3.1) A 2^{nd} order DE: $\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt})$ and $y = g(t)$ in known.

The LDE: $\begin{cases} y'' + P(t)y' + Q(t)y = g(t) \\ y(t_0) = y_0 \text{ and } y'(t_0) = y'_0 \end{cases}$

if $g(t) = 0$ then the DE is homogeneous

$ay'' + by' + c = 0$ is a 2^{nd} order homogeneous DE with constant coefficients.

$y = e^{rt}$ and r is roots of $ar^2 + br + c = 0$ so $r_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore$ $\boxed{y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}}$ is the solution

3.2) Soln of linear homogeneous DE: $y'' + P(t)y' + Q(t)y = 0$

$P(t)$ and $Q(t) \Rightarrow 2$ cont fns on an interval I (α, β)

$L[y] = y'' + P(t)y' + Q(t)y$ $\curvearrowright$ find existance and uniqueness of soln of $L[y]=0$ @ ic.

Thm I) IVP $\begin{cases} L[y] = y'' + P(t)y' + Q(t)y = g(t) \\ y(t_0) = y_0 \ ; \ y'(t_0) = y'_0 \end{cases}$ $\left(\begin{matrix} P, Q, g \text{ cont} \\ \text{over interval where } t_0 \end{matrix} \right)$

then there is a unique fn $y = \phi(t)$ that solves the IVP on interval

Thm II) Principle of superposition: let $y_1(t)$ and $y_2(t)$ be 2 solns to the DE $L[y] = y'' + P(t)y' + Q(t)y = 0$ then $c_1 y_1(t) + c_2 y_2(t)$ is also a soln to the DE for any choice of c_1 and c_2

Wronskian determinant: (has ic)

$$w(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix} \neq 0 \text{ to get solns}$$

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y'_0 & y'_2(t_0) \end{vmatrix}}{w} \ ; \quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y'_1(t_0) & y'_0 \end{vmatrix}}{w}$$

Thm III) if $y_1(t)$ and $y_2(t)$ are 2 solns to the DE and $w \neq 0$ then there exists a unique choice of c_1 and c_2 for which $y(t) = c_1 y_1 + c_2 y_2$ solves the IVP.

Thm IV) if $y_1(t)$ and $y_2(t)$ are 2 solns to the DE and there's a pt t_0 st $w(y_1, y_2)(t_0) \neq 0$ then the family of solns $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is the general soln to the DE and y_1 and y_2 form a fundamental set of solns

For $y'' + P(t) y' + Q(t) y = 0$ we assume $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ are solns

$w(y_1, y_2) = (r_2 - r_1) e^{(r_1 + r_2) t} \neq 0$ for $r_1 \neq r_2$

Thm V) let DE: $L[y] = y'' + P(t) y' + Q(t) y = 0$ when $P(t), Q(t)$ are cont over an open interval I and $t_0 \in I$ and y_1 and y_2 are 2 solns to the DE such that $\begin{cases} y_1(t_0) = 1 \; ; \; y_1'(t_0) = 0 \\ y_2(t_0) = 0 \; ; \; y_2'(t_0) = 1 \end{cases}$ then y_1 and y_2 form a fundamental set of solns to the DE on I

3.3) Complex roots; let $ay'' + by' + c = 0 \Rightarrow y = e^{rt}$ and $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

if $b^2 - 4ac < 0$ then $r_{1,2} = \lambda \pm i\mu$ $(i^2 = -1)$ ∴ complex solns

$y_1 = e^{(\lambda + i\mu)t}$ & $y_2 = e^{(\lambda - i\mu)t}$

conc $y(t) = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t$ is the general soln.

$\lambda < 0$ tends to 0

$\lambda = 0$ periodic

$\lambda > 0$ unbounded

Euler's formula: $e^{i\theta} = \cos\theta + i \sin\theta$

Euler eqn: $(t > 0)$

$t^2 \frac{d^2 y}{dt^2} + \alpha t \frac{dy}{dt} + \beta t = 0$ take $x = \ln t$ & $\frac{d^2 y}{dx^2} + (\alpha - 1) \frac{dy}{dx} + \beta y = 0$

get $y = e^{rx}$ and replace $x = \ln t$

3.4) let $ay''+by'+cy=0 \Rightarrow y=e^{rt}$ where $r_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

if $b^2-4ac=0$ then $r_1=r_2=\frac{-b}{2a}$ so $y_1=Ae^{\frac{-b}{2a}t}$

$\Rightarrow y=V(t)\,e^{\frac{-b}{2a}t}$ where $V(t)=c_1t+c_2$

so $\boxed{y(t)=c_1e^{\frac{-b}{2a}t}+c_2te^{\frac{-b}{2a}t}}$ is the general soln.

* Reduction of order: $y''+P(t)y'+Q(t)y=0$ and $y_1(t)$ is one known soln to the DE so we set $y_2(t)=V(t)\,y_1(t)$ and $w(t)=v'$

$y_1\frac{dw}{dt}+(2y_1'+P(t)y_1)\,w=0$ solve for w then $V=\int w\,dt$

$y(t)=v(t)y_1(t)$ is the G.S.

3.5) Thm I:

a) if Y_1 and Y_2 are 2 solns to the nonhomogenous eqn then Y_1-Y_2 is soln to the associated homogenous eq.

b) if y_1 and y_2 $\Rightarrow$ fundamental set of soln to $L[y]=0$ then $Y_1-Y_2=c_1y_1+c_2y_2$

Thm II:

The G.S of $y''+P(t)y'+Q(t)y=g(t)$ is $y(t)=c_1y_1+c_2y_2+Y$

method of undetermined coeffs for nonhoms eqs ($Q(t)$ and $P(t)$ are constants)

if $g(t)$ is exp. then $Y(t)$ exp. $\rightarrow$ substitute in DE to get constants

if sin/cos $\Rightarrow$ $Y(t)=A\sin+B\cos$

if polinomial $\Rightarrow$ $Y(t)$ polinomial (same degree)

($A, B\ldots$ should be constants)

if $g(t) = g_1(t) + g_2(t)$ then $Y(t) = Y_1(t) + Y_2(t)$

if $g(t) = 2e^{-t}$ and $Y(t) = Ae^{-t}$ won't work since $y_1(t) = e^{-t}$
so we try $Y(t) = Ate^{-t}$ if no then At^2e^{-t} . —

3.6) Variation of parameters $y'' + P(t)y' + Q(t)y = g(t)$ P, Q, g conts fns
If y_1 and y_2 are solns to $L[y] = 0$ then

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = g(t) \end{cases} \Rightarrow y(t) = u_1 y_1 + u_2 y_2$$

*(Attention coeff of $y'' = 1$)

$$u_1 = \int u_1' \, dt = \int -\frac{y_2 \, g(t)}{W(y_1, y_2)} \, dt$$

$$u_2 = \int u_2' \, dt = \int \frac{y_1 \, g(t)}{W(y_1, y_2)} \, dt$$

Method of reduction of order: $y_1(t)$ soln to $L[y] = 0$ then $L[y] = g(t)$ soln is $y(t) = v(t)\, y_1(t)$ (must be in form $y'' + P(t)y' + Q(t) = g(t)$)
$w' y_1 + (2y_1' + P(t)\, y_1)\, w = g(t)$ (solve method of $\int$ factors) then $v = \int w \, dt$

* Remember method of $\int$ factors: $w' + P_{new}(t)\, w = Q_{new}(t)$

① $\mu(t) = e^{\int P_n(t)\, dt}$

② $w(t) = \dfrac{1}{\mu(t)} \int \mu(t)\, Q(t) \, dt$

4.1) $\frac{d^n y}{dt^n} + P_1(t)\frac{d^{n-1}y}{dt^{n-1}} --- P_n y = g(t)$

* Thm I: if $P_1(t) ---- P_n(t)$, $g(t)$ are continuous on I, there exists a soln to IVP.

if $y_1 -- y_n$ are solns, $y(t) = c_1 y_1 + c_2 y_2 -- c_n y_n$ is soln

n eqns $\begin{cases} c_1 y_1(t_0) + c_2 y_2 + --- c_n y_n = y_0 \\ \vdots \\ c_1 y_1^{(n-1)}(t_0) + c_2 y_2^{(n-1)} --- c_n y_n^{(n-1)} = y_0^{n-1} \end{cases}$

the system has a unique soln if its wronskian $w = \begin{vmatrix} y_1(t_0) & --- & y_n(t_0) \\ \vdots & & \\ y_1^{(n-1)} & --- & y_n^{(n-1)} \end{vmatrix} \neq 0$

* Thm II: $y_1 -- y_n$ form a fundamental set of solns if $P_1 -- P_n$ are continuous and $y_1 -- y_n$ are solns with $w(y_1 - y_n) \neq 0$

* Non homogeneous: $\frac{d^n y}{dt^n} + P_1\frac{d^{n-1}y}{dt^{n-1}} --- P_n y = g(t)$

$y(t) = c_1 y_1 + c_2 y_2 -- c_n y_n + Y(t)$

* Reduction of order: $y''' + P_1 y'' + P_2 y' + P_3 y = 0$ set $y = y_1 v$ and substitute

$y_1 w'' + w'[3y_1' + P_1 y_1] + w[3y_1'' + 2P_1 y_1' + P_2 y_1] = 0$ then $v = \int w\,dt$

4.2) Homogeneous eqns with constant coefficients: $a_0 y^{(n)} + a_1 y^{(n-1)} --- a_n y = 0$

$y = e^{rt}$; $a_0 r^n + a_1 r^{n-1} --- a_n = 0$

If roots are real + distinct: $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} -- c_n e^{r_n t}$

if complex roots $r_{2,3} = \lambda \pm i\mu$ $y(t) = c_1 e^{rt} + c_2 e^{\lambda t}\cos\mu t + e^{\lambda t}\sin\mu t$ --

if repeated roots if r_k is repeated with multiplicity s ie $(r - r_k)^s = P(r)$

real: $e^{r_k t} --- t^{s-1} e^{r_k t}$

complex: $\begin{cases} e^{\lambda t}\cos\mu t, e^{\lambda t}\sin\mu t \\ \vdots \\ t^{s-1} e^{\lambda t}\cos\mu t, t^{s-1} e^{\lambda t}\sin\mu t \end{cases}$

real: $e^{r_k t} --- t^{s-1} e^{r_k t}$

complex: $\begin{cases} e^{\lambda t}\cos\mu t; e^{\lambda t}\sin\mu t \\ \vdots \\ t^{s-1} e^{\lambda t}\cos\mu t; t^{s-1} e^{\lambda t}\sin\mu t \end{cases}$

5.2) 2^{nd} order: $a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0$; $y'' + P_1(x)y' + P_2(x)y = 0$

$y = \sum_{n=0}^{\infty} C_n (x-x_0)^n$; $\frac{dy}{dx} = \sum_{n=1}^{\infty} n C_n x^{n-1}$; $\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$ if $x_0=0$

analytic if $f(x) = \sum_{n=0}^{\infty} f^{(n)}(x_0)\frac{(x-x_0)^n}{n!}$ ' ex polynomial, sin, cos, exp. etc

x_0 is ordinary if P_1 and P_2 are analytic at x_0

substitute y, y', y'' in DE and change all to x^n like;

$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$ $m = n-2$, $n = m+2$ if $n=2$ $m=0$; $\underset{n\to\infty}{m\to\infty}$

$\sum_{m=0}^{\infty} (m+2)(m+1) C_{m+2} x^m \rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} x^n$

then change all to same $\sum_{n=}^{\infty}$ (take highest)

$\underbrace{(\text{constants})}_{=0} + \underbrace{(\text{---})}_{=0}x + \text{---} \underbrace{\sum_{n=}^{\infty}[\text{---}]x^n}_{=0} = 0$ to get eqn for constants find first few

in terms of C_0 and C_1 then get soln. $y = \sum_{n=0}^{\infty} C_n (x)^n$

take x_0 at ic, if ic at 2 $\rightarrow \sum_{n=0}^{\infty} C_n (x-2)^n$ set $t = x-2$ substitute in DE

6.1) $L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt$

6.2) $L\{f'(t)\} = sL\{f(t)\} - f(0)$; $sY - y(0)$

$L\{f''(t)\} = s^2 L\{f(t)\} - s f(0) - f'(0)$; $s^2 Y - s y(0) - y'(0)$

$L\{f^{(n)}(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) \text{ --- } - f^{(n-1)}(0)$

set $Y(s) = L\{y(t)\}$

6.3) $f(t) = \begin{cases} A & \text{if } 0 \le t < a \\ B & \text{if } a \le t < b \\ C & \text{if } t \ge b \end{cases}$ $\quad f(t) = A + (B-A)\,U_a(t) + (C-B)\,U_b(t)$

$U_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \ge c \end{cases}$

$L\{U_c(t)\} = \frac{e^{-sc}}{s}$ for $s > 0$

Thm I: $L\{U_c(t)\,f(t-c)\} = e^{-cs}\,L\{f(t)\} = e^{-cs}\,F(s)$

and $L^{-1}\{e^{-cs}\,F(s)\} = U_c(t)\,f(t-c)$

Thm II: $L\{e^{ct}f(t)\} = F(s-c)$ where $F(s) = L\{f(t)\}$

and $L^{-1}\{F(s-c)\} = e^{ct}\,f(t)$

6.4) $\int_0^{\infty} g(t)\,\delta(t-a)\,dt = g(a)$; $L\{\delta(t-a)\} = e^{-as}$

6.5) $(f*g)(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$; $H(s) = F(s)\,G(s) = L\{h(t)\}$

where $h(t) = (f*g)(t)$

7.1) Change $ay'' + by' + cy = 0$ to system $x_1 = y$ $\quad x_2 = \frac{dx_1}{dt} = y'$ & $\frac{dx_2}{dt} = y''$

$y'' = \frac{-by' - cy}{a}$ is $\begin{cases} \frac{dx_2}{dt} = \frac{-b}{a}x_2 - \frac{c}{a}x_1 \\ \frac{dx_1}{dt} = x_2 \end{cases}$ ~~and~~

7.4) system: $\begin{cases} \frac{dx_1}{dt} = P_{11}x_1 \cdots P_{1n}x_n + g_1(t) \\ \vdots \\ \frac{dx_n}{dt} = P_{n1}x_1 + P \cdots P_{nn}x_n + g_n(t) \end{cases}$ $\Rightarrow \frac{dx}{dt} = P\cdot x + g$

where $x\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is soln

$x_1 \cdots x_n$ are linearly independent if the $\det(x_1 \cdots x_n) \neq 0$ then they form a fundamental set of soln. we must show $w(x^{(1)} \cdots x^{(n)}) \neq 0$

7.5) $x(t) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\quad A$ $n\times n$ constant matrix : $\frac{dx}{dt} = A\cdot x$ if $\det(A) \neq 0$ then $x\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ is only soln

Solving homogeneous systems of form $x' = A \cdot x$

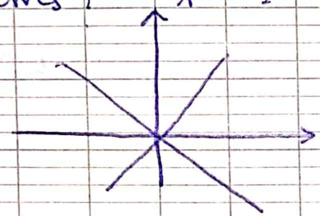
$x = \xi e^{rt}$ where $\det(A - rI) = 0$ (subtract r from diag of A) to get r and solve $A\xi = r\xi$

for each r to get ξ (2 redundant eqns set $\xi_1 = 1$) so $x^{(1)} = \xi^{(1)} e^{r_1 t}$ & $x^{(2)} = \xi^{(2)} e^{r_2 t}$

then must check for $W(x^{(1)}, x^{(2)}) \neq 0$ for $x(t) = c_1 x^{(1)} + c_2 x^{(2)}$

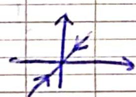
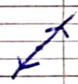
To visualize curves: $x^{(1)} = \begin{pmatrix} a \\ 2a \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ so $x_2 = 2x_1$ strt line

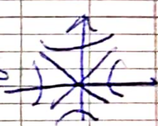
same for $x^{(2)}$



as $t \to \infty$

x_1 and $x_2 \to \infty$ or 0

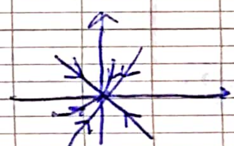
if $r < 0$ then  if $r > 0$ 

if ~~both~~ $r_1 > 0$ $r_2 < 0$ then x_1 dominates $t \to \infty$ and x_2 for $t \to -\infty$ so 

origin is saddle point since its only eq. soln

if r_1 & $r_2 < 0$ then highest dominates $x^{(1)} = e^{-t}$ $x^{(2)} = e^{-4t}$

here its asymptotically stable



7.6) when soln of $\det(A - rI) = 0$ gives complex $r_1 = \lambda + i\mu = \overline{r_2}$ leads to

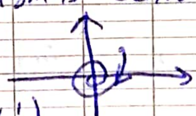
real soln: $x(t) = c_1 u(t) + c_2 v(t)$ and $\xi^{(1)} = a + ib = \overline{\xi^{(2)}}$

$u(t) = e^{\lambda t}[a \cos \mu t - b \sin \mu t]$; $v(t) = e^{\lambda t}[a \sin \mu t + b \cos \mu t]$

here soln isn't strt line;

if $\lambda > 0$ spiral away from origin  origin is unstable

if $\lambda = 0$ closed curve (origin is center)

if $\lambda < 0$ spiral towards origin  or  origin stable

to know directions $A \cdot x$ at $x(1)$ so we get vector tangent to spiral (left/right, up/down)

6.7) solve system using Laplace: Laplace both systems and solve $Y_1(s)$ and $Y_2(s)$

normally then find $y_1(t) = L^{-1}\{Y_1(s)\}$ and y_2