Comment on Occurrence

In an ODE the transform $R(s)$ of the right side $r(t)$ is known from Step 1. Solving the subsidiary equation algebraically for $Y(s)$ causes the transform $R(s)$ to be multiplied by the reciprocal of the factor of $Y(s)$ on the left (the transfer function $Q(s)$; see Sec. 6.2). This calls for the convolution theorem, unless one sees some other way or shortcut.

Very Short Courses. This section can be omitted.

Problem Set 6.5

This set concerns integrations needed to obtain convolutions (Probs. 1–7), the use of the latter in solving a certain class of integrations (Probs. 8–14), the effect of varying a parameter in an integral equation (CAS Experiment 15), a problem (Prob. 16) on general properties of convolution.

Problems 17–26 show how to obtain inverse transforms by evaluating integrals that define convolutions.

SOLUTIONS TO PROBLEM SET 6.5, page 237

1. $-t$

2. $1 * \sin \omega t = \displaystyle\int_0^t \sin \omega\tau \, d\tau = -\frac{\cos \omega\tau}{\omega}\Big|_0^t = \frac{1 - \cos \omega t}{\omega}$

3. $\frac{1}{2}(e^t + e^{-t}) = \sinh t$

4. We obtain

$$\int_0^t \cos \omega\tau \cos (\omega t - \omega\tau) \, d\tau = \frac{1}{2}\int_0^t [\cos \omega t + \cos (2\omega\tau - \omega t)] \, d\omega$$

$$= \frac{1}{2}\left[t \cos \omega t + \frac{\sin \omega t - \sin (-\omega t)}{2\omega}\right] = \frac{1}{2} t \cos \omega t + \frac{1}{2\omega} \sin \omega t.$$

5. $\frac{\sin \omega t}{\omega}$

6. $e^{at} * e^{bt} = \displaystyle\int_0^t e^{a\tau} e^{b(t-\tau)} \, d\tau = e^{bt} \int_0^t e^{(a-b)\tau} \, d\tau = \frac{e^{at} - e^{bt}}{a - b}$

7. $te^{-t} - te^{-2t}$.

8. The integral equation can be written

$$y(t) + 4y(t) * t = 2t.$$

By the convolution theorem it has the transform

$$Y + 4Y/s^2 = 2/s^2.$$

The solution is $Y = 2/(s^2 + 4)$. Its inverse transform is $y = \sin 2t$.

9. $y + 1 * y = 2$; $Y = \frac{2}{s + 1}$; $y = 2e^{-t}$

10. In terms of convolution the given integral equation is

$$y(t) - y(t) * \sin 2t = \sin 2t.$$

By the convolution theorem its transform is

$$Y - 2Y/(s^2 + 4) = 2/(s^2 + 4).$$

Multiplication by $s^2 + 4$ gives

$$Y(s^2 + 2) = 2; \qquad \text{thus} \qquad Y = \frac{2}{s^2 + 2}.$$

The inverse transform (the solution of the integral equation) is

$$y = \sqrt{2} \sin (t\sqrt{2}).$$

11. $Y = \frac{s}{s^2 - 1}$; $y = \cosh t$.

12. The integral equation can be written

$$y(t) + y(t) * \cosh t = t + e^t.$$

This implies, by the convolution theorem, that its transform is

$$Y + \frac{s}{s^2 - 1} Y = \frac{1}{s^2} + \frac{1}{s - 1}.$$

The solution is

$$Y = \frac{s^2 - 1}{s^2 + s - 1}\left(\frac{1}{s^2} + \frac{1}{s - 1}\right) = \frac{1}{s^2} + \frac{1}{s}.$$

Hence its inverse transform gives the *answer* $y(t) = t + 1$. This result can easily be checked by substitution into the given equation and integration.

14. $Y\left(1 - \frac{1}{s^2}\right) = \frac{2}{s} - \frac{1}{s^3}$, hence

$$Y = \frac{2s^2 - 1}{s(s^2 - 1)} = \frac{1}{s} + \frac{s}{s^2 - 1}.$$

The *answer* is $y = 1 + \cosh t$.

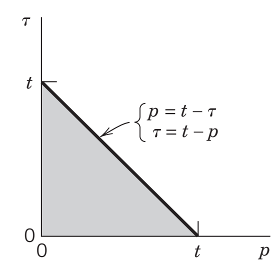
16. Team Project. (a) Setting $t - \tau = p$, we have $\tau = t - p$, $d\tau = -dp$, and p runs from t to 0; thus

$$f * g = \int_0^t f(\tau)g(t - \tau)\, d\tau = \int_t^0 g(p)f(t - p)(-dp)$$

$$= \int_0^t g(p)f(t - p)\, dp = g * f.$$

(b) Interchanging the order of integration and noting that we integrate over the shaded triangle in the figure, we obtain

$$
\begin{aligned}
(f * g) * v &= v * (f * g) \\
&= \int_0^t v(p) \int_0^{t-p} f(\tau) g(t - p - \tau)\, d\tau\, dp \\
&= \int_0^t f(\tau) \int_0^{t-\tau} g(t - \tau - p) v(p)\, dp\, d\tau \\
&= f * (g * v).
\end{aligned}
$$



Section 6.5. Team Project 16(b)

(c) This is a simple consequence of the additivity of the integral.

(d) Let $t > k$. Then $(f_k * f)(t) = \displaystyle\int_0^k \frac{1}{k} f(t - \tau)\, d\tau = f(t - \tilde{t})$ for some $\tilde{t}$ between 0 and k. Now let $k \to 0$. Then $\tilde{t} \to 0$ and $f_k(t - \tilde{t}) \to \delta(t)$, so that the formula follows.

(e) $s^2 Y - sy(0) - y'(0) + \omega^2 Y = \mathcal{L}(r)$ has the solution

$$Y = \frac{1}{\omega}\left(\frac{\omega}{s^2 + \omega^2}\right)\mathcal{L}(r) + y(0)\frac{s}{s^2 + \omega^2} + \frac{y'(0)}{\omega}\frac{\omega}{s^2 + \omega^2}$$

etc.

18. $e^{at} * e^{at} = \displaystyle\int_0^t a^{a\tau} e^{a(t-\tau)}\, d\tau = e^{at} \int_0^t d\tau = te^{at}$

20. $9 * e^{-3t} = 9\displaystyle\int_0^t e^{-3\tau}\, d\tau = 3 - 3e^{-3t}$

21. $-\frac{t}{w} + \frac{\sinh(wt)}{w^2}$.

22. $u(t - a) * e^{2t} = \displaystyle\int_a^t e^{2(t-\tau)}\, d\tau = e^{2t} \int_a^t e^{-2\tau}\, d\tau = \frac{1}{2}(e^{2(t-a)} - 1)$ if $t > a$ and 0 if $t < a$

24. $48\,(\sin t) * (\sin 5t) = 48\int_0^t \sin\tau \sin 5(t-\tau)\,d\tau$. Using formula (11) in App. 3.1, convert the product in the integrand to a sum and integrate, obtaining

$$\begin{aligned} &24\int_0^t [-\cos(5t-4\tau)+\cos(\tau-5t+5\tau)]\,d\tau \\ &= 24[-\tfrac{1}{4}\sin(-5t+4\tau)+\tfrac{1}{6}\sin(6\tau-5t)]\Big|_0^t \\ &= 3(\sin t-\sin 5t)+2(\sin t+\sin 5t) \\ &= 10\sin t - 2\sin 5t. \end{aligned}$$

SECTION 6.6. Differentiation and Integration of Transforms. ODEs with Variable Coefficients, page 238

Purpose. To show that, ***roughly***, differentiation and integration of transforms (not of functions, as before!) correspond to multiplication and division, respectively, of functions by t, with application to the derivation of further transforms and to the solution of Laguerre's differential equation.

Comment on Application to Variable-Coefficient Equations

This possibility is rather limited; our Example 3 is perhaps the best elementary example of practical interest.

Very Short Courses. This section can be omitted.

Problem Set 6.6

Problems 2–11 concern single or twofold applications of differentiation with respect to s, as the only method wanted for solving these problems, whereas in Probs. 14–20 the student has first to select a suitable one among several possible methods suggested.

Problem 13 is an invitation to study Laguerre polynomials in somewhat more detail, in particular, to compare the locations of the extrema depending on the parameter n.

SOLUTIONS TO PROBLEM SET 6.6, page 241

2. $F(s) = -3\left(\dfrac{4}{s^2-16}\right)' = \dfrac{24s}{(s^2-16)^2}$

3. $\frac{1}{4}(s+2)^{-2}$

4. We have

$$\mathcal{L}(e^{-t}\cos t) = \frac{s+1}{(s+1)^2+1} = \frac{s+1}{s^2+2s+2}.$$

Differentiation and simplification gives the answer

$$\begin{aligned} F'(s) = -\left(\frac{s+1}{s^2+2s+2}\right)' &= -\frac{s^2+2s+2-(s+1)(2s+2)}{(s^2+2s+2)^2} \\ &= \frac{s^2+2s}{(s^2+2s+2)^2}. \end{aligned}$$