2.2. Separable Equations

A first-order ODE $\frac{dy}{dt} = f(t,y)$ is called separable if $\boxed{f(t,y) = g(t) \cdot h(y)}$

Exs: ① $\frac{dy}{dt} = \frac{t^2}{1-y^2} = (t^2)\left(\frac{1}{1-y^2}\right) = g(t) \cdot h(y)$

② $t\frac{dy}{dt} = \sqrt{(1-y^2)} \Rightarrow \frac{dy}{dt} = \frac{1}{t} \cdot \sqrt{1-y^2} = g(t) \cdot h(y)$

③ $\frac{dy}{dt} = (3t^2-1)(3+2y)$, already in the form $g(t)h(y)$.

How to solve a separable ODE?

General approach

$$\frac{dy}{dt} = g(t)\,h(y)$$

$$\Rightarrow \frac{1}{h(y)}\,dy = g(t)\,dt \quad \text{, provided } h(y) \neq 0.$$

$$\Rightarrow \int \frac{1}{h(y)}\,dy = \int g(t)\,dt$$

↓

this gives us an expression for y.

Then we solve for y, if possible to get an explicit form of the solution

If it is not possible to solve for y explicitly, we keep the family of solutions in an implicit form

Remark: When dividing by $h(y)$ in the first step, we require $h(y) \neq 0$; this may exclude possible values of y.

To find all possible solutions, we substitute the values of y that have been excluded in the original ODE and check if the ODE is satisfied, in which case the excluded value is also a solution.

Application

Example 3: $\frac{dy}{dt} = (3t^2-1)(3+2y) \Rightarrow \frac{1}{3+2y}\,dy = (3t^2-1)dt$ if $y \neq -\frac{3}{2}$

$$\Rightarrow \frac{1}{2}\int \frac{2}{3+2y}\,dy = \int (3t^2-1)\,dt$$

$$\Rightarrow \frac{1}{2}\ln|3+2y| = t^3 - t + C$$ ← this is an implicit family of solutions.

We can find an explicit family of solutions here:

$$\ln|3+2y| = 2t^3 - 2t + 2C$$

$$\Rightarrow e^{\ln|3+2y|} = e^{2t^3-2t+2C}$$

$$\Rightarrow |3+2y| = e^{2t^3-2t} \cdot \underbrace{e^{2c}}_{=K}$$, some positive constant $K>0$

$$\Rightarrow 3+2y = \underbrace{\pm K}_{=A} e^{2t^3-2t}$$

= A, some non-zero constant

$$\Rightarrow 2y = Ae^{2t^3-2t} - 3 \Rightarrow y = \frac{A}{2} e^{2t^3-2t} - \frac{3}{2}$$

$$\Rightarrow \boxed{y = Be^{2t^3-2t} - \frac{3}{2}}; \quad B \neq 0 \quad \left(B = \frac{A}{2}\right).$$

But initially we excluded the possibility that $y = \frac{-3}{2}$.

Is $y = -\frac{3}{2}$ a solution? We check if it satisfies the ODE $\frac{dy}{dt} \overset{?}{=} (3t^2-1)(3+2y)$

indeed $\frac{d}{dt}\left(-\frac{3}{2}\right) = 0$ and $(3t^2-1)(3+2\cdot\frac{-3}{2}) = 0$ ✓

∴ The family of solutions is:

$$\boxed{y = Be^{2t^3-2t} - \frac{3}{2}, \text{ for any choice of } B}$$

($B=0$ implies $y = -\frac{3}{2}$)

Example 2: $\frac{dy}{dt} = \frac{1}{t}\sqrt{1-y^2} \Rightarrow \frac{1}{\sqrt{1-y^2}}\,dy = \frac{1}{t}\,dt;\ y \neq \pm 1$

$\Rightarrow \sin^{-1} y = \ln|t| + C$; we can keep it in this implicit form.

Now, when $y = \pm 1$, the ODE is satisfied

$\therefore$ The family of solutions consist of:

① $y = \pm 1$

② $\sin^{-1} y = \ln|t| + C$

Example 1: $\frac{dy}{dt} = t^2\left(\frac{1}{1-y^2}\right)$ [Here, for the ODE to make sense, we need $y \neq \pm 1$]

$\Rightarrow (1-y^2)\,dy = t^2\,dt$

$\Rightarrow \boxed{y - \frac{y^3}{3} = \frac{t^3}{3} + C}$; an implicit family of solutions.

Suppose now we are given an IC: $y(0) = 1$

$\Rightarrow 1 - \frac{1}{3} = C \Rightarrow C = \frac{2}{3}$

$\therefore$ the particular solution to this IVP is:

$$\boxed{y - \frac{y^3}{3} = \frac{t^3}{3} + \frac{2}{3}}$$