3.5. Nonhomogeneous Equations - Method of Undetermined Coefficients.

The focus in this section is on linear non-homogeneous ODE's: $y'' + p(t)y' + q(t)y = g(t)$. $(+)$

We shall find the general form of the solution to such ODE's.

There are two main theorems:

Theorem 1: If $Y_1(t)$ and $Y_2(t)$ are two solutions for the non-homogeneous equation $(+)$, then $(Y_1 - Y_2)$ is a solution to the corresponding homogeneous equation: $y'' + p(t)y' + q(t) = 0$.

Proof: Substitute $(Y_1 - Y_2)$ for $y(t)$ in the ~~non-homo~~ right-hand side of $(+)$:

$$(Y_1 - Y_2)'' + p(t)(Y_1 - Y_2)' + q(t)(Y_1 - Y_2)$$

$$= Y_1'' - Y_2'' + p(t)Y_1' - p(t)Y_2' + q(t)Y_1 - q(t)Y_2$$

$$= (Y_1'' + p(t)Y_1' + q(t)Y_1) - (Y_2'' + p(t)Y_2' + q(t)Y_2)$$

$$= g(t) - g(t) = 0$$

Theorem 2: The general solution of the non-homogeneous equation (+) can be written in the form: $\boxed{y = c_1 y_1(t) + c_2 y_2(t) + Y(t),}$ where $\{y_1, y_2\}$ is a fundamental solution set for the corresponding homogeneous equation, and $Y(t)$ is any solution of the non-homogeneous equation

Proof: Because y is a solution of the non-homog. equation and so is $Y \Rightarrow y - Y$ is a solution of the homog. equation $\Rightarrow y - Y = c_1 y_1 + c_2 y_2$

$$\Rightarrow y = c_1 y_1 + c_2 y_2 + Y(t)$$

Remark: The previous theorem (2) says that in order to find the general solution of (+) we need:

(1) the fundamental solution set of the homog. equation (can be found for constant coeffs)

(2) ~~A~~ Some solution $Y(t)$ for the non-homog. case (we shall guess using undetermined coefficients)

Exs:

① Solve: $y'' - 3y' - 4y = 3e^{2t}$.

the corresponding homog. equation is: $y'' - 3y' - 4y = 0$

Characteristic equation is: $r^2 - 3r - 4 = 0$ or $(r-4)(r+1) = 0$

$\Rightarrow r_1 = 4$ and $r_2 = -1$.

$\therefore$ A fundamental solution set is $\{e^{4t}, e^{-t}\}$.

For the particular solution of the non-homogeneous case, we need to make a guess:

here: $y = Ae^{2t}$, A to be determined

$y' = 2Ae^{2t}$ and $y'' = 4Ae^{2t}$.

We substitute in the non-homog. equation to find A:

$$4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} = 3e^{2t}$$

$$\Rightarrow -6A = 3 \Rightarrow \cancel{A = \tfrac{3}{2}} \quad A = -\frac{1}{2}$$

$\therefore$ General solution is: $y = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2} e^{2t}$.

Ex 2: $y'' - 3y' - 4y = 2\sin t$.

$\{e^{4t}, e^{-t}\}$ forms a fundamental solution set.

As for the guess of a particular solution for the non_homogeneous case,

$$y = A\sin t + B\cos t.$$

$\Rightarrow y' = A\cos t - B\sin t; \quad y'' = -A\sin t - B\cos t.$

Substitute and we obtain $A = -\frac{5}{17}$; $B = \frac{3}{17}$

$\therefore$ General solution is:

$$y = c_1 e^{4t} + c_2 e^{-t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t.$$

Ex 3: $y'' - 3y' - 4y = 4t^2 - 1$

Particular solution is: $y = At^2 + Bt + C$

$y' = 2At + B; \quad y'' = 4A.$

Substitute and we obtain:

$$4A - 3(2At + B) - 4(At^2 + Bt + C) = 4t^2 - 1$$

$\Rightarrow 4A - 6At - 3B - 4At^2 - 4Bt - 4C = 4t^2 - 1$

$\Rightarrow -4At^2 + (-6A - 4B)t + 4A - 3B - 4C = 4t^2 - 1$

$\therefore \quad -4A = 4 \Rightarrow \boxed{A = -1}$

$-6A - 4B = 0 \Rightarrow 6 - 4B = 0 \Rightarrow \boxed{B = \frac{6}{4} = \frac{3}{2}}$

and $\quad 4A - 3B - 4C = -1$

$\Rightarrow -4 - \frac{9}{2} - 4C = -1 \Rightarrow 4C = -4 - \frac{9}{2} + 1$

$= -3 - \frac{9}{2} = -\frac{15}{2}$

$\Rightarrow \boxed{C = -\frac{15}{8}}.$

Summary:

$g(t)$	$y(t)$
$e^{\alpha t}$	$Ae^{\alpha t}$
$\sin\alpha t$, $\cos\beta t$, combination	$A\sin\alpha t + B\cos\alpha t$
polynomial of degree n	$A_1 t^n + A_2 t^{n-1} + \dots + A_{n-1} t + A_n$.

What about Combinations?

Ex 4: $y'' - 3y' - 4y = -8e^{t}\cos(2t)$

Guess is: $y = e^{t}(A\sin 2t + B\cos 2t)$.

(of course, solving this is quite tedious). (see example 3 on page 178 of textbook)

Ex 5: $y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^{t}\cos 2t$

$3e^{2t} \to y$ ex 1

$2\sin t \to y$ ex 2

$8e^{t}\cos 2t \to y$ ex 4

Here, we use superposition principle!

$\therefore$ General solution is:

$$y = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2}e^{2t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t + \frac{10}{13}e^{t}\cos 2t + \frac{2}{13}e^{t}\sin 2t$$