Chapter 6 – The Laplace Transform.

6.1 The Definition

Given a function $f(t)$, the Laplace of f is given by $\mathcal{L}(f)(s) = \boxed{\int_0^\infty e^{-st} f(t)\,dt = F(s)}$, if the improper integral exists.

Exs:

① $f(t) = 1;\ t \geq 0.$

$$\mathcal{L}(1) = \int_0^\infty e^{-st}\,dt = \lim_{A\to\infty} \int_0^A e^{-st}\,dt$$

$$= \lim_{A\to\infty} -\frac{1}{s} e^{-st}\Big|_0^A$$

$$= \lim_{A\to\infty} \left[-\frac{1}{s} e^{-sA} + \frac{1}{s}\right].$$

If $s<0$, then $e^{-sA} \to \infty$ if $A\to\infty$ ∴ integral diverges.

If $s>0$, then $e^{-sA} \to 0$ as $A\to\infty$.

Hence $\boxed{\mathcal{L}(1) = \frac{1}{s};\ s>0}$

Similarly, $\boxed{\mathcal{L}(k) = \frac{k}{s};\ s>0}$

② Let $f(t) = e^{at}$; $t \geq 0$.

$$\mathcal{L}(e^{at}) = \lim_{A\to\infty} \int_0^A e^{-st} e^{at}\, dt = \lim_{A\to\infty} \int_0^A e^{(a-s)t}\, dt$$

$$= \lim_{A\to\infty} \frac{1}{a-s}\left[e^{(a-s)t}\right]_0^A$$

$$= \lim_{A\to\infty} \frac{1}{a-s}\left[e^{(a-s)A} - 1\right].$$

If $a-s>0$, then ~~the~~ $e^{(a-s)A} \to \infty$ as $A \to \infty$

If $a-s<0$, then $e^{(a-s)A} \to 0$ as $A\to\infty$.

Hence $\boxed{\mathcal{L}(e^{at}) = \frac{-1}{a-s} = \frac{1}{s-a} \text{ ; for } s>a.}$

③ Let $f(t) = \sin(at)$; $t \geq 0$.

$$\mathcal{L}(\sin(at)) = \lim_{A\to\infty} \int_0^A e^{-st} \sin(at)\, dt.$$

Integration by parts twice yields the following result: $\boxed{\mathcal{L}(\sin(at)) = \frac{a}{s^2+a^2};\ s>0}$

④ Similarly, $\boxed{\mathcal{L}(\cos(at)) = \frac{s}{s^2+a^2};\ s>0}$

Remark: The Laplace transform is a linear operator, in the sense that

$$\boxed{\begin{aligned} &\mathcal{L}(f_1+f_2) = \mathcal{L}(f_1)+\mathcal{L}(f_2) \\ \text{and } &\mathcal{L}(kf) = k\mathcal{L}(f) \end{aligned}}$$

Ex: Find the Laplace transform of $f(t) = 5e^{-2t} + 3\sin(4t)$.

Answer:
$$\begin{aligned} \mathcal{L}(f) &= \mathcal{L}\left(5e^{-2t}+3\sin(4t)\right) \\ &= \mathcal{L}\left(5e^{-2t}\right)+\mathcal{L}\left(3\sin(4t)\right) \\ &= 5\underbrace{\mathcal{L}\left(e^{-2t}\right)}_{\frac{1}{s+2};\ s>-2}+3\underbrace{\mathcal{L}\left(\sin(4t)\right)}_{\frac{4}{s^2+16};\ s>0} \end{aligned}$$

$$\therefore \boxed{\mathcal{L}(f) = \frac{5}{s+2}+\frac{12}{s^2+16};\ s>0}.$$

How to use Laplace transforms in ODE?

These turn out to be useful for IVPs.

6.2. Solution of IVP

Some preliminary properties:

① $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$

② $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$

$\downarrow$

ex: $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$

$\mathcal{L}(f''') = s^3\mathcal{L}(f) - s^2 f(0) - sf'(0) - f''(0)$

... etc ...

Ex: Consider the IVP: $y'' - y' - 2y = 0$; $y(0) = 1$ and $y'(0) = 0$.

This is easy to solve: $r^2 - r - 2 = 0 \Rightarrow (r-2)(r+1) = 0 \begin{cases} r = 2 \\ r = -1 \end{cases}$

$\therefore y = k_1 e^{2t} + k_2 e^{-t}$

But $\left.\begin{array}{l} y(0) = 1 \Rightarrow k_1 + k_2 = 1 \\ y'(0) = 0 \Rightarrow 2k_1 - k_2 = 0 \end{array}\right\} \Rightarrow 3k_1 = 1 \Rightarrow \boxed{k_1 = \frac{1}{3}}$ and $\boxed{k_2 = \frac{2}{3}}$

$$\therefore \boxed{y = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}}$$

We can also use Laplace transforms for that

Given $y'' - y' - 2y = 0$

$$\Rightarrow \mathcal{L}(y'' - y' - 2y) = \mathcal{L}(0)$$

$$\Rightarrow \mathcal{L}(y'') - \mathcal{L}(y') - 2\mathcal{L}(y) = 0.$$

$$\Rightarrow \left[s^2\mathcal{L}(y) - s\,y(0) - y'(0)\right] - \left[s\mathcal{L}(y) - y(0)\right] - 2\mathcal{L}(y) = 0.$$

Let $F(s) = \mathcal{L}(y)$

$$\Rightarrow \left(s^2F - s\right) - sF + 1 - 2F = 0$$

$$\Rightarrow \left(s^2 - s - 2\right)F + (1 - s) = 0$$

$$\Rightarrow F(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}.$$

$$\therefore \mathcal{L}(y) = \frac{s-1}{(s-2)(s+1)} = \cdots = \frac{1/3}{s-2} + \frac{2/3}{s+1}.$$

<u>Now</u>: If $\mathcal{L}(y) = F$, we say that $y = \mathcal{L}^{-1}(F)$

For example, $\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

$$\mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin(at)$$

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos(at)$$

$$\mathcal{L}^{-1}\left(\frac{k}{s}\right) = k.$$

Furthermore, $\mathcal{L}^{-1}$ is a linear operator:

$$\mathcal{L}^{-1}(F_1 + F_2) = \mathcal{L}^{-1}(F_1) + \mathcal{L}^{-1}(F_2)$$

$$\mathcal{L}^{-1}(kF) = k\,\mathcal{L}^{-1}(F).$$

Back to example:

$$\mathcal{L}(y) = \frac{1/3}{s-2} + \frac{2/3}{s+1}$$

$$\Rightarrow \quad y = \mathcal{L}^{-1}\left(\frac{1/3}{s-2} + \frac{2/3}{s+1}\right)$$

$$= \frac{1}{3}\,\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) + \frac{2}{3}\,\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$= \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t},$$ as desired.