Review of Chapter 3 :

① Homogeneous second-order linear differential equations with constant coefficients:

$$ay'' + by' + cy = 0.$$

We look at the corresponding characteristic equation:

$$ar^2 + br + c = 0.$$

Three cases arise.

Case 1: $b^2 - 4ac > 0 \Rightarrow$ 2 roots : r_1 and r_2

$\therefore$ General solution of ODE is: $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$

Case 2: $b^2 - 4ac = 0 \Rightarrow$ 1 double root $r_1 = r_2$

$\therefore$ General solution of ODE is: $y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}.$

Case 3: $b^2 - 4ac < 0 \Rightarrow$ 2 complex roots : $r = \lambda \pm i\mu.$

$\therefore$ General solution is: $y = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t).$

② Non-homogeneous second-order linear differential equations with constant coefficients:

$$ay'' + by' + cy = g(t).$$

General solution is: $y = \underbrace{c_1 y_1 + c_2 y_2}_{\text{solution of homog. equation (see ①)}} + Y(t)$

For $Y(t)$, we make a guess.

<u>Three Special Cases</u>

<u>Case 1</u>: $g(t) = ae^{\alpha t} \Rightarrow Y(t) = At^{\text{A}}e^{\alpha t}$, where

(+) A is to be determined

(++) A = least positive integer such that no term of $t^{\text{A}}e^{\alpha t}$ solves the homog. ODE.

<u>Case 2</u>: $g(t) = a\sin(\alpha t) + b\cos(\alpha t)$

$\Rightarrow Y(t) = t^{\text{A}}(A\sin(\alpha t) + B\cos(\alpha t))$, where

(+) A and B are to be determined

(++) Same as in case 1.

<u>Case 3</u>: $g(t) = a_n t^n + a_{n-1}t^{n-1} + \dots + a_1 t + a_0$, then

$Y(t) = t^{\text{A}}(A_n t^n + A_{n-1}t^{n-1} + \dots + A_1 t + A_0)$, where

(+) $A_n, A_{n-1}, \dots, A_1, A_0$ are to be determined

(++) Same as in case 1.

<u>Other Special cases</u>: Combinations of case 1/2/and 3.

$\therefore Y$ = combination of Guesses.

(3) Homogeneous second order linear differential equations with variable coefficients:

$$y'' + p(t)\,y' + q(t)\,y = 0.$$

Here we use the method of reduction of order to find the general solution.

This requires that you are given one solution y_1;

to find the second solution y_2, we assume that $y_2 = v(t)\,y_1$, for some $v(t)$.

To find $v(t)$, substitute in the original ODE.

Your equation should turn into a 1^{st} order ODE by letting $w(t) = v'(t)$.

Once you find $w(t)$, you conclude $v(t) = \int w(t)\,dt$.

General solution is:

$$y = c_1 y_1 + c_2 v(t)\, y_1.$$

(4) Non-homogeneous second order ODE's with ~~constant~~ coefficients: $ay'' + by' + cy = g(t)$ where $g(t)$ is <u>Not</u> exponential, <u>Not</u> sine/cosine, and <u>Not</u> a polynomial.

Here, we begin by finding y_1 and y_2 (see 3 cases on page 3.1), then we calculate:

$$u_1 = -\int \frac{y_2(t)\, g(t)}{W[y_1, y_2]}\, dt \quad ; \quad u_2 = \int \frac{y_1(t)\, g(t)}{W[y_1, y_2]}\, dt.$$

General solution is:

$$y = c_1 y_1 + c_2 y_2 + u_1(t)\, y_1 + u_2(t)\, y_2.$$

(5) Finally, remember that $W[y_1, y_2] = y_1 y_2' - y_1' y_2$.

Also, $W[y_1, y_2] = C e^{-\int p(t)\, dt}$, given that $y'' + p(t) y' + q(t) y = 0$