Maths4bio - Tutorial 01

1. An autocatalytic reaction uses its resulting product for the formation of a new product, as in the reaction

$$A + X \to X$$

If we assume that this reaction occurs in a closed vessel, then the reaction rate is given by

$$R(x) = kx(a - x)$$

for $0 \leq x \leq a$, where a is the initial concentration of A and x is the concentration of X.

   (a) Show that $R(x)$ is a polynomial and determine its degree.

   (b) Without using any graphic visualiser, graph $R(x)$ for $k = 2$ and $a = 6$. Find the value of x at which the reaction rate is maximal.

2. Suppose that a fungal disease originates in the middle of an orchard, initially affecting only one tree. The disease spreads out radially at a constant speed of 3 meters per day. What area will be affected after 2 days, 4 days, and 8 days? Write an equation that expresses the affected area as a function of time, measured in days, and show that this function is a polynomial of degree 2. If time is measured in weeks, how will your equation be written?

(*Hint*: Use the area of the circle $S = \pi R^2$, where R is the radius of the circle.)

3. There is a function that is frequently used to describe the per capita growth rate of organisms when the rate depends on the concentration of some nutrient and becomes saturated for large enough nutrient concentrations. If we denote the concentration of the nutrient by N, then the per capita growth rate $r(N)$ is given by the *Monod growth function*

$$r(N) = \frac{aN}{k + N}, \quad N \geq 0$$

where a and k are positive constants.

   (a) Use Python or R to investigate the Monod growth function. Graph $r(N)$ for (i) $a = 5$ and $k = 1$, (ii) $a = 5$ and $k = 3$, and (iii) $a = 8$ and k = 1. Place all three graphs in the same coordinate system.

   (b) On the basis of the graphs in (a), describe in words what happens when you change a.

   (c) On the basis of the graphs in (a), describe in words what happens when you change k.

4. The function

$$f(x) = \frac{x^n}{b^n + x^n}, \quad x \geq 0$$

where n is a positive integer and b is a positive real number, is used in biochemistry to model reaction rates as a function of the concentration of some reactants.

(a) Use Python or R to graph $f(x)$ for $n = 1, 2,$ and 3 in the same coordinate system when $b = 2$.

(b) Where do the three graphs in (a) intersect?

(c) What happens to $f(x)$ as x gets larger?

(d) For an arbitrary positive value of b, show that $f(b) = 1/2$. On the basis of this demonstration and your answer in (c), explain why b is called the half-saturation constant.

5. In the problems below, use Python or R to sketch each scaling relation (K. J. Niklas (1994). *Plant Allometry: The Scaling of Form and Process*. University of Chicago Press).

   (a) In a sample based on 46 species, leaf area was found to be proportional to $(stem\ diameter)^{1.84}$. On the basis of your graph, as stem diameter increases, does leaf area increase or decrease?

   (b) In a sample based on 28 species, the volume fraction of spongy mesophyll was found to be proportional to $(leaf\ thickness)^{-0.49}$. (The spongy mesophyll is part of the internal tissue of a leaf blade.) On the basis of your graph, as leaf thickness increases, does the volume fraction of spongy mesophyll increase or decrease?

6. Assume that a population size at time t is $N(t)$ and that

$$N(t) = 40 \cdot 2^t, \quad t \geq 0$$

   (a) Find the population size at time t = 0.

   (b) Show that
$$N(t) = 40\,\mathrm{e}^{t\,ln 2}, \quad t \geq 0$$

   (c) How long will it take until the population size reaches 1000?
   (*Hint*: Find t so that $N(t) = 1000$.)

7. (*Adapted from Moss, 1980*) Hall (1964) investigated the change in population size of the zooplankton species *Daphnia galeata mendota* in Base Line Lake, Michigan. The population size $N(t)$ at time t was modeled by the equation

$$N(t) = N_0 \mathrm{e}^{rt}$$

   where N_0 denotes the population size at time 0. The constant r is called the **intrinsic rate of growth**.

   (a) Plot $N(t)$ as a function of t if $N_0 = 100$ and $r = 2$. Compare your graph against the graph of $N(t)$ when $N_0 = 100$ and $r = 3$. Which population grows faster?

   (b) The constant r is an important quantity because it describes how quickly the population changes. Suppose that you determine the size of the population at the beginning and at the end of a period of length 1, and you find that at the beginning there were 200 individuals and after one unit of time there were 250 individuals. Determine r. (*Hint*: Consider the ratio $N(t+1)/N(t)$.)

8. Fish are indeterminate growers; that is, they grow throughout their lifetime. The growth of fish can be described by the von Bertalanffy function

$$L(x) = L_\infty(1 - \mathrm{e}^{-kx})$$

   for $x \geq 0$, where $L(x)$ is the length of the fish at age x and k and L_∞ are positive constants.

   (a) Using Python or R, graph $L(x)$ for $L_\infty = 20$, for (i) $k = 1$ and (ii) $k = 0.1$.

(b) For $k = 1$, find x so that the length is 90% of L_∞. Repeat for 99% of L_∞. Can the fish ever attain length L_∞? Interpret the meaning of L_∞.

(c) Compare the graphs obtained in (a). Which growth curve reaches 90% of L_∞ faster? Can you explain what happens to the curve of $L(x)$ when you vary k (for fixed L_∞)?

9. (I) Simplify the following expressions:
   (a) $3^{4\log_3 x}$ (b) $4^{-\log_{1/2} x}$ (c) $\log_{1/2} 4^x$ (d) $\log_3 9^{-x}$

   (II) Show that the function $y = (1/2)^x$ can be written in the form $y = \mathrm{e}^{-\mu x}$, where μ is a positive constant. Determine μ.

   (*Hint*: Remember the following identities:

$$\begin{aligned} \log_a x^r &= r \log_a x \\ \log_a a^x &= x \\ a^{\log_a x} &= x \\ a^{-r} &= \frac{1}{a^r} \\ (a^r)^s &= a^{rs}, \end{aligned}$$

   assuming a positive and different from 1)

10. A community measure that takes both species abundance and species richness into account is the Shannon diversity index H. To calculate H, the proportion p_i of species i in the community is used. Assume that the community consists of S species. Then

$$H = -\sum_{i=1}^{S} p_i \ln p_i = -(p_1 \ln p_1 + p_2 \ln p_2 + \cdots + p_S \ln p_S)$$

   (a) Assume that $S = 5$ and that all species are equally abundant; that is, $p_1 = p_2 = \cdots = p_5$. Compute H.

   (b) Assume that $S = 10$ and that all species are equally abundant; that is, $p_1 = p_2 = \cdots = p_{10}$. Compute H.

   (c) A measure of equitability (or evenness) of the species distribution can be measured by dividing the diversity index H by $\ln S$. Compute $H/\ln S$ for $S = 5$ and $S = 10$.

   (d) Show that, in general, if there are $S = N$ species and all species are equally abundant, then

$$\frac{H}{\ln S} = 1$$