2.6) $M(x,y) + N(x,y)\dfrac{dy}{dx} = 0 \Rightarrow$ soln is $\Psi(x,y) = c$

if exact where $\Psi_x = M$ and $\Psi_y = N$ ($M_y = N_x$ exact)

Thm I) M, N, M_y, N_x are continous over $R \begin{cases} \alpha < x < \beta \\ \gamma < y < \delta \end{cases}$

the DE is exact if $M_y = N_x$ then $\Psi = \int M dx = \dots + \phi(y)$

$\Psi_y = N$ to find $\phi'(y)$ then $\int \Rightarrow \phi(y)$

Thm II) Integrating factors if $M_y \neq N_x$ we find $\mu(x,y)$ and multiply the DE with $\mu(x,y)$ to become exact

if $\dfrac{M_y - N_x}{N}$ depends on x only then $\dfrac{1}{\mu}\dfrac{d\mu}{dx} = \dfrac{M_y - N_x}{N} \Rightarrow \mu(x)$ (set $K=1$)

if $\dfrac{N_x - M_y}{M}$ depends on y only then $\dfrac{1}{\mu}\dfrac{d\mu}{dy} = \dfrac{N_x - M_y}{M} \Rightarrow \mu(y)$

2.7) IVP: $\begin{cases} \dfrac{dy}{dt} = f(t,y) \\ y(t_0) = y_0 \end{cases}$ $y = \phi(t)$ solves the IVP but $y(t_1)$ should be $= \phi(t_1)$

the tangent $L(t)$ to $y = \phi(t)$ at $(t_0, y_0) \approx y = \phi(t)$

$L(t)$: $y - y_0 = m(t - t_0)$ where $m = f(t_0, y_0)$

at $t = t_1$: $\boxed{y_1 = y_0 + f(t_0, y_0)(t_1 - t_0) \approx y(t_1)}$

in General: $\boxed{y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n) \approx y(t_{n+1})}$

if $\underset{\substack{\downarrow \\ h}}{\Delta t} = t_{n+1} - t_n = c$ then $y_{n+1} = y_n + \Delta t\, f(t_n, y_n)$

3.1) A 2^{nd} order DE: $\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt})$ and $y = g(t)$ un known.

The LDE: $\begin{cases} y'' + P(t)y' + Q(t)y = g(t) \\ y(t_0) = y_0 \text{ and } y'(t_0) = y'_0 \end{cases}$

if $g(t) = 0$ then the DE is homogeneous

$ay'' + by' + c = 0$ is a 2^{nd} order homogeneous DE with constant coefficients.

$y = e^{rt}$ and r is roots of $ar^2 + br + c = 0$ so $r_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore$ $\boxed{y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}}$ is the solution

3.2) Soln of linear homogeneous DE: $y'' + P(t)y' + Q(t)y = 0$

$P(t)$ and $Q(t)$ => 2 cont fns on an interval I (α, β)

$L[y] = y'' + P(t)y' + Q(t)y$ → find existance and uniqueness of soln of $L[y]=0$ & i.c.

Thm I) IVP $\begin{cases} L[y] = y'' + P(t)y' + Q(t)y = g(t) \\ y(t_0) = y_0 \; ; \; y'(t_0) = y'_0 \end{cases}$ (P, Q, g cont over interval where $t_0 \in I$)

then there is a unique fn $y = \phi(t)$ that solves the IVP on interval

Thm II) Principle of superposition: let $y_1(t)$ and $y_2(t)$ be 2 solns to the DE $L[y] = y'' + P(t)y' + Q(t)y = 0$ then $c_1 y_1(t) + c_2 y_2(t)$ is also a soln to the DE for any choice of c_1 and c_2

Wronskian determinant: (has ic)

$w(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix} \neq 0$ to get solns

$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y'_0 & y'_2(t_0) \end{vmatrix}}{w}$; $c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y'_1(t_0) & y'_0 \end{vmatrix}}{w}$

Thm III) if $y_1(t)$ and $y_2(t)$ are 2 solns to the DE and $w \neq 0$ then there exists a unique choice of c_1 and c_2 for which $y(t) = c_1 y_1 + c_2 y_2$ solves the IVP.

Thm IV) if $y_1(t)$ and $y_2(t)$ are 2 solns to the DE and there's a pt t_0 st $w(y_1, y_2)(t_0) \neq 0$ then the family of solns $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is the general soln to the DE and y_1 and y_2 form a fundamental set of solns

For $y'' + P(t) y' + Q(t) y = 0$ we assume $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ are solns

$w(y_1, y_2) = (r_2 - r_1) e^{(r_1 + r_2)t} \neq 0$ for $r_1 \neq r_2$

Thm V) let DE: $L[y] = y'' + P(t) y' + Q(t) y = 0$ when $P(t), Q(t)$ are cont over an open interval I and $t_0 \in I$ and y_1 and y_2 are 2 solns to the DE such that $\begin{cases} y_1(t_0) = 1 \; ; \; y_1'(t_0) = 0 \\ y_2(t_0) = 0 \; ; \; y_2'(t_0) = 1 \end{cases}$ then y_1 and y_2 form a fundamental set of solns to the DE on I

3.3) Complex roots; let $ay'' + by' + c = 0 \Rightarrow y = e^{rt}$ and $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

if $b^2 - 4ac < 0$ then $r_{1,2} = \lambda \pm i\mu$ $(i^2 = -1)$ $\therefore$ complex solns

$y_1 = e^{(\lambda + i\mu)t}$ & $y_2 = e^{(\lambda - i\mu)t}$

so $y(t) = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t$ is the general soln.

$\lambda < 0$ tends to 0

$\lambda = 0$ periodic

$\lambda > 0$ unbounded

Euler's formula: $e^{i\theta} = \cos\theta + i \sin\theta$

Euler eqn: $(t > 0)$

$t^2 \frac{d^2 y}{dt^2} + \alpha t \frac{dy}{dt} + \beta t = 0$ take $x = \ln t$ & $\frac{d^2 y}{dx} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0$

get $y = e^{rx}$ and replace $x = \ln t$

3.4) let $ay''+by'+cy=0 \Rightarrow y=e^{rt}$ where $r_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

if $b^2-4ac=0$ then $r_1=r_2=\frac{-b}{2a}$ so $y_1=Ae^{\frac{-b}{2a}t}$

$\Rightarrow y=v(t)\,e^{\frac{-b}{2a}t}$ where $v(t)=c_1t+c_2$

so $\boxed{y(t)=c_1e^{\frac{-b}{2a}t}+c_2te^{\frac{-b}{2a}t}}$ is the general soln.

* Reduction of order: $y''+P(t)y'+Q(t)y=0$ and $y_1(t)$ is one known soln to the DE so we set $y_2(t)=v(t)\,y_1(t)$ and $w(t)=v'$

$y_1\frac{dw}{dt}+(2y_1'+P(t)y_1)\,w=0$ solve for w then $v=\int w\,dt$

$y(t)=v(t)y_1(t)$ is the G.S.

3.5) Thm I:

a) if Y_1 and Y_2 are 2 solns to the nonhomogeneous eqn then Y_1-Y_2 is soln to the associated homogeneous eq.

b) if y_1 and $y_2 \Rightarrow$ fundamental set of soln to $L[y]=0$ then $Y_1-Y_2=c_1y_1+c_2y_2$

Thm II:

The G.S of $y''+P(t)y'+Q(t)y=g(t)$ is $y(t)=c_1y_1+c_2y_2+Y$

method of undetermined coeffs for nonhomogeneous eqs ($Q(t)$ and $P(t)$ are constants)

if $g(t)$ is exp. then $Y(t)$ exp. $\rightarrow$ substitute in DE to get constants

if sin/cos $\Rightarrow Y(t)=A\sin+B\cos$

if polinomial $\Rightarrow Y(t)$ polinomial (same degree)

($A, B\ldots$ should be constants)

if $g(t) = g_1(t) + g_2(t)$ then $Y(t) = Y_1(t) + Y_2(t)$

if $g(t) = 2e^{-t}$ and $Y(t) = Ae^{-t}$ won't work since $y_1(t) = e^{-t}$
so we try $Y(t) = Ate^{-t}$ if no then At^2e^{-t}. —

3.6) variation of parameters $y'' + P(t)y' + Q(t)y = g(t)$ P, Q, g cont. fns
If y_1 and y_2 are solns to $L[y] = 0$ then

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = g(t) \end{cases} \Rightarrow y(t) = u_1 y_1 + u_2 y_2$$

*(Attention coeff of $y'' = 1$)

$$u_1 = \int u_1' \, dt = \int - \frac{y_2 g(t)}{w(y_1, y_2)} \, dt$$

$$u_2 = \int u_2' \, dt = \int \frac{y_1 g(t)}{w(y_1, y_2)} \, dt$$

Method of reduction of order: $y_1(t)$ soln to $L[y] = 0$ then $L[y] = g(t)$ soln
is $y(t) = v(t) y_1(t)$ (must be in form $y'' + P(t)y' + Q(t) = g(t)$)
$w' y_1 + (2y_1' + P(t) y_1) w = g(t)$ (solve method of $\int$ factors) then $v = \int w \, dt$

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