3.2 Solutions of Linear homogeneous equations.

We look at more general linear equations:

$$y'' + p(t)y' + q(t)y = g(t)$$

Where $p(t)$, $q(t)$ and $g(t)$ are functions of t.

If $g(t)=0$, the equation is homogeneous

If $g(t) \neq 0$, the equation is non-homogeneous

If $p(t)$ and $q(t)$ are constant functions, then the ODE is with constant coefficients (and can be homogeneous or non-homogeneous).

The Superposition Principle

If y_1 and y_2 are two solutions of $y'' + p(t)y' + q(t)y = 0$, then $c_1 y_1 + c_2 y_2$ is also a solution for any values of c_1 and c_2.

Proof: $(c_1 y_1 + c_2 y_2)'' + p(t)[c_1 y_1 + c_2 y_2]' + q(t)[c_1 y_1 + c_2 y_2]$

$$= c_1 y_1'' + c_2 y_2'' + c_1 p(t) y_1' + c_2 p(t) y_2' + c_1 q(t) y_1 + c_2 q(t) y_2$$

$$= c_1 \underbrace{[y_1'' + p(t) y_1' + q(t) y_1]}_{=0} + c_2 \underbrace{[y_2'' + p(t) y_2' + q(t) y_2]}_{=0}$$

since y_1 and y_2 are solutions

Can we say that $c_1 y_1 + c_2 y_2$ exhaust all the possible solutions?

Definition: Let $y_1(t)$ and $y_2(t)$ denote two solutions of $y'' + p(t)y' + q(t)y = 0$.

We define the Wronskian of y_1 and y_2 to be:

$$W[y_1, y_2](t) = y_1 y_2' - y_1' y_2.$$

Theorem: (No proof) If y_1 and y_2 are two solutions of $y'' + p(t)y' + q(t)y = 0$ such that $W[y_1, y_2](t_0) \neq 0$ for some value t_0, then any solution to the ODE takes the form: $c_1 y_1 + c_2 y_2$.

In this case, $\{y_1, y_2\}$ is called a fundamental solution set for the ODE and $y(t) = c_1 y_1 + c_2 y_2$ is the general solution of the ODE.

Ex1: In a previous example (Notes p. 61) we showed that $y_1 = e^t$ and $y_2 = e^{-t}$ are two solutions of $y'' - y = 0$. We also showed that $c_1 e^t + c_2 e^{-t}$ form a set of solutions. Are those all the solutions?

We look at $W[e^t, e^{-t}] = -e^t e^{-t} - (+e^{-t}e^t) = -2 \neq 0$

for any choise of t

$\therefore \{e^t, e^{-t}\}$ forms a fundamental solution set

and hence <u>any</u> solution of $y'' - y = 0$ takes

the form: $y = c_1 e^t + c_2 e^{-t}$.

<u>Ex 2</u>: We can show that $y_1 = \sqrt{t}$ and $y_2 = \frac{1}{t}$ are

two solutions of: $2t^2 y'' + 3ty' - y = 0$; $t > 0$.

Do they form a fundamental solution set?

<u>Answer</u>

$$W[y_1, y_2](t) = -\left(\frac{1}{2\sqrt{t}} \cdot \frac{1}{t} - \sqrt{t}\left(-\frac{1}{t^2}\right)\right) = -\frac{3}{2} \cdot \frac{1}{\sqrt{t^3}} \neq 0$$

$\therefore$ Any solution of this differential equation

takes the form: $c_1\sqrt{t} + \frac{c_2}{t}$.

<u>Theorem</u>: (Abel's Theorem).

If y_1 and y_2 are solutions of $y'' + p(t)y' + q(t)y = 0$, where p and q are continuous on some open interval I, then

$$W[y_1, y_2](t) = c\, e^{-\int p(t)\,dt} \quad ; c \text{ is some constant}$$

This means that either $W[y_1, y_2] \neq 0$ for any t (when $c \neq 0$) or else $W[y_1, y_2] = 0$ for all t (when $c = 0$).

Existence and Uniqueness

Consider the IVP $y'' + p(t)y' + q(t)y = g(t)$; $y(t_0) = y_0$
$y'(t_0) = y_1$

Where p, q, and g are continuous on an open interval I that contains the point t_0, then there exists exactly one solution $y = \phi(t)$ of this ODE, and the solution exists throughout I.

Ex: Find the longest interval in which the solution of the IVP $(t^2 - 3t)y'' + ty' - (t+3)y = 0$; $y(1) = 2$, $y'(1) = 1$ is certain to exist.

Answer: In this ODE, $p(t) = \dfrac{t}{t^2 - 3t} = \dfrac{1}{t-3}$ $(t \neq 0)$ and

$$q(t) = -\frac{t+3}{t^2 - 3t}.$$

$\therefore$ the points of discontinuity of $p(t)$ and $q(t)$ are $t = 0$ and $t = 3$.

the largest open interval containing $t_0 = 1$ that will guarantee existence and uniqueness of this IVP is: $0 < t < 3$.