Chapter 2 - First Order Differential Equations.

$$\boxed{\frac{dy}{dt} = f(t,y)}$$

2.1. Linear Equations; Methods of Integrating factor.

Definition: A general first order linear ODE takes the form: $\frac{dy}{dt} + p(t)y = g(t)$ $\left[\text{ie } f(t,y) = -p(t)y + g(t)\right]$.

Here $p(t)$ and $g(t)$ are arbitrary functions that depend on t only. Also notice that y appears to the power of 1 (no other powers or no other ways).

Exs: ① $\frac{dy}{dt} + e^t y = t^2$ is linear $\leftarrow p(t) = e^t$; $g(t) = t^2$.

② $\frac{dy}{dt} + ty^2 = 1$ is not linear because of y^2.

③ $\frac{dy}{dt} + e^t . e^y = t^2$ is not linear because of e^y.

④ $t\frac{dy}{dt} + 2y = 4t^2$ is linear but it is not written in standard form. We rewrite it:

$$\frac{dy}{dt} + \frac{2}{t}y = 4t \rightarrow p(t) = \frac{2}{t} \text{ and } g(t) = 4t$$

How to solve a linear ODE?

Given $\dfrac{dy}{dt} + p(t)y = g(t)$; define $\mu(t) = e^{\int p(t)\,dt}$. Notice

then that $\dfrac{d\mu}{dt} = \dfrac{d}{dt}\left(\int p(t)\,dt\right) e^{\int p(t)\,dt} = p(t)\,\mu(t)$.

Multiply the original ODE by $\mu(t)$:

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t)$$

$$\Rightarrow \quad \mu(t)\frac{dy}{dt} + y\frac{d\mu}{dt} = \mu(t)g(t)$$

$$\Rightarrow \quad \frac{d}{dt}\left[\mu(t)y\right] = \mu(t)g(t)$$

$$\Rightarrow \quad \mu(t)y = \int \mu(t)g(t)\,dt$$

$$\Rightarrow \quad \boxed{y = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)\,dt\right].}$$

$\mu(t)$ is called an <u>integrating factor</u>.

Ex 1: $\frac{dy}{dt} - 2y = 4-t$; here $p(t) = -2$ and $g(t) = 4-t$.

An integrating factor is: $\mu(t) = e^{\int -2dt} = e^{-2t}$.

Hence the family of solutions take the form:

$$y = \frac{1}{e^{-2t}} \int e^{-2t}(4-t)dt.$$

By parts: ~~$u = e^{-2t} \to du = -\frac{1}{2}e^{-2t}$~~

$$u = 4-t \to \frac{du}{dt} = -1$$

$$\frac{dv}{dt} = e^{-2t} \to v = -\frac{1}{2}e^{-2t}$$

$$\therefore \int e^{-2t}(4-t)dt = -\frac{1}{2}e^{-2t}(4-t) - \int \frac{1}{2}e^{-2t}$$

$$= -\frac{1}{2}e^{-2t}(4-t) + \frac{1}{4}e^{-2t} + C$$

$$\Rightarrow y = e^{2t}\left[-\frac{1}{2}e^{-2t}(4-t) + \frac{1}{4}e^{-2t} + C\right]$$

$$\Rightarrow \boxed{y = -\frac{1}{2}(4-t) + \frac{1}{4} + Ce^{2t}}$$

Initial Value Problem (IVP). This is the problem of finding one particular solution out of the family of solutions. This is based on an Initial Condition (IC)

Suppose the IC is $y(0) = -7/4$ ie $y = -7/4$ when $t = 0$.

Replace: $-\frac{7}{4} = -\frac{1}{2}(4) + \frac{1}{4} + C = -\frac{7}{4} + C \Rightarrow C = 0$

$\therefore$ The particular solution for this IVP is:

$$y = -\frac{7}{4} + \frac{t}{2}, \text{ which is a polynomial.}$$

If we change the initial Condition to: $y(0) = \frac{7}{4}$, then we

Obtain: $+\frac{7}{4} = -\frac{7}{4} + C \Rightarrow C = -7$

$\therefore$ the particular solution is now:

$$y = -\frac{7}{4} + \frac{t}{2} - 7e^{2t}$$ $\leftarrow$ which is a combination of a polynomial and an exponential function.

This is where a visual / qualitative approach becomes useful. [see Figure 2.1.2 in textbook on page 35].

<u>Ex 2</u>. Solve the IVP: $ty' + 2y = 4t^2$; $y(1) = 2$.

Here $p(t) = \frac{2}{t}$ and $g(t) = 4t$.

$\Rightarrow$ One integrating factor is $\mu(t) = e^{\int \frac{2}{t} dt} = e^{2\ln|t|} = t^2$.

$$\therefore \; y = \frac{1}{t^2} \int t^2 \cdot 4t \, dt = \frac{1}{t^2}\left[\frac{4t^4}{4} + c\right] = t^2 + \frac{c}{t^2}.$$

But $y(1) = 2 \Rightarrow 2 = 1 + C \Rightarrow C = 1$

$\therefore \; y = t^2 + \frac{1}{t^2}$. / See figure 2.1.3 in textbook on page 37.

Ex 3. Solve: $2y' + ty = 2$;

We write it first in standard form: $y' + \frac{t}{2}y = 1$; $y(0) = 1$.

$\therefore$ $p(t) = \frac{t}{2}$ and $g(t) = 1$.

Integrating factor is: $\mu(t) = e^{\int \frac{t}{2} dt} = e^{t^2/4}$

$\therefore$ $y = \frac{1}{e^{t^2/4}} \underbrace{\int e^{t^2/4} dt}$.

Notice that this integral can't be evaluated using the standard integration techniques; we can solve it however using power series!

In this course, we will keep the solution in this form, that we call an implicit form of the solution.

Of course, we can't understand this family of solutions unless we do the direction field (see figure 2.1.4 in textbook on page 39).