On the other hand, in Example 3 the general solution of the differential equation is

$$y = -\tfrac{7}{4} + \tfrac{1}{2}t + ce^{2t}, \tag{18}$$

and this is a diverging family. Note that solutions corresponding to two nearby values of c become arbitrarily far apart as t increases. In Example 3 we are trying to follow the solution for $c = 11/4$, but in the use of Euler's method we are actually at each step following another solution that separates from the desired one faster and faster as t increases. This explains why the errors in Example 3 are so much larger than those in Example 2.

In using a numerical procedure such as the Euler method, you must always keep in mind the question of whether the results are accurate enough to be useful. In the preceding examples, the accuracy of the numerical results could be determined directly by a comparison with the solution obtained analytically. Of course, usually the analytical solution is not available if a numerical procedure is to be employed, so what we usually need are bounds for, or at least estimates of, the error that do not require a knowledge of the exact solution. You should also keep in mind that the best that we can expect, or hope for, from a numerical approximation is that it reflects the behavior of the actual solution. Thus a member of a diverging family of solutions will always be harder to approximate than a member of a converging family.

If you wish to read more about numerical approximations to solutions of initial value problems, you may go directly to Chapter 8 at this point. There we present some information on the analysis of errors and also discuss several algorithms that are computationally much more efficient than the Euler method.

PROBLEMS

Most of the problems in this section call for fairly extensive numerical computations. To handle these problems you need suitable computing hardware and software. Keep in mind that numerical results may vary somewhat, depending on how your program is constructed and on how your computer executes arithmetic steps, rounds off, and so forth. Minor variations in the last decimal place may be due to such causes and do not necessarily indicate that something is amiss. Answers in the back of the book are recorded to six digits in most cases, although more digits were retained in the intermediate calculations.

In each of Problems 1 through 4:

(a) Find approximate values of the solution of the given initial value problem at $t = 0.1, 0.2, 0.3$, and 0.4 using the Euler method with $h = 0.1$.

(b) Repeat part (a) with $h = 0.05$. Compare the results with those found in (a).

(c) Repeat part (a) with $h = 0.025$. Compare the results with those found in (a) and (b).

(d) Find the solution $y = \phi(t)$ of the given problem and evaluate $\phi(t)$ at $t = 0.1, 0.2, 0.3$, and 0.4. Compare these values with the results of (a), (b), and (c).

1. $y' = 3 + t - y, \quad y(0) = 1$
2. $y' = 2y - 1, \quad y(0) = 1$
3. $y' = 0.5 - t + 2y, \quad y(0) = 1$
4. $y' = 3\cos t - 2y, \quad y(0) = 0$

In each of Problems 5 through 10, draw a direction field for the given differential equation and state whether you think that the solutions are converging or diverging.

5. $y' = 5 - 3\sqrt{y}$
6. $y' = y(3 - ty)$
7. $y' = (4 - ty)/(1 + y^2)$
8. $y' = -ty + 0.1y^3$
9. $y' = t^2 + y^2$
10. $y' = (y^2 + 2ty)/(3 + t^2)$

In each of Problems 11 through 14, use Euler's method to find approximate values of the solution of the given initial value problem at $t = 0.5, 1, 1.5, 2, 2.5$, and 3:

(a) With $h = 0.1$.
(b) With $h = 0.05$.
(c) With $h = 0.025$.
(d) With $h = 0.01$.

11. $y' = 5 - 3\sqrt{y}, \qquad y(0) = 2$
12. $y' = y(3 - ty), \qquad y(0) = 0.5$
13. $y' = (4 - ty)/(1 + y^2), \qquad y(0) = -2$
14. $y' = -ty + 0.1y^3, \qquad y(0) = 1$
15. Consider the initial value problem

$$y' = 3t^2/(3y^2 - 4), \qquad y(1) = 0.$$

(a) Use Euler's method with $h = 0.1$ to obtain approximate values of the solution at $t = 1.2, 1.4, 1.6$, and 1.8.

(b) Repeat part (a) with $h = 0.05$.

(c) Compare the results of parts (a) and (b). Note that they are reasonably close for $t = 1.2$, 1.4, and 1.6 but are quite different for $t = 1.8$. Also note (from the differential equation) that the line tangent to the solution is parallel to the y-axis when $y = \pm 2/\sqrt{3} \cong \pm 1.155$. Explain how this might cause such a difference in the calculated values.

16. Consider the initial value problem

$$y' = t^2 + y^2, \qquad y(0) = 1.$$

Use Euler's method with $h = 0.1$, 0.05, 0.025, and 0.01 to explore the solution of this problem for $0 \le t \le 1$. What is your best estimate of the value of the solution at $t = 0.8$? At $t = 1$? Are your results consistent with the direction field in Problem 9?

17. Consider the initial value problem

$$y' = (y^2 + 2ty)/(3 + t^2), \qquad y(1) = 2.$$

Use Euler's method with $h = 0.1$, 0.05, 0.025, and 0.01 to explore the solution of this problem for $1 \le t \le 3$. What is your best estimate of the value of the solution at $t = 2.5$? At $t = 3$? Are your results consistent with the direction field in Problem 10?

18. Consider the initial value problem

$$y' = -ty + 0.1y^3, \qquad y(0) = \alpha,$$

where α is a given number.

(a) Draw a direction field for the differential equation (or reexamine the one from Problem 8). Observe that there is a critical value of α in the interval $2 \le \alpha \le 3$ that separates converging solutions from diverging ones. Call this critical value α_0.

(b) Use Euler's method with $h = 0.01$ to estimate α_0. Do this by restricting α_0 to an interval $[a, b]$, where $b - a = 0.01$.

19. Consider the initial value problem

$$y' = y^2 - t^2, \qquad y(0) = \alpha,$$

where α is a given number.

(a) Draw a direction field for the differential equation. Observe that there is a critical value of α in the interval $0 \le \alpha \le 1$ that separates converging solutions from diverging ones. Call this critical value α_0.

(b) Use Euler's method with $h = 0.01$ to estimate α_0. Do this by restricting α_0 to an interval $[a, b]$, where $b - a = 0.01$.

20. **Convergence of Euler's Method.** It can be shown that under suitable conditions on f, the numerical approximation generated by the Euler method for the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$ converges to the exact solution as the step size h decreases. This is illustrated by the following example. Consider the initial value problem

$$y' = 1 - t + y, \qquad y(t_0) = y_0.$$

(a) Show that the exact solution is $y = \phi(t) = (y_0 - t_0)e^{t-t_0} + t$.

(b) Using the Euler formula, show that

$$y_k = (1 + h)y_{k-1} + h - ht_{k-1}, \qquad k = 1, 2, \ldots.$$

(c) Noting that $y_1 = (1 + h)(y_0 - t_0) + t_1$, show by induction that

$$y_n = (1 + h)^n(y_0 - t_0) + t_n \tag{i}$$

for each positive integer n.

(d) Consider a fixed point $t > t_0$ and for a given n choose $h = (t - t_0)/n$. Then $t_n = t$ for every n. Note also that $h \to 0$ as $n \to \infty$. By substituting for h in Eq. (i) and letting $n \to \infty$, show that $y_n \to \phi(t)$ as $n \to \infty$.

Hint: $\lim_{n\to\infty} (1 + a/n)^n = e^a$.

In each of Problems 21 through 23, use the technique discussed in Problem 20 to show that the approximation obtained by the Euler method converges to the exact solution at any fixed point as $h \to 0$.

21. $y' = y, \qquad y(0) = 1$
22. $y' = 2y - 1, \qquad y(0) = 1 \qquad$ *Hint:* $y_1 = (1 + 2h)/2 + 1/2$
23. $y' = \frac{1}{2} - t + 2y, \qquad y(0) = 1 \qquad$ *Hint:* $y_1 = (1 + 2h) + t_1/2$

2.8 The Existence and Uniqueness Theorem

In this section we discuss the proof of Theorem 2.4.2, the fundamental existence and uniqueness theorem for first order initial value problems. This theorem states that under certain conditions on $f(t, y)$, the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0 \tag{1}$$

has a unique solution in some interval containing the point t_0.

In some cases (for example, if the differential equation is linear) the existence of a solution of the initial value problem (1) can be established directly by actually solving the problem and exhibiting a formula for the solution. However, in general, this approach is not feasible because there is no method of solving the differential equation that applies in all cases. Therefore, for the general case, it is necessary to adopt an indirect approach that demonstrates the existence of a solution of Eqs. (1)