Lecture 13b

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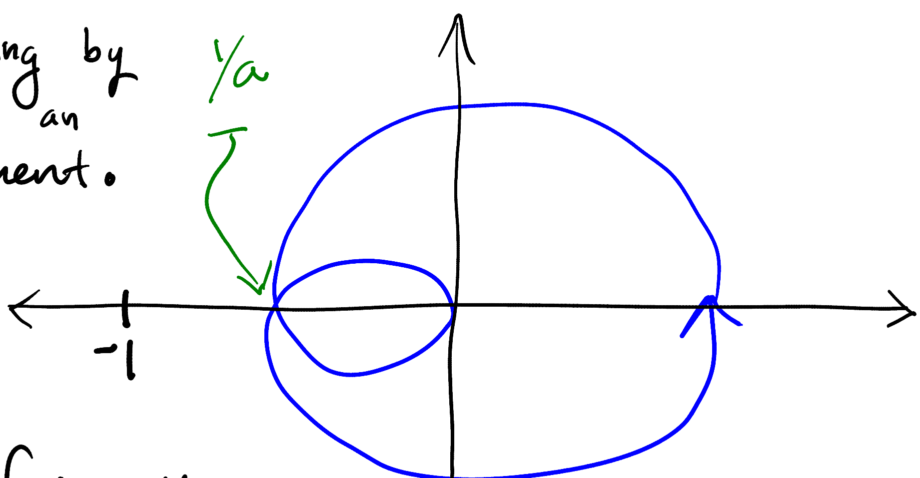
Lecture 13b : Gain and Phase Margin

... in which we use the Nyquist and Bode plots to understand robustness.

I. Gain and Phase Margin

Multiplying by a gives an encirclement.



We define the gain margin to be

$20 \log_{10} a = M_{gm}$

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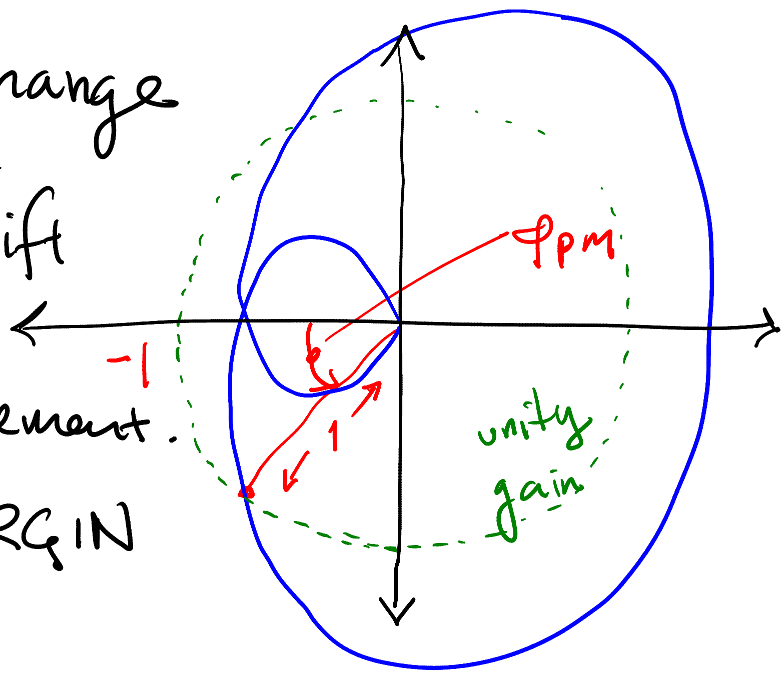
φ_{pm} is the change in the open loop phase shift required to change encirclement.
= PHASE MARGIN



Recall that the phase of $G(s)$ is

$$\varphi = \tan^{-1}\left(\frac{Im(G(j\omega))}{Re(G(j\omega))}\right)$$

Thus, the phase margin is the amount φ can change before instability.

Example: $$G(s) = \frac{s-1}{s^2 + \frac{1}{2}s + 2}$$

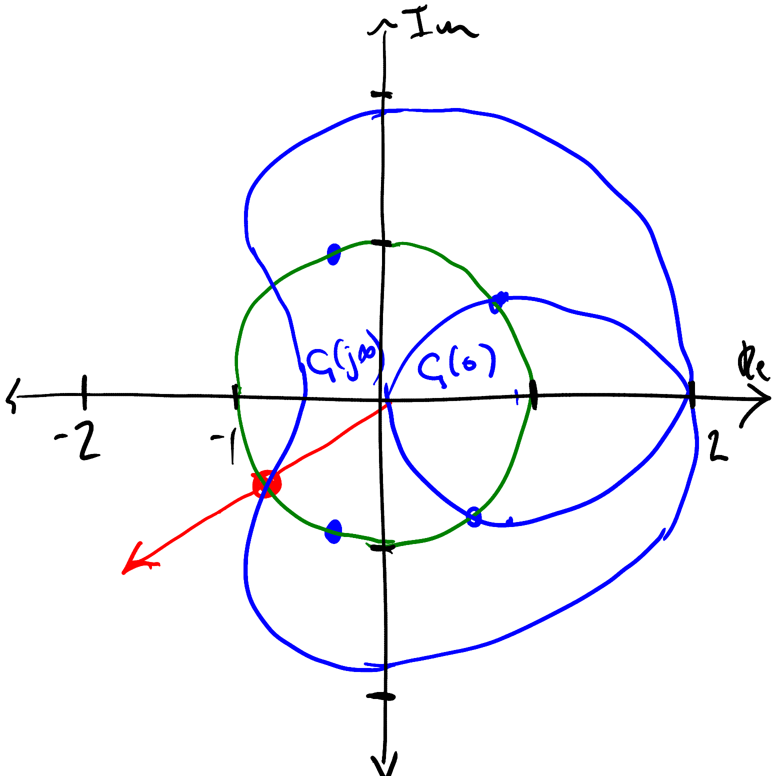
- open loop poles

$$s = \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - 4 \times 2}}{2} = \frac{1}{4} \pm \frac{\sqrt{31}}{4}$$

ω_s 1.39

- nyquist plot

ω	$G(j\omega)$
$-\infty$	0
-2	$4/5 + 3/5\, j$
-1	$-2/5 - 6/5\, j$
0	$-1/2$
1	$-2/5 + 6/5\, j$
2	$4/5 - 3/5\, j$
∞	0



$P = 0$
$N = 0 \Rightarrow Z = 0$

GAIN MARGIN: $20 \log_{10} \frac{1}{2} = 6\,\mathrm{dB}$

PHASE MARGIN: To compute the phase margin, we have to find the point where

$$|G(j\omega)| = 1$$

$$\left| \frac{j\omega - 1}{(j\omega)^2 + \frac{1}{2}j\omega + 2} \right| = \frac{|j\omega - 1|}{|-\omega^2 + \frac{1}{2}j\omega + 2|} = 1$$

$$\Rightarrow \quad \omega^2 + 1 = (2-\omega^2)^2 + \frac{1}{4}\omega^2$$

$$\omega^2 + 1 = 4 - 4\omega^2 + \omega^4 + \frac{1}{4}\omega^2$$

$$4\omega^2 + 4 = 16 - 16\omega^2 + 4\omega^4 + \omega^2$$

$$19\omega^2 - 4\omega^4 - 12 = 0$$

$$\omega = \pm 2, \underbrace{\pm \frac{\sqrt{3}}{2}}$$

we care about this one

At this point:

$$G\left(-\frac{\sqrt{3}}{2}j\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}j.$$

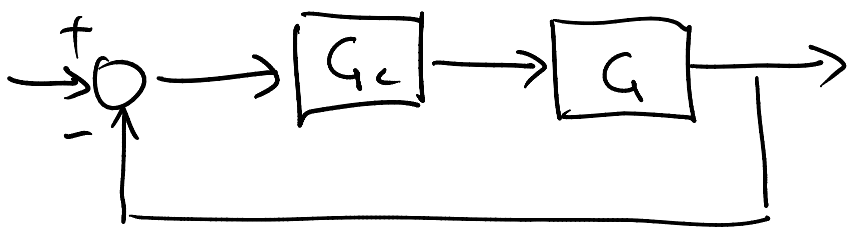
Thus, the phase margin is

$$\varphi_{pm} = \angle\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} j\right) - 180$$

$$= 30°$$

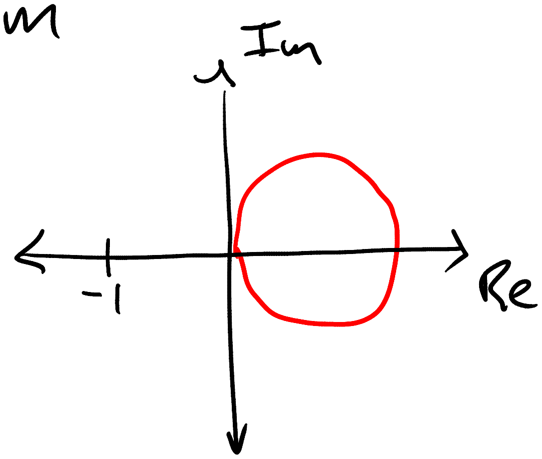
II. Interpretation

Gain and phase margin are measures of robustness to model uncertainty. For example, if we have a controlled system



then we are guaranteed stability even when G is replaced by $\alpha \cdot G$, $\alpha > 0$, as long as the gain margin allows it.

If you have a nyquist plot of the form



for example, you are guaranteed stability for <u>any</u> $\alpha > 0$. This is a <u>robust</u> <u>system</u>.

The phase margin has a similar interpretation. If $G(s)$ has phase margin φ_{pm} and is stable, then so is

$$e^{\theta j} G_p(s)$$

for all $\theta \in (-\varphi_{pm}, \varphi_{pm})$.

One important branch of control theory is called **robust control**. In the frequency domain, the problem is to find a controller that minimizes the gain margin and/or maximizes the phase margin. Powerful tools exist to determine such controllers automatically.

For more information see

Doyle, Francis & Tannenbaum,

Feedback Control Theory

Macmillan, 1990.

This is available free online (try Google Scholar).