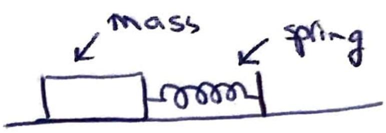
3.7 Mechanical and Electrical Vibrations (unforced)

Linear equations with constant coefficients are important because they serve as mathematical models of some important physical processes, such as the motion of a mass on a vibrating string.

Here is the model of a simple harmonic motion:



If m denotes the mass;

k = spring constant; γ = damping constant
Lb/ft

m, k and γ are all positive constants.

note about mass:
$W = m.g$
$\Rightarrow m = W/g$
W = Lbs; g = ft/sec^2
$\hookrightarrow = 32$ ft/sec^2

then the ODE modeling the motion of the mass attached to the spring is given by: $my'' + \gamma y' + ky = f(t)$,

where $y(t)$ = position of mass at time t

$f(t)$ = sum of external forces.

Some special cases:

$\gamma = 0 \rightarrow$ undamped motion

$f(t) = 0 \rightarrow$ unforced motion

$f(t) \neq 0 \rightarrow$ forced motion.

To solve the model, we first need to look at the characteristic equation: $mr^2 + \gamma r + k = 0$.

Existence / number of roots depends on: $\underline{\underline{\gamma^2 - 4km}}$.

Case 1 : $\gamma^2-4km>0 \Rightarrow$ two roots: $r=\dfrac{-\gamma\pm\sqrt{\gamma^2-4km}}{2m}$ $(=r_1,r_2)$
(overdamped)

Notice here that $\gamma^2-4km<\gamma^2$ since k and m are positive

Hence $-\gamma+\sqrt{\gamma^2-4km}<0$. Clearly $-\gamma-\sqrt{\gamma^2-4km}<0$ also

Hence the two roots r_1 and r_2 are negative roots.

∴ the general solution to the homogeneous ODE is:

$$y=C_1e^{r_1t}+C_2e^{r_2t}.$$

We notice that $\lim_{t\to\infty} y=0$, and that the approach

to 0 is exponential (fast).

Case 2 : $\gamma^2-4km=0 \Rightarrow$ one double root $r=-\dfrac{\gamma}{2m}<0$.
(critically damped)

∴ General solution is: $y=C_1e^{rt}+C_2te^{rt}$.

As $t\to\infty$, $y\to 0$ also; but the product te^{rt} slows down the approach to zero

Case 3 : $\gamma^2-4km<0 \Rightarrow$ the roots are complex roots:
(underdamped)

$r=\lambda+i\mu$, where $\lambda=-\dfrac{\gamma}{2m}<0$

and $\mu=\dfrac{\sqrt{4km-\gamma^2}}{2m}$.

the general solution is then: $y=C_1e^{\lambda t}\cos(\mu t)$
$+C_2e^{\lambda t}\sin(\mu t)$

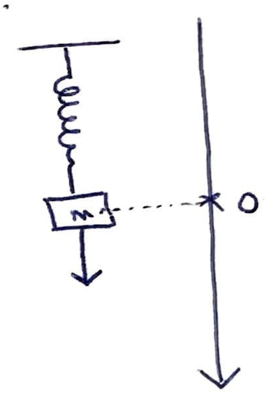
Since $\lambda < 0$, then the ~~osc~~ amplitude of the oscillation will get smaller and smaller as $t \to \infty$.

Finally, ~~in the~~ a sub special case of the underdamped motion is when $\gamma = 0$ ie $\lambda = 0 \Rightarrow$ general solution is:

$$y = c_1 \cos(\mu t) + c_2 \sin(\mu t).$$

Examples

1) A mass weighing 10 Lbs stretches a spring 2 inches. If the mass is displaced an additional 2 inches and then set in motion with an initial upward velocity of 1 ft/sec, determine the position of the mass at any time t, assuming no damping: $\gamma = 0$. [Note: 1 ft = 12 inches].



Solution: we construct a vertical axis for the position of the mass pointing downwards, and let the origin ~~at~~ on this axis be the position of the mass at rest. ~~The~~

If $y(t)$ = position of mass at time t in foot,

then: $y(0) = 2/12 = \frac{1}{6}$ ft.

Because the motion initially is upwards, then

$$y'(0) = -1 \text{ ft/sec}.$$

→ the spring constant can be calculated using the fact that the weight of 10lbs stretches it 2 inches or $\frac{1}{6}$ ft; $k = \frac{10}{1/6} = 60$ lbs/ft

→ the mass is: $\frac{w}{g} = 10/32$ *

This problem is then modeled by:

$$\cancel{\tfrac{1}{6}y'' + 0.y' + 60y} \quad \frac{10}{32}y'' + 0.y' + 60y = 0; y(0) = \frac{1}{6}$$

$$y'(0) = -1$$

Or: $y'' + \frac{32\times 60}{10}y = 0$ ie $y'' + 192y = 0.$

the characteristic equation is: $r^2 + 192 = 0$

$\Rightarrow r = \pm i\sqrt{192}$

$\therefore \quad y = C_1\cos(\sqrt{192}\,t) + C_2\sin(\sqrt{192}\,t).$

But $y(0) = 1/6 \Rightarrow C_1 = 1/6.$

Now, $y' = -\frac{1}{6}\cdot\sqrt{192}\sin(\sqrt{192}\,t) + C_2\sqrt{192}\cos(\sqrt{192}\,t).$

$y'(0) = -1 \Rightarrow -1 = C_2\sqrt{192} \Rightarrow C_2 = -1/\sqrt{192}.$

[Note: $192 = 32\times 6 = 2\times 16\times 6 = 2\times 4^2\times 3\times 2 = 2^2\times 4^2\times 3$

$\Rightarrow \sqrt{192} = 8\sqrt{3}.$]

$$\Rightarrow \boxed{y = \frac{1}{6}\cos(8\sqrt{3}\,t) - \frac{1}{8\sqrt{3}}\sin(8\sqrt{3}\,t).}$$

<u>Remark:</u> Given that $y = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$. This equation can be rewritten in the form $\boxed{y = R\cos(\omega_0 t - \delta)}$.

Indeed,

$$R\cos(\omega_0 t - \delta) = R\cos(\omega_0 t)\cos(\delta) + R\sin(\omega_0 t)\sin(\delta).$$

By comparison, we see that

$$c_1 = R\cos(\delta) \quad \text{and} \quad c_2 = R\sin(\delta) \; \boxed{\Rightarrow \tan(\delta) = \frac{c_2}{c_1} \text{ and } R^2 = c_1^2 + c_2^2}$$

Once we have c_1 and c_2 we can find R and δ.

<u>Back to previous example:</u>

we obtained: $y = \underbrace{\frac{1}{6}}_{c_1} \cos(\overbrace{8\sqrt{3}}^{\omega_0} t) - \underbrace{\frac{1}{8\sqrt{3}}}_{c_2} \sin(\overbrace{8\sqrt{3}}^{\omega_0} t)$

$$\Rightarrow \left.\begin{array}{l} \frac{1}{6} = R\cos(\delta) \\ -\frac{1}{8\sqrt{3}} = R\sin(\delta) \end{array}\right\} \Rightarrow \tan(\delta) = \frac{-6}{8\sqrt{3}} = \frac{-3}{4\sqrt{3}} = -\frac{\sqrt{3}}{4}$$

$$\therefore \delta = \tan^{-1}\left(-\frac{\sqrt{3}}{4}\right) \approx -0.41 \text{ radians}$$

$$\text{and } R^2 = \frac{1}{36} + \frac{1}{192} = \frac{19}{576} \approx 0.181$$

$$\Rightarrow y \approx 0.181\cos(8\sqrt{3}t - 0.41).$$