The sum of this inverse and (7) is the solution of the problem for $0 < t < \pi$, namely (the sines cancel),

$$y(t) = 3e^{-t}\cos t - 2\cos 2t - \sin 2t \qquad \text{if } 0 < t < \pi. \tag{9}$$

In the second fraction in (6), taken with the minus sign, we have the factor $e^{-\pi s}$, so that from (8) and the second shifting theorem (Sec. 6.3) we get the inverse transform of this fraction for $t > 0$ in the form

$$\begin{aligned} &+2\cos(2t - 2\pi) + \sin(2t - 2\pi) - e^{-(t-\pi)}[2\cos(t-\pi) + 4\sin(t-\pi)] \\ &= 2\cos 2t + \sin 2t + e^{-(t-\pi)}(2\cos t + 4\sin t). \end{aligned}$$

The sum of this and (9) is the solution for $t > \pi$,

$$y(t) = e^{-t}[(3 + 2e^{\pi})\cos t + 4e^{\pi}\sin t] \qquad \text{if } t > \pi. \tag{10}$$

Figure 136 shows (9) (for $0 < t < \pi$) and (10) (for $t > \pi$), a beginning vibration, which goes to zero rapidly because of the damping and the absence of a driving force after $t = \pi$. ■

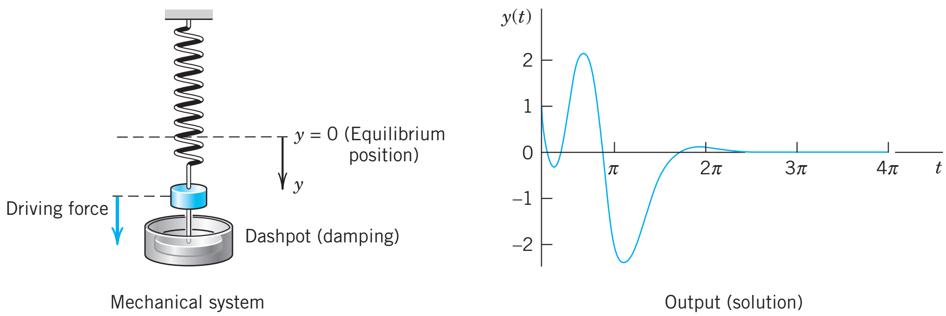

Fig. 136. Example 4

The case of repeated complex factors $[(s - a)(s - \bar{a})]^2$, which is important in connection with resonance, will be handled by "convolution" in the next section.

PROBLEM SET 6.4

1. CAS PROJECT. Effect of Damping. Consider a vibrating system of your choice modeled by

$$y'' + cy' + ky = \delta(t).$$

(a) Using graphs of the solution, describe the effect of continuously decreasing the damping to 0, keeping k constant.

(b) What happens if c is kept constant and k is continuously increased, starting from 0?

(c) Extend your results to a system with two δ-functions on the right, acting at different times.

2. CAS EXPERIMENT. Limit of a Rectangular Wave. Effects of Impulse.

(a) In Example 1 in the text, take a rectangular wave of area 1 from 1 to $1 + k$. Graph the responses for a sequence of values of k approaching zero, illustrating that for smaller and smaller k those curves approach the curve shown in Fig. 134. *Hint:* If your CAS gives no solution for the differential equation, involving k, take specific k's from the beginning.

(b) Experiment on the response of the ODE in Example 1 (or of another ODE of your choice) to an impulse $\delta(t - a)$ for various systematically chosen a (> 0); choose initial conditions $y(0) \neq 0$, $y'(0) = 0$. Also consider the solution if no impulse is applied. Is there a dependence of the response on a? On b if you choose $b\delta(t - a)$? Would $-\delta(t - \tilde{a})$ with $\tilde{a} > a$ annihilate the effect of $\delta(t - a)$? Can you think of other questions that one could consider experimentally by inspecting graphs?

3–12 EFFECT OF DELTA (IMPULSE) ON VIBRATING SYSTEMS

Find and graph or sketch the solution of the IVP. Show the details.

3. $y'' + 4y = \delta(t - \pi)$, $\quad y(0) = 8,\ y'(0) = 0$

4. $y'' + 16y = 4\delta(t - 3\pi), \quad y(0) = 2, y'(0) = 0$

5. $y'' + y = \delta(t - \pi) - \delta(t - 2\pi),$
$y(0) = 0, y'(0) = 1$

6. $y'' + 4y' + 5y = \delta(t - 1), \quad y(0) = 0, y'(0) = 3$

7. $4y'' + 24y' + 37y = 17e^{-t} + \delta(t - \tfrac{1}{2}),$
$y(0) = 1, y'(0) = 1$

8. $y'' + 3y' + 2y = 10(\sin t + \delta(t - 1)), \quad y(0) = 1,$
$y'(0) = -1$

9. $y'' + 4y' + 5y = [1 - u(t - 10)]e^t - e^{10}\delta(t - 10),$
$y(0) = 0, y'(0) = 1$

10. $y'' + 5y' + 6y = \delta(t - \tfrac{1}{2}\pi) + u(t - \pi)\cos t,$
$y(0) = 0, y'(0) = 0$

11. $y'' + 5y' + 6y = u(t - 1) + \delta(t - 2),$
$y(0) = 0, y'(0) = 1$

12. $y'' + 2y' + 5y = 25t - 100\delta(t - \pi), \quad y(0) = -2,$
$y'(0) = 5$

13. PROJECT. Heaviside Formulas. (a) Show that for a simple root a and fraction $A/(s - a)$ in $F(s)/G(s)$ we have the *Heaviside formula*

$$A = \lim_{s \to a} \frac{(s - a)F(s)}{G(s)}.$$

(b) Similarly, show that for a root a of order m and fractions in

$$\frac{F(s)}{G(s)} = \frac{A_m}{(s - a)^m} + \frac{A_{m-1}}{(s - a)^{m-1}} + \cdots + \frac{A_1}{s - a} + \text{further fractions}$$

we have the *Heaviside formulas* for the first coefficient

$$A_m = \lim_{s \to a} \frac{(s - a)^m F(s)}{G(s)}$$

and for the other coefficients

$$A_k = \frac{1}{(m - k)!} \lim_{s \to a} \frac{d^{m-k}}{ds^{m-k}} \left[\frac{(s - a)^m F(s)}{G(s)} \right], \qquad k = 1, \cdots, m - 1.$$

14. TEAM PROJECT. Laplace Transform of Periodic Functions

(a) Theorem. *The Laplace transform of a piecewise continuous function $f(t)$ with period p is*

$$\mathcal{L}(f) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t)\, dt \qquad (s > 0). \tag{11}$$

Prove this theorem. *Hint:* Write $\int_0^\infty = \int_0^p + \int_p^{2p} + \cdots$. Set $t = (n - 1)p$ in the nth integral. Take out $e^{-(n-1)p}$ from under the integral sign. Use the sum formula for the geometric series.

(b) Half-wave rectifier. Using (11), show that the half-wave rectification of $\sin \omega t$ in Fig. 137 has the Laplace transform

$$\mathcal{L}(f) = \frac{\omega(1 + e^{-\pi s/\omega})}{(s^2 + \omega^2)(1 - e^{-2\pi s/\omega})} = \frac{\omega}{(s^2 + \omega^2)(1 - e^{-\pi s/\omega})}.$$

(A *half-wave rectifier* clips the negative portions of the curve. A *full-wave rectifier* converts them to positive; see Fig. 138.)

(c) Full-wave rectifier. Show that the Laplace transform of the full-wave rectification of $\sin \omega t$ is

$$\frac{\omega}{s^2 + \omega^2} \coth \frac{\pi s}{2\omega}.$$

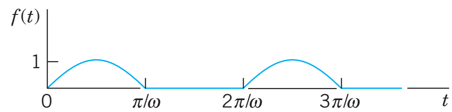

Fig. 137. Half-wave rectification

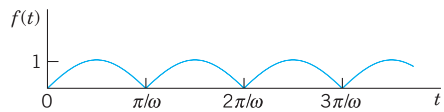

Fig. 138. Full-wave rectification

(d) Saw-tooth wave. Find the Laplace transform of the saw-tooth wave in Fig. 139.

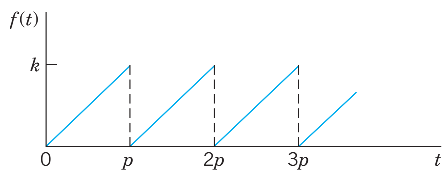

Fig. 139. Saw-tooth wave

15. Staircase function. Find the Laplace transform of the staircase function in Fig. 140 by noting that it is the difference of kt/p and the function in 14(d).

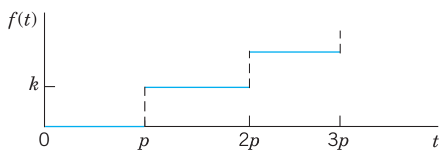

Fig. 140. Staircase function