Characteristic Equation with double root

Given $ay'' + by' + cy = 0$ and given that $b^2 - 4ac = 0$.

Hence the characteristic equation $ar^2 + br + c = 0$

yields one root $r = \frac{-b}{2a}$ and hence one solution

$y_1 = e^{-\frac{b}{2a}t}$ for the ODE.

In order to find the general solution of the ODE,

we need a second solution y_2 such that $W[y_1, y_2] \neq 0$.

Clearly, $y_2 \neq Ce^{-\frac{b}{2a}t}$ (otherwise $W[y_1, y_2] = 0$).

Then maybe $y_2(t) = v(t)\, y_1(t)$ with a proper choice

of $v(t)$. Let us see if we can determine such $v(t)$.

$y_2(t) = v(t)\, y_1(t) \Rightarrow y_2'(t) = v' y_1 + v y_1'$

and $y_2'' = v'' y_1 + v' y_1' + v' y_1' + v y_1''$

$= v'' y_1 + 2v' y_1' + v y_1''$.

~~We plug in into the ODE and~~

Now, $y_1 = e^{-\frac{b}{2a}t} \Rightarrow y_1' = \frac{-b}{2a} e^{-\frac{b}{2a}t}$ and $y_1'' = \frac{b^2}{4a^2} e^{-\frac{b}{2a}t}$

We plug in into the ODE, and after some algebra,

We obtain:

$$e^{-\frac{b}{2a}t}\left(av'' - \frac{1}{4a}\underbrace{(b^2-4ac)}_{=0}v\right) = 0.$$

$$\Rightarrow \quad v'' = 0 \quad \Rightarrow v' = c \text{ and } v(t) = ct + k.$$

Hence we have many choices for v; we choose the simplest: $c = 1$ and $k = 0$

$\therefore$ A second solution for the ODE is: $y_2(t) = te^{-\frac{b}{2a}t}$.

(You can easily check that $W[y_1, y_2] \neq 0$).

Ex: $y'' - 4y' + 4y = 0$; $y(0) = 12$ and $y'(0) = -3$.

Solution: $r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0 \Rightarrow r = 2$ is a double root.

$\therefore$ General solution is: $y(t) = c_1 e^{2t} + c_2 te^{2t}$.

But $y(0) = 12 \Rightarrow c_1 = 12$.

$y'(0) = -3$; $\quad y' = 24e^{2t} + c_2 e^{2t} + 2c_2 te^{2t}$

$$\Rightarrow -3 = 24 + c_2 \Rightarrow c_2 = -27.$$

$\therefore$ particular solution is: $y = 12e^{2t} - 27te^{2t}$.

The procedure we used to guess a second solution for the linear homogeneous equation with constant coefficients having a double root can be used in other instances.

Ex: Given that $y_1 = \frac{1}{t}$ is a solution of $2t^2y'' + 3ty' - y = 0$; $t>0$, find a fundamental solution set.

Solution: Set $y_2 = v(t)\,y_1$, where $v(t)$ is to be determined.

$$\Rightarrow y_2 = \frac{1}{t}v \text{ and } y_2' = -\frac{1}{t^2}v + \frac{1}{t}v'$$

$$y_2'' = +\frac{2}{t^3}v - \frac{1}{t^2}v' - \frac{1}{t^2}v' + \frac{1}{t}v''$$

$$= \frac{2}{t^3}v - \frac{2}{t^2}v' + \frac{1}{t}v''.$$

Substituting in the ODE we obtain:

$$2t^2\left[\frac{2}{t^3}v - \frac{2}{t^2}v' + \frac{1}{t}v''\right] + 3t\left[-\frac{1}{t^2}v + \frac{1}{t}v'\right] - \frac{1}{t}v = 0$$

$$\Rightarrow \cancel{\frac{4}{t}v} - 4v' + 2tv'' - \cancel{\frac{3}{t}v} + 3v' - \cancel{\frac{1}{t}v} = 0.$$

$$\Rightarrow 2tv'' - v' = 0.$$

Let $W = v'$ $\Rightarrow 2tW' - W = 0$. this is now a first order ODE which in fact is separable.

$$2tw' = w \Rightarrow \frac{w'}{w} = \frac{1}{2t}$$

$$\Rightarrow \ln|w| = \frac{1}{2}\ln|t| + C$$

$\Rightarrow W = c\, t^{1/2}$ ie $v' = c\, t^{1/2}$

$$\Rightarrow v(t) = \int c\, t^{1/2}\, dt = c \frac{t^{3/2}}{3/2} + k.$$

We have found many choices for $v(t)$: choose $c = 1$ and $k = 0$

$$\therefore v(t) = \frac{2}{3} t^{3/2}.$$

A second solution is therefore $y_2 = \frac{2}{3} t^{3/2} \times \frac{1}{t} = \frac{2}{3} t^{1/2}$.

For reassurance, we check that $W[y_1, y_2] \neq 0$.

$$W[y_1, y_2] = y_1 y_2' - y_1' y_2 = \frac{1}{t}\left[\frac{2}{3} \cdot \frac{1}{2} t^{-1/2}\right] - \left[\frac{-1}{t^2} \cdot \frac{2}{3} t^{1/2}\right]$$

$$= \frac{1}{3} t^{-3/2} + \frac{2}{3} t^{-3/2} = t^{-3/2} \neq 0$$ (hopefully, no algebra mistake!)

$\therefore$ General solution of this ODE is:

$$\boxed{y = c_1 \frac{1}{t} + c_2 \sqrt{t}.}$$

Remark: this method we used is called the method of reduction of order, because to find $v(t)$, we ~~were~~ eventually worked with a first order ODE, where the unknown is w $(= v')$.