3.3 - Complex Roots of the characteristic Equation

We continue the discussion of $ay'' + by' + cy = 0$, where a, b, c are real numbers, but we consider the case where the characteristic equation

$$ar^2 + br + c = 0$$

has complex roots; ie $b^2 - 4ac < 0$.

As a reminder, every complex number z can be written in the form: $z = x + iy$, where $i^2 = -1$; x is a real number called the real part of z and y is a real number also called the imaginary part of z

In case $b^2 - 4ac < 0$, then the complex roots of the characteristic equation are:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-b + i\sqrt{4ac - b^2}}{2a} = \frac{-b}{2a} + i\frac{\sqrt{4ac - b^2}}{2a}$$

and

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-b - i\sqrt{4ac - b^2}}{2a} = \frac{-b}{2a} - i\frac{\sqrt{4ac - b^2}}{2a}$$

Notice that r_1 and r_2 are complex conjugates of each other ($x+iy$ and $x-iy$ are said to be cpx conjugates)

Let $\lambda = \frac{-b}{2a}$ and $\mu = \frac{\sqrt{4ac-b^2}}{2a}$, then

$r_1 = \lambda + i\mu$ while $r_2 = \lambda - i\mu$.

If we were to follow the same ~~p~~ mode of thinking as in the case of real roots for the characteristic polynomial, then ~~...~~ solutions to the ODE taken the form:

$$y_1 = e^{r_1 t} = e^{(\lambda+i\mu)t} = e^{\lambda t + i\mu t} = e^{\lambda t} \cdot e^{i\mu t}$$

$$\text{and } y_2 = e^{r_2 t} = e^{(\lambda - i\mu)t} = e^{\lambda t - i\mu t} = e^{\lambda t} \cdot e^{-i\mu t}$$

Theorem: $e^{i\theta} = \cos\theta + i\sin\theta$. (Proof!)

Hence, $y_1 = e^{\lambda t}\left[\cos(\mu t) + i\sin(\mu t)\right]$

and $y_2 = e^{\lambda t}\left[\cos(\mu t) - i\sin(\mu t)\right]$.

Ex: $y'' + y' + 9.25y = 0$

$$\Rightarrow r^2 + r + 9.25 = 0 \Rightarrow r = \frac{-1 \pm \sqrt{1-37}}{2} = \frac{-1 \pm \sqrt{-(6^2)}}{2}$$

$$\Rightarrow r = -\frac{1}{2} \pm i\frac{6}{2} = -\frac{1}{2} \pm 3i.$$

$\therefore$ 2 solutions to the ODE are:

$$y_1 = e^{(-\frac{1}{2}+3i)t} = e^{-\frac{t}{2}} e^{i(3t)} = e^{-\frac{t}{2}}[\cos 3t + i \sin 3t]$$

and $$y_2 = e^{(-\frac{1}{2}-3i)t} = e^{-\frac{t}{2}} e^{i(-3t)} = e^{-\frac{t}{2}}[\cos(-3t) + i\sin(-3t)]$$

or $$y_2 = e^{-\frac{t}{2}}[\cos(3t) - i\sin(3t)]$$

What do these solutions "mean" physically?

Naturally, the solutions make sense if they were ~~real~~ real! But here they are not!

Can we extract real solutions from the complex solutions?

Theorem: If $z(t) = x(t) + i y(t)$ is a complex solution for the ODE $ay'' + by' + cy = 0$, then $x(t)$ and $y(t)$ are two real solutions of the same ODE.

Proof: $z = x(t) + i y(t)$

$$z' = x(t) + i y'(t); \quad z'' = x''(t) + i y''(t).$$

Substitute:

$$a\left(x''(t) + i y''(t)\right) + b\left(x'(t) + i y'(t)\right) + c\left(x(t) + i y(t)\right) = 0$$

$$\Rightarrow \left(a x'' + b x' + c x\right) + i\left(a y'' + b y'(t) + c y(t)\right) = 0$$

$$\Rightarrow \quad a x'' + b x' + c x = 0 \text{ and } a y'' + b y' + c y = 0$$

and the result follows.

This theorem is implying that from each complex solution of an ODE, we can deduce two real solutions.

Ex1 Back to $y'' + y' + 9.25y = 0$

We established that

$$y_1 = e^{-\frac{t}{2}}\cos(3t) \mp i\, e^{-\frac{t}{2}}\sin(3t)$$ is one complex solution.

Hence, $e^{-\frac{t}{2}}\cos(3t)$ and $e^{-\frac{t}{2}}\sin(3t)$ are two real solutions.

If $W\left[e^{-\frac{t}{2}}\cos(3t),\ e^{-\frac{t}{2}}\sin(3t)\right] \neq 0$, then they form a fundamental solution set.

Indeed,

~~We~~

$$\left[\begin{aligned} e^{-\frac{t}{2}}\cos(3t) &= y_1 \\ -\tfrac{1}{2}e^{-\frac{t}{2}}\cos(3t) - 3e^{-\frac{t}{2}}\sin(3t &= y_1' \end{aligned}\right] \quad \left[\begin{aligned} y_2 &= e^{-\frac{t}{2}}\sin(3t) \\ y_2' &= -\tfrac{1}{2}e^{-\frac{t}{2}}\sin(3t) \\ &\quad + 3e^{-\frac{t}{2}}\cos(3t) \end{aligned}\right]$$

$$y_1 y_2' - y_1' y_2 =$$

$$e^{-t}\left[-\tfrac{1}{2}\cos(3t)\sin(3t) + 3\cos^2(3t)\right] - e^{-t}\left[-\tfrac{1}{2}\sin(3t)\cos(3t) - 3\sin^2(3t)\right]$$

$$= 3\sin^2(3t) + 3\cos^2(3t) = 3 \neq 0.$$

Hence, a general solution to

$$y'' + y' + 9.25y = 0$$

is:

$$\boxed{y = C_1 e^{-\frac{t}{2}}\cos(3t) + C_2 e^{-\frac{t}{2}}\sin(3t)}$$

Remark: We had found a second complex solution for the ODE:

$$y_2 = \underbrace{e^{-\frac{t}{2}}\cos(3t)} - \underbrace{i e^{-\frac{t}{2}}\sin(3t)}$$

the real and imaginary parts of y_2 will not produce a different family of real solutions. (try it).

So it is enough to focus on one complex solution.

Moreover, it is always the case that the Wronskian of the real and imaginary parts are $\neq 0$

Ex 2: Solve $y'' + 9y = 0$

Solution: Characteristic equation is: $r^2 + 9 = 0$

$$\Rightarrow \quad r = \pm 3i = \underbrace{0}_{\lambda} \pm \underbrace{3}_{\mu} i$$

$\therefore$ the complex root $e^{(0+3i)t} = e^{i(3t)}$

$$= \cos(3t) + i\sin(3t)$$

Shall produce two real solutions:

$$y_1 = \cos(3t) \quad \text{and} \quad y_2 = \sin(3t)$$

And the general solution is:

$$y = C_1 \cos(3t) + C_2 \sin(3t).$$

Ex 3: Solve: $16y'' - 8y' + 145y = 0; \; y(0) = -2; y'(0) = -1$

Solution: $16r^2 - 8r + 145 = 0$

$$\Rightarrow \quad r = \frac{8 \pm \sqrt{8^2 - (16)(145)4}}{2 \times 16} = \cancel{8 \pm i\sqrt{2256}}$$

$$= \frac{8 \pm \sqrt{-9216}}{32} = \frac{8 \pm i(96)}{32}$$

$$= \frac{1}{4} \pm 3i$$

Hence the real solutions of the ODE are:

$$y_1 = e^{\frac{1}{4}t}\cos(3t) \text{ and } y_2 = e^{\frac{1}{4}t}\sin(3t).$$

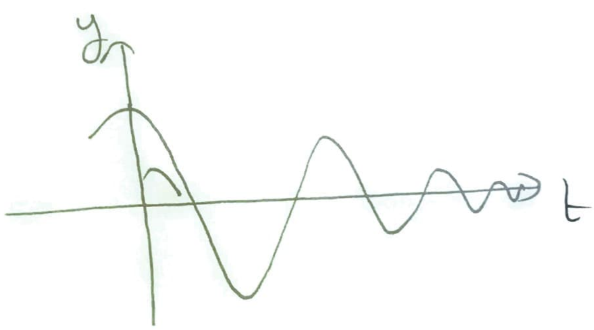
And general solution is:

$$y = C_1 e^{\frac{t}{4}}\cos(3t) + C_2 e^{\frac{t}{4}}\sin(3t).$$

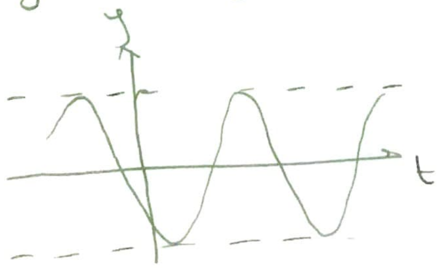
We use $y(0) = -2$, $y'(0) = -1$ to conclude $C_1 = -2$, $C_2 = \frac{1}{2}$.

Some graphs

Ex 1: $y'' + y' + 9.25y = 0 \longrightarrow y = C_1 e^{-\frac{t}{2}}\cos(3t) + C_2 e^{-\frac{t}{2}}\sin(3t)$



Ex 2: $y'' + 9y = 0 \longrightarrow y = C_1\cos(3t) + C_2\sin(3t)$



Ex 3: $16y'' - 8y' + 145y = 0$

$$y = C_1 e^{\frac{t}{4}}\cos(3t) + C_2 e^{\frac{t}{4}}\sin(3t)$$