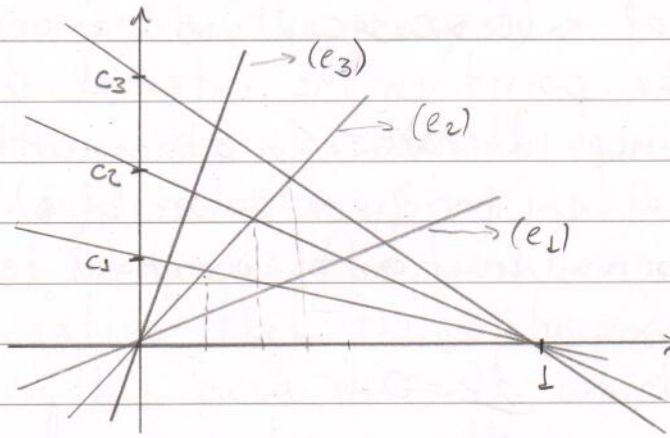
Maths 4b10 - Tutorial 06

1.a) For $c>0$ and $e>0$, we have:



Since both c and e are positive constants, that means that the gain from colonization and the loss from extiction are increasing and decreasing functions of p, respectively. As two straight lines (1^{st}-order degree polynomial) defined for the interval $0 \leq p \leq 1$, they will intersect for a given point p in this interval

The intersection point p^* will be given by:

$$f(p^*) = g(p^*)$$
$$c(1-p^*) = ep^*$$
$$c - cp^* = ep^*$$
$$c = (c+e)p^*$$
$$\boxed{p^* = \frac{c}{c+e} = \frac{1}{1+\left(\frac{e}{c}\right)}}$$

When $f(p^*) = g(p^*)$, then $\frac{dp}{dt} = 0$. This means that p^* gives the equilibrium value for the fraction of occupied islands in this model.

b) As seen from the graph in part (a), for a fixed value of e (say, for example, e_2), increasing the value of c ($c_3 > c_2 > c_1$) yield increasing values of p^*, the point in the interval $0 \leq p \leq 1$ where the two lines intersect. This means that the larger the value of c (for fixed values of e), the larger the equilibrium fraction of occupied islands.

2. a)

$$\frac{dS}{dt} = 0$$

$$S \underbrace{\left[b\left(1 - \frac{S}{K}\right) - a \right]}_{=0} = 0$$

$S = 0$ or

$$\Downarrow$$

$$b\left(1 - \frac{S}{K}\right) - a = 0$$

$$1 - \frac{S}{K} = \frac{a}{b}$$

$$1 - \frac{a}{b} = \frac{S}{K}$$

$$\boxed{S = K\left(1 - \frac{a}{b}\right)}$$

Thus the two possible equilibrium points are $S = 0$ and $S = K\left(1 - \frac{a}{b}\right)$.

b) This statement by Valentine is explained by the analysis of the non-trivial equilibrium point $S = K\left(1 - \frac{a}{b}\right)$. With $a > 0$ (and also assuming $a < b$), the factor $\left(1-\frac{a}{b}\right)$ always correspond to a value between 0 and 1. Therefore, the equilibrium value of S is always a fraction of K, the total number for species in the ecosystem.

c) If $a \geq b$, then the non-trivial equilibriom assumes a negative value. Since the biology of the model restricts the number of species to be non-negative, then the only possible equilibrium in this case is $S=0$, which means complete local extinction.

3. The slope of the tangent line will be given by the derivative of the function:

$$y = 1-3x^2 \implies y' = -6x^2$$

In order to be parallel to $y=-x$, the tangent needs a slope equal to -1;

$$-6x^2 = -1$$

$$x^2 = \frac{1}{6}$$

$$x = \pm\sqrt{\frac{1}{6}}$$

Thus, at points $x=-\sqrt{\frac{1}{6}}$ and $x=\sqrt{\frac{1}{6}}$, the tangent is parallel to the line $y=-x$.

4. As before, we have:

$$y = 2x^3 - 4x + 1 \Rightarrow y' = 6x^2 - 4$$

Also, the line

$$y - 2x = 1 \Rightarrow y = 2x + 1$$

has slope equal to 2. Thus:

$$6x^2 - 4 = 2$$
$$6x^2 = 6$$
$$x^2 = 1$$
$$x = \pm 1$$

Therefore, at points $x = -1$ and $x = 1$ the tangent is parallel to the line $y - 2x = 1$

5. a) $f'(x) = 2x(10x^3 + 9x - 2)$

b) $h'(s) = 12x(2x-1)(4x^3 - 3x^2 + 4)$

c) $f'(x) = \dfrac{\sqrt{3}(5x^2 - 1)}{2\sqrt{x}}$

d) $g'(t) = 8(9t^2 + 3t - 1)$

e) $f'(x) = -6x(15x^4 + 12x^2 - 8)$

f) $g'(x) = \dfrac{8x^3 - 12x^2 + 1}{(1-x)^2}$

g) $h'(x) = \dfrac{-20x^8 + 18x^6 + 19x^4 - 9x^2}{(3x^3 - 5x^5)^2}$

h) $f'(t) = -\dfrac{2(t+3)}{(1+t)^3}$

i) $h'(s) = \dfrac{-2s^4 + 8s^3 - 33s^2 + 52s - 15}{(s^2-3)^3}$

j) $g'(s) = \dfrac{2(s^{2/7} - s^{1/7} - 1)}{7(1+s^{1/7})^2 s^{9/7}}$

6. a) $N'(t) = \dfrac{0.\{1 + [(\frac{K}{N_0}) - 1]\}e^{-rt} - K[-r((\frac{K}{N_0}) - 1)e^{-rt}]}{[1 + [(\frac{K}{N_0}) - 1]e^{-rt}]^2}$

$$= \frac{Kr[(\frac{K}{N_0}) - 1]e^{-rt}}{\{1 + [(\frac{K}{N_0}) - 1]e^{-rt}\}^2}$$

b) If $N(t) = \dfrac{K}{1 + [(\frac{K}{N_0}) - 1]e^{-rt}}$, then:

$$rN\left(1 - \frac{N}{K}\right) = r\left\{\frac{K}{1 + [(\frac{K}{N_0}) - 1]e^{-rt}}\right\}.\left(1 - \frac{1}{1 + [(\frac{K}{N_0}) - 1]e^{-rt}}\right)$$

$$= rK.\frac{1}{1 + [(\frac{K}{N_0}) - 1]e^{-rt}}.\left(\frac{\cancel{1} + [(\frac{K}{N_0}) - 1]e^{-rt} - \cancel{1}}{1 + [(\frac{K}{N_0}) - 1]e^{-rt}}\right)$$

$$= \frac{rK[(\frac{K}{N_0}) - 1]e^{-rt}}{\{1 + [(\frac{K}{N_0}) - 1]e^{-rt}\}^2}$$

$$= N'(t)$$

c) The per-capita growth rate is given by:

$$\frac{1}{N}\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)$$

which is a 1^{st}-order degree polynomial with slope $\left(-\frac{r}{K}\right)$. Therefore, as N increases, the per-capita growth rate decreases in a linear fashion.

As N stands for the population density, the decrease in per-capita growth with the increase in density is called <u>density-dependent growth</u>, which can be seen as a consequence of <u>intra-specific competition</u>.

7. a) $f'(x) = (1+2e^{2x})\cos(e^{2x}+x)$

b) $h'(x) = 2\exp[x^2 - 2\cos(x)](x + \sin(x))$

c) $f'(x) = \dfrac{1+4^x - 2^x(1+x^2)\ln 2}{(1+x2^x)^2}$

d) $g'(t) = \dfrac{3}{(x-1)(1+2x)\ln 10}$

e) $f'(t) = \dfrac{\cos[\ln(3t)]}{t}$

f) $g'(x) = \mathrm{Tan}(1-x)$

g) $h'(x) = \dfrac{2x\left(1+\sqrt{1+x^2}\right)}{\sqrt{x^2-1}\cdot\sqrt{1+x^2}\left(2+\sqrt{1+x^2}\right)^2}$

h) $f'(t) = \dfrac{\frac{1}{\sqrt{1-t}} - 4t}{6\left(\sqrt[3]{t^2+\sqrt{1-t}}\right)^2}$

8. a) $\dfrac{\partial f}{\partial N} = \dfrac{b^2N(2+cN)T}{(1+cN+bT_hN^2)^2}$ (Note: we are always assuming <u>non-negative</u> N, T, T_h)

We see that $\dfrac{\partial f}{\partial N} > 0$, for all values of N, T, and T_h.

Therefore, f is an increasing function of N. It follows that the number of prey attacked per predator always increases with the increase in prey density.

b) $\dfrac{\partial f}{\partial T} = \dfrac{b^2N^2}{1+cN+bT_hN^2}$

Again $\dfrac{\partial f}{\partial T} > 0$, for all values of N, T, and T_h.

Therefore, f is an increasing function of T, and the number of prey attacked per predator increases with the increase in available time for searching.

c) $\dfrac{\partial f}{\partial T_h} = -\dfrac{b^3N^4T}{(1+cN+bT_hN^2)^2} < 0$, for all N, T, and T_h.

Therefore, f is a decreasing function of T_h, so the number of prey attacked per predator decreases with an increase in the handling time of each prey.