Power series solutions of differential equations

Power series solutions are sought primarily when the coefficients of the ODE are non-constant.

For the sake of simplicity, we shall use the independent variable x rather than t:

$$p(x)y'' + q(x)y' + r(x)y = g(x).$$

Definition: We say that x_0 is an ordinary point for the differential equation if $p(x_0) \neq 0$.

(If a point is not an ordinary point, we say that it is a singular point).

Theorem: We can always find power series solutions of ODE's if they are centered around an ordinary point.

Exs

① Determine a series solution for $y'' + y = 0$.

Solution: $p(x) = 1 \Rightarrow$ any point is an ordinary point. We choose $x_0 = 0$ $\therefore$ we find a power series solution centered at 0: $\sum_{n=0}^{\infty} a_n x^n$, a_n to be determined.

If $y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substitute:

$$\underbrace{\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}}_{\text{do a shift here}} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} + a_n\right] x^n = 0.$$

$$\Rightarrow (n+1)(n+2) a_{n+2} + a_n = 0, \text{ for all } n \geq 0$$

$$\Rightarrow \boxed{a_{n+2} = \frac{-a_n}{(n+1)(n+2)}}$$

(This is called a reccurrence relation).

$n=0 \to a_2 = \frac{-a_0}{2}$ | $n=1 \to a_3 = \frac{-a_1}{2.3}$

$n=2 \to a_4 = \frac{-a_2}{3.4}$ | $n=3 \to a_5 = \frac{-a_3}{4.5}$

$= + \frac{a_0}{2.3.4}$ | $= + \frac{a_1}{2.3.4.5}$

We continue in this fashion and we notice

that $a_{2n} = a_0 \frac{(-1)^n}{(2n)!}$ while $a_{2n+1} = (-1)^n \frac{a_1}{(2n+1)!}$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = a_0 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}}_{\cos x} + a_1 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}_{\sin x}$$

$$\therefore y = a_0 \cos x + a_1 \sin x.$$

[Test: $y'' + y = 0 \Rightarrow r^2 + 1 = 0 \Rightarrow r = \pm i$

$\therefore y = k_1 \cos x + k_2 \sin x$]

② Use power series to solve:

$$(x^2+1)y'' - 4xy' + 6y = 0.$$

Solution: Choose $x_0 = 0 \rightarrow y = \sum_{n=0}^{\infty} a_n x^n$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

$$\Rightarrow (x^2+1)y'' = (x^2+1)\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n x^{n} + \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n x^{n} + \underbrace{\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n}}_{①}$$

and

$$xy' = x\sum_{n=1}^{\infty} n a_n x^{n-1} = \underbrace{\sum_{n=1}^{\infty} n a_n x^{n}}_{②}$$

$$\cancel{\therefore (x^2+1)y'' - 4xy' + 6y = \sum_{n=2}^{\infty} \Big[n(n-1)a_n + (n+2)(n+1)a_{n+2} - 4na_n + 6a_n \Big] x^n}$$

$$\cancel{+ \Big(\underbrace{\underbrace{2a_2}_{n=0} + \underbrace{3.2.a_3 x}_{n=1}}_{①}\Big) - \underbrace{4.1.a_1.x}_{② \; n=1}}$$

① $$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n = \underbrace{2a_2x^0}_{n=0} + \underbrace{6a_3x}_{n=1} + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

② $$\sum_{n=1}^{\infty} n a_n x^n = a_1 x + \sum_{n=2}^{\infty} n a_n x^n.$$

Hence,

$(x^2+1)y'' - 4xy' + 6y = 0$ becomes:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + 2a_2 + \underbrace{6a_3x} + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n \underbrace{- 4a_1x} - 4\sum_{n=2}^{\infty} n a_n x^n + \underbrace{6\sum_{n=0}^{\infty} a_n x^n}_{6\left[a_0 + a_1x + \sum_{n=2}^{\infty} a_n x^n\right]} = 0.$$

Therefore:

$$\sum_{n=2}^{\infty}\left[n(n-1)a_n + (n+2)(n+1)a_{n+2} \overset{-4na_n}{\uparrow} + 6a_n\right]x^n + (6a_3 - 4a_1 + 6a_1)x + (2a_2 + 6a_0) = 0.$$

$\therefore$ $2a_2 + 6a_0 = 0 \Rightarrow \boxed{a_2 = -3a_0}$

$6a_3 + 2a_1 = 0 \Rightarrow \boxed{a_3 = -\frac{a_1}{3}}$

$$n(n-1)a_n + (n+2)(n+1)a_{n+2} - 4na_n + 6a_n = 0$$

$$\Rightarrow (n^2 a_n - na_n - 4na_n + 6a_n) + (n+2)(n+1)a_{n+2} = 0$$

$$\Rightarrow (n^2 - 5n + 6)\,a_n + (n+2)(n+1)a_{n+2} = 0$$

$$\Rightarrow (n-2)(n-3)\,a_n + (n+2)(n+1)\,a_{n+2} = 0$$

$$\Rightarrow a_{n+2} = -\frac{(n-2)(n-3)}{(n+2)(n+1)}\,a_n \; ; \; n \geq 2.$$

$\therefore$ $n = 2 \rightarrow a_4 = 0$

$n = 3 \rightarrow a_5 = 0$

$n = 4 \rightarrow a_6 = -\frac{2.(-1)}{6.5}\,a_4 = 0 = a_8 = a_{10} = a_{12} \cdots$

$n = 5 \rightarrow a_7 = 0 = a_9 = a_{11} \cdots$

$\therefore$ The series solution is a finite sum:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + 0$$

$$= a_0 + a_1 x - 3a_0 x^2 - \frac{a_1}{3} x^3$$

$$= \boxed{a_0 (1 - 3x^2) + a_1 (x - \tfrac{1}{3}x^3)}$$