Result 3 - The case of complex eigenvalues/eigenvectors

~~Research~~

Consider the system $\begin{cases} \dfrac{dx}{dt} = 2x - 5y \\ \dfrac{dy}{dt} = x - 2y \end{cases}$

The coefficients a, b, c, d in this system form a matrix

$$A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}.$$

For this matrix, we have found 2 complex eigenvalues

$$\lambda = \pm i$$

For $\lambda_1 = +i$, a corresponding eigenvector is $\vec{v}_1 = \langle 2-i, 1\rangle$.

If we were to follow the same reasoning as in the preceding cases, then we would have that the general solution is:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = k_1 e^{it} \begin{bmatrix} 2-i \\ 1 \end{bmatrix} + k_2 e^{-it} \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$ where

$\vec{v}_2 = \langle \alpha, \beta \rangle$ is an eigenvector corresponding to $\lambda_2 = -i$.

The issue/problem with this solution is that it is a complex one.

Let us look at $e^{it}\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$:

$$e^{it} = \cos t + i\sin t$$

$$\Rightarrow e^{it}\begin{bmatrix} 2-i \\ 1 \end{bmatrix} = (\cos t + i\sin t)\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} (\cos t + i\sin t)(2-i) \\ \cos t + i\sin t \end{bmatrix}$$

$$= \begin{bmatrix} 2\cos t + 2i\sin t - i\cos t + \sin t \\ \cos t + i\sin t \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 2\cos t + \sin t \\ \cos t \end{bmatrix}}_{\text{Real part}} + i\underbrace{\begin{bmatrix} 2\sin t - \cos t \\ \sin t \end{bmatrix}}_{\text{imaginary part.}}$$

<u>Theorem</u>: The General real solution of this system is:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = k_1\begin{bmatrix} 2\cos t + \sin t \\ \cos t \end{bmatrix} + k_2\begin{bmatrix} 2\sin t - \cos t \\ \sin t \end{bmatrix}.$$

Here is another example:

$$\frac{dx}{dt} = -2x - 3y$$

$$\frac{dy}{dt} = 3x - 2y$$

$$\Rightarrow A = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix}; \quad \det(A) = 4 + 9 = 13$$

$$\operatorname{tr}(A) = -4$$

$\therefore$ Eigenvalues are solutions to: $\lambda^2 + 4\lambda + 13 = 0$

$$\lambda = \frac{-4 \pm \sqrt{16 - 52}}{2} = \frac{-4 \pm i\sqrt{36}}{2} = -2 \pm i(3)$$

$$\Rightarrow \lambda_1 = -2 + 3i \quad ; \quad \lambda_2 = -2 - 3i.$$

For $\lambda_1 = -2 + 3i$, we have:

$$\begin{cases} -2x - 3y = (-2 + 3i)x \\ 3x - 2y = (-2 + 3i)y \end{cases} \Rightarrow \begin{cases} -2x - 3y = -2x + 3ix \\ 3x - 2y = -2y + 3iy \end{cases}$$

$$\Rightarrow \begin{cases} -3y = -3ix \text{ or } y = ix \\ 3x = 3iy \Rightarrow x = iy \Rightarrow ix = -y \Rightarrow y = -ix \end{cases}$$

$\therefore$ An eigenvector is $\vec{v}_1 \langle 1, -i \rangle$.

Now, $e^{\lambda_1 t}\vec{v}_1 = \left(e^{(-2+3i)t}\right)\begin{bmatrix}1\\-i\end{bmatrix}$

$$= e^{-2t}(\cos 3t + i\sin 3t)\begin{bmatrix}1\\-i\end{bmatrix}$$

$$= \begin{bmatrix} e^{-2t}\cos(3t) + i e^{-2t}\sin(3t) \\ -e^{-2t}\sin 3t - i e^{-2t}\cos(3t) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} e^{-2t}\cos(3t) \\ e^{-2t}\sin(3t) \end{bmatrix}}_{\text{Real}} + i\underbrace{\begin{bmatrix} e^{-2t}\sin(3t) \\ -e^{-2t}\cos(3t) \end{bmatrix}}_{\text{imaginary}}$$

$\therefore$ General <u>real</u> solution of this system is:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = k_1 e^{-2t}\begin{bmatrix} \cos(3t) \\ \sin(3t) \end{bmatrix} + k_2 e^{-2t}\begin{bmatrix} \sin(3t) \\ -\cos(3t) \end{bmatrix}.$$

~~In this same example, suppose we were given the following~~

A notation:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \vec{Y}(t) \quad (\text{or } \vec{X}(t)) \qquad \Bigg| \qquad \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \vec{Y}'(t) \text{ or } \frac{d\vec{Y}}{dt}.$$

A system like:

$$\begin{cases} \dfrac{dx}{dt} = -2x - 3y \\ \dfrac{dy}{dt} = 3x - 2y \end{cases}$$

we write as: $\dfrac{d\vec{Y}}{dt} = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix} \vec{Y}$.

In the previous example, suppose we are given the following conditions: $\vec{Y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

we find k_1, k_2.

$$\vec{Y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow k_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} k_1 \\ -k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow k_1 = 0 \text{ and } k_2 = -1$$

$$\therefore \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = -e^{-2t} \begin{bmatrix} \sin(3t) \\ -\cos(3t) \end{bmatrix}.$$