In determining the roots of the characteristic equation, it may be necessary to compute the cube roots, the fourth roots, or even higher roots of a (possibly complex) number. This can usually be done most conveniently by using Euler's formula $e^{it} = \cos t + i \sin t$ and the algebraic laws given in Section 3.3. This is illustrated in the following example.

EXAMPLE 4

Find the general solution of

$$y^{(4)} + y = 0. \tag{20}$$

The characteristic equation is

$$r^4 + 1 = 0.$$

To solve the equation, we must compute the fourth roots of -1. Now -1, thought of as a complex number, is $-1 + 0i$. It has magnitude 1 and polar angle π. Thus

$$-1 = \cos \pi + i \sin \pi = e^{i\pi}.$$

Moreover, the angle is determined only up to a multiple of 2π. Thus

$$-1 = \cos(\pi + 2m\pi) + i \sin(\pi + 2m\pi) = e^{i(\pi+2m\pi)},$$

where m is zero or any positive or negative integer. Thus

$$(-1)^{1/4} = e^{i(\pi/4+m\pi/2)} = \cos\left(\frac{\pi}{4} + \frac{m\pi}{2}\right) + i \sin\left(\frac{\pi}{4} + \frac{m\pi}{2}\right).$$

The four fourth roots of -1 are obtained by setting $m = 0, 1, 2,$ and 3; they are

$$\frac{1+i}{\sqrt{2}}, \qquad \frac{-1+i}{\sqrt{2}}, \qquad \frac{-1-i}{\sqrt{2}}, \qquad \frac{1-i}{\sqrt{2}}.$$

It is easy to verify that, for any other value of m, we obtain one of these four roots. For example, corresponding to $m = 4$, we obtain $(1 + i)/\sqrt{2}$. The general solution of Eq. (20) is

$$y = e^{t/\sqrt{2}}\left(c_1 \cos \frac{t}{\sqrt{2}} + c_2 \sin \frac{t}{\sqrt{2}}\right) + e^{-t/\sqrt{2}}\left(c_3 \cos \frac{t}{\sqrt{2}} + c_4 \sin \frac{t}{\sqrt{2}}\right). \tag{21}$$

In conclusion, we note that the problem of finding all the roots of a polynomial equation may not be entirely straightforward, even with computer assistance. For instance, it may be difficult to determine whether two roots are equal or merely very close together. Recall that the form of the general solution is different in these two cases.

If the constants $a_0, a_1, \ldots, a_n$ in Eq. (1) are complex numbers, the solution of Eq. (1) is still of the form (4). In this case, however, the roots of the characteristic equation are, in general, complex numbers, and it is no longer true that the complex conjugate of a root is also a root. The corresponding solutions are complex-valued.

PROBLEMS

In each of Problems 1 through 6, express the given complex number in the form $R(\cos\theta + i \sin\theta) = Re^{i\theta}$.

1. $1 + i$
2. $-1 + \sqrt{3}i$
3. -3
4. $-i$
5. $\sqrt{3} - i$
6. $-1 - i$

In each of Problems 7 through 10, follow the procedure illustrated in Example 4 to determine the indicated roots of the given complex number.

7. $1^{1/3}$
8. $(1-i)^{1/2}$
9. $1^{1/4}$
10. $[2(\cos \pi/3 + i \sin \pi/3)]^{1/2}$

In each of Problems 11 through 28, find the general solution of the given differential equation.

11. $y''' - y'' - y' + y = 0$
12. $y''' - 3y'' + 3y' - y = 0$
13. $2y''' - 4y'' - 2y' + 4y = 0$
14. $y^{(4)} - 4y''' + 4y'' = 0$
15. $y^{(6)} + y = 0$
16. $y^{(4)} - 5y'' + 4y = 0$
17. $y^{(6)} - 3y^{(4)} + 3y'' - y = 0$
18. $y^{(6)} - y'' = 0$
19. $y^{(5)} - 3y^{(4)} + 3y''' - 3y'' + 2y' = 0$
20. $y^{(4)} - 8y' = 0$
21. $y^{(8)} + 8y^{(4)} + 16y = 0$
22. $y^{(4)} + 2y'' + y = 0$
23. $y''' - 5y'' + 3y' + y = 0$
24. $y''' + 5y'' + 6y' + 2y = 0$
25. $18y''' + 21y'' + 14y' + 4y = 0$
26. $y^{(4)} - 7y''' + 6y'' + 30y' - 36y = 0$
27. $12y^{(4)} + 31y''' + 75y'' + 37y' + 5y = 0$
28. $y^{(4)} + 6y''' + 17y'' + 22y' + 14y = 0$

In each of Problems 29 through 36, find the solution of the given initial value problem, and plot its graph. How does the solution behave as $t \to \infty$?

29. $y''' + y' = 0; \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 2$
30. $y^{(4)} + y = 0; \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = -1, \quad y'''(0) = 0$
31. $y^{(4)} - 4y''' + 4y'' = 0; \quad y(1) = -1, \quad y'(1) = 2, \quad y''(1) = 0, \quad y'''(1) = 0$
32. $y''' - y'' + y' - y = 0; \quad y(0) = 2, \quad y'(0) = -1, \quad y''(0) = -2$
33. $2y^{(4)} - y''' - 9y'' + 4y' + 4y = 0; \quad y(0) = -2, \quad y'(0) = 0, \quad y''(0) = -2, \quad y'''(0) = 0$
34. $4y''' + y' + 5y = 0; \quad y(0) = 2, \quad y'(0) = 1, \quad y''(0) = -1$
35. $6y''' + 5y'' + y' = 0; \quad y(0) = -2, \quad y'(0) = 2, \quad y''(0) = 0$
36. $y^{(4)} + 6y''' + 17y'' + 22y' + 14y = 0; \quad y(0) = 1, \quad y'(0) = -2, \quad y''(0) = 0,$
$y'''(0) = 3$
37. Show that the general solution of $y^{(4)} - y = 0$ can be written as

$$y = c_1 \cos t + c_2 \sin t + c_3 \cosh t + c_4 \sinh t.$$

Determine the solution satisfying the initial conditions $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$, $y'''(0) = 1$. Why is it convenient to use the solutions $\cosh t$ and $\sinh t$ rather than e^t and e^{-t}?

38. Consider the equation $y^{(4)} - y = 0$.
(a) Use Abel's formula [Problem 20(d) of Section 4.1] to find the Wronskian of a fundamental set of solutions of the given equation.
(b) Determine the Wronskian of the solutions e^t, e^{-t}, $\cos t$, and $\sin t$.
(c) Determine the Wronskian of the solutions $\cosh t$, $\sinh t$, $\cos t$, and $\sin t$.

39. Consider the spring–mass system, shown in Figure 4.2.4, consisting of two unit masses suspended from springs with spring constants 3 and 2, respectively. Assume that there is no damping in the system.
(a) Show that the displacements u_1 and u_2 of the masses from their respective equilibrium positions satisfy the equations

$$u_1'' + 5u_1 = 2u_2, \qquad u_2'' + 2u_2 = 2u_1. \tag{i}$$

(b) Solve the first of Eqs. (i) for u_2 and substitute into the second equation, thereby obtaining the following fourth order equation for u_1:

$$u_1^{(4)} + 7u_1'' + 6u_1 = 0. \tag{ii}$$

Find the general solution of Eq. (ii).

(c) Suppose that the initial conditions are

$$u_1(0) = 1, \qquad u_1'(0) = 0, \qquad u_2(0) = 2, \qquad u_2'(0) = 0. \tag{iii}$$

Use the first of Eqs. (i) and the initial conditions (iii) to obtain values for $u_1''(0)$ and $u_1'''(0)$. Then show that the solution of Eq. (ii) that satisfies the four initial conditions on u_1 is $u_1(t) = \cos t$. Show that the corresponding solution u_2 is $u_2(t) = 2\cos t$.

(d) Now suppose that the initial conditions are

$$u_1(0) = -2, \qquad u_1'(0) = 0, \qquad u_2(0) = 1, \qquad u_2'(0) = 0. \tag{iv}$$

Proceed as in part (c) to show that the corresponding solutions are $u_1(t) = -2\cos\sqrt{6}\,t$ and $u_2(t) = \cos\sqrt{6}\,t$.

(e) Observe that the solutions obtained in parts (c) and (d) describe two distinct modes of vibration. In the first, the frequency of the motion is 1, and the two masses move in phase, both moving up or down together; the second mass moves twice as far as the first. The second motion has frequency $\sqrt{6}$, and the masses move out of phase with each other, one moving down while the other is moving up, and vice versa. In this mode the first mass moves twice as far as the second. For other initial conditions, not proportional to either of Eqs. (iii) or (iv), the motion of the masses is a combination of these two modes.

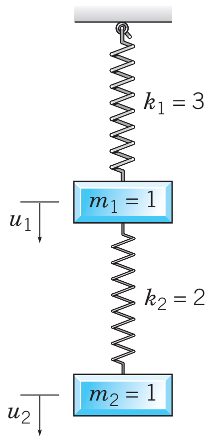

FIGURE 4.2.4 A two-spring, two-mass system.

40. In this problem we outline one way to show that if $r_1, \ldots, r_n$ are all real and different, then $e^{r_1 t}, \ldots, e^{r_n t}$ are linearly independent on $-\infty < t < \infty$. To do this, we consider the linear relation

$$c_1 e^{r_1 t} + \cdots + c_n e^{r_n t} = 0, \qquad -\infty < t < \infty \tag{i}$$

and show that all the constants are zero.

(a) Multiply Eq. (i) by $e^{-r_1 t}$ and differentiate with respect to t, thereby obtaining

$$c_2(r_2 - r_1)e^{(r_2 - r_1)t} + \cdots + c_n(r_n - r_1)e^{(r_n - r_1)t} = 0.$$

(b) Multiply the result of part (a) by $e^{-(r_2-r_1)t}$ and differentiate with respect to t to obtain

$$c_3(r_3-r_2)(r_3-r_1)e^{(r_3-r_2)t}+\cdots+c_n(r_n-r_2)(r_n-r_1)e^{(r_n-r_2)t}=0.$$

(c) Continue the procedure from parts (a) and (b), eventually obtaining

$$c_n(r_n-r_{n-1})\cdots(r_n-r_1)e^{(r_n-r_{n-1})t}=0.$$

Hence $c_n=0$, and therefore,

$$c_1e^{r_1t}+\cdots+c_{n-1}e^{r_{n-1}t}=0.$$

(d) Repeat the preceding argument to show that $c_{n-1}=0$. In a similar way it follows that $c_{n-2}=\cdots=c_1=0$. Thus the functions $e^{r_1t},\dots,e^{r_nt}$ are linearly independent.

41. In this problem we indicate one way to show that if $r=r_1$ is a root of multiplicity s of the characteristic polynomial $Z(r)$, then $e^{r_1t},\ te^{r_1t},\dots,\ t^{s-1}e^{r_1t}$ are solutions of Eq. (1). This problem extends to nth order equations the method for second order equations given in Problem 22 of Section 3.4. We start from Eq. (2) in the text

$$L[e^{rt}]=e^{rt}Z(r) \tag{i}$$

and differentiate repeatedly with respect to r, setting $r=r_1$ after each differentiation.

(a) Observe that if r_1 is a root of multiplicity s, then $Z(r)=(r-r_1)^sq(r)$, where $q(r)$ is a polynomial of degree $n-s$ and $q(r_1)\neq 0$. Show that $Z(r_1), Z'(r_1),\dots,Z^{(s-1)}(r_1)$ are all zero, but $Z^{(s)}(r_1)\neq 0$.

(b) By differentiating Eq. (i) repeatedly with respect to r, show that

$$\frac{\partial}{\partial r}L[e^{rt}]=L\left[\frac{\partial}{\partial r}e^{rt}\right]=L[te^{rt}],$$
$$\vdots$$
$$\frac{\partial^{s-1}}{\partial r^{s-1}}L[e^{rt}]=L[t^{s-1}e^{rt}].$$

(c) Show that $e^{r_1t}, te^{r_1t},\dots,t^{s-1}e^{r_1t}$ are solutions of Eq. (1).

4.3 The Method of Undetermined Coefficients

A particular solution Y of the nonhomogeneous nth order linear equation with constant coefficients

$$L[y]=a_0y^{(n)}+a_1y^{(n-1)}+\cdots+a_{n-1}y'+a_ny=g(t) \tag{1}$$

can be obtained by the method of undetermined coefficients, provided that $g(t)$ is of an appropriate form. Although the method of undetermined coefficients is not as general as the method of variation of parameters described in the next section, it is usually much easier to use when it is applicable.

Just as for the second order linear equation, when the constant coefficient linear differential operator L is applied to a polynomial $A_0t^m+A_1t^{m-1}+\cdots+A_m$, an