Solution. By (1) we can write $y - (1 + t) * y = 1 - \sinh t$. Writing $Y = \mathcal{L}(y)$, we obtain by using the convolution theorem and then taking common denominators

$$Y(s)\left[1 - \left(\frac{1}{s} + \frac{1}{s^2}\right)\right] = \frac{1}{s} - \frac{1}{s^2 - 1}, \qquad \text{hence} \qquad Y(s) \cdot \frac{s^2 - s - 1}{s^2} = \frac{s^2 - 1 - s}{s(s^2 - 1)}.$$

$(s^2 - s - 1)/s$ cancels on both sides, so that solving for Y simply gives

$$Y(s) = \frac{s}{s^2 - 1} \qquad \text{and the solution is} \qquad y(t) = \cosh t.$$

PROBLEM SET 6.5

1–7 CONVOLUTIONS BY INTEGRATION

Find:

1. $1 * 1$
2. $1 * \sin \omega t$
3. $e^t * e^{-t}$
4. $(\cos \omega t) * (\cos \omega t)$
5. $(\sin \omega t) * (\cos \omega t)$
6. $e^{at} * e^{bt} \, (a \neq b)$
7. $t * e^t$

8–14 INTEGRAL EQUATIONS

Solve by the Laplace transform, showing the details:

8. $y(t) + 4\int_0^t y(\tau)(t - \tau)\, d\tau = 2t$
9. $y(t) - \int_0^t y(\tau)\, d\tau = 1$
10. $y(t) - \int_0^t y(\tau) \sin 2(t - \tau)\, d\tau = \sin 2t$
11. $y(t) + \int_0^t (t - \tau) y(\tau)\, d\tau = 1$
12. $y(t) + \int_0^t y(\tau) \cosh(t - \tau)\, d\tau = t + e^t$
13. $y(t) + 2e^t \int_0^t y(\tau) e^{-\tau}\, d\tau = te^t$
14. $y(t) - \int_0^t y(\tau)(t - \tau)\, d\tau = 2 - \frac{1}{2}t^2$

15. **CAS EXPERIMENT. Variation of a Parameter.**
(a) Replace 2 in Prob. 13 by a parameter k and investigate graphically how the solution curve changes if you vary k, in particular near $k = -2$.
(b) Make similar experiments with an integral equation of your choice whose solution is oscillating.

16. **TEAM PROJECT. Properties of Convolution.** Prove:
(a) Commutativity, $f * g = g * f$
(b) Associativity, $(f * g) * v = f * (g * v)$
(c) Distributivity, $f * (g_1 + g_2) = f * g_1 + f * g_2$
(d) Dirac's delta. Derive the sifting formula (4) in Sec. 6.4 by using f_k with $a = 0$ [(1), Sec. 6.4] and applying the mean value theorem for integrals.
(e) Unspecified driving force. Show that forced vibrations governed by

$$y'' + \omega^2 y = r(t), \quad y(0) = K_1, \quad y'(0) = K_2$$

with $\omega \neq 0$ and an unspecified driving force $r(t)$ can be written in convolution form,

$$y = \frac{1}{\omega} \sin \omega t * r(t) + K_1 \cos \omega t + \frac{K_2}{\omega} \sin \omega t.$$

17–26 INVERSE TRANSFORMS BY CONVOLUTION

Showing details, find $f(t)$ if $\mathcal{L}(f)$ equals:

17. $\dfrac{5.5}{(s + 1.5)(s - 4)}$
18. $\dfrac{1}{(s - a)^2}$
19. $\dfrac{2\pi s}{(s^2 + \pi^2)^2}$
20. $\dfrac{9}{s(s + 3)}$
21. $\dfrac{\omega}{s^2(s^2 + \omega^2)}$
22. $\dfrac{e^{-as}}{s(s - 2)}$
23. $\dfrac{40.5}{s(s^2 - 9)}$
24. $\dfrac{240}{(s^2 + 1)(s^2 + 25)}$
25. $\dfrac{18s}{(s^2 + 36)^2}$

26. **Partial Fractions.** Solve Probs. 17, 21, and 23 by partial fraction reduction.