Maths4Bio - Tutorial 01

1. a) Expanding the factored expression:

$$R(x) = Kx(a-x) = Kxa - Kx^2 = (-K)x^2 + (aK)x$$

which shows that $R(x)$ is a polynomial of degree 2.

b) The function $R(x) = 2x(6-x)$ have two real roots: $x=0$ and $x=6$ (since $R(0) = R(6) = 0$).

Expanding the polynomial we have:

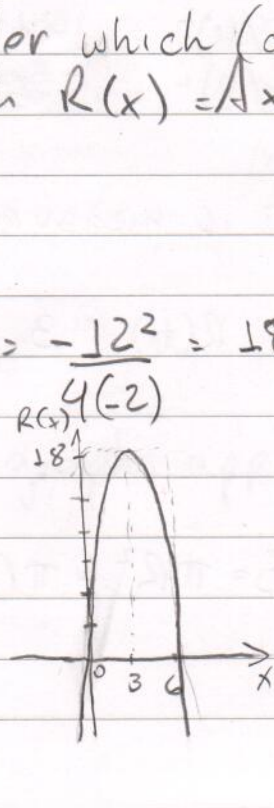
$$R(x) = -2x^2 + 12x.$$

Since the leading term is negative, this represents a concave-down parabola. The value of x for which the reaction rate is maximal corresponds to the vertix, for which (considering a polynomial of the form $R(x) = Ax^2 + Bx + C$):

$$x_v = -\frac{B}{2A} = -\frac{12}{2(-2)} = 3$$

$$y_v = -\frac{\Delta}{4A} = -\frac{(B^2-4AC)}{4A} = -\frac{12^2}{4(-2)} = 18.$$

The graph is then:



2. The radius of the disease are grows at a constant daily rate:

Time (days)	Radius (m)
0	0
1	3
2	6
⋮	⋮
t	$3t$

Assuming the spread to be circular, the affected area, S, at day t will be:

$$S = \pi R^2 = \pi(3t)^2 = (9\pi)t^2$$

which is a polynomial of degree 2.

At days 2, 4 and 8, we have:

$$S(2) = 9\pi(2)^2 = 36\pi \ m^2$$
$$S(4) = 9\pi(4)^2 = 144\pi \ m^2$$
$$S(8) = 9\pi(8)^2 = 576\pi \ m^2$$

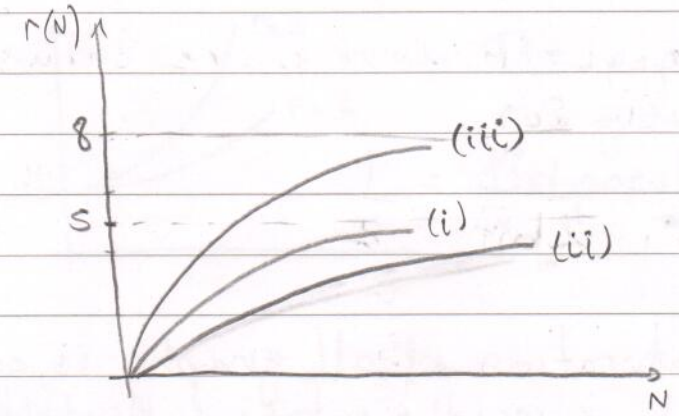
If time t is measured in weeks, then:

$$R(t) = 3.(7t) = 21t$$

Therefore:

$$S = \pi R^2 = \pi(21t)^2 = 441\pi t^2$$

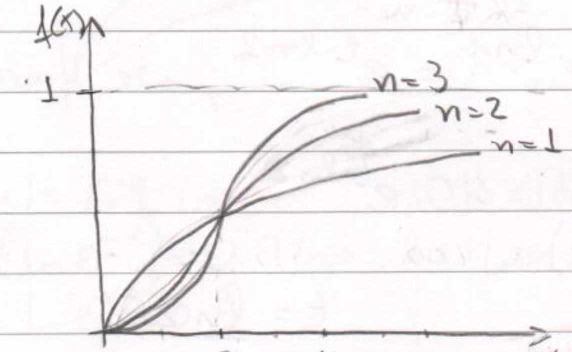
3. a)



b) When you increase a the function tends to saturate at a higher value. In order words, a seems to control the value approached by the function as N increases (compare graphs (i) and (iii)).

c) When you change K (keeping the same value of a) you change how fast the function approaches the saturation value (compare graphs (i) and (ii)). Smaller K leads to faster approach.

4. a)



b) The graphs intersect at $(x, y) = (2, 0.5)$

c) As x gets larger all graphs approach the value 1.

d) We have:

$$f(b) = \frac{b^n}{b^n + b^n} = \frac{b^n}{2b^n} = \frac{1}{2}$$

Since the saturation of all graphs is at 1, we see that $x=b$ gives the value of the independent variable for which the image of the function assumes half of the value of the saturation. Therefore, b is called half-saturation.

5. a) As stem diameter increases, leaf area increases

b) As leaf thickness increases, volume fraction of spongy mesophyll decreases.

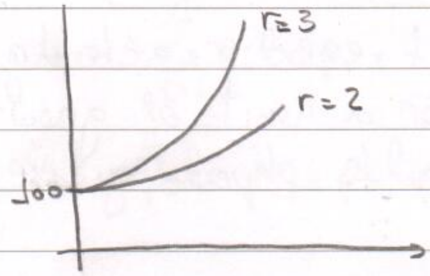
6. a) $N(0) = 40 \cdot 2^0 = 40 \cdot 1 = 40$

b) Since $2^t = e^{\ln 2^t} = e^{t \cdot \ln 2}$, we have:

$$N(t) = 40 \cdot e^{t \cdot \ln 2}$$

c)
$$N(t) = 1000$$
$$40 \cdot e^{t\ln 2} = 1000$$
$$e^{t\ln 2} = \frac{1000}{40}$$
$$e^{t\ln 2} = 25$$
$$\ln(e^{t\ln 2}) = \ln 25$$
$$t\ln 2 = \ln 25$$

$$t = \frac{\ln 25}{\ln 2}$$
$$t \cong 4.64$$

7. a) 

The population with r=3 grows faster than the population with r=2

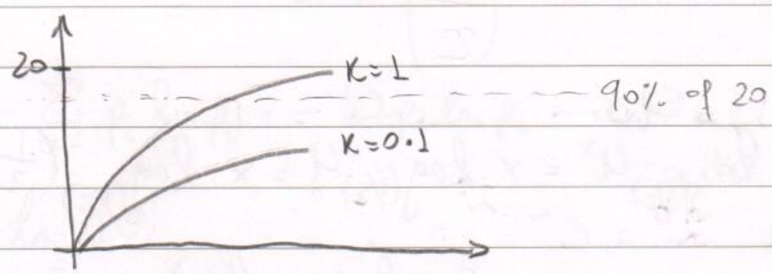
b) $\dfrac{N(t+1)}{N(t)} = \dfrac{250}{200}$

$\dfrac{\cancel{N_0}e^{r(t+1)}}{\cancel{N_0}e^{rt}} = \dfrac{250}{200}$

$\dfrac{\cancel{e^{rt}} \cdot e^{r \cdot 1}}{\cancel{e^{rt}}} = \dfrac{5}{4}$

$e^{r} = \dfrac{5}{4}$

$r = \ln\left(\dfrac{5}{4}\right) \cong 0.22$

8. a) 

b) $L(x) = 0.9 \cdot L_\infty$

$\cancel{L_\infty}(1-e^{-x}) = 0.9\cancel{L_\infty}$

$1-e^{-x} = 0.9$

$e^{-x} = 0.1$

$-x = \ln(0.1)$

$x = -\ln(0.1)$

$x \cong 2.30$

$L(x) = 0.99 L_\infty$

$\cancel{L_\infty}(1-e^{-x}) = 0.99\cancel{L_\infty}$

$1-e^{-x} = 0.99$

$e^{-x} = 0.01$

$-x = \ln(0.01)$

$x = -\ln(0.01)$

$x \cong 4.60$

The parameter L_∞ is the horizontal asymtote of $L(x)$ and the graph approaches this value as x increases, without ever reaching it. Biologically, L_∞ stands for a limit of growth for the fish, due possibly to physiological or developmental constraints.

c) The growth curve with $K=1$ reaches 90% of L_∞ faster than the growth curve with $K=0.1$. The larger the value of K, the faster the curve approaches L_∞.

9. I) a) $3^{4\log_3 x} = 3^{\log_3 x^4} = x^4$

b) $4^{-\log_{(1/2)} x} = (2^2)^{\log_{(1/2)} x^{-1}} =$
$$= \left[\left(\frac{1}{2}\right)^{-2}\right]^{\log_{(1/2)} x^{-1}} = \left(\frac{1}{2}\right)^{-2 \cdot \log_{(1/2)} x^{-1}}$$
$$= \left(\frac{1}{2}\right)^{\log_{(1/2)} x^2} = x^2$$

c) $\log_{(1/2)} 4^x = x \cdot \log_{(1/2)} 4 = x \cdot \log_{(1/2)} \left(\frac{1}{2}\right)^{-2}$
$$= -2x \log_{(1/2)} (1/2) = -2x$$

d) $\log_3 9^{-x} = \log_3 (3^2)^{-x} = \log_3 3^{-2x}$
$$= -2x \log_3 3 = -2x$$

II) $y = \left(\frac{1}{2}\right)^x = e^{\ln(1/2)^x} = e^{x \cdot \ln(1/2)} = e^{x \ln 2^{-1}}$
$= e^{-x\ln 2} = e^{-\mu x}$, where $\mu = \ln 2 > 0$

10. a) $H = -\sum_{i=1}^{5} p_i \ln p_i =$

$$= -p_1 \ln p_1 - p_2 \ln p_2 - p_3 \ln p_3 - p_4 \ln p_4 - p_5 \ln p_5$$

Since $p_1 = p_2 = p_3 = \dots = p_5$, we have:

$$H = -5 p_1 \ln p_1$$

Also, since all species are equally abundant:
$p_1 = p_2 = p_3 = \dots = p_5 = \frac{1}{5}$. Thus:

$$H = -\cancel{5} \cdot \frac{1}{\cancel{5}} \ln\left(\frac{1}{5}\right) = -\ln 5^{-1} = \ln 5$$

b) Since $p_1 = p_2 = \dots = p_9 = p_{10} = \frac{1}{10}$

$$H = -\sum_{i=1}^{10} p_i \ln p_i = -10 p_1 \ln p_1 = -\cancel{10} \cdot \frac{1}{\cancel{10}} \ln\left(\frac{1}{10}\right)$$
$$= \ln 10$$

c) $S = 5$: $H = \ln 5 \Rightarrow \frac{H}{\ln S} = \frac{\ln 5}{\ln 5} = 1$

$S = 10$: $H = \ln 10 \Rightarrow \frac{H}{\ln S} = \frac{\ln 10}{\ln 10} = 1$

d) If there are N equally abundant species:

$$p_1 = p_2 = \dots = p_{N-1} = p_N = \frac{1}{N}$$

Then:

$$H = -\sum_{i=1}^{N} p_i \ln p_i = -p_1 \ln p_1 - p_2 \ln p_2 - \cdots - p_N \ln p_N$$

$$= -N p_1 \ln p_1 = -N \cdot \left(\frac{1}{N}\right) \ln\left(\frac{1}{N}\right)$$

$$= -\ln N^{-1} = \ln N$$

Thus:

$$\frac{H}{\ln S} = \frac{\ln N}{\ln N} = 1$$