Elimination (or Cramer's rule in Sec. 7.7) yields the solution, which we can expand in terms of partial fractions,

$$Y_1 = \frac{(s + \sqrt{3k})(s^2 + 2k) + k(s - \sqrt{3k})}{(s^2 + 2k)^2 - k^2} = \frac{s}{s^2 + k} + \frac{\sqrt{3k}}{s^2 + 3k}$$

$$Y_2 = \frac{(s^2 + 2k)(s - \sqrt{3k}) + k(s + \sqrt{3k})}{(s^2 + 2k)^2 - k^2} = \frac{s}{s^2 + k} - \frac{\sqrt{3k}}{s^2 + 3k}.$$

Hence the solution of our initial value problem is (Fig. 147)

$$y_1(t) = \mathcal{L}^{-1}(Y_1) = \cos \sqrt{k}t + \sin \sqrt{3k}t$$

$$y_2(t) = \mathcal{L}^{-1}(Y_2) = \cos \sqrt{k}t - \sin \sqrt{3k}t.$$

We see that the motion of each mass is harmonic (the system is undamped!), being the superposition of a "slow" oscillation and a "rapid" oscillation. ■

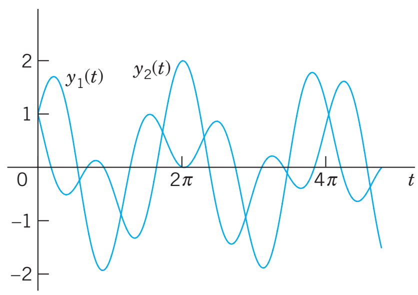

Fig. 147. Solutions in Example 3

PROBLEM SET 6.7

1. **TEAM PROJECT. Comparison of Methods for Linear Systems of ODEs**

   (a) Models. Solve the models in Examples 1 and 2 of Sec. 4.1 by Laplace transforms and compare the amount of work with that in Sec. 4.1. Show the details of your work.

   (b) Homogeneous Systems. Solve the systems (8), (11)–(13) in Sec. 4.3 by Laplace transforms. Show the details.

   (c) Nonhomogeneous System. Solve the system (3) in Sec. 4.6 by Laplace transforms. Show the details.

2–15 SYSTEMS OF ODES

Using the Laplace transform and showing the details of your work, solve the IVP:

2. $y_1' + y_2 = 0, \quad y_1 + y_2' = 2 \cos t,$
   $y_1(0) = 1, \quad y_2(0) = 0$
3. $y_1' = -y_1 + 4y_2, \quad y_2' = 3y_1 - 2y_2,$
   $y_1(0) = 3, \quad y_2(0) = 4$
4. $y_1' = 4y_2 - 8 \cos 4t, \quad y_2' = -3y_1 - 9 \sin 4t,$
   $y_1(0) = 0, \quad y_2(0) = 3$
5. $y_1' = y_2 + 1 - u(t - 1), \quad y_2' = -y_1 + 1 - u(t - 1),$
   $y_1(0) = 0, \quad y_2(0) = 0$
6. $y_1' = 5y_1 + y_2, \quad y_2' = y_1 + 5y_2,$
   $y_1(0) = 1, \quad y_2(0) = -3$
7. $y_1' = 2y_1 - 4y_2 + u(t - 1)e^t,$
   $y_2' = y_1 - 3y_2 + u(t - 1)e^t, \quad y_1(0) = 3, \quad y_2(0) = 0$
8. $y_1' = -2y_1 + 3y_2, \quad y_2' = 4y_1 - y_2,$
   $y_1(0) = 4, \quad y_2(0) = 3$
9. $y_1' = 4y_1 + y_2, \quad y_2' = -y_1 + 2y_2, \quad y_1(0) = 3,$
   $y_2(0) = 1$
10. $y_1' = -y_2, \quad y_2' = -y_1 + 2[1 - u(t - 2\pi)] \cos t,$
    $y_1(0) = 1, \quad y_2(0) = 0$
11. $y_1'' = y_1 + 3y_2, \quad y_2'' = 4y_1 - 4e^t,$
    $y_1(0) = 2, \quad y_1'(0) = 3, \quad y_2(0) = 1, \quad y_2'(0) = 2$
12. $y_1'' = -2y_1 + 2y_2, \quad y_2'' = 2y_1 - 5y_2,$
    $y_1(0) = 1, \quad y_1'(0) = 0, \quad y_2(0) = 3, \quad y_2'(0) = 0$
13. $y_1'' + y_2 = -101 \sin 10t, \quad y_2'' + y_1 = 101 \sin 10t,$
    $y_1(0) = 0, \quad y_1'(0) = 6, \quad y_2(0) = 8, \quad y_2'(0) = -6$

14. $4y_1' + y_2' - 2y_3' = 0, \quad -2y_1' + y_3' = 1,$
$2y_2' - 4y_3' = -16t$
$y_1(0) = 2, \quad y_2(0) = 0, \quad y_3(0) = 0$

15. $y_1' + y_2' = 2 \sinh t, \quad y_2' + y_3' = e^t,$
$y_3' + y_1' = 2e^t + e^{-t}, \quad y_1(0) = 1, \quad y_2(0) = 1,$
$y_3(0) = 0$

FURTHER APPLICATIONS

16. Forced vibrations of two masses. Solve the model in Example 3 with $k = 4$ and initial conditions $y_1(0) = 1$, $y_1'(0) = 1$, $y_2(0) = 1$, $y_2' = -1$ under the assumption that the force $11 \sin t$ is acting on the first body and the force $-11 \sin t$ on the second. Graph the two curves on common axes and explain the motion physically.

17. CAS Experiment. Effect of Initial Conditions. In Prob. 16, vary the initial conditions systematically, describe and explain the graphs physically. The great variety of curves will surprise you. Are they always periodic? Can you find empirical laws for the changes in terms of continuous changes of those conditions?

18. Mixing problem. What will happen in Example 1 if you double all flows (in particular, an increase to 12 gal/min containing 12 lb of salt from the outside), leaving the size of the tanks and the initial conditions as before? First guess, then calculate. Can you relate the new solution to the old one?

19. Electrical network. Using Laplace transforms, find the currents $i_1(t)$ and $i_2(t)$ in Fig. 148, where $v(t) = 390 \cos t$ and $i_1(0) = 0$, $i_2(0) = 0$. How soon will the currents practically reach their steady state?

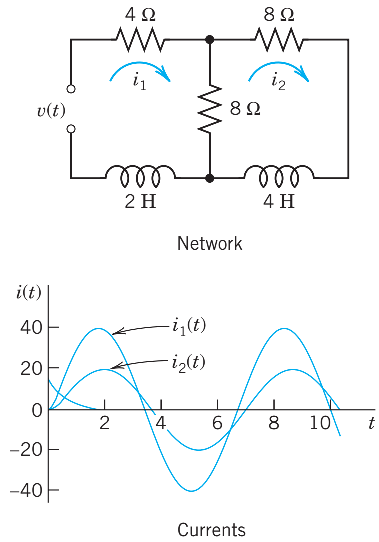

Fig. 148. Electrical network and currents in Problem 19

20. Single cosine wave. Solve Prob. 19 when the EMF (electromotive force) is acting from 0 to 2π only. Can you do this just by looking at Prob. 19, practically without calculation?