equation can be expressed as a linear combination of a fundamental set of solutions $y_1, \ldots, y_n$, it follows that any solution of Eq. (2) can be written as

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) + Y(t), \tag{16}$$

where Y is some particular solution of the nonhomogeneous equation (2). The linear combination (16) is called the general solution of the nonhomogeneous equation (2).

Thus the primary problem is to determine a fundamental set of solutions $y_1, \ldots, y_n$ of the homogeneous equation (4). If the coefficients are constants, this is a fairly simple problem; it is discussed in the next section. If the coefficients are not constants, it is usually necessary to use numerical methods such as those in Chapter 8 or series methods similar to those in Chapter 5. These tend to become more cumbersome as the order of the equation increases.

To find a particular solution $Y(t)$ in Eq. (16), the methods of undetermined coefficients and variation of parameters are again available. They are discussed and illustrated in Sections 4.3 and 4.4, respectively.

The method of reduction of order (Section 3.4) also applies to nth order linear equations. If y_1 is one solution of Eq. (4), then the substitution $y = v(t)y_1(t)$ leads to a linear differential equation of order $n - 1$ for v' (see Problem 26 for the case when $n = 3$). However, if $n \geq 3$, the reduced equation is itself at least of second order, and only rarely will it be significantly simpler than the original equation. Thus, in practice, reduction of order is seldom useful for equations of higher than second order.

PROBLEMS

In each of Problems 1 through 6, determine intervals in which solutions are sure to exist.

1. $y^{(4)} + 4y''' + 3y = t$
2. $ty''' + (\sin t)y'' + 3y = \cos t$
3. $t(t-1)y^{(4)} + e^t y'' + 4t^2 y = 0$
4. $y''' + ty'' + t^2 y' + t^3 y = \ln t$
5. $(x-1)y^{(4)} + (x+1)y'' + (\tan x)y = 0$
6. $(x^2 - 4)y^{(6)} + x^2 y''' + 9y = 0$

In each of Problems 7 through 10, determine whether the given functions are linearly dependent or linearly independent. If they are linearly dependent, find a linear relation among them.

7. $f_1(t) = 2t - 3, \quad f_2(t) = t^2 + 1, \quad f_3(t) = 2t^2 - t$
8. $f_1(t) = 2t - 3, \quad f_2(t) = 2t^2 + 1, \quad f_3(t) = 3t^2 + t$
9. $f_1(t) = 2t - 3, \quad f_2(t) = t^2 + 1, \quad f_3(t) = 2t^2 - t, \quad f_4(t) = t^2 + t + 1$
10. $f_1(t) = 2t - 3, \quad f_2(t) = t^3 + 1, \quad f_3(t) = 2t^2 - t, \quad f_4(t) = t^2 + t + 1$

In each of Problems 11 through 16, verify that the given functions are solutions of the differential equation, and determine their Wronskian.

11. $y''' + y' = 0; \qquad 1, \quad \cos t, \quad \sin t$
12. $y^{(4)} + y'' = 0; \qquad 1, \quad t, \quad \cos t, \quad \sin t$
13. $y''' + 2y'' - y' - 2y = 0; \qquad e^t, \quad e^{-t}, \quad e^{-2t}$
14. $y^{(4)} + 2y''' + y'' = 0; \qquad 1, \quad t, \quad e^{-t}, \quad te^{-t}$
15. $xy''' - y'' = 0; \qquad 1, \quad x, \quad x^3$
16. $x^3 y''' + x^2 y'' - 2xy' + 2y = 0; \qquad x, \quad x^2, \quad 1/x$
17. Show that $W(5, \sin^2 t, \cos 2t) = 0$ for all t. Can you establish this result without direct evaluation of the Wronskian?
18. Verify that the differential operator defined by

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y$$

is a linear differential operator. That is, show that

$$L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2],$$

where y_1 and y_2 are n-times-differentiable functions and c_1 and c_2 are arbitrary constants. Hence, show that if $y_1, y_2, \dots, y_n$ are solutions of $L[y] = 0$, then the linear combination $c_1y_1 + \cdots + c_ny_n$ is also a solution of $L[y] = 0$.

19. Let the linear differential operator L be defined by

$$L[y] = a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny,$$

where $a_0, a_1, \dots, a_n$ are real constants.

(a) Find $L[t^n]$.

(b) Find $L[e^{rt}]$.

(c) Determine four solutions of the equation $y^{(4)} - 5y'' + 4y = 0$. Do you think the four solutions form a fundamental set of solutions? Why?

20. In this problem we show how to generalize Theorem 3.2.7 (Abel's theorem) to higher order equations. We first outline the procedure for the third order equation

$$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0.$$

Let y_1, y_2, and y_3 be solutions of this equation on an interval I.

(a) If $W = W(y_1, y_2, y_3)$, show that

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

Hint: The derivative of a 3-by-3 determinant is the sum of three 3-by-3 determinants obtained by differentiating the first, second, and third rows, respectively.

(b) Substitute for y_1''', y_2''', and y_3''' from the differential equation; multiply the first row by p_3, multiply the second row by p_2, and add these to the last row to obtain

$$W' = -p_1(t)W.$$

(c) Show that

$$W(y_1, y_2, y_3)(t) = c\exp\left[-\int p_1(t)\,dt\right].$$

It follows that W is either always zero or nowhere zero on I.

(d) Generalize this argument to the nth order equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0$$

with solutions $y_1, \dots, y_n$. That is, establish Abel's formula

$$W(y_1, \dots, y_n)(t) = c\exp\left[-\int p_1(t)\,dt\right]$$

for this case.

In each of Problems 21 through 24, use Abel's formula (Problem 20) to find the Wronskian of a fundamental set of solutions of the given differential equation.

21. $y''' + 2y'' - y' - 3y = 0$

22. $y^{(4)} + y = 0$

23. $ty''' + 2y'' - y' + ty = 0$

24. $t^2y^{(4)} + ty''' + y'' - 4y = 0$

25. (a) Show that the functions $f(t) = t^2|t|$ and $g(t) = t^3$ are linearly dependent on $0 < t < 1$ and on $-1 < t < 0$.
    (b) Show that $f(t)$ and $g(t)$ are linearly independent on $-1 < t < 1$.
    (c) Show that $W(f, g)(t)$ is zero for all t in $-1 < t < 1$.
26. Show that if y_1 is a solution of

$$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0,$$

then the substitution $y = y_1(t)v(t)$ leads to the following second order equation for v':

$$y_1 v''' + (3y_1' + p_1 y_1)v'' + (3y_1'' + 2p_1 y_1' + p_2 y_1)v' = 0.$$

In each of Problems 27 and 28, use the method of reduction of order (Problem 26) to solve the given differential equation.

27. $(2 - t)y''' + (2t - 3)y'' - ty' + y = 0, \quad t < 2; \qquad y_1(t) = e^t$
28. $t^2(t + 3)y''' - 3t(t + 2)y'' + 6(1 + t)y' - 6y = 0, \quad t > 0; \qquad y_1(t) = t^2, \quad y_2(t) = t^3$

4.2 Homogeneous Equations with Constant Coefficients

Consider the nth order linear homogeneous differential equation

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = 0, \tag{1}$$

where $a_0, a_1, \ldots, a_n$ are real constants and $a_0 \neq 0$. From our knowledge of second order linear equations with constant coefficients, it is natural to anticipate that $y = e^{rt}$ is a solution of Eq. (1) for suitable values of r. Indeed,

$$L[e^{rt}] = e^{rt}(a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1}r + a_n) = e^{rt}Z(r) \tag{2}$$

for all r, where

$$Z(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1}r + a_n. \tag{3}$$

For those values of r for which $Z(r) = 0$, it follows that $L[e^{rt}] = 0$ and $y = e^{rt}$ is a solution of Eq. (1). The polynomial $Z(r)$ is called the **characteristic polynomial**, and the equation $Z(r) = 0$ is the **characteristic equation** of the differential equation (1). Since $a_0 \neq 0$, we know that $Z(r)$ is a polynomial of degree n and therefore has n zeros,[1] say, $r_1, r_2, \ldots, r_n$, some of which may be equal. Hence we can write the characteristic polynomial in the form

$$Z(r) = a_0(r - r_1)(r - r_2)\cdots(r - r_n). \tag{4}$$

[1] An important question in mathematics for more than 200 years was whether every polynomial equation has at least one root. The affirmative answer to this question, the fundamental theorem of algebra, was given by Carl Friedrich Gauss (1777–1855) in his doctoral dissertation in 1799, although his proof does not meet modern standards of rigor. Several other proofs have been discovered since, including three by Gauss himself. Today, students often meet the fundamental theorem of algebra in a first course on complex variables, where it can be established as a consequence of some of the basic properties of complex analytic functions.