Chapter 3 - Second-Order Linear Equations

3.1 Homogeneous linear equations with constant coefficients

Definition: Such equations take the form:

$$ay'' + by' + cy = 0 \quad \text{homogeneous}$$

constant coefficients

[In general linear 2nd order equations take the form:

$$y'' + p(t)y' + q(t)y = g(t)].$$

Naturally, a solution of a homogeneous linear 2nd order with constant coefficients must be (a form of) an exponential function.

Ex: Consider $y'' - y = 0$.

(a) Show that $y_1(t) = c_1 e^t$ is a family of solutions.

Proof: $y_1 = c_1 e^t \Rightarrow y_1' = y_1'' = c_1 e^t$

$\therefore c_1 e^t - c_1 e^t = 0$ ✓

(b) Show that $y_2(t) = c_2 e^{-t}$ is another family of solutions.

Proof: $y_2' = -c_2 e^{-t}$ and $y_2'' = c_2 e^{-t}$

$\Rightarrow y_2'' - y_2 = c_2 e^{-t} - c_2 e^{-t} = 0$ ✓

(c) Show that $y(t) = \overbrace{c_1 e^{t}}^{y_1} + \overbrace{c_2 e^{-t}}^{y_2}$ is also a more general family of solutions.

Proof: This is obvious: $y'' - y = (y_1 + y_2)'' - (y_1 + y_2)$

$$= y_1'' + y_2'' - y_1 - y_2 = (y_1'' - y_1) + (y_2'' - y_2)$$

$$= 0 + 0 \checkmark$$

(d) Knowing that $y(0) = 2$ and $y'(0) = -1$, find the particular solution satisfying these 2 initial conditions.

Solution: $y(0) = 2 \implies c_1 + c_2 = 2.$

Now, $y' = c_1 e^{t} - c_2 e^{-t}$

$y'(0) = -1 \implies c_1 - c_2 = -1$

2 equations into 2 unknowns

We find that $c_1 = \frac{1}{2}$ and $c_2 = \frac{3}{2}$.

$\therefore$ the particular solution is:

$$\boxed{y = \frac{1}{2} e^{t} - \frac{3}{2} e^{-t}}$$

How do we guess / find y_1 and y_2 ?

Actually, we already established that a solution is some exponential function $y = e^{rx}$

$\Rightarrow y' = re^{rx}$ and $y'' = r^2 e^{rx}$

Substituting in: $ay'' + by' + cy = 0$ we obtain:

$$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$$

or $(ar^2 + br + c)\, e^{rx} = 0$

$\Rightarrow r$ must satisfy $\boxed{ar^2 + br + c = 0}$, called the characteristic equation.

Ex. In the previous example $y'' - y = 0$, $a = 1$, $b = 0$, $c = -1$

$\therefore$ characteristic equation is $r^2 - 1 = 0$ or $r = \pm 1$

$\therefore$ 2 possible solutions are e^{t} and e^{-t}.

We should remark here that an IVP requires two initial conditions: $y(x_0) = y_0$ and $y'(x_0) = y_1$.

Ex: Solve: $4y'' - 8y' + 3y = 0$; $y(0) = 2$ and $y'(0) = \frac{1}{2}$.

Solution: Characteristic equation is: $4r^2 - 8r + 3 = 0$.

$$\Delta = 64 - 48 = 16 \Rightarrow r = \frac{8 \pm 4}{8} = \begin{cases} \frac{12}{8} = \frac{3}{2} \\ \frac{4}{8} = \frac{1}{2} \end{cases}$$

$$\therefore \; y = c_1 e^{\frac{3}{2}t} + c_2 e^{\frac{1}{2}t}.$$

But $y(0) = 2 \Rightarrow \boxed{c_1 + c_2 = 2}$.

And $y' = \frac{3}{2} c_1 e^{\frac{3}{2}t} + \frac{1}{2} c_2 e^{\frac{1}{2}t}$

$$y'(0) = \frac{1}{2} \Rightarrow \frac{3}{2} c_1 + \frac{1}{2} c_2 = \frac{1}{2} \text{ or } \boxed{3c_1 + c_2 = 1}$$

Solving this system, we obtain $c_1 = -\frac{1}{2}$ and $c_2 = \frac{5}{2}$.