Application to mass–spring systems and electric networks

More on partial fractions (Example 4)

For the beginning of the discussion of partial fractions in the present context, see Sec. 6.2.

Problem Set 6.4

CAS Project 1 involves exploring the nature of solutions with the damping or spring constant being changed in the presence of one or two Dirac delta functions on the right.

CAS Experiment 2 models a wave of constant area acting for shorter and shorter times.

Problems 3–12 model vibrating systems with driving forces consisting of unit step functions, Dirac's delta functions, and other functions.

Problems 14–15 concern rectifiers, sawtooth waves, and staircase functions.

SOLUTIONS TO PROBLEM SET 6.4, page 230

2. CAS Experiment. Students should become aware that careful observation of graphs may lead to discoveries or to more information about conjectures that they may want to prove or disprove. The curves branch from the solution of the homogeneous ODE at the instant at which the impulse is applied, which by choosing, say $a = 1, 2, 3, \cdots$, gives an interesting joint graph.

3. $y = 2\cos(3t) + 1/3u(t - 1/2\pi)\cos(3t)$

4. The subsidiary equation is

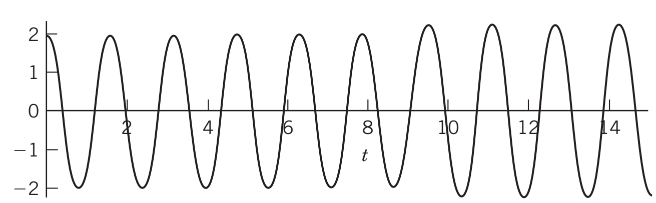
$$(s^2 + 16)Y = 2s + 4e^{-3\pi s}$$

where $2s$ comes from $y(0)$. The solution is

$$Y = \frac{2s + 4e^{-3\pi s}}{s^2 + 16}.$$

The inverse transform (the solution of the IVP) is

$$y = 2\cos 4t + u(t - 3\pi)\sin 4t.$$



Section 6.4. Problem 4

5. $$y = \begin{cases} 1/2\sin(2t) & 0 < t < \pi \\ \sin(2t) & \pi < t \le 2\pi \\ 1/2\sin(2t) & t > 2\pi \end{cases}$$

6. The subsidiary equation is

$$(s^2 + 4s + 5)Y = 3 + e^{-s}$$

where 3 comes from $y'(0)$. The solution is

$$Y = \frac{3 + e^{-s}}{(s + 2)^2 + 1}.$$

The inverse transform (the solution of the initial value problem) is

$$y = 3e^{-2t} \sin t + u(t - 1)e^{-2(t-1)} \sin (t - 1).$$

7. $y = 3/5\, \mathrm{e}^{-t} + 1/2\, u\,(t - 1/4)\, \mathrm{e}^{-2t+1/2} \sin (1/2t - 1/8)$

8. The subsidiary equation is

$$(s^2 + 3s + 2)Y = s - 1 + 3 + \frac{10}{s^2 + 1} + 10e^{-s}.$$

In terms of partial fractions, its solution is

$$Y = \frac{-2}{s + 2} + \frac{6}{s + 1} - \frac{3s - 1}{s^2 + 1} + 10\left(\frac{1}{s + 1} - \frac{1}{s + 2}\right)e^{-s}.$$

Its inverse transform is

$$y = -2e^{-2t} + 6e^{-t} - 3 \cos t + \sin t + 10u(t - 1)[e^{-t+1} - e^{-2(t-1)}].$$

Without the δ-term, the solution is $-3 \cos t + \sin t - 2e^{-2t} + 6e^{-t}$ and approaches a harmonic oscillation fairly soon. With the δ-term the first half-wave has a maximum amplitude of about 5, but from about $t = 8$ or 10 on its graph practically coincides with the graph of that harmonic oscillation (whose maximum amplitude is $\sqrt{10}$). This is physically understandable, since the system has damping that eventually consumes the additional energy due to the δ-term.

9. $y(t) = 1/5\, \mathrm{e}^{t} u(2 - t) + 1/5\, \mathrm{e}^{4-t} u\,(t - 2)\,(\cos\,(t - 2) - 3 \sin\,(t - 2))$
$\quad + 1/5(-\cos\,(t) + 3 \sin\,(t))\, \mathrm{e}^{-t}$

10. The subsidiary equation is

$$(s^2 + 5s + 6)Y = e^{-\pi s/2} - \frac{s}{s^2 + 1} e^{-\pi s}.$$

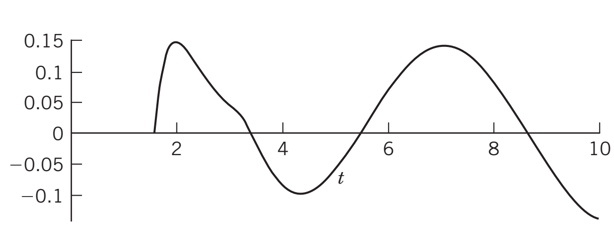
Its solution is

$$Y = \left(\frac{1}{s + 2} - \frac{1}{s + 3}\right)e^{-\pi s/2} - \left(-\frac{0.4}{s + 2} + \frac{0.3}{s + 3} + \frac{0.1(s + 1)}{s^2 + 1}\right)e^{-\pi s}.$$

The inverse transform of Y is

$$\begin{aligned} y = {} & u(t - \tfrac{1}{2}\pi)\left[e^{-2t+\pi} - e^{-3t+3\pi/2}\right] \\ & - 0.1u(t - \pi)\left[-4e^{-2t+2\pi} + 3e^{-3t+3\pi} - \cos t - \sin t\right]. \end{aligned}$$

This solution is zero from 0 to $\frac{1}{2}\pi$ and then increases rapidly. Its first negative half-wave has a smaller maximum amplitude (about 0.1) than the continuation as a harmonic oscillation with maximum amplitude of about 0.15.



Section 6.4. Problem 10

11. $y(t) = \mathrm{e}^{-t} - \mathrm{e}^{-2t} + u(t-2)\,(\mathrm{e}^{2-t} - \mathrm{e}^{-2t+4})$
$+ 1/2(1 - 2\mathrm{e}^{1-t} + \mathrm{e}^{-2t+2})u(t-1).$

12. The subsidiary equation is

$$(s^2 + 2s + 5)Y = 1 - 2s + \frac{25}{s^2} - 100e^{-\pi s}.$$

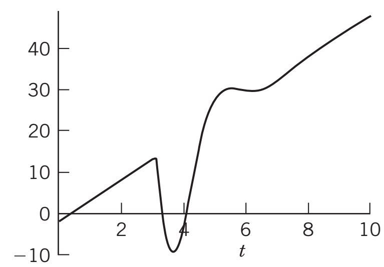
Its solution is

$$Y = \frac{-s^2 + 2s^3 - 25 + 100s^2e^{-\pi s}}{s^2((s+1)^2 + 4)}.$$

Its inverse transform (the solution of the IVP) is

$$y = 5t - 2 - 50u(t-\pi)e^{-t+\pi}\sin 2t.$$

This is essentially a straight line, sharply deformed between π and about 8.



Section 6.4. Problem 12

14. TEAM PROJECT. (a) If $f(t)$ is piecewise continuous on an interval of length p, then its Laplace transform exists, and we can write the integral from zero to infinity as the series of integrals over successive periods:

$$\mathscr{L}(f) = \int_0^\infty e^{-st}f(t)\,dt = \int_0^p e^{-st}f\,dt + \int_p^{2p} e^{-st}f\,dt + \int_{2p}^{3p} e^{-st}f\,dt + \cdots.$$

If we substitute $t = \tau + p$ in the second integral, $t = \tau + 2p$ in the third integral, $\cdots$, $t = \tau + (n-1)p$ in the nth integral, $\cdots$, then the new limits in every integral are 0 and p. Since

$$f(\tau + p) = f(\tau), \qquad f(\tau + 2p) = f(\tau),$$

etc., we thus obtain

$$\mathscr{L}(f) = \int_0^p e^{-s\tau}f(\tau)\,d\tau + \int_0^p e^{-s(\tau+p)}f(\tau)\,d\tau + \int_0^p e^{-s(\tau+2p)}f(\tau)\,d\tau + \cdots.$$

The factors that do not depend on τ can be taken out from under the integral signs; this gives

$$\mathscr{L}(f) = [1 + e^{-sp} + e^{-2sp} + \cdots]\int_0^p e^{-s\tau}f(\tau)\,d\tau.$$

The series in brackets $[\cdots]$ is a geometric series whose sum is $1/(1 - e^{-ps})$. The theorem now follows.

(b) From (11) we obtain

$$\mathcal{L}(f) = \frac{1}{1 - e^{-2\pi s/\omega}} \int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt.$$

Using $1 - e^{-2\pi s/\omega} = (1 + e^{-\pi s/\omega})(1 - e^{-\pi s/\omega})$ and integrating by parts or noting that the integral is the imaginary part of the integral

$$\int_0^{\pi/\omega} e^{(-s+i\omega)t} \, dt = \frac{1}{-s + i\omega} e^{(-s+i\omega)t} \Big|_0^{\pi/\omega} = \frac{-s - i\omega}{s^2 + \omega^2} (-e^{-s\pi/\omega} - 1)$$

we obtain the result.

(c) From (11) we obtain the following equation by using $\sin \omega t$ from 0 to π/ω and $-\sin \omega t$ from π/ω to $2\pi/\omega$:

$$\begin{aligned} \frac{\omega}{s^2 + \omega^2} \frac{1 + e^{\pi s/\omega}}{e^{\pi s/\omega} - 1} &= \frac{\omega}{s^2 + \omega^2} \frac{e^{-\pi s/2\omega} + e^{\pi s/2\omega}}{e^{\pi s/2\omega} - e^{-\pi s/2\omega}} \\ &= \frac{\omega}{s^2 + \omega^2} \frac{\cosh(\pi s/2\omega)}{\sinh(\pi s/2\omega)}. \end{aligned}$$

This gives the result.

(d) The sawtooth wave has the representation

$$f(t) = \frac{k}{p} t \qquad \text{if} \quad 0 < t < p, \qquad f(t + p) = f(t).$$

Integration by parts gives

$$\begin{aligned} \int_0^p e^{-st} t \, dt &= -\frac{t}{s} e^{-st} \Big|_0^p + \frac{1}{s} \int_0^p e^{-st} \, dt \\ &= -\frac{p}{s} e^{-sp} - \frac{1}{s^2} (e^{-sp} - 1) \end{aligned}$$

and thus from (11) we obtain the result

$$\mathcal{L}(f) = \frac{k}{ps^2} - \frac{ke^{-ps}}{s(1 - e^{-ps})} \qquad (s > 0).$$

SECTION 6.5. Convolution. Integral Equations, page 232

Purpose. To find the inverse $h(t)$ of a product $H(s) = F(s)G(s)$ of transforms whose inverses are known.

Main Content, Important Concepts

Convolution $f * g$, its properties

Convolution theorem

Application to ODEs and integral equations