Chapter 1.

1.1. Direction fields.

Here we shall focus on a geometric / qualitative technique for finding solutions.

A general 1^{st} order ODE can be represented by the general form: $\boxed{\frac{dy}{dt} = f(t, y)}$. (*)

If a function $y(t)$ is some solution to (*) passing through the point (t_1, y_1), where $y_1 = y(t_1)$, then the ODE ~~tells us~~ tells us that $\left.\frac{dy}{dt}\right|_{t=t_1} = f(t_1, y_1)$. (**).

If we were to read this equation (**) with meaning, then it says:

the slope of the solution function at the point (t_1, y_1) is equal to $f(t_1, y_1)$.

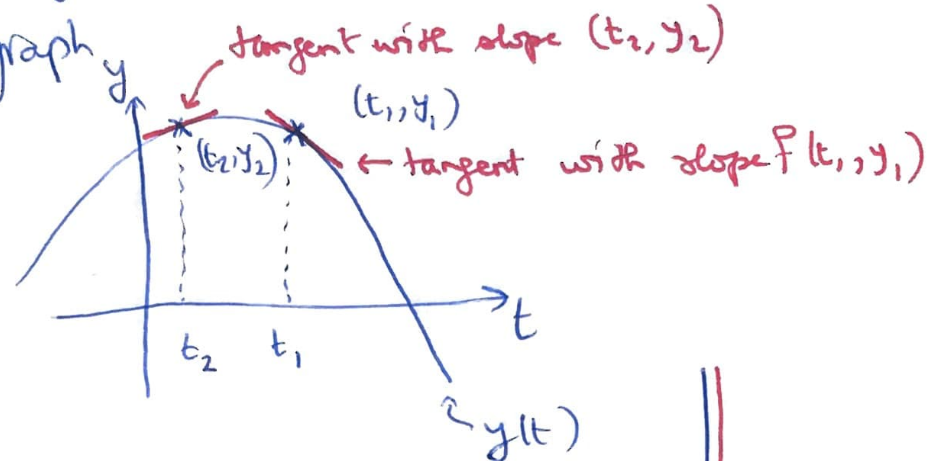
This gives us also the slope of the line tangent to the solution function at (t_1, y_1).

Since, we do <u>not</u> know the solution function, we can instead draw the tangent line at (t_1, y_1) [for a line, we need a point $= (t_1, y_1)$ and a slope $= f(t_1, y_1)$].

Of course this line is tangent to the solution function at (t_1, y_1).

If we change the point, the tangent will change as well.

Let us say, the solution is known - below is its graph



A sketch of the tangent lines (also called mini_tangents) is a main tool for visualizing solutions to ODEs. It is called a <u>slope</u> or a <u>direction field</u>.

Sketching a direction field is best done using computers.

Starting at any point of the ty-plane and <u>flowing</u> through the field gives a picture of a solution through that point, enabling the learner to discuss properties of a particular solution.

Examples

(A) Using technology: geogebra.org/m/ZGeeGQbp (slope field viewer).

(B) Slope fields by hand (a very tedious job of course).

We will start with simple examples: $\frac{dy}{dt} = f(t,y)$

(1) $\frac{dy}{dt} = \cos t$. ← we know the solutions to this ODE.

They are: $y = \sin t + C$.

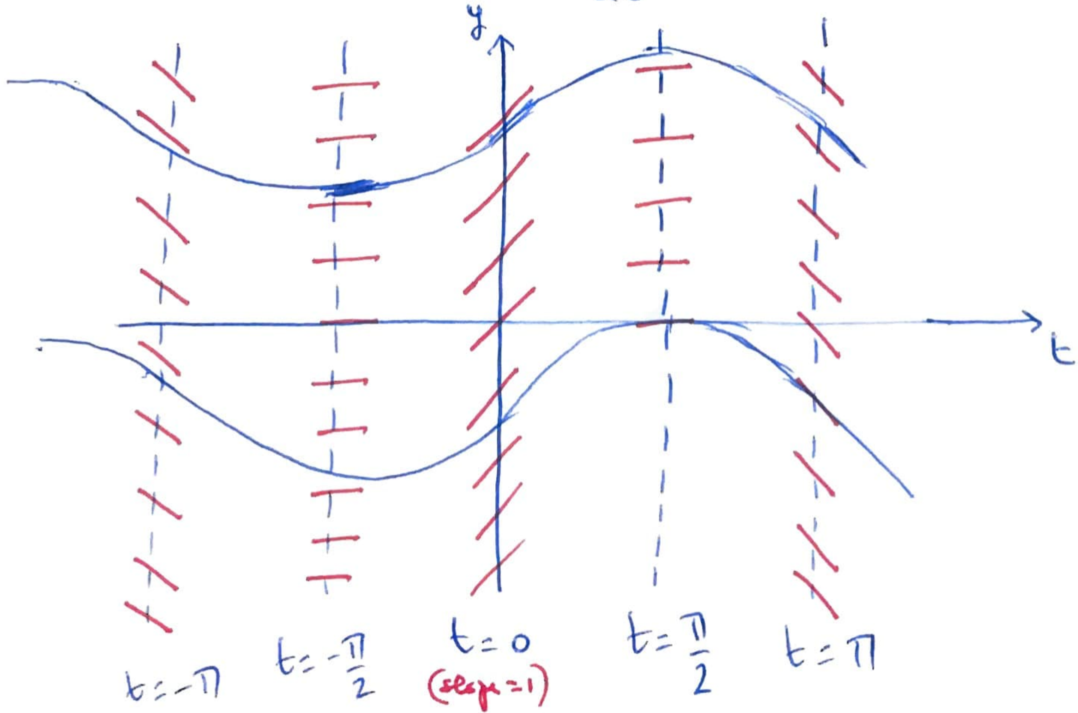
Since $f(t,y) = \cos t$ depends only on t, then the values of $\frac{dy}{dt}$ do not change for a fixed value of t.

e.g: At the points $(0,y)$, $\left.\frac{dy}{dt}\right|_{(0,y)} = \cos 0 = 1$, for all y.

At the points $(\frac{\pi}{2}, y)$, $\left.\frac{dy}{dt}\right|_{(\frac{\pi}{2},0)} = \cos\frac{\pi}{2} = 0$, for all y.

At the points $(-\frac{\pi}{2}, y)$, $\frac{dy}{dt} = \cos(-\frac{\pi}{2}) = 0$, for all y

At the points (π, y), $\frac{dy}{dt} = \cos(\pi) = \cos(\pi) = -1$, for all y.



these plots look indeed look like sine functions

Notice that these solutions are ~~horizontal~~ Vertical translations (or ~~horizontal~~ Vertical translates) of each other.

This is true for any ODE of the form $\frac{dy}{dt} = f(t)$.

2) $\frac{dy}{dt} = 5y$. From a previous example, we expect the solutions to take the form $y = Ce^{5t}$.

Here $f(t,y) = 5y = f(y)$ $\therefore$ the values of $\frac{dy}{dt}$ do not change at t changes, but for a given fixed value of y.

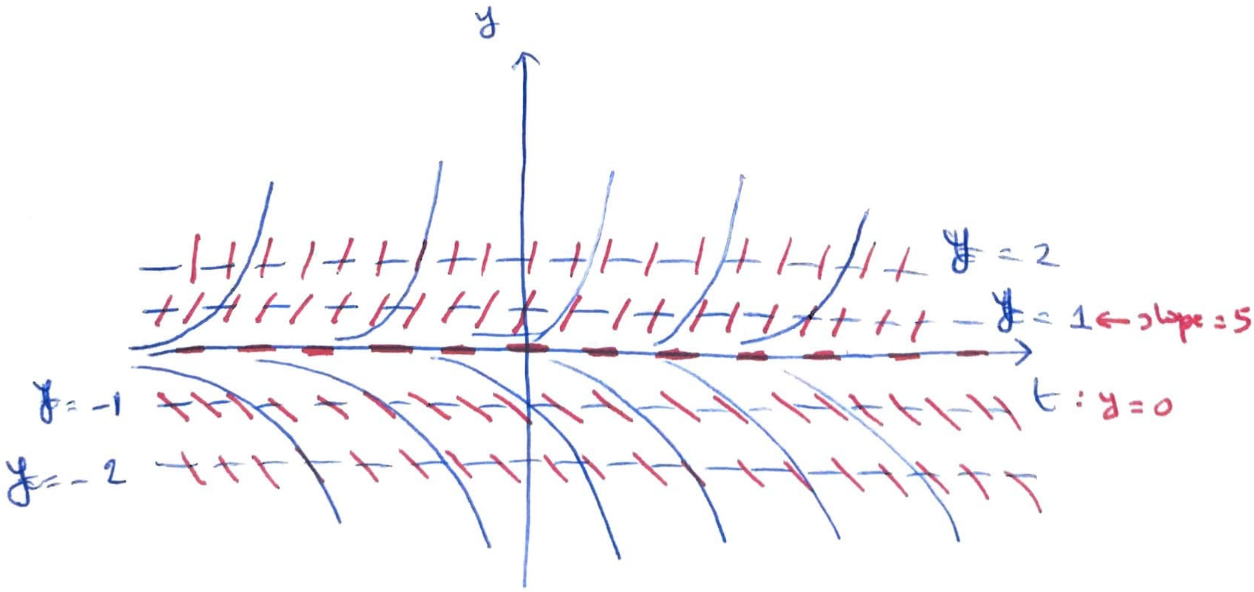
e.g: $(t,0) \rightarrow \frac{dy}{dt} = 0, \ \forall t$

$(t,1) \rightarrow \frac{dy}{dt} = 5, \ \forall t$

$(t,2) \rightarrow \frac{dy}{dt} = 10, \ \forall t$

$(t,-1) \rightarrow \frac{dy}{dt} = -5, \ \forall y$

$(t,-2) \rightarrow \frac{dy}{dt} = -10, \ \forall y$.



these plots look indeed like Ce^{5t} $(C>0, C<0)$

for $C=0$, we obtain $y=0$, which is the t-axis.

These solutions are ~~vertical~~ horizontal translates of each other.

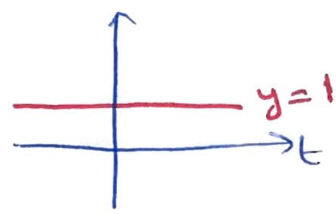
This is the case for any ODE of the form $\frac{dy}{dt} = f(y)$.

We also notice that one solution is $y=0$ (a horizontal solution); for this solution $\frac{dy}{dt} = 0$ ie y does not change.

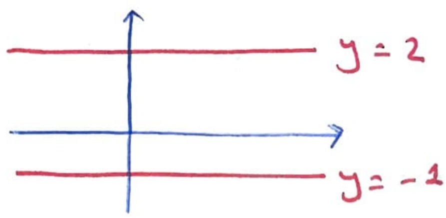
We call it an equilibrium solution

(3) Other Examples of equilibrium solutions:

(a) $\frac{dy}{dt} = y-1 = 0 \Rightarrow y=1$ is equilibrium



(b) $\frac{dy}{dt} = (y+1)(2-y) = 0 \Rightarrow y=-1$ and $y=2$ are equilibrium



(c) $\frac{dy}{dt} = \cos y = 0 \Rightarrow y = \pm\frac{\pi}{2} + 2k\pi$.