Lecture 8a

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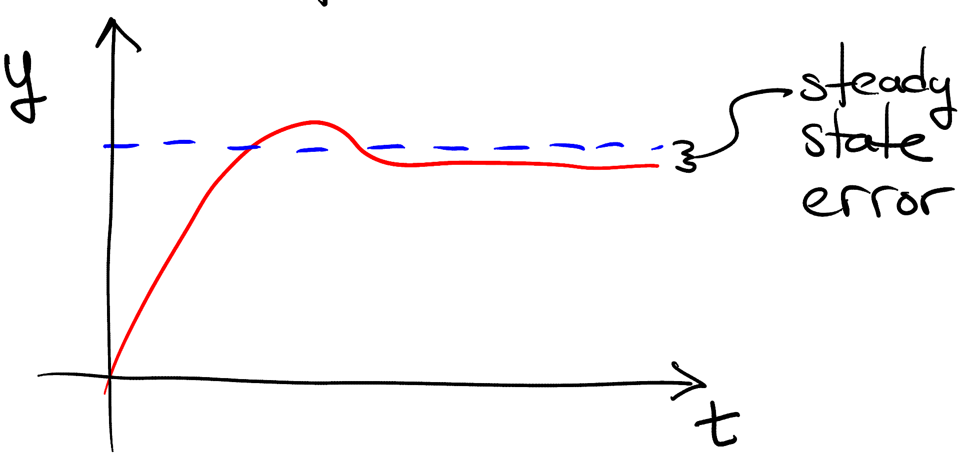
Lecture 8a: Steady State Error

... in which we examine in detail the effects of various control schemes on the steady state error.

I. Steady State Error

The steady state error is the difference between the input and the output after the system has come to equilibrium.



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In the frequency domain, the error is

$$\underbrace{E(s)}_{\text{error}} = \underbrace{R(s)}_{\text{input}} - \underbrace{Y(s)}_{\text{output}}$$

Using the final value theorem,

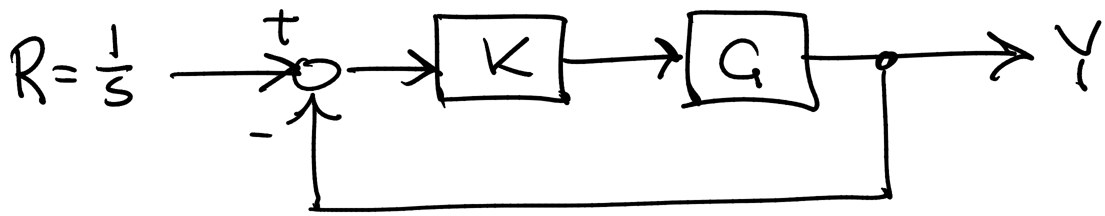
$$\boxed{\begin{aligned} e(\infty) &= r(\infty) - y(\infty) \\ &= \lim_{s \to 0} s\left(R(s) - Y(s)\right). \end{aligned}}$$

Example: Open Loop with Unit Step Input

$$R \longrightarrow \boxed{G} \longrightarrow Y$$

$$e(\infty) = \lim_{s \to 0} s\left(\frac{1}{s} - \frac{1}{s}G\right) = 1 - \underbrace{G(0)}_{\text{this is called the DC Gain.}}$$

Example: Closed Loop w/ step input



First, determine Y and E

$$Y(s) = R(s) \frac{KG(s)}{1 + KG(s)}$$

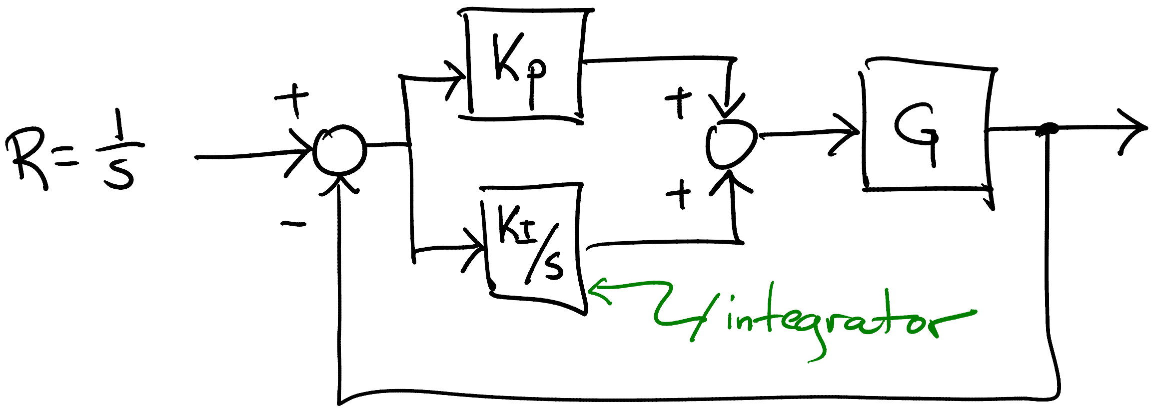
$$E(s) = R(s) - Y(s) = \frac{1}{s}\left(1 - \frac{KG}{1+KG}\right)$$

$$= \frac{1}{s}\left(\frac{1}{1+KG}\right)$$

Then use the F.V.Θ.

$$e(\infty) = \lim_{s \to \infty} sE(s) = \underbrace{\frac{1}{1+KG(0)}}_{\text{This can be made arbitrarily small}}$$

Example: Let's try



$$T(s) = \frac{\frac{K_p s + K_I}{s} G(s)}{1 + \frac{K_p s + K_I}{s} G(s)}$$

$$= \frac{(K_p s + K_I) G(s)}{s + (K_p s + K_I) G(s)}$$

$$\lim_{s \to 0} s\left(R(s) - Y(s)\right) = \lim_{s \to 0} s\left(\frac{1}{s} - \frac{1}{s} T(s)\right)$$

$$= \lim_{s \to 0}\left(1 - T(s)\right) = 1 - \frac{K_I G(0)}{K_I G(0)} = \boxed{0}.$$

cool!

II. Steady State Error in S.S.

For a state space system of the form

$$\dot{\vec{x}} = A\vec{x} + Bu$$

$$y = C\vec{x}$$

we have that the error is

$$e(t) = u(t) - y(t)$$

$$= u(t) - C\underbrace{\vec{x}_{ss}}_{\text{this is the state at equilibrium}}$$

To find $\vec{x}_{ss}$, we use

$$\dot{\vec{x}}_{ss} = 0 = A\vec{x}_{ss} + Bu$$

So $$\vec{x}_{ss} = -A^{-1}Bu$$

Putting this in the above gives:

$$e(t) = u(t) + CA^{-1}Bu(t)$$
$$= u(t)\left[1 + CA^{-1}B\right]$$

Taking the limit gives

$$\boxed{e(\infty) = u(\infty)\left[1 + CA^{-1}B\right]}.$$

Example: Say that

$$\dot{\vec{x}} = \begin{pmatrix} 0 & 1 \\ -2-K & -3 \end{pmatrix}\vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}u$$

$$y = \begin{pmatrix} K & 0 \end{pmatrix}\vec{x}$$

And suppose $u(t) = 1$.

$$e(\infty) = 1 + CA^{-1}B$$

$$= 1 + (K \;\; 0)\,\frac{1}{2+K}\begin{pmatrix} -3 & -1 \\ 2+K & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= 1 + \frac{1}{2+K}(K \;\; 0)\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= 1 + \frac{-K}{2+K} = \boxed{\frac{2}{2+K}}.$$

So choosing K large decreases the steady state error to 0.