All you have to do is integrate in order to find y.

$$y(t) = \int 5e^{-t}\, dt = -5e^{-t} + C.$$

As you can see we have found infinitely many solutions to this ODE. We call it a **family of solutions**.

Other examples

* $\dfrac{dy}{dt} = ky$ ← Here we cannot simply integrate, why?

$$\int \frac{dy}{dt}\,dt = \int ky\,dt$$

$$y(t) = k\int y\,dt$$ ← ? we do not know y !

However we can predict for this example what the solutions look like.

If we try to read this equation with meaning, it is saying that the derivative of y is a multiple of itself.

Q: What function when differentiated produces a multiple of itself?

A: Exponential functions.

More particularly here: $y(t) = e^{kt}$.

Indeed, if $y = e^{kt} \Rightarrow \frac{dy}{dt} = k e^{kt} = ky$,

exactly what the ODE says $\left(\frac{dy}{dt} = ky\right)$.

Moreover, if $y = Ce^{kt}$, for any constant C,

then $\frac{dy}{dt} = C.k.e^{t} = kCe^{t} = ky$, also therefore

satisfying the ODE.

Hence, for this example, the family of

solutions is: $\boxed{y = Ce^{kt}}$.

* Not all equations are as simple: $\frac{d^2y}{dt^2} = 5\frac{dy}{dt} - 6y$.

Here a family of solutions is: $\underline{\underline{y(t) = Ae^{2t} + Be^{3t}}}$.

Indeed, $y' = 2Ae^{2t} + 3Be^{3t}$

$y'' = 4Ae^{2t} + 9Be^{3t}$

$$5y' - 6y = \underbrace{10Ae^{2t}} + \overbrace{15Be^{3t}} \underbrace{- 6Ae^{2t}} \overbrace{- 6Be^{3t}}$$

$$= 4Ae^{2t} + 9Be^{3t} = y''(t) \checkmark$$

$\boxed{\text{Q. How do we find such solutions?}}$

$$* \quad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

Let $z = Ae^x \cos y + Be^x \sin y$.

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = Ae^x \cos y + Be^x \sin y.$$

$$\frac{\partial z}{\partial y} = -Ae^x \sin y + Be^x \cos y.$$

$$\frac{\partial^2 z}{\partial y^2} = -Ae^x \cos y - Be^x \sin y.$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

$\therefore$ Family of solutions is $\boxed{z = Ae^x \cos y + Be^x \sin y}$

Again, the question is <u>How</u> to find these solutions?

The Course is about Finding these solutions - fortunately, only for ODE's and <u>not</u> for PDE's.

There are <u>3 ways</u> for finding solutions:

1. **Analytic**: This involves finding the actual formulas for the dependent variable

2. **Qualitative**: This involves obtaining a rough sketch of the graph of the ~~in~~dependent variable.

3. **Numeric**: This involves doing arithmetic that approximates a numerical value of the dependent variable at some point in time.

Examples of Analytic Solutions (done previously)

① $\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 0 \Rightarrow y(t) = Ae^{2t} + Be^{3t}$

(Much of the course is about finding ways to solve analytically an ODE).

② $\frac{dy}{dt} = ky \Rightarrow y(t) = Ce^{kt}$.

Examples of Qualitative Solutions

(in much of the course, we will try to understand the qualitative behavior of the solutions)

① $\frac{dy}{dt} = ky$; $k > 0$.

How do we analyze this equation qualitatively?

(i) if $y = 0 \Rightarrow \frac{dy}{dt} = 0 \Rightarrow y$ is constant $= 0$.

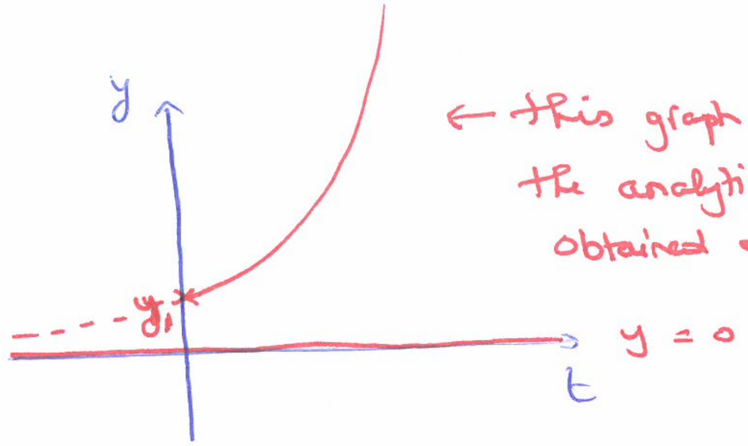
$\therefore$ one solution is the constant function $y = 0$.

(ii) if $y > 0$, say $y = y_1 > 0$ at time $t = 0$.

$\Rightarrow \frac{dy}{dt} = ky_1 > 0 \Rightarrow y$ is increasing; hence the quantity that begins at y_1 increases with time.

Notice that $\frac{d^2y}{dt^2} = k\frac{dy}{dt} = k^2y > 0 \Rightarrow y$ is concave up.

Graphically,



← this graph agrees with the analytic solution we obtained earlier.

(2) The above is sometimes called the exponential growth model.

How good is this model?

Suppose that in 1790, the US population was 3.9 million. In the year 1800, the population was 5.3 million. Assuming t = years and 1790 is our starting year ($t_0 = 1790$), and assuming an exponential growth model ($\frac{dp}{dt} = kp$; $p(t)$ = population of US in year t), estimate the population in 2020.

Solution: We know that $p(t) = Ce^{kt}$.

Here, there are two unknowns: C and k.

We know that $p(0) = 3.9 \Rightarrow \boxed{3.9 = C}$

We also know that $p(10) = 5.3$ $(1800 - 1790 = 10)$

$$\Rightarrow \quad 5.3 = 3.9\, e^{10k} \Rightarrow 10k = \ln\left(\frac{5.3}{3.9}\right)$$

$$\Rightarrow k = \frac{1}{10}\ln\left(\frac{5.3}{3.9}\right) \approx 0.03067.$$

Hence in year 2020; ~~p(t)~~ $t = 2020 - 1790 = 230$

$$\Rightarrow P(230) = \underbrace{3.9}_{\text{in millions}} e^{230k} \approx 4,514,621,149.$$

That is almost 4 and a half billion!

Currently, the US population is around 328 millions.

This model therefore is not at all perfect, mainly we are not considering other external factors.

3 The Logistic Population Model

We shall adjust the earlier population model to take into account more external factors.

We make the following new assumptions:

(a) If the population is small, the rate of growth is proportional to the size.

(b) If population is too large to be supported by the environment and its resources, the population will decrease.

For this assumption, we introduce a new parameter N, called the carrying capacity.

Mathematically, our model take the following form:

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P.$$

~~If $P < N$ (P is small) $\Rightarrow$~~

If P is very small (compared to N), then $1 - \frac{P}{N} \approx 1$

and $\frac{dP}{dt} \approx kP$ (assumption 1).

If $P > N \Rightarrow 1 - \frac{P}{N} < 0 \Rightarrow \frac{dP}{dt} < 0$ and $P(t)$ decreases (assumption 2)

Now if $P < N$ (but not too small), then

$1 - \frac{P}{N} > 0$, $\frac{dP}{dt} > 0$ and $P(t) \nearrow$.

We analyse this model qualitatively:

We shall plot the graph of $\frac{dP}{dt}$, the derivative of $P(t)$, and conclude from it the qualitative behavior of $P(t)$ ($\nearrow$, $\searrow$, long-term, ---etc.--).

$$\frac{dP}{dt} = \underbrace{k\left(1 - \frac{P}{N}\right)P = f(P)}$$

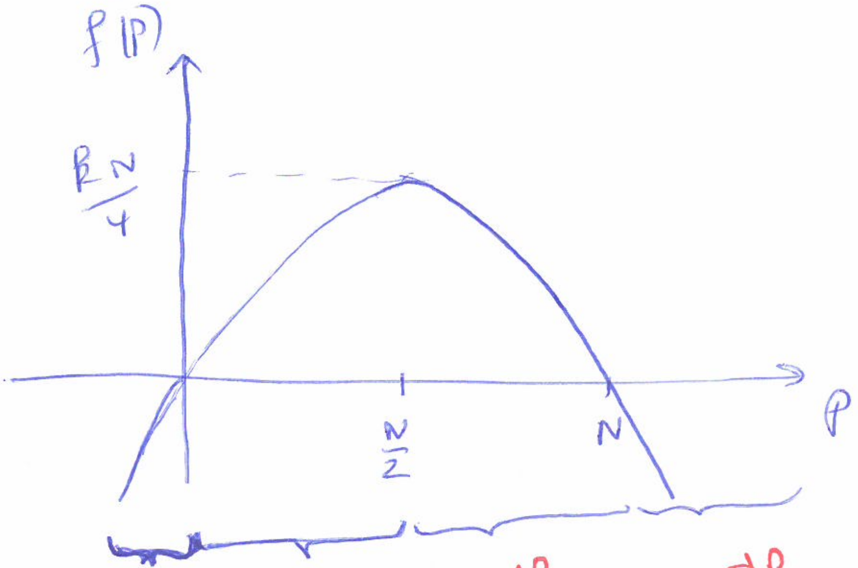
we plot this parabola in the $P - f(P)$ plane.

$$f'(P) = k\left(1 - \frac{P}{N}\right) - \frac{k}{N}P = k\left(1 - \frac{2P}{N}\right) = 0 \Rightarrow P = \frac{N}{2}$$

$P = \frac{N}{2} \Rightarrow f\left(\frac{N}{2}\right) = k\left(1 - \frac{1}{2}\right)\frac{N}{2} = \frac{kN}{4}$

$\therefore \left(\frac{N}{2}, \frac{kN}{4}\right)$ is a max.

Furthermore, $f(P) = 0$ if $P = 0$ or $P = N$.



$\frac{dP}{dt} < 0$ $\quad$ $\frac{dP}{dt} > 0$ and $\nearrow$ $\quad$ $\frac{dP}{dt} > 0$ and $\searrow$ $\quad$ $\frac{dP}{dt} < 0$.

$\therefore$ If $P < 0$, $\frac{dP}{dt} < 0$ and $P(t) \searrow$.

the more $P < 0$, the more $\frac{dP}{dt}$ is negative, and hence the decrease is concave up.

If ~~P~~ $P > N$, $\frac{dP}{dt} < 0 \Rightarrow P(t) \searrow$.

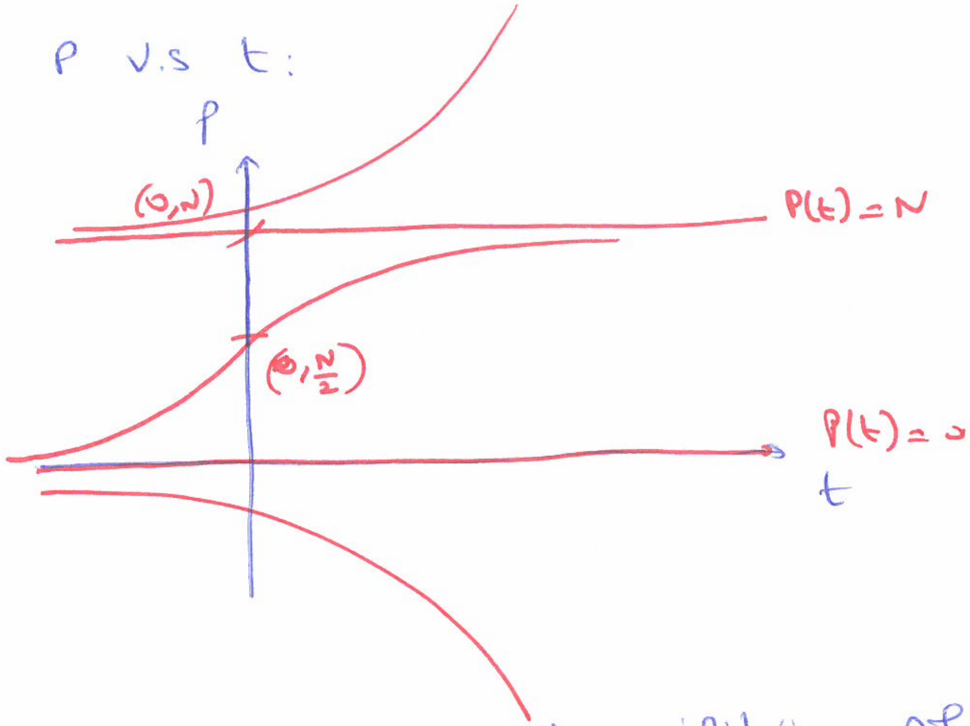
the bigger P, the more $\frac{dP}{dt}$ is negative, and also the decrease is concave up.

If $0 < P < \frac{N}{2}$, $\frac{dP}{dt} > 0$ and increase $\Rightarrow P(t) \nearrow$ and concave up

If $\frac{N}{2} < P < N$, $\frac{dP}{dt} > 0$ and decreases $\Rightarrow P(t) \nearrow$ and concave down

Finally, if $P(t) = 0$ or $P(t) = N$, $\frac{dP}{dt} = 0$ and $P(t)$ will remain constant.

$\therefore$ Graph of P v.s t:



$P(t) = 0$ and $P(t) = N$ are called equilibrium solutions.