CHAPTER

4

Higher Order Linear Equations

4.1

2. We will first rewrite the equation as $y''' + (\sin t/t)y'' + (4/t)y = \cos t/t$. Since the coefficient functions $p_1(t) = \sin t/t$, $p_2(t) = 4/t$ and $g(t) = \cos t/t$ are continuous for all $t \neq 0$, the solution is sure to exist in the intervals $(-\infty, 0)$ and $(0, \infty)$.

4. The coefficients are continuous everywhere, but the function $g(t) = 2\ln t$ is defined and continuous only on the interval $(0, \infty)$. Hence solutions are defined for positive reals.

8. We have

$$W(f_1, f_2, f_3) = \begin{vmatrix} 2t-3 & 4t^2+2 & 3t^2+t \\ 2 & 8t & 6t+1 \\ 0 & 8 & 6 \end{vmatrix} = 0$$

for all t. Thus by the extension of Theorem 3.3.1 the given functions are linearly dependent. To find a linear relation we have $c_1(2t-3) + c_2(4t^2+2) + c_3(3t^2+t) = (4c_2+3c_3)t^2 + (2c_1+c_3)t + (-3c_1+2c_2) = 0$, which is zero when the coefficients are zero. Solving, we find $c_1 = 1$, $c_2 = 3/2$ and $c_3 = -2$. This implies that $(2t-3) + (3/2)(4t^2+2) - 2(3t^2+t) = 0$.

13. By direct substitution, for $y_1 = e^t$ we get $y_1''' - 3y_1'' - y_1' + 3y_1 = e^t - 3e^t - e^t + 3e^t = 0$, for $y_2 = e^{-t}$ we get $y_2''' - 3y_2'' - y_2' + 3y_2 = -e^{-t} - 3e^{-t} + e^{-t} + 3e^{-t} = 0$ and for $y_3 = e^{3t}$ we get $y_3''' - 3y_3'' - y_3' + 3y_3 = 27e^{3t} - 27e^{3t} - 3e^{3t} + 3e^{3t} = 0$.

Therefore, y_1, y_2, y_3 are all solutions of the differential equation. We now compute their Wronskian. We have

$$W(y_1, y_2, y_3) = \begin{vmatrix} e^t & e^{-t} & e^{3t} \\ e^t & -e^{-t} & 3e^{3t} \\ e^t & e^{-t} & 9e^{3t} \end{vmatrix} = e^{3t} \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 1 & 1 & 9 \end{vmatrix} = -16e^{3t}.$$

17. We note first that $(\cos^2 t)' = -2\sin t \cos t = -\sin 2t$. Then

$$W(5, \cos^2 t, \cos 2t) = \begin{vmatrix} 5 & \cos^2 t & \cos 2t \\ 0 & -\sin 2t & -2\sin 2t \\ 0 & -2\cos 2t & -4\cos 2t \end{vmatrix} = 5(4\sin 2t\cos 2t - 4\cos 2t \sin 2t) = 0.$$

Also, $\cos^2 t = (1 + \cos 2t)/2 = (1/10)5 + (1/2)\cos 2t$ and hence $\cos^2 t$ is a linear combination of 5 and $\cos 2t$. Thus the functions are linearly dependent and their Wronskian is zero.

19.(a) Note that $d^k(t^n)/dt^k = n(n-1)\ldots(n-k+1)t^{n-k}$, for $k = 1, 2, \ldots, n$. Thus $L[t^n] = a_0\, n! + a_1\,[n(n-1)\ldots 2]\, t + \ldots\, a_{n-1}\, n\, t^{n-1} + a_n\, t^n$.

(b) We have $d^k(e^{rt})/dt^k = r^k e^{rt}$, for $k = 0, 1, 2, \ldots$. Hence $L[e^{rt}] = a_0\, r^n e^{rt} + a_1 r^{n-1} e^{rt} + \ldots + a_{n-1} r\, e^{rt} + a_n\, e^{rt} = [a_0\, r^n + a_1 r^{n-1} + \ldots + a_{n-1}\, r\, + a_n\,]e^{rt}$.

(c) Set $y = e^{rt}$, and substitute into the ODE. It follows that $r^4 - 5r^2 + 4 = 0$, with $r = \pm 1, \pm 2$. Furthermore, $W(e^t, e^{-t}, e^{2t}, e^{-2t}) = 72$.

23. After writing the equation in standard form, observe that $p_1(t) = 2/t$. Based on the results in Problem 20, we find that $W' = (-2/t)W$, and hence $W = c/t^2$.

25.(a) On the interval $(-1, 0)$, $f(t) = t^2|t| = -t^3 = -g(t)$, and on the interval $(0, 1)$, $f(t) = t^2|t| = t^3 = g(t)$. This shows that on these intervals the functions are linearly dependent.

(b) On the interval $(-1, 1)$ these two functions are linearly independent, because if $c_1 f(t) + c_2 g(t) = 0$ for every t, then for $t = 1/2$ we obtain $c_1 + c_2 = 0$ and for $t = -1/2$ we get $c_1 - c_2 = 0$, which implies that $c_1 = c_2 = 0$.

(c) The Wronskian is

$$W(f, g)(t) = \begin{vmatrix} t^2|t| & t^3 \\ 3t|t| & 3t^2 \end{vmatrix} = 3t^4|t| - 3t^4|t| = 0.$$

27. Differentiating e^t and substituting into the differential equation we verify that $y = e^t$ is a solution: $(2-t)e^t + (2t-3)e^t - te^t + e^t = 0$. Now, as in Problem 26, we let $y = v(t)e^t$. Differentiating three times and substituting into the differential equation yields $(2-t)e^t v''' + (3-t)e^t v'' = 0$. Dividing by $(2-t)e^t$ and letting $w = v''$ we obtain the first order separable equation $w' = -(t-3)w/(t-2) = (-1 + 1/(t-2))w$. Separating t and w, integrating, and then solving for w yields $w = v'' = c_1(t-2)e^{-t}$. Integrating this twice the gives $v = c_1 te^{-t} + c_2 t + c_3$ so that

$y = ve^t = c_1 t + c_2 te^t + c_3 e^t$, which is the complete solution, since it contains the given $y_1(t)$ and three constants.

4.2

2. The magnitude of $-2+2\sqrt{3}\,i$ is $R = \sqrt{16}\ = 4$ and the polar angle is $2\pi/3$. Hence the polar form is given by $-2+2\sqrt{3}\,i = 4\,e^{2\pi/3\,i}$. The angle θ is only determined up to an additive integer multiple of 2π.

8. Writing $1+i$ in the form $Re^{i\theta}$, we have $R=\sqrt{2}$ and $\theta = \pi/4$. Thus $1+i = \sqrt{2}\,e^{i(\pi/4+2m\pi)}$ (where m is any integer), and hence $(1+i)^{1/2} = \sqrt[4]{2}\,e^{i(\pi/8+m\pi)}$. We obtain the two square roots by setting $m=0,1$. They are $\sqrt[4]{2}\,e^{i\pi/8}$ and $\sqrt[4]{2}\,e^{i9\pi/8}$.

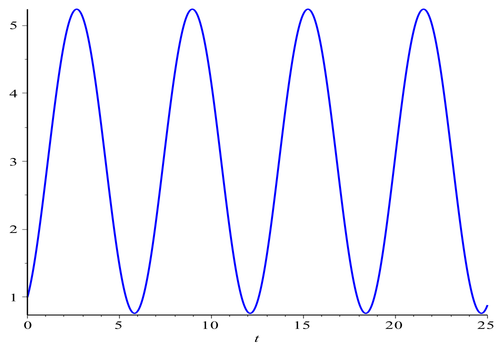
12. The characteristic equation is $r^3-6r^2+12r-8=(r-2)^3=0$. The roots are $r=2,2,2$. The roots are repeated, hence $y = c_1e^{2t}+c_2te^{2t}+c_3t^2e^{2t}$.

15. The characteristic equation is $r^6+1=0$. The roots are given by $r=(-1)^{1/6}$, that is, the six sixth roots of -1. They are $e^{-\pi i/6+m\pi i/3}$, $m=0,1,\ldots,5$. Explicitly, $r=(\sqrt{3}-i)/2$, $(\sqrt{3}+i)/2$, i, $-i$, $(-\sqrt{3}+i)/2$, $(-\sqrt{3}-i)/2$. Note that there are three pairs of conjugate roots. Thus $y = e^{\sqrt{3}\,t/2}\,[c_1\cos(t/2)+c_2\sin(t/2)] + c_3\cos t + c_4\sin te^{-\sqrt{3}\,t/2}\,[c_5\cos(t/2)+c_6\sin(t/2)]$.

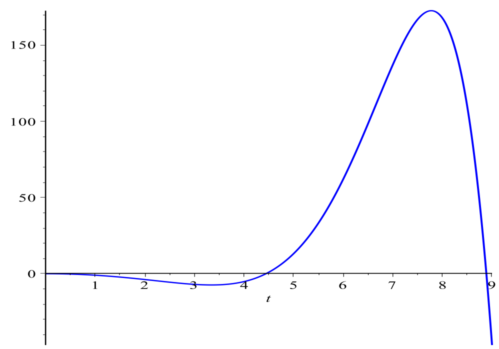
23. The characteristic equation is $r^3-3r^2+r+1=0$. Using the procedure suggested following Eq.(12) we try, since $a_n=a_0=1$, $r=1$ as a root and find that indeed it is. Factoring out $r-1$ we are then left with $r^2-2r-1=0$, which has the roots $1\pm\sqrt{2}$. Hence the general solution is $y=c_1e^t+c_2e^{(1+\sqrt{2})t}+c_3e^{(1-\sqrt{2})t}$.

27. The characteristic equation is $12r^4+31r^3+75r^2+37r+5=0$. It can be shown (with the aid of a mathematical software) that $12r^4+31r^3+75r^2+37r+5=(3r+1)(4r+1)(r^2+2r+5)$. This implies that the roots are $r=-1/3$, $-1/4$, and $-1\pm2i$. The solution is $y=c_1e^{-t/3}+c_2e^{-t/4}+c_3e^{-t}\cos 2t+c_4e^{-t}\sin 2t$.

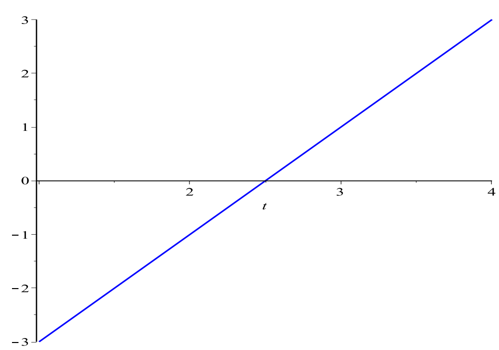
29. The characteristic equation is $r^3+r=0$, with roots $r=0$, $\pm i$. Hence the general solution is $y(t)=c_1+c_2\cos t+c_3\sin t$. Invoking the initial conditions, we obtain the system of equations $c_1+c_2=1$, $c_3=1$, $-c_2=2$, with solution $c_1=3$, $c_2=-2$, $c_3=1$. Therefore the solution of the initial value problem is $y(t)=3-2\cos t+\sin t$, which oscillates about $y=3$ as $t\to\infty$.



30. The characteristic equation is $r^4+1=0$, with roots $r=\pm\sqrt{2}/2\pm i\sqrt{2}/2$, Hence the general solution is $y(t)=c_1e^{\sqrt{2}t/2}\cos(\sqrt{2}t/2)+c_2e^{\sqrt{2}t/2}\sin(\sqrt{2}t/2)+c_3e^{-\sqrt{2}t/2}\cos(\sqrt{2}t/2)+c_4e^{-\sqrt{2}t/2}\sin(\sqrt{2}t/2)$. Invoking the initial conditions, we obtain that the solution of the initial value problem is $y(t)=-e^{\sqrt{2}t/2}\sin(\sqrt{2}t/2)+e^{-\sqrt{2}t/2}\sin(\sqrt{2}t/2)$, which oscillates with an exponentially growing amplitude as $t\to\infty$.

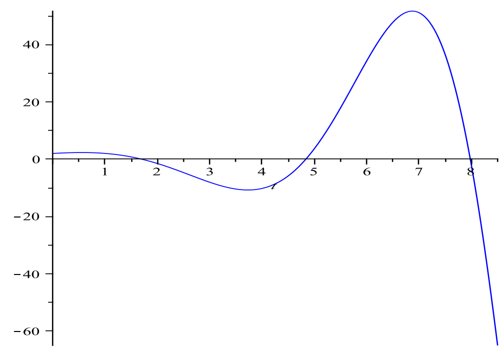


31. The characteristic equation is $r^4-4r^3+r^2=0$, with roots $r=0,\ 0,\ 2,\ 2$. Hence the general solution is $y(t)=c_1+c_2t+c_3e^{2t}+c_4te^{2t}$. Invoking the initial conditions, we obtain that the solution of the initial value problem is $y(t)=-5+2t$, which grows without bound as $t\to\infty$.



34. The characteristic equation is $4r^3+r+5=0$, with roots $r=-1,\ 1/2\pm i$.

Hence the general solution is $y(t) = c_1 e^{-t} + c_2 e^{t/2} \cos t + c_3 e^{t/2} \sin t$. Invoking the initial conditions, we obtain that the solution of the initial value problem is $y(t) = (2/13)e^{-t} + e^{t/2}[(24/13) \cos\, t + (3/13) \sin\, t]$, which oscillates with an exponentially growing amplitude as $t \to \infty$.



37. The approach for solving the differential equation would normally yield $y(t) = c_1 \cos t + c_2 \sin t + c_5 e^t + c_6 e^{-t}$ as the solution. Since $\cosh t = (e^t + e^{-t})/2$ and $\sinh t = (e^t - e^{-t})/2$, $y(t)$ can be written as $y(t) = c_1 \cos t + c_2 \sin t + c_3 \cosh t + c_4 \sinh t$, where $c_3 = c_5 + c_6$ and $c_4 = c_5 - c_6$. It is more convenient to use this form because the initial conditions are given at $t = 0$, and the functions $\cosh t$ and $\sinh t$ and all their derivatives are 0 or 1 at $t = 0$, so the algebra is simplified. If $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$ and $y'''(0) = 1$, then the resulting system of equations is $c_1 + c_3 = 0$, $c_2 + c_4 = 0$, $-c_1 + c_3 = 1$, and $-c_2 + c_4 = 1$, which yields immediately that $c_1 = -1/2$, $c_3 = 1/2$, $c_2 = -1/2$ and $c_4 = 1/2$, so the solution is $y(t) = -(1/2)(\cos t + \sin t) + (1/2)(\cosh t + \sinh t)$

38.(a) Since $p_1(t) = 0$, $W = ce^{-\int 0\, dt} = c$.

(b) $W(e^t, e^{-t}, \cos\, t, \sin\, t) = -8$.

(c) $W(\cosh\, t, \sinh\, t, \cos\, t, \sin\, t) = 4$.

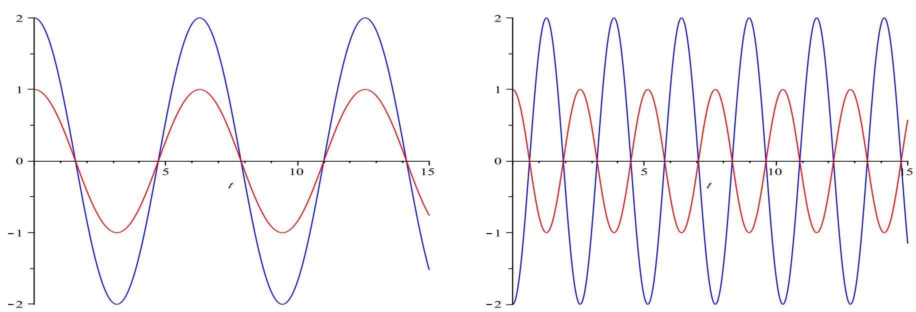
39.(a) As in Section 3.7, the force that the spring designated by k_1 exerts on mass m_1 is $-3u_1$. By an analysis similar to that shown in Section 3.7, the middle spring exerts a force of $-2(u_1 - u_2)$ on mass m_1 and a force of $-2(u_2 - u_1)$ on mass m_2. Thus Newton's law gives $m_1 u_1'' = -3u_1 - 2(u_1 - u_2)$ and $m_2 u_2'' = -2(u_2 - u_1)$, where u_1 and u_2 are measured from their equilibrium positions. Setting the masses equal to 1 and rewriting each equation yields Eq.(i). In all cases the positive direction is taken in the direction shown in Figure 4.2.4.

(b) Clearly, $u_2 = u_1''/2 + (5/2)u_1$, so by differentiating this twice and using the other equation $u_2'' + 2u_2 = 2u_1$ we get that $u_1''''/2 + (5/2)u_1'' + u_1'' + 5u_1 = 2u_1$, which turns into $u_1'''' + 7u_1'' + 6u_1 = 0$ after a multiplication by 2. The characteristic equation is $r^4 + 7r^2 + 6 = 0$, or $(r^2 + 1)(r^2 + 6) = 0$. Thus the general solution of Eq.(ii) is $u_1(t) = c_1 \cos t + c_2 \sin t + c_3 \cos \sqrt{6}t + c_4 \sin \sqrt{6}t$.

(c) We see that $u_1'' = 2u_2 - 5u_1$, so $u_1''(0) = 2\cdot 2 - 5\cdot 1 = -1$ and by differentiating the previous equation, $u_1''' = 2u_2' - 5u_1'$, so $u_1'''(0) = 0$. Substituting these initial conditions into the previous general solution we obtain the solution $u_1(t) = \cos t$. Also, $2u_2 = u_1'' + 5u_1 = 4\cos t$ so $u_2(t) = 2\cos t$.

(d) As in part (c), $u_1'' = 2u_2 - 5u_1$, so $u_1''(0) = 2\cdot 1 - 5\cdot(-2) = 12$ and $u_1''' = 2u_2' - 5u_1'$, so $u_1'''(0) = 0$. Substituting these initial conditions into the general solution we obtain the solution $u_1(t) = -2\cos\sqrt{6}t$. Then $2u_2 = u_1'' + 5u_1 = 2\cos\sqrt{6}t$ so $u_2(t) = \cos\sqrt{6}t$.

(e)



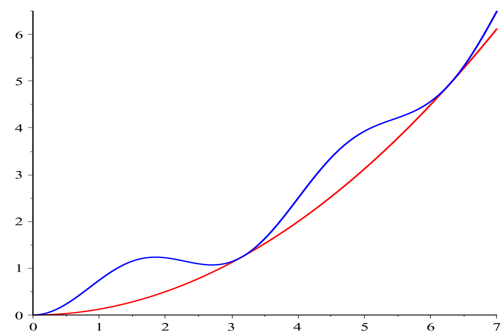
(a) Solutions from part (c) (b) Solutions from part (d)

4.3

1. First solve the homogeneous equation. The characteristic equation for this is $r^3 - r^2 - r + 1 = 0$, the roots are $r = -1, 1, 1$, so $y_c(t) = c_1e^{-t} + c_2e^t + c_3te^t$. Using the superposition principle, we can write a particular solution as the sum of the particular solutions corresponding to the differential equations $y''' - y'' - y' + y = 4e^{-t}$ and $y''' - y'' - y' + y = 3$. Our initial choice for $Y_1(t)$ is Ae^{-t}, but because this is a solution of the homogeneous equation we need $Y_1(t) = Ate^{-t}$. The second equation gives us $Y_2(t) = B$. The constants A and B can be determined by substituting into the individual equations. We obtain $A = 1$ and $B = 3$. Thus the general solution is $y(t) = c_1e^{-t} + c_2e^t + c_3te^t + te^{-t} + 3$.

5. The characteristic equation is $r^4 - 4r^2 = r^2(r^2 - 4) = 0$, so $y_c(t) = c_1 + c_2t + c_3e^{-2t} + c_4e^{2t}$. For the particular solution corresponding to t^2 we assume $Y_1(t) = t^2(At^2 + Bt + C)$ and for the particular solution corresponding to $4e^t$ we assume $Y_2(t) = De^t$. The constants A, B, C, and D can be determined by substituting into the individual equations. We obtain that the general solution is $y(t) = c_1 + c_2t + c_3e^{-2t} + c_4e^{2t} - t^4/48 - t^2/16 - 4e^t/3$.

9. The characteristic equation for the related homogeneous differential equation is $r^3 + 4r = 0$ with roots $r = 0, \pm 2i$. Hence $y_c(t) = c_1 + c_2 \cos 2t + c_3 \sin 2t$. The initial choice for $Y(t)$ is $At + B$, but because B is a solution of the homogeneous equation we assume $Y(t) = t(At + B)$. A and B are found by substituting this into the differential equation, which gives us $A = 1/8$ and $B = 0$. Thus the general solution is $y = c_1 + c_2 \cos 2t + c_3 \sin 2t + t^2/8$. Applying the initial conditions at this point we obtain that $y(0) = c_1 + c_2 = 0$, $y'(0) = 2c_3 = 0$ and $y''(0) = -4c_2 + 1/4 = 2$. This gives $c_2 = -7/16$, $c_1 = 7/16$ and $c_3 = 0$. The solution is $y = 7/16 - (7/16)\cos 2t + t^2/8$. We can see that for $t = \pi, 2\pi, \ldots$ the graph will be tangent to $t^2/8$ and for large t values the graph will be approximated by $t^2/8$.



13. The characteristic equation for the homogeneous equation is $r^3 - 2r^2 + r = 0$, with roots $r = 0, 1, 1$. Hence the complementary solution is $y_c(t) = c_1 + c_2 e^t + c_3 t e^t$. We consider the differential equations $y''' - 2y'' + y' = 3t^3$ and $y''' - 2y'' + y' = 2e^t$ separately. Our initial choice for a particular solution Y_1 of the first equation is $A_0 t^3 + A_1 t^2 + A_2 t + A_3$; but since a constant is a solution of the homogeneous equation we must multiply this by t. Thus $Y_1(t) = t(A_0 t^3 + A_1 t^2 + A_2 t + A_3)$. For the second equation we first choose $Y_2(t) = Be^t$, but since both e^t and te^t are solutions of the homogeneous equation, we multiply by t^2 to obtain $Y_2(t) = Bt^2 e^t$. Then $Y(t) = Y_1(t) + Y_2(t)$ by the superposition principle and $y(t) = y_c(t) + Y(t)$.

17. The characteristic equation for the homogeneous equation is $r^4 - r^3 - r^2 + r = r(r-1)(r^2-1) = 0$, with roots $r = 0, 1, 1, -1$. Hence the complementary solution is $y_c(t) = c_1 + c_2 e^{-t} + c_3 e^t + c_4 t e^t$. We consider the differential equations $y^{(4)} - y''' - y'' + y' = t^2 + 8$ and $y^{(4)} - y''' - y'' + y' = t \sin t$ separately. Our initial choice for a particular solution Y_1 of the first equation is $A_0 t^2 + A_1 t + A_2$; but since a constant is a solution of the homogeneous equation we must multiply this by t. Thus $Y_1(t) = t(A_0 t^2 + A_1 t + A_2)$. For the second equation our initial choice $Y_2(t) = (B_0 t + B_1)\cos t + (C_0 t + C_1)\sin t$ does not need to be modified. Thus $Y(t) = Y_1(t) + Y_2(t)$ by the superposition principle and $y(t) = y_c(t) + Y(t)$.

20. We get $(D-a)(D-b)f = (D-a)(Df - bf) = D^2 f - (a+b)Df + abf$ and $(D-b)(D-a)f = (D-b)(Df - af) = D^2 f - (b+a)Df + baf$. Thus we find that the given equation holds for any function f.

22. (13) The equation in Problem 13 can be written as $D(D-1)^2y = t^3 + 2e^t$. Since D^4 annihilates t^3 and $D-1$ annihilates $2e^t$, we have $D^5(D-1)^3y = 0$, which corresponds to Eq.(ii) of Problem 21. The solution of this equation is $y(t) = A_1t^4 + A_2t^3 + A_3t^2 + A_4t + A_5 + (B_1t^2 + B_2t + B_3)e^t$. Since A_5 and $(B_2t + B_3)e^t$ are solutions of the homogeneous equation related to the original differential equation, they may be deleted and thus $Y(t) = A_1t^4 + A_2t^3 + A_3t^2 + A_4t + B_1t^2e^t$.

22. (14) If $y = te^{-t}$, then $Dy = -te^{-t} + e^{-t}$ and $D^2y = te^{-t} - 2e^{-t}$, which means $(D+1)^2y = (D^2 + 2D + 1)y = 0$ and thus $(D+1)^2$ annihilates te^{-t}. Likewise, $D^2 - 1$ annihilates $2\cos t$. Thus $(D+1)^2(D^2+1)$ annihilates the right side of the differential equation.

22. (17) $D^3(D^2+1)^2$ annihilates the right side of the differential equation.

4.4

1. The characteristic equation is $r(r^2+1) = 0$. Hence the homogeneous solution is $y_c(t) = c_1 + c_2\cos t + c_3\sin t$. The Wronskian is evaluated as $W(1, \cos t, \sin t) = 1$. Now compute the three determinants

$$W_1(t) = \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{vmatrix} = 1, \quad W_2(t) = \begin{vmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & 1 & -\sin t \end{vmatrix} = -\cos t,$$

$$W_3(t) = \begin{vmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & 1 \end{vmatrix} = -\sin t.$$

The solution of the system of Equations (11) is

$$u_1'(t) = \frac{2\tan t\, W_1(t)}{W(t)} = 2\tan t, \quad u_2'(t) = \frac{2\tan t\, W_2(t)}{W(t)} = -2\sin t,$$

$$u_3'(t) = \frac{2\tan t\, W_3(t)}{W(t)} = -2\sin^2 t/\cos t.$$

Hence $u_1(t) = -2\ln(\cos t)$, $u_2(t) = 2\cos t$, $u_3(t) = 2\sin t - 2\ln(\sec t + \tan t)$. The particular solution becomes $Y(t) = -2\ln(\cos t) + 2 - 2\sin t\ln(\sec t + \tan t)$, since $\sin^2 t + \cos^2 t = 1$. The constant is a solution of the homogeneous equation, therefore the general solution is

$$y(t) = c_1 + c_2\cos t + c_3\sin t - 2\ln(\cos t) - 2\sin t\ln(\sec t + \tan t).$$

4. Similarly to Problem 1, the characteristic equation is $r(r^2+1) = 0$. Hence the homogeneous solution is $y_c(t) = c_1 + c_2\cos t + c_3\sin t$. The Wronskian is evaluated

as $W(1, \cos t, \sin t) = 1$. Now compute the three determinants

$$W_1(t) = \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{vmatrix} = 1, \quad W_2(t) = \begin{vmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & 1 & -\sin t \end{vmatrix} = -\cos t,$$

$$W_3(t) = \begin{vmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & 1 \end{vmatrix} = -\sin t.$$

The solution of the system of Equations (11) is

$$u_1'(t) = \frac{\sec t\, W_1(t)}{W(t)} = \sec t, \quad u_2'(t) = \frac{\sec t\, W_2(t)}{W(t)} = -1,$$

$$u_3'(t) = \frac{\sec t\, W_3(t)}{W(t)} = -\sin t/\cos t.$$

Hence $u_1(t) = \ln(\sec t + \tan t)$, $u_2(t) = -t$, $u_3(t) = \ln(\cos t)$. The particular solution becomes $Y(t) = \ln(\sec t + \tan t) - t\cos t + \sin t \ln(\cos t)$.

5. The characteristic equation is $r^3 - r^2 + r - 1 = (r-1)(r^2+1) = 0$. Hence the homogeneous solution is $y_c(t) = c_1 e^t + c_2 \cos t + c_3 \sin t$. The Wronskian is evaluated as $W(e^t, \cos t, \sin t) = 2e^t$. (This also can be found by using Abel's identity: $W(t) = ce^{-\int p_1(t)\, dt} = ce^t$, where $W(0) = 2$, so $c = 2$ and again $W(t) = 2e^t$.) Now compute the three determinants

$$W_1(t) = \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{vmatrix} = 1, \quad W_2(t) = \begin{vmatrix} e^t & 0 & \sin t \\ e^t & 0 & \cos t \\ e^t & 1 & -\sin t \end{vmatrix} = e^t(\sin t - \cos t),$$

$$W_3(t) = \begin{vmatrix} e^t & \cos t & 0 \\ e^t & -\sin t & 0 \\ e^t & -\cos t & 1 \end{vmatrix} = -e^t(\sin t + \cos t).$$

The solution of the system of equations (10) is

$$u_1'(t) = \frac{2e^{-t}\sin t\, W_1(t)}{W(t)} = e^{-2t}\sin t, \; u_2'(t) = \frac{2e^{-t}\sin t\, W_2(t)}{W(t)} = e^{-t}(\sin^2 t - \sin t \cos t),$$

$$u_3'(t) = \frac{2e^{-t}\sin t\, W_3(t)}{W(t)} = -e^{-t}(\sin^2 t + \sin t\cos t).$$

Hence $u_1(t) = -e^{-2t}(\cos t + 2\sin t)/5$, $u_2(t) = -e^{-t}/2 + 3e^{-t}\cos 2t/10 - \sin 2t/10$, $u_3(t) = e^{-t}/2 + e^{-t}\cos 2t/10 + 3e^{-t}\sin 2t/10$. Substitution into $Y = u_1 e^t + u_2 \cos t + u_3 \sin t$ yields the desired particular solution.

7. Similarly to Problem 5, the characteristic equation for the differential equation is $r^3 - r^2 + r - 1 = (r-1)(r^2+1) = 0$. Hence the homogeneous solution is $y_c(t) = c_1 e^t + c_2\cos t + c_3 \sin t$. The Wronskian is evaluated as $W(e^t, \cos t, \sin t) =$

$2e^t$. (Also, as in Problem 5, this can be found by using Abel's identity.) Now compute the three determinants

$$W_1(t) = \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{vmatrix} = 1, \quad W_2(t) = \begin{vmatrix} e^t & 0 & \sin t \\ e^t & 0 & \cos t \\ e^t & 1 & -\sin t \end{vmatrix} = e^t(\sin t - \cos t),$$

$$W_3(t) = \begin{vmatrix} e^t & \cos t & 0 \\ e^t & -\sin t & 0 \\ e^t & -\cos t & 1 \end{vmatrix} = -e^t(\sin t + \cos t).$$

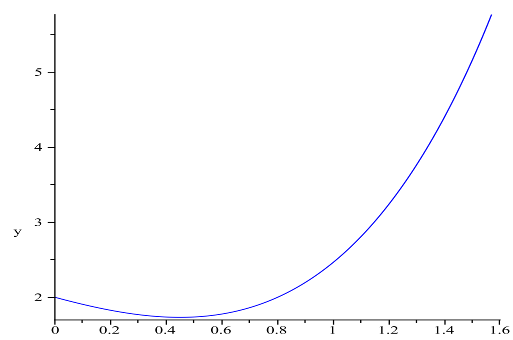
The solution of the system of equations (10) is

$$u_1'(t) = \frac{\sec t\, W_1(t)}{W(t)} = \frac{e^{-t}\sec t}{2}, \quad u_2'(t) = \frac{\sec t\, W_2(t)}{W(t)} = \frac{\sec t(\sin t - \cos t)}{2},$$

$$u_3'(t) = \frac{\sec t\, W_3(t)}{W(t)} = -\frac{\sec t(\sin t + \cos t)}{2}.$$

Hence $u_1(t) = (1/2)\int_{t_0}^t e^{-s}\sec s\, ds$, $u_2(t) = -t/2 - \ln(\cos t)/2$, and $u_3(t) = -t/2 + \ln(\cos t)/2$. Substitution into $Y = u_1 e^t + u_2\cos t + u_3 \sin t$ yields the desired particular solution.

11. Since the differential equation is the same as in Problem 7. we may use the complete solution from there, with $t_0 = 0$. Thus $y(0) = c_1 + c_2 = 2$, $y'(0) = c_1 + c_3 - 1/2 + 1/2 = -1$ and $y''(0) = c_1 - c_2 + 1/2 - 1 + 1/2 = 1$. A computer algebra system may be used to find the respective derivatives. Note that the solution is valid only for $0 \le t < \pi/2$, where we see the vertical asymptote.



14. Using Problem 7 (or Problem 5) again, we get that $Y = u_1 e^t + u_2 \cos t + u_3 \sin t$, where

$$u_1'(t) = \frac{g(t)\, W_1(t)}{W(t)} = \frac{g(t)e^{-t}}{2}, \quad u_2'(t) = \frac{g(t)\, W_2(t)}{W(t)} = \frac{g(t)(\sin t - \cos t)}{2},$$

$$u_3'(t) = \frac{g(t)\, W_3(t)}{W(t)} = -\frac{g(t)(\sin t + \cos t)}{2}.$$

Thus we obtain that

$$Y(t) = \frac{1}{2}[e^t \int_{t_0}^t e^{-s} g(s)\, ds + \cos t \int_{t_0}^t (\sin s - \cos s) g(s)\, ds$$
$$- \sin t \int_{t_0}^t (\sin s + \cos s) g(s)\, ds].$$

We can move e^t, $\cos t$ and $\sin t$ inside the integrals and use trigonometric identities to obtain the desired formula.

16. The characteristic equation for the differential equation is $r^3 - 3r^2 + 3r - 1 = (r-1)^3 = 0$. Hence the homogeneous solution is $y_c(t) = c_1 e^t + c_2 te^t + c_3 t^2 e^t$. The Wronskian is evaluated as $W(e^t, te^t, t^2e^t) = 2e^{3t}$. Now compute the three determinants

$$W_1(t) = \begin{vmatrix} 0 & te^t & t^2e^t \\ 0 & e^t + te^t & 2te^t + t^2e^t \\ 1 & 2e^t + te^t & 2e^t + 4te^t + t^2e^t \end{vmatrix} = t^2e^{2t},$$

$$W_2(t) = \begin{vmatrix} e^t & 0 & t^2e^t \\ e^t & 0 & 2te^t + t^2e^t \\ e^t & 1 & 2e^t + 4te^t + t^2e^t \end{vmatrix} = -2te^{2t},$$

$$W_3(t) = \begin{vmatrix} e^t & te^t & 0 \\ e^t & e^t + te^t & 0 \\ e^t & 2e^t + te^t & 1 \end{vmatrix} = e^{2t}.$$

The solution of the system of equations (10) is

$$u_1'(t) = \frac{g(t)\, W_1(t)}{W(t)} = \frac{g(t)t^2e^{-t}}{2}, \quad u_2'(t) = \frac{g(t)\, W_2(t)}{W(t)} = -g(t)te^{-t},$$

$$u_3'(t) = \frac{g(t)\, W_3(t)}{W(t)} = \frac{g(t)e^{-t}}{2}.$$

Thus we obtain that

$$Y(t) = e^t \int_{t_0}^t \frac{g(s)s^2e^{-s}}{2}\, ds - te^t \int_{t_0}^t g(s)se^{-s}\, ds + t^2e^t \int_{t_0}^t \frac{g(s)e^{-s}}{2}\, ds =$$
$$= \int_{t_0}^t \frac{g(s)e^{t-s}(s^2 - 2ts + t^2)}{2}\, ds = \int_{t_0}^t \frac{g(s)e^{t-s}(s-t)^2}{2}\, ds.$$

If $g(t) = t^{-2}e^t$, then this formula gives

$$Y(t) = \int_{t_0}^t \frac{s^{-2}e^se^{t-s}(s-t)^2}{2}\, ds = e^t \int_{t_0}^t \frac{s^{-2}(s-t)^2}{2}\, ds = e^t \int_{t_0}^t \frac{1}{2} - \frac{t}{s} + \frac{t^2}{2s^2}\, ds.$$

Note that terms involving t_0 become part of the complementary solution, so we obtain that $Y(t) = -te^t \ln t$ only.