Lecture 2b: A Taxonomy of Linear Systems

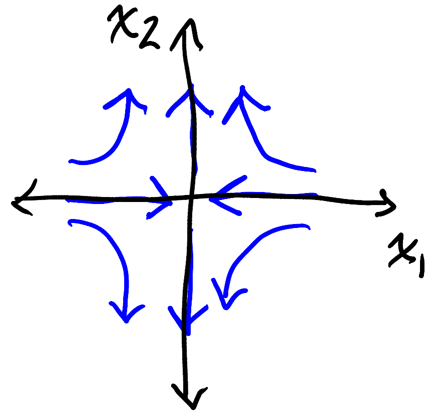
... in which we see that stability all comes down to the eigenvalues.

I. The Various Cases

$\boxed{\text{Case 1:}}$ The eigenvalues of A are real and have opposite signs. Then $\dot{\vec{x}} = A\vec{x}$ is a saddle:

Ex: $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$

$A\vec{x} = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}$



$\boxed{\text{Case 2:}}$ All eigenvalues have negative real parts. Then $\dot{\vec{x}} = A\vec{x}$ is a sink.

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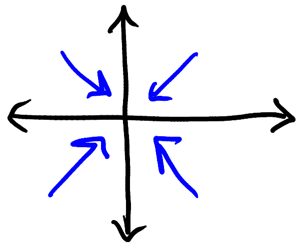
Sinks come in varieties:

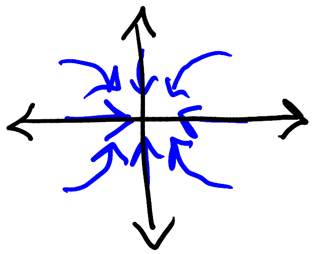
(2a) Real

$A = \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix}$



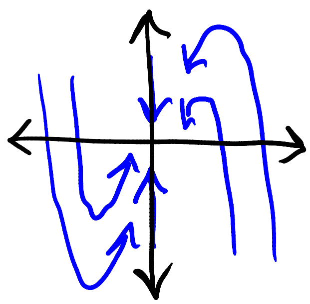
"Focus"

$A = \begin{pmatrix} -\lambda & 0 \\ 0 & -\mu \end{pmatrix}$



"Node"

$A = \begin{pmatrix} -\lambda & 0 \\ 1 & -\lambda \end{pmatrix}$

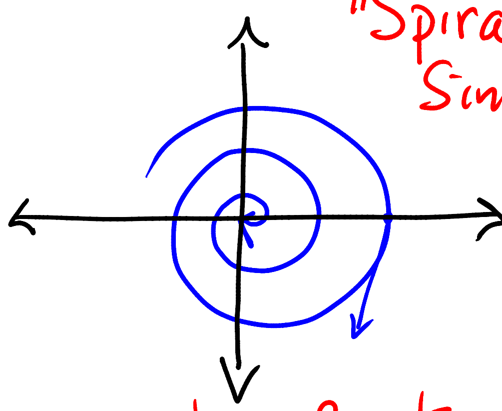


"Improper Node"

$A = \begin{pmatrix} -a & -b \\ b & -a \end{pmatrix}$

$\lambda = -a \pm bj$

$a > 0$



"Spiral Sink"

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} -a \\ b \end{pmatrix}$

$b > 0 \Rightarrow$ counterclockwise

$b < 0 \Rightarrow$ clockwise

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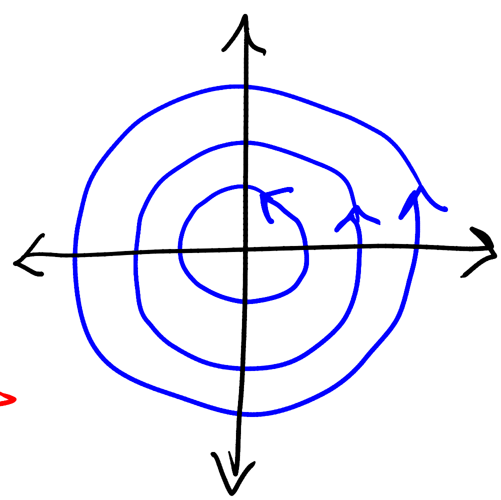
$\boxed{\text{Case 3:}}$ Eigenvalues have positive real parts. Then the system is a source. The sub-cases are like case 2 but with the arrows reversed.

$\boxed{\text{Case 4:}}$ The eigenvalues are pure imaginary: Then the system is a center.

eg $A = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}$



$b > 0 \Rightarrow$ counter clockwise

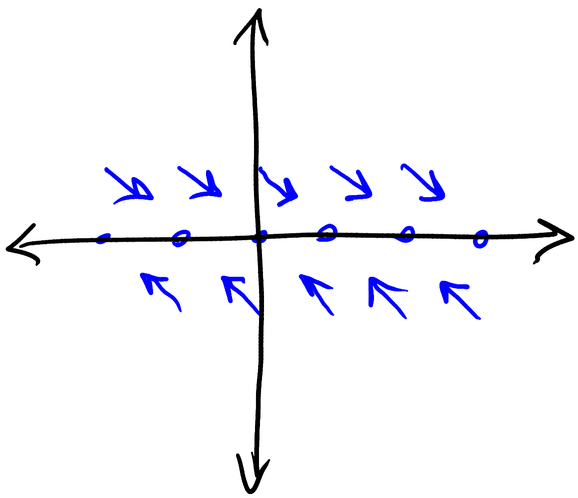
$b < 0 \Rightarrow$ clockwise

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ b \end{pmatrix}$

$\boxed{\text{Case 5:}}$ One eigenvalue is 0.

Then there is an attracting or repelling subspace depending on whether the other eig. is positive or negative.

e.g.



$$\dot{\vec{x}} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \vec{x} = \begin{pmatrix} x_2 \\ -x_2 \end{pmatrix}$$

$\lambda = 0, -1$

$\hookrightarrow$ vec. $= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Notes:

- sinks are called stable.

$$\boxed{\vec{x}(t) \longrightarrow \vec{0} \quad \text{as} \quad t \to \infty}$$

- centers are called marginally stable.
- $\lambda = 0$, neg. are also marginally stable.

$$\boxed{\vec{x}(t) \text{ does not explode}}$$

- all other situations are unstable

$$\vec{x}(t) \longrightarrow \infty \quad \text{as} \quad t \to \infty.$$

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II. An Example

Say that we wish to draw the phase portrait of the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ (x_1-1)(x_1+1) - x_2 \end{pmatrix}$$

To find equilibrium points, solve

$$\vec{0} = f(\vec{x}) = \begin{pmatrix} x_2 \\ (x_1-1)(x_1+1) - x_2 \end{pmatrix}$$

$$\Rightarrow x_2 = 0$$

$$\Rightarrow (x_1-1)(x_2+1) = 0$$

$$\Rightarrow x_1 = \pm 1.$$

Next, we figure out what kind of equilibrium each point is.

The Jacobian is

$$J = \begin{pmatrix} \frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial x_2} \\ \frac{\partial (x_1-1)(x_1+1)-x_2}{\partial x_1} & \frac{\partial (x_1-1)(x_1+1)-x_2}{x_2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 2x_1 & -1 \end{pmatrix}.$$

We next determine how the system looks near each point.

$$\boxed{\vec{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}} \Rightarrow A = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 2 & \lambda+1 \end{vmatrix} = \lambda(\lambda+1)+2$$

$$= \lambda^2 + \lambda + 2$$

$$\lambda = \frac{-1 \pm \sqrt{1-8}}{2} = \frac{-1 \pm \sqrt{7}\, i}{2}$$

Note: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -2 \end{pmatrix}$

$\Rightarrow$ A clockwise spiral sink!

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$$\boxed{\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}} \qquad A = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ -2 & \lambda+1 \end{vmatrix} = \lambda(\lambda+1) - 2$$

$$= \lambda^2 - \lambda - 2$$

$$\lambda = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = 1, -2$$

(saddle).

$$\boxed{\lambda = 1} \Rightarrow A\vec{v} = \vec{v} \iff \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\iff \left.\begin{matrix} x_1 = x_2 \\ 2x_1 - x_2 = x_2 \end{matrix}\right\} \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boxed{\lambda = 2} \Rightarrow A\vec{v} = -2\vec{v} \iff \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2x_1 \\ -2x_2 \end{pmatrix}$$

$$\iff x_2 = -2x_1 \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

So the final phase portrait is approximately: