Lecture 10a

Sunday, November 06, 2005
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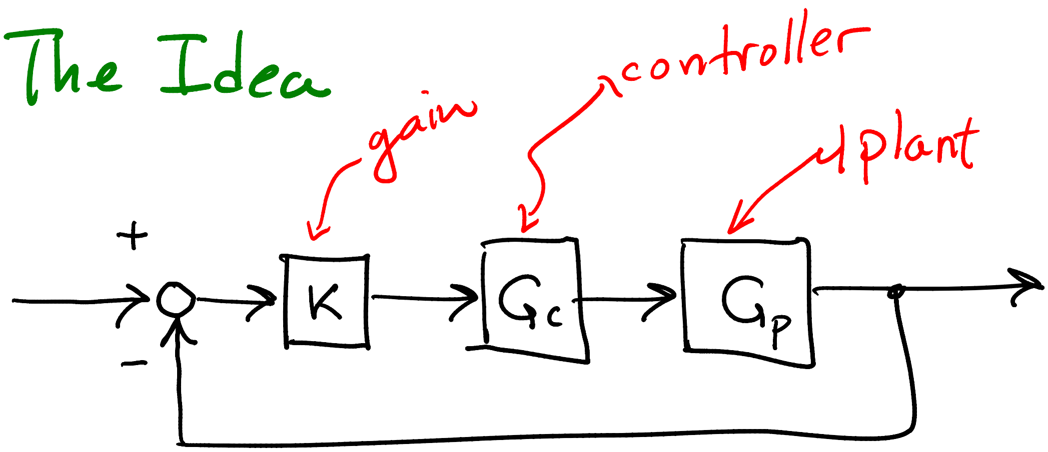
Lecture 10a: The Root Locus

... in which we develop a technique for designing controllers based on the relationship between gain and pole location.

I. The Idea



Changing K changes the poles of the closed-loop system.

The root locus is a plot of pole location versus K in the imaginary plane.

Example: $$KG_cG = \frac{K(s+1)}{s^2+1}$$

Then $$T(s) = \frac{K(s+1)}{s^2 + Ks + K + 1}$$

critically damped

$K^2 = 4K+4$

$K^2 - 4K - 4$

$K = \frac{4 \pm \sqrt{32}}{2}$

$= 2 \pm 2\sqrt{2}$

$\Rightarrow s = \frac{1}{2}(2+\sqrt{2})$

$= 1 + \frac{\sqrt{2}}{2}$

$= 2.41$

The poles are

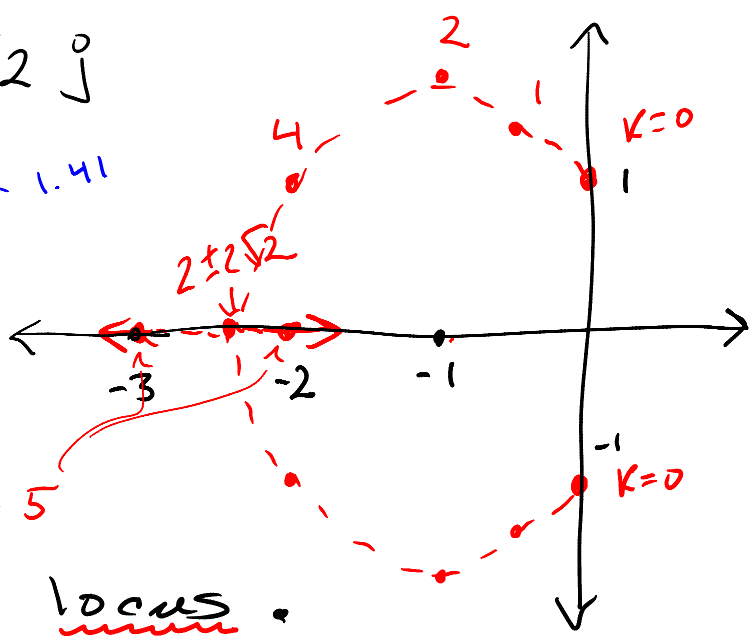
$$s = \frac{1}{2}\left(-K \pm \sqrt{K^2 - 4(K+1)}\right)$$

Table

K	s
0	$\pm j$
1	$-1/2 \pm \sqrt{7}/2\, j$
2	$-1 \pm \sqrt{2}\, j$
4	$-2 \pm j$
$2 \pm 2\sqrt{2}$	2.41
5	$-2, -3$

1.33

1.41



The curve is the root locus.

Say $G = G_c G_p = \dfrac{p(s)}{q(s)}$ where $p(s)$ and $q(s)$ are polynomials is s. Then the closed loop system has the transfer function

$$T(s) = \frac{Kp(s)/q(s)}{1 + Kp(s)/q(s)}$$

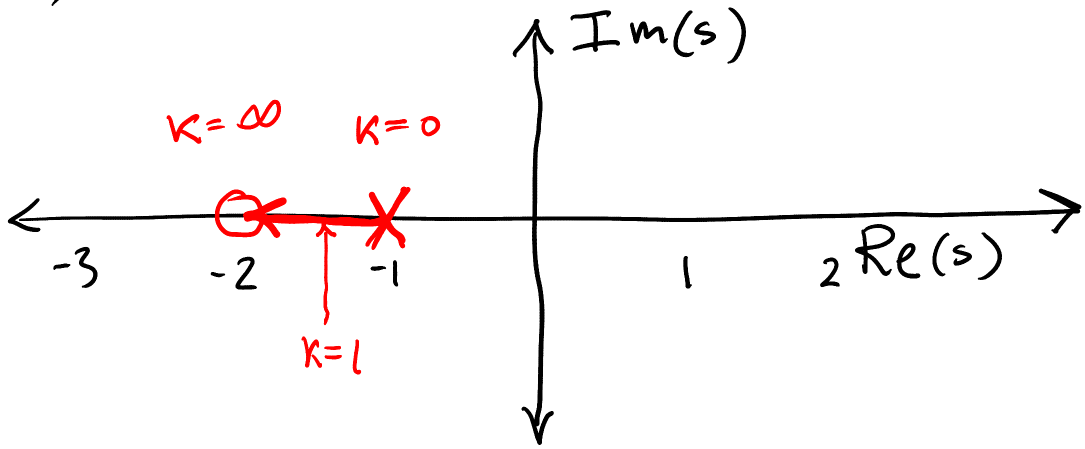
PROPERTY ONE

$$= \frac{Kp(s)}{q(s) + Kp(s)}$$

- when $K \approx 0$, the poles are the roots of $q(s)$. That is, the poles of $T(s)$ are the poles of $G(s)$.
- when $K \to \infty$, the $Kp(s)$ term dominates, and the poles of $T(s)$ are the roots of $p(s)$, or the zeros of $G(s)$

Example: Say $G = \frac{s+2}{s+1}$.

Then, the root locus looks like



As you turn up the gain, the pole of $T(s)$ goes from -1 to -2.

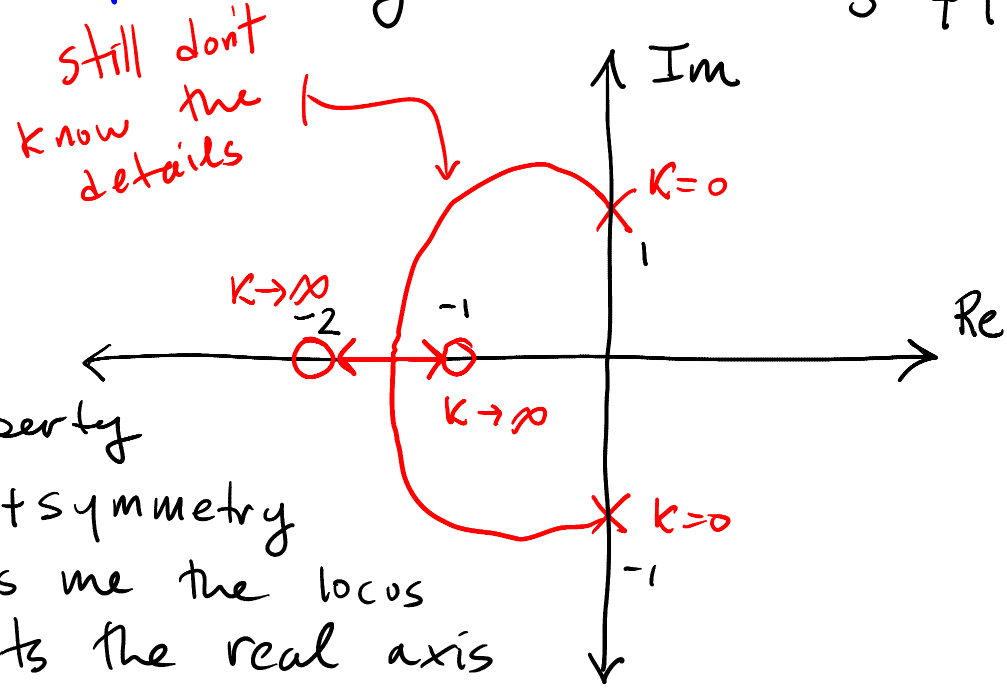
Note that we drew this without knowing $T(s)$ exactly. But just to check

$$T(s) = \frac{K(s+2)}{s+1+K(s+2)} = \frac{K(s+2)}{(K+1)s + 2K+1}$$

which has pole $s = -\frac{2K+1}{K+1}$

PROPERTY TWO The root locus is symmetric about the real axis. This is because imaginary poles come in complex conjugate pairs. This is how I knew that the locus in the last example was on the real line between -1 and -2.

Example: Say $G(s) = \dfrac{(s+1)(s+2)}{s^2+1}$



property one + symmetry tells me the locus meets the real axis between -1 and -2.

II. The angle property

Before we look at the next property, we remind ourselves of some basic math. Say

$$F(s) = \frac{p(s)}{q(s)} = \frac{(s+z_1)(s+z_2)\cdots\cdots}{(s+p_1)(s+p_2)\cdots}$$

If we evaluate $F(s)$ at any point s, we get that

$$F(s) = M \angle \theta$$

where

$$M = \frac{|s+z_1||s+z_2|\cdots}{|s+p_1||s+p_2|\cdots}$$

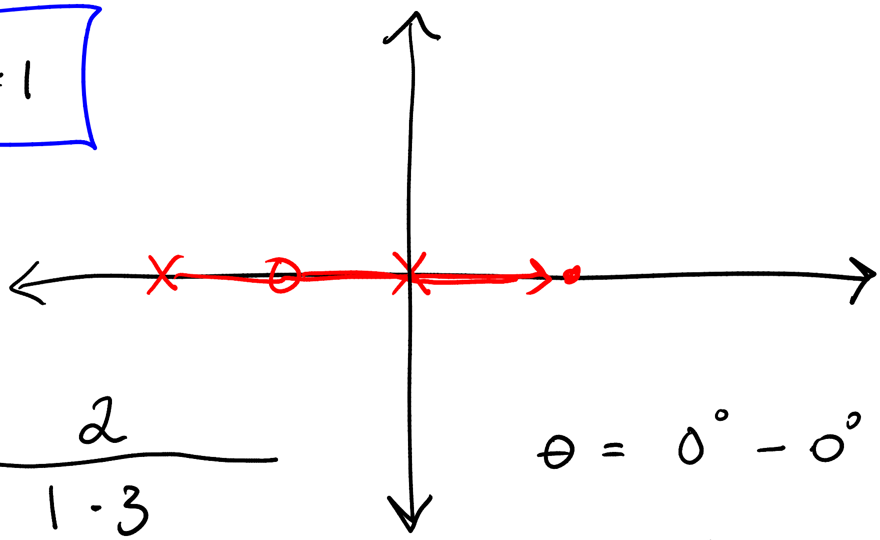
$$\theta = \sum_{i=1}^{m} \angle(s+z_i) - \sum_{j=1}^{n} \angle(s+p_j)$$

$$\boxed{\frac{a+bi}{d+ci} = \frac{|z_1|e^{i\theta_1}}{|z_2|e^{i\theta_2}} = \frac{|z_1|}{|z_2|}e^{i(\theta_1-\theta_2)}}$$

Reminder

Example: $$F(s) = \frac{s+1}{s(s+2)}$$

$\boxed{\text{If } s = 1}$

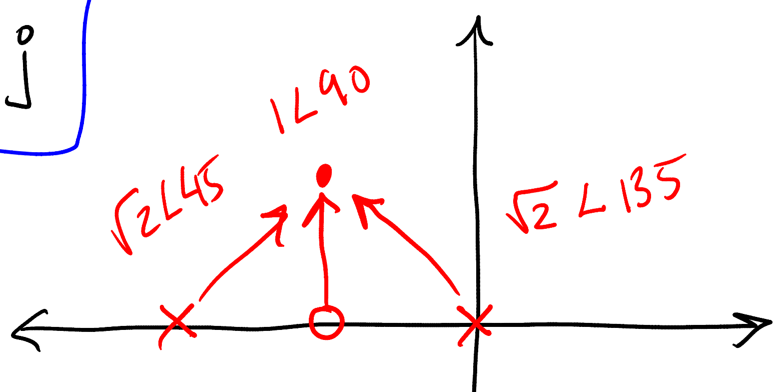


$$M = \frac{2}{1 \cdot 3} = \frac{2}{3}$$

$$\theta = 0^\circ - 0^\circ - 0^\circ = 0^\circ$$

so $F(1) = \frac{2}{3}$.

$\boxed{\text{If } s = -1 + j}$



$$M = \frac{1}{\sqrt{2}\sqrt{2}}$$

$$\theta = 90 - 135 - 45 = -90$$

so $F(-1+j) = \frac{1}{2}\angle{-90} = -j/2$.

Now consider

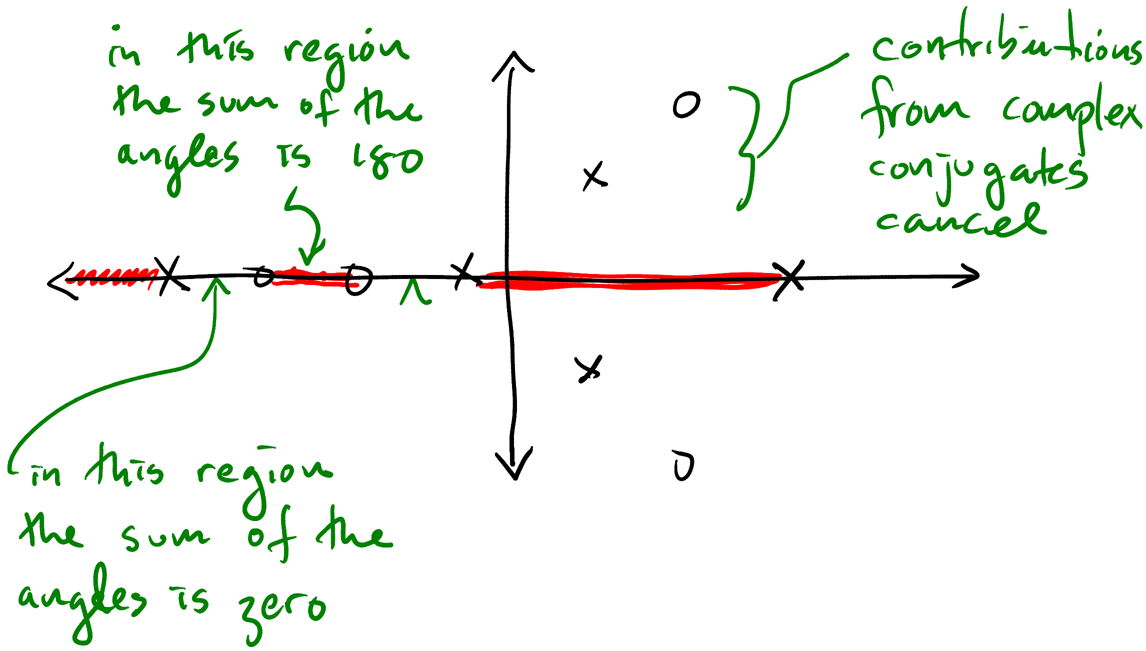
$$T(s) = \frac{KG(s)}{1 + KG(s)}$$

This has a pole when

$$KG(s) = -1 \quad = +180^\circ \text{ too.}$$

or when $\angle KG(s) = -180^\circ$. If s is a real root then

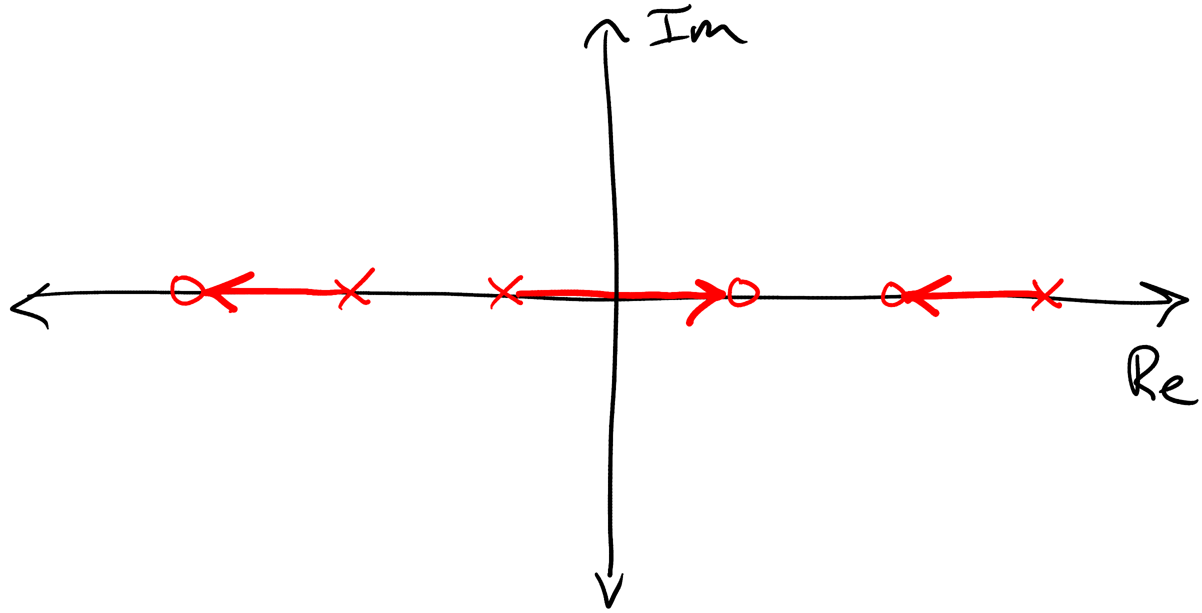
$$\angle G(s) = \sum \angle \text{zeros of } G - \sum \angle \text{poles of } G = 180^\circ$$



PROPERTY THREE

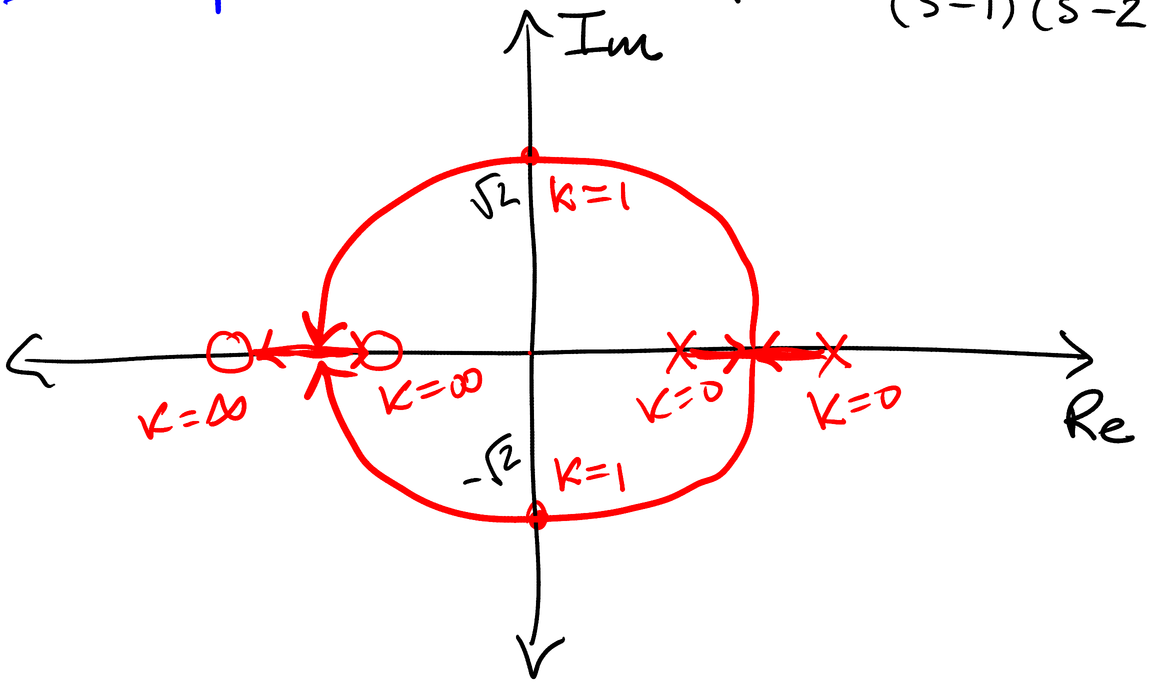
The result is that there must be an odd number of poles or zeros to the left of any part of the root locus on the real line.

Example: $$KG(s) = K\frac{(s+3)(s-1)(s-2)}{(s+1)(s+2)(s-3)}$$



Example:

$$G(s) = \frac{(s+1)(s+2)}{(s-1)(s-2)}$$



The crossing of the imaginary axis is important because that's when the system becomes stable:

$$T(s) = \frac{K(s+1)(s+2)}{K(s+1)(s+2) + (s-1)(s-2)}$$

$$= \frac{''}{(K+1)s^2 + (3K-3)s + 2K+2}$$

$$Re(s) = \frac{3K-3}{2} = 0$$

$$K = 1$$

$$\Rightarrow 2s^2 = 4$$

$$s = \pm\sqrt{2}\,j$$

II. Zeros at Infinity

Consider a system with more poles than zeros.

$$KG(s) = \frac{K}{(s+1)(s+2)}$$

If the root locus goes from the open loop poles to the open loop zeros, where does this one go?

We want

$$KG = -1 = \frac{K}{s^2+3s+2}$$

$$\Rightarrow \quad s^2+3s \approx -K \qquad \left(\text{when } s \text{ is big, this}\right)$$

$$\Leftrightarrow \quad s^2\left(1+\frac{3}{s}\right) = -K$$

$$\Leftrightarrow \quad s\left(1+\frac{3}{s}\right)^{1/2} = (-K)^{1/2}$$

Note $\left(1+\frac{3}{2s}\right)^2 \approx 1 + 2\frac{3}{2s} = 1 + \frac{3}{s}$

$$\Rightarrow \quad 1 + \frac{3}{2s} \approx \left(1+\frac{3}{s}\right)^{1/2}.$$

So $s\left(1+\frac{3}{s}\right)^{1/2} = (-K)^{1/2}$

$$\Rightarrow \quad s\left(1+\frac{3}{2s}\right) = K^{1/2}(-1)^{1/2}$$

$$s + \frac{3}{2} = K^{1/2} e^{(2k+1)\pi/2}.$$

Substituting $s = \sigma + jw$:

$$\sigma + jw + \frac{3}{2} = K^{1/2}\left(\cos\frac{(2k+1)\pi}{2} + j\sin\frac{(2k+1)\pi}{2}\right)$$

So

$$\sigma + \frac{3}{2} = K^{1/2}\cos\frac{(2k+1)\pi}{2}$$

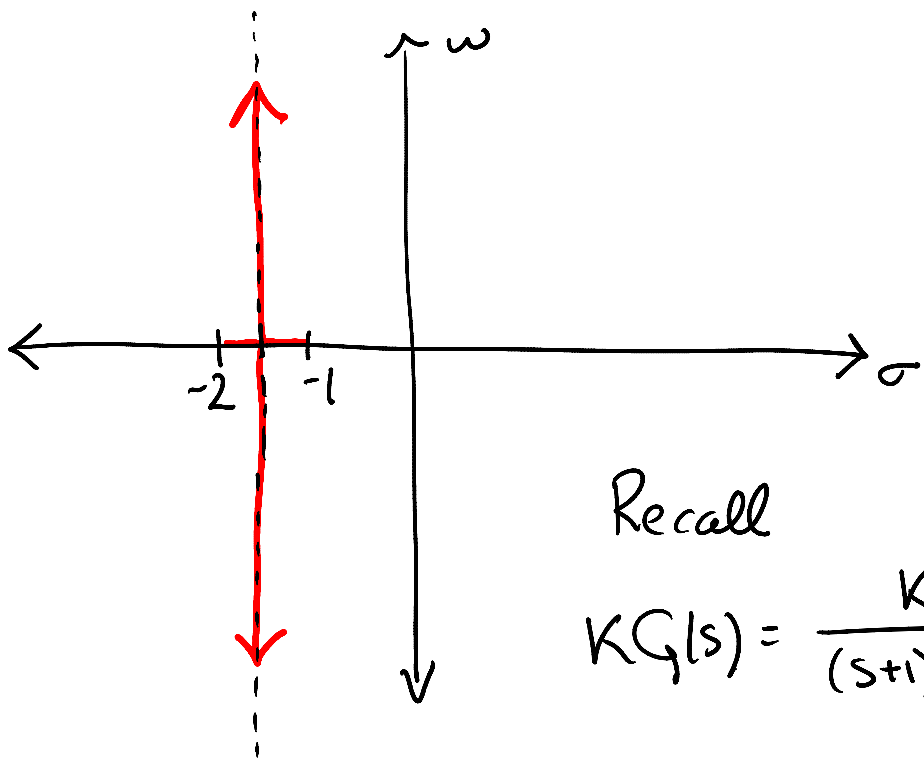
$$w = K^{1/2}\sin\frac{(2k+1)\pi}{2}$$

Then

$$\frac{\omega}{\sigma + \frac{3}{2}} = \tan \frac{(2k+1)\pi}{2}$$

$$\Rightarrow \quad \omega = \underbrace{\tan \frac{(2k+1)\pi}{2}}_{\text{slope} = \pm\infty} \left(\sigma + \frac{3}{2}\right)$$

- σ-intercept



Recall

$$KG(s) = \frac{K}{(s+1)(s+2)}$$

Don't worry, you don't have to do this every time! There is a general formula.

$\boxed{\text{Rule 4}}$ The zeros at infinity lay on the asymptotes with real axis intercept σ_0 and angle θ where.

$$\sigma_0 = \frac{\sum \text{finte poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}}$$

$$\theta = \frac{(2K+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}}$$

Example: $KG_cG_p = \dfrac{K}{(s+1)(s^2+1)}$

$$\sigma_0 = \frac{(-1+j-j)-0}{3-0} = -1/3$$

$$\theta = \frac{(2k+1)\pi}{3} = \frac{1}{3}\pi, \pi, \frac{5}{3}\pi, \ldots$$

$$= 60^\circ, 180^\circ, 300^\circ, \ldots$$