Lecture 16a: Optimal Control

... in which we devise a new way to represent performance specifications and then come up with an automatic way to realize them.

I. Performance

Suppose we are concerned with regulating a system so that the state $\vec{x}$ stays at $\vec{0}$.

One way to specify this is to require that

$$\int_0^\infty \vec{x}^T \vec{x}\, dt$$

is as small as possible. Note

$$\vec{x}^T \vec{x} = x_1^2 + x_2^2 + \cdots + x_n^2$$

$$= \vec{x} \cdot \vec{x}.$$

We might also want to weight different states differently. For example, say

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We might want

$$\int_0^\infty (x_1^2 + 2x_2^2)\,dt$$

to be small. This would drive x_2 to 0 faster than x_1. Note

$$x_1^2 + x_2^2 = (x_1 \quad x_2)\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \vec{x}^T Q \vec{x}.$$

So the general form is

$$\int_0^\infty \vec{x}^T Q \vec{x}\,dt.$$

Of course, the best way to minimize this integral is to set

$$u = -K\vec{x}$$

where $|K|$ is <u>huge</u>. But this is impractical.

Another goal we might have is to minimize the effort (e.g. fuel) required to stablize the system. We can specify this by requiring that

$$\int_0^\infty u^2 \, dt$$

be as small as possible.

Now the trouble is that we could just set K as small as possible, which means T_s is big.

A comprimise is to minimize the "cost function"

$$J = \int_0^\infty (\vec{x}^T Q \vec{x} + Ru^2)\,dt$$

Here $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}$ weigh the relative importance of minimizing the various states quickly and minimizing the effort.

II. The 1-dimensional Case

Suppose $x \in \mathbb{R}$ and

$$\dot{x} = ax + bu.$$

The goal is to find $u = -kx$ so that $J = \int_0^\infty (qx^2 + ru^2)\,dt$

is small.

- First, note that

$$\dot{x} = ax + bu = \underbrace{(a-bk)}_{\text{define } h = a-bk} x = hx.$$

- Next, consider the equation

$$\frac{d}{dt}(px^2) = -(q + rk^2)x^2$$

which we will need shortly.
Applying the derivative gives

$$2px\dot{x} = 2phx^2 = -(q+rk^2)x^2$$

So $p = -\dfrac{q+rk^2}{2(a-bk)}$.

- Now, consider

$$J = \int_0^\infty (qx^2 + ru^2)\,dt$$

$$= \int_0^\infty (qx^2 + rk^2x^2)\,dt$$

$$= \int_0^\infty (q + rk^2) x^2 \, dt$$

$$= -\int_0^\infty \frac{d}{dt}(p x^2) \, dt$$

$$= -p x^2 \Big|_0^\infty$$

Note that we assume that $x(\infty) = 0$ since we plan to choose $u = -kx$ to ensure this. Thus,

$$J = p \, x(0)^2$$

To minimize J, we need to find k to minimize p, which we could do by setting the derivative to 0.

$$\frac{d}{dt}\left(\frac{q+rk^2}{2(a-bk)}\right)=0$$

$$\Rightarrow \quad 2rk\,2(a-bk)+(q+rk^2)2b=0$$

$$\Rightarrow \quad rbk^2-2ark-qb=0$$

$$\Rightarrow \quad k=\frac{2ar\pm\sqrt{4a^2r^2+4rbrq}}{2rb}$$

$$=\frac{ar\pm\sqrt{a^2r^2+rqb^2}}{rb}.$$

Note that $a-bk$ must be negative for $x\to 0$, so $k>a/b$. We choose whichever solution makes this true.

<u>Example</u>: Consider

$$\dot{x} = x + u$$

$$J = \int_0^\infty (x^2 + u^2)\,dt.$$

The optimal k is

$$k = \frac{ar \pm \sqrt{a^2 r^2 + rqb^2}}{rb}$$

$$= \frac{1 \pm \sqrt{1+1}}{1} = \boxed{1+\sqrt{2}}.$$

III. The SISO Case

Now, suppose $\vec{x} \in \mathbb{R}^n$,

$$\dot{\vec{x}} = A\vec{x} + Bu$$

and $$J = \int_0^\infty (x^T Q x + Ru^2)\,dt.$$

Once again, suppose that there is a matrix P such that

$$\frac{d}{dt}\left(\vec{x}^T P \vec{x}\right) = -\vec{x}^T\left(Q + K^T R K\right)\vec{x}.$$

Then,

$$\dot{\vec{x}}^T P \vec{x} + \vec{x}^T P \dot{\vec{x}}$$

$$= \left[(A-BK)\vec{x}\right]^T P \vec{x} + \vec{x}^T P (A-BK)\vec{x}.$$

$$= \vec{x}^T (A-BK)^T P \vec{x} + \vec{x} P (A-BK)\vec{x}$$

$$= -\vec{x}^T\left(Q + K^T R K\right)\vec{x}$$

Or,

$$(A-BK)^T P + P(A-BK) + Q + K^T R K = 0. \quad \text{(EQN 1)}$$

Now,

$$J = \int_0^\infty (\vec{x}^T Q \vec{x} + R u^2) dt$$

$$= \int_0^\infty \left(\vec{x}^T Q \vec{x} + R (K\vec{x})^T K\vec{x} \right) dt$$

$$= \int_0^\infty \vec{x}^T (Q + K^T R K^T) \vec{x} \, dt$$

$$= \vec{x}^T(0) \, P \, \vec{x}(0).$$

Thus, we need to find K to minimize P subject to the constraint $\left(\begin{smallmatrix}\text{EQN} \\ 1\end{smallmatrix}\right)$. It turns out:

$$K = \frac{1}{R} B^T P$$

and P solves the "Algebraic Riccati Eqn"

$$A^T P + PA - PBR^{-1}B^T P + Q = 0.$$

Furthermore,

- $A-BK$ has ∞ gain margin.
- $A-BK$ has $60°$ phase margin.

It turns out you can do this with a computer pretty easily.

In MATLAB:

$$K = \text{lqr}(A, B, Q, R)$$

where

- Q is positive semidefinite (i.e. $|Q| \geq 0$)
- $R > 0$

and A and B are the system description.