shall see in Section 5.3 that even without knowing the formula for a_n, it is possible to establish that the two series in Eq. (23) converge for all x. Further, they define functions y_3 and y_4 that are a fundamental set of solutions of the Airy equation (15). Thus

$$y = a_0 y_3(x) + a_1 y_4(x)$$

is the general solution of Airy's equation for $-\infty < x < \infty$.

It is worth emphasizing, as we saw in Example 3, that if we look for a solution of Eq. (1) of the form $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$, then the coefficients $P(x)$, $Q(x)$, and $R(x)$ in Eq. (1) must also be expressed in powers of $x - x_0$. Alternatively, we can make the change of variable $x - x_0 = t$, obtaining a new differential equation for y as a function of t, and then look for solutions of this new equation of the form $\sum_{n=0}^{\infty} a_n t^n$. When we have finished the calculations, we replace t by $x - x_0$ (see Problem 19).

In Examples 2 and 3 we have found two sets of solutions of Airy's equation. The functions y_1 and y_2 defined by the series in Eq. (20) are a fundamental set of solutions of Eq. (15) for all x, and this is also true for the functions y_3 and y_4 defined by the series in Eq. (23). According to the general theory of second order linear equations, each of the first two functions can be expressed as a linear combination of the latter two functions, and vice versa—a result that is certainly not obvious from an examination of the series alone.

Finally, we emphasize that it is not particularly important if, as in Example 3, we are unable to determine the general coefficient a_n in terms of a_0 and a_1. What is essential is that we can determine *as many coefficients as we want.* Thus we can find as many terms in the two series solutions as we want, even if we cannot determine the general term. While the task of calculating several coefficients in a power series solution is not difficult, it can be tedious. A symbolic manipulation package can be very helpful here; some are able to find a specified number of terms in a power series solution in response to a single command. With a suitable graphics package we can also produce plots such as those shown in the figures in this section.

PROBLEMS

In each of Problems 1 through 14:

(a) Seek power series solutions of the given differential equation about the given point x_0; find the recurrence relation.

(b) Find the first four terms in each of two solutions y_1 and y_2 (unless the series terminates sooner).

(c) By evaluating the Wronskian $W(y_1, y_2)(x_0)$, show that y_1 and y_2 form a fundamental set of solutions.

(d) If possible, find the general term in each solution.

1. $y'' - y = 0, \quad x_0 = 0$
2. $y'' - xy' - y = 0, \quad x_0 = 0$
3. $y'' - xy' - y = 0, \quad x_0 = 1$
4. $y'' + k^2x^2y = 0, \quad x_0 = 0, \quad k$ a constant
5. $(1 - x)y'' + y = 0, \quad x_0 = 0$
6. $(2 + x^2)y'' - xy' + 4y = 0, \quad x_0 = 0$
7. $y'' + xy' + 2y = 0, \quad x_0 = 0$
8. $xy'' + y' + xy = 0, \quad x_0 = 1$
9. $(1 + x^2)y'' - 4xy' + 6y = 0, \quad x_0 = 0$
10. $(4 - x^2)y'' + 2y = 0, \quad x_0 = 0$

11. $(3 - x^2)y'' - 3xy' - y = 0, \qquad x_0 = 0$
12. $(1 - x)y'' + xy' - y = 0, \qquad x_0 = 0$
13. $2y'' + xy' + 3y = 0, \qquad x_0 = 0$
14. $2y'' + (x + 1)y' + 3y = 0, \qquad x_0 = 2$

In each of Problems 15 through 18:

(a) Find the first five nonzero terms in the solution of the given initial value problem.

(b) Plot the four-term and the five-term approximations to the solution on the same axes.

(c) From the plot in part (b) estimate the interval in which the four-term approximation is reasonably accurate.

15. $y'' - xy' - y = 0, \qquad y(0) = 2, \quad y'(0) = 1;$ see Problem 2
16. $(2 + x^2)y'' - xy' + 4y = 0, \qquad y(0) = -1, \quad y'(0) = 3;$ see Problem 6
17. $y'' + xy' + 2y = 0, \qquad y(0) = 4, \quad y'(0) = -1;$ see Problem 7
18. $(1 - x)y'' + xy' - y = 0, \qquad y(0) = -3, \quad y'(0) = 2;$ see Problem 12
19. (a) By making the change of variable $x - 1 = t$ and assuming that y has a Taylor series in powers of t, find two series solutions of

$$y'' + (x - 1)^2 y' + (x^2 - 1)y = 0$$

in powers of $x - 1$.

(b) Show that you obtain the same result by assuming that y has a Taylor series in powers of $x - 1$ and also expressing the coefficient $x^2 - 1$ in powers of $x - 1$.

20. Show directly, using the ratio test, that the two series solutions of Airy's equation about $x = 0$ converge for all x; see Eq. (20) of the text.
21. **The Hermite Equation.** The equation

$$y'' - 2xy' + \lambda y = 0, \qquad -\infty < x < \infty,$$

where λ is a constant, is known as the Hermite[5] equation. It is an important equation in mathematical physics.

(a) Find the first four terms in each of two solutions about $x = 0$ and show that they form a fundamental set of solutions.

(b) Observe that if λ is a nonnegative even integer, then one or the other of the series solutions terminates and becomes a polynomial. Find the polynomial solutions for $\lambda = 0$, 2, 4, 6, 8, and 10. Note that each polynomial is determined only up to a multiplicative constant.

(c) The Hermite polynomial $H_n(x)$ is defined as the polynomial solution of the Hermite equation with $\lambda = 2n$ for which the coefficient of x^n is 2^n. Find $H_0(x), \ldots, H_5(x)$.

22. Consider the initial value problem $y' = \sqrt{1 - y^2}$, $y(0) = 0$.

(a) Show that $y = \sin x$ is the solution of this initial value problem.

(b) Look for a solution of the initial value problem in the form of a power series about $x = 0$. Find the coefficients up to the term in x^3 in this series.

[5]Charles Hermite (1822–1901) was an influential French analyst and algebraist. An inspiring teacher, he was professor at the École Polytechnique and the Sorbonne. He introduced the Hermite functions in 1864 and showed in 1873 that e is a transcendental number (that is, e is not a root of any polynomial equation with rational coefficients). His name is also associated with Hermitian matrices (see Section 7.3), some of whose properties he discovered.

In each of Problems 23 through 28, plot several partial sums in a series solution of the given initial value problem about $x = 0$, thereby obtaining graphs analogous to those in Figures 5.2.1 through 5.2.4.

23. $y'' - xy' - y = 0, \quad y(0) = 1, \quad y'(0) = 0;$ see Problem 2
24. $(2 + x^2)y'' - xy' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0;$ see Problem 6
25. $y'' + xy' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 1;$ see Problem 7
26. $(4 - x^2)y'' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 1;$ see Problem 10
27. $y'' + x^2 y = 0, \quad y(0) = 1, \quad y'(0) = 0;$ see Problem 4
28. $(1 - x)y'' + xy' - 2y = 0, \quad y(0) = 0, \quad y'(0) = 1$

5.3 Series Solutions Near an Ordinary Point, Part II

In the preceding section we considered the problem of finding solutions of

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{1}$$

where P, Q, and R are polynomials, in the neighborhood of an ordinary point x_0. Assuming that Eq. (1) does have a solution $y = \phi(x)$ and that ϕ has a Taylor series

$$y = \phi(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{2}$$

that converges for $|x - x_0| < \rho$, where $\rho > 0$, we found that the a_n can be determined by directly substituting the series (2) for y in Eq. (1).

Let us now consider how we might justify the statement that if x_0 is an ordinary point of Eq. (1), then there exist solutions of the form (2). We also consider the question of the radius of convergence of such a series. In doing this, we are led to a generalization of the definition of an ordinary point.

Suppose, then, that there is a solution of Eq. (1) of the form (2). By differentiating Eq. (2) m times and setting x equal to x_0, we obtain

$$m! a_m = \phi^{(m)}(x_0).$$

Hence, to compute a_n in the series (2), we must show that we can determine $\phi^{(n)}(x_0)$ for $n = 0, 1, 2, \ldots$ from the differential equation (1).

Suppose that $y = \phi(x)$ is a solution of Eq. (1) satisfying the initial conditions $y(x_0) = y_0, y'(x_0) = y'_0$. Then $a_0 = y_0$ and $a_1 = y'_0$. If we are solely interested in finding a solution of Eq. (1) without specifying any initial conditions, then a_0 and a_1 remain arbitrary. To determine $\phi^{(n)}(x_0)$ and the corresponding a_n for $n = 2, 3, \ldots$, we turn to Eq. (1). Since ϕ is a solution of Eq. (1), we have

$$P(x)\phi''(x) + Q(x)\phi'(x) + R(x)\phi(x) = 0.$$

For the interval about x_0 for which P is nonzero, we can write this equation in the form

$$\phi''(x) = -p(x)\phi'(x) - q(x)\phi(x), \tag{3}$$

where $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$. Setting x equal to x_0 in Eq. (3) gives

$$\phi''(x_0) = -p(x_0)\phi'(x_0) - q(x_0)\phi(x_0).$$