Chapter 4 - Higher Order ODE's

4.2 / 4.3 Nothing really new in these two sections; we just generalize the concepts and theories ~~of~~ discussed in ~~sections~~ chapter 3 (without touching on the method of variation of parameters).

Examples of homog. equations

① Solve the homogeneous equation:

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0.$$

Solution: Characteristic equation is: $r^4 + r^3 - 7r^2 - r + 6 = 0$.

Knowing that $r = 1$, $r_2 = -1$; $r_3 = 2$, and $r_4 = -3$ are the four roots of this polynomial, then the general solution of the ODE is:

$$y = c_1 e^{t} + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-3t}.$$

(we will discuss how to find these roots later).

② Solve: $y^{(4)} - y = 0$

Solution: Characteristic equation is: $r^4 - 1 = 0$

$\Rightarrow (r^2 - 1)(r^2 + 1) = 0 \Rightarrow (r-1)(r+1)(r^2+1) = 0.$

$\therefore$ roots are: $r_1 = 1$; $r_2 = -1$; $\underbrace{r_3 = i \text{ and } r_4 = -i}_{r = \pm i}$

$\Rightarrow$ general solution is:

$$y(t) = c_1 e^{t} + c_2 e^{-t} + c_3 \cos t + c_4 \sin t.$$

③ ~~Solve: $y^{(4)}+2y''+y=0 \Rightarrow r^4+1$~~ =

③ Solve $y^{(4)}+2y''+y=0$.

<u>Solution:</u> Characteristic equation is: $r^4+2r^2+1=0$

or $(r^2+1)^2=0 \Rightarrow r=\pm i$ are 2 repeated roots.

$\therefore\ y = c_1\cos t + c_2\sin t + c_3 t\cos t + c_4 t\sin t$.

<u>Examples of non-homog. equations</u>

① Solve $y''' - 3y'' + 3y' - y = 5e^{2t}$.

Characteristic equation is: $r^3 - 3r^2 + 3r - 1 = 0$

or $(r-1)^3 = 0 \Rightarrow r = 1$ is a repeated root ~~twice~~. 3 times

$\therefore\ y_1 = e^t,\ y_2 = te^t,\ \underline{y_3 = t^2e^t}$ are the 3 solutions

of the homog. equation.

Now, our guess for the solution of the non-homog. case should be: $y = Ae^{2t} \rightarrow y' = 2Ae^{2t},\ y'' = 4Ae^{2t},$

and $y''' = 8Ae^{2t}$.

Substitute in the original ODE and solve for A:

$$8Ae^{2t} - 12Ae^{2t} + 6Ae^{2t} - Ae^{2t} = 5e^{2t}$$

$$\Rightarrow Ae^{2t} = 5e^{2t} \Rightarrow A = 5.$$

② Solve : $y''' - 3y'' + 3y' - y = 4e^t$.

<u>Solution</u>: the characteristic equation is as in previous example

$\Rightarrow y_1 = e^t,\ y_2 = te^t,\ y_3 = t^2 e^t$ are the solutions to the homog. equation.

Now for the non_homog. case, $y = At^n e^t$, where n is the smallest integer ~~into~~ such that $At^n e^t$ does not solve the homog. ODE. Here $n = 3$.

$\therefore$ $y = At^3 e^t$ is the guess for the non_homog. case.

Calculation yields $A = \frac{2}{3}$.

$\therefore$ General solution is:

$$y = c_1 e^t + c_2 t e^t + c_3 t^2 e^t + \frac{2}{3} t^3 e^t.$$

<u>Remark</u>: how to find roots of polynomials of degree ≥ 2 ?

<u>Ex</u>: $r^4 + r^3 - 7r^2 - r + 6 = 0$ (see example 1) on page 93)

<u>Proposition</u>: If a root exists and is an integer, then it has to divide the constant term.

Here the constant term is 6 $\Rightarrow$ divisors are: $\pm 1, \pm 2, \pm 3, \pm 6$.

We try to see which one works:

$r = 1 \rightarrow 1^4 + 1^3 - 7 - 1 + 6 = 0$ ✓

$r = -1 \rightarrow (-1)^4 + (-1)^3 - 7(-1)^2 - (-1) + 6 = 1 - 1 - 7 + 1 + 6 = 0$ ✓

$r = 2 \rightarrow 2^4 + 2^3 - 7(2^2) - 2 + 6 = 16 + 8 - 28 - 2 + 6 = 0$ ✓

$r = -2 \rightarrow (-2)^4 + (-2)^3 - 7(-2^2) - (-2) + 6 = 16 - 8 - 28 + 2 + 6 \neq 0$ X

$r = 3 \rightarrow (3)^4 + 3^3 - 7(3^2) - 3 + 6 = 81 + 27 - 63 + 3 \neq 0$ X.

$r = -3 \rightarrow +81 - 27 - 63 + 3 + 6 = 0$ ✓

this polynomial is of degree 4, so it has 4 roots.

We found them:

$r_1 = 1, r_2 = -1; r_3 = 2$ and $r_4 = -3$.