Two-dimensional simulation of viscoplastic Drucker–Prager free surface flows, application to granular collapse

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Drucker–Prager model

- 2-dimensional **viscoplastic** flow with free surface.
- Time dependant domain Ω_t.
- Incompressible non-Newtonian dynamics:

$$\begin{aligned} \rho(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla_{\vec{X}})\mathbf{u}) - \mathrm{div}_{\vec{X}}\,\boldsymbol{\sigma} &= \rho\boldsymbol{f} \quad \text{in } (0,T)\times\Omega_t,\\ \mathrm{div}_{\vec{X}}\,\mathbf{u} &= 0 \quad \text{in } (0,T)\times\Omega_t, \end{aligned}$$

where $\mathbf{u}$ is the material velocity, $\boldsymbol{f}$ an external force (gravity), $\rho>0$ the mass density, and $\boldsymbol{\sigma}$ the total stress tensor defined by the Drucker-Prager rheology $\boldsymbol{\sigma} = \boldsymbol{\sigma}' - p\mathrm{Id}$, with the viscoplastic constitutive equation:

$$\mathrm{trace}(\boldsymbol{\sigma}') = 0, \quad \begin{cases} \boldsymbol{\sigma}' = 2\eta D\mathbf{u} + \kappa \dfrac{D\mathbf{u}}{\|D\mathbf{u}\|} & \text{if } D\mathbf{u} \neq 0,\\ \|\boldsymbol{\sigma}'\| \le \kappa & \text{if } D\mathbf{u} = 0. \end{cases}$$

- The plasticity $\kappa \equiv \kappa(p)$ stands for the yield limit with a **Drucker–Prager** criterion:

$$\kappa(p) = \sqrt{2}\mu[p]_+,$$

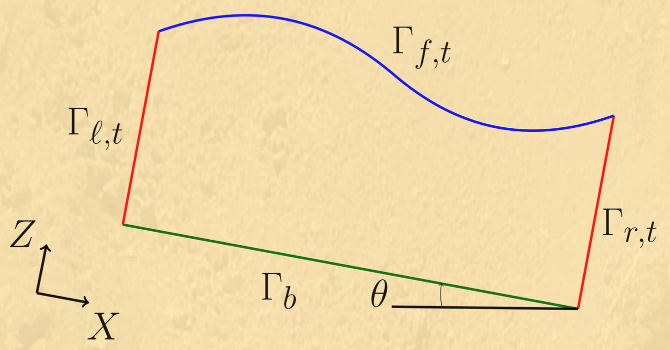
with μ an internal friction coefficient.

- The force $\boldsymbol{f}$ is the gravity force acting on the material. Taking into account the slope θ, it is given by $\boldsymbol{f} = (g\sin\theta, -g\cos\theta)$ in the coordinates $(\boldsymbol{X},\boldsymbol{Z})$, with g the gravity constant.

Domain geometry and boundary conditions

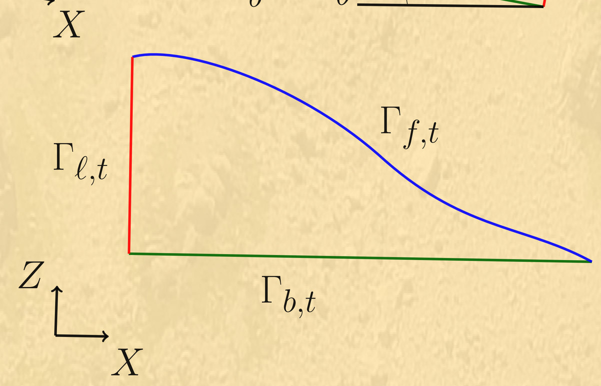
(a) periodic flow over an inclined bed:

- no-slip condition at the bottom Γ_b.
- periodicity condition on the lateral side $\Gamma_{l,t} \cup \Gamma_{r,t}$.



(b) collapse of a granular mass over a rigid or erodible bed:

- no-penetration condition on $\Gamma_{b,t} \cup \Gamma_{\ell,t}$.
- Coulomb friction condition on $\Gamma_{b,t} \cup \Gamma_{\ell,t}$.



(a)-(b)

- no-stress condition at the free surface $\Gamma_{f,t}$.
- kinematic condition at the free surface: $N_t + \vec{N}\cdot\mathbf{u}(t,\vec{X}) = 0$ on $\Gamma_{f,t}$.
- initial condition: $\mathbf{u}(0,\vec{X}) = \mathbf{u}_0(\vec{X})$.

Regularization and variational formulation

Regularized constitutive equation:

$$\boldsymbol{\sigma}'_\epsilon = 2\eta D\mathbf{u} + \kappa(p)\frac{D\mathbf{u}}{\sqrt{\|D\mathbf{u}\|^2+\epsilon^2}}, \qquad 0<\epsilon\ll 1.$$

Associated variational formulation:

$$\begin{aligned} &\int_{\Omega_t}\rho(\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u})\cdot\mathbf{v} + \int_{\Omega_t} 2\eta D\mathbf{u} : D\mathbf{v} + \int_{\Omega_t}\kappa(p)\frac{D\mathbf{u}}{\sqrt{\|D\mathbf{u}\|^2+\epsilon^2}} : D\mathbf{v} - \int_{\Omega_t} p\,\mathrm{div}\,\mathbf{v}\\ &+ \int_{\Gamma_{b,t}} \mu_b \frac{\mathbf{u}_T\cdot\mathbf{v}}{\sqrt{|\mathbf{u}_T|^2+\epsilon_f{}^2}}[p-\sigma'_N]_+ + \int_{\Gamma_{\ell,t}} \mu_\ell \frac{\mathbf{u}_T\cdot\mathbf{v}}{\sqrt{|\mathbf{u}_T|^2+\epsilon_f{}^2}}[p-\sigma'_N]_+ = \int_{\Omega_t}\rho\boldsymbol{f}\cdot\mathbf{v} + \int_{\Gamma_{f,t}} \gamma\mathcal{C}\mathbf{v}\cdot\vec{N},\\ &\int_{\Omega_t} q\,\mathrm{div}\,\mathbf{u} = 0. \end{aligned}$$

(a): $\mu_b = \mu_l = 0$.

Displacement of the domain: ($\mathbf{w}$ = mesh velocity)

$$-\mathrm{div}(D\mathbf{w}) = 0 \quad \text{in } \Omega_t,$$

boundary conditions in case (a)

$$\begin{aligned} (\mathbf{w}-\mathbf{u})\cdot\vec{N} &= 0 \quad \text{on } \Gamma_b\cup\Gamma_{f,t},\\ \mathbf{w}\cdot\vec{N} &= 0 \quad \text{on } \Gamma_{\ell,t}\cup\Gamma_{r,t},\\ (D\mathbf{w}\vec{N})_T &= 0 \quad \text{on } \Gamma. \end{aligned}$$

boundary conditions in case (b)

$$\begin{aligned} (\mathbf{w}-\mathbf{u})\cdot\vec{N} &= 0 \quad \text{on } \Gamma,\\ (D\mathbf{w}\vec{N})_T &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Discrete algorithm

Time discretization: semi-implicit finite difference scheme of first order.
Space discretization: $\mathbb{P}_2/\mathbb{P}_1$ finite element (Taylor-Hood).
Mesh mapping:

$$\begin{aligned} \mathcal{A}_{n,n+1} : \Omega_h^n &\to \Omega_h^{n+1}\\ x &\mapsto y = x + \Delta t_n\, \mathbf{w}_h^n(x) \end{aligned}$$

Update algorithm:
1) We suppose that Ω_h^n, $\mathbf{u}_h^n$, p_h^n are known and we compute $\mathbf{w}_h^n$.
2) We move the nodes of the mesh according to $\mathcal{A}_{n,n+1}$, thus we obtain Ω_h^{n+1}.
3) Finally, we compute $(\mathbf{u}_h^{n+1}, p_h^{n+1})$ on Ω_h^{n+1}.

Uniform flow with plug

Evaluation of the accuracy of the 2-dimensional regularization method (case (a)):

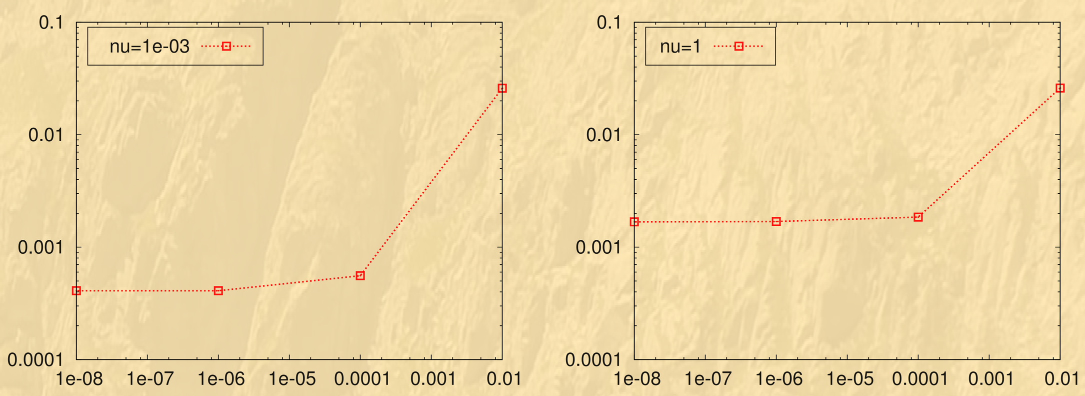
- ⋆ We consider a test case with a 1-dimensional solution $U(t,\boldsymbol{Z})$ depending only on the normal variable $\boldsymbol{Z}$.
- ⋆ The height of domain h is constant, the domain does not depend on time and the pressure is hydrostatic.

Comparaison between the 1-dimensional and the 2-dimensional solution:

- ⋆ We evaluate the error between the 1-dimensional profile U and the longitudinal component of the 2-dimensional velocity $\mathbf{u} = (U,0)$, by extending the 1-dimensional solution on the 2-dimensional mesh.
- ⋆ We compute the relative error on the velocity in the L^2-norm.

Numerical results:

- Convergence of order 1 in space and time.
- $\epsilon = 10^{-2}$: regularization error dominates.
- $\epsilon = 10^{-6}$: regularization error dominated by discretization error.



ϵ	10^{-2}	10^{-4}	10^{-6}	10^{-8}
total iterations	824	2958	5086	5272

$\epsilon = 10^{-8} \Longrightarrow$ saturation of iterations cost.

Figure 1: Velocity error with respect to ϵ.

Granular collapse: comparison with augmented Lagrangian [2]

Collapse of a trapezoidal mass over a rigid bed (case (b))

Static/flowing interface comparison:

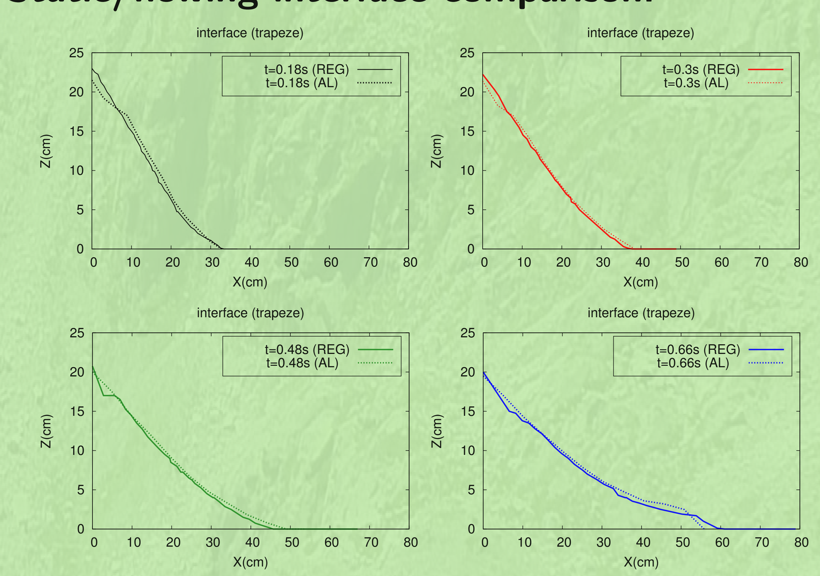


Figure 2: Comparison of the static/flowing interface.

The regularization method leads to:
- very similar profiles (with respect to AL method),
- a slightly lower position of the interface.

Freesurface comparison:

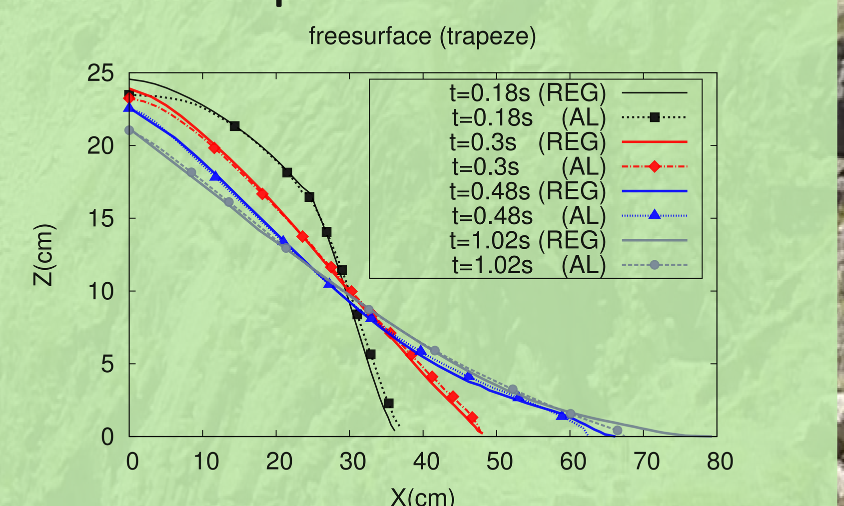


Figure 3: Evolution of the free surface ($\epsilon = 10^{-6}s^{-1}$).

The regularization method leads to:
- more decreasing thickness on the left side,
- a slightly faster front propagation (on the right),
- running 7 times faster.

Granular collapse: comparison with laboratory experiments [1]

Collapse of a trapezoidal mass over a rigid bed (case (b))

Freesurface comparison:

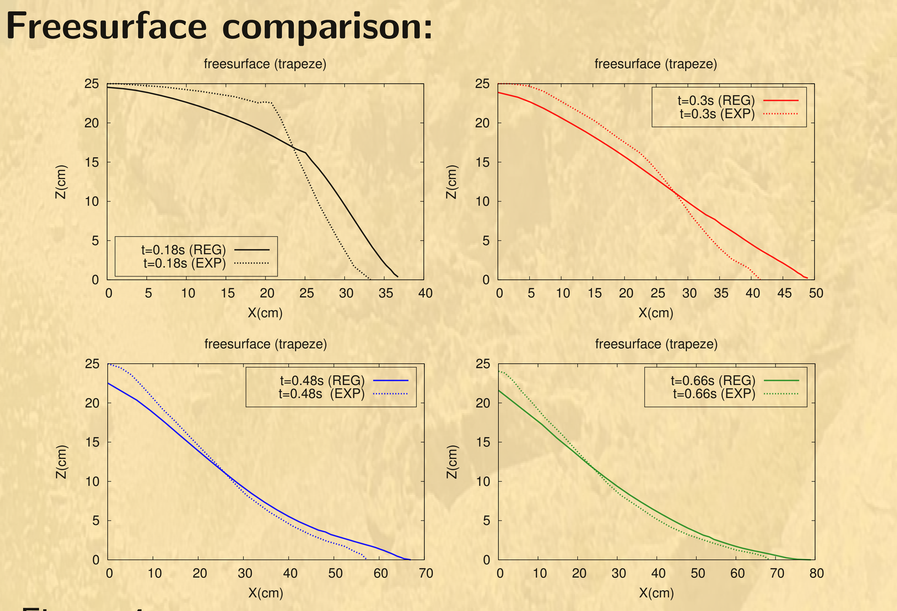


Figure 4: Evolution of the free surface for a trapezoidal mass over a rigid bed.

The regularization method leads to:
- more decreasing thickness on the left side,
- overestimated front position (on the right),
- the shape of the final deposit is very well reproduced.

Static/flowing interface comparison:

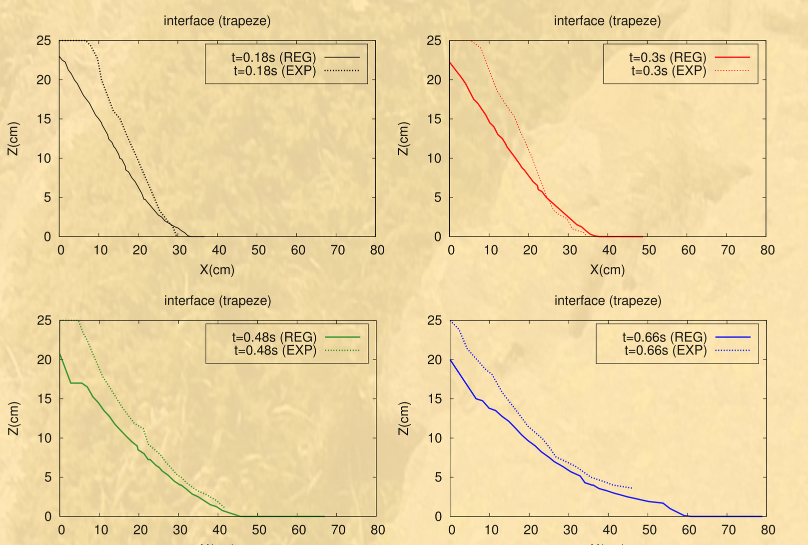


Figure 5: Comparison of the static/flowing interface.

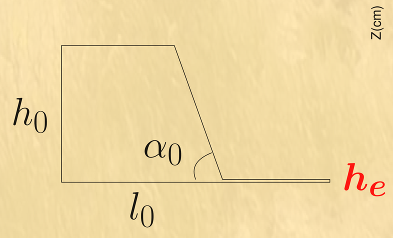
The regularization method leads to:
- pretty well predicted static/flowing interface,
- position underestimated on the top of the left side.

Numerical simulations $\Rightarrow$ faster dynamics than in experiments.

Collapse of a trapezoidal mass over an erodible bed (case (b),[3])

Goal: to model erosion process in nature.

Simulation of a granular collapse over an erodible bed made of the same material represented by a thin layer of thickness $h_e = 5$mm under the trapezoidal column.



Mean: local surface tension effects.

The regularization method with surface tension leads to:
- comparable profiles all along the simulation,
- the final shape of the deposit is very well reproduced.

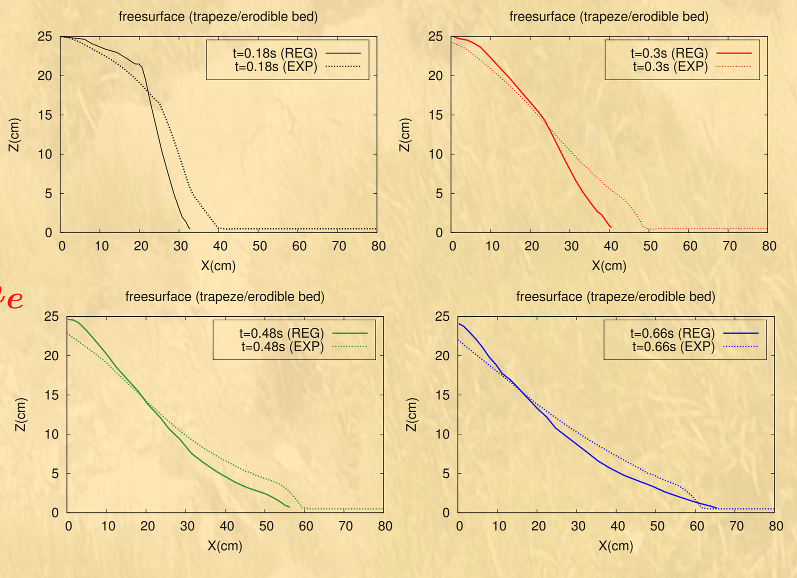


Figure 6: Evolution of the free surface for a trapezoidal mass over an erodible bed of thickness $h_e = 5$mm.

References

[1] M. Farin, A. Mangeney, O. Roche, *Fundamental changes of granular flow dynamics, deposition, and erosion processes at high slope angles: Insights from laboratory experiments*, J. Geophys. Res. Earth Surf. 119 (2014).
[2] I.R. Ionescu, A. Mangeney, F. Bouchut, O. Roche, *Viscoplastic modelling of granular column collapse with pressure dependent rheology*, hal-01080456, (2014).
[3] C. Lusso, F. Bouchut, A. Ern, A. Mangeney, M. Farin, O. Roche, *Two-dimensional simulation of viscoplastic Drucker–Prager free surface flows by regularization, and application to granular collapse* (2015).