Exact Equations

Given $\frac{dy}{dt} = f(t,y)$, we shall rewrite this equation in the form of $M(t,y) + N(t,y)\frac{dy}{dt} = 0$.

Remark: For this section, it is customary to use x for the independent variable rather than t.

Ex: Consider the ODE: $\frac{dy}{dx} = \frac{9x^2 - 2xy}{(2y + x^2 + 1)} \leftarrow = f(x,y)$

then this can be re-written as:

$$(2y + x^2 + 1)\frac{dy}{dx} = 9x^2 - 2xy$$

or $\underbrace{(2xy - 9x^2)}_{M(x,y)} + \underbrace{(2y + x^2 + 1)}_{N(x,y).}\frac{dy}{dx} = 0$

Definition: An ODE $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ is called exact if: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Ex: In the example above:

$\frac{\partial M}{\partial y} = 2x$; $\frac{\partial N}{\partial x} = 2x$ ✓ i.e. exact.

What is so special about an exact equation?

Suppose there exists a function $\psi(x,y)$ such that $\frac{\partial \psi}{\partial x} = M(x,y)$ and $\frac{\partial \psi}{\partial y} = N(x,y)$.

Then the ODE becomes:

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \cdot \frac{dy}{dx} = 0.$$

By the chain rule of differentiation for a function of several variables:

$$\frac{d}{dx}\left[\psi(x,y)\right] = \frac{\partial \psi}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial \psi}{\partial y} \cdot \frac{dy}{dx}$$

$$= \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \cdot \frac{dy}{dx}$$

$\Rightarrow$

$$\frac{d}{dx}\left[\psi(x,y)\right] = 0 \Rightarrow \boxed{\psi(x,y) = \text{constant}}$$ is the family of solutions to the ODE.

<u>Back to example</u>: $(2xy - 9x^2) + (2y + x^2 + 1)\frac{dy}{dx} = 0$

is exact $\Rightarrow$ family of solutions is:

$$\psi(x,y) = c$$

where: $\frac{\partial \psi}{\partial x} = M(x,y)$ and $\frac{\partial \psi}{\partial y} = N(x,y)$ (*)

The 2 equations in $(*)$ allow us to find $\psi(x,y)$:

$$\frac{\partial \psi}{\partial x} = M(x,y) = (2xy - 9x^2)$$

$$\Rightarrow \psi(x,y) = \int (2xy - 9x^2)\,dx = x^2y - 3x^3 + \underbrace{C(y)}_{\text{may depend on } y.}$$

But $\dfrac{\partial \psi}{\partial y} = (2y + x^2 + 1)$

$$\Rightarrow \frac{\partial}{\partial y}\left[x^2y - 3x^3 + C(y)\right] = 2y + x^2 + 1$$

$$\Rightarrow x^2 + C'(y) = 2y + x^2 + 1 \Rightarrow C'(y) = 2y + 1$$

$$\Rightarrow C(y) = y^2 + y + K.$$

$\therefore$ family of solutions is:

$$x^2y - 3x^3 + y^2 + y + K = C$$

or $\boxed{x^2y - 3x^3 + y^2 + y = C}$

Ex: Is $\dfrac{2xy}{x^2+1} - 2x - \underbrace{(2 - \ln(x^2+1))}\dfrac{dy}{dx} = 0$ exact?

If so, solve the IVP $y(5) = 0$.

Solution: Here $M(x,y) = \dfrac{2xy}{x^2+1} - 2x \Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{2x}{x^2+1}$

$N(x,y) = -\left(2 - \ln(x^2+1)\right) \Rightarrow \dfrac{\partial N}{\partial x} = \dfrac{2x}{x^2+1}$

exact.

Hence we need to find $\psi(x,y)$:

$$\frac{\partial \psi}{\partial x} = \frac{2xy}{x^2+1} - 2x \Rightarrow \psi(x,y) = y\ln(x^2+1) - x^2 + C(y)$$

$$\frac{\partial \psi}{\partial y} = -2 + \ln(x^2+1) \Rightarrow \ln(x^2+1) + C'(y) = -2 + \ln(x^2+1)$$

$$\Rightarrow C'(y) = -2 \Rightarrow C(y) = -2y + K.$$

$\therefore$ Family of solutions is:

$$y\ln(x^2+1) - x^2 - 2y = C$$

But $y(5) = 0 \Rightarrow -25 = C$ ✓

Example: Consider the equation $x^2y^3 + x(1+y^2)\dfrac{dy}{dx} = 0.$ (1)

Here $M(x,y) = x^2y^3 \Rightarrow \dfrac{\partial M}{\partial y} = 3x^2y^2.$

$N(x,y) = x(1+y^2) \Rightarrow \dfrac{\partial N}{\partial x} = (1+y^2).$

Hence this is not exact.

Let $\mu(x,y) = \dfrac{1}{xy^3}$, and multiply the ODE by μ:

$$\frac{1}{xy^3}\left[x^2y^3 + x(1+y^2)\frac{dy}{dx}\right] = 0. \quad (2)$$

Of course if y is a solution that satisfies (1), then it satisfies (2) and vice-versa.

In (2), $M(x,y) = \dfrac{x^2y^3}{xy^3} = x \Rightarrow \dfrac{\partial M}{\partial y} = 0$

$$N(x,y) = \frac{x(1+y^2)}{xy^3} = \frac{1+y^2}{y^3} \Rightarrow \frac{\partial N}{\partial x} = 0$$

Hence (2) is exact and its solution is:

$\Psi(x,y) = C$, where

$$\frac{\partial \Psi}{\partial x} = M = x \Rightarrow \Psi(x,y) = \frac{x^2}{2} + C(y)$$

and $\dfrac{\partial \Psi}{\partial y} = N \Rightarrow C'(y) = \dfrac{1+y^2}{y^3} = y^{-3} + y^{-1}$

$$\Rightarrow C(y) = -\frac{1}{2y^2} + \ln|y| + K$$

$\therefore$ family of solutions is:

$$\boxed{\frac{x^2}{2} - \frac{1}{2y^2} + \ln|y| = C}$$

__Remark:__

If $M(x,y) + N(x,y)\dfrac{dy}{dx} = 0$ is <u>not</u> exact But:

(a) $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)$ is a function of <u>x only</u>, then

An integrating factor that turns the ODE into an exact equation is $\mu(x)$ satisfying:

$$\frac{d\mu}{\mu} = \frac{M_y - N_x}{N}; \quad M_y = \frac{\partial M}{\partial y}; \quad N_x = \frac{\partial N}{\partial x}$$

(b) $\dfrac{1}{M}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)$ is a function of <u>y only</u>, then

An integrating factor that turns the ODE into an exact equation is $\mu(y)$ satisfying:

$$\frac{d\mu}{\mu} = \frac{N_x - M_y}{M}$$

__Exs:__ ① See example 4 on page 99

② ~~Exercise~~ Consider the ODE:

$$\underbrace{(x+y)\sin y}_{M} + \underbrace{(x\sin y + \cos y)}_{N}\frac{dy}{dx} = 0$$

Here $\dfrac{\partial M}{\partial y} = \sin y + (x+y)\cos y$, $\dfrac{\partial N}{\partial x} = \sin y.$ } this equation is not exact

But $\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = (x+y)\cos y$

$$\Rightarrow -\frac{\partial M/\partial y - \partial N/\partial x}{M} = -\frac{(x+y)\cos y}{(x+y)\sin y} = -\cot y,$$ which

is a function of y only.

Hence an integrating factor is $\mu(y)$, where

$$\int \frac{d\mu}{\mu} = \int -\cot y \Rightarrow \ln|\mu(y)| = -\ln|\sin y|$$

$\Rightarrow$ one integrating factor is:

$$\boxed{\mu(y) = \frac{1}{\sin y}}$$

We multiply the ODE by $\mu(y)$ to obtain:

$$\underbrace{(x+y)}_{\text{new } M} + \underbrace{(x+\cot y)}_{\text{New } N}\frac{dy}{dx} = 0$$

Now, $\dfrac{\partial M}{\partial y} = 1$ and $\dfrac{\partial N}{\partial x} = 1$ $\therefore$ the new equation

is exact, and its family of solutions is $\psi(x,y) = C$

where $\dfrac{\partial \psi}{\partial x} = M \Rightarrow \psi(x,y) = \int (x+y)\,dx = \dfrac{x^2}{2} + xy + C(y)$

But also $\dfrac{\partial \psi}{\partial y} = N \Rightarrow x + C'(y) = x + \cot(y)$

$\Rightarrow C'(y) = \cot(y) \Rightarrow C(y) = \ln|\sin y| + C$

$\therefore$ Family of solutions is: $\boxed{\dfrac{x^2}{2} + xy + \ln|\sin y| = C}$