Exam III

Overview

Exam III

When?: We have an exam scheduled for November 5, 2021

Which Sections?: It will cover sections 8.5, 9.1, 9.2, 9.3, 9.4, and 9.5.

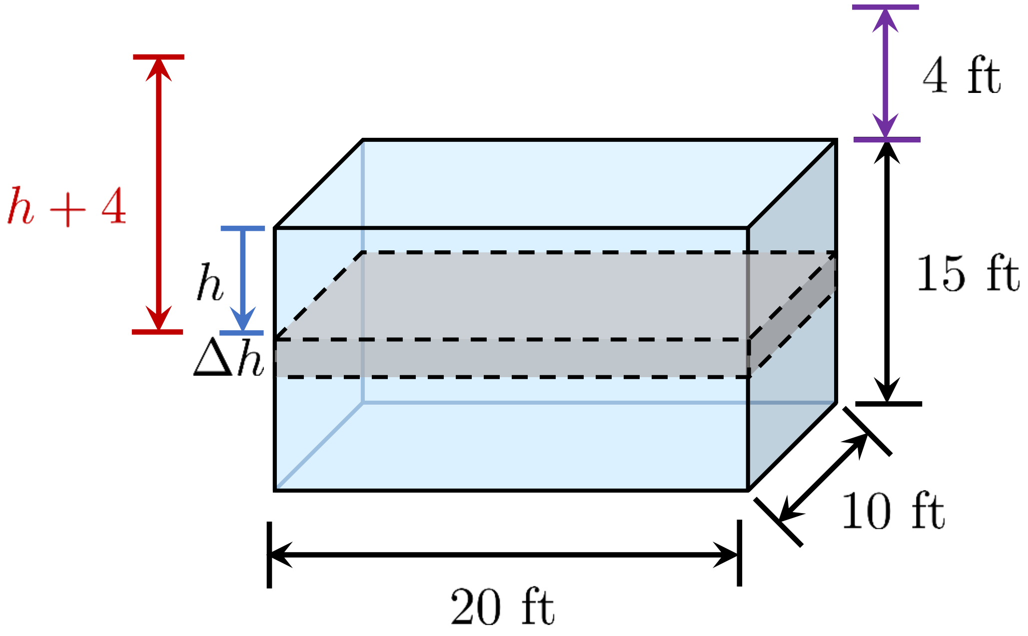
Exam Details: **Notes will NOT be allowed**.

- The exam will have about 10 problems.
- The exam will be out of 80 points, and
- 50 minutes in duration

Use of the free **Desmos Test Mode App** is permitted but you **MUST** inform the instructor before beginning the exam.

8.5 Work

A rectangular water tank has length 20 ft, width 10 ft, and depth 15 ft. If the tank is full, how much work does it take to pump all but 8 ft of the water to a height of 4ft above the top of the tank? (Note that 1 cubic foot of water weighs 62.4 lb.)



$$W_{\text{total}} = F(\text{water}) \cdot d(\text{water has to travel})$$

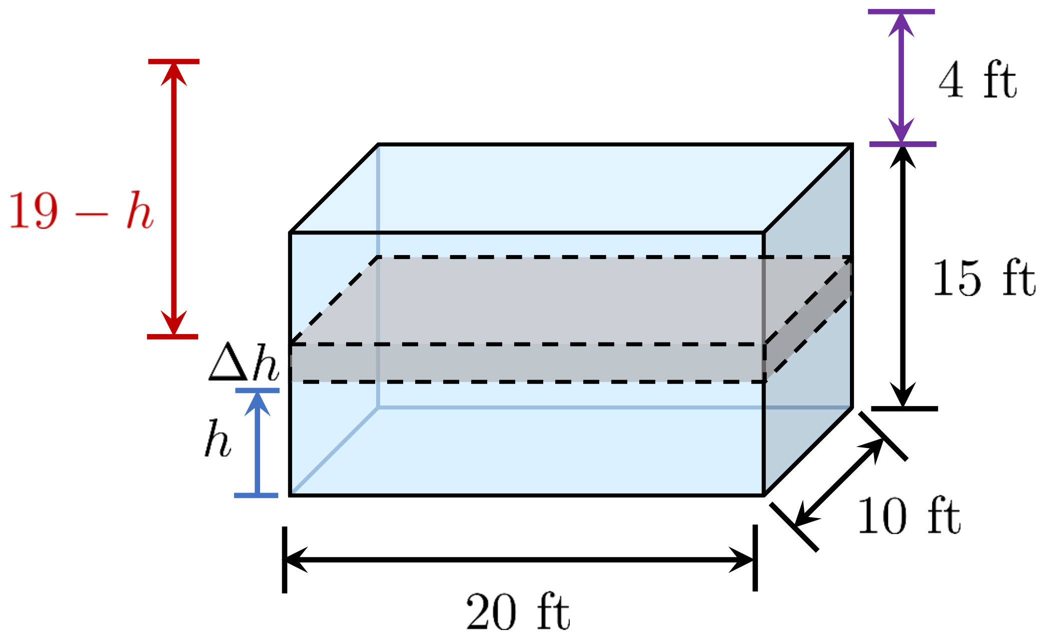
$$\begin{aligned}
W(\text{slice}) &= F(\text{slice}) \cdot d(\text{slice has to travel}) \\
&\approx [\text{Density} \cdot V(\text{slice})] \cdot d(\text{slice has to travel}) \\
&\approx \left[62.4 \text{ lb/ft}^3 \cdot (20 \text{ ft} \cdot 10 \text{ ft} \cdot \Delta h \text{ ft})\right](h+4) \text{ ft} \\
&\approx 12480(h+4)\Delta h \text{ ft-lb}
\end{aligned}$$

$$W_{\text{total}} \approx \sum_{\text{slices}} 12480(h+4)\Delta h \text{ ft-lb}$$

$$W_{\text{total}} = \int_0^{15-8} 12480(h+4)dh \text{ ft-lb} = \int_0^7 12480(h+4)dh \text{ ft-lb} = 655200 \text{ ft-lb}$$

8.5 Work (Approach 2)

A rectangular water tank has length 20 ft, width 10 ft, and depth 15 ft. If the tank is full, how much work does it take to pump all but 8 ft of the water to a height of 4ft above the top of the tank? (Note that 1 cubic foot of water weighs 62.4 lb.)



$$W_{\text{total}} = F(\text{water}) \cdot d(\text{water has to travel})$$

$$\begin{aligned} W(\text{slice}) &= F(\text{slice}) \cdot d(\text{slice has to travel}) \\ &\approx [\text{Density} \cdot V(\text{slice})] \cdot d(\text{slice has to travel}) \\ &\approx \left[62.4 \text{ lb/ft}^3 \cdot (20 \text{ ft} \cdot 10 \text{ ft} \cdot \Delta h \text{ ft})\right](19-h) \text{ ft} \\ &\approx 12480(19-h)\Delta h \text{ ft-lb} \end{aligned}$$

$$W_{\text{total}} \approx \sum_{\text{slices}} 12480(19-h)\Delta h \text{ ft-lb}$$

$$W_{\text{total}} = \int 12480(19-h)dh \text{ ft-lb} = \int_8^{15} 12480(19-h)dh \text{ ft-lb} = 655200 \text{ ft-lb}$$

9.1 Sequences

You are deciding whether to buy a new or a two-year-old car (of the same make) based on which will have cost you less when you resell it at the end of three years. Your cost consists of two parts: the loss in value of he car and the repairs. A new car costs \$20,000 and loses 12% of its value each year. Repairs are \$400 the first year and increase by 18% each subsequent year.

(a) For a new car, find the first three terms of the sequence v_n giving the value of the car in dollars at the end of n years. Give a formula for v_n.

$v_1 = 20000(0.88)$ (Losing %12 of value is the same as retaining %88 of the value)

$v_2 = v_1 \cdot (0.88) = [20000(0.88)](0.88) = 20000(0.88)^2$

$v_3 = v_2 \cdot (0.88) = [20000(0.88)^2](0.88) = 20000(0.88)^3$

$v_n = 20000(0.88)^n$

(b) Find the first three terms of the sequence d_n giving the depreciation (loss of value) in dollars in year n. Give a formula for d_n.

$d_1 = v_0 \cdot (0.12) = 20000(0.12)$

$d_2 = v_1 \cdot (0.12) = [20000(0.88)^1](0.12)$

$d_3 = v_2 \cdot (0.12) = [20000(0.88)^2](0.12)$

$d_n = v_{n-1} \cdot (0.12)$

$d_n = 20000(0.88)^{n-1}(0.12)$

9.1 Sequences

You are deciding whether to buy a new or a two-year-old car (of the same make) based on which will have cost you less when you resell it at the end of three years. Your cost consists of two parts: the loss in value of he car and the repairs. A new car costs \$20,000 and loses 12% of its value each year. Repairs are \$400 the first year and increase by 18% each subsequent year.

(c) Find the first three terms of the sequence r_n, the repair cost in dollars for a new car in year n. Give a formula for r_n.

$$r_1 = 400$$

$$r_2 = 400 + 400(0.18) = 400(1.18)$$

$$r_3 = 400(1.18) + 400(1.18)(0.18) = 400(1.18)^2$$

$$r_n = 400(1.18)^{n-1}$$

(d) Find the total cost of owning a new car for three years.

$$\text{Total cost} = d_1 + d_2 + d_3 + r_1 + r_2 + r_3 = \$8056.73$$

(e) Find the total cost of owning the two-year old car for three years. Which should you buy?

$$\text{Total cost} = d_3 + d_4 + d_5 + r_3 + r_4 + r_5 = \$7281.19$$

9.2 Series: Finite vs Infinite Geometric Series

Geometric Series:

- A **finite** geometric series has the form

$$\sum_{i=1}^{n} ax^{i-1} = a + ax + ax^2 + \cdots + ax^{n-2} + ax^{n-1} \quad = \frac{a(1-x^n)}{1-x} \quad \text{where } x \neq 1$$

(can be done by hand, there is a last sum)

- An **infinite** geometric series has the form

$$\sum_{i=1}^{\infty} ax^{i-1} = a + ax + ax^2 + \cdots + ax^{n-2} + ax^{n-1} + \cdots \; = \frac{a}{1-x} \quad \text{where } |x| < 1$$

(would take infinite time to sum, no last sum)

Note: In both cases x is called the common ratio.

9.2 Series: Finite vs Infinite Geometric Series

(a) $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64}$

$$\implies \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} + \frac{1}{2^5} - \frac{1}{2^6} = \frac{1}{2}\left(1 - \frac{1}{2^1} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5}\right) = \frac{1}{2}\left(\frac{1-\left(-\frac{1}{2}\right)^6}{1-\left(-\frac{1}{2}\right)}\right)$$

$$\sum\left(-\frac{1}{2}\right)^n \qquad = \frac{1}{2}\left(\frac{1-\frac{1}{2^6}}{1+\frac{1}{2}}\right)$$

(b) $-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots$

$$\implies -2 + 1 - \frac{1}{2^1} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} + \cdots = -2 + \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n = -2 + \frac{1}{1+\frac{1}{2}}$$

$$= -2 + \frac{2}{3}$$

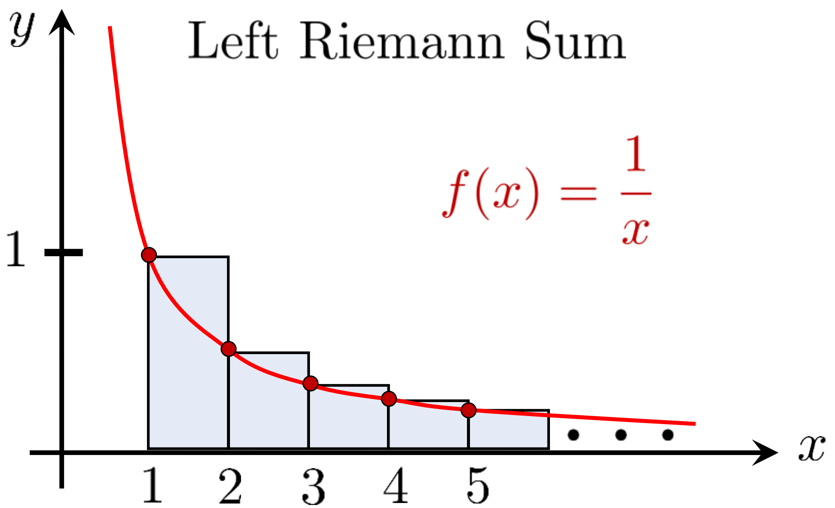
$$= -\frac{4}{3}$$

9.3 Convergence of Series

Example: Investigate the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \quad = \infty \ \text{(Diverges)}$$

$$\int_1^{\infty} \frac{1}{x} dx \ = \infty \ \text{(Diverges)}$$



The Integral Test:
Suppose $a_n = f(n)$, where $f(x)$ is decreasing and positive.

- If $\int_1^{\infty} f(x)dx$ **converges**, then $\sum a_n$ **converges**.
- If $\int_1^{\infty} f(x)dx$ **diverges**, then $\sum a_n$ **diverges**.

9.3 Convergence of Series

Example: Investigate the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

The p-series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p > 1 \text{ and diverges if } p \leq 1.$$

Solution:

We use the integral test.

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^2} dx = \lim_{b \to \infty} -\frac{1}{x}\Big|_1^b = \lim_{b \to \infty} \frac{1}{x}\Big|_b^1 = \lim_{b \to \infty} \left(\frac{1}{1} - \cancel{\frac{1}{b}} \right) = 1 \quad \text{(Converges)}$$

Since $\int_1^{\infty} \frac{1}{x^2} dx$ converges, **the series** $\sum_{n=1}^{\infty} \frac{1}{n^2}$ also converges by **the integral test**.

QUESTION: True or false?

$$\cancel{\sum_{n=1}^{\infty} \frac{1}{n^2} = \int_1^{\infty} \frac{1}{x^2} dx}?$$

FALSE!

Euler proved that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

The integral test only tells us about converges, NOT what a series converges to!

9.4: Tests for Convergence of Series (Summary)

The Integral Test (9.3):

Suppose $a_n = f(n)$, where $f(x)$ is decreasing and positive.

- If $\int_1^{\infty} f(x)dx$ **converges**, then $\sum a_n$ **converges**.
- If $\int_1^{\infty} f(x)dx$ **diverges**, then $\sum a_n$ **diverges**.

The p-series (9.3):

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Comparison Test:

Supposed that $0 \leq a_n \leq b_n$ for all n.

- If $\sum_{n=1}^{\infty} b_n$ **converges**, then $\sum_{n=1}^{\infty} a_n$ **converges**.
- If $\sum_{n=1}^{\infty} a_n$ **diverges**, then $\sum_{n=1}^{\infty} b_n$ **diverges**.

The Limit Comparison Test:

Supposed that $a_n > 0$ and $b_n > 0$ for all n.

$$\text{If } \lim_{n\to\infty} \frac{a_n}{b_n} = c \qquad \text{where } c > 0,$$

then the two series $\sum a_n$ and $\sum b_n$ both converge or both diverge.

The (Geometric) Ratio Test:

For a series $\sum a_n$, suppose that the sequence of rations $\left|\frac{a_{n+1}}{a_n}\right|$ has a limit:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L.$$

- If $L < 1$, then the series $\sum a_n$ **converges**.
- If $L > 1$, or if $L \to \infty$, then $\sum a_n$ **diverges**.
- If $L = 1$, then the test does not tell us anything about the convergence of the series $\sum a_n$. (Use a different test).

Convergence of Absolute Values Implies Convergence:

If $\sum |a_n|$ converges, then so does the series $\sum a_n$.

Absolute and Conditional Convergence:

We say that the series $\sum a_n$ is

- **Absolutely convergent** if $\sum a_n$ and $\sum |a_n|$ both converge.
- **Conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

L'Hospital's Rule:

Supposed that we have one of the following indeterminant forms,

$$\lim_{n\to\infty} f(n)/g(n) = 0/0 \qquad \text{OR} \qquad \lim_{n\to\infty} f(n)/g(n) = \pm\infty/\pm\infty,$$

we have,

$$\lim_{n\to\infty} f(n)/g(n) = \lim_{n\to\infty} f'(n)/g'(n)$$

9.5: Power Series and Intervals of Convergence

Important ideas from Section 9.5:

- Recognize a power series and its center
- Two views of convergence
  - Consider a single value of x (regular series)
  - Consider it as a function of x (look at intervals)

- Computing the radius of convergence (Ratio Test)

9.5: Power Series and Intervals of Convergence

Method for Computing Radius of Convergence:

To calculate the radius of convergence, R, for the power series $\sum_{n=0}^{\infty} C_n \cdot (x-a)^n$, use the ratio test with $a_n = C_n \cdot (x-a)^n$.

- If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ is infinite, then $R = 0$.
- If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0$, then $R = \infty$.
- If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = K|x-a|$, where K is finite and non-zero, then $R = \frac{1}{K}$.

9.5: Power Series and Intervals of Convergence

Example: Study the convergence of the power series about $x = 2$

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n} \qquad \text{Here } C_n = \frac{1}{3^n}$$

$$\left(\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L \begin{cases} L < 1 & \sum a_n \ \textbf{converges} \\ L > 1 & \sum a_n \ \textbf{diverges} \\ L = 1 & \text{inconclusive} \end{cases} \right)$$

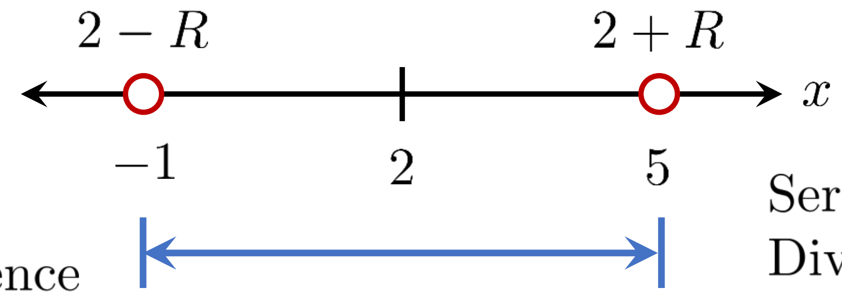
$$\lim_{n\to\infty} \frac{\left|\frac{(x-2)^{n+1}}{3^{n+1}}\right|}{\left|\frac{(x-2)^n}{3^n}\right|} = \lim_{n\to\infty} \left|\frac{(x-2)^{n+1}}{3^{n+1}}\right| \left|\frac{3^n}{(x-2)^n}\right| = \lim_{n\to\infty} \frac{|x-2|^{\cancel{n}+1}}{3^{\cancel{n}+1}} \frac{\cancel{3^n}}{\cancel{|x-2|^n}}$$

Center

$$= \lim_{n\to\infty} \frac{|x-2|}{3} = \frac{|x-2|}{3} < 1 \implies |x-2| < 3$$

Radius of convergence

$$R = 3$$



Series Divergence

Series Divergence

Interval of convergence

$= (-1, 5)$

Check end points!

If $x = -1$

$$\sum_{n=0}^{\infty} (-1)^n \quad \textbf{diverges}$$

If $x = 5$

$$\sum_{n=0}^{\infty} 1^n \quad \textbf{diverges}$$