Modeling Problems

1) A mixing problem: A 1500 gallon initially contains 600 gallons of water with 5 lbs of salt dissolved in it. ①

Water ~~ends the~~ enters the tank at a rate of 9 gal/hr with a salt concentration of $\frac{1}{5}(1+\cos t)$ lbs/gal. ②

The well-mixed solution leaves the tank at a rate of 6 gal/hr. ③

How much salt is in the tank when it overflows?] ④.

Solution: Let $Q(t)$ denote the amount of salt in the liquid at time t.

① is equivalent to saying that $\boxed{Q(0) = 5}$

② is equivalent to: Rate-in $= \left(9 \text{ gal/h}\right) \times \left(\frac{1}{5}(1+\cos t)\right)$ lbs/gal

$$\Rightarrow \boxed{\text{Rate-in} = \frac{9}{5}(1+\cos t) \text{ lbs/hr}}$$

③ is equivalent to: $\cancel{\text{Rate-out} = \left[\frac{Q(t)}{600+3t} \text{ gal/hr}\right]}$.

$$\text{Rate-out} = \left(6 \text{ gal/hr}\right) \times \left(\frac{Q(t)}{600+3t}\right) \text{ lbs/gal}$$

$$\Rightarrow \boxed{\text{Rate-out} = \frac{2Q(t)}{200+t} \text{ lbs/hr}}$$

Note that the volume of the liquid inside the tank is increasing by 3 gallons per hour

$\therefore$ after t-minutes, the volume of liquid is $600+3t$.

(4) is equivalent to finding $Q(300)$.

$$\left[\ 300 = \frac{1500 - 600}{3}\ \right].$$

We now need to find an expression for $Q(t)$.

$$\frac{dQ}{dt} = (\text{Rate in}) - (\text{Rate out})$$

$$= \frac{9}{5}(1 + \cos t) - \frac{2}{200+t} Q(t)$$

$$\Rightarrow \frac{dQ}{dt} + \frac{2}{200+t} Q = \frac{9}{5}(1 + \cos t).$$

This is linear where $p(t) = \frac{2}{200+t}$ and $g(t) = \frac{9}{5}(1 + \cos t)$

The integrating factor is:

$$\mu(t) = e^{\int p(t)\,dt} = e^{\int \frac{2}{200+t}\,dt} = e^{2\ln|200+t|}$$

$$\Rightarrow \mu(t) = e^{\ln(200+t)^2} = (200+t)^2.$$

Hence, $$Q(t) = \frac{1}{(200+t)^2} \int (200+t)^2 . \frac{9}{5}(1 + \cos t)\, dt.$$

$$= \frac{9}{5(200+t)^2} \int (200+t)^2 (1 + \cos t)\, dt.$$

This integral is done by parts (twice!)

Let $u = (200+t)^2 \longrightarrow du = 2(200+t)$

$dv = (1+\cos t) \longrightarrow v = t + \sin t$

$$\therefore \int (200+t)^2 (1+\cos t)\,dt = (200+t)^2 (t+\sin t) - \int 2(200+t)(t+\sin t)\,dt$$

$$= (200+t)^2 (t+\sin t) - 2\int (200+t).t\,dt - 2\int (200+t)\sin t\,dt$$

$$= (200+t)^2 (t+\sin t) - 2\left[\frac{200t^2}{2} + \frac{t^3}{3}\right] - 2\underbrace{\int (200+t)\sin t\,dt}_{\text{by parts again}}$$

Now, Let $u = 200+t \longrightarrow du = dt$

$dv = \sin t \longrightarrow v = -\cos t.$

$$\therefore \int (200+t)\sin t\,dt = -(200+t)\cos t + \int \cos t\,dt$$

$$= -(200+t)\cos t + \sin t + C$$

$$\therefore Q(t) = \frac{1}{(200+t)^2}\left[(200+t)^2(t+\sin t) - 200t^2 - \frac{2}{3}t^3 + 2(200+t)\cos t - 2\sin t - 2C\right].$$

$= \cancel{(t+2\sin t)}$ $= \dots$

Now, $Q(0) = 5$

$$\Rightarrow 5 = \frac{1}{(200)^2}\left[(200)^2(0+0) - 0 + 2(200)\cos 0 - 2\sin 0 - 2C\right]$$

$$\Rightarrow 5 = \frac{1}{(200)^2}\left[400 - 2C\right] \Rightarrow (200)^2.5 = 400 - 2C$$

$$\Rightarrow 2C = 400 - 5(200)^2.$$

2) Newton's Law of Cooling

Let $T(t)$ denote the temperature of some object at time t.

Newton's law of cooling states the rate of change of the object's temperature is proportional to the difference between the temperature of the object itself and the temperature of its surrounding:

$$\frac{dT}{dt} = k(T_s - T) \; ; \quad T_s = \text{surrounding temperature}, \; k > 0.$$

Section 1.1 # 23

Suppose that the surrounding temperature is $70°F$ and $k = 0.05$. Find a formula for ~~the~~ $T(t)$.

Here $\frac{dT}{dt} = (0.05)(70 - T)$: This is separable

$$\therefore \int \frac{1}{70 - T}\, dT = \int (0.05)\, dt$$

$$\Rightarrow -\ln|70 - T| = 0.05t + C \Rightarrow \ln\left|\frac{1}{70 - T}\right| = 0.05t + C$$

$$\Rightarrow \frac{1}{70 - T} = e^{0.05t + C} = Ke^{0.05t}$$

$$\Rightarrow 70 - T = \frac{1}{K} e^{-0.05t} = \mathcal{E} e^{-0.05t}$$

$$\Rightarrow T = 70 - \mathcal{E} e^{-0.05t}$$

Suppose $T(0) = T_0$

$$\Rightarrow T_0 = 70 - C \Rightarrow C = 70 - T_0.$$

$$\therefore \quad T(t) = 70 - (70 - T_0)e^{-0.05t}$$

or $\boxed{T(t) = 70 + (T_0 - 70)e^{-0.05t}}$

In general, $\boxed{T(t) = T_s + (T_0 - T_s)e^{-kt}}$

Note: $\lim\limits_{t \to \infty} T(t) = \lim\limits_{t \to \infty} \left[T_s + \underbrace{(T_0 - T_s)e^{-kt}}_{\to 0} \right]$

since $e^{-kt} \to 0$

$\therefore \lim\limits_{t \to \infty} T(t) = T_s$. (Does this make sense? Justify).

Exercise: Choose your own T_s ~~and T...~~ and use software to plot the direction field for the example above. Also plot solutions for various values of T_0 ($T_0 > T_s$ and $T_0 < T_s$). Make some observations.