6.4. Differential Equations with discontinuous forcing functions.

Examples

① Solve: $2y'' + y' + 2y = g(t)$, where

$$g(t) = u_5(t) - u_{20}(t) = \begin{cases} 1; & 5 \le t < 20 \\ 0; & 0 \le t < 5;\ t \ge 20 \end{cases}$$

and $y(0) = 0$; $y'(0) = 0$

Solution: $\mathcal{L}(2y'' + y' + 2y) = \mathcal{L}(u_5 - u_{20})$

$$\therefore 2\mathcal{L}(y'') + \mathcal{L}(y') + 2\mathcal{L}(y) = \mathcal{L}(u_5) - \mathcal{L}(20)$$

$$\therefore 2\Big[s^2 Y - s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=0}\Big] + \Big[sY - \underbrace{y(0)}_{=0}\Big] + 2Y = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$$

$$\therefore Y\left[2s^2 + s + 2\right] = \frac{e^{-5s} - e^{-20s}}{s}$$

$$\text{or} \quad Y = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)}$$

$$\therefore y = \mathcal{L}^{-1}\left[e^{-5s} \cdot \frac{1}{s(2s^2 + s + 2)} - e^{-20s} \cdot \frac{1}{s(2s^2 + s + 2)}\right]$$

It is enough to find $\mathcal{L}^{-1}\left[\dfrac{1}{s(2s^2 + s + 2)}\right]$

(see theorem on page 120)

$$\frac{1}{s(2s^2+s+2)} = \frac{A}{s} + \frac{Bs+C}{2s^2+s+2}$$

$$= \frac{A}{s} + \frac{Bs+C}{2\left[s^2+\frac{1}{2}s+1\right]} \leftarrow s^2+\frac{1}{2}s+\left(\frac{1}{4}\right)^2-\left(\frac{1}{4}\right)^2+1$$

$$= \frac{A}{s} + \frac{1}{2}\cdot\frac{Bs+C}{\left(s+\frac{1}{4}\right)^2+1-\frac{1}{16}}$$

$$= \frac{A}{s} + \frac{1}{2}\cdot\frac{Bs+C}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}.$$

$\Rightarrow A = \frac{1}{2}$; $B = -1$ and $C = -\frac{1}{2}$ (exercise)

$$\therefore \mathcal{L}^{-1}\left(\frac{A}{s}\right) = \mathcal{L}^{-1}\left(\frac{1/2}{s}\right) = \frac{1}{2}.$$

$$\mathcal{L}^{-1}\left[\frac{Bs+C}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\right] = -\mathcal{L}^{-1}\left[\frac{s}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\right]$$

$$\cancel{= -\mathcal{L}^{-1}\left[\frac{s+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\left(\frac{\sqrt{15}}{4}\right)^2}\right] - \frac{4}{\sqrt{15}}\frac{1}{2}\mathcal{L}^{-1}\left[\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\right]}$$

$$\cancel{= -e^{-\frac{1}{4}t}\cos\frac{\sqrt{15}}{4}t}$$

$$= -\mathcal{L}^{-1}\left[\frac{s+\frac{1}{4}}{(s+\frac{1}{4})^2+\left(\frac{\sqrt{15}}{4}\right)^2}\right]+\frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{(s+\frac{1}{4})^2+\left(\frac{\sqrt{15}}{4}\right)^2}\right]$$

$$-\frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{(s+\frac{1}{4})^2+\left(\frac{\sqrt{15}}{4}\right)^2}\right]$$

$$= -e^{-\frac{1}{4}t}\cos\left(\frac{\sqrt{15}}{4}t\right)-\frac{1}{4}\cdot\frac{4}{\sqrt{15}}\mathcal{L}^{-1}\left[\frac{\sqrt{15}/4}{(s+\frac{1}{4})^2+\left(\frac{\sqrt{15}}{4}\right)^2}\right]$$

$$= -e^{-\frac{1}{4}t}\cos\left(\frac{\sqrt{15}}{4}t\right)-\frac{1}{\sqrt{15}}e^{-\frac{1}{4}t}\sin\left(\frac{\sqrt{15}}{4}t\right)$$

$$\therefore \mathcal{L}^{-1}\left[\frac{1}{s(2s^2+s+2)}\right]=\frac{1}{2}-\frac{1}{2}e^{-\frac{1}{4}t}\cos\left(\frac{\sqrt{15}}{4}t\right)-\frac{1}{2\sqrt{15}}e^{-\frac{1}{4}t}\sin\left(\frac{\sqrt{15}}{4}t\right).$$

Hence $\mathcal{L}^{-1}\left[e^{-5s}\cdot\frac{1}{s(2s^2+s+2)}\right]=$

$$u_5(t)\cdot\left[\frac{1}{2}-\frac{1}{2}e^{-\frac{1}{4}(t-5)}\cdot\cos\left[\frac{\sqrt{15}}{4}(t-5)\right]-\frac{1}{2\sqrt{15}}e^{-\frac{1}{4}(t-5)}\cdot\sin\left[\frac{\sqrt{15}}{4}(t-5)\right]\right].$$

Similarly for $\mathcal{L}^{-1}\left[e^{-20s}\cdot\frac{1}{s(2s^2+s+2)}\right]$.