2.3 Modeling with first-order equations

We have already done some modeling, mainly modeling population growth:

The exponential growth model $\frac{dP}{dt} = kP$

The logistic ~~po~~ growth model $\frac{dP}{dt} = kP(1 - \frac{P}{N})$.

In some exercises, we modeled radioactive decay and another problem on ambient temperature.

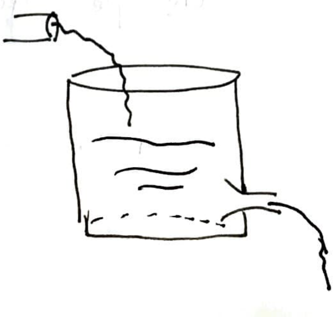
Now we look at "mixing" problems. We do the modeling via an example.

<u>Example 1</u> (Textbook page 52).

At time $t=0$, a tank contains Q_0 lb (pounds) of salt dissolved in 100 gal of water.

Assume that water containing $\frac{1}{4}$ lb of salt/gal is entering the tank at a rate of r gal/min, and that the mixed solution leaves the tank at the <u>same rate</u>.

Set up an IVP that describes the rate of change of ~~the~~ salt ~~of~~ inside the tank.



Solution: Let $Q(t)$ denote the amount of salt inside the tank at time t.

Initially there was Q_0 lb of salt inside the tank.

ie $Q(0) = Q_0$.

Now as salt water pours inside the tank, and as the mixed solution leaves the tank, the amount of salt is changing inside the tank.

We claim: $\dfrac{dQ}{dt} = \underbrace{\text{(rate at which salt is flowing in)}}^{\text{rate in}}$

$- \underbrace{\text{(rate at which salt is flowing out)}}_{\text{rate out}}$

Now, rate in $= \left(\frac{1}{4}\ \text{lb/gal}\right) \cdot \left(r\ \text{gal/min}\right) = \frac{r}{4}\ \text{lb/min}$

$\therefore \frac{r}{4}$ Lbs of salt are flowing in per minute (rate in).

We will write a similar expression for the rate out.

The main difference is that the ~~rate~~ amount of lbs per gallon is changing, ~~it~~ because $Q(t)$ = amount of salt inside tank is changing (is unknown) $\Rightarrow$ amount of lbs per gallon is $\dfrac{Q(t)}{100}$ (100 being the volume of tank)

$\Rightarrow$ rate out $= \left(\frac{Q(t)}{100}\ \text{lbs/gall}\right) \cdot r\left(\text{gal/min}\right) = \frac{rQ}{100}\ \text{lbs/min}$

$$\Rightarrow \quad \frac{dQ}{dt} = \frac{r}{4} - \frac{rQ}{100} \quad \text{or} \quad \boxed{\frac{dQ}{dt} + \frac{r}{100} Q = \frac{r}{4}}$$

this is a linear ODE where $p(t) = \frac{r}{100}$ and $g(t) = \frac{r}{4}$.

the solutions take the form: $Q(t) = 25 + k.e^{\frac{-r}{100}t}$.

But $Q(0) = Q_0 \Rightarrow Q_0 = 25 + k \Rightarrow k = Q_0 - 25$.

$$\therefore \boxed{Q(t) = 25 + (Q_0 - 25)e^{\frac{-r}{100}t}}.$$

Remark: Since the liquid pouring in has a concentration of $\frac{1}{4}$ lbs of ~~gallons~~ salt per gallon, then we expect after some time for the concentration inside the tank to reach that same amount: 0.25 lbs/gallon

$\Rightarrow$ Since the tank has 100 gallons $\Rightarrow$ eventually (i.e after some time), we expect the amount of salt to be: $\frac{1}{4} \times 100 = 25$ lbs. Does the solution satisfy this property?

Take $\lim_{t\to\infty} Q(t) = \lim_{t\to\infty} 25 + (Q_0 - 25)\underbrace{e^{\frac{-r}{100}t}}_{\to 0}$

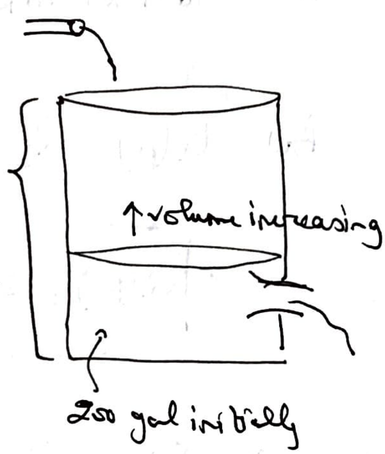
$= 25$, as expected.

Example 2 (Exercise 4 on page 60).

A tank with capacity 500 gal contains originally 200 gal of water with 100 lb of salt in the solution.

If $Q(t)$ = amount of salt at time t

$\Rightarrow Q(0) = 100$ lb.



Water containing 1 lb of salt per gallon is entering at the rate of 3 gal/min

$\therefore$ rate in $= \left(1 \text{ lb}/\text{gal}\right) \cdot \left(3 \text{ gal}/\text{min}\right) = 3 \text{ lbs}/\cancel{\text{gal}}\ \text{min}$

The mixture is flowing out at the rate of 2 gallons per minute.

Because the liquid is flowing in at the rate of 3 gallons per minute, and flowing out at the rate of 2 gallons per minute $\Rightarrow$ volume of liquid inside the tank is not constant but rather increasing by 1 gallon per minute.

$\therefore$ the volume of liquid inside tank after t minutes is : $200 + t$.

$\therefore$ rate out $= \left(\dfrac{Q(t)}{200+t}\right) \times (2) = \dfrac{2Q}{200+t}$ lbs/$\cancel{\text{gal}}$ min

$\Rightarrow$ the ODE modeling this problem is:

$$\frac{dQ}{dt} = 3 - \frac{2}{200+t} Q \; ; \; Q(0) = 100$$

or $$\boxed{\frac{dQ}{dt} + \frac{2}{200+t} Q = 3; \; Q(0) = 100}$$

This again is a linear ODE: $p(t) = \frac{2}{200+t}$; $g(t) = 3$.

$$\mu(t) = e^{\int \frac{2}{200+t} dt} = e^{2\ln|200+t|} = (200+t)^2$$

$$\Rightarrow \quad Q(t) = \frac{1}{(200+t)^2} \int (200+t)^2 . 3 \, dt$$

$$= \frac{1}{(200+t)^2} \left[(200+t)^3 + C \right] = (200+t) + \frac{C}{(200+t)^2}$$

But $Q(0) = 100 \Rightarrow 100 = 200 + \frac{C}{(200)^2} \Rightarrow C = -(200)^2 \times (100)$.

$$C = -4 \times 10^6$$

Find the amount of salt inside the tank when the solution begins to overflow.

Since the volume is increasing by 1 gallon per minute, then after 300 minutes the tank will start overflowing.

$$Q(300) = (200+300) - \frac{4\times 10^6}{(500)^2} = \frac{(500)^3 - 4\times 10^6}{(500)^2} = \frac{121\times 10^6}{25\times 10^4}$$

$$= \frac{121}{25} \times 10^2$$

Now, the concentration of salt ~~at~~ inside the tank

at the time when it overflows $= \dfrac{Q(300)}{\text{Volume}} = \dfrac{\frac{121}{25} \times 10^2}{500}$

$$= \boxed{\frac{121}{125} \text{ Lbs/gal}}$$

2.5. Autonomous ODE's.

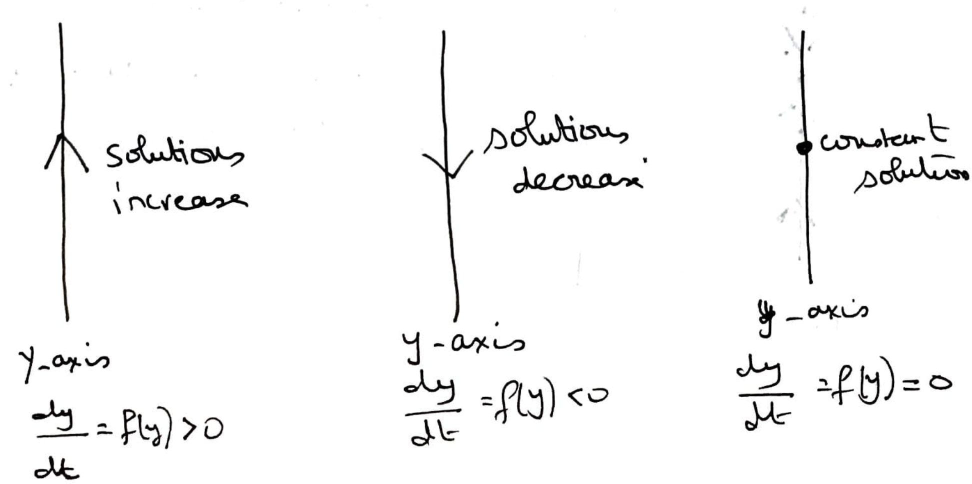
A 1^{st} order ODE $\frac{dy}{dt} = f(y)$ is called **automous**.

We have seen earlier that solutions to autonomous ODE's are horizontal translates of each other.

Furthermore, we define equilibrium solutions to be those solutions y, where $f(y)=0$. Those are horizontal solutions.

Since $f(y)$ depends only on y, then the sign of $\frac{dy}{dt}$ depends also on y only: where $f(y)>0$, $\frac{dy}{dt}>0$ and $y \nearrow$ and where $f(y)<0$, $\frac{dy}{dt}<0$ and $y \searrow$; Of course where $f(y)=0$, $\frac{dy}{dt}=0$ and y is a constant ($y=$ equilibrium solution).

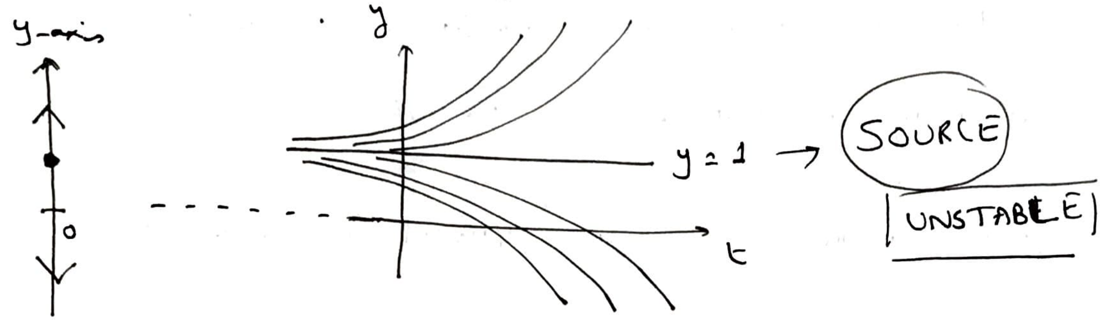
This can be summarized on a vertical line /a y-axis, called the phase line:



This can be better understood through examples.

Ex1: $\frac{dy}{dt} = y - 1$

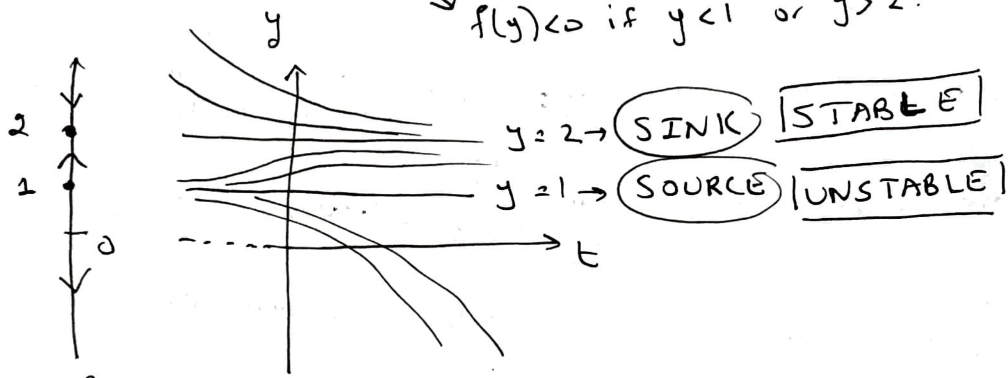
- $f(y) > 0$ if $y > 1 \rightarrow y \nearrow$
- $f(y) = 0$ if $y = 1 \rightarrow y = 1$ is equilibrium
- $f(y) < 0$ if $y < 1 \rightarrow y \searrow$



Phase line

Ex2 $\frac{dy}{dt} = (y-1)(2-y)$

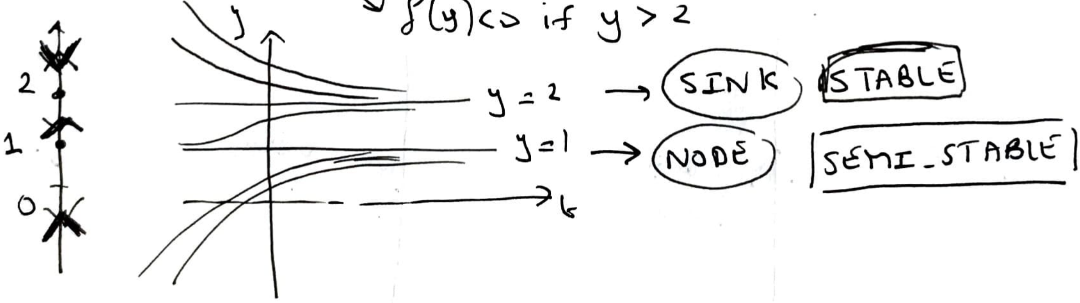
- $f(y) > 0$ if $1 < y < 2$
- $f(y) = 0$ if $y = 1$ or $y = 2$
- $f(y) < 0$ if $y < 1$ or $y > 2$.



Phase line

Ex3. $\frac{dy}{dt} = (y-1)^2(2-y)$

- $f(y) > 0$ if $y < 2$
- $f(y) = 0$ if $y = 1$ or $y = 2$
- $f(y) < 0$ if $y > 2$



Phase line