## PROBLEMS

In each of Problems 1 through 6, sketch the graph of the given function on the interval $t \geq 0$.

1. $g(t) = u_1(t) + 2u_3(t) - 6u_4(t)$
2. $g(t) = (t-3)u_2(t) - (t-2)u_3(t)$
3. $g(t) = f(t-\pi)u_\pi(t), \quad \text{where } f(t) = t^2$
4. $g(t) = f(t-3)u_3(t), \quad \text{where } f(t) = \sin t$
5. $g(t) = f(t-1)u_2(t), \quad \text{where } f(t) = 2t$
6. $g(t) = (t-1)u_1(t) - 2(t-2)u_2(t) + (t-3)u_3(t)$

In each of Problems 7 through 12:

(a) Sketch the graph of the given function.

(b) Express $f(t)$ in terms of the unit step function $u_c(t)$.

7. $f(t) = \begin{cases} 0, & 0 \leq t < 3, \\ -2, & 3 \leq t < 5, \\ 2, & 5 \leq t < 7, \\ 1, & t \geq 7. \end{cases}$

8. $f(t) = \begin{cases} 1, & 0 \leq t < 1, \\ -1, & 1 \leq t < 2, \\ 1, & 2 \leq t < 3, \\ -1, & 3 \leq t < 4, \\ 0, & t \geq 4. \end{cases}$

9. $f(t) = \begin{cases} 1, & 0 \leq t < 2, \\ e^{-(t-2)}, & t \geq 2. \end{cases}$

10. $f(t) = \begin{cases} t^2, & 0 \leq t < 2, \\ 1, & t \geq 2. \end{cases}$

11. $f(t) = \begin{cases} t, & 0 \leq t < 1, \\ t-1, & 1 \leq t < 2, \\ t-2, & 2 \leq t < 3, \\ 0, & t \geq 3. \end{cases}$

12. $f(t) = \begin{cases} t, & 0 \leq t < 2, \\ 2, & 2 \leq t < 5, \\ 7-t, & 5 \leq t < 7, \\ 0, & t \geq 7. \end{cases}$

In each of Problems 13 through 18, find the Laplace transform of the given function.

13. $f(t) = \begin{cases} 0, & t < 2 \\ (t-2)^2, & t \geq 2 \end{cases}$

14. $f(t) = \begin{cases} 0, & t < 1 \\ t^2 - 2t + 2, & t \geq 1 \end{cases}$

15. $f(t) = \begin{cases} 0, & t < \pi \\ t - \pi, & \pi \leq t < 2\pi \\ 0, & t \geq 2\pi \end{cases}$

16. $f(t) = u_1(t) + 2u_3(t) - 6u_4(t)$

17. $f(t) = (t-3)u_2(t) - (t-2)u_3(t)$

18. $f(t) = t - u_1(t)(t-1), \qquad t \geq 0$

In each of Problems 19 through 24, find the inverse Laplace transform of the given function.

19. $F(s) = \dfrac{3!}{(s-2)^4}$

20. $F(s) = \dfrac{e^{-2s}}{s^2 + s - 2}$

21. $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2 - 2s + 2}$

22. $F(s) = \dfrac{2e^{-2s}}{s^2 - 4}$

23. $F(s) = \dfrac{(s-2)e^{-s}}{s^2 - 4s + 3}$

24. $F(s) = \dfrac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}$

25. Suppose that $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$.
   (a) Show that if $c$ is a positive constant, then

$$\mathcal{L}\{f(ct)\} = \frac{1}{c} F\left(\frac{s}{c}\right), \qquad s > ca.$$

(b) Show that if $k$ is a positive constant, then

$$\mathcal{L}^{-1}\{F(ks)\} = \frac{1}{k} f\left(\frac{t}{k}\right).$$

(c) Show that if $a$ and $b$ are constants with $a > 0$, then

$$\mathcal{L}^{-1}\{F(as+b)\} = \frac{1}{a} e^{-bt/a} f\left(\frac{t}{a}\right).$$

In each of Problems 26 through 29, use the results of Problem 25 to find the inverse Laplace transform of the given function.

26. $F(s) = \dfrac{2^{n+1} n!}{s^{n+1}}$

27. $F(s) = \dfrac{2s+1}{4s^2+4s+5}$

28. $F(s) = \dfrac{1}{9s^2 - 12s + 3}$

29. $F(s) = \dfrac{e^2 e^{-4s}}{2s-1}$

In each of Problems 30 through 33, find the Laplace transform of the given function. In Problem 33, assume that term-by-term integration of the infinite series is permissible.

30. $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$

31. $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \\ 1, & 2 \le t < 3 \\ 0, & t \ge 3 \end{cases}$

32. $f(t) = 1 - u_1(t) + \cdots + u_{2n}(t) - u_{2n+1}(t) = 1 + \sum_{k=1}^{2n+1} (-1)^k u_k(t)$

33. $f(t) = 1 + \sum_{k=1}^{\infty} (-1)^k u_k(t)$. See Figure 6.3.7.

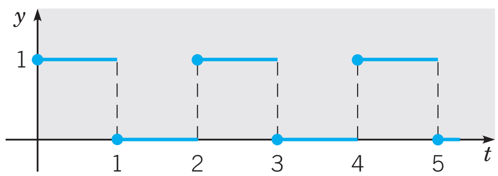


**FIGURE 6.3.7** The function $f(t)$ in Problem 33; a square wave.

34. Let $f$ satisfy $f(t+T) = f(t)$ for all $t \ge 0$ and for some fixed positive number $T$; $f$ is said to be periodic with period $T$ on $0 \le t < \infty$. Show that

$$\mathcal{L}\{f(t)\} = \frac{\displaystyle\int_0^T e^{-st} f(t)\, dt}{1 - e^{-sT}}.$$

In each of Problems 35 through 38, use the result of Problem 34 to find the Laplace transform of the given function.

35. $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & 1 \le t < 2; \end{cases}$
$f(t+2) = f(t)$.
Compare with Problem 33.

36. $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ -1, & 1 \le t < 2; \end{cases}$
$f(t+2) = f(t)$.
See Figure 6.3.8.

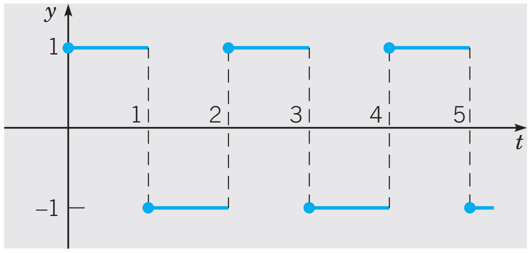


**FIGURE 6.3.8** The function $f(t)$ in Problem 36; a square wave.

37. $f(t) = t, \quad 0 \le t < 1;$
$f(t+1) = f(t).$
See Figure 6.3.9.

38. $f(t) = \sin t, \quad 0 \le t < \pi;$
$f(t+\pi) = f(t).$
See Figure 6.3.10.

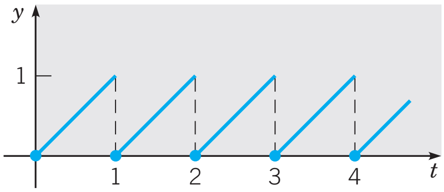


**FIGURE 6.3.9** The function $f(t)$ in Problem 37; a sawtooth wave.

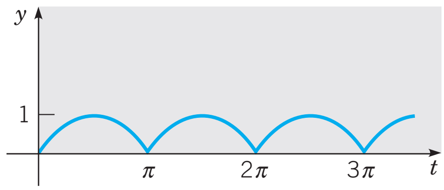


**FIGURE 6.3.10** The function $f(t)$ in Problem 38; a rectified sine wave.

39. (a) If $f(t) = 1 - u_1(t)$, find $\mathcal{L}\{f(t)\}$; compare with Problem 30. Sketch the graph of $y = f(t)$.

(b) Let $g(t) = \int_0^t f(\xi)\, d\xi$, where the function $f$ is defined in part (a). Sketch the graph of $y = g(t)$ and find $\mathcal{L}\{g(t)\}$.

(c) Let $h(t) = g(t) - u_1(t)g(t-1)$, where $g$ is defined in part (b). Sketch the graph of $y = h(t)$ and find $\mathcal{L}\{h(t)\}$.

40. Consider the function $p$ defined by

$$p(t) = \begin{cases} t, & 0 \le t < 1, \\ 2 - t, & 1 \le t < 2; \end{cases} \qquad p(t+2) = p(t).$$

(a) Sketch the graph of $y = p(t)$.

(b) Find $\mathcal{L}\{p(t)\}$ by noting that $p$ is the periodic extension of the function $h$ in Problem 39(c) and then using the result of Problem 34.

(c) Find $\mathcal{L}\{p(t)\}$ by noting that

$$p(t) = \int_0^t f(t)\, dt,$$

where $f$ is the function in Problem 36, and then using Theorem 6.2.1.