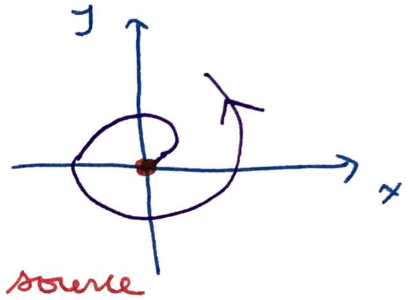
② Linear systems with complex eigenvalues:

$$\lambda_1 = \alpha + i\beta \quad (\text{and } \lambda_2 = \alpha - i\beta)$$

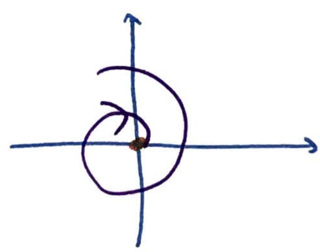
In this case, there are no straight-line solutions. Rather, all solutions spiral around the origin. There are 3 possible scenarios:

(a) $\alpha > 0$: all solutions spiral away from the origin



(0,0) = spiral source

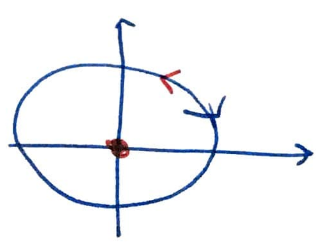
(b) $\alpha < 0$: all solutions spiral towards the origin



(0,0) = spiral sink

(c) $\alpha = 0$: all solutions are centered at the origin

(0,0) = center



Some Examples

① $\dfrac{d\vec{Y}}{dt} = A\vec{Y}$; $\vec{Y}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$; $A = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$.

(a) Find the eigenvalues and eigenvectors of this system

(b) Find the straight-line solutions if any.

(c) Draw the phase plane/portrait

(d) Classify the equilibrium solution.

(we will answer these questions all together).

Solution: $\cancel{\lambda^2 - \det(A)\lambda + \operatorname{tr}(A) = 6 \Rightarrow ; \lambda^2 + 6\lambda - 1 = 0}$.

$$\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0 \Rightarrow \lambda^2 + \lambda - 6 = 0$$

$$\Rightarrow (\lambda + 3)(\lambda - 2) = 0 \quad \begin{cases} \lambda_1 = -3 \\ \lambda_2 = 2. \end{cases}$$ are eigenvalues

$\lambda_1 = -3 \rightarrow \quad -3x + 0\cdot y = -3x \rightarrow x$ is arbitrary

$0\cdot x + 2y = -3y \rightarrow 5y = 0 \Rightarrow y = 0$

$\therefore$ One eigenvector is: $\vec{v}_1 = \langle 1, 0 \rangle$ (x can be anything)

$\lambda_2 = 2 \rightarrow \quad -3x + 0\cdot y = 2x \rightarrow x = 0$

$0\cdot x + 2y = 2y \rightarrow y$ is arbitrary

$\therefore$ one eigenvector is: $\vec{v}_2 = \langle 0, 1 \rangle$.

Hence the straight line solutions are the ones through: $\vec{v}_1 = \langle 1, 0 \rangle \Rightarrow$ x-axis

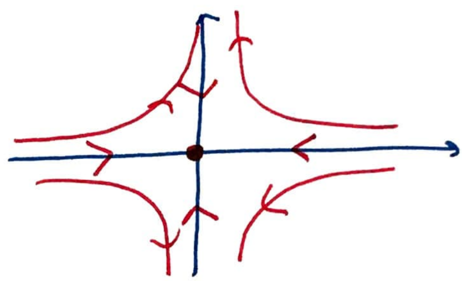
$\vec{v}_2 = \langle 0, 1 \rangle \Rightarrow$ y-axis.

Because $\lambda_1 = -3 < 0 \Rightarrow$ orientation on the corresponding straight line solution is towards the origin

Because $\lambda_2 = +2 > 0 \Rightarrow$ orientation on the corresponding straight line solution is away from the origin.

$\therefore$ the origin (the only equilibrium solution) is a saddle.

Phase portrait:



(See figures 1 attached) - using the Slopes App)

(2) Same questions as before.

$$A = \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix}.$$

$$\lambda^2 - (-5)\lambda + (6-2) = 0 \Rightarrow \lambda^2 + 5\lambda + 4 = 0$$

$$\Rightarrow (\lambda+1)(\lambda+4) = 0 \begin{cases} \lambda_1 = -1 \\ \lambda_2 = -4 \end{cases}$$

$$\lambda_1 = -1 \Rightarrow \left.\begin{matrix} -2x - 2y = -x \\ -x - 3y = -y \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} -x - 2y = 0 \\ -x - 2y = 0 \end{matrix}\right\} \Rightarrow x = -2y$$

$\therefore \vec{v}_1 = \langle -2, 1 \rangle$ is an eigenvector corresponding to λ_1.

the orientation or the straight line solution through $\vec{v}_1$ is towards the origin $(\lambda_1 < 0)$.

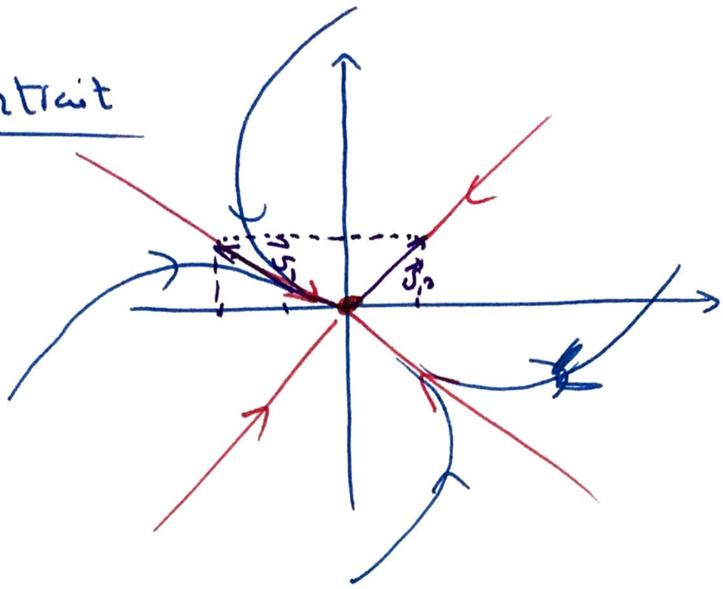
$$\lambda_2 = -4 \Rightarrow \left.\begin{matrix} -2x - 2y = -4x \\ -x - 3y = -4y \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} 2x - 2y = 0 \\ -x + y = 0 \end{matrix}\right\} \Rightarrow y = x$$

$\therefore \vec{v}_2 = \langle 1, 1 \rangle$ is an eigenvector corresponding to λ_2.

the orientation or the straight-line solution through $\vec{v}_2$ is also towards the origin

$\therefore$ $(0,0)$ is a sink.

Phase portrait



In absolute values $|\lambda_1| < |\lambda_2|$ $\therefore$ the solutions approach the origin in a way that is tangent to the straight line solution through $\vec{v}_1$.

(see figures 2 attached).

3. $A = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix}$

On page 133, we found that the eigenvalues are complex: $\lambda = -2 \pm 3i$.

Since $\alpha = -2 < 0 \Rightarrow$ we expect the solutions to spiral towards the origin.

(See figures 3 attached)

4. $A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$

On page 131, we found that $\lambda = \pm i$ are the eigenvalues.

Since $\alpha = 0 \Rightarrow$ the origin is a center.

(See figures 4 attached).