short. Finally, the same result can be established on the basis of somewhat weaker but more complicated hypotheses, so the theorem as stated is not the most general one known, and the given conditions are sufficient, but not necessary, for the conclusion to hold.

If each of the functions $F_1, \dots, F_n$ in Eqs. (11) is a linear function of the dependent variables $x_1, \dots, x_n$, then the system of equations is said to be **linear**; otherwise, it is **nonlinear**. Thus the most general system of n first order linear equations has the form

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t), \\ x_2' &= p_{21}(t)x_1 + \cdots + p_{2n}(t)x_n + g_2(t), \\ &\ \vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t). \end{aligned} \tag{14}$$

If each of the functions $g_1(t), \dots, g_n(t)$ is zero for all t in the interval I, then the system (14) is said to be **homogeneous**; otherwise, it is **nonhomogeneous**. Observe that the systems (1) and (2) are both linear. The system (1) is nonhomogeneous unless $F_1(t) = F_2(t) = 0$, while the system (2) is homogeneous. For the linear system (14), the existence and uniqueness theorem is simpler and also has a stronger conclusion. It is analogous to Theorems 2.4.1 and 3.2.1.

Theorem 7.1.2

If the functions $p_{11}, p_{12}, \dots, p_{nn}, g_1, \dots, g_n$ are continuous on an open interval I: $\alpha < t < \beta$, then there exists a unique solution $x_1 = \phi_1(t), \dots, x_n = \phi_n(t)$ of the system (14) that also satisfies the initial conditions (13), where t_0 is any point in I, and $x_1^0, \dots, x_n^0$ are any prescribed numbers. Moreover, the solution exists throughout the interval I.

Note that, in contrast to the situation for a nonlinear system, the existence and uniqueness of the solution of a linear system are guaranteed throughout the interval in which the hypotheses are satisfied. Furthermore, for a linear system the initial values $x_1^0, \dots, x_n^0$ at $t = t_0$ are completely arbitrary, whereas in the nonlinear case the initial point must lie in the region R defined in Theorem 7.1.1.

The rest of this chapter is devoted to systems of linear first order equations (nonlinear systems are included in the discussion in Chapters 8 and 9). Our presentation makes use of matrix notation and assumes that you have some familiarity with the properties of matrices. The basic facts about matrices are summarized in Sections 7.2 and 7.3, and some more advanced material is reviewed as needed in later sections.

PROBLEMS

In each of Problems 1 through 4, transform the given equation into a system of first order equations.

1. $u'' + 0.5u' + 2u = 0$
2. $u'' + 0.5u' + 2u = 3\sin t$
3. $t^2u'' + tu' + (t^2 - 0.25)u = 0$
4. $u^{(4)} - u = 0$

In each of Problems 5 and 6, transform the given initial value problem into an initial value problem for two first order equations.

5. $u'' + 0.25u' + 4u = 2\cos 3t, \qquad u(0) = 1, \quad u'(0) = -2$
6. $u'' + p(t)u' + q(t)u = g(t), \qquad u(0) = u_0, \quad u'(0) = u'_0$
7. Systems of first order equations can sometimes be transformed into a single equation of higher order. Consider the system

$$x'_1 = -2x_1 + x_2, \qquad x'_2 = x_1 - 2x_2.$$

(a) Solve the first equation for x_2 and substitute into the second equation, thereby obtaining a second order equation for x_1. Solve this equation for x_1 and then determine x_2 also.

(b) Find the solution of the given system that also satisfies the initial conditions $x_1(0) = 2$, $x_2(0) = 3$.

(c) Sketch the curve, for $t \geq 0$, given parametrically by the expressions for x_1 and x_2 obtained in part (b).

In each of Problems 8 through 12, proceed as in Problem 7.

(a) Transform the given system into a single equation of second order.

(b) Find x_1 and x_2 that also satisfy the given initial conditions.

(c) Sketch the graph of the solution in the x_1x_2-plane for $t \geq 0$.

8. $x'_1 = 3x_1 - 2x_2, \qquad x_1(0) = 3$
   $x'_2 = 2x_1 - 2x_2, \qquad x_2(0) = \frac{1}{2}$
9. $x'_1 = 1.25x_1 + 0.75x_2, \qquad x_1(0) = -2$
   $x'_2 = 0.75x_1 + 1.25x_2, \qquad x_2(0) = 1$
10. $x'_1 = x_1 - 2x_2, \qquad x_1(0) = -1$
    $x'_2 = 3x_1 - 4x_2, \qquad x_2(0) = 2$
11. $x'_1 = 2x_2, \qquad x_1(0) = 3$
    $x'_2 = -2x_1, \qquad x_2(0) = 4$
12. $x'_1 = -0.5x_1 + 2x_2, \qquad x_1(0) = -2$
    $x'_2 = -2x_1 - 0.5x_2, \qquad x_2(0) = 2$
13. Transform Eqs. (2) for the parallel circuit into a single second order equation.
14. Show that if a_{11}, a_{12}, a_{21}, and a_{22} are constants with a_{12} and a_{21} not both zero, and if the functions g_1 and g_2 are differentiable, then the initial value problem

$$\begin{aligned} x'_1 &= a_{11}x_1 + a_{12}x_2 + g_1(t), \qquad & x_1(0) &= x_1^0 \\ x'_2 &= a_{21}x_1 + a_{22}x_2 + g_2(t), \qquad & x_2(0) &= x_2^0 \end{aligned}$$

can be transformed into an initial value problem for a single second order equation. Can the same procedure be carried out if $a_{11}, \ldots, a_{22}$ are functions of t?

15. Consider the linear homogeneous system

$$\begin{aligned} x' &= p_{11}(t)x + p_{12}(t)y, \\ y' &= p_{21}(t)x + p_{22}(t)y. \end{aligned}$$

Show that if $x = x_1(t)$, $y = y_1(t)$ and $x = x_2(t)$, $y = y_2(t)$ are two solutions of the given system, then $x = c_1x_1(t) + c_2x_2(t)$, $y = c_1y_1(t) + c_2y_2(t)$ is also a solution for any constants c_1 and c_2. This is the principle of superposition.

16. Let $x = x_1(t)$, $y = y_1(t)$ and $x = x_2(t)$, $y = y_2(t)$ be any two solutions of the linear nonhomogeneous system

$$x' = p_{11}(t)x + p_{12}(t)y + g_1(t),$$

$$y' = p_{21}(t)x + p_{22}(t)y + g_2(t).$$

Show that $x = x_1(t) - x_2(t)$, $y = y_1(t) - y_2(t)$ is a solution of the corresponding homogeneous system.

17. Equations (1) can be derived by drawing a free-body diagram showing the forces acting on each mass. Figure 7.1.3*a* shows the situation when the displacements x_1 and x_2 of the two masses are both positive (to the right) and $x_2 > x_1$. Then springs 1 and 2 are elongated and spring 3 is compressed, giving rise to forces as shown in Figure 7.1.3*b*. Use Newton's law ($F = ma$) to derive Eqs. (1).

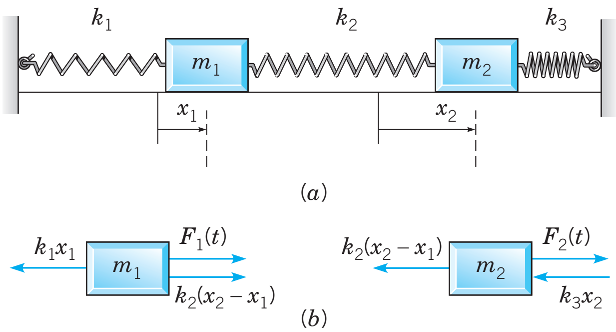

FIGURE 7.1.3 (a) The displacements x_1 and x_2 are both positive. (b) The free-body diagram for the spring–mass system.

18. Transform the system (1) into a system of first order equations by letting $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1'$, and $y_4 = x_2'$.

Electric Circuits. The theory of electric circuits, such as that shown in Figure 7.1.2, consisting of inductors, resistors, and capacitors, is based on Kirchhoff's laws: (1) The net flow of current into each node (or junction) is zero, and (2) the net voltage drop around each closed loop is zero. In addition to Kirchhoff's laws, we also have the relation between the current I in amperes through each circuit element and the voltage drop V in volts across the element:

$$V = RI, \qquad R = \text{resistance in ohms;}$$

$$C\frac{dV}{dt} = I, \qquad C = \text{capacitance in farads}^1;$$

$$L\frac{dI}{dt} = V, \qquad L = \text{inductance in henrys.}$$

Kirchhoff's laws and the current–voltage relation for each circuit element provide a system of algebraic and differential equations from which the voltage and current throughout the circuit can be determined. Problems 19 through 21 illustrate the procedure just described.

[1]Actual capacitors typically have capacitances measured in microfarads. We use farad as the unit for numerical convenience.

19. Consider the circuit shown in Figure 7.1.2. Let I_1, I_2, and I_3 be the currents through the capacitor, resistor, and inductor, respectively. Likewise, let V_1, V_2, and V_3 be the corresponding voltage drops. The arrows denote the arbitrarily chosen directions in which currents and voltage drops will be taken to be positive.

(a) Applying Kirchhoff's second law to the upper loop in the circuit, show that

$$V_1 - V_2 = 0. \tag{i}$$

In a similar way, show that

$$V_2 - V_3 = 0. \tag{ii}$$

(b) Applying Kirchhoff's first law to either node in the circuit, show that

$$I_1 + I_2 + I_3 = 0. \tag{iii}$$

(c) Use the current–voltage relation through each element in the circuit to obtain the equations

$$CV_1' = I_1, \qquad V_2 = RI_2, \qquad LI_3' = V_3. \tag{iv}$$

(d) Eliminate V_2, V_3, I_1, and I_2 among Eqs. (i) through (iv) to obtain

$$CV_1' = -I_3 - \frac{V_1}{R}, \qquad LI_3' = V_1. \tag{v}$$

Observe that if we omit the subscripts in Eqs. (v), then we have the system (2) of this section.

20. Consider the circuit shown in Figure 7.1.4. Use the method outlined in Problem 19 to show that the current I through the inductor and the voltage V across the capacitor satisfy the system of differential equations

$$\frac{dI}{dt} = -I - V, \qquad \frac{dV}{dt} = 2I - V.$$

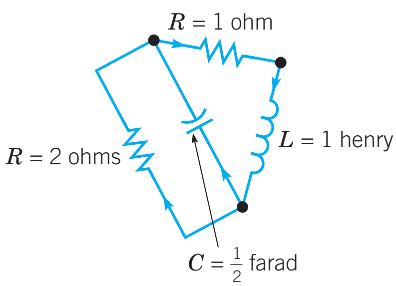

FIGURE 7.1.4 The circuit in Problem 20.

21. Consider the circuit shown in Figure 7.1.5. Use the method outlined in Problem 19 to show that the current I through the inductor and the voltage V across the capacitor satisfy the system of differential equations

$$L\frac{dI}{dt} = -R_1 I - V, \qquad C\frac{dV}{dt} = I - \frac{V}{R_2}.$$

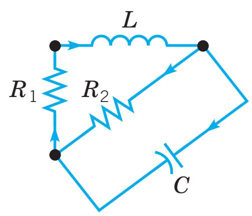

FIGURE 7.1.5 The circuit in Problem 21.

22. Consider the two interconnected tanks shown in Figure 7.1.6. Tank 1 initially contains 30 gal of water and 25 oz of salt, and Tank 2 initially contains 20 gal of water and 15 oz of salt. Water containing 1 oz/gal of salt flows into Tank 1 at a rate of 1.5 gal/min. The mixture flows from Tank 1 to Tank 2 at a rate of 3 gal/min. Water containing 3 oz/gal of salt also flows into Tank 2 at a rate of 1 gal/min (from the outside). The mixture drains from Tank 2 at a rate of 4 gal/min, of which some flows back into Tank 1 at a rate of 1.5 gal/min, while the remainder leaves the system.

(a) Let $Q_1(t)$ and $Q_2(t)$, respectively, be the amount of salt in each tank at time t. Write down differential equations and initial conditions that model the flow process. Observe that the system of differential equations is nonhomogeneous.

(b) Find the values of Q_1 and Q_2 for which the system is in equilibrium—that is, does not change with time. Let Q_1^E and Q_2^E be the equilibrium values. Can you predict which tank will approach its equilibrium state more rapidly?

(c) Let $x_1 = Q_1(t) - Q_1^E$ and $x_2 = Q_2(t) - Q_2^E$. Determine an initial value problem for x_1 and x_2. Observe that the system of equations for x_1 and x_2 is homogeneous.

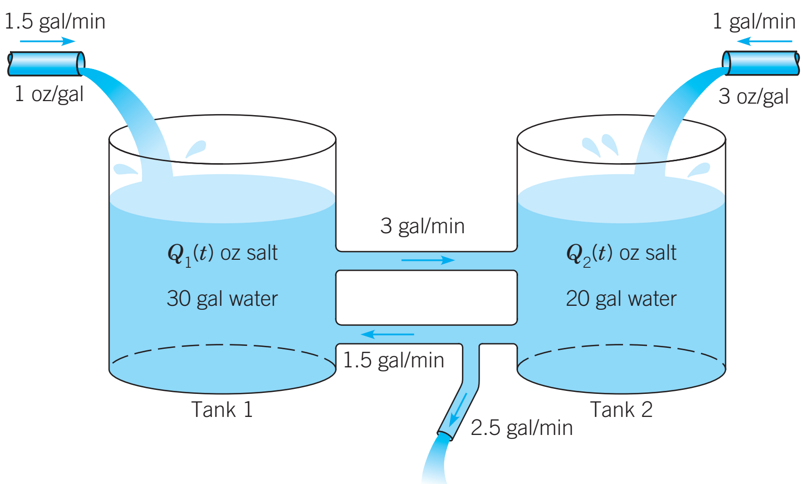

FIGURE 7.1.6 Two interconnected tanks (Problem 22).