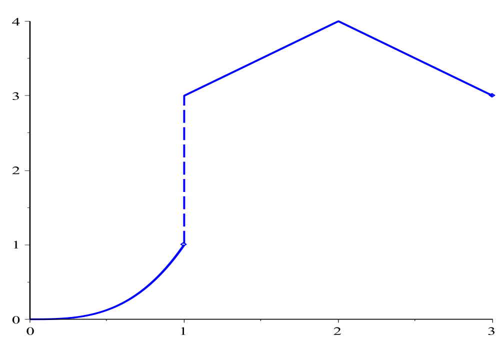
CHAPTER

6

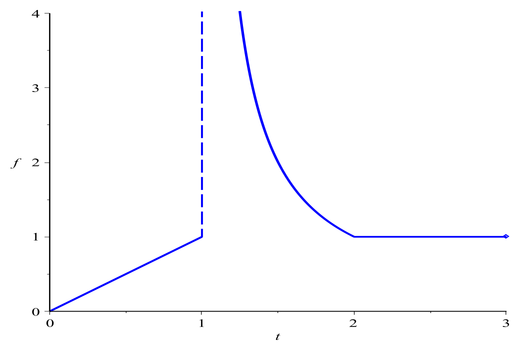
The Laplace Transform

6.1

1. The graph of $f(t)$ is shown. Since the function is continuous on each interval, but has a jump discontinuity at $t = 1$, $f(t)$ is piecewise continuous.



2. The function is neither continuous nor piecewise continuous. Observe that $\lim_{t\to 1^+}(t-1)^{-1} = \infty$.



5.(a) Since t is continuous for $0 \le t \le A$ for any positive A and since $t \le e^{at}$ for any $a > 0$ and for t sufficiently large, it follows from Theorem 6.1.2 that the Laplace transform of $f(t) = t$ exists for $s > 0$. Using integration by parts we obtain

$$F(s) = \int_0^\infty te^{-st}\,dt = \lim_{M\to\infty}\int_0^M te^{-st}\,dt = \lim_{M\to\infty}\left[\frac{-te^{-st}}{s}\bigg|_0^M + \frac{1}{s}\int_0^M e^{-st}\,dt\right] =$$

$$= \lim_{M\to\infty}\frac{-Me^{-sM}}{s} + \frac{1}{s}\lim_{M\to\infty}\left[\frac{-e^{-st}}{s}\bigg|_0^M\right] = 0 + \frac{1}{s}\lim_{M\to\infty}\left[\frac{-e^{-sM}}{s} + \frac{1}{s}\right] = \frac{1}{s^2}.$$

From this point forward, we do not check the assumptions of Theorem 6.1.2; they stand for all the functions given. Also, we omit the limit finding process - the student should check these.

(b) For $f(t) = t^2$, the Laplace transform (using integration by parts twice) is

$$F(s) = \int_0^\infty e^{-st}t^2\,dt = -\frac{e^{-st}s^2t^2 + 2e^{-st}st + 2e^{-st}}{s^3}\bigg|_{t=0}^\infty = \frac{2}{s^3}.$$

(c) For $f(t) = t^n$, the Laplace transform (using integration by parts and induction) is

$$F(s) = \int_0^\infty e^{-st}t^n\,dt = \frac{n!}{s^{n+1}}.$$

6. $f(t) = \cos at$ satisfies the hypothesis of Theorem 6.1.2 because $|\cos at| \le 1$ for all t. Using integration by parts twice, we obtain that

$$\int_0^\infty e^{-st}\cos at\,dt = \left[-\frac{e^{-st}\cos at}{s} + \frac{ae^{-st}\sin at}{s^2}\right]\bigg|_{t=0}^\infty - \frac{a^2}{s^2}\int_0^\infty e^{-st}\cos at\,dt.$$

This implies that

$$\int_0^\infty e^{-st}\cos at\,dt = \frac{1}{s} - \frac{a^2}{s^2}\int_0^\infty e^{-st}\cos at\,dt,$$

and hence (after rearrangement of terms and division by $(1 + a^2/s^2)$) we get that

$$\int_0^\infty e^{-st}\cos at\,dt = \frac{s}{a^2 + s^2}.$$

9. We note that $e^{at}\cosh(bt) = (e^{(a+b)t} + e^{(a-b)t})/2$. Using the linearity property of the Laplace transform, we get that

$$\mathcal{L}[e^{at}\cosh bt] = \frac{1}{2}\mathcal{L}[e^{(a+b)t}] + \frac{1}{2}\mathcal{L}[e^{(a-b)t}] = \frac{1/2}{s-(a+b)} + \frac{1/2}{s-(a-b)} =$$

$$= \frac{s-a}{(s-a)^2 - b^2},$$

as long as $s - a > |b|$.

13. Since the Laplace transform is a linear operator, we have

$$\mathcal{L}[e^{at}\sin(bt)] = \frac{1}{2i}\mathcal{L}[e^{(a+ib)t}] - \frac{1}{2i}\mathcal{L}[e^{(a-ib)t}].$$

Now

$$\int_0^\infty e^{(a\pm ib)t}e^{-st}\,dt = \frac{1}{s-a\mp ib}.$$

Therefore (for $s > a$),

$$\mathcal{L}[e^{at}\cos(bt)] = \frac{1}{2i}\left[\frac{1}{s-a-ib} - \frac{1}{s-a+ib}\right] = \frac{b}{(s-a)^2+b^2}.$$

16. We will use the fact that $\cos(at) = (e^{iat} + e^{-iat})/2$. Therefore,

$$\int_0^A t\cos(at)e^{-st}\,dt = \frac{1}{2}\left[\int_0^A te^{(ia-s)t}\,dt + \int_0^A te^{-(ia+s)t}\,dt\right] =$$

$$= \frac{1}{2}\left[-\left.\frac{te^{(ia-s)t}}{s-ia}\right|_0^A + \int_0^A \frac{1}{s-ia}e^{(ia-s)t}\,dt\right]$$

$$+\frac{1}{2}\left[-\left.\frac{te^{(-ia-s)t}}{s+ia}\right|_0^A + \int_0^A \frac{1}{s+ia}e^{(-ia-s)t}\,dt\right] =$$

$$= \frac{1}{2}\left[-\frac{Ae^{(ia-s)A}}{s-ia} - \frac{Ae^{(-ia-s)A}}{s+ia} + \frac{1}{(s-ia)^2} + \frac{1}{(s+ia)^2}\right].$$

Taking the limit as $A \to \infty$, we have

$$\int_0^\infty t\cos(at)e^{-st}\,dt = \frac{1}{2}\left[\frac{(s+ia)^2 + (s-ia)^2}{(s-ia)^2(s+ia)^2}\right] = \frac{s^2-a^2}{(s^2+a^2)^2}.$$

21. Using the fact that $f(t) = 0$ when $t \geq 2\pi$ we obtain that

$$\mathcal{L}\left[f(t)\right] = \int_0^\infty e^{-st}f(t)\,dt = \int_0^{2\pi} e^{-st}\,dt = \left[-\frac{e^{-st}}{s}\right]_0^{2\pi} = -\frac{e^{-2\pi s}}{s} + \frac{1}{s}.$$

24. Using the fact that $f(t) = 0$ when $t \geq 2$ we obtain (after integration by parts and some simplification) that

$$\mathcal{L}\left[f(t)\right] = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 te^{-st}\, dt + \int_1^2 (2-t)e^{-st}\, dt =$$

$$= \left[-\frac{e^{-st}(1+st)}{s^2}\right]_0^1 + \left[\frac{e^{-st}(1-2s+st)}{s^2}\right]_1^2 = \frac{e^{-2s}(-1+e^s)^2}{s^2}.$$

25. First, we note that

$$\int_0^A (t^2+1)^{-1}\, dt = \arctan t|_0^A = \arctan(A) - \arctan(0) = \arctan(A).$$

Therefore, as $A \to \infty$, we have

$$\int_0^\infty (t^2+1)^{-1}\, dt = \lim_{A\to\infty} \arctan(A) = \frac{\pi}{2}.$$

Therefore, the integral converges.

29. If we let $u = f$ and $dv = e^{-st}\, dt$, then

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt = \lim_{A\to\infty} -\frac{1}{s} e^{-st} f(t)\Big|_0^A + \frac{1}{s}\int_0^\infty e^{-st} f'(t)\, dt =$$

$$= \frac{1}{s} f(0) + \frac{1}{s}\int_0^\infty e^{-st} f'(t)\, dt.$$

This last integral converges (and is thus finite) using an argument similar to that given to establish Theorem 6.1.2. Hence $\lim_{s\to\infty} F(s) = 0$.

31.(a) By definition,

$$\mathcal{L}[t^p] = \int_0^\infty e^{-st} t^p\, dt.$$

Now letting $x = st$, we see that $dx = s\, dt$. Therefore,

$$\mathcal{L}[t^p] = \int_0^\infty e^{-st} t^p\, dt = \int_0^\infty e^{-st}\left(\frac{x}{s}\right)^p \frac{dx}{s}. = \frac{1}{s^{p+1}}\int_0^\infty e^{-st} x^p\, dx = \frac{\Gamma(p+1)}{s^{p+1}}.$$

(b) Using part (a) and the fact that for n a positive integer, $\Gamma(n+1) = n!$, we conclude that $\mathcal{L}[t^n] = n!/s^{n+1}$.

(c) By definition,

$$\mathcal{L}[t^{-1/2}] = \int_0^\infty e^{-st} t^{-1/2}\, dt.$$

Making the change of variables $x = \sqrt{st}$, we see that $x^2 = st$ implies $2x\, dx = s\, dt$. Therefore,

$$\int_0^\infty e^{-st} t^{-1/2}\, dt = \int_0^\infty e^{-x^2}\left(\frac{x^2}{s}\right)^{-1/2} \frac{2x\, dx}{s} = \frac{2}{\sqrt{s}}\int_0^\infty e^{-x^2}\, dx.$$

(d) Using the result from part (a), we see that

$$\mathcal{L}[t^{1/2}] = \frac{\Gamma(3/2)}{s^{3/2}}.$$

Then, using the result from Problem 30, we know that $\Gamma(3/2) = \sqrt{\pi}/2$. Therefore,

$$\mathcal{L}[t^{1/2}] = \frac{\sqrt{\pi}}{2s^{3/2}}.$$

6.2

Problems 1 through 10 are solved by using partial fractions and algebra to manipulate the given function into a form matching one of the functions appearing in the middle column of Table 6.2.1.

2. Writing the function as

$$\frac{5}{(s-1)^3} = \frac{5}{2}\frac{2!}{(s-1)^3},$$

we see that $\mathcal{L}^{-1}(Y(s)) = 5t^2e^t/2$, using line 11.

4. Using partial fractions, we write

$$\frac{2s}{s^2-s-6} = \frac{6}{5}\frac{1}{s-3} + \frac{4}{5}\frac{1}{s+2}.$$

Therefore, we see from line 2 that $\mathcal{L}^{-1}(Y(s)) = (6/5)e^{3t} + (4/5)e^{-2t}$.

7. Completing the square in the denominator, we have

$$\frac{2s+3}{s^2-2s+2} = \frac{2s+3}{(s-1)^2+1} = \frac{2(s-1)}{(s-1)^2+1} + \frac{5}{(s-1)^2+1}.$$

Therefore, we see from lines 9 and 10 that $\mathcal{L}^{-1}(Y(s)) = 2e^t\cos t + 5e^t\sin t$.

In each of Problems 11 through 23 it is assumed that the initial value problem has a solution $y = \phi(t)$ which, with its first two derivatives, satisfies the conditions of Corollary 6.2.2.

11. Taking the Laplace transform of the differential equation, using Eq.(1) and Eq.(2), we obtain $[s^2Y(s) - sy(0) - y'(0)] - [sY(s) - y(0)] - 6Y(s) = 0$. Using the initial conditions and solving for $Y(s)$ we obtain $Y(s) = (2s-3)/(s^2-s-6)$. Using partial fractions, we write

$$Y(s) = \frac{3}{5}\frac{1}{s-3} + \frac{7}{5}\frac{1}{s+2},$$

which implies that $y(t) = (3/5)e^{3t} + (7/5)e^{-2t}$.

14. Taking the Laplace transform of the differential equation we obtain $[s^2Y(s) - sy(0) - y'(0)] - 4[sY(s) - y(0)] + 4Y(s) = 0$. Using the initial conditions and solving for $Y(s)$ we obtain $Y(s) = (s-1)/(s^2-4s+4)$. Therefore, we have

$$Y(s) = \frac{s-1}{(s-2)^2} = \frac{s-2}{(s-2)^2} + \frac{1}{(s-2)^2} = \frac{1}{s-2} + \frac{1}{(s-2)^2}$$

which implies that $y(t) = e^{2t} + te^{2t}$.

15. Taking the Laplace transform of the differential equation we obtain $[s^2Y(s) - sy(0) - y'(0)] - 2[sY(s) - y(0)] + 4Y(s) = 0$. Using the initial conditions and solving for $Y(s)$ we obtain $Y(s) = (3s-6)/(s^2-2s+4)$. Completing the square in the denominator, we have

$$Y(s) = \frac{3s-6}{(s-1)^2+3} = \frac{3(s-1)}{(s-1)^2+3} - \frac{3}{(s-1)^2+3}$$

which (using line 14 in Table 6.2.1) gives $y(t) = 3e^t \cos(\sqrt{3}t) - \sqrt{3}e^t \sin(\sqrt{3}t)$.

17. Taking the Laplace transform of the ODE, we obtain

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - 4\left[s^3Y(s) - s^2y(0) - sy'(0) - y''(0)\right] +$$
$$+6\left[s^2Y(s) - sy(0) - y'(0)\right] - 4\left[sY(s) - y(0)\right] + Y(s) = 0.$$

Applying the initial conditions,

$$s^4Y(s) - 4s^3Y(s) + 6s^2\,Y(s) - 4s\,Y(s) + Y(s) - s^2 + 4s - 7 = 0\,.$$

Solving for the transform of the solution,

$$Y(s) = \frac{s^2-4s+7}{s^4-4s^3+6s^2-4s+1} = \frac{s^2-4s+7}{(s-1)^4}\,.$$

Using partial fractions,

$$\frac{s^2-4s+7}{(s-1)^4} = \frac{4}{(s-1)^4} - \frac{2}{(s-1)^3} + \frac{1}{(s-1)^2}.$$

Using line 11 and line 14 from Table 6.2.1, we obtain that the solution of the initial value problem is

$$y(t) = \mathcal{L}^{-1}\left[\frac{s^2-4s+7}{(s-1)^4}\right] = \frac{2}{3}\,t^3e^t - t^2e^t + t\,e^t.$$

20. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + \omega^2\,Y(s) = \frac{s}{s^2+4}\,.$$

Applying the initial conditions,

$$s^2\,Y(s) + \omega^2\,Y(s) - s = \frac{s}{s^2+4}\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{s}{(s^2+\omega^2)(s^2+4)} + \frac{s}{s^2+\omega^2}\,.$$

Using partial fractions on the first term,

$$\frac{s}{(s^2+\omega^2)(s^2+4)} = \frac{1}{4-\omega^2}\left[\frac{s}{s^2+\omega^2} - \frac{s}{s^2+4}\right].$$

First note that

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+\omega^2}\right] = \cos\omega t \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] = \cos 2t\,.$$

Hence the solution of the IVP is

$$\begin{aligned} y(t) &= \frac{1}{4-\omega^2}\cos\omega t - \frac{1}{4-\omega^2}\cos 2t + \cos\omega t = \\ &= \frac{5-\omega^2}{4-\omega^2}\cos\omega t - \frac{1}{4-\omega^2}\cos 2t\,. \end{aligned}$$

22. Taking the Laplace transform of both sides of the ODE, we obtain

$$[s^2Y(s) - sy(0) - y'(0)] - 2[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+1}.$$

Solving for $Y(s)$ we get

$$Y(s) = \frac{s-1}{s^2-2s+2} + \frac{1}{(s^2-2s+2)(s+1)}.$$

Using partial fractions on the second term, we have

$$\frac{1}{(s^2-2s+2)(s+1)} = \frac{1}{5}\frac{1}{s+1} + \frac{1}{5}\frac{3-s}{s^2-2s+2}.$$

Therefore, we can write

$$Y(s) = \frac{1}{5}\frac{1}{s+1} + \frac{2}{5}\frac{2s-1}{s^2-2s+2}.$$

Completing the square in the denominator for the last term, we have

$$\frac{2s-1}{s^2-2s+2} = \frac{2(s-1)+1}{(s-1)^2+1}.$$

Therefore,

$$Y(s) = \frac{1}{5}\frac{1}{s+1} + \frac{2}{5}\frac{2(s-1)+1}{(s-1)^2+1},$$

which implies that $y = (e^{-t} + 4e^t\cos t + 2e^t\sin t)/5$.

24. Using Problem 21 from the previous section, we obtain

$$[s^2Y(s) - sy(0) - y'(0)] + 4Y(s) = -\frac{e^{-2\pi s}}{s} + \frac{1}{s}.$$

Therefore, using the initial conditions and solving for $Y(s)$ we get

$$Y(s) = \frac{s^2+1-e^{-2\pi s}}{s(s^2+4)}.$$

27. Using Problem 24 from the previous section, we obtain

$$[s^2Y(s) - sy(0) - y'(0)] + Y(s) = \frac{e^{-2s}(-1+e^s)^2}{s^2}.$$

Therefore, using the initial conditions and solving for $Y(s)$ we get

$$Y(s) = \frac{e^{-2s}(-1+e^s)^2}{s^2(s^2+1)}.$$

28.(a)

$$\mathcal{L}[\sin t] = \int_0^\infty \sin(t)e^{-st}\,dt = \int_0^\infty \sum_{n=0}^\infty \frac{(-1)^n t^{2n+1}}{(2n+1)!}e^{-st}\,dt =$$

$$= \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}\int_0^\infty t^{2n+1}e^{-st}\,dt = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}\left[\frac{(2n+1)!}{s^{2n+2}}\right] =$$

$$= \sum_{n=0}^\infty \frac{(-1)^n}{s^{2n+2}} = \frac{1}{s^2}\sum_{n=0}^\infty \left(\frac{-1}{s^2}\right)^n = \frac{1}{s^2}\left[\frac{1}{1+1/s^2}\right] = \frac{1}{s^2+1},$$

where we used the fact that the sum of the geometric series $\sum_{n=0}^\infty r^n = 1/(1-r)$.

(b) From part (a) we see that the Taylor series for $f(t)$ is given by

$$\sum_{n=0}^\infty \frac{(-1)^n t^{2n}}{(2n+1)!}.$$

Then

$$\mathcal{L}[f(t)] = \int_0^\infty \sum_{n=0}^\infty \frac{(-1)^n t^{2n}}{(2n+1)!}e^{-st}\,dt = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}\int_0^\infty t^{2n}e^{-st}\,dt =$$

$$= \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}\left[\frac{(2n)!}{s^{2n+1}}\right] = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)s^{2n+1}} = \arctan(1/s),$$

where we have used the Taylor series for $\arctan t$.

(c)

$$\mathcal{L}[J_0(t)] = \int_0^\infty J_0(t)e^{-st}\,dt = \int_0^\infty \sum_{n=0}^\infty \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}e^{-st}\,dt =$$

$$= \sum_{n=0}^\infty \frac{(-1)^n}{2^{2n}(n!)^2}\int_0^\infty t^{2n}e^{-st}\,dt = \sum_{n=0}^\infty \frac{(-1)^n}{2^{2n}(n!)^2}\left[\frac{(2n)!}{s^{2n+1}}\right].$$

Finding the power series expansion (or using the generalized binomial theorem) for $(1+s^2)^{-1/2}$, we realize that the two series match.

$$\mathcal{L}[J_0(\sqrt{t})] = \int_0^\infty J_0(\sqrt{t})e^{-st}\,dt = \int_0^\infty \sum_{n=0}^\infty \frac{(-1)^n t^n}{2^{2n}(n!)^2}e^{-st}\,dt =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} \int_0^{\infty} t^n e^{-st}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} \left[\frac{n!}{s^{n+1}}\right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)s^{n+1}}.$$

Again, finding the power series for $s^{-1}e^{-1/4s}$, we realize that the two series match.

31. We know that

$$\mathcal{L}\left[\sin bt\right] = \frac{b}{s^2+b^2}\,.$$

Based on Problem 29,

$$\mathcal{L}\left[t^2 \sin bt\right] = \frac{d^2}{ds^2}\left[\frac{b}{s^2+b^2}\right] = \frac{d}{ds}\left[\frac{-2bs}{(s^2+b^2)^2}\right] = \frac{2b(3s^2-b^2)}{(s^2+b^2)^3}.$$

33. Using the result of Problem 29 repeatedly, we have

$$\mathcal{L}[te^{at}] = -\frac{d}{ds}(s-a)^{-1} = (s-a)^{-2},$$

$$\mathcal{L}[t^2e^{at}] = -\frac{d}{ds}(s-a)^{-2} = 2(s-a)^{-3},$$

$$\mathcal{L}[t^3e^{at}] = -\frac{d}{ds}2(s-a)^{-3} = 3!(s-a)^{-4}.$$

Using induction we obtain that $\mathcal{L}[t^n e^{at}] = n!(s-a)^{-(n+1)}$.

37.(a) Taking the Laplace transform of the given Airy equation,

$$\mathcal{L}\left[y''\right] - \mathcal{L}\left[t\,y\right] = \mathcal{L}\left[y''\right] + \mathcal{L}\left[-t\,y\right] = s^2Y(s) - sy(0) - y'(0) + Y'(s) = 0\,.$$

Hence $Y(s)$ satisfies $Y' + s^2Y = s$.

(b) Taking the Laplace transform of the given Legendre equation,

$$\mathcal{L}\left[\,y''\right] - \mathcal{L}\left[t^2\,y''\right] - 2\,\mathcal{L}\left[\,t\,y'\right] + \alpha(\alpha+1)\mathcal{L}\left[\,y\,\right] = 0\,.$$

Using the differentiation property of the transform,

$$\mathcal{L}\left[\,y''\right] - \frac{d^2}{ds^2}\mathcal{L}\left[\,y''\right] + 2\frac{d}{ds}\mathcal{L}\left[\,y'\right] + \alpha(\alpha+1)\mathcal{L}\left[\,y\,\right] = 0\,.$$

That is,

$$\left[s^2Y(s) - s\,y(0) - y'(0)\right] - \frac{d^2}{ds^2}\left[s^2Y(s) - s\,y(0) - y'(0)\right] +$$

$$+\, 2\frac{d}{ds}\left[s\,Y(s) - \,y(0)\right] + \alpha(\alpha+1)Y(s) = 0\,.$$

Invoking the initial conditions, we have

$$s^2Y(s) - 1 - \frac{d^2}{ds^2}\left[s^2Y(s) - 1\right] + 2\frac{d}{ds}\left[s\,Y(s)\right] + \alpha(\alpha+1)Y(s) = 0,$$

which simplifies to

$$\frac{d^2}{ds^2}\left[s^2Y(s)\right] - 2\frac{d}{ds}\left[s\,Y(s)\right] - \left[s^2 + \alpha(\alpha+1)\right]Y(s) = -1\,.$$

After carrying out the differentiation, we obtain

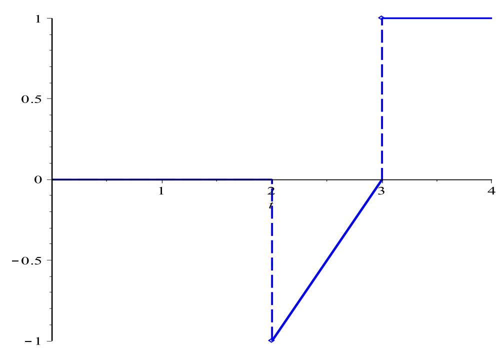
$$s^2 \frac{d^2}{ds^2} Y(s) + 2s \frac{d}{ds} Y(s) - \left[s^2 + \alpha(\alpha+1)\right] Y(s) = -1 \,.$$

39.(a) Form Eq.(i) we have $A_k = \lim_{s \to r_k} (s - r_k) P(r_k)/Q(r_k)$, since Q has distinct zeros. Thus $A_k = P(r_k) \lim_{s \to r_k} (s - r_k)/Q(r_k) = P(r_k)/Q'(r_k)$ by L'Hospital's rule.

(b) Since $\mathcal{L}^{-1}[1/(s - r_k)] = e^{r_k t}$, the result follows.

6.3

2.



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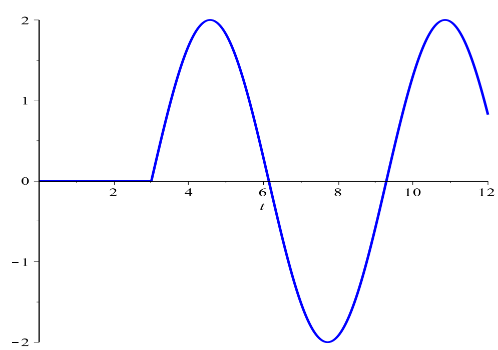
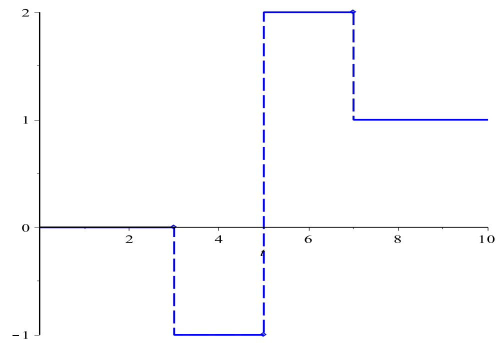
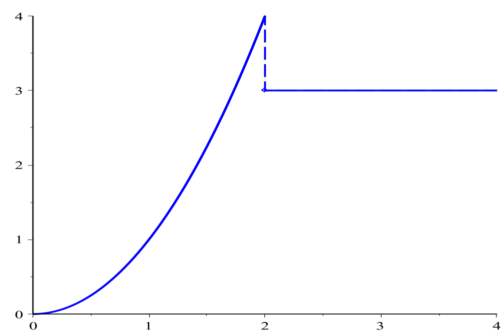


7.(a)



(b) There are step changes at $t = 3$, 5, and 7, thus we use $u_3(t)$, $u_5(t)$ and $u_7(t)$ multiplied by the appropriate step size to obtain $f(t) = -u_3(t) + 3u_5(t) - u_7(t)$.

10.



(b) $f(t) = t^2 + (3 - t^2)u_2(t)$.

14. Using the Heaviside function and completing the square, we can write $f(t) = ((t-1)^2 + 2)\,u_1(t)$. The Laplace transform has the property that $\mathcal{L}\left[\,u_c(t)f(t-c)\right] = e^{-cs}\mathcal{L}\left[f(t)\right]$. Hence

$$\mathcal{L}\left[\,((t-1)^2 + 2)\,u_1(t)\right] = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s}\right).$$

20. First consider the function

$$G(s) = \frac{1}{s^2 + s - 2}\,.$$

Factoring the denominator,

$$G(s) = \frac{1}{(s-1)(s+2)} = \frac{1}{3}\frac{1}{s-1} - \frac{1}{3}\frac{1}{s+2}\,.$$

It follows that

$$\mathcal{L}^{-1}\left[G(s)\right] = \frac{1}{3}e^{t} - \frac{1}{3}e^{-2t}$$

Hence

$$\mathcal{L}^{-1}\left[e^{-3s}G(s)\right] = \left(\frac{1}{3}e^{t-3} - \frac{1}{3}e^{-2(t-3)}\right) u_3(t)\,.$$

21. First consider the function

$$G(s) = \frac{2(s-1)}{s^2 - 2s + 2}\,.$$

Completing the square in the denominator,

$$G(s) = \frac{2(s-1)}{(s-1)^2 + 1}\,.$$

It follows that

$$\mathcal{L}^{-1}\left[G(s)\right] = 2\,e^{t}\,\cos\,t\,.$$

Hence

$$\mathcal{L}^{-1}\left[e^{-s}G(s)\right] = 2\,e^{t-1}\cos\left(t-1\right)u_1(t)\,.$$

27. By completing the square in the denominator of F we can write

$$F(s) = \frac{2s+1}{(2s+1)^2+4}.$$

This has the form $G(2s+1)$ where $G(u) = u/(u^2+4)$. Applying the result of Problem 25 part (c) with $a = 2$ and $b = 1$, we have $\mathcal{L}^{-1}\left[F(s)\right] = (1/2)e^{-t/2}\cos t$.

28. With the approach of Problem 27, we find $f(t) = (1/3)e^{2t/3}\sinh(t/3)$. Using the definition of $\sinh t$, this is equivalent to the given answer.

33. Assuming that term-by-term integration of the infinite series is permissible and recalling that $\mathcal{L}[u_c(t)] = e^{-cs}/s$ for $s > 0$, we have

$$\mathcal{L}[f(t)] = \frac{1}{s} + \sum_{k=1}^{\infty}(-1)^k\frac{e^{-ks}}{s} = \frac{1}{s}\sum_{k=0}^{\infty}(-1)^k(e^{-s})^k.$$

We recognize that the last series is a geometric series, which converges to $1/(1+e^{-s})$ when $s > 0$. Hence $\mathcal{L}[f(t)] = 1/s(1+e^{-s})$.

34. Let $f_T(t) = f(t)$ when $0 \le t \le T$, and 0 otherwise. The Laplace transform of $f_T(t)$ is $F_T(s) = \int_0^T e^{-st}f(t)\,dt$. Then we can write the entire periodic function as $f(t) = \sum_{k=0}^{\infty} f_T(t-kT)u_{kT}(t)$. Clearly, $\mathcal{L}[f_T(t-kT)u_{kT}(t)] = e^{-kTs}\mathcal{L}[f_T(t)] = e^{-kTs}F_T(s)$. Consider now the function $\sum_{k=0}^{n-1} f_T(t-kT)u_{kT}(t)$. Using linearity we obtain that

$$\mathcal{L}\left[\sum_{k=0}^{n-1} f_T(t-kT)u_{kT}(t)\right] = \sum_{k=0}^{n-1} e^{-kTs}F_T(s) = F_T(s)\sum_{k=0}^{n-1}(e^{-sT})^k = F_T(s)\frac{1-(e^{-sT})^n}{1-e^{-sT}},$$

using the sum formula for the first n terms of a geometric series. Since $e^{-sT} < 1$ for $s > 0$, it follows that

$$F(s) = \lim_{n\to\infty} F_T(s)\frac{1-(e^{-sT})^n}{1-e^{-sT}} = \frac{F_T(s)}{1-e^{-sT}}.$$

36. Using Problem 34, we know that

$$\mathcal{L}[f(t)] = \frac{F_T(s)}{1-e^{-sT}}$$

where T is the period and

$$F_T(s) = \int_0^T e^{-st} f(t)\,dt.$$

Here, $T = 2$. Therefore,

$$F_T(s) = \int_0^1 e^{-st}\,dt - \int_1^2 e^{-st}\,dt = \frac{1-e^{-s}}{s} - \frac{e^{-s}-e^{-2s}}{s} = \frac{1-2e^{-s}+e^{-2s}}{s}.$$

Therefore,

$$\mathcal{L}[f(t)] = \frac{1-2e^{-s}+e^{-2s}}{s(1-e^{-2s})} = \frac{(1-e^{-s})^2}{s(1-e^{-s})(1+e^{-s})} = \frac{1-e^{-s}}{s(1+e^{-s})}.$$

6.4

1.(a) Let $Y(s) = \mathcal{L}(y)$. Applying the Laplace transform to the equation, we obtain

$$[s^2Y(s) - sy(0) - y'(0)] + 9Y(s) = \mathcal{L}(f(t)).$$

Applying the initial conditions, we have

$$s^2Y(s) - 1 + 9Y(s) = \mathcal{L}(f(t)).$$

The forcing function $f(t)$ can be written as $f(t) = 1 - u_{3\pi}(t)$. Therefore,

$$\mathcal{L}(f(t)) = \mathcal{L}(1) - \mathcal{L}(u_{3\pi}(t)) = \frac{1-e^{-3\pi s}}{s}.$$

Therefore, the equation for Y becomes

$$[s^2+9]Y(s) = 1 + \frac{1-e^{-3\pi s/2}}{s},$$

which gives us

$$Y(s) = \frac{1}{s^2+9} + \frac{1-e^{-3\pi s}}{s(s^2+9)}.$$

Using partial fractions, we write the second term as

$$(1-e^{-3\pi s})\frac{1}{9}\left[\frac{1}{s} - \frac{s}{s^2+9}\right].$$

Therefore,

$$Y(s) = \frac{1}{s^2+9} + \frac{1}{9}\frac{1}{s} - \frac{1}{9}\frac{s}{s^2+9} - \frac{1}{9}\frac{e^{-3\pi s}}{s} + \frac{1}{9}\frac{se^{-3\pi s}}{s^2+9}.$$

Then, using the fact that

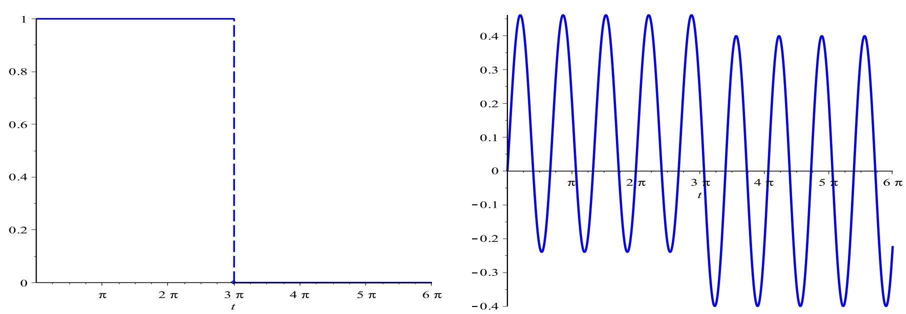
$$\mathcal{L}^{-1}(e^{-cs}G(s)) = u_c(t)g(t-c),$$

we conclude that

$$y(t) = \frac{1}{3}\sin 3t + \frac{1}{9} - \frac{1}{9}\cos 3t - \frac{1}{9}u_{3\pi}(t) + \frac{1}{9}u_{3\pi}(t)\cos(3t-3\pi) =$$

$$= \frac{1}{3}\sin 3t + \frac{1}{9} - \frac{1}{9}\cos 3t - \frac{1}{9}u_{3\pi}(t)[1+\cos 3t].$$

(b)



The graph of the solution is composed of two segments. The first is an oscillation about 1/9, with amplitude $\sqrt{10}/9$. For $t \geq 3\pi$, the forcing function is zero and the system response is $y = \sin 3t/3 - 2\cos 3t/9$, which is an oscillation with amplitude $\sqrt{13}/9$ about 0.

3.(a) Let $Y(s) = \mathcal{L}(y)$. Applying the Laplace transform to the equation, we obtain

$$[s^2Y(s) - sy(0) - y'(0)] + 4Y(s) = \frac{1-e^{-2\pi s}}{s^2+1}.$$

Applying the initial conditions, we have

$$s^2Y(s) + 4Y(s) = \frac{1-e^{-2\pi s}}{s^2+1}.$$

Therefore,

$$Y(s) = \frac{1-e^{-2\pi s}}{(s^2+4)(s^2+1)}.$$

Using partial fractions, we can write

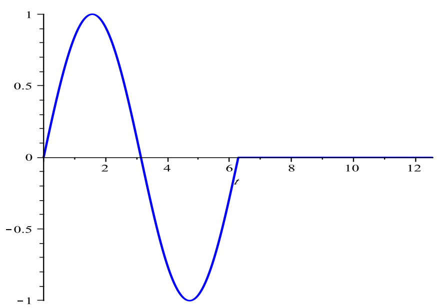
$$Y(s) = \frac{1}{3}(1-e^{-2\pi s})\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right].$$

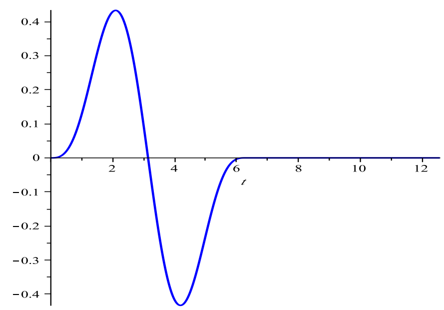
Therefore, we conclude that

$$y(t) = \frac{1}{3}\left[\sin t - \frac{1}{2}\sin 2t - u_{2\pi}(t)\sin(t-2\pi) + \frac{1}{2}u_{2\pi}(t)\sin(2(t-2\pi))\right] =$$

$$= \frac{1}{3}\left[\sin t - \frac{1}{2}\sin 2t\right] - \frac{1}{3}u_{2\pi}(t)\left[\sin t - \frac{1}{2}\sin 2t\right] = \frac{1}{6}(1 - u_{2\pi}(t))[2\sin t - \sin 2t].$$

(b)





8.(a) Let $Y(s) = \mathcal{L}(y)$. Applying the Laplace transform to the equation, we have

$$[s^2Y(s) - sy(0) - y'(0)] + [sY(s) - y(0)] + \frac{5}{4}Y(s) = \frac{1 - e^{-\pi s/2}}{s^2}.$$

Applying the initial conditions, we obtain

$$s^2Y(s) + sY(s) + \frac{5}{4}Y(s) = 1 + \frac{1 - e^{-\pi s/2}}{s^2}.$$

Therefore, the equation for Y becomes

$$Y(s) = \frac{1}{s^2 + s + 5/4} + \frac{1 - e^{-\pi s/2}}{s^2(s^2 + s + 5/4)}.$$

Using partial fractions, we can write

$$Y(s) = \frac{1}{(s + 1/2)^2 + 1} + (1 - e^{-\pi s/2})\left[\frac{64s - 16}{25(4s^2 + 4s + 5)} + \frac{4}{5s^2} - \frac{16}{25s}\right].$$

Now, completing the square in the denominator of the second term on the right-hand side above, we have

$$\frac{64s - 16}{25(4s^2 + 4s + 5)} = \frac{64s - 16}{100((s + 1/2)^2 + 1)} = \frac{16}{25}\left[\frac{s + 1/2}{(s + 1/2)^2 + 1} - \frac{3/4}{(s + 1/2)^2 + 1}\right].$$

Therefore,

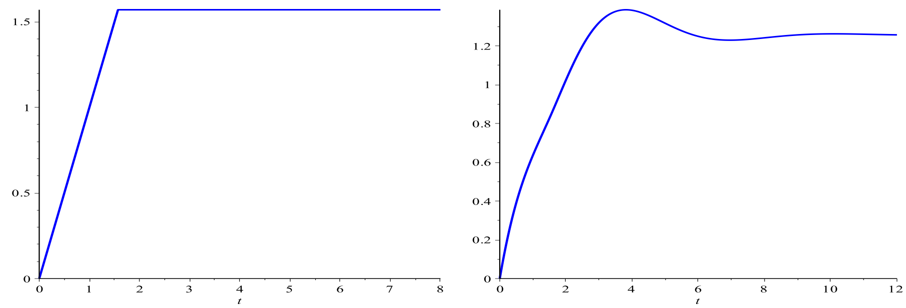
$$Y(s) = \frac{1}{(s + 1/2)^2 + 1} + (1 - e^{-\pi s/2}) \cdot \frac{16}{25}\left[\frac{s + 1/2}{(s + 1/2)^2 + 1} - \frac{3/4}{(s + 1/2)^2 + 1} + \frac{5}{4s^2} - \frac{1}{s}\right].$$

We conclude that

$$y(t) = e^{-t/2}\sin t + \frac{16}{25}\left[e^{-t/2}\cos t - \frac{3}{4}e^{-t/2}\sin t + \frac{5}{4}t - 1\right]$$

$$-\frac{16}{25}u_{\pi/2}(t)\left[e^{-(t-\pi/2)/2}\cos(t - \pi/2) - \frac{3}{4}e^{-(t-\pi/2)/2}\sin(t - \pi/2) + \frac{5}{4}(t - \pi/2) - 1\right].$$

(b)



There is an initial oscillation with an exponentially decreasing amplitude, then the solution approaches the steady state $y_s(t) = 4\pi/10$.

10. First, we can write the nonhomogeneous term as $g(t) = \sin t + u_\pi(t)\sin(t-\pi)$. Then, let $Y(s) = \mathcal{L}(y)$. Applying the Laplace transform to the equation, we have

$$[s^2Y(s) - sy(0) - y'(0)] + [sY(s) - y(0)] + \frac{5}{4}Y(s) = \frac{1+e^{-\pi s}}{s^2+1}.$$

Applying the initial conditions, we get

$$s^2Y(s) + sY(s) + \frac{5}{4}Y(s) = s + 1 + \frac{1+e^{-\pi s}}{s^2+1}.$$

Therefore, the equation for Y becomes

$$Y(s) = \frac{s+1}{(s+1/2)^2+1} + \frac{1+e^{-\pi s}}{(s^2+s+5/4)(s^2+1)}.$$

Using partial fractions, we can write

$$Y(s) = \frac{s+1}{(s+1/2)^2+1} + (1+e^{-\pi s})\left[\frac{64s+48}{17(4s^2+4s+5)} + \frac{-16s+4}{17(s^2+1)}\right] =$$

$$= \frac{s+1/2}{(s+1/2)^2+1} + \frac{1/2}{(s+1/2)^2+1} + \frac{1}{17}(1+e^{-\pi s})\left[\frac{16(s+1/2)+4}{(s+1/2)^2+1} - \frac{16s}{s^2+1} + \frac{4}{s^2+1}\right].$$

Therefore, we conclude that

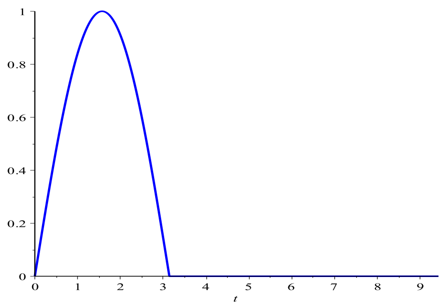
$$y(t) = e^{-t/2}\cos t + \frac{1}{2}e^{-t/2}\sin t + \frac{1}{17}\left[16e^{-t/2}\cos t + 4e^{-t/2}\sin t - 16\cos t + 4\sin t\right] +$$

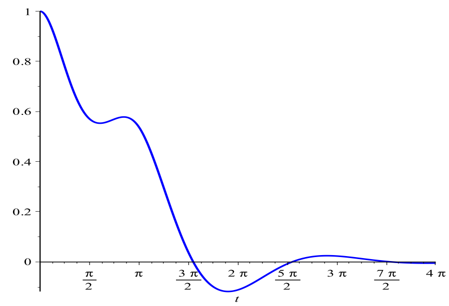
$$+\frac{1}{17}u_\pi(t)\left[16e^{-(t-\pi)/2}\cos(t-\pi) + 4e^{-(t-\pi)/2}\sin(t-\pi) - 16\cos(t-\pi) + 4\sin(t-\pi)\right] =$$

$$= e^{-t/2}\cos t + \frac{1}{2}e^{-t/2}\sin t + \frac{1}{17}[16\cos t(e^{-t/2}-1) + 4\sin t(e^{-t/2}+1)] +$$

$$+\frac{1}{17}u_\pi(t)[16\cos t(1-e^{-(t-\pi)/2}) - 4\sin t(1+e^{-(t-\pi)/2})].$$

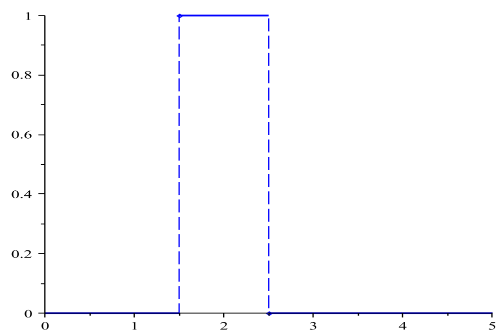
(b)





The initial oscillation is a response to the forcing term, but after the forcing term is taken away, the solution decays to zero due to the damping term.

16.(a)



(b) Let $U(s) = \mathcal{L}(u)$. Applying the Laplace transform to the equation, we have

$$[s^2U(s) - su(0) - u'(0)] + \frac{1}{4}[sU(s) - u(0)] + U(s) = k\frac{e^{-3s/2} - e^{-5s/2}}{s}.$$

Applying the initial conditions, we have

$$s^2U(s) + \frac{1}{4}sU(s) + U(s) = k\frac{e^{-3s/2} - e^{-5s/2}}{s}.$$

Therefore, the equation for U becomes

$$U(s) = k\frac{e^{-3s/2} - e^{-5s/2}}{s(s^2 + 1/4s + 1)}.$$

Using partial fractions, we can write

$$U(s) = k(e^{-3s/2} - e^{-5s/2})\left[\frac{1}{s} - \frac{4s+1}{4s^2 + s + 4}\right] =$$

$$= k(e^{-3s/2} - e^{-5s/2})\left[\frac{1}{s} - \frac{(s+1/8) + 1/8}{(s+1/8)^2 + 63/64}\right].$$

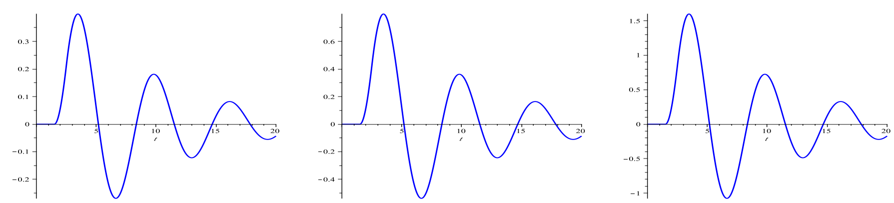
If we now let $H(s)$ be the terms in the square brackets, then

$$h(t) = 1 - e^{-t/8}\left[\frac{\sqrt{7}}{21}\,\sin\left(\frac{3\sqrt{7}\,t}{8}\right) + \cos\left(\frac{3\sqrt{7}\,t}{8}\right)\right],$$

and then

$$u(t) = k\,u_{3/2}(t)\,h\left(t - \frac{3}{2}\right) - k\,u_{5/2}(t)\,h\left(t - \frac{5}{2}\right).$$

(c) In all cases the plot will be zero for $0 \le t < 3/2$. For $3/2 \le t < 5/2$ the plot will be the system response to a step input of magnitude k. For $t \ge 5/2$, the plot will be the system response to the initial condition $u(5/2)$, $u'(5/2)$ with no forcing function. Varying k affects the amplitude.



(a) $k = 1/2$ (b) $k = 1$ (c) $k = 2$

(d) From part (c) the solution is

$$u(t) = k\,u_{3/2}(t)\,h\left(t - \frac{3}{2}\right) - k\,u_{5/2}(t)\,h\left(t - \frac{5}{2}\right),$$

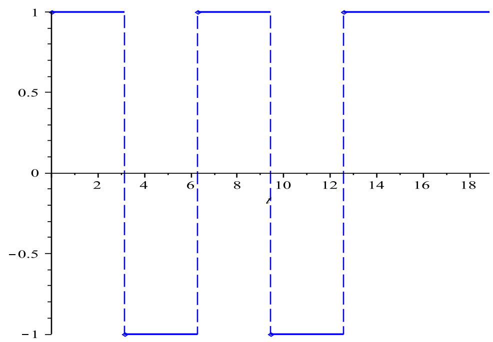
where

$$h(t) = 1 - e^{-t/8}\left[\frac{\sqrt{7}}{21}\,\sin\left(\frac{3\sqrt{7}\,t}{8}\right) + \cos\left(\frac{3\sqrt{7}\,t}{8}\right)\right]$$

Due to the damping term, the solution will decay to zero. The maximum will occur shortly after the forcing ceases. By plotting the various solutions, it appears that the solution will reach a value of $y = 2$, as long as $k > 2.51$.

(e) Based on the graph, and numerical calculation, $|u(t)| < 0.1$ for $t > 25.6773$.

19.(a) The graph below is for $n = 4$.



(b) Let $Y(s) = \mathcal{L}(y)$. Applying the Laplace transform to the equation, we get

$$[s^2Y(s) - sy(0) - y'(0)] + Y(s) = \frac{1 + 2\sum_{k=1}^{n}(-1)^k e^{-k\pi s}}{s}.$$

Applying the initial conditions, we obtain

$$Y(s) = \frac{1 + 2\sum_{k=1}^{n}(-1)^k e^{-k\pi s}}{s(s^2+1)}.$$

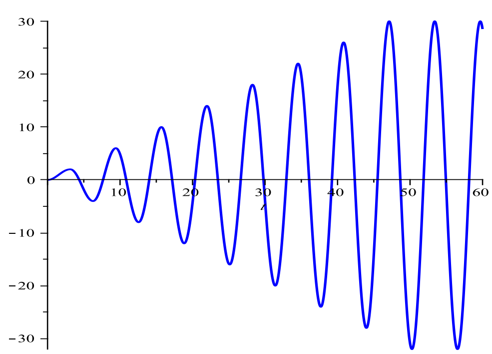
Using partial fractions, we see that

$$Y(s) = \left(1 + 2\sum_{k=1}^{n}(-1)^k e^{-k\pi s}\right)\left[\frac{1}{s} - \frac{s}{s^2+1}\right].$$

Therefore, we conclude that

$$y(t) = 1 - \cos t + 2\sum_{k=1}^{n}(-1)^k[1 - \cos(t - k\pi)]u_{k\pi}(t).$$

(c)



For $0 \le t < \pi$, $y(t) = 1 - \cos t$, which peaks at $t = \pi$, just when the forcing term changes from 1 to -1. Thus the forcing term "reinforces" the natural motion, creating a "resonance". This occurs at each π interval until $t > 15\pi$, at which time the forcing function no longer changes and the solution continues oscillating about -1.

(d) Since $\cos(t - k\pi) = (-1)^k \cos t$, the solution in part (b) can be written as

$$u(t) = 1 - \cos\, t + 2\sum_{k=1}^{n} (-1)^k \left[1 - \cos(t - k\pi)\right] u_{k\pi}(t) =$$

$$= 1 - \cos t - 2n\cos t + 2\sum_{k=1}^{n} (-1)^k u_{k\pi}(t),$$

which diverges for $n \to \infty$, since the nth term does not approach zero.

20.(a) Let $Y(s) = \mathcal{L}(y)$. Applying the Laplace transform to the equation, we obtain

$$[s^2Y(s) - sy(0) - y'(0)] + 0.1[sY(s) - y(0)] + Y(s) = \frac{1 + 2\sum_{k=1}^{n}(-1)^k e^{-k\pi s}}{s}.$$

Applying the initial conditions, we get

$$Y(s) = \frac{1 + 2\sum_{k=1}^{n}(-1)^k e^{-k\pi s}}{s(s^2 + 0.1s + 1)}.$$

Using partial fractions, we see that

$$Y(s) = \left(1 + 2\sum_{k=1}^{n}(-1)^k e^{-k\pi s}\right)\left[\frac{1}{s} - \frac{s + 0.1}{s^2 + 0.1s + 1}\right] =$$

$$= \left(1 + 2\sum_{k=1}^{n}(-1)^k e^{-k\pi s}\right)\left[\frac{1}{s} - \frac{s + 0.05}{(s + 0.05)^2 + .9975} - \frac{0.05}{(s + 0.05)^2 + .9975}\right].$$

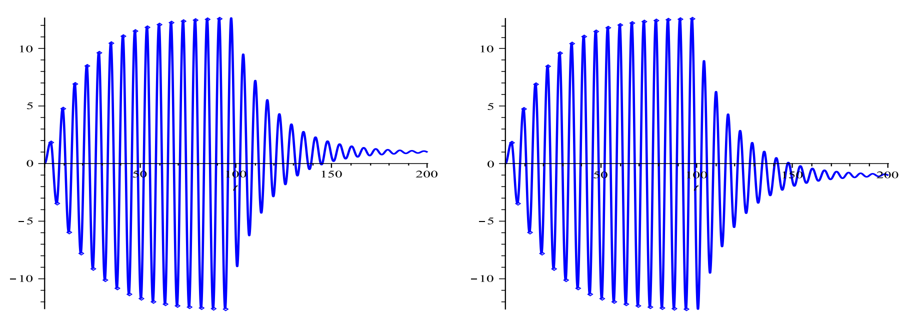
Therefore, we conclude that

$$y(t) = h(t) + 2\sum_{k=1}^{n}(-1)^k u_{k\pi}(t) h(t - k\pi),$$

where

$$h(t) = 1 - e^{-0.05t}\cos(\sqrt{.9975}t) - \frac{0.05}{\sqrt{.9975}} e^{-0.05t}\sin(\sqrt{.9975}t).$$

The figure below shows the behavior for $n = 30$ and $n = 31$.



(b) From the graph of part (a), $A = 12.5$ and the frequency is 2π.

(c) From the graph (or analytically), $A = 10$ and the frequency is 2π.

6.5

1.(a) Let $Y(s) = \mathcal{L}(y)$ and take the Laplace transform of the ODE. We arrive at

$$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 2Y(s) = e^{-\pi s}.$$

Applying the initial conditions, we have

$$[s^2Y(s) - s - 1] + 2[sY(s) - 1] + 2Y(s) = e^{-\pi s},$$

which can be rewritten as

$$[s^2 + 2s + 2]Y(s) - s - 3 = e^{-\pi s}.$$

Therefore,

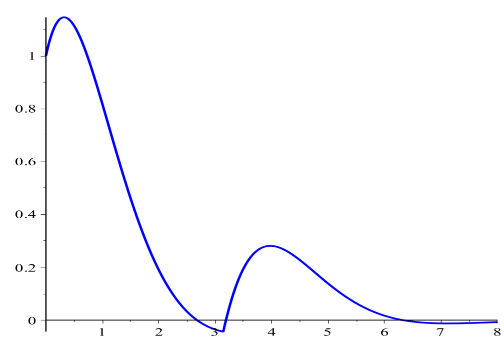
$$Y(s) = \frac{s + 3 + e^{-\pi s}}{s^2 + 2s + 2} = \frac{s + 3 + e^{-\pi s}}{(s+1)^2 + 1} =$$

$$= \frac{s+1}{(s+1)^2 + 1} + \frac{2}{(s+1)^2 + 1} + \frac{e^{-\pi s}}{(s+1)^2 + 1}.$$

We obtain that

$$y(t) = e^{-t}\cos t + 2e^{-t}\sin t + u_\pi(t)e^{-(t-\pi)}\sin(t - \pi).$$

(b)



3.(a) Let $Y(s) = \mathcal{L}(y)$ and take the Laplace transform of the ODE. We arrive at

$$[s^2Y(s) - sy(0) - y'(0)] + 3[sY(s) - y(0)] + 2Y(s) = e^{-4s} + \frac{e^{-8s}}{s}.$$

Applying the initial conditions, we get

$$Y(s) = \frac{1/2 + e^{-4s}}{s^2 + 3s + 2} + \frac{e^{-8s}}{s(s^2 + 3s + 2)}.$$

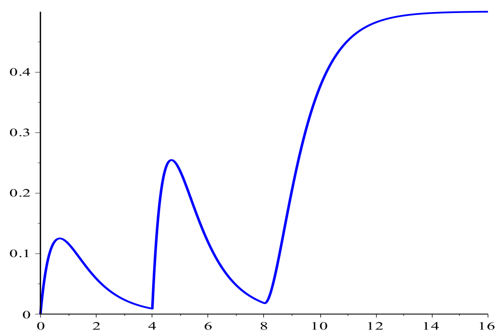
Therefore,

$$Y(s) = (\frac{1}{2} + e^{-4s}) \left[\frac{1}{s+1} - \frac{1}{s+2} \right] + e^{-8s} \left[\frac{1}{2(s+2)} - \frac{1}{s+1} + \frac{1}{2s} \right].$$

Thus

$$y(t) = \frac{1}{2}[e^{-t} - e^{-2t}] + u_4(t)[e^{-(t-4)} - e^{-2(t-4)}] + u_8(t) \left[\frac{1}{2} e^{-2(t-8)} - e^{-(t-8)} + \frac{1}{2} \right].$$

(b)



5.(a) Let $Y(s) = \mathcal{L}(y)$ and take the Laplace transform of the ODE. We arrive at

$$[s^2 Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 3Y(s) = \frac{1}{s^2+1} + e^{-2\pi s}.$$

Applying the initial conditions, we have

$$[s^2 + 2s + 3]Y(s) = \frac{1}{s^2+1} + e^{-2\pi s}.$$

Therefore,

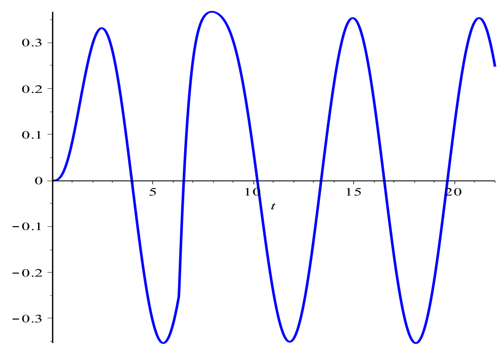
$$Y(s) = \frac{1}{(s^2+1)(s^2+2s+3)} + \frac{e^{-2\pi s}}{s^2+2s+3} =$$

$$= \frac{1}{4} \left[\frac{-s+1}{s^2+1} + \frac{s+1}{(s+1)^2+2} \right] + \frac{e^{-2\pi s}}{(s+1)^2+2}.$$

Thus

$$y(t) = \frac{1}{4} \left[-\cos t + \sin t + e^{-t} \cos(\sqrt{2}t) \right] + \frac{\sqrt{2}}{2} u_{2\pi}(t) e^{-(t-2\pi)} \sin(\sqrt{2}(t - 2\pi)).$$

(b)



7.(a) Let $Y(s) = \mathcal{L}(y)$ and take the Laplace transform of the ODE. We arrive at

$$[s^2Y(s) - sy(0) - y'(0)] + Y(s) = e^{-2\pi s},$$

where the transform on the right side was obtained using Eq.(16):

$$\int_0^\infty e^{-st}\delta(t-2\pi)\cos t\,dt = \int_{-\infty}^\infty e^{-st}\delta(t-2\pi)\cos t\,dt = e^{-2\pi s}\cos 2\pi = e^{-2\pi s}.$$

Applying the initial conditions, we have

$$[s^2Y(s) - s - 1] + Y(s) = e^{-2\pi s},$$

which can be rewritten as

$$[s^2+1]Y(s) - s - 1 = e^{-2\pi s}.$$

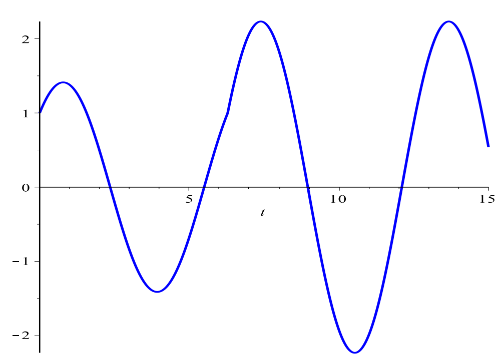
Therefore,

$$Y(s) = \frac{s+1}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1}.$$

Thus

$$y(t) = \sin t + \cos t + u_{2\pi}(t)\sin(t-2\pi) = \cos t + [1 + u_{2\pi}(t)]\sin t.$$

(b)



13.(a) Our equation will be of the form

$$2y'' + y' + 2y = \delta(t-5) + k\delta(t-t_0).$$

We need to find k and t_0 so that the system will rest again after exactly one cycle. Applying the Laplace transform and using the initial conditions, this equation becomes

$$[2s^2 + s + 2]Y(s) = e^{-5s} + ke^{-t_0 s}.$$

Therefore,

$$Y(s) = \frac{e^{-5s} + ke^{-t_0 s}}{2s^2 + s + 2} = (e^{-5s} + ke^{-t_0 s})\frac{1}{2}\left[\frac{1}{(s+1/4)^2 + (15/16)}\right].$$

Thus

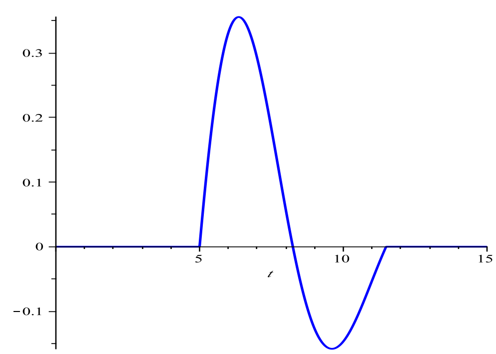
$$y(t) = u_5(t)\frac{2}{\sqrt{15}}e^{-(t-5)/4}\sin\left(\frac{\sqrt{15}}{4}(t-5)\right) + ku_{t_0}(t)\frac{2}{\sqrt{15}}e^{-(t-t_0)/4}\sin\left(\frac{\sqrt{15}}{4}(t-t_0)\right).$$

In order for this system to return to equilibrium after one cycle, we need the coefficients of the sinusoidal waves to be equal with opposite signs. Therefore, we need $e^{5/4} = -ke^{t_0/4}$, and we need the phase to differ by 2π. That is, we need

$$\frac{\sqrt{15}}{4}(t-5) = \frac{\sqrt{15}}{4}(t-t_0) + 2\pi.$$

Solving this equation, we see that $t_0 = 5 + 8\pi/\sqrt{15}$. This means that the period of this one completed cycle is $T = 8\pi/\sqrt{15}$. Solving the equation above for k, we see that $k = -e^{-2\pi/\sqrt{15}} = -e^{-T/4}$.

(b) From the analysis above, we see that the solution will be zero after time t_0.



17.(b) Applying the Laplace transform to the equation and using the initial conditions, we have

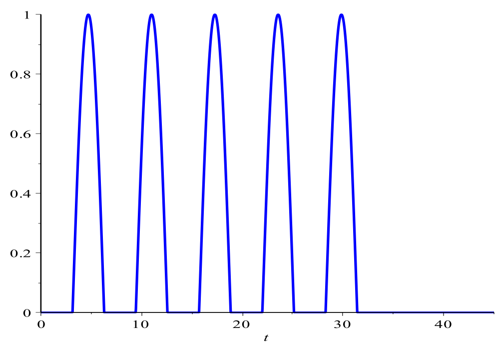
$$[s^2 + 1]Y(s) = \sum_{k=1}^{10} e^{-k\pi s}.$$

Therefore,

$$Y(s) = \frac{1}{s^2+1}\sum_{k=1}^{10} e^{-k\pi s},$$

which implies

$$y(t) = \sum_{k=1}^{10} \sin(t-k\pi)u_{k\pi}(t).$$



(c) After the sequence of impulses ends, the oscillator returns to equilibrium.

21.(b) Applying the Laplace transform to the equation and using the initial conditions, we have

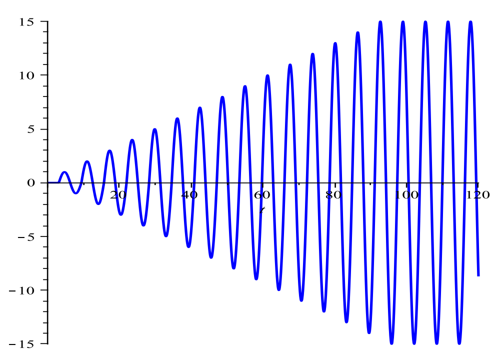
$$[s^2+1]Y(s) = \sum_{k=1}^{15} e^{-(2k-1)\pi s}.$$

Therefore,

$$Y(s) = \frac{1}{s^2+1}\sum_{k=1}^{15} e^{-(2k-1)\pi s},$$

which implies

$$y(t) = \sum_{k=1}^{15} \sin(t-(2k-1)\pi)u_{(2k-1)\pi}(t).$$



(c) After the sequence of impulses ends, the oscillator continues to oscillate at a constant amplitude.

25.(a) A fundamental set of solutions is $y_1(t) = e^{-t}\cos t$ and $y_2(t) = e^{-t}\sin t$. Based on Problem 22, in Section 3.6, a particular solution is given by

$$y_p(t) = \int_0^t \frac{y_1(s)y_2(t) - y_1(t)y_2(s)}{W(y_1\,, y_2)(s)} f(s)ds\,.$$

In the given problem,

$$y_p(t) = \int_0^t \frac{e^{-s-t}\left[\cos s \sin t - \sin s \cos t\right]}{e^{-2s}} f(s)ds =$$

$$= \int_0^t e^{-(t-s)} \sin(t-s) f(s)ds$$

Leibniz's rule for differentiating an integral in this form is given by

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,s)\,ds = f(t,b(t))b'(t) - f(t,a(t))a'(t) + \int_{a(t)}^{b(t)} \frac{\partial f(t,s)}{\partial t}\,ds.$$

Using this rule with the specified initial conditions,

$$y(t) = \int_0^t e^{-(t-s)} \sin(t-s) f(s)ds\,.$$

(b) Let $f(t) = \delta(t-\pi)$. It is easy to see that if $t < \pi$, $y(t) = 0$. If $t > \pi$,

$$\int_0^t e^{-(t-s)} \sin(t-s)\delta(s-\pi)ds = e^{-(t-\pi)} \sin(t-\pi)\,.$$

Setting $t = \pi + \varepsilon$, and letting $\varepsilon \to 0$, we find that $y(\pi) = 0$. Hence

$$y(t) = e^{-(t-\pi)} \sin(t-\pi)\, u_\pi(t)\,.$$

(c) The Laplace transform of the solution is

$$Y(s) = \frac{e^{-\pi s}}{s^2+2s+2} = \frac{e^{-\pi s}}{(s+1)^2+1},$$

hence the solutions agree.

6.6

1.(a) The convolution integral is defined as

$$f * g\,(t) = \int_0^t f(t-\tau)g(\tau)d\tau\,.$$

Consider the change of variable $u = t - \tau$. It follows that

$$\int_0^t f(t-\tau)g(\tau)d\tau = \int_t^0 f(u)g(t-u)(-du) = \int_0^t g(t-u)f(u)du = g * f(t).$$

(b) Based on the distributive property of the real numbers, the convolution is also distributive.

(c) By definition,

$$f * (g * h)(t) = \int_0^t f(t-\tau)\,[g * h\,(\tau)]\,d\tau =$$

$$= \int_0^t f(t-\tau)\left[\int_0^\tau g(\tau-\eta)h(\eta)d\eta\right]d\tau = \int_0^t \int_0^\tau f(t-\tau)g(\tau-\eta)h(\eta)\,d\eta d\tau\,.$$

The region of integration in the double integral is the area between the straight lines $\eta = 0\,, \eta = \tau$ and $\tau = t$. Interchanging the order of integration,

$$\int_0^t \int_0^\tau f(t-\tau)g(\tau-\eta)h(\eta)\,d\eta d\tau = \int_0^t \int_\eta^t f(t-\tau)g(\tau-\eta)h(\eta)\,d\tau d\eta =$$

$$= \int_0^t \left[\int_\eta^t f(t-\tau)g(\tau-\eta)d\tau\right] h(\eta)\,d\eta\,.$$

Now let $\tau - \eta = u$. Then

$$\int_\eta^t f(t-\tau)g(\tau-\eta)d\tau = \int_0^{t-\eta} f(t-\eta-u)g(u)du = f * g\,(t-\eta)\,.$$

Hence

$$\int_0^t f(t-\tau)\,[g * h\,(\tau)]\,d\tau = \int_0^t [f * g\,(t-\tau)]\,h(\tau)\,d\tau = (f * g) * h(t).$$

4. $\mathcal{L}[t^2] = 2/s^3$ and $\mathcal{L}[\cos(3t)] = s/(s^2+9)$. Therefore,

$$\mathcal{L}\left[\int_0^t (t-\tau)^2\cos(3\tau)\,d\tau\right] = \left(\frac{2}{s^3}\right)\cdot\left(\frac{s}{s^2+9}\right) = \frac{2}{s^2(s^2+9)}.$$

8. $\mathcal{L}^{-1}[1/s^4] = t^3/6$ and $\mathcal{L}^{-1}[1/(s^2+4)] = \sin 2t/2$. Therefore,

$$\mathcal{L}^{-1}\left[\frac{1}{s^4(s^2+4)}\right] = \int_0^t \frac{1}{12}(t-\tau)^3\sin 2\tau\,d\tau.$$

14. Applying the Laplace transform to the equation, we have

$$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 2Y(s) = \frac{\alpha}{s^2+\alpha^2}.$$

Applying the initial conditions, we have

$$[s^2+2s+2]Y(s) - 1 = \frac{\alpha}{s^2+\alpha^2}.$$

Therefore,

$$Y(s) = \frac{1}{s^2+2s+2} + \frac{1}{s^2+2s+2} \cdot \frac{\alpha}{s^2+\alpha^2} = \frac{1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1} \cdot \frac{\alpha}{s^2+\alpha^2}.$$

Therefore,

$$y(t) = e^{-t}\sin t + \int_0^t e^{-(t-\tau)}\sin(t-\tau)\sin(\alpha\,\tau)\,d\tau.$$

16. Applying the Laplace transform to the equation, we have

$$[s^2Y(s) - sy(0) - y'(0)] + [sY(s) - y(0)] + \frac{5}{4}Y(s) = \frac{1}{s} - \frac{e^{-2\pi s}}{s}.$$

Applying the initial conditions, we have

$$[s^2 + s + \frac{5}{4}]Y(s) = s + \frac{1}{s} - \frac{e^{-2\pi s}}{s}.$$

Therefore,

$$\begin{aligned} Y(s) &= \frac{s}{s^2+s+5/4} + \frac{1}{s^2+s+5/4} \cdot \frac{1-e^{-2\pi s}}{s} \\ &= \frac{s+1/2}{(s+1/2)^2+1} - \frac{1/2}{(s+1/2)^2+1} + \frac{1}{(s+1/2)^2+1} \cdot \frac{1-e^{-2\pi s}}{s} \end{aligned}$$

Therefore,

$$y(t) = e^{-t/2}\cos t - \frac{1}{2}e^{-t/2}\sin t + \int_0^t e^{-(t-\tau)/2}\sin(t-\tau)(1-u_{2\pi}(\tau))\,d\tau.$$

18. Applying the Laplace transform to the equation, we have

$$[s^2Y(s) - sy(0) - y'(0)] + 3[sY(s) - y(0)] + 2Y(s) = \frac{s}{s^2+\alpha^2}$$

Applying the initial conditions, we have

$$[s^2 + 3s + 2]Y(s) = 2s + 5 + \frac{s}{s^2+\alpha^2}.$$

Therefore,

$$Y(s) = \frac{2s+5}{s^2+3s+2} + \frac{s}{(s^2+3s+2)(s^2+\alpha^2)}.$$

Using partial fractions, we write

$$\frac{2s+5}{s^2+3s+2} = \frac{3}{s+1} - \frac{1}{s+2},$$

and

$$\frac{1}{s^2+3s+2} = \frac{1}{s+1} - \frac{1}{s+2}.$$

Therefore, we can conclude that

$$y(t) = 3e^{-t} - e^{-2t} + \int_0^t (e^{-(t-\tau)} - e^{-2(t-\tau)})\cos(\alpha\,\tau)\,d\tau.$$

21. Taking the Laplace transform of the integral equation, we have

$$\mathcal{L}[\phi(t)] + \mathcal{L}\left[\int_0^t k(t-\xi)\phi(\xi)\,d\xi\right] = \mathcal{L}[f(t)],$$

which implies

$$\mathcal{L}[\phi(t)] + \mathcal{L}[k(t)] \cdot \mathcal{L}[\phi(t)] = \mathcal{L}[f(t)].$$

Therefore,

$$\Phi(s) = \mathcal{L}[\phi(t)] = \frac{\mathcal{L}[f(t)]}{1 + \mathcal{L}[k(t)]} = \frac{F(s)}{1 + K(s)}.$$

24.(a) Taking the Laplace transform of the equation, we have

$$\Phi(s) - \frac{1}{s^2}\Phi(s) = \frac{1}{s}.$$

Therefore,

$$\Phi(s) = \frac{s}{s^2 - 1},$$

which implies $\phi(t) = \cosh t$.

(b) Using Leibniz's rule (see Problem 25 in the previous section) and differentiating the equation once, we have

$$\phi'(t) - \int_0^t \phi(\xi)\,d\xi = 0.$$

Differentiating the equation again, we have

$$\phi''(t) - \phi(t) = 0.$$

Using the original equation, we see that $\phi(0) = 1$. Further, using the equation for the first derivative, we see that $\phi'(0) = 0$.

(c) Taking the Laplace transform of the equation in part (b), we have

$$[s^2\Phi(s) - s\phi(0) - \phi'(0)] - \Phi(s) = 0.$$

Applying the initial conditions, we have $[s^2 - 1]\Phi(s) = s$. Therefore,

$$\Phi(s) = \frac{s}{s^2 - 1},$$

which implies $\phi(t) = \cosh t$.

26.(a) Taking the Laplace transform of the equation, we have

$$s\Phi(s) + \frac{1}{s^2}\Phi(s) = \frac{1}{s^2}.$$

Therefore,

$$\Phi(s) = \frac{1}{s^3 + 1} = \frac{2 - s}{3(s^2 - s + 1)} + \frac{1}{3}\frac{1}{s+1} =$$

$$= -\frac{1}{3}\left[\frac{s - 1/2}{(s - 1/2)^2 + 3/4}\right] + \frac{1/2}{(s - 1/2)^2 + 3/4} + \frac{1}{3}\frac{1}{s+1}$$

which implies

$$\phi(t) = -\frac{1}{3}e^{t/2}\cos(\sqrt{3}t/2) + \frac{1}{\sqrt{3}}e^{t/2}\sin(\sqrt{3}t/2) + \frac{1}{3}e^{-t}.$$

(b) Differentiating the equation once, we have

$$\phi''(t) + \int_0^t \phi(\xi)\,d\xi = 1.$$

Differentiating the equation again, we have

$$\phi'''(t) + \phi(t) = 0.$$

Using the equation for ϕ', we see that $\phi'(0) = 0$. Further, using the equation for ϕ'', we see that $\phi''(0) = 1$.

(c) Taking the Laplace transform of the equation in part (b), we have

$$[s^3\Phi(s) - s^2\phi(0) - s\phi'(0) - \phi''(0)] + \Phi(s) = 0.$$

Applying the initial conditions, we have $[s^3 + 1]\Phi(s) = 1$. Therefore,

$$\Phi(s) = \frac{1}{s^3 + 1}.$$

Using the same analysis as in part (a), we see that

$$\phi(t) = -\frac{1}{3}e^{t/2}\cos(\sqrt{3}t/2) + \frac{1}{\sqrt{3}}e^{t/2}\sin(\sqrt{3}t/2) + \frac{1}{3}e^{-t}.$$