## 3.6 Method of Variation of Parameters

We continue the discussion of non-homogeneous linear $2^{nd}$ order ODE's: $y'' + p(t)y' + q(t)y = g(t)$, where $g(t)$ is neither an exponential function, nor sin/cosine, nor a polynomial. This means that we cannot make a guess for a particular solution and use the method of undetermined coefficients.

Main result: If $y_1(t)$ and $y_2(t)$ are 2 solutions of the the corresponding homog. ODE that form an fundamental ~~independent~~ solution set, then the general solution is:

$$y(t) = c_1 y_1 + c_2 y_2 + u_1(t) y_1 + u_2(t) y_2 \text{ , where}$$

$$u_1(t) = -\int \frac{y_2(t) g(t)}{W[y_1, y_2]} dt \quad \text{and} \quad u_2(t) = \int \frac{y_1(t) g(t)}{W[y_1, y_2]} dt$$

Ex: Find the general solution of: $y'' + 4y = \csc(t)$.

Solution: Characteristic equation is $r^2 + 4 = 0 \Rightarrow r = \pm 2i$

$\therefore$ A fundamental solution set for the corresponding homogeneous equation consists of:

$$y_1 = \cos(2t) \quad \text{and} \quad y_2 = \sin(2t)$$

Now, because the right hand side is $\csc(t)$, we cannot make a proper guess; so we use the main result of this section; We find $u_1(t)$ and $u_2(t)$. In both we need $W[y_1, y_2] = y_1 y_2' - y_1' y_2$

$$= \cos(2t)(2\cos(2t)) - (-2\sin(2t))\sin(2t)$$

$$= 2\left(\cos^2(2t) + \sin^2(2t)\right) = 2.$$

$$\Rightarrow u_1(t) = -\int \frac{\sin(2t).\csc(t)}{2}\,dt = -\frac{1}{2}\int 2\sin t\cos t\,\frac{1}{\sin t}\,dt$$

$$= -\int \cos t\,dt = -\sin t$$

and $$u_2(t) = \int \frac{\cancel{\sin 2}\cos(2t)\cdot\csc(t)}{2}\,dt$$

$$= \frac{1}{2}\int \frac{1 - 2\sin^2 t}{\sin t}\,dt = \frac{1}{2}\int(\csc t - 2\sin t)\,dt$$

$$= \frac{1}{2}\left[-\ln|\csc t + \cot(t)| + \cos t\right]$$

$\therefore$ General solution is:

$$y(t) = c_1\cos(2t) + c_2\sin(2t) - \sin(t)\cos(2t) + \sin(2t)\left[\frac{\cos t - \ln|\csc t + \cot t|}{2}\right]$$

This is called the method of variation of parameters.