Maths 4B10 - Tutorial 05

1. a) $\begin{cases} 5x - y + 2z = 6 \\ x + 2y - z = -1 \\ 3x + 2y - 2z = 1 \end{cases}$

Changing order of rows $\begin{cases} x + 2y - z = -1 & (R1) \\ 3x + 2y - 2z = 1 & (R2) \\ 5x - y + 2z = 6 & (R3) \end{cases}$

$\begin{array}{r} \\ (R2) - 3(R1) \\ (R3) - 5(R1) \end{array} \begin{cases} x + 2y - z = -1 & \\ -4y + z = 4 & (R4) \\ -11y + 7z = 11 & (R5) \end{cases}$

$\begin{array}{r} \\ \\ 4(R5) - 11(R4) \end{array} \begin{cases} x + 2y - z = -1 \\ -4y + z = 4 \\ 17z = 0 \end{cases}$

$17z = 0$, $\boxed{z = 0}$

$-4y + z = 4$, $-4y = 4$, $\boxed{y = -1}$

$x + 2y - z = -1$, $x - 2 - 0 = -1$, $\boxed{x = 1}$

Tip: When the system is consistent (unique solution) you can substitute the values of x, y, and z on the original system to check your answer.

The solution of the system is:

$\{(1, -1, 0)\}$

b) $\begin{cases} 2x - y + 3z = 3 & (R_1) \\ 2x + y + 4z = 4 & (R2) \\ 2x - 3y + 2z = 2 & (R3) \end{cases}$

$\begin{array}{l} \\ (R2)-(R1) \\ (R3)-(R1) \end{array} \begin{cases} 2x - y + 3z = 3 \\ 2y + z = 1 & (R4) \\ -2y - z = -1 & (R5) \end{cases}$

$\begin{array}{l} \\ \\ (R5)+(R4) \end{array} \begin{cases} 2x - y + 3z = 3 \\ 2y + z = 1 \\ 0 = 0 \end{cases}$

In this case, we will not be able to find an unique solution. Since the last equation does not present an inconsistency (for example, $1=0$), the system is consistent and has infinite solutions. The solutions for the system can be written as:

$2y + z = 1$

$\boxed{z = 1 - 2y}$

$2x - y + 3z = 3$

$2x - y + 3(1-2y) = 3$

$2x - y + \cancel{3} - 6y = \cancel{3}$

$2x = 7y$

$\boxed{x = \frac{7y}{2}}$

The solution of the system is:

$\left\{ \left(\frac{7t}{2}, t, 1-2t \right) ; t \in \mathbb{R} \right\}$

2. x = number of individuals of species 1
y = number of individuals of species 2
z = number of individuals of species 3

A system of linear equations will be defined summing the total daily requirements for food A and food B.

$$\begin{cases} 3x + 2y + z = 500 & (R1) \\ 5x + 3y + 2z = 900 & (R2) \end{cases}$$

$$\begin{array}{l} \\ 3(R2)-5(R1) \end{array} \begin{cases} 3x + 2y + z = 500 \\ \quad -y + z = 200 \end{cases}$$

Given that the number of variables is greater than the number of equations, the system is under-determined. Since there are no inconsistencies, there could be, in principle, infinite solutions:

$$\boxed{y = z - 200} \quad (i) \begin{cases} 3x + 2y + z = 500 \\ 3x + 2z - 400 + z = 500 \\ 3x = 900 - 3z \\ \boxed{x = 300 - z} \quad (ii) \end{cases}$$

The biology of the problem, however, imposes two additional constraints:

- Numbers of individuals have to be non-negative integers (0, 1, 2, 3, ...)
- If the three species have to be reared together, the ranges of possible values have to be restricted, so that none of the numbers of individuals is negative (since, for example, $x = 300 - z$, z can not

assume a value of 400, otherwise we would have $x = -100$).

Given (i) and (ii), and the additional constraints, the solution will be given by:

$$\{(300-t, t-200, t) ; 200 \leq t \leq 300, t \in \mathbb{Z}_{\geq 0}\}$$

If another food type is added we have the following system:

$$\begin{cases} 3x + 2y + z = 500 & (R1) \\ 5x + 3y + 2z = 900 & (R2) \\ 2x + 4y + z = 550 & (R3) \end{cases}$$

$$\begin{array}{r} \\ 3(R2)-5(R1) \\ 3(R3)-2(R1) \end{array} \begin{cases} 3x + 2y + z = 500 & \\ -y + z = 200 & (R4) \\ 8y + z = 650 & (R5) \end{cases}$$

$$\begin{array}{r} \\ \\ (R5)-(R4) \end{array} \begin{cases} 3x + 2y + z = 500 \\ -y + z = 200 \\ 9y = 450 \end{cases}$$

$$9y = 450 \quad \boxed{y = 50}$$

$$-y + z = 200 \quad \boxed{z = 250}$$

$$3x + 2y + z = 500$$
$$3x + 100 + 250 = 500$$
$$3x = 150$$
$$\boxed{x = 50}$$

In this case, there is an unique solution:

$$\{(50, 50, 250)\}$$

3. a) $AB = \begin{pmatrix} 7 & 5 & 9 & -1 \\ -4 & -2 & -6 & 0 \end{pmatrix}$

b) The multiplication BA cannot be performed, since the number of columns of B (4 columns) differ from the number of rows of A (2 rows)

4. a) $AB = (1)$

b) $BA = \begin{pmatrix} -1 & -4 & 2 \\ 2 & 8 & -4 \\ 3 & 12 & -6 \end{pmatrix}$

5. $A^2 = \begin{pmatrix} 3 & -1 \\ 1 & 8 \end{pmatrix}$

$A^3 = \begin{pmatrix} 7 & 6 \\ -6 & -23 \end{pmatrix}$

$A^4 = \begin{pmatrix} 8 & -11 \\ 11 & 63 \end{pmatrix}$

6. a) $\begin{cases} 2x_2 - x_1 = x_3 \\ 4x_1 + x_3 = 7x_2 \\ x_2 - x_1 = x_3 \end{cases}$

$$\begin{cases} -x_1 + 2x_2 - x_3 = 0 \\ 4x_1 - 7x_2 + x_3 = 0 \\ -x_1 + x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{pmatrix} -1 & 2 & -1 \\ 4 & -7 & 1 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

b) $\begin{cases} 2x_1 - 3x_2 = 4 \\ -x_1 + x_2 = 3 \\ 3x_1 = 4 \end{cases}$

$\begin{cases} 2x_1 - 3x_2 = 4 \\ -x_1 + x_2 = 3 \\ 3x_1 + 0x_2 = 4 \end{cases} \Rightarrow \begin{pmatrix} 2 & -3 \\ -1 & 1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 4 \end{pmatrix}$

7. a) $\det A = \det \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$

$= 2 \cdot 6 - 3 \cdot 4 = 12 - 12 = 0$

Since $\det A = 0$, A is singular, and therefore, non-invertible.

b) $AX = B \Rightarrow \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

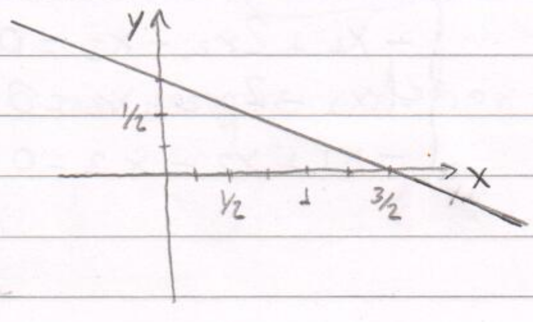
$\Rightarrow \begin{cases} 2x + 4y = b_1 \\ 3x + 6y = b_2 \end{cases}$

c) $\begin{cases} 2x + 4y = 3 & (R1) \\ 3x + 6y = 9/2 & (R2) \end{cases}$

$\begin{cases} 2x + 4y = 3 \\ 0 = 0 \end{cases}$ $3(R1) - 2(R2)$

$2x + 4y = 3$
$4y = 3 - 2x$
$\boxed{y = \frac{3}{4} - \frac{1}{2}x}$

In this case, the two lines coincide:



d) If the two straight lines corresponding to the rows of the system $AX=B$ are parallel, the system will have no solutions.

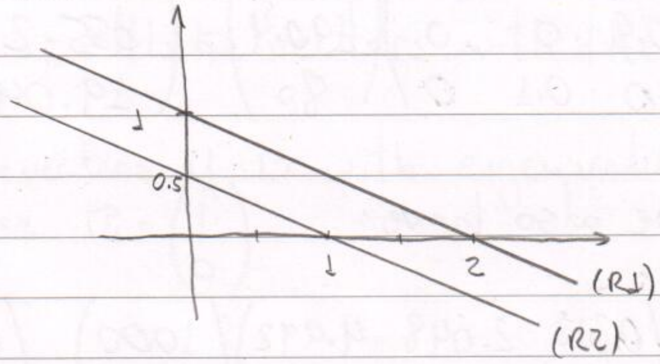
Since for rows R1 and R2 of A we have $3(R1) = 2(R2)$, any vector B for which $3b_1 \neq 2b_2$ will yield parallel straight lines.

If $B = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, we have:

$$\begin{cases} 2x + 4y = 4 & (R1) \\ 3x + 6y = 3 & (R2) \end{cases}$$

$$3(R1) - 2(R2) \begin{cases} 2x + 4y = 4 \\ \quad 0 = 6 \Rightarrow \text{inconsistency!} \end{cases}$$

If we plot the two lines we see that they are parallel:



8. The Leslie matrix will be given by:

$$P = \begin{pmatrix} 0 & 1.6 & 3.9 \\ 0.8 & 0 & 0 \\ 0 & 0.1 & 0 \end{pmatrix}$$

Starting at time 0 with an age distribution of $(1000, 100, 20)$, the age distribution at time 3 ($\vec{X}(3)$) can be found in two ways:

$$\vec{X}(1) = P\vec{X}(0) = \begin{pmatrix} 0 & 1.6 & 3.9 \\ 0.8 & 0 & 0 \\ 0 & 0.1 & 0 \end{pmatrix}\begin{pmatrix} 1000 \\ 100 \\ 20 \end{pmatrix} = \begin{pmatrix} 238 \\ 800 \\ 10 \end{pmatrix}$$

$$\vec{X}(2) = P\vec{X}(1) = \begin{pmatrix} 0 & 1.6 & 3.9 \\ 0.8 & 0 & 0 \\ 0 & 0.1 & 0 \end{pmatrix}\begin{pmatrix} 238 \\ 800 \\ 10 \end{pmatrix} = \begin{pmatrix} 1319 \\ 190.4 \\ 80 \end{pmatrix}$$

$$\vec{X}(3) = P\vec{X}(2) = \begin{pmatrix} 0 & 1.6 & 3.9 \\ 0.8 & 0 & 0 \\ 0 & 0.1 & 0 \end{pmatrix}\begin{pmatrix} 1319 \\ 190.4 \\ 80 \end{pmatrix} = \begin{pmatrix} 616.64 \\ 1055.2 \\ 19.04 \end{pmatrix}$$

Alternatively, we also have:

$$\vec{X}(3) = P^3\vec{X}(0) = \begin{pmatrix} 0.312 & 2.048 & 4.992 \\ 1.024 & 0.312 & 0 \\ 0 & 0.128 & 0.312 \end{pmatrix}\begin{pmatrix} 1000 \\ 100 \\ 20 \end{pmatrix} = \begin{pmatrix} 616.64 \\ 1055.2 \\ 19.04 \end{pmatrix}$$

9. Write the code to calculate successive population age distributions.

10. a) $\det(B - \lambda I) = \det\begin{pmatrix} 2-\lambda & 3 \\ 0 & -1-\lambda \end{pmatrix} =$

$= (2-\lambda)(-1-\lambda) - 0$

$(2-\lambda)(-1-\lambda) = 0 \Rightarrow \boxed{\lambda_1 = 2}$ $\boxed{\lambda_2 = -1}$

For $\lambda_1 = 2$:

$$\begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} 2x + 3y = 2x \\ -y = 2y \end{cases}$$

$$\boxed{y = 0} \quad \left\{ \begin{array}{l} 2x + 0 = 2x \\ \boxed{x = x} \end{array} \right.$$

Thus, all the vectors with the form $\begin{pmatrix} x \\ 0 \end{pmatrix}$ (with $x \neq 0$) a

eigenvectors of B with eigenvalue $\lambda_1 = 2$. Let us choose $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

For $\lambda_2 = -1$:

$$\begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} 2x + 3y = -x \\ -y = -y \end{cases}$$

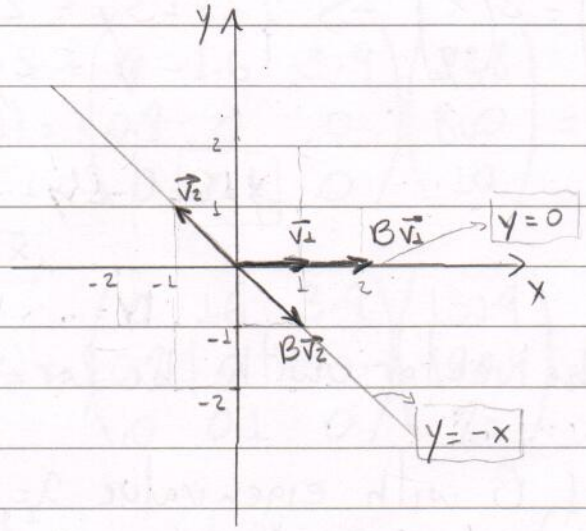
$$\boxed{y = y} \quad \left\{ \begin{array}{l} 2x + 3y = -x \\ \boxed{x = -y} \end{array} \right.$$

Thus, all vectors with the form $\begin{pmatrix}-y\\y\end{pmatrix}$ (with $y\neq 0$) are eigenvectors of B with eigenvalue $\lambda_2 = -1$. Let us choose $\vec{v_2} = \begin{pmatrix}-1\\1\end{pmatrix}$

We have:

$$B\vec{v_1} = \begin{pmatrix}2 & 3\\0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} = 2\begin{pmatrix}1\\0\end{pmatrix} = \lambda_1\vec{v_1}$$

$$B\vec{v_2} = \begin{pmatrix}2 & 3\\0 & -1\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} = -1\begin{pmatrix}-1\\1\end{pmatrix} = \lambda_2\vec{v_2}$$



b) $\det\begin{pmatrix}3-\lambda & 6\\-1 & -4-\lambda\end{pmatrix} = (3-\lambda)(-4-\lambda)+6$

$(3-\lambda)(-4-\lambda)+6=0 \Rightarrow \lambda^2+\lambda-6=0$

$\Rightarrow \boxed{\begin{matrix}\lambda_1 = 2\\ \lambda_2 = -3\end{matrix}}$

$-12-3\lambda+4\lambda+\lambda^2+6$

$\lambda^2+\lambda-6$

For $\lambda_1 = 2$:

$$\begin{pmatrix} 3 & 6 \\ -1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} 3x + 6y = 2x \\ -x - 4y = 2y \end{cases}$$

$$\boxed{x = -6y}$$

Thus, all vectors with the form $\begin{pmatrix} -6y \\ y \end{pmatrix}$ (with $y \neq 0$) are eigenvectors of B with eigenvalue $\lambda_1 = 2$. Let us choose $\vec{v_1} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$.

For $\lambda_2 = -3$:

$$\begin{pmatrix} 3 & 6 \\ -1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -3\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} 3x + 6y = -3x \\ -x - 4y = -3y \end{cases}$$

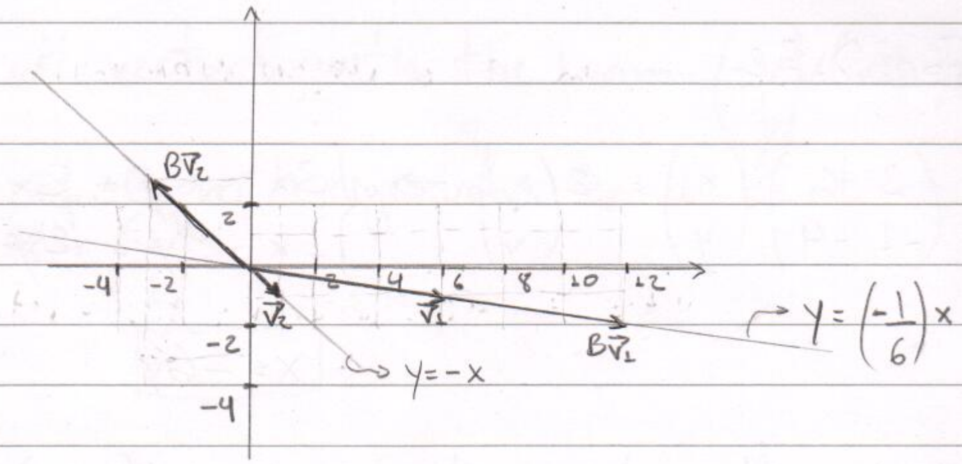
$$\boxed{y = -x}$$

Thus, all vectors with the form $\begin{pmatrix} x \\ -x \end{pmatrix}$ (with $x \neq 0$) are eigenvectors of B with eigenvalue $\lambda_2 = -3$. Let us choose $\vec{v_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

We have:

$$B\vec{v_1} = \begin{pmatrix} 3 & 6 \\ -1 & -4 \end{pmatrix}\begin{pmatrix} 6 \\ -1 \end{pmatrix} = \begin{pmatrix} 12 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 6 \\ -1 \end{pmatrix} = \lambda_1 \vec{v_1}$$

$$B\vec{v_2} = \begin{pmatrix} 3 & 6 \\ -1 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix} = -3\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \lambda_2 \vec{v_2}$$



11. The growth constant for the population will be given by the leading eigenvalue of the Leslie matrix. We have:

$$\det\begin{pmatrix} 0.5-\lambda & 2.3 \\ a & 0-\lambda \end{pmatrix} = -\lambda(0.5-\lambda) - 2.3a$$

$$-\lambda(0.5-\lambda) - 2.3a = 0 \Rightarrow \lambda^2 - 0.5\lambda - 2.3a = 0$$

$$\Delta = (-0.5)^2 - 4(-2.3a) = 0.25 + 9.2a$$

$$\lambda = \frac{0.5 \pm \sqrt{0.25+9.2a}}{2} = 0.25 \pm 0.5\sqrt{0.25+9.2a}$$

The population will grow if the leading eigenvalue is greater than 1. Thus:

$$0.25 + 0.5\sqrt{0.25+9.2a} > 1$$
$$0.5\sqrt{0.25+9.2a} > 0.75$$
$$\sqrt{0.25+9.2a} > 1.5$$
$$0.25+9.2a > 2.25$$
$$9.2a > 2$$
$$a > 0.217$$

We conclude that the population will grow if $0.217 < \alpha \leq 1$

12. a) The long-term survival of the population will be given by the leading eigenvalue of the Leslie matrix. We have:

$$\det\begin{pmatrix} 0.5-\lambda & 2.0 \\ 0.1 & -\lambda \end{pmatrix} = -\lambda(0.5-\lambda) - 0.2$$

$$-\lambda(0.5-\lambda) - 0.2 = 0 \Rightarrow \lambda^2 - 0.5\lambda - 0.2 = 0$$

$$\Delta = (-0.5)^2 - 4(-0.2) = 0.25 + 0.8 = 1.05 \quad \Bigg| \quad \lambda = \frac{0.5 \pm \sqrt{1.05}}{2} \Rightarrow \boxed{\begin{matrix} \lambda_1 = 0.76 \\ \lambda_2 = -0.26 \end{matrix}}$$

Since $\lambda_1 = 0.76 < 1$, in the long-term this population would decline and be extinct

b) Considering fecundity and survival of zero-year-olds as two free parameters f and s, we have:

$$\det\begin{pmatrix} f-\lambda & 2 \\ s & -\lambda \end{pmatrix} = -\lambda(f-\lambda) - 2s$$

$$-\lambda(f-\lambda) - 2s = 0 \Rightarrow \lambda^2 - f\lambda - 2s = 0$$

$$\Delta = f^2 + 8s \quad \Big\} \quad \lambda = \frac{f \pm \sqrt{f^2+8s}}{2}$$

As we see, the leading eigenvalue is given by
$$\lambda(f,s) = \frac{f + \sqrt{f^2+8s}}{2}$$

Let us investigate different scenarios:

- $f = 0.5$, $s = 0.4$

$$\lambda(0.5, 0.4) = \frac{0.5 + \sqrt{0.5^2 + 8 \cdot 0.4}}{2} \cong 1.18$$

- $f = 1.5$, $s = 0.1$

$$\lambda(1.5, 0.1) = \frac{1.5 + \sqrt{1.5^2 + 8 \cdot 0.1}}{2} \cong 1.62$$

- $f = 1.5$, $s = 0.4$

$$\lambda(1.5, 0.4) = \frac{1.5 + \sqrt{1.5^2 + 8 \cdot 0.4}}{2} \approx 1.92$$

The maximum achievable growth rate would be given by the maximum values for fecundity and survival ($f = 1.5$ and $s = 0.4$), and would be given by $\lambda = 1.92$.

c) There is not an unique answer for this question, and of course the factors would depend on the specifics of the system being managed. Two factors, however, tend to be common to real situations:

- There is usually a trade-off in life-history traits, meaning that sometimes is hard (or even impossible) to improving both traits (in this case, fecundity and survival) at the same time. Also, there is usually different costs associated to the improvement of each trait. A more complete strategy would take into account how the two parameters in the model quantitatively affect the value of the final growth rate (see "sensitivity analysis") for more information), and also how this would balance with the constraint in resources (money, time, personnel) to implement the chosen strategy.

- If the population being managed is subject to harvesting (logging or fisheries, for example), one thing to take into account is if the activity, by focusing on specific traits which are desirable for humans, does not end up selecting groups of individuals with systematically higher fecundity or survival. This non-random harvest of specific groups would end up reducing the average value of the parameters, with important impact on the long-term dynamics of the system.