PROBLEMS

In each of Problems 1 through 4, sketch the graph of the given function. In each case determine whether f is continuous, piecewise continuous, or neither on the interval $0 \le t \le 3$.

1. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 2+t, & 1 < t \le 2 \\ 6-t, & 2 < t \le 3 \end{cases}$

2. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ (t-1)^{-1}, & 1 < t \le 2 \\ 1, & 2 < t \le 3 \end{cases}$

3. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 1, & 1 < t \le 2 \\ 3-t, & 2 < t \le 3 \end{cases}$

4. $f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 3-t, & 1 < t \le 2 \\ 1, & 2 < t \le 3 \end{cases}$

5. Find the Laplace transform of each of the following functions:
   (a) $f(t) = t$
   (b) $f(t) = t^2$
   (c) $f(t) = t^n$, where n is a positive integer
6. Find the Laplace transform of $f(t) = \cos at$, where a is a real constant.

Recall that $\cosh bt = (e^{bt} + e^{-bt})/2$ and $\sinh bt = (e^{bt} - e^{-bt})/2$. In each of Problems 7 through 10, find the Laplace transform of the given function; a and b are real constants.

7. $f(t) = \cosh bt$
8. $f(t) = \sinh bt$
9. $f(t) = e^{at} \cosh bt$
10. $f(t) = e^{at} \sinh bt$

Recall that $\cos bt = (e^{ibt} + e^{-ibt})/2$ and that $\sin bt = (e^{ibt} - e^{-ibt})/2i$. In each of Problems 11 through 14, find the Laplace transform of the given function; a and b are real constants. Assume that the necessary elementary integration formulas extend to this case.

11. $f(t) = \sin bt$
12. $f(t) = \cos bt$
13. $f(t) = e^{at} \sin bt$
14. $f(t) = e^{at} \cos bt$

In each of Problems 15 through 20, use integration by parts to find the Laplace transform of the given function; n is a positive integer and a is a real constant.

15. $f(t) = te^{at}$
16. $f(t) = t \sin at$
17. $f(t) = t \cosh at$
18. $f(t) = t^n e^{at}$
19. $f(t) = t^2 \sin at$
20. $f(t) = t^2 \sinh at$

In each of Problems 21 through 24, find the Laplace transform of the given function.

21. $f(t) = \begin{cases} 1, & 0 \le t < \pi \\ 0, & \pi \le t < \infty \end{cases}$

22. $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & 1 \le t < \infty \end{cases}$

23. $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & 1 \le t < \infty \end{cases}$

24. $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 2-t, & 1 \le t < 2 \\ 0, & 2 \le t < \infty \end{cases}$

In each of Problems 25 through 28, determine whether the given integral converges or diverges.

25. $\displaystyle\int_0^\infty (t^2+1)^{-1}\, dt$
26. $\displaystyle\int_0^\infty te^{-t}\, dt$
27. $\displaystyle\int_1^\infty t^{-2}e^t\, dt$
28. $\displaystyle\int_0^\infty e^{-t} \cos t\, dt$

29. Suppose that f and f' are continuous for $t \geq 0$ and of exponential order as $t \to \infty$. Use integration by parts to show that if $F(s) = \mathcal{L}\{f(t)\}$, then $\lim_{s\to\infty} F(s) = 0$. The result is actually true under less restrictive conditions, such as those of Theorem 6.1.2.
30. **The Gamma Function.** The gamma function is denoted by $\Gamma(p)$ and is defined by the integral
$$\Gamma(p+1) = \int_0^\infty e^{-x}x^p\,dx. \tag{i}$$
The integral converges as $x \to \infty$ for all p. For $p < 0$ it is also improper at $x = 0$, because the integrand becomes unbounded as $x \to 0$. However, the integral can be shown to converge at $x = 0$ for $p > -1$.
(a) Show that, for $p > 0$,
$$\Gamma(p+1) = p\Gamma(p).$$
(b) Show that $\Gamma(1) = 1$.
(c) If p is a positive integer n, show that
$$\Gamma(n+1) = n!.$$
Since $\Gamma(p)$ is also defined when p is not an integer, this function provides an extension of the factorial function to nonintegral values of the independent variable. Note that it is also consistent to define $0! = 1$.
(d) Show that, for $p > 0$,
$$p(p+1)(p+2)\cdots(p+n-1) = \Gamma(p+n)/\Gamma(p).$$
Thus $\Gamma(p)$ can be determined for all positive values of p if $\Gamma(p)$ is known in a single interval of unit length—say, $0 < p \leq 1$. It is possible to show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Find $\Gamma\left(\frac{3}{2}\right)$ and $\Gamma\left(\frac{11}{2}\right)$.
31. Consider the Laplace transform of t^p, where $p > -1$.
(a) Referring to Problem 30, show that
$$\mathcal{L}\{t^p\} = \int_0^\infty e^{-st}t^p\,dt = \frac{1}{s^{p+1}}\int_0^\infty e^{-x}x^p\,dx$$
$$= \Gamma(p+1)/s^{p+1}, \qquad s > 0.$$
(b) Let p be a positive integer n in part (a); show that
$$\mathcal{L}\{t^n\} = n!/s^{n+1}, \qquad s > 0.$$
(c) Show that
$$\mathcal{L}\{t^{-1/2}\} = \frac{2}{\sqrt{s}}\int_0^\infty e^{-x^2}\,dx, \qquad s > 0.$$
It is possible to show that
$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2};$$
hence
$$\mathcal{L}\{t^{-1/2}\} = \sqrt{\pi/s}, \qquad s > 0.$$
(d) Show that
$$\mathcal{L}\{t^{1/2}\} = \sqrt{\pi}/(2s^{3/2}), \qquad s > 0.$$