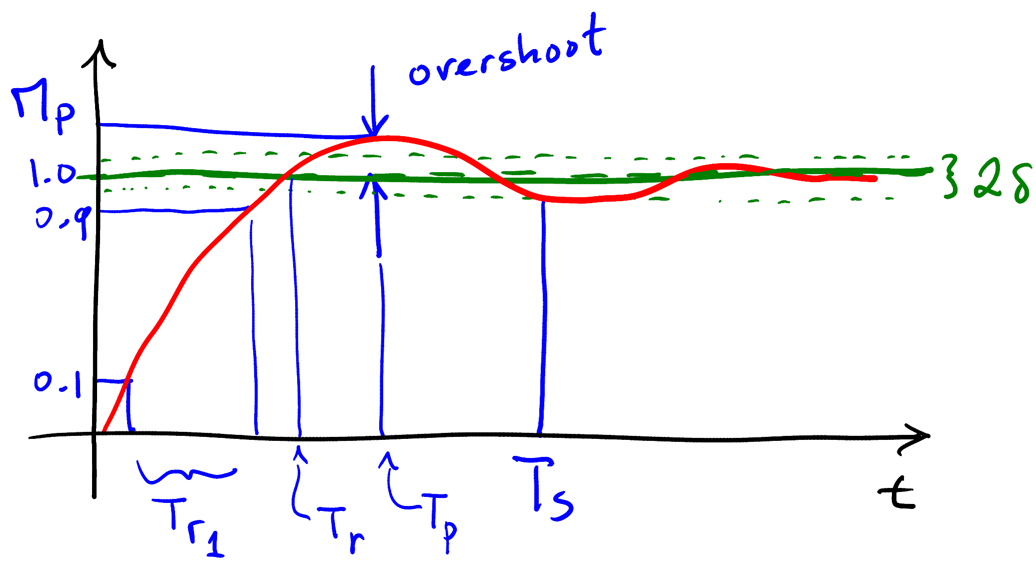
Lecture 9b

Wednesday, October 26, 2005
9:38 PM

Lecture 9b: Performance Measures

... in which we define several ways to characterize how well a system performs.

I. Performances Measures



T_s = settling time

T_p = peak time

T_r = rise time (T_{r_1} for damped)

M_p = peak response

P.O. = $M_p - 1$ = % overshoot

① Overshoot (T_p, M_p and P.O.)

The overshoot can be found by looking at the first critical point of $y(t)$. That is, we set

$$\dot{y}(t) = 0 = \frac{\omega}{\beta} e^{-\zeta \omega t} \sin(\beta \omega t).$$

This is 0 when $\sin(\beta\omega t) = 0$

or when

$$\omega\beta t = 0, \pi, 2\pi, \ldots$$

↳ first peak

So $$= \boxed{\frac{\pi}{\omega\sqrt{1-\zeta^2}}}$$

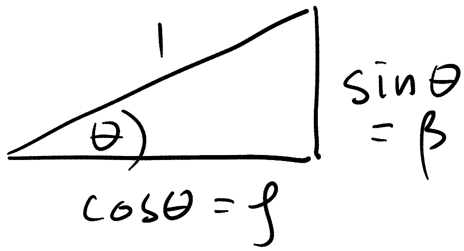
The maximum response is $y(T_p)$ or

$$M_p = 1 - \frac{1}{\beta} e^{-\omega \zeta T_p} \sin(\underbrace{\beta\omega T_p}_{\pi} + \theta)$$

Wednesday, October 26, 2005
10:04 PM

Now, $\sin(\pi + \theta) = -\sin\theta$

$$= -\sin(\cos^{-1}\zeta) = Q$$



$$\sin^{-1}(-Q) = \cos^{-1}\zeta.$$

Thus $Q = -\beta$.

So we get

$$M_p = 1 + \frac{1}{\cancel{\beta}} e^{-\omega\zeta T_p} \cancel{\beta}$$

$$= 1 + e^{-\cancel{\omega}\zeta\frac{\pi}{\cancel{\omega}\beta}} = \boxed{1 + e^{-\zeta\pi/\beta}}$$

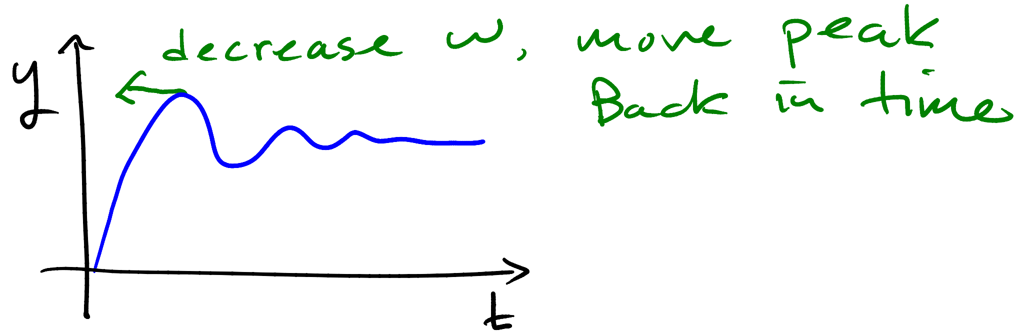
Finally,

$$P.O. = (M_p - 1) \times 100$$

$$= \boxed{e^{-\zeta\pi/\beta} \times 100}$$

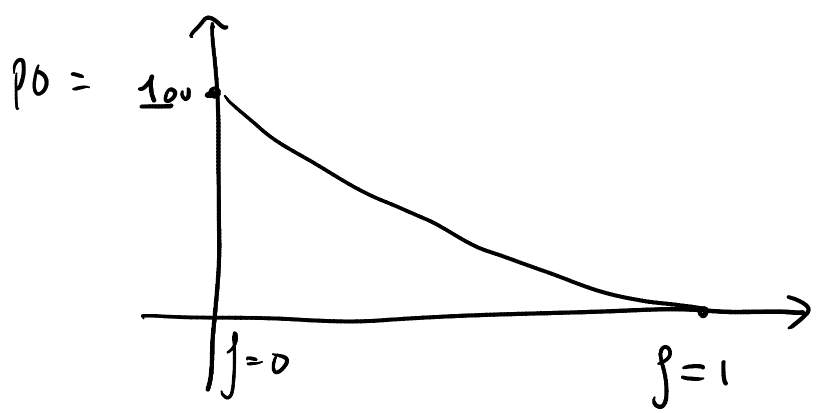
Wednesday, October 26, 2005
10:10 PM

Note that P.O. does not depend on ω, although T_p does.



Thus, increasing ω increases the aggressiveness of the system.

Also note that increasing ζ decreases P.O:



② Settling time

This is the time for the system to settle into $[1-\delta, 1+\delta]$. We have

$$y(t) \approx 1 - \frac{1}{\beta} e^{-\zeta\omega t}$$

when

$$e^{-\zeta\omega t} \lesssim \delta$$

$$\Rightarrow \quad -\zeta\omega t = \ln\delta$$

so

$$T_s = -\frac{\ln\delta}{\zeta\omega}.$$

We usually take $\delta = 0.02$. So

$$T_s = -\frac{\ln 0.2}{\zeta\omega} \approx \boxed{\frac{4}{\zeta\omega}}$$

Wednesday, October 26, 2005
10:18 PM

We call $\frac{1}{\zeta\omega}$ the time constant $\boxed{\tau}$ of the system.

So $\boxed{T_s = 4\tau}$.

Now we see the effect of decreasing ζ

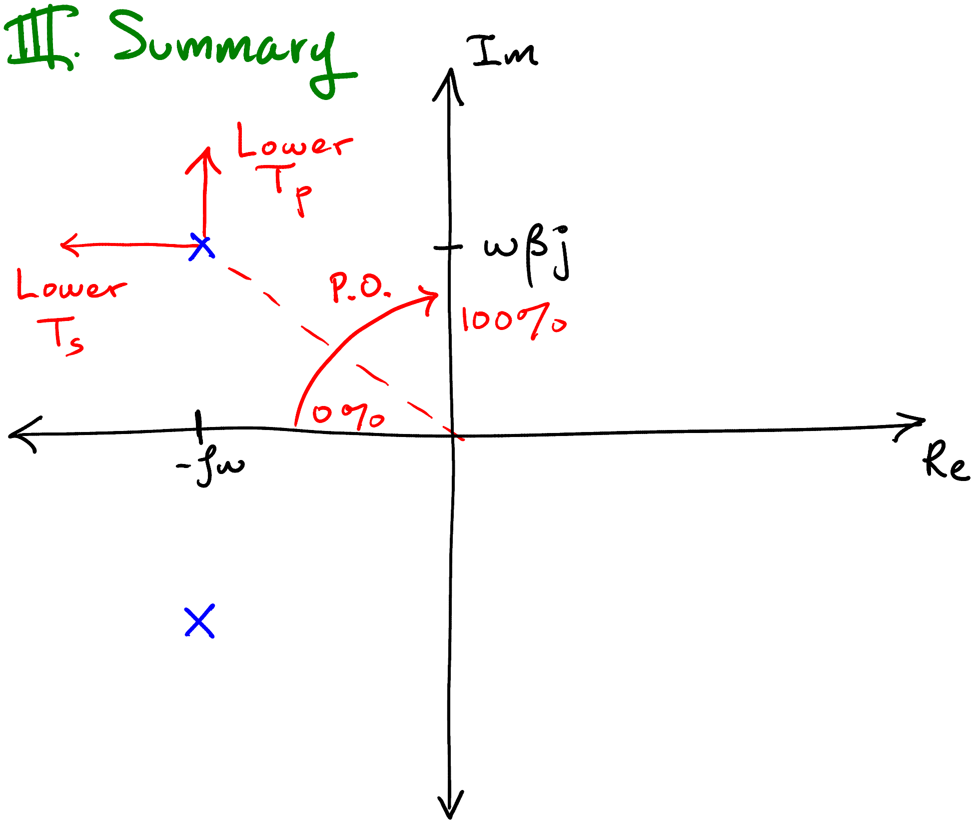
ζ goes down

$\Downarrow$

M_p goes up
and
T_s goes up

So you settle in slower and overshoot more when you decrease the damping.

III. Summary



$$\boxed{T_p = \frac{\pi}{\omega\beta}}$$

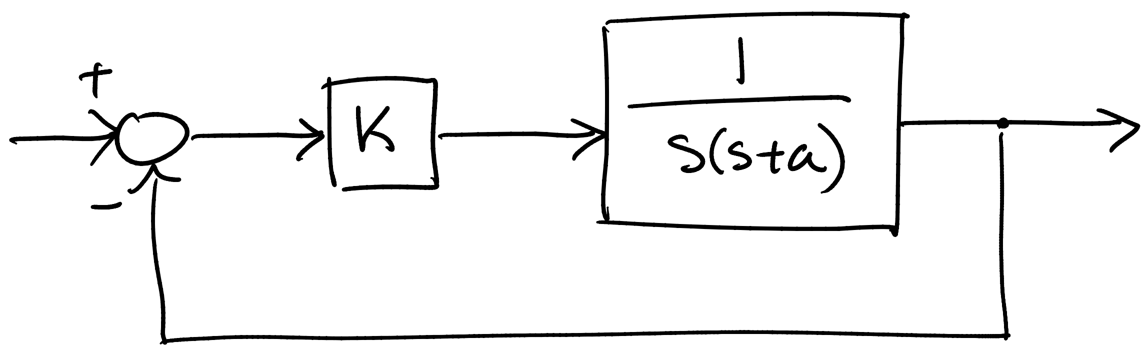
$$\boxed{M_p = 1 + e^{-\zeta\pi/\beta}}$$

$$\boxed{P.O. = e^{-\zeta\pi/\beta} \times 100}$$

$$\boxed{T_s = \frac{4}{\zeta\omega} = 4\tau}$$

IV. Control Design

Consider the system



where K is a tunable gain, and a is a design parameter.

Then

$$T(s) = \frac{K}{s^2 + as + K}$$

and

$$\omega = \sqrt{K}$$

$$2\zeta\omega = a \Rightarrow \zeta = \frac{a}{2\omega} = \frac{a}{2\sqrt{K}}.$$

We can choose K and a so that the resulting system behavior meet a given performance specification. For example, say we want

$$T_s = 5\,\text{sec}$$

$$P.O. = 25\%$$

Then we put

$$\frac{4}{\zeta\omega} = 5 \Rightarrow \frac{a}{2} = 5 \Rightarrow \boxed{a = 10}$$

$$e^{-\zeta\pi/\beta} = 0.25 \Rightarrow \frac{\zeta\pi}{\beta} = 1.4$$

$$\Rightarrow \frac{\frac{10\pi}{2\sqrt{K}}}{\sqrt{1 - \frac{100\pi^2}{4K}}} = 1.4 \Rightarrow \boxed{K \approx 150}$$