without solving the differential equation. Further, since under the conditions of Theorem 3.2.7 the Wronskian W is either always zero or never zero, you can determine which case actually occurs by evaluating W at any single convenient value of t.

EXAMPLE 7

In Example 5 we verified that $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ are solutions of the equation

$$2t^2y'' + 3ty' - y = 0, \qquad t > 0. \tag{29}$$

Verify that the Wronskian of y_1 and y_2 is given by Eq. (23).

From the example just cited we know that $W(y_1, y_2)(t) = -(3/2)t^{-3/2}$. To use Eq. (23), we must write the differential equation (29) in the standard form with the coefficient of y'' equal to 1. Thus we obtain

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0,$$

so $p(t) = 3/2t$. Hence

$$W(y_1, y_2)(t) = c\,\exp\left[-\int \frac{3}{2t}\,dt\right] = c\exp\left(-\frac{3}{2}\ln t\right)$$
$$= c\,t^{-3/2}. \tag{30}$$

Equation (30) gives the Wronskian of any pair of solutions of Eq. (29). For the particular solutions given in this example, we must choose $c = -3/2$.

Summary. We can summarize the discussion in this section as follows: to find the general solution of the differential equation

$$y'' + p(t)y' + q(t)y = 0, \qquad \alpha < t < \beta,$$

we must first find two functions y_1 and y_2 that satisfy the differential equation in $\alpha < t < \beta$. Then we must make sure that there is a point in the interval where the Wronskian W of y_1 and y_2 is nonzero. Under these circumstances y_1 and y_2 form a fundamental set of solutions, and the general solution is

$$y = c_1y_1(t) + c_2y_2(t),$$

where c_1 and c_2 are arbitrary constants. If initial conditions are prescribed at a point in $\alpha < t < \beta$, then c_1 and c_2 can be chosen so as to satisfy these conditions.

PROBLEMS

In each of Problems 1 through 6, find the Wronskian of the given pair of functions.

1. $e^{2t}, \quad e^{-3t/2}$
2. $\cos t, \quad \sin t$
3. $e^{-2t}, \quad te^{-2t}$
4. $x, \quad xe^x$
5. $e^t \sin t, \quad e^t \cos t$
6. $\cos^2\theta, \quad 1 + \cos 2\theta$

In each of Problems 7 through 12, determine the longest interval in which the given initial value problem is certain to have a unique twice-differentiable solution. Do not attempt to find the solution.

7. $ty'' + 3y = t, \qquad y(1) = 1, \quad y'(1) = 2$
8. $(t - 1)y'' - 3ty' + 4y = \sin t, \qquad y(-2) = 2, \quad y'(-2) = 1$
9. $t(t - 4)y'' + 3ty' + 4y = 2, \qquad y(3) = 0, \quad y'(3) = -1$
10. $y'' + (\cos t)y' + 3(\ln|t|)y = 0, \qquad y(2) = 3, \quad y'(2) = 1$

11. $(x-3)y'' + xy' + (\ln|x|)y = 0, \qquad y(1) = 0, \quad y'(1) = 1$
12. $(x-2)y'' + y' + (x-2)(\tan x)y = 0, \qquad y(3) = 1, \quad y'(3) = 2$

13. Verify that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are two solutions of the differential equation $t^2y'' - 2y = 0$ for $t > 0$. Then show that $y = c_1t^2 + c_2t^{-1}$ is also a solution of this equation for any c_1 and c_2.
14. Verify that $y_1(t) = 1$ and $y_2(t) = t^{1/2}$ are solutions of the differential equation $yy'' + (y')^2 = 0$ for $t > 0$. Then show that $y = c_1 + c_2t^{1/2}$ is not, in general, a solution of this equation. Explain why this result does not contradict Theorem 3.2.2.
15. Show that if $y = \phi(t)$ is a solution of the differential equation $y'' + p(t)y' + q(t)y = g(t)$, where $g(t)$ is not always zero, then $y = c\phi(t)$, where c is any constant other than 1, is not a solution. Explain why this result does not contradict the remark following Theorem 3.2.2.
16. Can $y = \sin(t^2)$ be a solution on an interval containing $t = 0$ of an equation $y'' + p(t)y' + q(t)y = 0$ with continuous coefficients? Explain your answer.
17. If the Wronskian W of f and g is $3e^{4t}$, and if $f(t) = e^{2t}$, find $g(t)$.
18. If the Wronskian W of f and g is t^2e^t, and if $f(t) = t$, find $g(t)$.
19. If $W(f, g)$ is the Wronskian of f and g, and if $u = 2f - g$, $v = f + 2g$, find the Wronskian $W(u, v)$ of u and v in terms of $W(f, g)$.
20. If the Wronskian of f and g is $t\cos t - \sin t$, and if $u = f + 3g$, $v = f - g$, find the Wronskian of u and v.
21. Assume that y_1 and y_2 are a fundamental set of solutions of $y'' + p(t)y' + q(t)y = 0$ and let $y_3 = a_1y_1 + a_2y_2$ and $y_4 = b_1y_1 + b_2y_2$, where a_1, a_2, b_1, and b_2 are any constants. Show that

$$W(y_3, y_4) = (a_1b_2 - a_2b_1)W(y_1, y_2).$$

Are y_3 and y_4 also a fundamental set of solutions? Why or why not?

In each of Problems 22 and 23, find the fundamental set of solutions specified by Theorem 3.2.5 for the given differential equation and initial point.

22. $y'' + y' - 2y = 0, \qquad t_0 = 0$
23. $y'' + 4y' + 3y = 0, \qquad t_0 = 1$

In each of Problems 24 through 27, verify that the functions y_1 and y_2 are solutions of the given differential equation. Do they constitute a fundamental set of solutions?

24. $y'' + 4y = 0; \qquad y_1(t) = \cos 2t, \quad y_2(t) = \sin 2t$
25. $y'' - 2y' + y = 0; \qquad y_1(t) = e^t, \quad y_2(t) = te^t$
26. $x^2y'' - x(x+2)y' + (x+2)y = 0, \quad x > 0; \qquad y_1(x) = x, \quad y_2(x) = xe^x$
27. $(1 - x\cot x)y'' - xy' + y = 0, \quad 0 < x < \pi; \qquad y_1(x) = x, \quad y_2(x) = \sin x$

28. Consider the equation $y'' - y' - 2y = 0$.
    (a) Show that $y_1(t) = e^{-t}$ and $y_2(t) = e^{2t}$ form a fundamental set of solutions.
    (b) Let $y_3(t) = -2e^{2t}$, $y_4(t) = y_1(t) + 2y_2(t)$, and $y_5(t) = 2y_1(t) - 2y_3(t)$. Are $y_3(t)$, $y_4(t)$, and $y_5(t)$ also solutions of the given differential equation?
    (c) Determine whether each of the following pairs forms a fundamental set of solutions: $[y_1(t), y_3(t)]$; $[y_2(t), y_3(t)]$; $[y_1(t), y_4(t)]$; $[y_4(t), y_5(t)]$.

In each of Problems 29 through 32, find the Wronskian of two solutions of the given differential equation without solving the equation.

29. $t^2y'' - t(t+2)y' + (t+2)y = 0$
30. $(\cos t)y'' + (\sin t)y' - ty = 0$
31. $x^2y'' + xy' + (x^2 - \nu^2)y = 0, \qquad$ Bessel's equation
32. $(1 - x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0, \qquad$ Legendre's equation

33. Show that if p is differentiable and $p(t) > 0$, then the Wronskian $W(t)$ of two solutions of $[p(t)y']' + q(t)y = 0$ is $W(t) = c/p(t)$, where c is a constant.
34. If the differential equation $ty'' + 2y' + te^t y = 0$ has y_1 and y_2 as a fundamental set of solutions and if $W(y_1, y_2)(1) = 2$, find the value of $W(y_1, y_2)(5)$.
35. If the differential equation $t^2y'' - 2y' + (3 + t)y = 0$ has y_1 and y_2 as a fundamental set of solutions and if $W(y_1, y_2)(2) = 3$, find the value of $W(y_1, y_2)(4)$.
36. If the Wronskian of any two solutions of $y'' + p(t)y' + q(t)y = 0$ is constant, what does this imply about the coefficients p and q?
37. If f, g, and h are differentiable functions, show that $W(fg, fh) = f^2W(g, h)$.

In Problems 38 through 40, assume that p and q are continuous and that the functions y_1 and y_2 are solutions of the differential equation $y'' + p(t)y' + q(t)y = 0$ on an open interval I.

38. Prove that if y_1 and y_2 are zero at the same point in I, then they cannot be a fundamental set of solutions on that interval.
39. Prove that if y_1 and y_2 have maxima or minima at the same point in I, then they cannot be a fundamental set of solutions on that interval.
40. Prove that if y_1 and y_2 have a common point of inflection t_0 in I, then they cannot be a fundamental set of solutions on I unless both p and q are zero at t_0.

41. **Exact Equations.** The equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

is said to be exact if it can be written in the form

$$[P(x)y']' + [f(x)y]' = 0,$$

where $f(x)$ is to be determined in terms of $P(x)$, $Q(x)$, and $R(x)$. The latter equation can be integrated once immediately, resulting in a first order linear equation for y that can be solved as in Section 2.1. By equating the coefficients of the preceding equations and then eliminating $f(x)$, show that a necessary condition for exactness is

$$P''(x) - Q'(x) + R(x) = 0.$$

It can be shown that this is also a sufficient condition.

In each of Problems 42 through 45, use the result of Problem 41 to determine whether the given equation is exact. If it is, then solve the equation.

42. $y'' + xy' + y = 0$
43. $y'' + 3x^2y' + xy = 0$
44. $xy'' - (\cos x)y' + (\sin x)y = 0, \quad x > 0$
45. $x^2y'' + xy' - y = 0, \quad x > 0$

46. **The Adjoint Equation.** If a second order linear homogeneous equation is not exact, it can be made exact by multiplying by an appropriate integrating factor $\mu(x)$. Thus we require that $\mu(x)$ be such that

$$\mu(x)P(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = 0$$

can be written in the form

$$[\mu(x)P(x)y']' + [f(x)y]' = 0.$$

By equating coefficients in these two equations and eliminating $f(x)$, show that the function μ must satisfy

$$P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0.$$

This equation is known as the adjoint of the original equation and is important in the advanced theory of differential equations. In general, the problem of solving the adjoint differential equation is as difficult as that of solving the original equation, so only occasionally is it possible to find an integrating factor for a second order equation.

In each of Problems 47 through 49, use the result of Problem 46 to find the adjoint of the given differential equation.

47. $x^2y'' + xy' + (x^2 - \nu^2)y = 0,$ Bessel's equation
48. $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$ Legendre's equation
49. $y'' - xy = 0,$ Airy's equation
50. For the second order linear equation $P(x)y'' + Q(x)y' + R(x)y = 0$, show that the adjoint of the adjoint equation is the original equation.
51. A second order linear equation $P(x)y'' + Q(x)y' + R(x)y = 0$ is said to be self-adjoint if its adjoint is the same as the original equation. Show that a necessary condition for this equation to be self-adjoint is that $P'(x) = Q(x)$. Determine whether each of the equations in Problems 47 through 49 is self-adjoint.

3.3 Complex Roots of the Characteristic Equation

We continue our discussion of the equation

$$ay'' + by' + cy = 0, \tag{1}$$

where a, b, and c are given real numbers. In Section 3.1 we found that if we seek solutions of the form $y = e^{rt}$, then r must be a root of the characteristic equation

$$ar^2 + br + c = 0. \tag{2}$$

We showed in Section 3.1 that if the roots r_1 and r_2 are real and different, which occurs whenever the discriminant $b^2 - 4ac$ is positive, then the general solution of Eq. (1) is

$$y = c_1e^{r_1t} + c_2e^{r_2t}. \tag{3}$$

Suppose now that $b^2 - 4ac$ is negative. Then the roots of Eq. (2) are conjugate complex numbers; we denote them by

$$r_1 = \lambda + i\mu, \qquad r_2 = \lambda - i\mu, \tag{4}$$

where λ and μ are real. The corresponding expressions for y are

$$y_1(t) = \exp[(\lambda + i\mu)t], \qquad y_2(t) = \exp[(\lambda - i\mu)t]. \tag{5}$$

Our first task is to explore what is meant by these expressions, which involve evaluating the exponential function for a complex exponent. For example, if $\lambda = -1$, $\mu = 2$, and $t = 3$, then from Eq. (5),

$$y_1(3) = e^{-3+6i}. \tag{6}$$