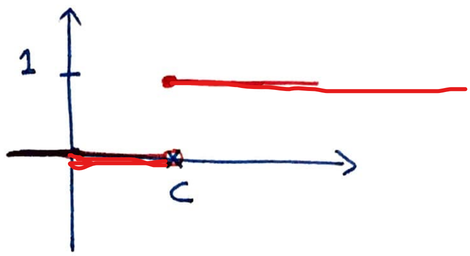
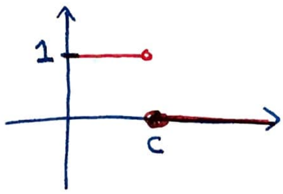
6.3. Step Functions

Definition: The unit step function (also called the Heaviside function) is defined by:

$$u_c(t) = \begin{cases} 0; & 0 \le t < c \\ 1; & t \ge c \end{cases} \quad ; \; c \ge 0.$$



Another related function is $y = 1 - u_c(t)$:

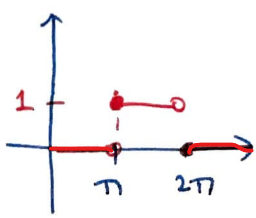


Exs:

① Sketch the graph of $y = u_\pi(t) - u_{2\pi}(t)$; $t \ge 0$.

$$u_\pi(t) = \begin{cases} 0; & 0 \le t < \pi \\ 1; & t \ge \pi \end{cases} \qquad u_{2\pi} = \begin{cases} 0; & 0 \le t < 2\pi \\ 1, & t \ge 2\pi \end{cases}$$

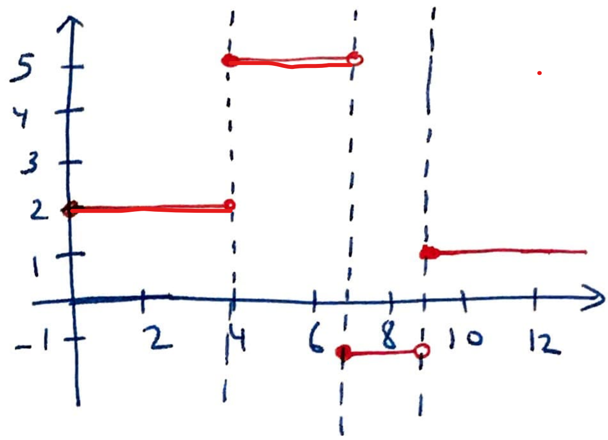
$$\Rightarrow \; u_\pi - u_{2\pi} = \begin{cases} 0; & 0 \le t < \pi \\ 1; & \pi \le t < 2\pi \\ 0; & t \ge 2\pi \end{cases}$$



② Consider the function $f(t) = \begin{cases} 2; & 0 \le t < 4 \\ 5; & 4 \le t < 7 \\ -1; & 7 \le t < 9 \\ 1, & t \geq 9. \end{cases}$

Express $f(t)$ in terms of $u_c(t)$.

Solution:



We start with:

$f_1(t) = 2$

this function agrees with $f(t)$ on $[0, 4)$.

To produce the jump of 3 units at $t=4$, we add $3u_4(t)$ to $f_1(t)$: $f_2(t) = 2 + 3u_4(t)$

$f_2(t)$ agrees with $f(t)$ on $[0, 7)$.

The negative jump of 6 units at $t=7$ corresponds to adding $-6u_7(t)$; $f_3(t) = 2 + 3u_4(t) - 6u_7(t)$

this agrees with $f(t)$ on $[0, 9)$.

Finally, we jump 2 units at $t=9$: $f_4(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t)$.

$$\Rightarrow f(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t)$$

Proposition: $\mathcal{L}(u_c(t)) = \dfrac{e^{-cs}}{s}$; $s > 0$.

Proof: $\mathcal{L}(u_c(t)) = \displaystyle\int_0^\infty e^{-st} u_c(t)\,dt$

$$= \int_0^c e^{-st} \underbrace{u_c(t)}_{=0} dt + \int_c^\infty e^{-st} \underbrace{u_c(t)}_{=1} dt$$

$$= \int_c^\infty e^{-st}\,dt = -\frac{1}{s} e^{-st}\Big|_c^\infty = \frac{e^{-cs}}{s} ; s > 0.$$

Ex: for the function $f(t)$ in example 2, find $\mathcal{L}(f)$.

Solution: $f(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t)$

$$\therefore \mathcal{L}(f(t)) = \mathcal{L}(2) + 3\mathcal{L}(u_4(t)) - 6\mathcal{L}(u_7(t)) + 2\mathcal{L}(u_9(t))$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

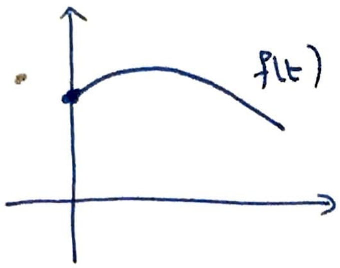
$$= \frac{2}{s} ; s \geq 0 \qquad \frac{e^{-4s}}{s} ; s > 0 \qquad \frac{e^{-7s}}{s} ; s > 0 \qquad \frac{e^{-9s}}{s} ; s > 0$$

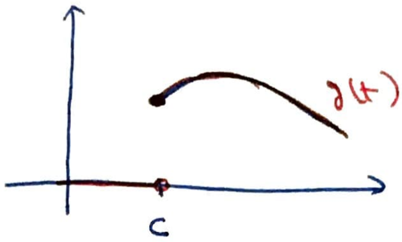
$$= \frac{2}{s} + 3\frac{e^{-4s}}{s} - 6\frac{e^{-7s}}{s} + 2\frac{e^{-9s}}{s} ; s > 0.$$

Remark:

Given a function $f(t)$; $t \geq 0$.

We often want to consider a translation of f a distance c in the positive t direction:





Hence:

$$g(t) = \begin{cases} 0; & t < c \\ f(t-c); & t \geq c \end{cases}$$

In fact, $g(t) = u_c(t) f(t-c)$.

Theorem: If $F(s) = \mathcal{L}(f)$, then:

$$\mathcal{L}\left(u_c(t) f(t-c)\right) = e^{-cs} F(s);\ s > 0.$$

Consequently, $\mathcal{L}^{-1}\left(e^{-cs} F(s)\right) = u_c(t) f(t-c)$, where

$$f(t) = \mathcal{L}^{-1}(F).$$

Ex: Let $f(t) = \begin{cases} \sin t & ; 0 \leq t < \pi/4 \\ \sin t + \cos(t - \pi/4); & t \geq \pi/4. \end{cases}$

Find $\mathcal{L}(f(t))$.

Solution: We can write $f(t)$ as follows:

$$f(t) = \sin t + g(t), \text{ where } g(t) = \begin{cases} 0; & 0 \leq t < \frac{\pi}{4} \\ \cos(t - \pi/4); & t \geq \pi/4 \end{cases}$$

$$\Rightarrow g(t) = u_{\frac{\pi}{4}}(t) \cos(t - \pi/4).$$

$$\Rightarrow \mathcal{L}(f(t)) = \mathcal{L}(\sin t) + \mathcal{L}\left(u_{\pi/4}(t)\cos(t - \pi/4)\right)$$

$$\Rightarrow \mathcal{L}(f(t)) = \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s}\mathcal{L}(\cos t)\ ;\ s > 0$$

$$= \frac{1}{s^2+1} + e^{-(\pi/4)s} \cdot \frac{s}{s^2+1}\ ;\ s > 0.$$

Ex: Find $\mathcal{L}^{-1}\left(\frac{1-e^{-2s}}{s^2}\right)$.

Solution: $\mathcal{L}^{-1}\left(\frac{1-e^{-2s}}{s^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - \mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2}\right)$

$$= t - u_2(t)\, f(t-2)$$

where $f(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t$

$$\therefore \mathcal{L}^{-1}\left(\frac{1-e^{-2s}}{s^2}\right) = t - u_2(t).\,(t-2)$$

$$= t - \begin{cases} 0; & 0 \le t < 2 \\ t-2; & t \geq 2 \end{cases}$$

$$= \begin{cases} t; & 0 \le t < 2 \\ 2; & t \geq 2. \end{cases}$$