Lecture 10a

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Lecture 10a: A Review via A Design Problem

... in which we examine a particular system in detail and apply all the methods we have learned so far.

I. A Nonlinear System

Consider the following system:

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 x_2 \\ a x_1 - x_2^2 + u \end{pmatrix}$$

$$y = x_1$$

where $a \approx 1/2$ is a parameter.
Our goal is to control the system near the origin.

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A. Phase Portrait

First, find the equilibrium points:

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 x_2 \\ a x_1 - x_2^2 \end{pmatrix}$$

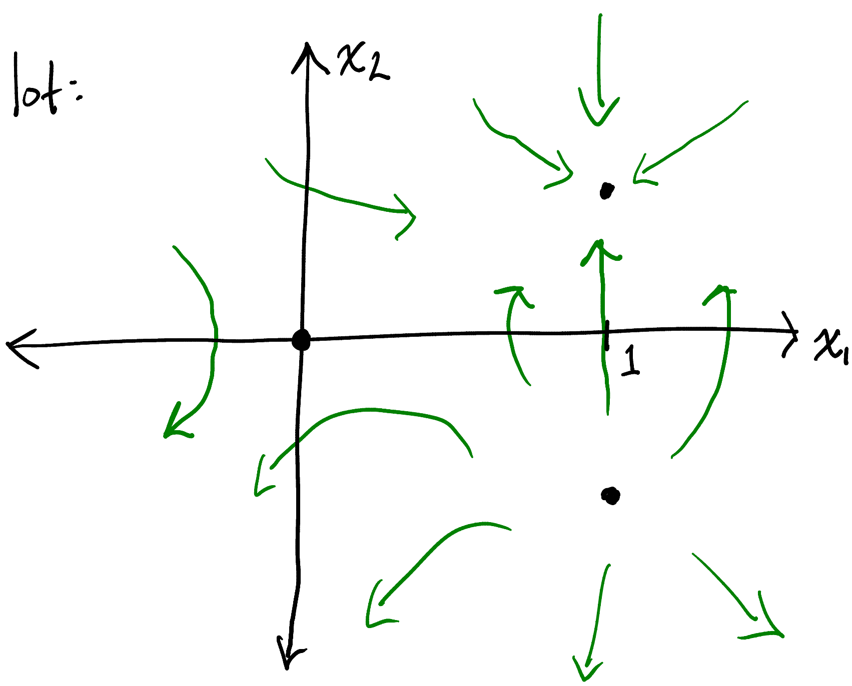
$$\Rightarrow \quad \begin{aligned} x_2 &= x_1 x_2 \\ a x_1 &= x_2^2 \end{aligned}$$

$$\Rightarrow \quad x_1 = x_2 = 0$$

or

$$x_1 = 1 \text{ and } x_2 = \pm\sqrt{a}$$

Then plot:



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II. Linearization

Now, find a linear model near (0,0).

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}\Bigg|_{\substack{\vec{x}=0\\u=0}} = \begin{pmatrix} -x_2 & 1-x_1 \\ a & -2x_2 \end{pmatrix}\Bigg|_{\substack{\vec{x}=0\\u=0}}$$

$$= \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

In the frequency domain:

$$G(s) = C(sI - A)^{-1}B$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s & -1 \\ -a & s \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

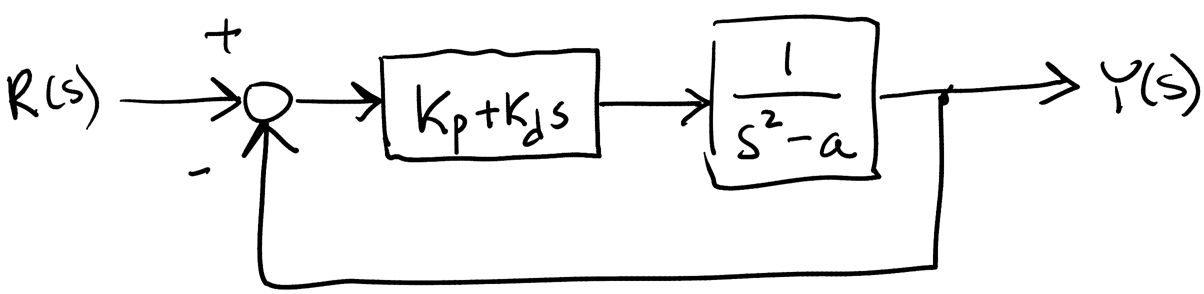
$$= \frac{1}{s^2-a} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s & 1 \\ a & s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{s^2-a} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ s \end{pmatrix} = \boxed{\frac{1}{s^2-a}}$$

We see that the poles of the system are at $\pm\sqrt{a}$. So the origin is a saddle point.

III. A PD Controller

Now let's explore a simple control scheme:



The closed loop transfer function is

$$T(s) = \frac{G_c G}{1 + G_c G}$$

$$= \frac{(K_p + K_D s)\frac{1}{s^2 - a}}{1 + (K_p + K_D s)\frac{1}{s^2 - a}}$$

$$= \boxed{\frac{K_p + K_D s}{s^2 + K_D s + K_p - a}}$$

We now have two knobs to tweak. Let's see what they do to various performance metrics.

A. Steady State Error For $R(s) = 1/s$

$$y(\infty) = \lim_{s \to 0} \cancel{s}\, T(s) \frac{1}{\cancel{s}} = \frac{K_p}{K_p - a}.$$

Thus, $e(\infty) = 1 - y(\infty) = 1 - \frac{K_P}{K_p - a}$

$$= \boxed{\frac{a}{K_p - a}}$$

which can be made small.

B. Damping + Frequency

The poles of the system are:

$$s = \frac{-K_D \pm \sqrt{K_D^2 - 4(K_p - a)}}{2}$$

so in terms of our standard model

$$\begin{aligned} \omega^2 &= K_p - a \\ 2\omega\zeta &= K_D \end{aligned} \quad \Rightarrow \quad \begin{aligned} \omega &= \sqrt{K_p - a} \\ \zeta &= \frac{K_D}{2\sqrt{K_p - a}} \end{aligned}$$

(note: we are ignoring the zero for simplicity)

To get a critically damped response (for example) we would set

$$\zeta = 1 = \frac{K_D}{2\sqrt{K_p - a}} \Rightarrow \boxed{K_D = 2\sqrt{K_p - a}}$$

This also affects settling time:

$$T_s = \frac{4}{\zeta\omega} = \frac{4}{\frac{K_D}{2\cancel{\sqrt{K_p - a}}} \cdot \cancel{\sqrt{K_p - a}}}$$

$$= \boxed{8/K_D}$$

Example: Say $a = 1/2$ and we want crit. damp. with a 1% error. Then

$$e(\infty) = 0.01 = \frac{1/2}{1/2 + K_p} \Rightarrow \frac{1}{2} + K_p = 50 \Rightarrow \boxed{K_p \approx 50}$$

$$\Rightarrow K_D = 2\sqrt{50} \approx \boxed{14} \Rightarrow T_s = \frac{8}{14} \approx \boxed{0.6s}$$

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IV. Sensitivity

$$\frac{gf' - g'f}{g^2}$$

How sensitive is the system to the parameter a?

$$\frac{\partial T}{\partial a} \cdot \frac{a}{T} = \left(\frac{\partial}{\partial a} \frac{K_p + K_D s}{s^2 + K_D s + K_p - a} \right) \cdot \frac{a}{T}$$

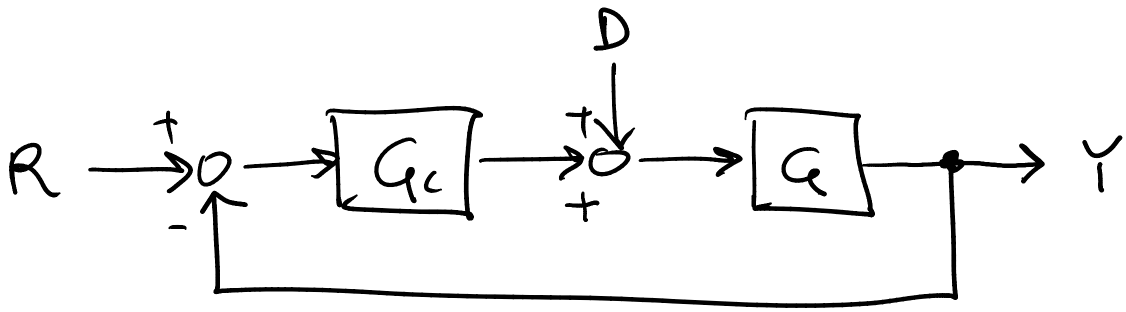
$$= \frac{0 - (-1)\cancel{(K_p + K_D s)}}{D^{\cancel{2}}} \cdot \frac{a\,\cancel{D}}{\cancel{N}}$$

$$= \frac{a}{s^2 + K_D s + K_p - a}$$

$\boxed{d = \text{denominator of } T(s)}$

This can be made small by choosing K_p large.

V. A Disturbance



Say $R=0$. What is $\frac{Y(s)}{D(s)}$?

$$Y = G(D + G_c E) = G(D - G_c Y)$$

so $$Y(1 + G_c G) = GD$$

$$\frac{Y(s)}{D(s)} = \frac{G(s)}{1 + G_c(s) G(s)}$$

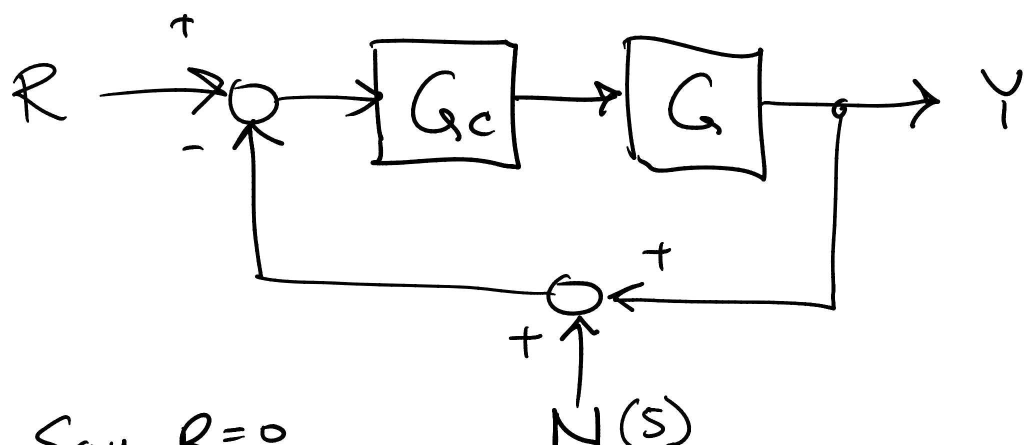
$$= \frac{\frac{1}{s^2-a}}{1 + (K_p + K_D s)\frac{1}{s^2-a}} = \frac{1}{s^2 + K_D s + K_p - a}$$

For example, if $D(s) = \frac{d}{s}$ then

$$y(\infty) = \lim_{s \to 0} s \frac{d}{s} \frac{1}{s^2 + K_D s + K_p - a} = \boxed{\frac{d}{K_p - a}}$$

So we see that we can decrease the effect of a disturbance by increasing K_p. (ex: $K_p = 50$, $d = 1$ $\Rightarrow \approx 1/50$)

VI. Noise



Say $R = 0$

$$Y = G_c G E = -G_c G (Y + N)$$

$$Y(1 + G_c G) = -G_c G N$$

$$\frac{Y}{N} = -\frac{G_c G N}{1 + G_c G}.$$

For example, with $N(s) = n/s$ we get

$$y(\infty) = \lim_{s \to 0} \cancel{s}\, \frac{n}{\cancel{s}} \frac{K_P + K_D s}{s^2 + K_D s + K_P - a}$$

$$= \boxed{n \cdot \frac{K_P}{K_P - a}}$$

Note that changing K_P really does not help. In fact, there is not much you can do - except go buy better sensors.

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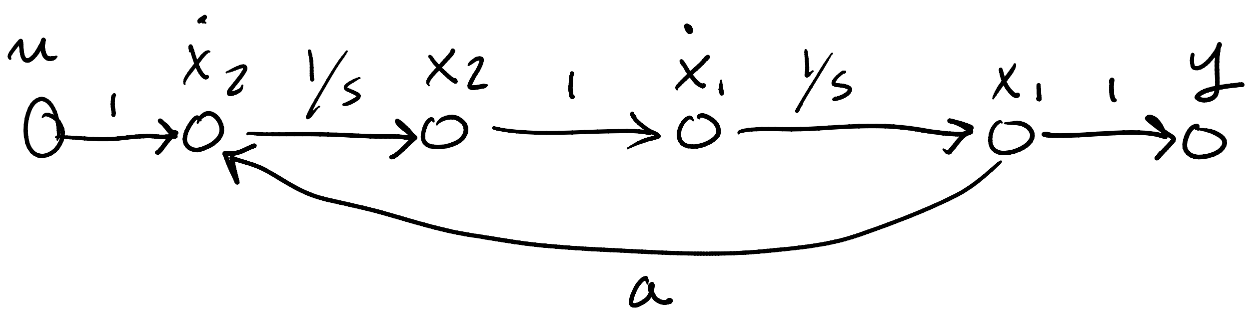
VII. State Space

First, let's translate $G(s) = \frac{1}{s^2 - a}$ into a signal flow graph. First, note that the system was obtained from

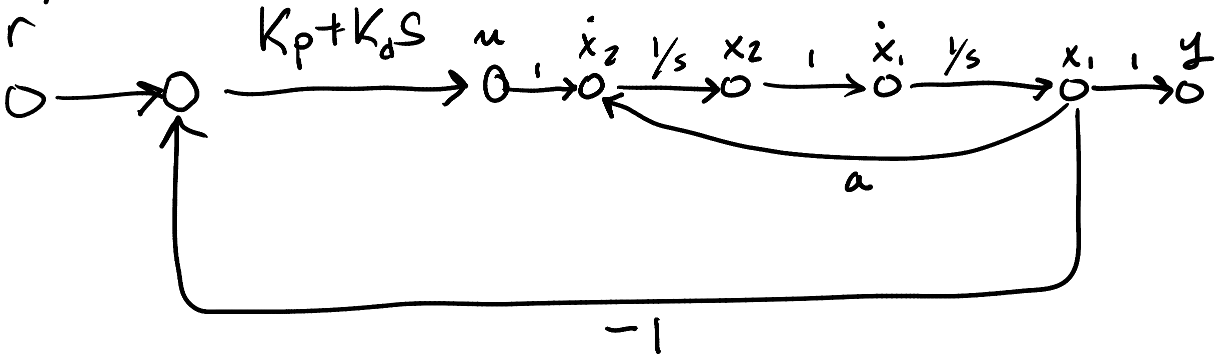
$$\dot{\vec{x}} = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \vec{x}$$

from which it is easy to get



Now we add the feedback path and the controller



What is this in state space?
We still have two states, x_1 & x_2, but the input is different:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -a x_1 + u$$

$$= -a x_1 + \left(K_p + K_d \frac{d}{dt}\right)(r - x_1)$$

$$= -a x_1 + K_p r + K_d \dot{r} - K_p x_1 - K_D \dot{x}_1$$

$$= -a x_1 + K_p (r - x_1) + K_D (\dot{r} - \dot{x}_1)$$

$$= \quad " \qquad " \qquad " \quad x_2$$

Note that both r and $\dot{r}$ are used!
We get

$$\dot{\vec{x}} = \begin{pmatrix} 0 & 1 \\ -(a+K_p) & K_D \end{pmatrix} \vec{x} + \begin{pmatrix} 0 & 0 \\ K_p & K_D \end{pmatrix} \begin{pmatrix} r \\ \dot{r} \end{pmatrix}$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \vec{x} .$$