Chapter 7 - Linear Systems of differential equations

(A brief view)

7.2 - 7.3

Matrices:

1) A 2×2 matrix takes the form: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

2) the determinant of A is: $\det(A) = ad - bc$.

3) the trace of A is: $tr(A) = a + d$.

4) The eigenvalues of A are the solutions to the quadratic polynomial: $\lambda^2 - tr(A)\lambda + \det(A) = 0$.

Exs ① $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$ $\therefore$ $\det(A) = 6 - 2 = 4$

$tr(A) = -5$

$\therefore$ Eigenvalues are solutions to: $\lambda^2 + 5\lambda + 4 = 0$

or $(\lambda+4)(\lambda+1) = 0$ $\begin{cases} \lambda_1 = -4 \\ \lambda_2 = -1 \end{cases}$

② $A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$ $\therefore$ $\det(A) = -4 + 5 = 1$

$tr(A) = 0$

$\therefore$ Eigenvalues are solutions to: $\lambda^2 + 1 = 0$

or $\lambda = \pm i$

③ $A = \begin{bmatrix} +1 & -1 \\ 1 & 3 \end{bmatrix} \Rightarrow \lambda^2 - 4\lambda + 4 = 0$

$\Rightarrow \lambda = 2$ is a repeated eigenvalue.

5) Given that λ is an eigenvalue for A, we define the vector $\vec{v} = \langle x, y \rangle$ to be an eigenvector if ~~a, b are~~ x and y are solutions to the system:

$$\begin{cases} ax + by = \lambda x \\ cx + dy = \lambda y \end{cases}$$

Exs: ① For $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$, we need to solve the system:

For: $\lambda = -4$: $\begin{cases} -3x + \sqrt{2}y = -4x \\ \sqrt{2}x - 2y = -4y \end{cases} \Rightarrow \begin{cases} x + \sqrt{2}y = 0 \to x = -\sqrt{2}y \\ \sqrt{2}x + 2y = 0. \end{cases}$

$\therefore \sqrt{2}(-\sqrt{2}y) + 2y = 0 \Rightarrow -2y + 2y = 0$

$\therefore y$ is arbitrary

Hence any vector of the form $\vec{v} = \langle -\sqrt{2}y, y \rangle$ is an eigenvector. For example $\vec{v} = \langle -\sqrt{2}, 1 \rangle$

For $\lambda_2 = -1$: $\left. \begin{array}{l} -3x + \sqrt{2}y = -x \\ \sqrt{2}x - 2y = -y \end{array} \right\} \Rightarrow \begin{cases} -2x + \sqrt{2}y = 0 \\ \sqrt{2}x - y = 0 \to y = \sqrt{2}x \end{cases}$

$\therefore -2x + \sqrt{2}(\sqrt{2}x) = 0 \Rightarrow -2x + 2x = 0$

$\therefore x$ is arbitrary.

Hence any vector of the form $\vec{v} = \langle x, \sqrt{2}x \rangle$ is an eigenvector, such as $\vec{v} = \langle 1, \sqrt{2} \rangle$.

② For $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$, there is one single eigenvalue $\lambda = 2$.

$$\therefore \left.\begin{aligned} x - y &= 2x \\ x + 3y &= 2y \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} -x - y &= 0 \\ x + y &= 0 \end{aligned}\right\} \Rightarrow x = y$$

$\therefore$ any vector of the form $\vec{v} = \langle x, x \rangle$, such as $\vec{v} = \langle 1, 1 \rangle$ is an eigenvector.

Before we proceed to $A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$ where the eigenvalues are complex, notice that for each found eigenvalue, we have an infinite collection of eigenvectors.

③ For $A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$, we found $\lambda = \pm i$ as eigenvalues.

$$\text{For } \lambda = i \rightarrow \left.\begin{aligned} 2x - 5y &= ix \\ x - 2y &= iy \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} (2-i)x - 5y &= 0 \\ x - (2+i)y &= 0 \rightarrow \end{aligned}\right.$$

$$x = (2+i)y$$

$\therefore\ (2-i)(2+i)y - 5y = 0$

$\Rightarrow (4+1)y - 5y = 0 \Rightarrow 0y = 0 \therefore$ true for all y.

Hence vectors of the form $\vec{v} = \langle (2-i)y, y \rangle$ are eigenvectors;

for example ~~$\vec{v} = \langle (2-i)y$~~ $\vec{v} = \langle 2-i, 1 \rangle$.

Systems of differential Equations:

Definition: A linear system of differential equations takes the form:

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases}$$

where x and y are two variables that depend on t (x and y do not depend on each other).

A solution to a system is a pair of functions $(x(t), y(t))$ satisfying both equations of the system.

Exs :

① For the system $\begin{cases} \dfrac{dx}{dt} = -3x + \sqrt{2}\,y \\ \dfrac{dy}{dt} = \sqrt{2}\,x - 2y \end{cases}$

the pairs of functions $(x(t), y(t))$, where:

$$x(t) = -k_1\sqrt{2}\,e^{-4t} + k_2 e^{-t}$$
$$y(t) = k_1 e^{-4t} + k_2\sqrt{2}\,e^{-t}$$

for any $k_1, k_2 \in \mathbb{R}$

satisfy the system.

Indeed,

(1) $\dfrac{dx}{dt} = +4k_1\sqrt{2}\,e^{-4t} - k_2 e^{-t}$

and $-3x + \sqrt{2}y = -3\left(-k_1\sqrt{2}\,e^{-4t} + k_2 e^{-t}\right) + \sqrt{2}\left(k_1 e^{-4t} + k_2\sqrt{2}\,e^{-t}\right)$

$= +3k_1\sqrt{2}\,e^{-4t} - 3k_2 e^{-t} + k_1\sqrt{2}\,e^{-4t} + 2k_2 e^{-t}$

$= 4k_1\sqrt{2}\,e^{-4t} - k_2 e^{-t}$ ✓

(2) $\dfrac{dy}{dt} = -4k_1 e^{-4t} - k_2\sqrt{2}\,e^{-t}$

and $\sqrt{2}x - 2y = \sqrt{2}\left(-k_1\sqrt{2}\,e^{-4t} + k_2 e^{-t}\right)$

$- 2\left(k_1 e^{-4t} + k_2\sqrt{2}\,e^{-t}\right)$

$= -2k_1 e^{-4t} + k_2\sqrt{2}\,e^{-t} - 2k_1 e^{-4t} - 2k_2\sqrt{2}\,e^{-t}$

$= -4k_1 e^{-4t} - k_2\sqrt{2}\,e^{-t}$ ✓

Notice in this example that the coefficients $a=-3$, $b=\sqrt{2}$, $c=\sqrt{2}$, $d=-2$ form the entries of the matrix $A=\begin{bmatrix}-3 & \sqrt{2}\\ \sqrt{2} & -2\end{bmatrix}$ of example 1 on page 125.

~~In~~ for that matrix we found that $\lambda_1=-4$ is an eigenvalue and $\vec{v}=\langle -\sqrt{2},1\rangle$ is a corresponding eigenvector; that $\lambda_2=-1$ is another eigenvalue and $\vec{v}=\langle 1,\sqrt{2}\rangle$ is a corresponding eigenvector.

Any relationship between the general solution of a system and the eigenvalues / eigenvectors ?

<u>Result 1</u>: Given
$$\frac{dx}{dt}=ax+by$$
$$\frac{dy}{dt}=cx+dy.$$

If the matrix $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ has two distinct eigenvalues λ_1, λ_2 with ~~the~~ corresponding eigenvectors $\vec{v}=\langle x_1, y_1\rangle$ and $\vec{v}_2=\langle x_2, y_2\rangle$, then the general pair of solutions is:

$$\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}=k_1 e^{\lambda_1 t}\begin{bmatrix}x_1\\ y_1\end{bmatrix}+k_2 e^{\lambda_2 t}\begin{bmatrix}x_2\\ y_2\end{bmatrix}$$

or $\begin{cases} x(t) = k_1 x_1 e^{\lambda_1 t} + k_2 x_2 e^{\lambda_2 t} \\ y(t) = k_1 y_1 e^{\lambda_1 t} + k_2 y_2 e^{\lambda_2 t}. \end{cases}$

Result 2: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has one repeated eigenvalue λ

and a corresponding eigenvector $\vec{v} = \langle x_1, y_1 \rangle$, then

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = k_1 e^{\lambda t} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + k_2 t e^{\lambda t} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

~~Result 3: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has complex eigenvalues~~

~~$\lambda = \alpha \pm i\beta$ and~~

[The case of complex eigenvalues is more complicated and cannot be summarized in a general result.]