Chapter 5 - Power Series Solutions of ODE's.

5.1 Review of Power series.

A power series centered at a takes the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n.$$

A Power series ~~have~~ satisfy exactly one of the following 3 properties:

1. It converges $\forall x$ (for all $x \in \mathbb{R}$).
2. It converges only when $x = a$.
3. It converges over an interval I centered at a. (The interval can be closed $[a-R, a+R]$; open $(a-R, a+R)$; half-open/closed $[a-R, a+R)$ or $(a-R, a+R]$).

I is called the interval of convergence

R is called the radius of convergence.

One can determine interval of convergence using the ratio or root test on the series of absolute values.

Ex: Find the interval and radius of convergence of the series $\sum_{n=1}^{\infty} (-1)^{n+1} n (x-2)^n$.

Solution: We begin by considering the series of absolute values: $\sum_{n=1}^{\infty} \left| (-1)^{n+1} n (x-2)^n \right| = \sum_{n=1}^{\infty} n|x-2|^n$

We do the Root test:

$$\rho = \lim_{n\to\infty} \sqrt[n]{n|x-2|^n} = \lim_{n\to\infty} \sqrt[n]{n}\, |x-2| = |x-2|$$

$$\left(\lim_{n\to\infty} \sqrt[n]{n} = 1\right) \therefore \rho = |x-2|.$$

For convergence, we need $\rho < 1$

$$\Rightarrow -1 < x-2 < 1$$

$$\Rightarrow -1+2 < x < 1+2 \quad (R = 1 \text{ is radius})$$

$$\Rightarrow \boxed{1 < x < 3}$$

We have divergence if $\rho > 1$ ie $x > 3$ or $x < 1$.

We do not know about $\rho = 1$ ie $x = 3$ and $x = 1$.

We do each separately:

$x = 3 \to$ series becomes $\sum (-1)^n n (3-2)^n = \sum (-1)^n n$
divergent

$x = 1 \to$ series becomes $\sum (-1)^n n (1-2)^n = \sum n$, div. also

Shifting the index of summation

Sometimes we find it convenient to make changes of summation indices in calculating series solutions of differential equations.

EX: Re-Write the series $\sum_{n=2}^{\infty}(n+2)(n+1)a_n(x-x_0)^{n-2}$ so that the sum starts at $n=0$.

Solution: If the series starts at 0, rather than 2, then n should be replaced with $n+2$

[$n=0 \rightarrow n+2=2$ which is the actual first value of n]

$$\therefore \sum_{n=2}^{\infty}(n+2)(n+1)a_n(x-x_0)^{n-2}$$

$$= \sum_{n=0}^{\infty}((n+2)+2)((n+2)+1)a_{n+2}(x-x_0)^{(n+2)-2}$$

$$= \sum_{n=0}^{\infty}(n+4)(n+3)a_{n+2}(x-x_0)^{n}.$$

Ex: Suppose $\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n$

What does that imply about the coefficients a_n?

Solution: We begin by shifting the index of summation of the series to the left: $n \to n+1$:

$$\sum_{n=0}^{\infty} \underbrace{(n+1) a_{n+1}} x^n = \sum_{n=0}^{\infty} \underbrace{a_n} x^n$$

$$\Rightarrow a_{n+1} = \frac{a_n}{n+1} \; ; \; n = 0, 1, 2, 3, \ldots$$

$\therefore$ $n = 0$ implies $a_1 = a_0 = \frac{a_0}{1!}$

$n = 1$ implies $a_2 = \frac{a_1}{2} = \frac{a_0}{2!}$

$n = 2$ implies $a_3 = \frac{a_2}{3} = \frac{a_0}{2.3} = \frac{a_0}{3!}$

$n = 3$ implies $a_4 = \frac{a_3}{4} = \frac{a_0}{3!\,4} = \frac{a_0}{4!}$

$\therefore$ In general $a_n = \frac{a_0}{n!}$, which gives a representation of all the coefficients in terms of a_0.

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{a_0}{n!} x^n = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Now, it is important to remember that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x, \text{ for all } x \leftarrow \text{this is called the Maclaurin series for } e^x$$

$\therefore$ In this example, $\sum_{n=0}^{\infty} a_n x^n = a_0 e^x$.

Other important Maclaurin series to remember:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \text{ for all } x$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \text{ for all } x$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \text{ for all } |x| < 1.$$

Other important results to remember:

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, for $x \in I$

then: $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$, for $x \in I$ (sort of!)

$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

... etc ...

Also, $\int f(x)\,dx = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1} + C$, for $x \in I$.