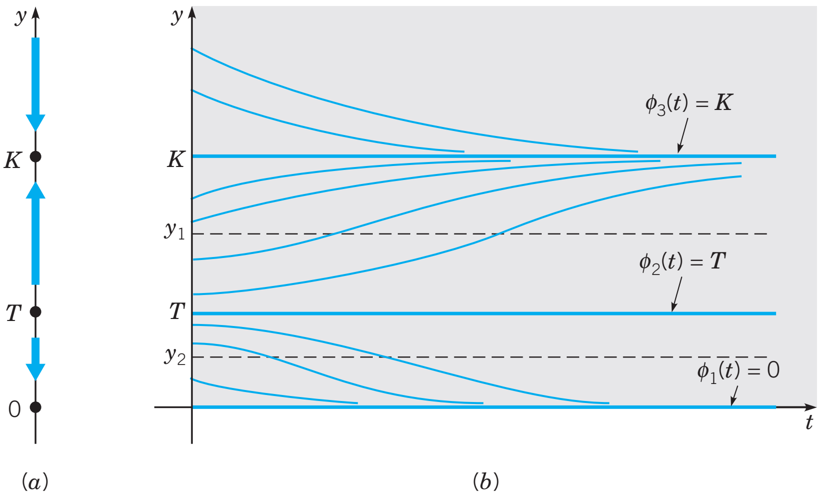

FIGURE 2.5.8 Logistic growth with a threshold: $dy/dt = -r(1 - y/T)(1 - y/K)y$. ($a$) The phase line. ($b$) Plots of y versus t.

A model of this general sort apparently describes the population of the passenger pigeon,[13] which was present in the United States in vast numbers until late in the nineteenth century. It was heavily hunted for food and for sport, and consequently its numbers were drastically reduced by the 1880s. Unfortunately, the passenger pigeon could apparently breed successfully only when present in a large concentration, corresponding to a relatively high threshold T. Although a reasonably large number of individual birds remained alive in the late 1880s, there were not enough in any one place to permit successful breeding, and the population rapidly declined to extinction. The last survivor died in 1914. The precipitous decline in the passenger pigeon population from huge numbers to extinction in a few decades was one of the early factors contributing to a concern for conservation in this country.

PROBLEMS

Problems 1 through 6 involve equations of the form $dy/dt = f(y)$. In each problem sketch the graph of $f(y)$ versus y, determine the critical (equilibrium) points, and classify each one as asymptotically stable or unstable. Draw the phase line, and sketch several graphs of solutions in the ty-plane.

1. $dy/dt = ay + by^2, \qquad a > 0, \quad b > 0, \quad y_0 \geq 0$
2. $dy/dt = ay + by^2, \qquad a > 0, \quad b > 0, \quad -\infty < y_0 < \infty$
3. $dy/dt = y(y-1)(y-2), \qquad y_0 \geq 0$
4. $dy/dt = e^y - 1, \qquad -\infty < y_0 < \infty$
5. $dy/dt = e^{-y} - 1, \qquad -\infty < y_0 < \infty$
6. $dy/dt = -2(\arctan y)/(1 + y^2), \qquad -\infty < y_0 < \infty$
7. **Semistable Equilibrium Solutions.** Sometimes a constant equilibrium solution has the property that solutions lying on one side of the equilibrium solution tend to approach it,

[13] See, for example, Oliver L. Austin, Jr., *Birds of the World* (New York: Golden Press, 1983), pp. 143–145.

whereas solutions lying on the other side depart from it (see Figure 2.5.9). In this case the equilibrium solution is said to be **semistable**.

(a) Consider the equation

$$dy/dt = k(1-y)^2, \tag{i}$$

where k is a positive constant. Show that $y = 1$ is the only critical point, with the corresponding equilibrium solution $\phi(t) = 1$.

(b) Sketch $f(y)$ versus y. Show that y is increasing as a function of t for $y < 1$ and also for $y > 1$. The phase line has upward-pointing arrows both below and above $y = 1$. Thus solutions below the equilibrium solution approach it, and those above it grow farther away. Therefore, $\phi(t) = 1$ is semistable.

(c) Solve Eq. (i) subject to the initial condition $y(0) = y_0$ and confirm the conclusions reached in part (b).

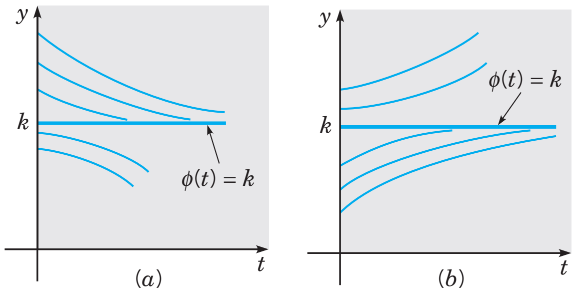

FIGURE 2.5.9 In both cases the equilibrium solution $\phi(t) = k$ is semistable. (a) $dy/dt \le 0$; (b) $dy/dt \ge 0$.

Problems 8 through 13 involve equations of the form $dy/dt = f(y)$. In each problem sketch the graph of $f(y)$ versus y, determine the critical (equilibrium) points, and classify each one asymptotically stable, unstable, or semistable (see Problem 7). Draw the phase line, and sketch several graphs of solutions in the ty-plane.

8. $dy/dt = -k(y-1)^2, \quad k > 0, \quad -\infty < y_0 < \infty$
9. $dy/dt = y^2(y^2-1), \quad -\infty < y_0 < \infty$
10. $dy/dt = y(1-y^2), \quad -\infty < y_0 < \infty$
11. $dy/dt = ay - b\sqrt{y}, \quad a > 0, \quad b > 0, \quad y_0 \ge 0$
12. $dy/dt = y^2(4-y^2), \quad -\infty < y_0 < \infty$
13. $dy/dt = y^2(1-y)^2, \quad -\infty < y_0 < \infty$
14. Consider the equation $dy/dt = f(y)$ and suppose that y_1 is a critical point—that is, $f(y_1) = 0$. Show that the constant equilibrium solution $\phi(t) = y_1$ is asymptotically stable if $f'(y_1) < 0$ and unstable if $f'(y_1) > 0$.
15. Suppose that a certain population obeys the logistic equation $dy/dt = ry[1-(y/K)]$.

    (a) If $y_0 = K/3$, find the time τ at which the initial population has doubled. Find the value of τ corresponding to $r = 0.025$ per year.

    (b) If $y_0/K = \alpha$, find the time T at which $y(T)/K = \beta$, where $0 < \alpha, \beta < 1$. Observe that $T \to \infty$ as $\alpha \to 0$ or as $\beta \to 1$. Find the value of T for $r = 0.025$ per year, $\alpha = 0.1$, and $\beta = 0.9$.

16. Another equation that has been used to model population growth is the Gompertz[14] equation

$$dy/dt = ry\ln(K/y),$$

where r and K are positive constants.

(a) Sketch the graph of $f(y)$ versus y, find the critical points, and determine whether each is asymptotically stable or unstable.

(b) For $0 \le y \le K$, determine where the graph of y versus t is concave up and where it is concave down.

(c) For each y in $0 < y \le K$, show that dy/dt as given by the Gompertz equation is never less than dy/dt as given by the logistic equation.

17. (a) Solve the Gompertz equation

$$dy/dt = ry\ln(K/y),$$

subject to the initial condition $y(0) = y_0$.
Hint: You may wish to let $u = \ln(y/K)$.

(b) For the data given in Example 1 in the text ($r = 0.71$ per year, $K = 80.5 \times 10^6$ kg, $y_0/K = 0.25$), use the Gompertz model to find the predicted value of $y(2)$.

(c) For the same data as in part (b), use the Gompertz model to find the time τ at which $y(\tau) = 0.75K$.

18. A pond forms as water collects in a conical depression of radius a and depth h. Suppose that water flows in at a constant rate k and is lost through evaporation at a rate proportional to the surface area.

(a) Show that the volume $V(t)$ of water in the pond at time t satisfies the differential equation

$$dV/dt = k - \alpha\pi(3a/\pi h)^{2/3}V^{2/3},$$

where α is the coefficient of evaporation.

(b) Find the equilibrium depth of water in the pond. Is the equilibrium asymptotically stable?

(c) Find a condition that must be satisfied if the pond is not to overflow.

19. Consider a cylindrical water tank of constant cross section A. Water is pumped into the tank at a constant rate k and leaks out through a small hole of area a in the bottom of the tank. From Torricelli's principle in hydrodynamics (see Problem 6 in Section 2.3) it follows that the rate at which water flows through the hole is $\alpha a\sqrt{2gh}$, where h is the current depth of water in the tank, g is the acceleration due to gravity, and α is a contraction coefficient that satisfies $0.5 \le \alpha \le 1.0$.

(a) Show that the depth of water in the tank at any time satisfies the equation

$$dh/dt = (k - \alpha a\sqrt{2gh})/A.$$

(b) Determine the equilibrium depth h_e of water, and show that it is asymptotically stable. Observe that h_e does not depend on A.

[14] Benjamin Gompertz (1779–1865) was an English actuary. He developed his model for population growth, published in 1825, in the course of constructing mortality tables for his insurance company.

Harvesting a Renewable Resource. Suppose that the population y of a certain species of fish (for example, tuna or halibut) in a given area of the ocean is described by the logistic equation

$$dy/dt = r(1 - y/K)y.$$

Although it is desirable to utilize this source of food, it is intuitively clear that if too many fish are caught, then the fish population may be reduced below a useful level and possibly even driven to extinction. Problems 20 and 21 explore some of the questions involved in formulating a rational strategy for managing the fishery.[15]

20. At a given level of effort, it is reasonable to assume that the rate at which fish are caught depends on the population y: the more fish there are, the easier it is to catch them. Thus we assume that the rate at which fish are caught is given by Ey, where E is a positive constant, with units of 1/time, that measures the total effort made to harvest the given species of fish. To include this effect, the logistic equation is replaced by

$$dy/dt = r(1 - y/K)y - Ey. \tag{i}$$

This equation is known as the **Schaefer model** after the biologist M. B. Schaefer, who applied it to fish populations.

(a) Show that if $E < r$, then there are two equilibrium points, $y_1 = 0$ and $y_2 = K(1 - E/r) > 0$.

(b) Show that $y = y_1$ is unstable and $y = y_2$ is asymptotically stable.

(c) A sustainable yield Y of the fishery is a rate at which fish can be caught indefinitely. It is the product of the effort E and the asymptotically stable population y_2. Find Y as a function of the effort E; the graph of this function is known as the yield–effort curve.

(d) Determine E so as to maximize Y and thereby find the **maximum sustainable yield** Y_m.

21. In this problem we assume that fish are caught at a constant rate h independent of the size of the fish population. Then y satisfies

$$dy/dt = r(1 - y/K)y - h. \tag{i}$$

The assumption of a constant catch rate h may be reasonable when y is large but becomes less so when y is small.

(a) If $h < rK/4$, show that Eq. (i) has two equilibrium points y_1 and y_2 with $y_1 < y_2$; determine these points.

(b) Show that y_1 is unstable and y_2 is asymptotically stable.

(c) From a plot of $f(y)$ versus y, show that if the initial population $y_0 > y_1$, then $y \to y_2$ as $t \to \infty$, but that if $y_0 < y_1$, then y decreases as t increases. Note that $y = 0$ is not an equilibrium point, so if $y_0 < y_1$, then extinction will be reached in a finite time.

(d) If $h > rK/4$, show that y decreases to zero as t increases, regardless of the value of y_0.

(e) If $h = rK/4$, show that there is a single equilibrium point $y = K/2$ and that this point is semistable (see Problem 7). Thus the maximum sustainable yield is $h_m = rK/4$, corresponding to the equilibrium value $y = K/2$. Observe that h_m has the same value as Y_m in Problem 20(d). The fishery is considered to be overexploited if y is reduced to a level below $K/2$.

[15] An excellent treatment of this kind of problem, which goes far beyond what is outlined here, may be found in the book by Clark mentioned previously, especially in the first two chapters. Numerous additional references are given there.

Epidemics. The use of mathematical methods to study the spread of contagious diseases goes back at least to some work by Daniel Bernoulli in 1760 on smallpox. In more recent years many mathematical models have been proposed and studied for many different diseases.[16] Problems 22 through 24 deal with a few of the simpler models and the conclusions that can be drawn from them. Similar models have also been used to describe the spread of rumors and of consumer products.

22. Suppose that a given population can be divided into two parts: those who have a given disease and can infect others, and those who do not have it but are susceptible. Let x be the proportion of susceptible individuals and y the proportion of infectious individuals; then $x + y = 1$. Assume that the disease spreads by contact between sick and well members of the population and that the rate of spread dy/dt is proportional to the number of such contacts. Further, assume that members of both groups move about freely among each other, so the number of contacts is proportional to the product of x and y. Since $x = 1 - y$, we obtain the initial value problem

$$dy/dt = \alpha y(1 - y), \qquad y(0) = y_0, \tag{i}$$

where α is a positive proportionality factor, and y_0 is the initial proportion of infectious individuals.

(a) Find the equilibrium points for the differential equation (i) and determine whether each is asymptotically stable, semistable, or unstable.

(b) Solve the initial value problem (i) and verify that the conclusions you reached in part (a) are correct. Show that $y(t) \to 1$ as $t \to \infty$, which means that ultimately the disease spreads through the entire population.

23. Some diseases (such as typhoid fever) are spread largely by *carriers*, individuals who can transmit the disease but who exhibit no overt symptoms. Let x and y denote the proportions of susceptibles and carriers, respectively, in the population. Suppose that carriers are identified and removed from the population at a rate β, so

$$dy/dt = -\beta y. \tag{i}$$

Suppose also that the disease spreads at a rate proportional to the product of x and y; thus

$$dx/dt = -\alpha xy. \tag{ii}$$

(a) Determine y at any time t by solving Eq. (i) subject to the initial condition $y(0) = y_0$.

(b) Use the result of part (a) to find x at any time t by solving Eq. (ii) subject to the initial condition $x(0) = x_0$.

(c) Find the proportion of the population that escapes the epidemic by finding the limiting value of x as $t \to \infty$.

24. Daniel Bernoulli's work in 1760 had the goal of appraising the effectiveness of a controversial inoculation program against smallpox, which at that time was a major threat to public health. His model applies equally well to any other disease that, once contracted and survived, confers a lifetime immunity.

Consider the cohort of individuals born in a given year ($t = 0$), and let $n(t)$ be the number of these individuals surviving t years later. Let $x(t)$ be the number of members of this cohort who have not had smallpox by year t and who are therefore still susceptible. Let β be the rate at which susceptibles contract smallpox, and let ν be the rate at which

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[16] A standard source is the book by Bailey listed in the references. The models in Problems 22, 23, and 24 are discussed by Bailey in Chapters 5, 10, and 20, respectively.

people who contract smallpox die from the disease. Finally, let $\mu(t)$ be the death rate from all causes other than smallpox. Then dx/dt, the rate at which the number of susceptibles declines, is given by

$$dx/dt = -[\beta + \mu(t)]x. \tag{i}$$

The first term on the right side of Eq. (i) is the rate at which susceptibles contract smallpox, and the second term is the rate at which they die from all other causes. Also

$$dn/dt = -\nu\beta x - \mu(t)n, \tag{ii}$$

where dn/dt is the death rate of the entire cohort, and the two terms on the right side are the death rates due to smallpox and to all other causes, respectively.

(a) Let $z = x/n$, and show that z satisfies the initial value problem

$$dz/dt = -\beta z(1 - \nu z), \qquad z(0) = 1. \tag{iii}$$

Observe that the initial value problem (iii) does not depend on $\mu(t)$.

(b) Find $z(t)$ by solving Eq. (iii).

(c) Bernoulli estimated that $\nu = \beta = \frac{1}{8}$. Using these values, determine the proportion of 20-year-olds who have not had smallpox.

Note: On the basis of the model just described and the best mortality data available at the time, Bernoulli calculated that if deaths due to smallpox could be eliminated ($\nu = 0$), then approximately 3 years could be added to the average life expectancy (in 1760) of 26 years, 7 months. He therefore supported the inoculation program.

Bifurcation Points. For an equation of the form

$$dy/dt = f(a, y), \tag{i}$$

where a is a real parameter, the critical points (equilibrium solutions) usually depend on the value of a. As a steadily increases or decreases, it often happens that at a certain value of a, called a **bifurcation point**, critical points come together, or separate, and equilibrium solutions may be either lost or gained. Bifurcation points are of great interest in many applications, because near them the nature of the solution of the underlying differential equation is undergoing an abrupt change. For example, in fluid mechanics a smooth (laminar) flow may break up and become turbulent. Or an axially loaded column may suddenly buckle and exhibit a large lateral displacement. Or, as the amount of one of the chemicals in a certain mixture is increased, spiral wave patterns of varying color may suddenly emerge in an originally quiescent fluid. Problems 25 through 27 describe three types of bifurcations that can occur in simple equations of the form (i).

25. Consider the equation

$$dy/dt = a - y^2. \tag{ii}$$

(a) Find all of the critical points for Eq. (ii). Observe that there are no critical points if $a < 0$, one critical point if $a = 0$, and two critical points if $a > 0$.

(b) Draw the phase line in each case and determine whether each critical point is asymptotically stable, semistable, or unstable.

(c) In each case sketch several solutions of Eq. (ii) in the ty-plane.

(d) If we plot the location of the critical points as a function of a in the ay-plane, we obtain Figure 2.5.10. This is called the **bifurcation diagram** for Eq. (ii). The bifurcation at $a = 0$ is called a **saddle–node** bifurcation. This name is more natural in the context of second order systems, which are discussed in Chapter 9.

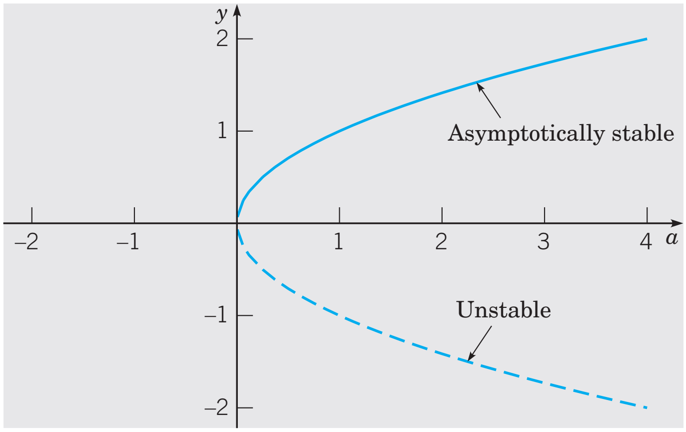

FIGURE 2.5.10 Bifurcation diagram for $y' = a - y^2$.

26. Consider the equation

$$dy/dt = ay - y^3 = y(a - y^2). \tag{iii}$$

(a) Again consider the cases $a < 0$, $a = 0$, and $a > 0$. In each case find the critical points, draw the phase line, and determine whether each critical point is asymptotically stable, semistable, or unstable.

(b) In each case sketch several solutions of Eq. (iii) in the ty-plane.

(c) Draw the bifurcation diagram for Eq. (iii)—that is, plot the location of the critical points versus a. For Eq. (iii) the bifurcation point at $a = 0$ is called a **pitchfork bifurcation**. Your diagram may suggest why this name is appropriate.

27. Consider the equation

$$dy/dt = ay - y^2 = y(a - y). \tag{iv}$$

(a) Again consider the cases $a < 0$, $a = 0$, and $a > 0$. In each case find the critical points, draw the phase line, and determine whether each critical point is asymptotically stable, semistable, or unstable.

(b) In each case sketch several solutions of Eq. (iv) in the ty-plane.

(c) Draw the bifurcation diagram for Eq. (iv). Observe that for Eq. (iv) there are the same number of critical points for $a < 0$ and $a > 0$ but that their stability has changed. For $a < 0$ the equilibrium solution $y = 0$ is asymptotically stable and $y = a$ is unstable, while for $a > 0$ the situation is reversed. Thus there has been an **exchange of stability** as a passes through the bifurcation point $a = 0$. This type of bifurcation is called a **transcritical bifurcation**.

28. **Chemical Reactions.** A second order chemical reaction involves the interaction (collision) of one molecule of a substance P with one molecule of a substance Q to produce one molecule of a new substance X; this is denoted by $P + Q \to X$. Suppose that p and q, where $p \neq q$, are the initial concentrations of P and Q, respectively, and let $x(t)$ be the concentration of X at time t. Then $p - x(t)$ and $q - x(t)$ are the concentrations of P and Q at time t, and the rate at which the reaction occurs is given by the equation

$$dx/dt = \alpha(p - x)(q - x), \tag{i}$$

where α is a positive constant.

(a) If $x(0) = 0$, determine the limiting value of $x(t)$ as $t \to \infty$ without solving the differential equation. Then solve the initial value problem and find $x(t)$ for any t.

(b) If the substances P and Q are the same, then $p = q$ and Eq. (i) is replaced by

$$dx/dt = \alpha(p - x)^2. \tag{ii}$$

If $x(0) = 0$, determine the limiting value of $x(t)$ as $t \to \infty$ without solving the differential equation. Then solve the initial value problem and determine $x(t)$ for any t.

2.6 Exact Equations and Integrating Factors

For first order equations there are a number of integration methods that are applicable to various classes of problems. The most important of these are linear equations and separable equations, which we have discussed previously. Here, we consider a class of equations known as exact equations for which there is also a well-defined method of solution. Keep in mind, however, that those first order equations that can be solved by elementary integration methods are rather special; most first order equations cannot be solved in this way.

EXAMPLE 1

Solve the differential equation

$$2x + y^2 + 2xyy' = 0. \tag{1}$$

The equation is neither linear nor separable, so the methods suitable for those types of equations are not applicable here. However, observe that the function $\psi(x, y) = x^2 + xy^2$ has the property that

$$2x + y^2 = \frac{\partial\psi}{\partial x}, \qquad 2xy = \frac{\partial\psi}{\partial y}. \tag{2}$$

Therefore, the differential equation can be written as

$$\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

Assuming that y is a function of x, we can use the chain rule to write the left side of Eq. (3) as $d\psi(x, y)/dx$. Then Eq. (3) has the form

$$\frac{d\psi}{dx}(x, y) = \frac{d}{dx}(x^2 + xy^2) = 0. \tag{4}$$

By integrating Eq. (4) we obtain

$$\psi(x, y) = x^2 + xy^2 = c, \tag{5}$$

where c is an arbitrary constant. The level curves of $\psi(x, y)$ are the integral curves of Eq. (1). Solutions of Eq. (1) are defined implicitly by Eq. (5).

In solving Eq. (1) the key step was the recognition that there is a function ψ that satisfies Eqs. (2). More generally, let the differential equation

$$M(x, y) + N(x, y)y' = 0 \tag{6}$$