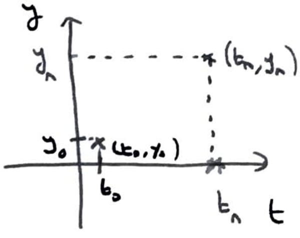
2.7 - Numerical Approximations: Euler's Method

This is the only section of the course where we do a numerical approximation of a solution to an ODE: $\frac{dy}{dt} = f(t,y)$

It is important to note the following:

a. We approximate the value of a solution at some time t_n (that is we do not find a solution or a formula for a solution):

For $t = t_n$, we approximate $y_n = y(t_n)$



b. We always need a starting point: (t_0, y_0) is an initial condition: $t_0 \to y_0$

In summary, we approximate the value of the solution at t_n to the IVP: $\frac{dy}{dt} = f(t,y); y(t_0) = y_0$.

We will use the mini-tangents of the slope field to find that approximate value: At (t_0, y_0), we find the equation of the tangent line L_1 to the solution; we move on that solution for some time interval until we reach a second pt (t_1, y_1);

At (t_1, y_1), we find the tangent line L_2 and we move on L_2 until we reach a point (t_3, y_3)

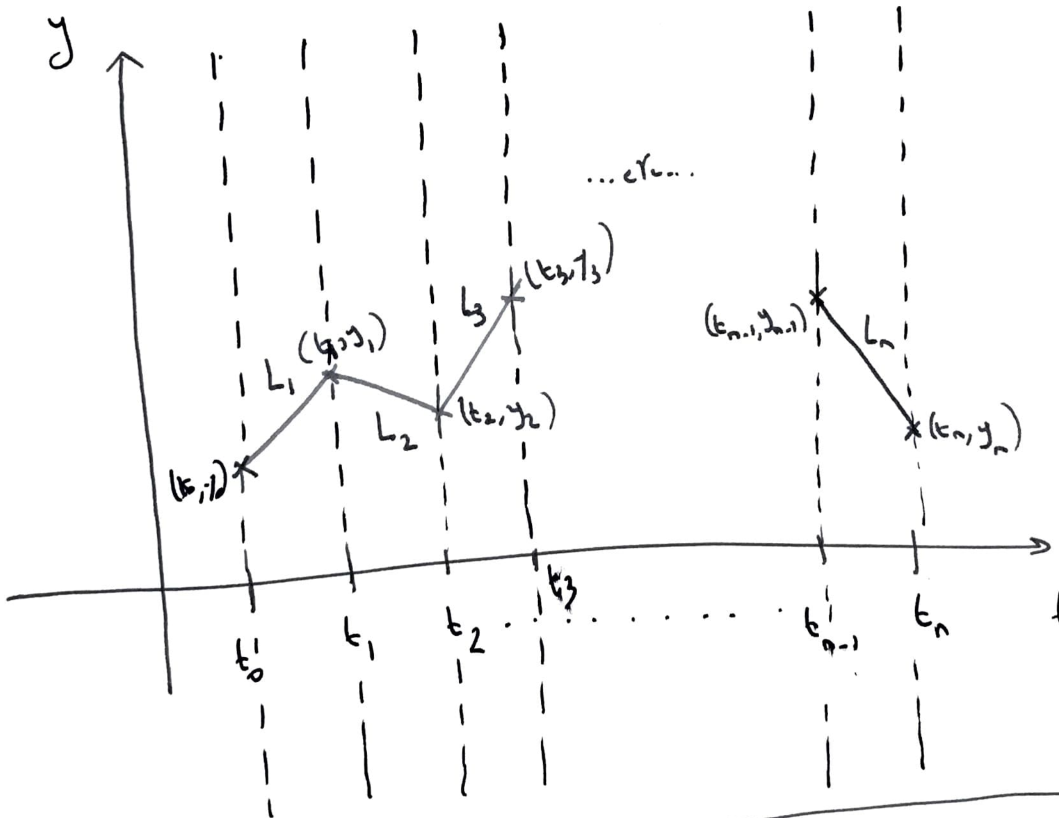
... etc ...

We do this a "number of times" until we reach the point (t_{n-1}, y_{n-1}). ~~We calculate the tan~~

We find the tangent line L_n at the point.

We use L_n to approximate y_n by replacing t in the equation of L_n by t_n.

We need to determine at first the number of steps we would like to do



It can be shown that L_m: $y = y_{m-1} + f(t_{m-1}, y_{m-1})(t - t_{m-1})$

Remark: We can do this approximation using any number of steps. The more the steps, the narrower the intervals, in which case the better the approximation is (in most cases).

If we do this approximation in r steps, then the length of each interval is: $\frac{t_n - t_0}{r}$; it is called step size and is denoted by h.

This method is called Euler's Method

Ex: Consider the IVP: $\frac{dy}{dt} = \underbrace{3 - 2t - \frac{1}{2}y}_{f(t,y)}$; $y(0) = 1$

Use Euler's Method with step size $h = 0.2$ to approximate the solution at $t = 1$.

Solution: $t_0 = 0$ and $t_1 = 1$; $h = 0.2 \Rightarrow$ this is done in 5 steps.

$t_0 = 0$ and $y_0 = 1 \Rightarrow L_1: y = y_0 + f(t_0, y_0)\cdot(t - t_0)$

$\cancel{\Rightarrow y = f(0,1) + t = 3 - \frac{1}{2} + t = \frac{5}{2} + t.}$

$\Rightarrow y = 1 + f(0,1)(t-0) = 1 + t f(0,1) = 1 + 2.5t.$

We use the equation of L_1 to find y_1:

$t_1 = 0.2 \Rightarrow y(0.2) = 1 + 2.5(0.2) = 1.5$

$\therefore (t_1, y_1) = (0.2, 1.5)$

We now find the equation of L_2:

$$L_2: \quad y = y_1 + f(t_1, y_1)(t - t_1)$$

$$= 1.5 + \left[3 - 2 \times 0.2 - \tfrac{1}{2}\cdot(1.5)\right](t - 0.2)$$

$$= 1.13 + 1.85t$$

We use L_2 to find y_2:

$$t_2 = 0.4 \rightarrow y_2 = y(0.4) = 1.13 + 1.85(0.4) = 1.87$$

... etc ...

Figure 2.7.3 on page 105.