Systems of Differential Equations - An Appendix

Definition: A system of 2 differential equations takes the form:

$$\begin{cases} \dfrac{dx}{dt} = f(t,x,y) \\ \\ \dfrac{dy}{dt} = g(t,x,y). \end{cases} \quad (*)$$

A solution to a system is a pair of functions $(x(t), y(t))$ that satisfies both equations simultaneously.

Definition: System $(*)$ is called autonomous if

$$\left.\begin{cases} \dfrac{dx}{dt} = f(x,y) \\ \\ \dfrac{dy}{dt} = g(x,y) \end{cases}\right\} \text{ independent of } t.$$

It is called linear if $f(x,y) = ax + by$ and $g(x,y) = cx + dy$

then the system is:
$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \\ \dfrac{dy}{dt} = cx + dy \end{cases}$$

<u>Definition</u>: A solution $(x(t), y(t))$ is called an equilibrium solution if $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$.

<u>Some Examples</u>:

1. $$\begin{cases} \frac{dx}{dt} = (2 - 1.2y)x \\ \frac{dy}{dt} = (-1 + 0.9x)y . \end{cases}$$

This is a <u>non-linear</u> system. Find its equilibrium solutions.

<u>Solutions</u>: $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$

$$(2 - 1.2y)x = 0 < \begin{matrix} x = 0 \\ \text{or } y = \frac{2}{1.2} = \frac{5}{3} \end{matrix}$$

and $(-1 + 0.9x)y = 0 \longrightarrow x = 0 \therefore -y = 0 \therefore (0,0)$ is one equilibrium solution

$\searrow y = \frac{5}{3} \Rightarrow (-1 + 0.9x) \cdot \frac{5}{3} = 0$

$\Rightarrow x = \frac{1}{0.9} = \frac{10}{9}$

$\therefore \left(\frac{10}{9}, \frac{5}{3}\right)$ is another equilibrium solution.

2 $\begin{cases} \dfrac{dx}{dt} = 2x\left(1-\dfrac{x}{2}\right) - xy \\[2ex] \dfrac{dy}{dt} = 3y\left(1-\dfrac{y}{3}\right) - 2xy \end{cases}$

this is also non-linear. Find its equilibrium solutions.

Solution: $\dfrac{dx}{dt} = 0 \Rightarrow x\left[2 - x - y\right] = 0 \left\langle \begin{array}{l} x = 0 \\ \text{or } x = 2-y. \end{array} \right.$

Substitute in : $3y\left(1-\frac{y}{3}\right) - 2xy = 0.$

$x = 0 \Rightarrow 3y\left(1-y/3\right) = 0 \left\langle \begin{array}{l} y = 0 \Rightarrow (0,0) \\ y = 3 \Rightarrow (0,3) \end{array} \right)$ equilibrium.

$x = 2-y \Rightarrow 3y\left(1-\frac{y}{3}\right) - 2(2-y)y = 0$

$\Rightarrow 3y - y^2 - 4y + 2y^2 = 0$

$\Rightarrow y^2 - y = 0 \Rightarrow y = 0$ or $y = 1$

$\qquad\qquad\qquad\qquad\qquad\downarrow \qquad\qquad \downarrow$

$\qquad\qquad\qquad\qquad\qquad x = 2 \qquad\quad x = 1$

$\therefore (1,1)$ and $(2,0)$ are 2 additional equilibrium solutions.

How about solving a system of differential equations?

Examples / Special

1 $\begin{cases} \frac{dx}{dt} = -2x & \rightarrow x = k_1 e^{-2t} \\ \frac{dy}{dt} = -y & \rightarrow y = k_2 e^{-t} \end{cases}$

$\therefore$ all solution pairs take the form $(k_1 e^{-2t}, k_2 e^{-t})$.

We call this system a completely decoupled system.

2 A partially decoupled system:

$$\begin{cases} \frac{dx}{dt} = xy \\ \frac{dy}{dt} = -y+1 \rightarrow \text{linear; } \mu(t) = e^{\int dt} = e^t \end{cases}$$

$$\Rightarrow y = e^{-t}\int e^t \cdot 1\,dt$$

$$= e^{-t}(e^t + c) = \boxed{1 + Ce^{-t}}.$$

Substitute in the first equation:

$$\frac{dx}{dt} = x(1 + Ce^{-t}) \Rightarrow \frac{dx}{\cancel{dx}} = (1 + Ce^{-t})dt$$

$$\therefore \ln|x| = t - Ce^{-t} + C'$$

$$\Rightarrow \boxed{x = k e^{t - Ce^{-t}}}.$$

$\therefore$ General solutions are the pairs $(k e^{t - Ce^{-t}}, 1 + Ce^{-t})$.

[Now, go back to 28].