CHAPTER

3

Second Order Linear Equations

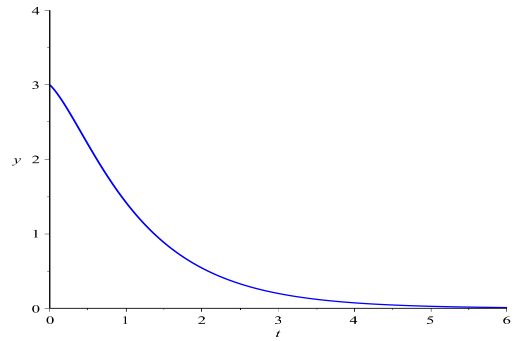
3.1

3. Let $y = e^{rt}$, so that $y' = r\,e^{rt}$ and $y'' = r^2\,e^{rt}$. Direct substitution into the differential equation yields $(12r^2 - r - 1)e^{rt} = 0$. Since $e^{rt} \neq 0$, the characteristic equation is $12r^2 - r - 1 = 0$. The roots of the equation are $r = -1/4\,, 1/3$. Hence the general solution is $y = c_1 e^{-t/4} + c_2 e^{t/3}$.

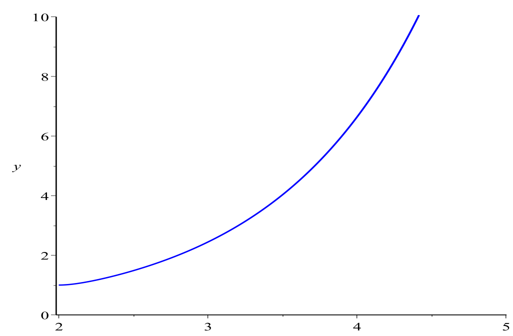
5. Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $r^2 + 6r = 0$. The roots of the equation are $r = 0\,, -6$. Hence the general solution is $y = c_1 e^{0t} + c_2 e^{-6t} = c_1 + c_2 e^{-6t}$.

7. The characteristic equation is $r^2 - 8r + 9 = 0$, with roots $r = 4 \pm \sqrt{7}$. Therefore the general solution is $y = c_1 e^{(4+\sqrt{7})t} + c_2 e^{(4-\sqrt{7})t}$.

10. Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $r^2 + 4r + 3 = 0$. The roots of the equation are $r = -1\,, -3$. Hence the general solution is $y = c_1 e^{-t} + c_2 e^{-3t}$. Its derivative is $y' = -c_1 e^{-t} - 3c_2 e^{-3t}$. Based on the first condition, $y(0) = 2$, we require that $c_1 + c_2 = 3$. In order to satisfy $y'(0) = -1$, we find that $-c_1 - 3c_2 = -1$. Solving for the constants, $c_1 = 4$ and $c_2 = -1$. Hence the specific solution is $y(t) = 4e^{-t} - e^{-3t}$. It clearly converges to 0 as $t \to \infty$.

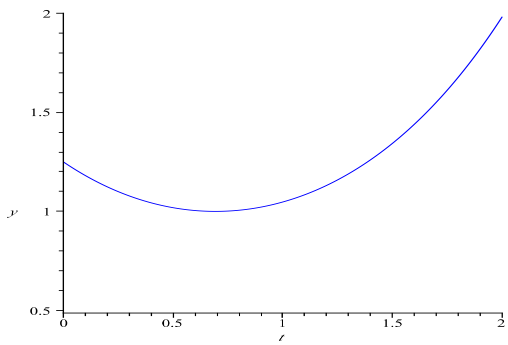


15. Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $r^2 + 8r - 9 = 0$. The roots of the equation are $r = 1, -9$. Hence the general solution is $y = c_1e^t + c_2e^{-9t}$. Its derivative is $y' = c_1e^t - 9c_2e^{-9t}$. Based on the first condition, $y(2) = 1$, we require that $c_1e^2 + c_2e^{-18} = 1$. In order to satisfy the condition $y'(2) = 0$, we find that $c_1e^2 - 9c_2e^{-18} = 0$. Solving for the constants, $c_1 = 9e^{-2}/10$ and $c_2 = e^{18}/10$. Hence the specific solution is $y(t) = 9e^{t-2}/10 + e^{18-9t}/10 = 9e^{(t-2)}/10 + e^{-9(t-2)}/10$. (Observe the shift on the time axis.) It clearly increases without bound as $t \to \infty$.



17. An algebraic equation with roots 4 and -3 is $(r-4)(r+3) = r^2 - r - 12 = 0$. This is the characteristic equation for the differential equation $y'' - y' - 12y = 0$.

19. The characteristic equation is $r^2 - 1 = 0$, with roots $r = 1, -1$. Therefore the general solution is $y = c_1e^t + c_2e^{-t}$, with derivative $y' = c_1e^t - c_2e^{-t}$. To satisfy the initial conditions, we require that $c_1 + c_2 = 5/4$ and $c_1 - c_2 = -3/4$. Solving for the coefficients, $c_1 = 1/4$ and $c_2 = 1$. This means that the specific solution is $y(t) = e^t/4 + e^{-t}$. From this, $y' = e^t/4 - e^{-t} = 0$ when $e^{2t} = 4$ or $t = \ln 2$. The value here is $y(\ln 2) = 2/4 + 1/2 = 1$. Since $y'' = y$ is positive at $t = \ln 2$, this is a minimum.

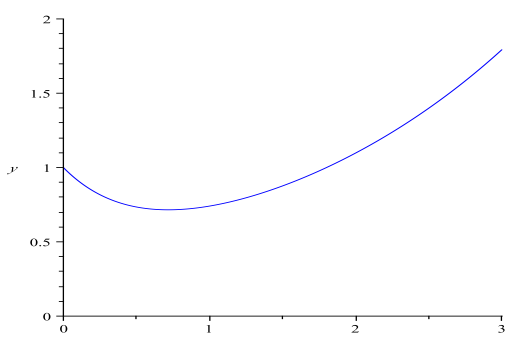


21. The general solution is $y = c_1 e^{-t} + c_2 e^{2t}$. Using the initial conditions we obtain $c_1 + c_2 = \alpha$ and $-c_1 + 2c_2 = 2$, so adding the two equations we find $3c_2 = \alpha + 2$. If y is to approach 0 as $t \to \infty$, c_2 must be zero. Thus $\alpha = -2$.

24. The characteristic equation is $r^2 + (3 - \alpha)r - 2(\alpha - 1) = 0$. Solving this equation, we see that the roots are $r = \alpha - 1, -2$. Therefore, the general solution is $y(t) = c_1 e^{(\alpha-1)t} + c_2 e^{-2t}$. In order for all solutions to tend to zero, we need $\alpha - 1 < 0$. Therefore, the solutions will all tend to zero as long as $\alpha < 1$. Due to the term $c_2 e^{-2t}$, we can never guarantee that all solutions will become unbounded as $t \to \infty$.

25.(a) The characteristic equation is $2r^2 + 3r - 2 = 0$, with roots $r = 1/2$ and $r = -2$. The initial conditions give $y(t) = (2\beta + 1)e^{-2t}/5 + (4 - 2\beta)e^{t/2}/5$.

(b) $y(t) = 2\,e^{t/2}/5 + 3\,e^{-2t}/5$.



We obtain that $y' = (-6e^{-2t} + e^{t/2})/5$. Setting this equal to zero and solving for t yields $t_0 = (2 \ln 6)/5$. At this point, $y_0 = \sqrt[5]{3/16} \approx 0.715485$.

(c) From part (a), if $\beta = 2$ then $y = e^{-2t}$ and the solution simply decays to zero. For $\beta > 2$, the solution becomes unbounded negatively, and again there is no minimum point. For $0 < \beta < 2$ there is always a minimum point, as found in part (b).

28. (a) The roots of the characteristic equation are $r = (-b \pm \sqrt{b^2 - 4ac})/2a$. For the roots to be real and different we must have $b^2 - 4ac > 0$. If they are to be

negative, then we must have $b > 0$ (since we are given that $a > 0$) and $c > 0$. This latter condition comes from the fact that if $c \leq 0$ then $\sqrt{b^2 - 4ac} \geq b$ and hence the numerator of r would give both positive and negative values, or a zero if $c = 0$.

(b) From part (a), this will happen when $b^2 - 4ac > 0$ and $c < 0$.

(c) Similarly to part (a), this happens when $b^2 - 4ac > 0$ and $b < 0$ and $c > 0$.

3.2

2.

$$W(\cos t, \sin t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1.$$

4.

$$W(x, xe^{2x}) = \begin{vmatrix} x & xe^{2x} \\ 1 & e^{2x} + 2xe^{2x} \end{vmatrix} = xe^{2x} + 2x^2e^{2x} - xe^{2x} = 2x^2e^{2x}.$$

8. Write the IVP as

$$y'' - \frac{3t}{t-1}y' + \frac{5}{t-1}y = \frac{\sin t}{t-1}.$$

Since the coefficient functions are continuous for all $t < 1$ and $t_0 = -3 < 1$, the IVP is guaranteed to have a unique solution for all $t < 1$. The longest interval of existence is $(-\infty, 1)$.

12. Write the IVP as

$$y'' + \frac{1}{x-2}y' + (\tan x)y = 0.$$

Since the coefficient functions are continuous for all x such that $x \neq 2$, $n\pi + \pi/2$ and $x_0 = 4$, the IVP is guaranteed to have a unique solution for all x such that $2 < x < 3\pi/2$.

14. For $y = 1$, $y' = 0$ and $y'' = 0$, so $yy'' + (y')^2 = 0$. For $y = t^{1/2}$, $y' = t^{-1/2}/2$ and $y'' = -t^{-3/2}/4$, thus $yy'' + (y')^2 = -t^{-1}/4 + t^{-1}/4 = 0$. If $y = c_1 \cdot 1 + c_2t^{1/2}$ is substituted into the differential equation, we get $(c_1 + c_2t^{1/2})(-c_2t^{-3/2}/4) + (c_2t^{-1/2}/2)^2 = -c_1c_2t^{-3/2}/4$, which is zero only if $c_1 = 0$ or $c_2 = 0$. Thus the linear combination of two solutions is not, in general, a solution. Theorem 3.2.2 is not contradicted however, since the differential equation is not linear.

15. $y = \phi(t)$ is a solution of the differential equation, so $L[\phi](t) = g(t)$. Since L is a linear operator, $L[c\phi](t) = cL[\phi](t) = cg(t)$. But, since $g(t) \neq 0$, $cg(t) = g(t)$ if and only if $c = 1$. This is not a contradiction of Theorem 3.2.2 since the linear differential equation is not homogeneous.

18. $W(t, g(t)) = tg'(t) - g(t) = 2t^2e^t$. Dividing both sides of the equation by t, we have $g' - g/t = 2te^t$. This a linear equation for g with an integrating factor $1/t$. Therefore, $g(t) = 2te^t + ct$.

22. The general solution is $y = c_1e^t + c_2e^{-3t}$. $W(e^t, e^{-3t}) = -4e^{-2t}$, and hence the exponentials form a fundamental set of solutions. On the other hand, the fundamental solutions must also satisfy the conditions $y_1(0) = 1$, $y_1'(0) = 0$; $y_2(0) = 0$, $y_2'(1) = 0$. For y_1, the initial conditions require $c_1 + c_2 = 1$, $c_1 - 3c_2 = 0$. The coefficients are $c_1 = 3/4$, $c_2 = 1/4$. For the solution y_2, the initial conditions require $c_1 + c_2 = 0$, $c_1 - 3c_2 = 1$. The coefficients are $c_1 = 1/4$, $c_2 = -1/4$. Hence the fundamental solutions are

$$y_1 = \frac{3}{4}e^t + \frac{1}{4}e^{-3t} \quad \text{and} \quad y_2 = \frac{1}{4}e^t - \frac{1}{4}e^{-3t}.$$

26. For $y_1 = x$, $y_1' = 1$ and $y_1'' = 0$. Therefore, $x^2y_1'' - x(x+2)y_1' + (x+2)y_1 = -x(x+2) + (x+2)x = 0$. For $y_2 = xe^x$, $y_2' = (1+x)e^x$ and $y_2'' = (2+x)e^x$. Hence $x^2y_2'' - x(x+2)y_2' + (x+2)y_2 = x^2(2+x)e^x - x(x+2)(1+x)e^x + (x+2)xe^x = 0$. Further, $W(x, xe^x) = x^2e^x \neq 0$ for $x > 0$. Therefore, the solutions form a fundamental set.

28.(a) For $y_1 = e^{-t}$, $y_1' = -e^{-t}$ and $y_1'' = e^{-t}$. Therefore, $y_1'' - y_1' - 2y_1 = e^{-t} + e^{-t} - 2e^{-t} = 0$. For $y_2 = e^{2t}$, $y_2' = 2e^{2t}$ and $y_2'' = 4e^{2t}$. Therefore, $y_2'' - y_2' - 2y_2 = 4e^{2t} - 2e^{2t} - 2e^{2t} = 0$. Further, $W(e^{-t}, e^{2t}) = 3e^t \neq 0$. Therefore, the functions form a fundamental set of solutions.

(b) Since the equation is linear and y_3, y_4, y_5 are all linear combinations of solutions, they are also solutions of the differential equation by Theorem 3.2.2.

(c) $W(y_1, y_3) = -6e^t$, $W(y_2, y_3) = 0$, $W(y_1, y_4) = 6e^t$ and $W(y_4, y_5) = 0$. Therefore, $\{y_1, y_3\}$ and $\{y_1, y_4\}$ form fundamental sets of solutions, but $\{y_2, y_3\}$ and $\{y_4, y_5\}$ do not.

29. Writing the differential equation in the form of Eq.(22), $p(t) = -(t+2)/t$. Thus Eq.(23) yields $W(t) = ce^{-\int -(t+2)/t\, dt} = ct^2e^t$.

34. From Eq.(23) we have $W(y_1, y_2) = ce^{-\int p(t)\, dt}$, where $p(t) = 2/t$ from the differential equation. Thus $W(y_1, y_2) = c/t^2$. We identify $c = 3$ from $W(y_1, y_2)(1) = 3$ and then $W(y_1, y_2)(5) = 3/25$.

38. Let c be the point in I at which both y_1 and y_2 vanish. Then $W(y_1, y_2)(c) = y_1(c)y_2'(c) - y_1'(c)y_2(c) = 0$. Since the Wronskian is zero the functions y_1 and y_2 cannot form a fundamental set.

40. Suppose that y_1 and y_2 have a point of inflection at t_0 and either $p(t_0) \neq 0$ or $q(t_0) \neq 0$. Since $y''(t_0) = 0$ it follows from the differential equation that $p(t_0)y_1'(t_0) + q(t_0)y_1(t_0) = 0$ and $p(t_0)y_2'(t_0) + q(t_0)y_2(t_0) = 0$. Now if $p(t_0) = 0$ and $q(t_0) \neq 0$, then $y_1(t_0) = y_2(t_0) = 0$, and $W(y_1, y_2)(t_0) = 0$, so the solutions cannot form a fundamental set. If $p(t_0) \neq 0$ and $q(t_0) = 0$, then $y_1'(t_0) = y_2'(t_0) =$

0, and $W(y_1, y_2)(t_0) = 0$, so again the solutions cannot form a fundamental set. If $p(t_0) \neq 0$ and $q(t_0) \neq 0$, then we obtain that $y_1'(t_0) = -q(t_0)y_1(t_0)/p(t_0)$ and $y_2'(t_0) = -q(t_0)y_2(t_0)/p(t_0)$ and thus

$$W(y_1, y_2)(t_0) = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) =$$

$$= y_1(t_0)(-q(t_0)y_2(t_0)/p(t_0)) - y_2(t_0)(-q(t_0)y_1(t_0)/p(t_0)) = 0.$$

41. Suppose that $P(x)y'' + Q(x)y' + R(x)y = [P(x)y']' + [f(x)y]'$. On expanding the right side and equating coefficients, we find $f'(x) = R(x)$ and $P'(x) + f(x) = Q(x)$. These two conditions on f can be satisfied if $R(x) = Q'(x) - P''(x)$ which gives the necessary condition $P''(x) - Q'(x) + R(x) = 0$.

44. $P = x,\ Q = -\cos x,\ R = \sin x$. We have $P'' - Q' + R = 0$. The equation is exact. Note that $[xy']' - [(1 + \cos x)y]' = 0$. Hence $xy' - (1 + \cos x)y = c_1$. This equation is linear, with integrating factor $\mu(x) = e^{-\int (1+\cos x)/x\, dx}$. Therefore the general solution is

$$y(x) = [\mu(x)]^{-1}(c_1 \int_{x_0}^{x} t^{-1}\mu(t)\, dt + c_2).$$

46. We want to choose $\mu(x)$ and $f(x)$ so that

$$\mu(x)P(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = [\mu(x)P(x)y']' + [f(x)y]'.$$

Expanding the right side and equating the coefficients of y, y' and y'' gives $\mu'(x)P(x) + \mu(x)P'(x) + f(x) = \mu(x)Q(x)$ and $f'(x) = \mu(x)R(x)$. Differentiate the first equation and then eliminate $f'(x)$ to obtain the adjoint equation $P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0$.

48. $P = 1 - x^2,\ Q = -2x,\ R = \alpha(\alpha + 1)$. Hence the coefficients are $2P' - Q = -4x + 2x = -2x$ and $P'' - Q' + R = -2 + 2 + \alpha(\alpha + 1) = \alpha(\alpha + 1)$. The adjoint of the original differential equation is given by $(1 - x^2)\mu'' - 2x\mu' + \alpha(\alpha + 1)\mu = 0$.

50. Write the adjoint as $\tilde{P}\mu'' + \tilde{Q}\mu' + \tilde{R}\mu = 0$ where $\tilde{P} = P,\ \tilde{Q} = 2P' - Q$ and $\tilde{R} = P'' - Q' + R$. The adjoint of this equation, namely, the adjoint of the adjoint is $\tilde{P}y'' + (2\tilde{P}' - \tilde{Q})y' + (\tilde{P}'' - \tilde{Q}' + \tilde{R})y = 0$. After substitution for $\tilde{P}$, $\tilde{Q}$ and $\tilde{R}$ and simplification we obtain $Py'' + Qy' + Ry = 0$. This is just the original equation.

51. The adjoint of $Py'' + Qy' + Ry = 0$ is $P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0$. The two equations are the same if $2P' - Q = Q$ and $P'' - Q' + R = R$. This will be true if $P' = Q$. For Problem 47, $P' = 2x \neq x = Q$, so the Bessel equation of order ν is not self-adjoint. In a similar manner we find that the equations in Problem 48 and 49 are self-adjoint.

3.3

1. $\exp(1 + 3i) = e^{1+3i} = e^1 e^{3i} = e(\cos 3 + i \sin 3)$.

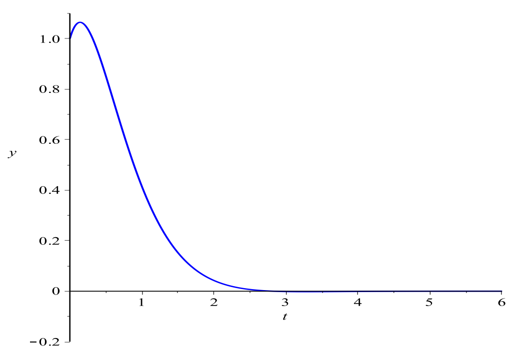
5. $2^{2-i} = e^{\ln 2^{2-i}} = e^{(2-i)\ln 2} = e^{2\ln 2}e^{-i\ln 2} = 4(\cos\ln 2 - i\sin\ln 2)$.

7. The characteristic equation is $r^2 - 4r + 5 = 0$. Therefore, the roots are $r = 2 \pm i$. Using Eq.(24), we arrive at the general solution $y(t) = c_1e^{2t}\cos t + c_2e^{2t}\sin t$.

11. The characteristic equation is $r^2 + 6r + 10 = 0$. Therefore, the roots are $r = -3 \pm i$. Thus we arrive at the general solution $y(t) = c_1e^{-3t}\cos t + c_2e^{-3t}\sin t$.

14. The characteristic equation is given by $9r^2 + 3r - 2 = 0$. Therefore, the roots are $r = -2/3, 1/3$. Therefore, the general solution is $y(t) = c_1e^{-2t/3} + c_2e^{t/3}$.

18. The characteristic equation is $r^2 + 4r + 5 = 0$, which has roots $r = -2 \pm i$. Therefore, the general solution is $y(t) = c_1e^{-2t}\cos t + c_2e^{-2t}\sin t$. The derivative of y is $y'(t) = c_1e^{-2t}(-2\cos t - \sin t) + c_2e^{-2t}(-2\sin t + \cos t)$. Using the initial conditions, we have $c_1 = 1$ and $-2c_1 + c_2 = 1$. Therefore, $c_1 = 1$ and $c_2 = 3$, and we conclude that the solution is $y(t) = e^{-2t}\cos t + 3e^{-2t}\sin t$. The solution oscillates as it decays to zero as $t \to \infty$. The oscillation is hard to see on this graph.



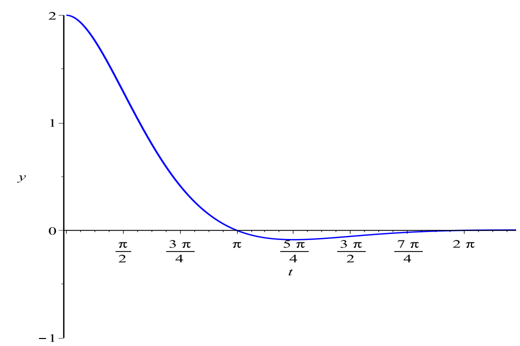
22. The characteristic equation is $r^2 + 2r + 2 = 0$, which has roots $r = -1 \pm i$. Therefore, the general solution is $y(t) = c_1e^{-t}\cos t + c_2e^{-t}\sin t$. The derivative of y is $y'(t) = c_1e^{-t}(-\cos t - \sin t) + c_2e^{-t}(-\sin t + \cos t)$. Using the initial conditions, we have

$$\frac{\sqrt{2}}{2}c_1e^{-\pi/4} + \frac{\sqrt{2}}{2}c_2e^{-\pi/4} = 2 \qquad \text{and} \qquad -\sqrt{2}c_1e^{-\pi/4} = 0.$$

Therefore, $c_1 = 0$ and $c_2 = 2\sqrt{2}e^{\pi/4}$, and we conclude that the solution is

$$y(t) = 2\sqrt{2}e^{-(t-\pi/4)}\sin t.$$

The solution oscillates as it decays to zero as $t \to \infty$.



23.(a) The characteristic equation is $3r^2 - r + 2 = 0$, which has roots $r = 1/6 \pm i\sqrt{23}/6$. Thus the general solution is $u(t) = c_1 e^{t/6}\cos\sqrt{23}t/6 + c_2 e^{t/6}\sin\sqrt{23}t/6$. We obtain $u(0) = c_1 = 2$ and $u'(0) = c_1/6 + \sqrt{23}c_2/6 = 0$. Solving for c_2 we find that

$$u(t) = e^{t/6}(2\cos\frac{\sqrt{23}}{6}t - \frac{2}{\sqrt{23}}\sin\frac{\sqrt{23}}{6}t).$$

(b) To estimate the first time that $|u(t)| = 10$ plot the graph of $u(t)$ as found in part (a). Use this estimate in an appropriate computer software program to find $t = 10.7598$.

25.(a) The characteristic equation is $r^2 + 2r + 6 = 0$, so $r = -1 \pm \sqrt{5}\,i$ and $y(t) = e^{-t}(c_1\cos\sqrt{5}\,t + c_2\sin\sqrt{5}\,t)$. Thus $y(0) = c_1 = 2$ and $y'(0) = -c_1 + \sqrt{5}c_2 = \alpha$ and hence $y(t) = e^{-t}(2\cos\sqrt{5}\,t + [(\alpha+2)/\sqrt{5}]\sin\sqrt{5}\,t)$.

(b) We can see that $y(1) = e^{-1}(2\cos\sqrt{5} + [(\alpha+2)/\sqrt{5}]\sin\sqrt{5}) = 0$, which gives $\alpha = -2 - 2\sqrt{5}\cot\sqrt{5} \approx 1.508$.

(c) For $y(t) = 0$ we must have $2\cos\sqrt{5}\,t + [(\alpha+2)/\sqrt{5}]\sin\sqrt{5}\,t = 0$ or $\tan\sqrt{5}\,t = -2\sqrt{5}/(\alpha+2)$. For $\alpha \geq 0$ this yields $\sqrt{5}t = \pi - \arctan(2\sqrt{5}/(\alpha+2))$ since $\arctan x$ is an odd function.

(d) From part (c), $\arctan(2\sqrt{5}/(\alpha+2)) \to 0$ as $\alpha \to \infty$, so $t \to \pi/\sqrt{5}$.

31. Let $r = \lambda + i\mu$, then

$$\frac{de^{rt}}{dt} = \frac{d[e^{\lambda t}(\cos\mu t + i\sin\mu t)]}{dt} = \lambda e^{\lambda t}(\cos\mu t + i\sin\mu t) +$$
$$+e^{\lambda t}(-\mu\sin\mu t + i\mu\cos\mu t) = \lambda e^{\lambda t}(\cos\mu t + i\sin\mu t) + i\mu e^{\lambda t}(i\sin\mu t + \cos\mu t) =$$
$$= e^{\lambda t}(\lambda + i\mu)(\cos\mu t + i\sin\mu t) = re^{rt}.$$

33. Suppose that $t = a$ and $t = b$ $(b > a)$ are consecutive zeros of y_1. We must show that y_2 vanishes once and only once in the interval $a < t < b$. Assume that it does not vanish. Then we can form the quotient y_1/y_2 on the interval $a \leq t \leq b$. Note that $y_2(a) \neq 0$ and $y_2(b) \neq 0$, otherwise y_1 and y_2 would not be a fundamental

set of solutions. Next, y_1/y_2 vanishes at $t=a$ and $t=b$ and has a derivative in $a<t<b$. By Rolle's theorem, the derivative must vanish at an interior point. But $(y_1/y_2)' = (y_1'y_2 - y_2y_1')/y_2^2 = -W(y_1,y_2)/y_2^2$, which cannot be zero since y_1 and y_2 are fundamental solutions. Hence we have a contradiction and conclude that y_2 must vanish at a point between a and b. Finally, we show that it can vanish at only one point between a and b. Suppose that it vanishes at two points c and d between a and b. By the argument we have just given we can show that y_1 must vanish between c and d. But this contradicts the assumption that a and b are consecutive zeros of y_1.

34.(a) Let $x=\ln t$. We differentiate, using the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}\frac{1}{t}$$

and

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{1}{t} + \frac{dy}{dx}\left(-\frac{1}{t^2}\right) = \frac{d^2y}{dx^2}\frac{1}{t^2} + \frac{dy}{dx}\left(-\frac{1}{t^2}\right).$$

(b) Using part (a), we can see that

$$t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0$$

transforms into

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + \alpha\frac{dy}{dx} + \beta y = \frac{d^2y}{dx^2} + (\alpha-1)\frac{dy}{dx} + \beta y = 0.$$

36. The equation transforms into $y'' + 4y' + 3y = 0$. The characteristic roots are $r=-1,\,-3$. The solution is

$$y = c_1e^{-x} + c_2e^{-3x} = c_1e^{-\ln t} + c_2e^{-3\ln t} = \frac{c_1}{t} + \frac{c_2}{t^3}.$$

40. The equation transforms into $y'' - 2y' + 5y = 0$. The characteristic roots are $r = 1 \pm 2i$. The solution is

$$y = c_1e^x\cos(2x) + c_2e^x\sin(2x) = c_1t\cos(2\ln t) + c_2t\sin(2\ln t).$$

44. We use the result of Problem 43. Note that $p(t)=t$ and $q(t)=e^{-t^2}>0$ for $-\infty<t<\infty$. Thus $(q'+2pq)/q^{3/2}=0$ and the differential equation can be transformed into an equation with constant coefficients by letting $x=u(t)=\int e^{-t^2/2}\,dt$. Substituting $x=u(t)$ in the differential equation found in part (b) of Problem 43 we obtain, after dividing by the coefficient of d^2y/dx^2, the differential equation $d^2y/dx^2+y=0$. Hence the general solution of the original differential equation is $y(t)=c_1\cos x(t)+c_2\sin x(t)$, where $x(t)=\int e^{-t^2/2}\,dt$.

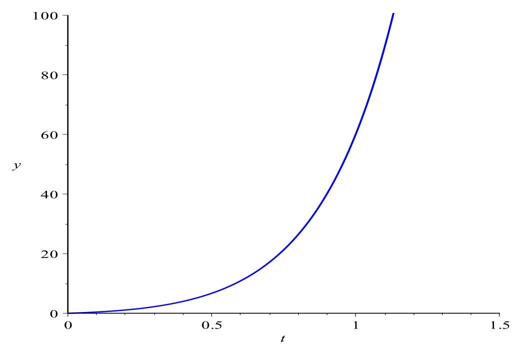
3.4

1. The characteristic equation is given by $r^2 + 2r + 1 = 0$. Therefore, we have one repeated root $r = -1$, and the general solution is given by $y(t) = c_1 e^{-t} + c_2 t e^{-t}$.

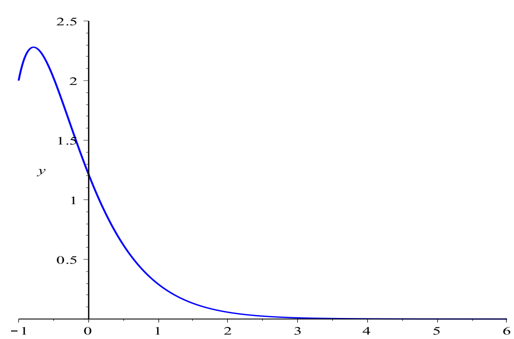
5. The characteristic equation is $r^2 + 2r + 10 = 0$, with complex roots $r = -1 \pm 3i$. The general solution is $y(t) = c_1 e^{-t} \cos 3t + c_2 e^{-t} \sin 3t$.

9. The characteristic equation is given by $25r^2 - 30r + 9 = 0$. Therefore, we have one repeated root $r = 3/5$, and the general solution is given by $y(t) = c_1 e^{3t/5} + c_2 t e^{3t/5}$.

12. The characteristic equation is given by $r^2 - 6r + 9 = 0$. Therefore, there is one repeated root, $r = 3$, and the general solution is given by $y(t) = c_1 e^{3t} + c_2 t e^{3t}$. After differentiation, $y'(t) = 3c_1 e^{3t} + c_2(1 + 3t)e^{3t}$. Now using the initial conditions, we need $c_1 = 0$ and $3c_1 + c_2 = 3$. The solution of this system of equations is $c_1 = 0$ and $c_2 = 3$, and the specific solution is $y(t) = 3te^{3t}$. The solution $y \to \infty$ as $t \to \infty$.



14. The characteristic equation is given by $r^2 + 4r + 4 = 0$. Therefore, there is one repeated root, $r = -2$, and the general solution is given by $y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$. We can see that $y'(t) = -2c_1 e^{-2t} + c_2(1 - 2t)e^{-2t}$. Now using the initial conditions, we need $c_1 e^2 - c_2 e^2 = 2$ and $-2c_1 e^2 + 3c_2 e^2 = 3$. The solution of this system of equations is $c_1 = 9e^{-2}$ and $c_2 = 7e^{-2}$, and the specific solution is $y(t) = 9e^{-2(t+1)} + 7te^{-2(t+1)}$. The solution $y \to 0$ as $t \to \infty$.



17.(a) The characteristic equation is $4r^2+4r+1=(2r+1)^2=0$, so we have $y(t)=(c_1+c_2t)e^{-t/2}$. Thus $y(0)=c_1=1$ and $y'(0)=-c_1/2+c_2=0$ and hence $c_2=5/2$ and $y(t)=(1+5t/2)e^{-t/2}$.

(b) From part (a), $y'=-(1/2)(1+5t/2)e^{-t/2}+(5/2)e^{-t/2}=0$, when $-1/2-5t/4+5/2=0$, so $t_M=8/5$ and $y_M=5e^{-4/5}$.

(c) From part (a), c_1 is the same and $y'(0)=-1/2+c_2=b$ or $c_2=b+1/2$ and $y(t)=[1+(b+1/2)t]e^{-t/2}$.

(d) From part (c), $y'=-(1/2)[1+(b+1/2)t]e^{-t/2}+(b+1/2)e^{-t/2}=0$ which yields $t_M=4b/(2b+1)\to 2$ as $b\to\infty$ and $y_M=(1+(2b+1)/2\cdot 4b/(2b+1))e^{-2b/(2b+1)}=(1+2b)e^{-2b/(2b+1)}$. Since $e^{-2b/(2b+1)}\to e^{-1}$ as $b\to\infty$, $y_M\to\infty$ as $b\to\infty$.

19. Suppose the roots are distinct, $r_1<r_2$. Then the solution is $y(t)=c_1e^{r_1t}+c_2e^{r_2t}$. Solving the equation $y(t)=0$, we see that we must have $c_1e^{r_1t}=-c_2e^{r_2t}$ which implies $e^{(r_1-r_2)t}=-c_2/c_1$. First, in order to guarantee any solution of this equation, we would need $c_2/c_1<0$. Then, applying the natural logarithm function to the equation, we see that $t=\ln(-c_2/c_1)/(r_1-r_2)$.

If the roots are not distinct, then the solution is given by $y(t)=c_1e^{rt}+c_2te^{rt}$. Therefore, $y(t)=0$ implies $(c_1+c_2t)e^{rt}=0$. Since $e^{rt}\neq 0$, we must have $c_1+c_2t=0$. Therefore, the solution will be zero only when $t=-c_1/c_2$.

21. If $r_1\neq r_2$, then $\phi(t;r_1,r_2)=(e^{r_2t}-e^{r_1t})/(r_2-r_1)$ is defined for all t. Note that ϕ is a linear combination of the fundamental solutions, e^{r_1t} and e^{r_2t}, so ϕ is a solution of the differential equation. The limit of ϕ as $r_2\to r_1$ is (by its definition) the derivative of the function e^{rt} with respect to r at the point r_1, hence $\phi(t;r_1,r_2)\to te^{r_1t}$ as $r_2\to r_1$.

25. Following the reduction of order technique given, $y_1=1/t$, $p(t)=3/t$, so the equation for v is $v''/t+v'/t^2=0$. After separating the variables the equation becomes $v''/v'=-1/t$, so $\ln v'=-\ln t+c$. We obtain that $v'=c/t$ and then $v=c\ln t$. Thus the second solution is $y_2=\ln t/t$.

27. Following the reduction of order technique given, $y_1=\sin(x^2)$, $p(x)=-1/x$, so the equation for v is $\sin(x^2)v''+(4x\cos(x^2)-\sin(x^2)/x)v'=0$. After separating the variables the equation becomes $v''/v'=1/x-4x\cot(x^2)$, so $\ln v'=\ln x-2\ln(\sin(x^2))$. We obtain that $v'=x/\sin^2(x^2)$ and then $v=-\cot(x^2)/2$. Thus the second solution is $y_2=\cos(x^2)$.

30. Let $y_2(x)=[x^{-1/2}\sin x]v(x)$. Substituting y_2 into the differential equation, we conclude that $\sin x\, v''+2\cos x\, v'=0$. This equation is linear in v'. Its solution is given by $v'(x)=c/\sin^2 x$. Integrating with respect to x, we have $v(x)=c_1\cot x+c_2$. Therefore, $y_2(x)=c_1x^{-1/2}\cos x+c_2x^{-1/2}\sin x$. Since we already have the solution $y_1(x)=x^{-1/2}\sin x$, we take the solution $y_2(x)=x^{-1/2}\cos x$.

32. $(y_2/y_1)'=(y_2'y_1-y_1'y_2)/y_1^2=W(y_1,y_2)/y_1^2$. Abel's formula is $W(y_1,y_2)=ce^{-\int_{t_0}^t p(r)\,dr}$. Hence $(y_2/y_1)'=cy_1^{-2}e^{-\int_{t_0}^t p(r)\,dr}$. Integrating and setting $c=1$

(since a solution y_2 can be multiplied by any constant) and taking the constant of integration to be zero we obtain $y_2 = y_1 \int_{t_0}^{t} [e^{-\int_{s_0}^{s} p(r)\,dr}/y_1^2(s)]\,ds$.

34. From Problem 32 and Abel's formula we have $(y_2/y_1)' = e^{\int (1/t)\,dt}/\sin^2 t^2 = e^{\ln t}/\sin^2 t^2 = t\csc^2 t^2$. Thus $y_2/y_1 = -(1/2)\cot t^2$ and hence we can choose $y_2 = \cos t^2$ since $y_1 = \sin t^2$.

37. The general solution of the differential equation is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ where $r_1, r_2 = (-b \pm \sqrt{b^2 - 4ac})/2a$, provided $b^2 - 4ac \neq 0$. In this case there are two possibilities. If $b^2 - 4ac > 0$ then $\sqrt{b^2 - 4ac} < b$ and r_1 and r_2 are real and negative. Consequently, $y \to 0$ as $t \to \infty$. If $b^2 - 4ac < 0$, then r_1 and r_2 are complex conjugates with negative real part. Again, $y \to 0$ as $t \to \infty$. Finally, if $b^2 - 4ac = 0$, then $y = c_1 e^{rt} + c_2 t e^{rt}$ where $r = -b/2a < 0$. Hence, again $y \to 0$ as $t \to \infty$. This conclusion does not hold if either $b = 0$ (since in this case $y(t) = c_1 \cos \omega t + c_2 \sin \omega t$, where $\omega^2 = c/a$) or $c = 0$ (since one of the solutions would be $y_1(t) = c_1$).

41. Letting $x = \ln t$, the equation transforms into $y'' + 3y' + (9/4)y = 0$. We get a repeated root $r = -3/2$, so the solution is $y = c_1 e^{-3x/2} + c_2 x e^{-3x/2} = c_1 e^{-3\ln t/2} + c_2 \ln t e^{-3\ln t/2} = c_1 t^{-3/2} + c_2 t^{-3/2} \ln t$.

45. Letting $x = \ln t$, the equation transforms into $y'' + 4y' + 29y = 0$. The characteristic roots are $r = -2 \pm 5i$. The solution is $y = c_1 e^{-2x}\cos(5x) + c_2 e^{-2x}\sin(5x) = c_1 t^{-2}\cos(5\ln t) + c_2 t^{-2}\sin(5\ln t)$.

3.5

1. The characteristic equation for the homogeneous problem is $r^2 - 2r - 3 = 0$, which has roots $r = 3, -1$. Therefore, the solution of the homogeneous problem is $y_h(t) = c_1 e^{3t} + c_2 e^{-t}$. To find a solution of the nonhomogeneous problem, we look for a solution of the form $y_p(t) = Ae^{2t}$. Substituting a function of this form into the differential equation, we have $4Ae^{2t} - 4Ae^{2t} - 3Ae^{2t} = 6e^{2t}$. This means that we need $-3A = 6$, or $A = -2$. Hence the general solution of the nonhomogeneous problem is $y(t) = c_1 e^{3t} + c_2 e^{-t} - 2e^{2t}$.

3. The characteristic equation for the homogeneous problem is $r^2 - r - 2 = 0$, with roots $r = -1,\ 2$. Hence $y_h(t) = c_1 e^{-t} + c_2 e^{2t}$. Set now $y_p = At^2 + Bt + C$. Substitution into the given differential equation, and comparing the coefficients, results in the system of equations $-2A = 4$, $-2A - 2B = 0$ and $2A - B - 2C = -3$. Hence $y_p = -2t^2 + 2t - 3/2$. The general solution is $y(t) = y_h(t) + y_p(t)$.

6. The characteristic equation for the homogeneous problem is $r^2 + 2r = 0$, which has roots $r = 0, -2$. Therefore, the solution of the homogeneous problem is $y_h(t) = c_1 + c_2 e^{-2t}$. To find a solution of the nonhomogeneous problem, we look for a solution of the form $y_p(t) = At + B\cos 2t + C\sin 2t$. Substituting a function of this form into the differential equation, and equating like terms, we have $2A = 5$, $-4B + 4C = 0$ and $-4B - 4C = 4$. The solution of these equations is $A = 5/2$,

$B = -1/2$ and $C = -1/2$. Therefore, the general solution of the nonhomogeneous problem is $y(t) = c_1 + c_2e^{-2t} + 5t/2 - \cos 2t/2 - \sin 2t/2$.

8. The characteristic equation for the homogeneous problem is $r^2 + 2r + 1 = 0$, which has the repeated root $r = -1$. Therefore, the solution of the homogeneous problem is $y_h(t) = c_1e^{-t} + c_2te^{-t}$. To find a solution of the nonhomogeneous problem, we look for a solution of the form $y_p(t) = At^2e^{-t}$. Substituting a function of this form into the differential equation, and equating like terms, we have $2A = 4$. Therefore, $A = 2$ and the general solution of the nonhomogeneous problem is $y(t) = c_1e^{-t} + c_2te^{-t} + 2t^2e^{-t}$.

10. The characteristic equation for the homogeneous problem is $r^2 + 1 = 0$, which has roots $r = \pm i$. Therefore, the solution of the homogeneous problem is $y_h(t) = c_1 \cos t + c_2 \sin t$. To find a solution of the nonhomogeneous problem, we look for a solution of the form $y_p(t) = A \cos 2t + B \sin 2t + Ct \cos 2t + Dt \sin 2t$. Substituting a function of this form into the differential equation, and equating like terms, we have $4D - 3A = 0$, $-3B - 4C = 4$, $-3C = 1$, and $-3D = 0$. The solution of these equations is $A = 0$, $B = -8/9$, $C = -1/3$ and $D = 0$. Therefore, the genaral solution of the nonhomogeneous problem is $y(t) = c_1 \cos t + c_2 \sin t - 8 \sin 2t/9 - t \cos 2t/3$.

13. The characteristic equation for the homogeneous problem is $r^2 + r + 4 = 0$, which has roots $r = (-1 \pm i\sqrt{15})/2$. Therefore, the solution of the homogeneous problem is $y_h(t) = e^{-t/2}(c_1 \cos(\sqrt{15}t/2) + c_2 \sin(\sqrt{15}t/2))$. To find a solution of the nonhomogeneous problem, we look for a solution of the form $y_p(t) = Ae^t + Be^{-t}$. Substituting a function of this form into the differential equation, and equating like terms, we have $6A = 2$ and $4B = -2$. The solution of these equations is $A = 1/3$ and $B = -1/2$. Therefore, the general solution of the nonhomogeneous problem is $y(t) = e^{-t/2}(c_1 \cos(\sqrt{15}t/2) + c_2 \sin(\sqrt{15}t/2)) + e^t/3 - e^{-t}/2$. In this case, we could also assume that y_p is a linear combination of $\sinh t$ and $\cosh t$.

15. The characteristic equation for the homogeneous problem is $r^2 + r - 2 = 0$, which has roots $r = 1, -2$. Therefore, the solution of the homogeneous problem is $y_h(t) = c_1e^t + c_2e^{-2t}$. To find a solution of the nonhomogeneous problem, we look for a solution of the form $y_p(t) = At + B$. Substituting a function of this form into the differential equation, and equating like terms, we have $-2A = 2$ and $A - 2B = 0$. The solution of these equations is $A = -1$ and $B = -1/2$. Therefore, the general solution of the nonhomogeneous problem is $y(t) = c_1e^t + c_2e^{-2t} - t - 1/2$. The initial conditions imply $c_1 + c_2 - 1/2 = 0$ and $c_1 - 2c_2 - 1 = 2$. Therefore, $c_1 = 4/3$ and $c_2 = -5/6$ which implies that the solution of the IVP is $y(t) = 4e^t/3 - 5e^{-2t}/6 - t - 1/2$.

18. The characteristic equation for the homogeneous problem is $r^2 - 2r - 3 = 0$, which has roots $r = 3, -1$. Therefore, the solution of the homogeneous problem is $y_h(t) = c_1e^{3t} + c_2e^{-t}$. To find a solution of the nonhomogeneous problem, we look for a solution of the form $y_p(t) = Ae^{2t} + Bte^{2t}$. Substituting a function of this form into the differential equation, and equating like terms, we have $y_p(t) = -2e^{2t}/3 - te^{2t}$. Therefore, the general solution of the nonhomogeneous problem is

$y(t) = c_1e^{3t} + c_2e^{-t} - 2e^{2t}/3 - te^{2t}$. The initial conditions imply $c_1 + c_2 - 2/3 = 3$ and $3c_1 - c_2 - 4/3 - 1 = 0$. Therefore, $c_1 = 3/2$ and $c_2 = 13/6$ which implies that the solution of the IVP is $y(t) = 3e^{3t}/2 + 13e^{-t}/6 - 2e^{2t}/3 - te^{2t}$.

21.(a) The characteristic equation for the homogeneous problem is $r^2 + 3r = 0$, which has roots $r = 0, -3$. Therefore, the solution of the homogeneous problem is $y_h(t) = c_1 + c_2e^{-3t}$. After inspection of the nonhomogeneous term, for $2t^4$ we must assume a fourth order polynomial, for t^2e^{-3t} we must assume a quadratic polynomial times the exponential, and for $\sin 3t$ we must assume $C\sin 3t + D\cos 3t$. However, since e^{-3t} and a constant are solutions of the homogeneous differential equation, we must multiply the coefficient of e^{-3t} and the polynomial by t. The correct form then is $Y(t) = t(A_0t^4 + A_1t^3 + A_2t^2 + A_3t + A_4) + t(B_0t^2 + B_1t + B_2)e^{-3t} + C\sin 3t + D\cos 3t$.

(b) Substituting a function of this form into the differential equation, and equating like terms, we have $A_0 = 2/15$, $A_1 = -2/9$, $A_2 = 8/27$, $A_3 = -8/27$, $A_4 = 16/81$, $B_0 = -1/9$, $B_1 = -1/9$, $B_2 = -2/27$, $C = -1/9$, $D = -1/9$ Therefore, the general solution of the nonhomogeneous problem is

$$y(t) = c_1 + c_2e^{-3t} + t(\frac{2}{15}t^4 - \frac{2}{9}t^3 + \frac{8}{27}t^2 - \frac{8}{27}t + \frac{16}{81}) +$$
$$+ t(-\frac{1}{9}t^2 - \frac{1}{9}t - \frac{2}{27})e^{-3t} - \frac{\sin 3t}{9} - \frac{\cos 3t}{9}.$$

24.(a) The characteristic equation for the homogeneous problem is $r^2 + 2r + 2 = 0$, which has roots $r = -1 \pm i$. Therefore, the solution of the homogeneous problem is $y_h(t) = e^{-t}(c_1\cos t + c_2\sin t)$. After inspection of the nonhomogeneous term, since $e^{-t}\cos t$ and $e^{-t}\sin t$ are solutions of the homogeneous differential equation, it is necessary to multiply again by t in the particular solution, so the desired form is $Y(t) = Ae^{-t} + t(B_0t^2 + B_1t + B_2)e^{-t}\cos t + t(C_0t^2 + C_1t + C_2)e^{-t}\sin t$.

(b) Substituting a function of this form into the differential equation, and equating like terms, we have $A = 2$, $B_0 = -2/3$, $B_1 = 0$, $B_2 = 1$, $C_0 = 0$, $C_1 = 1$, $C_2 = 1$. Therefore, the general solution of the nonhomogeneous problem becomes $y(t) = e^{-t}(c_1\cos t + c_2\sin t) + 2e^{-t} + t((-2/3)t^2 + 1)e^{-t}\cos t + t(t+1)e^{-t}\sin t$.

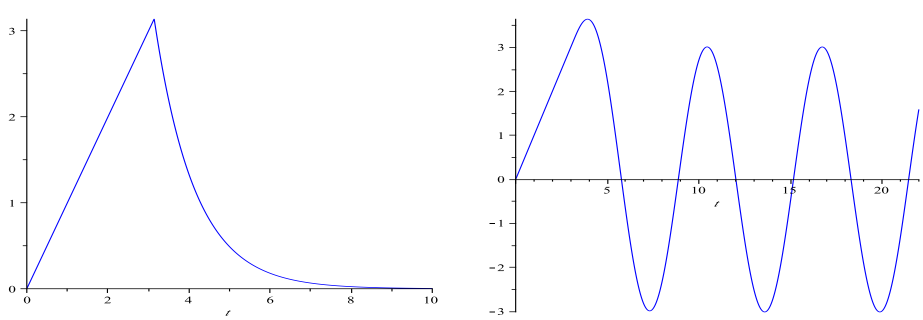
29.(a) For $Y = ve^{-t}$, we have $Y' = v'e^{-t} - ve^{-t}$ and $Y'' = v''e^{-t} - 2v'e^{-t} + ve^{-t}$. Then $Y'' - 3Y' - 4Y = 2e^{-t}$ implies that $v''e^{-t} - 2v'e^{-t} + ve^{-t} - 3v'e^{-t} + 3ve^{-t} - 4ve^{-t} = 2e^{-t}$. Simplifying this equation, we have $v'' - 5v' = 2$.

(b) We see that $(v')' - 5(v') = 2$. Therefore, letting $w = v'$, we see that w must satisfy $w' - 5w = 2$. This equation is linear with integrating factor $\mu(t) = e^{-5t}$. Therefore, we have $[e^{-5t}w]' = 2e^{-5t}$ which implies that $w = -2/5 + ce^{5t}$.

(c) Integrating w, we see that $v = (-2/5)t + c_1e^{5t}/5 + c_2$. Then using the fact that $Y = ve^{-t}$, we conclude that $Y(t) = -2te^{-t}/5 + c_1e^{4t}/5 + c_2e^{-t}$. Here the first term is y_p and the last two terms comprise y_h.

31. The solution of the homogeneous problem is $y_h(t) = c_1 \cos t + c_2 \sin t$. To solve the nonhomogeneous IVP (starting at $t = 0$), we begin by looking for a solution of the nonhomogeneous equation of the form $Y = A + Bt$. Substituting a function of this form into the ODE leads to the equation $A + Bt = t$. Therefore, $A = 0$ and $B = 1$, and the solution of the nonhomogeneous problem is $y(t) = c_1 \cos t + c_2 \sin t + t$. Now, we consider the initial conditions. The initial conditions imply that $c_1 = 0$ and $c_2 + 1 = 1$. Therefore, the solution of the nonhomogeneous problem for $0 \le t \le \pi$ is given by $y_1(t) = t$. Now we need to solve the nonhomogeneous problem starting at $t = \pi$. We look for a solution of the nonhomogeneous problem of the form $Y(t) = Ce^{-t}$. Substituting a function of this form into the ODE leads to the equation $Ae^{-t} + Ae^{-t} = \pi e^{\pi - t}$. Therefore, $2A = \pi e^{\pi}$, or $A = \pi e^{\pi}/2$. Therefore, a solution of the nonhomogeneous problem (starting at $t = \pi$) is given by $y(t) = d_1 \cos t + d_2 \sin t + \pi e^{\pi - t}/2$. Using the solution of the IVP starting at $t = 0$, $y_1(t) = t$, we see that at time $t = \pi$, $y_1(\pi) = \pi$ and $y_1'(\pi) = 1$. Using these as our new initial conditions for $t = \pi$, we see that d_1, d_2 must satisfy $-d_1 + \pi/2 = \pi$ and $-d_2 - \pi/2 = 1$. The solution of these equations is $d_1 = -\pi/2$, $d_2 = -1 - \pi/2$. Therefore, we conclude that the solution of the nonhomogeneous IVP is

$$y(t) = \begin{cases} t & 0 \le t \le \pi \\ -\frac{\pi}{2} \cos t - \left(1 + \frac{\pi}{2}\right) \sin t + \frac{\pi}{2} e^{\pi - t} & t > \pi. \end{cases}$$



(a) The nonhomogeneous term

(b) The solution $y(t)$

33. According to Theorem 3.5.1, the difference of any two solutions of the linear second order nonhomogeneous differential equation is a solution of the corresponding homogeneous differential equation. Hence $Y_1 - Y_2$ is a solution of $ay'' + by' + cy = 0$. In Problem 37 of Section 3.4 we showed that if $a > 0$, $b > 0$ and $c > 0$ then every solution of this differential equation goes to zero as $t \to \infty$. If $b = 0$, then y_c involves only sines and cosines, so $Y_1 - Y_2$ does not approach zero as $t \to \infty$.

36. We have $(D^2 - 3D - 4)y = (D - 4)(D + 1)y$. Let $u = (D + 1)y$, and consider the ODE $u' - 4u = 3e^{2t}$. The general solution is $u(t) = -3e^{2t}/2 + c\,e^{4t}$. We therefore have the first order equation $y' + y = -3e^{2t}/2 + c\,e^{4t}$. The general solution of the latter differential equation is $y(t) = -e^{2t}/2 + c_1 e^{4t} + c_2 e^{-t}$.

3.6

2. Two linearly independent solutions of the homogeneous differential equation are $y_1(t) = e^{2t}$ and $y_2(t) = e^{-t}$. Assume $Y = u_1(t)e^{2t} + u_2(t)e^{-t}$, then $Y'(t) = [2u_1(t)e^{2t} - u_2(t)e^{-t}] + [u_1'(t)e^{2t} + u_2'(t)e^{-t}]$. We set $u_1'(t)e^{2t} + u_2'(t)e^{-t} = 0$. Then $Y''(t) = 4u_1e^{2t} + u_2e^{-t} + 2u_1'e^{2t} - u_2'e^{-t}$ and substituting into the differential equation gives $2u_1'(t)e^{2t} - u_2'(t)e^{-t} = 4e^{-t}$ (the terms involving u_1 and u_2 add to zero since e^{-t} and e^{2t} are solutions of the homogeneous equation). Thus we have two algebraic equations for $u_1'(t)$ and $u_2'(t)$ with the solution $u_1'(t) = 4e^{-3t}/3$ and $u_2'(t) = -4/3$. Hence $u_1(t) = -4e^{-3t}/9$ and $u_2(t) = -4t/3$. Substituting in the expression for $Y(t)$ we obtain $Y(t) = -4e^{-t}/9 - 4te^{-t}/3$. Since e^{-t} is a solution of the homogeneous differential equation, we can choose $Y(t) = -4te^{-t}/3$.

5. Since $\cos t$ and $\sin t$ are solutions of the homogeneous differential equation, we assume $Y(t) = u_1(t)\cos t + u_2(t)\sin t$. Thus $Y' = -u_1(t)\sin t + u_2(t)\cos t$, after setting $u_1'(t)\cos t + u_2'(t)\sin t = 0$. Finding Y'' and substituting into the differential equation then yields $-u_1'(t)\sin t + u_2'(t)\cos t = 2\tan t$. The two equations for $u_1'(t)$ and $u_2'(t)$ have the solution $u_1'(t) = -2\sin^2 t/\cos t = -2\sec t + 2\cos t$ and $u_2'(t) = 2\sin t$. Thus $u_1(t) = 2\sin t - 2\ln(\sec t + \tan t)$ and $u_2(t) = -2\cos t$, which when substituted into the assumed form for Y, simplified, and added to the homogeneous solution yields $y(t) = c_1\cos t + c_2\sin t - 2\cos t\ln(\sec t + \tan t)$.

11. The solution of the homogeneous equation is $y_h(t) = c_1e^{5t} + c_2e^{2t}$. The functions $y_1(t) = e^{5t}$ and $y_2(t) = e^{2t}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = -3e^{7t}$. Using the method of variation of parameters, a particular solution is given by $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ where

$$u_1(t) = -\int \frac{e^{2s}(g(s))}{W(s)}\,ds = \frac{1}{3}\int e^{-5s}g(s)ds$$

$$u_2(t) = \int \frac{e^{5s}(g(s))}{W(s)}\,ds = -\frac{1}{3}\int e^{-2s}g(s)ds.$$

Therefore, a particular solution is $Y(t) = \frac{1}{3}\int e^{5(t-s)}g(s)ds - \frac{1}{3}\int e^{2(t-s)}g(s)ds$. Therefore, the general solution is $y(t) = c_1e^{5t} + c_2e^{2t} + \frac{1}{3}\int e^{5(t-s)}g(s)ds - \frac{1}{3}\int e^{2(t-s)}g(s)ds$.

14. By direct substitution, it can be verified that $y_1(t) = t$ and $y_2(t) = te^t$ are solutions of the homogeneous equation. The Wronskian of these functions is $W(y_1, y_2) = t^2e^t$. Rewriting the equation in standard form, we have

$$y'' - \frac{t(t+2)}{t^2}y' + \frac{t+2}{t^2}y = 6t.$$

Therefore, $g(t) = 6t$. Using the method of variation of parameters, a particular solution is given by $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ where

$$u_1(t) = -\int \frac{te^t(6t)}{W(t)}\,dt = -6t \quad \text{and} \quad u_2(t) = \int \frac{t(6t)}{W(t)}\,dt = -6e^{-t}.$$

Therefore, a particular solution is $Y(t) = -6t^2 - 6t$. (Since t is a solution of the homogeneous differential equation, we can choose only the $-6t^2$ part.)

18. By direct substitution, it can be verified that $y_1(x) = x^{-1/2}\sin x$ and $y_2(x) = x^{-1/2}\cos x$ are solutions of the homogeneous equations. The Wronskian of these functions is $W(y_1, y_2) = -1/x$. Rewriting the equation in standard form, we have

$$y'' + \frac{1}{x}y' + \frac{x^2 - 0.25}{x^2}y = 3\frac{\sin x}{x^{1/2}}.$$

Therefore, $g(x) = 3x^{-1/2}\sin x$. Using the method of variation of parameters, a particular solution is given by $Y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ where

$$u_1(x) = -\int \frac{x^{-1/2}\cos x(3x^{-1/2}\sin x)}{W(x)}\,dx = -\frac{3}{2}\cos^2 x$$
$$u_2(x) = \int \frac{x^{-1/2}\sin x(3x^{-1/2}\sin x)}{W(x)}\,dx = \frac{3}{2}\cos x\sin x - \frac{3x}{2}.$$

Therefore, a particular solution is

$$Y(x) = -\frac{3}{2}x^{-1/2}\cos^2 x\sin x + \left(\frac{3}{2}\cos x\sin x - \frac{3x}{2}\right)x^{-1/2}\cos x = -\frac{3}{2}x^{1/2}\cos x.$$

22. Equation (28) is

$$Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2)(s)}\,ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2)(s)}\,ds,$$

where t_0 is now considered the initial point. Bringing the terms $y_1(t)$ and $y_2(t)$ inside the integrals and using the fact that $W(y_1, y_2)(s) = y_1(s)y_2'(s) - y_1'(s)y_2(s)$, the desired result holds. To show that $Y(t)$ satisfies $L[y] = g(t)$ we must take the derivative using Leibniz's rule, which says that if $y(t) = \int_{t_0}^{t} G(t,s)\,ds$, then $Y'(t) = G(t,t) + \int_{t_0}^{t} G_t(t,s)\,ds$. Letting $G(t,s)$ be the above integrand, we have that $G(t,t) = 0$ and

$$\frac{\partial G}{\partial t} = \frac{y_1(s)y_2'(t) - y_1'(t)y_2(s)}{W(y_1, y_2)(s)}g(s).$$

Likewise,

$$Y'' = \frac{\partial G(t,t)}{\partial t} + \int_{t_0}^{t} \frac{\partial^2 G}{\partial t^2}(t,s)\,ds = g(t) + \int_{t_0}^{t} \frac{y_1(s)y_2''(t) - y_1''(t)y_2(s)}{W(y_1, y_2)(s)}g(s)\,ds.$$

Since y_1 and y_2 are solutions of $L[y] = 0$, we have $L[Y] = g(t)$ since all the terms involving the integral will add to zero. Clearly $y(t_0) = 0$ and $y'(t_0) = 0$.

25. The given linear operator $L[y] = [D^2 - 2\lambda D + (\lambda^2 + \mu^2)]y$ can be written as $L[y] = y'' - 2\lambda y' + (\lambda^2 + \mu^2)y$. Therefore, $p(t) = -2\lambda$ and $q(t) = \lambda^2 + \mu^2$. To solve the given nonhomogeneous problem, we first need to solve the associated homogeneous problem. In particular, we need to look for the general solution of

$$y'' - 2\lambda y' + (\lambda^2 + \mu^2)y = 0.$$

The roots of the associated equation are $\lambda \pm i\mu$. Therefore, the general solution of the homogeneous problem is $y(t) = e^{\lambda t}(c_1\cos(\mu t) + c_2\sin(\mu t))$. Letting $y_1(t) = e^{\lambda t}\cos(\mu t)$ and $y_2(t) = e^{\lambda t}\sin(\mu t)$, we now use the result in Problem 22. In particular, a solution of the indicated problem will be given by

$$\begin{aligned} Y(t) &= \int_{t_0}^{t} \frac{e^{\lambda s}\cos(\mu s)e^{\lambda t}\sin(\mu t) - e^{\lambda t}\cos(\mu t)e^{\lambda s}\sin(\mu s)}{\mu e^{2\lambda s}} g(s)\,ds \\ &= \frac{1}{\mu}\int_{t_0}^{t} e^{\lambda(t-s)}\sin(\mu(t-s))g(s)ds. \end{aligned}$$

29. First we put the differential equation in standard form by dividing by t^2; we obtain $y'' - 2y'/t + 2y/t^2 = 8$. Assuming that $y = tv(t)$ and substituting in the differential equation we obtain $tv'' = 8$. Hence $v' = 8\ln t + c_2$ and $v(t) = 8\int \ln t\,dt + c_2t + c_1 = 8(t\ln t - t) + c_2t + c_1$, using integration by parts. Thus $y = 8t^2\ln t + c_3t^2 + c_1t$, where $c_3 = c_2 - 8$. Since $y_1 = c_1t$, we can take $y_2 = 8t^2\ln t + c_3t^2$, where c_3t^2 represents the second fundamental solution of the related homogeneous equation and $8t^2\ln t$ is the particular solution.

3.7

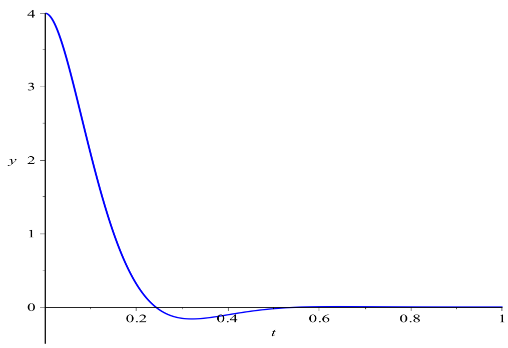
2. $R\cos\delta = -1$ and $R\sin\delta = \sqrt{3}$ implies $R = \sqrt{4} = 2$ and $\delta = \pi + \arctan(-3) = 2\pi/3$. Note that we have to add π to the inverse tangent value since δ must be a second quadrant angle. Therefore, $u = 2\cos(t - 2\pi/3)$.

6. The spring constant is $k = .98/.05 = 19.6$ N/m. The mass $m = 0.1$ kg. Therefore, the equation of motion is $0.1u'' + 19.6u = 0$, which can be simplified to $u'' + 196u = 0$. The initial conditions are $u(0) = 0$ cm, $u'(0) = 20$ cm/sec. The general solution of the differential equation is $u(t) = A\cos 14t + B\sin 14t$. The initial condition implies $A = 0$ and $B = 10/7$. Therefore, the solution is $u(t) = (10/7)\sin 14t$ cm. The period is $T = 2\pi/14$ seconds. Therefore, the mass will first return to its equilibrium position in half that time; that is, in $\pi/14$ seconds.

8. The inductance $L = 1$ henry. The resistance $R = 0$. The capacitance $C = 0.25 \times 10^{-6}$ farads. Therefore, the equation for charge Q is $Q'' + (4 \times 10^6)Q = 0$. The initial conditions are $Q(0) = 2 \times 10^{-6}$ coulombs, $Q'(0) = 0$ coulombs/sec. The general solution of this equation is $Q(t) = A\cos(2000t) + B\sin(2000t)$. The initial conditions imply the specific solution is $Q(t) = 2 \cdot 10^{-6}\cos(2000t)$ coulombs.

9. The spring constant is $k = .196/.05 = 3.92$ N/m. The mass $m = .02$ kg. The damping constant is $\gamma = 400$ dyne-sec/cm $= .4$ N-sec/m. Therefore, the equation of motion is $.02u'' + .4u' + 3.92u = 0$ or $u'' + 20u' + 196u = 0$, with initial conditions $u(0) = .04$ m, $u'(0) = 0$ m/sec. The solution of this equation is $u(t) = e^{-10t}(A\cos(4\sqrt{6}t) + B\sin(4\sqrt{6}t))$. The initial conditions imply that

$$y(t) = e^{-10t}\left[4\cos(4\sqrt{6}t) + (10/\sqrt{6})\sin(4\sqrt{6}t)\right] \text{cm}$$



The quasi frequency is $\nu = 4\sqrt{6}$ rad/sec. The quasi period is $T_d = \pi\sqrt{6}/12$. For undamped motion, the equation would be $y'' + 196y = 0$, which has the general solution $y(t) = A\cos(14t) + B\sin(14t)$. Therefore, the period for the undamped motion is $T = \pi/7$. The ratio of the quasi period to the period of the undamped motion is $T_d/T = 7\sqrt{6}/12 \approx 1.4289$. Using a computer software program we obtain that the solution y will satisfy $|y(\tau)| < 0.05$ for all $\tau > .4579$ seconds.

12. The inductance $L = 0.2$ henry. The resistance $R = 3 \times 10^2$ ohms. The capacitance $C = 10^{-5}$ farads. Therefore, the equation for charge Q is $0.2Q'' + 300Q' + 10^5 Q = 0$, which can be rewritten as $Q'' + 1500Q' + 500,000Q = 0$. The initial conditions are $Q(0) = 3 \cdot 10^{-6}$ coulombs, $Q'(0) = 0$ couloumbs/sec. The general solution of the differential equation is $Q(t) = Ae^{-500t} + Be^{-1000t}$. The initial conditions imply $A + B = 3 \cdot 10^{-6}$ and $-500A - 1000B = 0$. Therefore, $A = 6 \cdot 10^{-6}$ and $B = -3 \cdot 10^{-6}$ and the specific solution is $Q(t) = 10^{-6}(6e^{-500t} - 3e^{-1000t})$.

17. The spring constant is $k = 16/(3/12) = 64$ lb/ft. The mass is $m = 16/32 = 1/2$ lb-s^2/ft. Therefore, the equation of motion is $u''/2 + \gamma u' + 64u = 0$. Using the quadratic formula, the motion will experience critical damping when $\gamma^2 - 4km = 0$, i.e. $\gamma = 2\sqrt{km} = 8\sqrt{2}$ lb-s/ft.

19. If the system is critically damped or overdamped, then $\gamma \geq 2\sqrt{km}$. If $\gamma = 2\sqrt{km}$ (critically damped), then the solution is given by $u(t) = (A + Bt)e^{-\gamma t/2m}$. In this case, if $u = 0$, then we must have $A + Bt = 0$, that is, $t = -A/B$ (assuming $B \neq 0$). If $B = 0$, the solution is never zero (unless $A = 0$). If $\gamma > 2\sqrt{km}$ (overdamped), then the solution is given by $u(t) = Ae^{r_1 t} + Be^{r_2 t}$, where r_1, r_2 are given by equation (22) in the text. Assume for the moment, that $A, B \neq 0$. Then $u = 0$ implies $Ae^{r_1 t} = -Be^{r_2 t}$ which implies $e^{(r_1 - r_2)t} = -B/A$. There is only one solution to this equation. If $A = 0$ or $B = 0$, then there are no solutions to the equation $u = 0$ (unless they are both zero, in which case, the solution is identically zero).

20. If the system is critically damped, then the general solution is $u(t) = (A + Bt)e^{-\gamma t/2m}$. The initial conditions imply $A = u_0$ and $B = v_0 + (\gamma u_0/2m)$. If $v_0 = 0$, then $B = \gamma u_0/2m$. In this case, the specific solution is $u(t) = u_0(1 + \gamma t/2m)e^{-\gamma t/2m}$. As $t \to \infty$, $u \to 0$. In order for $u = 0$, we would need $1 + \gamma t/2m = 0$, but there are no positive times t satisfying this equation. If $v_0 \neq 0$, the specific solution is $u(t) = (u_0 + (v_0 + (\gamma u_0/2m))t)e^{-\gamma t/2m}$. In order to find a positive time such that $u(t) = 0$, we need $u_0 + (v_0 + (\gamma u_0/2m))t = 0$. The solution of this equa-

tion is $t = -u_0/[v_0 + (\gamma u_0/2m)]$. In order for there to be a positive time t satisfying this equation, we need $v_0 + (\gamma u_0/2m) < 0$. That is, we need $v_0 < -\gamma u_0/2m$.

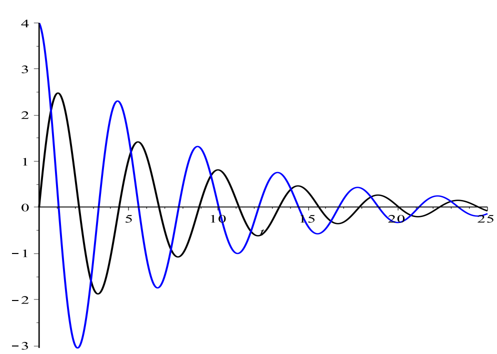
23. For Problem 17, the mass is $m = 1/4$ lb-s^2/ft. We suppose that $\Delta = 3$ and $T_d = 0.3$ seconds. From Problem 21, we know that $\Delta = \gamma T_d/2m$. Therefore, we have $3 = 0.3\gamma/[2(1/4)] = 0.6\gamma$. Therefore, the damping coefficient is $\gamma = 5$ lb-sec/ft.

24. The general solution of this equation is $u(t) = A\cos(\sqrt{2k/3}\,t) + B\sin(\sqrt{2k/3}\,t)$. The period of this system is $T = 2\pi/\sqrt{2k/3}$ sec. Therefore, if the period is $T = \pi$ seconds, then k must satisfy $2\pi/\sqrt{2k/3} = \pi$. The solution of this equation is $k = 6$. The initial conditions $u(0) = 2$ and $u'(0) = v$ imply $A = 2$ and $B = \sqrt{2k/3} = v$. Since $k = 6$, we know that $2B = v$. The amplitude of the system is $\sqrt{A^2 + B^2} = \sqrt{2^2 + (v/2)^2}$. Since the amplitude is assumed to be 3, we must have $\sqrt{2^2 + (v/2)^2} = 3$ which implies $v = \pm 2\sqrt{5}$.

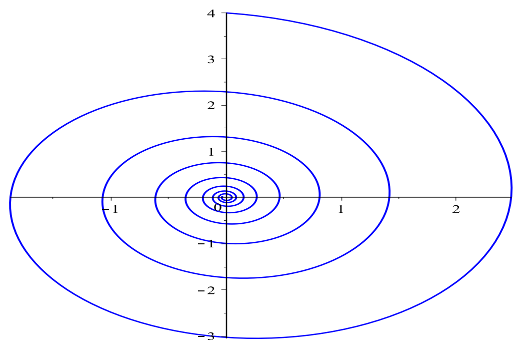
27. First, consider the static case (which is the equilibrium position). Let Δl denote the length of the block below the surface of the water. The weight of the block, which is a downward force, is $w = \rho l^3 g$. This is balanced by an equal and opposite buoyancy force B, which is equal to the weight of the displaced water. Thus $B = \rho_0 l^2 \Delta l g = \rho l^3 g$. Now let $u(t)$ be the displacement of the block from its equilibrium position. We take downward as the positive direction. In a displaced position the forces acting on the block are its weight, which acts downward and is unchanged, and the buoyancy force which is now $\rho_0 l^2(\Delta l + u)g$ and acts upward. The resultant force must be equal to the mass of the block times the acceleration, namely $\rho l^3 u''$. Hence $\rho l^3 g - \rho_0 l^2(\Delta l + u)g = \rho l^3 u''$. Hence the differential equation for the motion of the block is $\rho l^3 u'' + \rho_0 l^2 g u = 0$ or $u'' + \rho_0 g/(\rho l)u = 0$. This gives a simple harmonic motion with frequency $(\rho_0 g/\rho l)^{1/2}$ and natural period $T = 2\pi(\rho l/\rho_0 g)^{1/2}$.

29.(a) The characteristic equation is $4r^2 + r + 8 = 0$, so $r = (-1 \pm \sqrt{127})/8$ and hence $u(t) = e^{-t/8}(c_1\cos\sqrt{127}t/8 + c_2\sin\sqrt{127}t/8)$. The initial condition $u(0) = 0$ implies that $c_1 = 0$ and the initial condition $u'(0) = 4$ implies that $c_2 = 32/\sqrt{127}$. Thus $u(t) = (32/\sqrt{127})e^{-t/8}\sin\sqrt{127}t/8$.

(b)



(c)



The direction of motion is clockwise since the graph starts at $(0, 4)$ and u increases initially.

30.(a) The kinetic energy is given by $mv^2/2$, where v is velocity. Initially, $v = b$, therefore, the kinetic energy initially is $mb^2/2$. The work done deforming a spring an amount y from its undeformed state is stored in the spring and is known as the elastic potential energy. For our example, then, the potential energy is given by $\int_0^x F\,dy = \int_0^x ky\,dy = kx^2/2$. For $x = u(0) = a$, this becomes $ka^2/2$ as the initial potential energy.

(b) The general solution is $u(t) = A\cos(\sqrt{k/m}\;t) + B\sin(\sqrt{k/m}\;t)$. The initial conditions imply that $A = a$ and $B = b\sqrt{m/k}$. Therefore, the specific solution is $u(t) = a\cos(\sqrt{k/m}\;t) + b\sqrt{m/k}\,\sin(\sqrt{k/m}\;t)$.

(c) By the equation for u, we see that

$$\frac{ku^2}{2} = \frac{k}{2}(A^2\cos^2(\sqrt{k/m}\,t) + 2AB\cos(\sqrt{k/m}\,t)\sin(\sqrt{k/m}\,t) + B^2\sin^2(\sqrt{k/m}\,t)).$$

Further, $u'(t) = -A\sqrt{k/m}\sin(\sqrt{k/m}\,t) + B\sqrt{k/m}\cos(\sqrt{k/m}\,t)$ implies

$$\begin{aligned}\frac{m(u')^2}{2} = \frac{m}{2}(&A^2(k/m)\sin^2(\sqrt{k/m}\,t) - 2AB(k/m)\sin(\sqrt{k/m}\,t)\cos(\sqrt{k/m}\,t)\\ &+ B^2(k/m)\cos^2(\sqrt{k/m}\,t)).\end{aligned}$$

Hence the total energy satisfies

$$\frac{ku^2}{2} + \frac{m(u')^2}{2} = \frac{k}{2}(A^2 + B^2) = \frac{k}{2}(a^2 + b^2(m/k)) = \frac{ka^2}{2} + \frac{mb^2}{2}.$$

Therefore, energy is conserved.

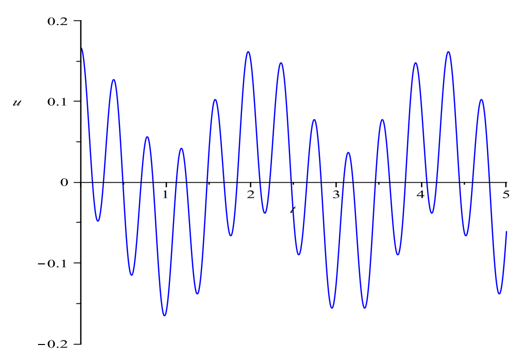
3.8

1. Consider the trigonometric identities $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$. Subtracting these two identities we obtain $\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin\alpha\sin\beta$. Here, our expression is $\cos 11t - \cos 7t$. Therefore, we let $\alpha - \beta = 11t$ and $\alpha + \beta = 7t$. Solving this system of equations, we have $\alpha = 9t$ and $\beta = -2t$. Therefore, we can write $\cos 11t - \cos 7t = 2\sin 9t \sin(-2t) = -2\sin 9t \sin 2t$.

5. The spring constant is $k = 4/(1.5/12) = 32$ lb/ft. The mass is $m = 4/32 = 1/8$ lb-s^2/ft. Assuming no damping, but an external force, $F(t) = 2\cos 3t$, the equation describing the motion is $u''/8 + 32u = 2\cos 3t$ which can be rewritten as $u'' + 256u = 16\cos 3t$. The initial conditions are $u(0) = 3/12 = 1/4$ ft. and $u'(0) = 0$, where u is measured in feet and t in seconds.

7.(a) The solution of the homogeneous problem is $u_h(t) = c_1\cos 16t + c_2\sin 16t$. To find a solution of the nonhomogeneous problem, we look for a solution of the form $U(t) = A\cos 3t$ (since there is no first derivative term, we may exclude the $\sin 3t$ function). Looking for a solution of this form, we arrive at the equation $-9A\cos 3t + 256A\cos 3t = 16\cos 3t$. Therefore, we need A to satisfy $247A = 16$ or $A = 16/247$. Therefore, the solution of the nonhomogeneous problem is $u(t) = c_1\cos 16t + c_2\sin 16t + 16\cos 3t/247$. The initial conditions are $u(0) = 1/6$ and $u'(0) = 0$. Therefore, c_1, c_2 must satisfy $c_1 + 16/247 = 1/6$ and $16c_2 = 0$. So $c_1 = 151/1482$ and $c_2 = 0$. The solution is $u(t) = (151/1482)\cos 16t + (16/247)\cos 3t$.

(b)



(c) Resonance occurs when the frequency ω of the forcing function $4\sin\omega t$ is the same as the natural frequency ω_0 of the system. Since $\omega_0 = 16$, the system will resonate when $\omega = 16$ rad/sec.

10. The spring constant is $k = 8/(1/2) = 16$ lb/ft and the mass is $8/32 = 1/4$ lb-s^2/ft. The forcing term is $8\sin 8t$. Therefore, the equation of motion is $u''/4 + 16u = 8\sin 8t$, which can be simplified to $u'' + 64u = 32\sin 8t$. The solution of the homogeneous problem is $u_h(t) = c_1\cos 8t + c_2\sin 8t$. Then we look for a particular solution of the form $u_p(t) = At\sin 8t + Bt\cos 8t$. Substituting u_p into the ODE, we conclude that $A = 0$ and $B = -2$. Therefore, the general solution of this differential

equation is $u(t) = c_1 \cos 8t + c_2 \sin 8t - 2t \cos 8t$. The initial conditions $u(0) = 1/2$ ft and $u'(0) = 0$ ft/sec imply that $c_1 = 1/2$ and $c_2 = 1/4$. Therefore, the solution of this IVP is $u(t) = (1/2) \cos 8t + (1/4) \sin 8t - 2t \cos 8t$. Solving for u', we see that $u'(t) = (-4 + 16t) \sin 8t$. We see that $u'(t) = 0$ when $t = 1/4$ or $\sin 8t = 0$. Therefore, the first four times the velocity is zero are $t = 1/4, \pi/8, \pi/4, 3\pi/8$.

11.(a) The spring constant is $k = 8/(1/2) = 16$ lb/ft and the mass is $8/32 = 1/4$ lb-s^2/ft. The damping constant is $\gamma = 0.25$ lb-sec/ft. The external force is $4 \cos 2t$ lbs. Therefore, the equation of motion is $u''/4 + u'/4 + 16u = 4 \cos 2t$ which can be simplified to $u'' + u' + 64u = 16 \cos 2t$. The roots of the characteristic equation are $r = (-1 \pm \sqrt{255}i)/2$. Therefore, the solution of the homogeneous equation will be transient. To find the steady-state solution, we look for a particular solution of the form $u_p(t) = A \cos 2t + B \sin 2t$. Substituting u_p into the ODE, we conclude that $A = 240/901$ and $B = 8/901$. Therefore, the steady-state response is $u(t) = (240/901) \cos 2t + (8/901) \sin 2t$.

(b) With a forcing term of the form $F_0 \cos(\omega t)$, the steady-state response can be written as $U(t) = R \cos(\omega t - \delta)$ where the amplitude

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \gamma^2\omega^2}}.$$

The amplitude will be maximized when the denominator is minimized. This will occur when $k = m\omega^2$; that is, when $m = k/\omega^2 = 16/4 = 4$ slugs.

14. Since $U(t) = R \cos(\omega t - \delta)$ we have $U'(t) = -(F_0\omega/\Delta) \sin(\omega t - \delta)$, where Δ is given by Eq.(12). Since F_0 is a constant, differentiate ω/Δ with respect to ω and set it equal to zero. Alternatively, we can minimize $(\Delta/\omega)^2$, which simplifies the differentiation.

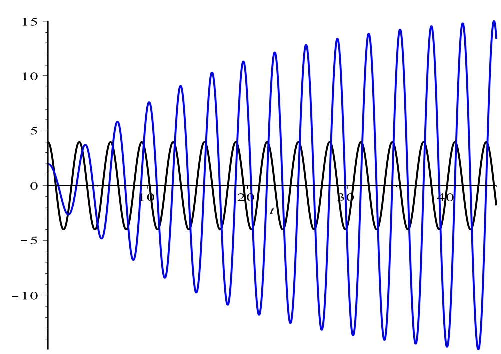
15. First, for $0 \le t \le \pi$, the solution of the homogeneous problem is $u_h(t) = c_1 \cos t + c_2 \sin t$. Then we look for a particular solution of the form $u_p(t) = At$. We see that a particular solution is given by $u_p(t) = F_0 t$. Therefore, the general solution for $0 \le t \le \pi$ is $u(t) = c_1 \cos t + c_2 \sin t + F_0 t$. The initial condition $u(0) = 0$ implies $c_1 = 0$. Then $u'(t) = -c_1 \sin t + c_2 \cos t + F_0$. Therefore, $u'(0) = c_2 + F_0 = 0$ implies $c_2 = -F_0$. Therefore, the solution of this IVP is $u(t) = -F_0 \sin t + F_0 t$ for $0 \le t \le \pi$. Then at time $t = \pi$, $u(\pi) = F_0\pi$ and $u'(\pi) = 2F_0$. In this time interval, the forcing term is $F(t) = F_0(2\pi - t)$. Therefore, we look for a particular solution of the form $u_p(t) = B + Ct$. Substituting a function of this form into the ODE, we see that $B = 2\pi F_0$ and $C = -F_0$. Therefore, the general solution for $\pi < t - 2\pi$ is $u(t) = c_1 \cos t + c_2 \sin t + 2\pi F_0 - F_0 t$. Now considering the initial conditions $u(\pi) = F_0\pi$ and $u'(\pi) = 2F_0$, we need $-c_1 + \pi F_0 = F_0\pi$ and $-c_2 - F_0 = 2F_0$. Therefore, $c_1 = 0$ and $c_2 = -3F_0$. Therefore, the solution of the IVP for $\pi < t \le 2\pi$ is $u(t) = -3F_0 \sin t + 2\pi F_0 - F_0 t$. Then at time $t = 2\pi$, $u(2\pi) = 0$ and $u'(2\pi) = -4F_0$. In this time interval, the forcing term is $F(t) = 0$. Therefore, the general solution is given by the solution of the homogeneous problem, $u(t) = c_1 \cos t + c_2 \sin t$. Considering our initial conditions, we need $c_1 = 0$ and $c_2 = -4F_0$. Therefore, the solution of the IVP for $2\pi < t$ is given by $u(t) =$

$-4F_0 \sin t$. To summarize, we conclude that the solution is given by

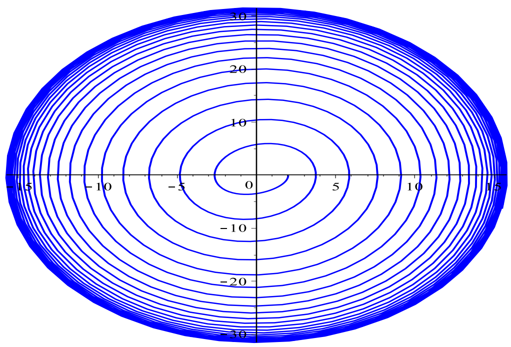
$$u(t) = \begin{cases} -F_0 \sin t + F_0 t & 0 \le t \le \pi \\ -3F_0 \sin t + 2\pi F_0 - F_0 t & \pi < t \le 2\pi \\ -4F_0 \sin t & 2\pi < t. \end{cases}$$

16. The inductance is $L = 1$ henry. The resistance $R = 5 \times 10^3$ ohms. The capacitance $C = 0.25 \times 10^{-6}$ farads. The forcing term is due to the 12−volt battery. Therefore, the equation for charge Q is $Q'' + 5000Q' + (4 \times 10^6)Q = 12$. The initial conditions are $Q(0) = 0$, $Q'(0) = 0$. The solution of the homogeneous problem is $Q_h(t) = c_1 e^{-1000t} + c_2 e^{-4000t}$. The particular solution is of the form $Q_p(t) = A$, so $A = 3 \times 10^{-6}$. Therefore, the general solution is given by $Q(t) = c_1 e^{-1000t} + c_2 e^{-4000t} + 3 \times 10^{-6}$. Considering our initial conditions, we conclude that $c_1 = -4 \times 10^{-6}$ and $c_2 = 10^{-6}$. Therefore, the solution of the IVP is $Q(t) = 10^{-6}(-4e^{-1000t} + e^{-4000t} + 3)$. At $t = 0.001,\ 0.01$, we have $Q(0.001) \approx 1.5468 \times 10^{-6}$ and $Q(0.01) \approx 2.9998 \times 10^{-6}$, respectively. From our function Q, we see that $Q(t) \to 3 \times 10^{-6}$ as $t \to \infty$.

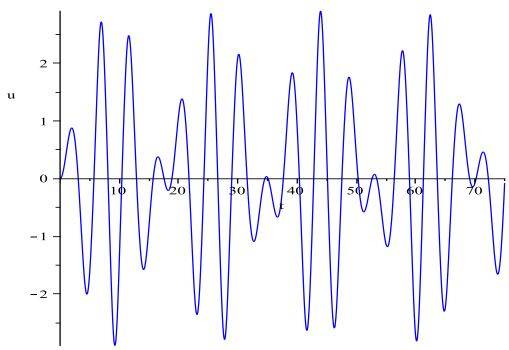
22.(a) The steady-state response is $16 \sin 2t$ and thus the amplitude of the steady-state response is four times the amplitude of the forcing term. This large an increase is due to the fact that the forcing function has the same frequency as the natural frequency, $\omega_0 = 2$, of the system. The graph also shows a phase lag of approximately 1/4 of a period. That is, the maximum of the response occurs 1/4 of a period after the maximum of the forcing function. Both these results are substantially different than those of either Problems 21 or 23.



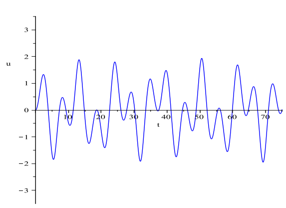
(b) Phase plot - u' vs u :



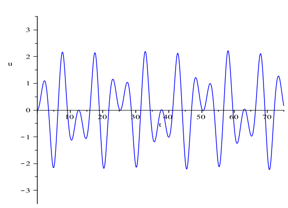
24.(a) For $\omega = 1$:



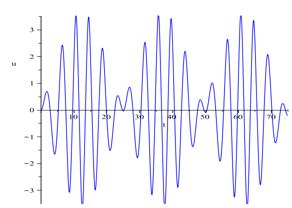
(b)



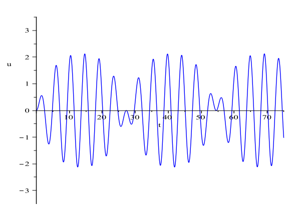
(a) $\omega = 0.5$



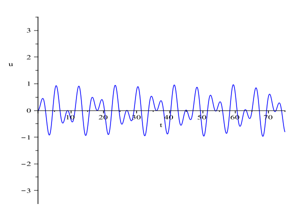
(b) $\omega = 0.75$



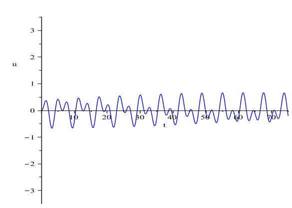
(c) $\omega = 1.25$



(d) $\omega = 1.5$



(e) $\omega = 1.75$



(f) $\omega = 2$

From viewing the above graphs, it appears that the system exhibits a beat near $\omega = 1.5$, while the pattern for $\omega = 1$ is more irregular. However, the system exhibits the resonance characteristic of the linear system for ω near 1, as the amplitude of the response is the largest here.