The most important elementary applications of the Laplace transform are in the study of mechanical vibrations and in the analysis of electric circuits; the governing equations were derived in Section 3.7. A vibrating spring–mass system has the equation of motion

$$m\frac{d^2u}{dt^2} + \gamma\frac{du}{dt} + ku = F(t), \tag{31}$$

where m is the mass, γ the damping coefficient, k the spring constant, and $F(t)$ the applied external force. The equation that describes an electric circuit containing an inductance L, a resistance R, and a capacitance C (an LRC circuit) is

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t), \tag{32}$$

where $Q(t)$ is the charge on the capacitor and $E(t)$ is the applied voltage. In terms of the current $I(t) = dQ(t)/dt$, we can differentiate Eq. (32) and write

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}(t). \tag{33}$$

Suitable initial conditions on u, Q, or I must also be prescribed.

We have noted previously, in Section 3.7, that Eq. (31) for the spring–mass system and Eqs. (32) or (33) for the electric circuit are identical mathematically, differing only in the interpretation of the constants and variables appearing in them. There are other physical problems that also lead to the same differential equation. Thus, once the mathematical problem is solved, its solution can be interpreted in terms of whichever corresponding physical problem is of immediate interest.

In the problem lists following this and other sections in this chapter are numerous initial value problems for second order linear differential equations with constant coefficients. Many can be interpreted as models of particular physical systems, but usually we do not point this out explicitly.

PROBLEMS

In each of Problems 1 through 10, find the inverse Laplace transform of the given function.

1. $F(s) = \dfrac{3}{s^2 + 4}$
2. $F(s) = \dfrac{4}{(s-1)^3}$
3. $F(s) = \dfrac{2}{s^2 + 3s - 4}$
4. $F(s) = \dfrac{3s}{s^2 - s - 6}$
5. $F(s) = \dfrac{2s + 2}{s^2 + 2s + 5}$
6. $F(s) = \dfrac{2s - 3}{s^2 - 4}$
7. $F(s) = \dfrac{2s + 1}{s^2 - 2s + 2}$
8. $F(s) = \dfrac{8s^2 - 4s + 12}{s(s^2 + 4)}$
9. $F(s) = \dfrac{1 - 2s}{s^2 + 4s + 5}$
10. $F(s) = \dfrac{2s - 3}{s^2 + 2s + 10}$

In each of Problems 11 through 23, use the Laplace transform to solve the given initial value problem.

11. $y'' - y' - 6y = 0; \qquad y(0) = 1, \quad y'(0) = -1$
12. $y'' + 3y' + 2y = 0; \qquad y(0) = 1, \quad y'(0) = 0$

13. $y'' - 2y' + 2y = 0; \qquad y(0) = 0, \quad y'(0) = 1$
14. $y'' - 4y' + 4y = 0; \qquad y(0) = 1, \quad y'(0) = 1$
15. $y'' - 2y' + 4y = 0; \qquad y(0) = 2, \quad y'(0) = 0$
16. $y'' + 2y' + 5y = 0; \qquad y(0) = 2, \quad y'(0) = -1$
17. $y^{(4)} - 4y''' + 6y'' - 4y' + y = 0; \qquad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 1$
18. $y^{(4)} - y = 0; \qquad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1, \quad y'''(0) = 0$
19. $y^{(4)} - 4y = 0; \qquad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2, \quad y'''(0) = 0$
20. $y'' + \omega^2 y = \cos 2t, \quad \omega^2 \neq 4; \qquad y(0) = 1, \quad y'(0) = 0$
21. $y'' - 2y' + 2y = \cos t; \qquad y(0) = 1, \quad y'(0) = 0$
22. $y'' - 2y' + 2y = e^{-t}; \qquad y(0) = 0, \quad y'(0) = 1$
23. $y'' + 2y' + y = 4e^{-t}; \qquad y(0) = 2, \quad y'(0) = -1$

In each of Problems 24 through 27, find the Laplace transform $Y(s) = \mathcal{L}\{y\}$ of the solution of the given initial value problem. A method of determining the inverse transform is developed in Section 6.3. You may wish to refer to Problems 21 through 24 in Section 6.1.

24. $y'' + 4y = \begin{cases} 1, & 0 \le t < \pi, \\ 0, & \pi \le t < \infty; \end{cases} \qquad y(0) = 1, \quad y'(0) = 0$

25. $y'' + y = \begin{cases} t, & 0 \le t < 1, \\ 0, & 1 \le t < \infty; \end{cases} \qquad y(0) = 0, \quad y'(0) = 0$

26. $y'' + 4y = \begin{cases} t, & 0 \le t < 1, \\ 1, & 1 \le t < \infty; \end{cases} \qquad y(0) = 0, \quad y'(0) = 0$

27. $y'' + y = \begin{cases} t, & 0 \le t < 1, \\ 2 - t, & 1 \le t < 2, \\ 0, & 2 \le t < \infty; \end{cases} \qquad y(0) = 0, \quad y'(0) = 0$

28. The Laplace transforms of certain functions can be found conveniently from their Taylor series expansions.
    (a) Using the Taylor series for $\sin t$

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!},$$

and assuming that the Laplace transform of this series can be computed term by term, verify that

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}, \qquad s > 1.$$

(b) Let

$$f(t) = \begin{cases} (\sin t)/t, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

Find the Taylor series for f about $t = 0$. Assuming that the Laplace transform of this function can be computed term by term, verify that

$$\mathcal{L}\{f(t)\} = \arctan(1/s), \qquad s > 1.$$

(c) The Bessel function of the first kind of order zero, J_0, has the Taylor series (see Section 5.7)

$$J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}.$$

Assuming that the following Laplace transforms can be computed term by term, verify that

$$\mathcal{L}\{J_0(t)\} = (s^2+1)^{-1/2}, \qquad s > 1$$

and

$$\mathcal{L}\{J_0(\sqrt{t})\} = s^{-1}e^{-1/(4s)}, \qquad s > 0.$$

Problems 29 through 37 are concerned with differentiation of the Laplace transform.

29. Let

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt.$$

It is possible to show that as long as f satisfies the conditions of Theorem 6.1.2, it is legitimate to differentiate under the integral sign with respect to the parameter s when $s > a$.

(a) Show that $F'(s) = \mathcal{L}\{-tf(t)\}$.

(b) Show that $F^{(n)}(s) = \mathcal{L}\{(-t)^n f(t)\}$; hence differentiating the Laplace transform corresponds to multiplying the original function by $-t$.

In each of Problems 30 through 35, use the result of Problem 29 to find the Laplace transform of the given function; a and b are real numbers and n is a positive integer.

30. $f(t) = te^{at}$
31. $f(t) = t^2 \sin bt$
32. $f(t) = t^n$
33. $f(t) = t^n e^{at}$
34. $f(t) = te^{at} \sin bt$
35. $f(t) = te^{at} \cos bt$
36. Consider Bessel's equation of order zero

$$ty'' + y' + ty = 0.$$

Recall from Section 5.7 that $t = 0$ is a regular singular point for this equation, and therefore solutions may become unbounded as $t \to 0$. However, let us try to determine whether there are any solutions that remain finite at $t = 0$ and have finite derivatives there. Assuming that there is such a solution $y = \phi(t)$, let $Y(s) = \mathcal{L}\{\phi(t)\}$.

(a) Show that $Y(s)$ satisfies

$$(1+s^2)Y'(s) + sY(s) = 0.$$

(b) Show that $Y(s) = c(1+s^2)^{-1/2}$, where c is an arbitrary constant.

(c) Writing $(1+s^2)^{-1/2} = s^{-1}(1+s^{-2})^{-1/2}$, expanding in a binomial series valid for $s > 1$, and assuming that it is permissible to take the inverse transform term by term, show that

$$y = c\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2} = cJ_0(t),$$

where J_0 is the Bessel function of the first kind of order zero. Note that $J_0(0) = 1$ and that J_0 has finite derivatives of all orders at $t = 0$. It was shown in Section 5.7 that the second solution of this equation becomes unbounded as $t \to 0$.

37. For each of the following initial value problems, use the results of Problem 29 to find the differential equation satisfied by $Y(s) = \mathcal{L}\{\phi(t)\}$, where $y = \phi(t)$ is the solution of the given initial value problem.

(a) $y'' - ty = 0; \quad y(0) = 1, \quad y'(0) = 0$ (Airy's equation)

(b) $(1 - t^2)y'' - 2ty' + \alpha(\alpha + 1)y = 0; \quad y(0) = 0, \quad y'(0) = 1$ (Legendre's equation)

Note that the differential equation for $Y(s)$ is of first order in part (a), but of second order in part (b). This is due to the fact that t appears at most to the first power in the equation of part (a), whereas it appears to the second power in that of part (b). This illustrates that the Laplace transform is not often useful in solving differential equations with variable coefficients, unless all the coefficients are at most linear functions of the independent variable.

38. Suppose that

$$g(t) = \int_0^t f(\tau)\,d\tau.$$

If $G(s)$ and $F(s)$ are the Laplace transforms of $g(t)$ and $f(t)$, respectively, show that

$$G(s) = F(s)/s.$$

39. In this problem we show how a general partial fraction expansion can be used to calculate many inverse Laplace transforms. Suppose that

$$F(s) = P(s)/Q(s),$$

where $Q(s)$ is a polynomial of degree n with distinct zeros $r_1, \dots, r_n$, and $P(s)$ is a polynomial of degree less than n. In this case it is possible to show that $P(s)/Q(s)$ has a partial fraction expansion of the form

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s - r_1} + \cdots + \frac{A_n}{s - r_n}, \tag{i}$$

where the coefficients $A_1, \dots, A_n$ must be determined.

(a) Show that

$$A_k = P(r_k)/Q'(r_k), \qquad k = 1, \dots, n. \tag{ii}$$

Hint: One way to do this is to multiply Eq. (i) by $s - r_k$ and then to take the limit as $s \to r_k$.

(b) Show that

$$\mathcal{L}^{-1}\{F(s)\} = \sum_{k=1}^{n} \frac{P(r_k)}{Q'(r_k)} e^{r_k t}. \tag{iii}$$

6.3 Step Functions

In Section 6.2 we outlined the general procedure involved in solving initial value problems by means of the Laplace transform. Some of the most interesting elementary applications of the transform method occur in the solution of linear differential equations with discontinuous or impulsive forcing functions. Equations of this type frequently arise in the analysis of the flow of current in electric circuits or the vibrations of mechanical systems. In this section and the following ones, we develop some additional properties of the Laplace transform that are useful in the solution of such problems. Unless a specific statement is made to the contrary, all functions appearing