Lecture 3b

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Lecture 3b: A Preview!

... in which we make a simple controller for the pendulum.

I. Proportional Control of a Pendulum

Recall the pendulum model

$$\begin{pmatrix} \dot{\theta} \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} \omega \\ \sin\theta - \omega + u \end{pmatrix}$$

torque

$$y = \theta$$

The problem is to

1. Find the linearized system at $\begin{pmatrix} \theta \\ \omega \end{pmatrix} = \vec{0}$ and $u = 0$.

2. Use feedback to control the pendulum.

① $A = \frac{\partial f}{\partial \vec{x}} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ (see lecture 2a)

$$B = \frac{\partial f}{\partial u} = \begin{pmatrix} \frac{\partial \omega}{\partial u} \\ \frac{\partial (\sin\theta - \omega + u)}{\partial u} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

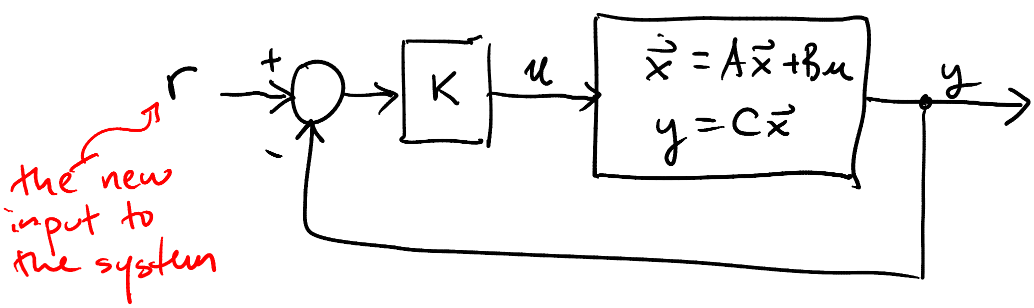
$$C = \frac{\partial g}{\partial \vec{x}} = \begin{pmatrix} \frac{\partial \theta}{\partial \theta} & \frac{\partial \theta}{\partial \omega} \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

So the linearized system is

$$\begin{pmatrix} \dot{\theta} \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \theta \\ \omega \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \theta \\ \omega \end{pmatrix}$$

(2) To use feedback to control the system so that $\begin{pmatrix}\theta\\ \omega\end{pmatrix} = \vec{0}$ is stable, we feed the output through a negative gain back to the input:



Thus, $u = -Ky$ where $K > 0$ is a gain. This gives

$$\begin{pmatrix}\dot{\theta}\\ \dot{\omega}\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\theta\\ \omega\end{pmatrix} \quad \begin{pmatrix}0\\ 1\end{pmatrix} K \overbrace{\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\theta\\ \omega\end{pmatrix}}^{y}$$

$$= \left[\begin{pmatrix}0 & 1\\ 1 & -1\end{pmatrix} - \begin{pmatrix}0 & 0\\ K & 0\end{pmatrix}\right]\begin{pmatrix}\theta\\ \omega\end{pmatrix}$$

$$= \begin{bmatrix}0 & 1\\ 1-K & -1\end{bmatrix}\begin{pmatrix}\theta\\ \omega\end{pmatrix}.$$

To see how this works, lets find the response when $K=1$. Then

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \Rightarrow |\lambda I - A|$$

$$= \begin{vmatrix} \lambda & -1 \\ 0 & \lambda+1 \end{vmatrix}$$

$$\Rightarrow \lambda = 0, -1.$$

This means the system is neutrally stable.

Now try a stronger gain: $K = \frac{3}{2}$. Then

$$A = \begin{pmatrix} 0 & 1 \\ -1/2 & -1 \end{pmatrix} \Rightarrow |\lambda I - A|$$

$$= \begin{vmatrix} \lambda & -1 \\ 1/2 & \lambda+1 \end{vmatrix}$$

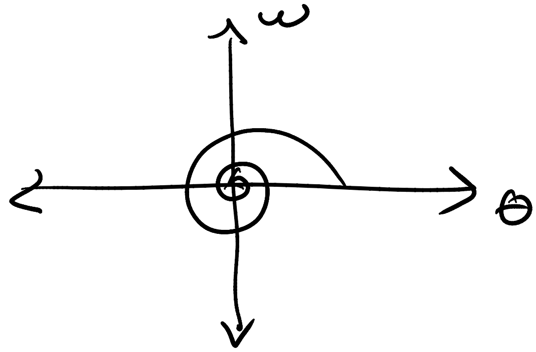
$$= \lambda(\lambda+1) + 1/2$$

$$= \lambda^2 + \lambda + 1/2$$

stable spiral $\Leftarrow$ $\Rightarrow \lambda = -\frac{1}{2} \pm \frac{j}{2}.$

Furthermore, because we know that a stable system converges to its input, we have that as $t \to \infty$, the natural dynamics die out, leaving only the input to the closed-loop system:

$\boxed{r=0}$



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