The integral on the right side of Eq. (45) cannot be evaluated in terms of the usual elementary functions, so we leave the integral unevaluated. However, by choosing the lower limit of integration as the initial point $t = 0$, we can replace Eq. (45) by

$$e^{t^2/4}y = \int_0^t e^{s^2/4}\,ds + c, \tag{46}$$

where c is an arbitrary constant. It then follows that the general solution y of Eq. (41) is given by

$$y = e^{-t^2/4}\int_0^t e^{s^2/4}\,ds + ce^{-t^2/4}. \tag{47}$$

The initial condition (42) requires that $c = 1$.

The main purpose of this example is to illustrate that sometimes the solution must be left in terms of an integral. This is usually at most a slight inconvenience, rather than a serious obstacle. For a given value of t, the integral in Eq. (47) is a definite integral and can be approximated to any desired degree of accuracy by using readily available numerical integrators. By repeating this process for many values of t and plotting the results, you can obtain a graph of a solution. Alternatively, you can use a numerical approximation method, such as those discussed in Chapter 8, that proceed directly from the differential equation and need no expression for the solution. Software packages such as Maple and Mathematica readily execute such procedures and produce graphs of solutions of differential equations.

Figure 2.1.4 displays graphs of the solution (47) for several values of c. From the figure it may be plausible to conjecture that all solutions approach a limit as $t \to \infty$. The limit can be found analytically (see Problem 32).

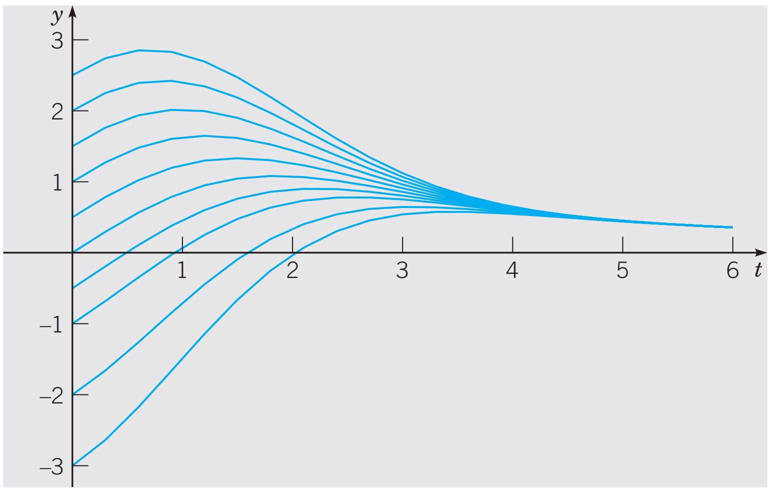

FIGURE 2.1.4 Integral curves of $2y' + ty = 2$.

PROBLEMS

In each of Problems 1 through 12:

(a) Draw a direction field for the given differential equation.

(b) Based on an inspection of the direction field, describe how solutions behave for large t.

(c) Find the general solution of the given differential equation, and use it to determine how solutions behave as $t \to \infty$.

1. $y' + 3y = t + e^{-2t}$
2. $y' - 2y = t^2 e^{2t}$
3. $y' + y = te^{-t} + 1$
4. $y' + (1/t)y = 3\cos 2t, \qquad t > 0$
5. $y' - 2y = 3e^t$
6. $ty' + 2y = \sin t, \qquad t > 0$
7. $y' + 2ty = 2te^{-t^2}$
8. $(1 + t^2)y' + 4ty = (1 + t^2)^{-2}$
9. $2y' + y = 3t$
10. $ty' - y = t^2 e^{-t}, \qquad t > 0$
11. $y' + y = 5\sin 2t$
12. $2y' + y = 3t^2$

In each of Problems 13 through 20, find the solution of the given initial value problem.

13. $y' - y = 2te^{2t}, \qquad y(0) = 1$
14. $y' + 2y = te^{-2t}, \qquad y(1) = 0$
15. $ty' + 2y = t^2 - t + 1, \qquad y(1) = \frac{1}{2}, \quad t > 0$
16. $y' + (2/t)y = (\cos t)/t^2, \qquad y(\pi) = 0, \quad t > 0$
17. $y' - 2y = e^{2t}, \qquad y(0) = 2$
18. $ty' + 2y = \sin t, \qquad y(\pi/2) = 1, \quad t > 0$
19. $t^3 y' + 4t^2 y = e^{-t}, \qquad y(-1) = 0, \quad t < 0$
20. $ty' + (t + 1)y = t, \qquad y(\ln 2) = 1, \quad t > 0$

In each of Problems 21 through 23:

(a) Draw a direction field for the given differential equation. How do solutions appear to behave as t becomes large? Does the behavior depend on the choice of the initial value a? Let a_0 be the value of a for which the transition from one type of behavior to another occurs. Estimate the value of a_0.

(b) Solve the initial value problem and find the critical value a_0 exactly.

(c) Describe the behavior of the solution corresponding to the initial value a_0.

21. $y' - \frac{1}{2}y = 2\cos t, \qquad y(0) = a$
22. $2y' - y = e^{t/3}, \qquad y(0) = a$
23. $3y' - 2y = e^{-\pi t/2}, \qquad y(0) = a$

In each of Problems 24 through 26:

(a) Draw a direction field for the given differential equation. How do solutions appear to behave as $t \to 0$? Does the behavior depend on the choice of the initial value a? Let a_0 be the value of a for which the transition from one type of behavior to another occurs. Estimate the value of a_0.

(b) Solve the initial value problem and find the critical value a_0 exactly.

(c) Describe the behavior of the solution corresponding to the initial value a_0.

24. $ty' + (t + 1)y = 2te^{-t}, \qquad y(1) = a, \quad t > 0$
25. $ty' + 2y = (\sin t)/t, \qquad y(-\pi/2) = a, \quad t < 0$
26. $(\sin t)y' + (\cos t)y = e^t, \qquad y(1) = a, \quad 0 < t < \pi$
27. Consider the initial value problem

$$y' + \tfrac{1}{2}y = 2\cos t, \qquad y(0) = -1.$$

Find the coordinates of the first local maximum point of the solution for $t > 0$.

28. Consider the initial value problem

$$y' + \tfrac{2}{3}y = 1 - \tfrac{1}{2}t, \qquad y(0) = y_0.$$

Find the value of y_0 for which the solution touches, but does not cross, the t-axis.

29. Consider the initial value problem

$$y' + \tfrac{1}{4}y = 3 + 2\cos 2t, \qquad y(0) = 0.$$

(a) Find the solution of this initial value problem and describe its behavior for large t.
(b) Determine the value of t for which the solution first intersects the line $y = 12$.

30. Find the value of y_0 for which the solution of the initial value problem

$$y' - y = 1 + 3\sin t, \qquad y(0) = y_0$$

remains finite as $t \to \infty$.

31. Consider the initial value problem

$$y' - \tfrac{3}{2}y = 3t + 2e^t, \qquad y(0) = y_0.$$

Find the value of y_0 that separates solutions that grow positively as $t \to \infty$ from those that grow negatively. How does the solution that corresponds to this critical value of y_0 behave as $t \to \infty$?

32. Show that all solutions of $2y' + ty = 2$ [Eq. (41) of the text] approach a limit as $t \to \infty$, and find the limiting value.
Hint: Consider the general solution, Eq. (47), and use L'Hôpital's rule on the first term.

33. Show that if a and λ are positive constants, and b is any real number, then every solution of the equation

$$y' + ay = be^{-\lambda t}$$

has the property that $y \to 0$ as $t \to \infty$.
Hint: Consider the cases $a = \lambda$ and $a \neq \lambda$ separately.

In each of Problems 34 through 37, construct a first order linear differential equation whose solutions have the required behavior as $t \to \infty$. Then solve your equation and confirm that the solutions do indeed have the specified property.

34. All solutions have the limit 3 as $t \to \infty$.
35. All solutions are asymptotic to the line $y = 3 - t$ as $t \to \infty$.
36. All solutions are asymptotic to the line $y = 2t - 5$ as $t \to \infty$.
37. All solutions approach the curve $y = 4 - t^2$ as $t \to \infty$.
38. **Variation of Parameters.** Consider the following method of solving the general linear equation of first order:

$$y' + p(t)y = g(t). \tag{i}$$

(a) If $g(t) = 0$ for all t, show that the solution is

$$y = A\exp\left[-\int p(t)\,dt\right], \tag{ii}$$

where A is a constant.
(b) If $g(t)$ is not everywhere zero, assume that the solution of Eq. (i) is of the form

$$y = A(t)\exp\left[-\int p(t)\,dt\right], \tag{iii}$$

where A is now a function of t. By substituting for y in the given differential equation, show that $A(t)$ must satisfy the condition

$$A'(t) = g(t) \exp\left[\int p(t)\,dt\right]. \tag{iv}$$

(c) Find $A(t)$ from Eq. (iv). Then substitute for $A(t)$ in Eq. (iii) and determine y. Verify that the solution obtained in this manner agrees with that of Eq. (33) in the text. This technique is known as the method of **variation of parameters**; it is discussed in detail in Section 3.6 in connection with second order linear equations.

In each of Problems 39 through 42, use the method of Problem 38 to solve the given differential equation.

39. $y' - 2y = t^2 e^{2t}$
40. $y' + (1/t)y = 3\cos 2t, \qquad t > 0$
41. $ty' + 2y = \sin t, \qquad t > 0$
42. $2y' + y = 3t^2$

2.2 Separable Equations

In Section 1.2 we used a process of direct integration to solve first order linear equations of the form

$$\frac{dy}{dt} = ay + b, \tag{1}$$

where a and b are constants. We will now show that this process is actually applicable to a much larger class of equations.

We will use x, rather than t, to denote the independent variable in this section for two reasons. In the first place, different letters are frequently used for the variables in a differential equation, and you should not become too accustomed to using a single pair. In particular, x often occurs as the independent variable. Further, we want to reserve t for another purpose later in the section.

The general first order equation is

$$\frac{dy}{dx} = f(x, y). \tag{2}$$

Linear equations were considered in the preceding section, but if Eq. (2) is nonlinear, then there is no universally applicable method for solving the equation. Here, we consider a subclass of first order equations that can be solved by direct integration.

To identify this class of equations, we first rewrite Eq. (2) in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0. \tag{3}$$

It is always possible to do this by setting $M(x, y) = -f(x, y)$ and $N(x, y) = 1$, but there may be other ways as well. If it happens that M is a function of x only and N is a function of y only, then Eq. (3) becomes

$$M(x) + N(y)\frac{dy}{dx} = 0. \tag{4}$$