- log(0) = negative infinity
- infinity + 1 | / | infinity -1 | = 1
  - properties of logs

6. The improper integral converges to

$$\int_4^{\infty} \frac{1}{x^2-1}\,dx$$
$$= \lim_{b\to\infty} \int_4^{b} \frac{1}{x^2-1}\,dx$$
$$= \lim_{b\to\infty} \frac{1}{2}\left(\ln|x-1| - \ln|x+1|\right)\Big|_4^b$$
$$= \lim_{b\to\infty} \left(\frac{1}{2}\ln\left|\frac{b-1}{b+1}\right| - \frac{1}{2}\ln\left|\frac{3}{5}\right|\right)$$
$$= -\frac{1}{2}\ln\left(\frac{3}{5}\right) = \frac{1}{2}\ln\left(\frac{5}{3}\right)$$

7.5 Numerical Methods for Definite Integrals

`deltax` = (right bound - left bound) / intervals

Left: `deltax * f(x0) + f(xn-1)`

Right: `deltax * f(x1) + f(xn)`

Trapezoid Rule

- While the midpoint rule uses the midpoint between each sub-interval, the trapezoid just averages the left and right results
  - `TRAP(n) = (LEFT(n) + RIGHT(n)) / 2`

Over or Under Estimates Rules and Relationships

How to Use Rules: - f negative AND integral is `a -> b` where `a` $<$ `b` - reverse each of the below rules - because values of sums will be negative - right is always opposite of left - midpoints is always opposite of trapezoid - Left and Right will always be more of an over or under estimate than midpoint and trapezoid

Rules: - f increasing - right = over - f decreasing - right = under - f concave down (looks like an `n` - `n` is in word "down") - trapezoial = under - f concave up (looks up a `u`, `u` is in word "up") - trapezoidal = over

Signs of Functions

Negative Functions

- To get the area under the curve value, swap the lower and upper bound (turn negative to positive)
  - In this case, the overestimate/underestimate rule still applies
  - If bounds are not swapped `Integrate[f(x), {x, a, b}] where a < b`, then the left-hand rule becomes the over-estimate and right-hand rule becomes under-estaimte, because the result will always be a negative value

Mixed Functions

- If the function takes on both positive and negative values, then the Riemann sum is the sum of the areas of the rectangles that lie above the x-axis and the negative of the areas of the recatangles that lie blow the x-axis
  - I.e., the areas of the rectangles where f is positive minus the areas of the rectangles where f is negative

Error

`Error = Actual Value - Approximate Value`

- The error in both the left and right rules decreases by afactor of about 5 as n increases by a factor of 5
  - 10 times work for every digit
- The error for the midpoint rule, in absolute value, seems to be about half the error of the trapezoid rule
- The error in trapezoid and midpoint decreases by a factor of about 25 as n increases by a factor of 5.
  - This squaring relationship holds for any factor
  - 10 times work for every 2 digits

Simpsons Rule

- As n increases by a factor of 5, the errors decrease by a factor of about 600, or 5^4
  - This behavior holds for any factor

- In Simpson's rule, each extra 4 digits of accuracy requires about 10 times the work

Approximating by Lines and Parabolas

Alternate Appraoch to Numerical Integration

These rules for numerical integration can be obtained by approximating `f(x)` on subintervals by a function: - The left and right rules use constant functions - The trapezoid and midpoint rules use linear functions - Simpson's rule use quadratic functions

7.6 Improper Integrals

- 1/ln(x) as x approaches 1 = infinity
- when the limit of integration is infinite
- when the integrand becomes infinite
  - the function is unbounded near some points in the interval

find the indefinite integral using the FTC and plugging in b for infinity -> then apply limit as b approaches infinity

Example:

1. integral 2 -> infinity of 1/x^2
2. compute indefinite integral: -1/x + C
3. plug-in b for upper bound and calculate definite integral using FTC
4. i.e., find definite integral of -1/x from 2 to b
5. FTC: (-1/b) - (-1/2)
6. apply limit as b approaches upper bound (infinity)
7. limit b -> inifnity of (-1/b) - (-1/2)
8. -1/infinity = 0, so limity b -> infinity = -(-1/2) = 1/2
9. The integral is imrproper because the upper limity of integration is infinity
10. therefore the improper integral converges to 1/2

Process 1. if the integral is imrpoper at both or one endpoints, we at an arbitrary point 1. If limits are infinity > infinity, split at arbitrary value c - if either diverge, the original integral diverges as well. If both converge, add them to get the integral 1. if the function is unbounded at a point ***between*** *the bounds, break into two integrals split at that point. If the limits of integration approaching the split point are finte, it converges – add to get integral. Else, the integral diverges 2. compute definite integral 2. examine limits as b approaches value 2. if the function is unbounded near some points in the interval, it is improper 2. if the integrand tends to infinity* inside* the interval of integration (rather than an endpoint), it is improper 2. if the limit does not exist (unbounded function), it diverges 2. if the limit of integration if inifinte, it diverges 2. if the integral diverges (integral tends to infinity as x approaches upper limit), it is improper 2. else if the limit of integration is finite, it converges 2. if the integral converges, define Integral a->b f(x) = lim c > b- of Integral a->c f(x)

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$

Suppose that $f(x)$ is positive and continuous on $[a,b]$ except at the point c. If $f(x)$ tends to infinity as $x \to c$, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

If *either* of the two new improper integrals diverges, we say the original integral diverges. Only if *both* of the new integrals have a finite value do we add the values to get a finite value for the original integral.

Suppose $f(x)$ is positive for $x \geq a$.
If $\lim_{b\to\infty} \int_a^b f(x)\,dx$ is a finite number, we say that $\int_a^\infty f(x)\,dx$ **converges** and define

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

Otherwise, we say that $\int_a^\infty f(x)\,dx$ **diverges**. We define $\int_{-\infty}^b f(x)\,dx$ similarly.

- Both functions approach 0 as x grows, so as b grows larger, smaller bits of area are being addded to the definite integral. The differrence between the functinos is subtle: the values of the function 1/sqrt(x) *dont shrink fast enough* for the integral to have a finite value. Of the two functions, 1/x^2 drops to 0 much faster than 1/sqrt(x) and this feature keeps the area under 1/x^2 from growing beyond 1.

For a positive function $f(x)$, we can use any (finite) number c to define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx.$$

If *either* of the two new improper integrals diverges, we say the original integral diverges. Only if both of the new integrals have a finite value do we add the values to get a finite value for the original integral.

7.7 Comparison of Improper Integrals

- We first look at the behavior of the integrand as x -> infinite because the convergence or divergence of the integral is determined by what happens as x approaches inifinity

The Comparison Test for $\int_a^{\infty} f(x)\, dx$

Assume $f(x)$ is positive. Making a comparison involves two stages:

1. Guess, by looking at the behavior of the integrand for large x, whether the integral converges or not. (This is the "behaves like" principle.)
2. Confirm the guess by comparison with a positive function $g(x)$:
   - If $f(x) \le g(x)$ and $\int_a^{\infty} g(x)\, dx$ converges, then $\int_a^{\infty} f(x)\, dx$ converges.
   - If $g(x) \le f(x)$ and $\int_a^{\infty} g(x)\, dx$ diverges, then $\int_a^{\infty} f(x)\, dx$ diverges.

Useful Integrals for Comparison

- $\displaystyle\int_1^{\infty} \frac{1}{x^p} dx$ converges for $p > 1$ and diverges for $p \le 1$.
- $\displaystyle\int_0^{1} \frac{1}{x^p} dx$ converges for $p < 1$ and diverges for $p \ge 1$.
- $\displaystyle\int_0^{\infty} e^{-ax} dx$ converges for $a > 0$.

Of course, we can use any function for comparison, provided we can determine its behavior.

If f is positive and continuous on $[a, b]$,

$$\int_a^{\infty} f(x)\, dx \text{ and } \int_b^{\infty} f(x)\, dx$$

either both converge or both diverge.

In particular, when the comparison test is applied to $\int_a^{\infty} f(x)\, dx$, the inequalities for $f(x)$ and $g(x)$ do not need to hold for all $x \ge a$ but only for x greater than some value, say b.

8.1 Areas and Volumes

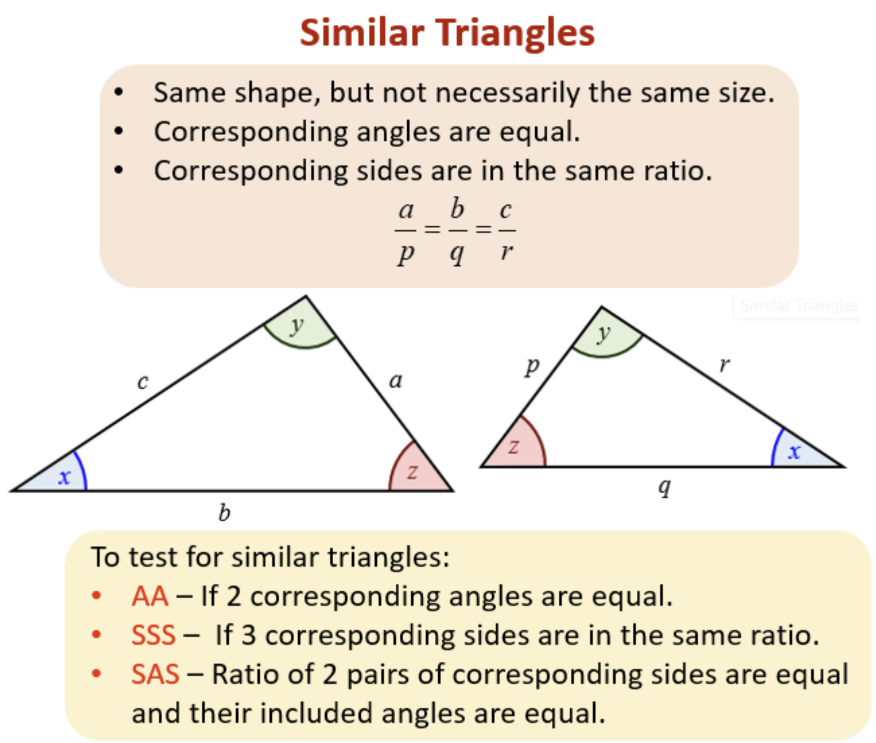
Process

1. Choose x-axis (eg height) by which the solid/curve can be split into small piecces along
2. find formula for volume/area of slice (pythagorean or similar triangle)

3. get riemann sum of the slices which represetns the sum of all the slices
4. take the limit as the number of terms in the sum tends to infinity, giving a definite integral for the total volume
5. find indefinite integral between bounds defined by the x-axis (e.g., if the x-axis is height, from the starting height to the finishing height of the geometric object)
6. compute the integral for the given bounds

Area

Isosceles Triangle



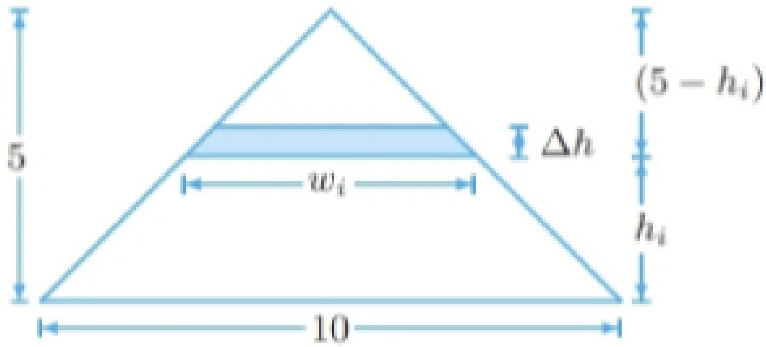


Figure 8.2: Horizontal slices of isosceles triangle

$$\text{Area of strip } \approx w_i \Delta h \text{ cm}^2.$$

To get w_i in terms of h_i, the height above the base, use the similar triangles in Figure 8.2:

$$\frac{w_i}{10} = \frac{5-h_i}{5}$$
$$w_i = 2(5-h_i) = 10 - 2h_i.$$

Summing the areas of the strips gives the Riemann sum approximation:

$$\text{Area of triangle } \approx \sum_{i=1}^{n} w_i \Delta h = \sum_{i=1}^{n} (10-2h_i)\Delta h \text{ cm}^2.$$

Taking the limit as $n \to \infty$, the change in h shrinks and we get the integral:

$$\text{Area of triangle } = \lim_{n\to\infty} \sum_{i=1}^{n} (10-2h_i)\Delta h = \int_0^5 (10-2h)\, dh \text{ cm}^2.$$

Evaluating the integral gives

$$\text{Area of triangle } = \int_0^5 (10-2h)\, dh = (10h - h^2)\Big|_0^5 = 25 \text{ cm}^2.$$

Notice that the limits in the definite integral are the limits for the variable h. Once we decide to slice the triangle horizontally, we know that a typical slice has thickness Δh, so h is the variable in our definite integral, and the limits must be values of h.

Circle

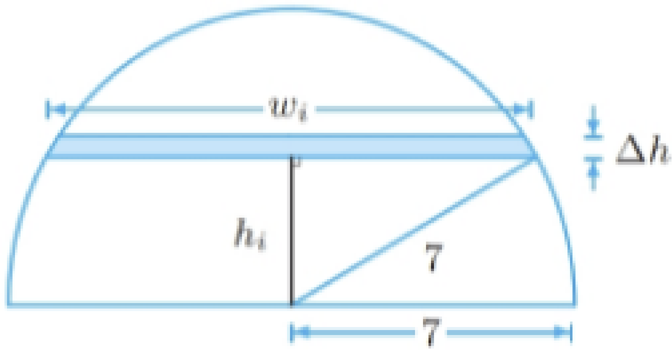


Figure 8.4: Horizontal slices of semicircle

$$h_i^2 + \left(\frac{w_i}{2}\right)^2 = 7^2,$$

$$w_i = \sqrt{4(7^2 - h_i^2)} = 2\sqrt{49 - h_i^2}.$$

Volume

Cone

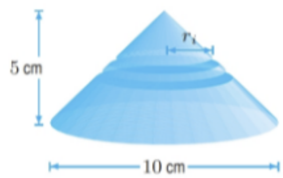


Figure 8.7: Cone

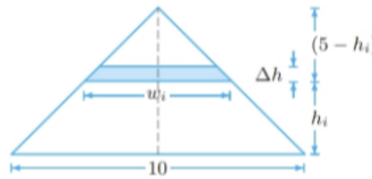


Figure 8.8: Vertical cross-section of cone in Figure 8.7

Each slice is a circular disk of thickness Δh. See Figure 8.7. The disk at height h_i above the base has radius $r_i = \frac{1}{2}w_i$. From Figure 8.8 and the previous example, we have

$$w_i = 10 - 2h_i \quad \text{so} \quad r_i = 5 - h_i.$$

Each slice is approximately a cylinder of radius r_i and thickness Δh, so

$$\text{Volume of slice } \approx \pi r_i^2 \Delta h = \pi(5-h_i)^2 \Delta h \text{ cm}^3.$$

Summing over all slices, we have

$$\text{Volume of cone } \approx \sum_{i=1}^{n} \pi(5-h_i)^2 \Delta h \text{ cm}^3.$$

Taking the limit as $n \to \infty$, so $\Delta h \to 0$, gives

$$\text{Volume of cone } = \lim_{n\to\infty} \sum_{i=1}^{n} \pi(5-h_i)^2 \Delta h = \int_0^5 \pi(5-h)^2\, dh \text{ cm}^3.$$

The integral can be evaluated using the substitution $u = 5 - h$ or by multiplying out $(5-h)^2$. Using the substitution, we have

$$\text{Volume of cone } = \int_0^5 \pi(5-h)^2 dh = -\frac{\pi}{3}(5-h)^3\Big|_0^5 = \frac{125}{3}\pi \text{ cm}^3.$$

Sphere

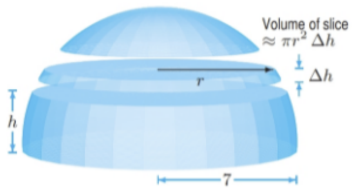


Figure 8.9: Slicing to find the volume of a hemisphere

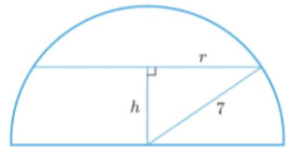


Figure 8.10: Vertical cut through center of hemisphere showing relation between r_i and h_i

Solution

We will not use the formula $\frac{4}{3}\pi r^3$ for the volume of a sphere. However, our approach can be used to derive that formula.

Divide the hemisphere into horizontal slices of thickness Δh cm. (See Figure 8.9.) Each slice is circular. Let r be the radius of the slice at height h, so

$$\text{Volume of slice} \approx \pi r^2 \Delta h \text{ cm}^3.$$

We express r in terms of h using the Pythagorean Theorem as in Example 2. From Figure 8.10, we have

$$h^2 + r^2 = 7^2,$$

so

$$r = \sqrt{7^2 - h^2} = \sqrt{49 - h^2}.$$

Thus,

$$\text{Volume of slice} \approx \pi r^2 \, \Delta h = \pi(7^2 - h^2)\, \Delta h \text{ cm}^3.$$

Summing the volumes of all slices gives:

$$\text{Volume} \approx \sum \pi r^2 \, \Delta h = \sum \pi(7^2 - h^2)\, \Delta h \text{ cm}^3.$$

Pyramid

$$\text{Volume of slice} \approx s^2\,\Delta h\ \text{ft}^3.$$

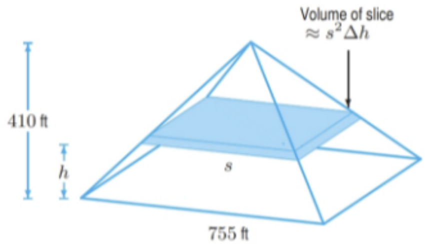


Figure 8.11: The Great Pyramid

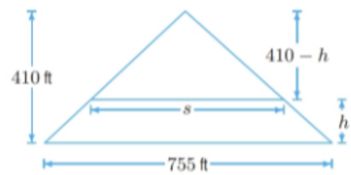


Figure 8.12: Cross-section relating s and h

We express s as a function of h using the vertical cross-section in Figure 8.12. By similar triangles, we get

$$\frac{s}{755} = \frac{(410-h)}{410}.$$

Thus,

$$s = \left(\frac{755}{410}\right)(410-h),$$

and the total volume, V, is approximated by adding the volumes of the n layers:

$$V \approx \sum s^2\,\Delta h = \sum \left(\left(\frac{755}{410}\right)(410-h)\right)^2 \Delta h\ \text{ft}^3.$$

8.2 Applications to Geometry

Volumes of Revolution

1. slice perpindicular to x-axis
2. create circular disks of thickness detla-x
3. the radius of the disk is the function
4. thus the volume of a slice = pi$y\hat{}2$ detla-x
5. as the thickness tends to zero -> compute definite integral across domain of the formula for a slice

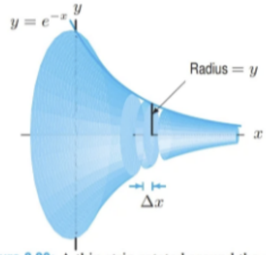


Figure 8.20: A thin strip rotated around the x-axis to form a circular slice

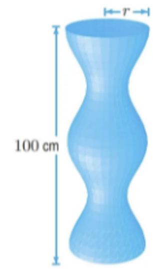


Figure 8.21: A table leg

Example 2 A table leg in Figure 8.21 has a circular cross section with radius r cm at a height of y cm above the ground given by $r = 3 + \cos(\pi y/25)$. Find the volume of the table leg.

Solution The table leg is formed by rotating the curve $r = 3 + \cos(\pi y/25)$ around the y-axis. Slicing the table leg horizontally gives circular disks of thickness Δy and radius $r = 3 + \cos(\pi y/25)$.

To set up a definite integral for the volume, we find the volume of a typical slice:

$$\text{Volume of slice } \approx \pi r^2 \Delta y = \pi \left(3 + \cos\left(\frac{\pi}{25} y\right)\right)^2 \Delta y.$$

Summing over all slices gives the Riemann sum approximation:

$$\text{Total volume } = \sum \pi \left(3 + \cos\left(\frac{\pi}{25} y\right)\right)^2 \Delta y.$$

Taking the limit as $\Delta y \to 0$ gives the definite integral:

$$\text{Total volume } = \lim_{\Delta y \to 0} \sum \pi \left(3 + \cos\left(\frac{\pi}{25} y\right)\right)^2 \Delta y = \int_0^{100} \pi \left(3 + \cos\left(\frac{\pi}{25} y\right)\right)^2 dy.$$

Evaluating the integral numerically gives:

$$\text{Total volume } = \int_0^{100} \pi \left(3 + \cos\left(\frac{\pi}{25} y\right)\right)^2 dy = 2984.5 \text{ cm}^3.$$

Bounded Intersection

1. find intersection to find upper bound
2. determine volumes of revolution for each function
3. volume of difference = volume of outer (function that bounds inner) - volume of inner
4. computer indefinite integral from first intersection to second intersection of volume of difference (outer slice minus inner slice)

3 The region bounded by the curves $y = x$ and $y = x^2$ is rotated about the line $y = 3$. Compute the volume of the resulting solid.

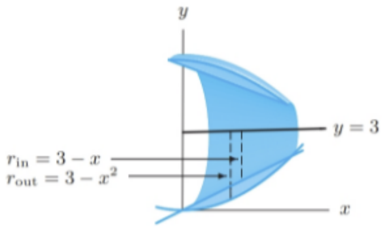


Figure 8.22: Cutaway view of volume showing inner and outer radii

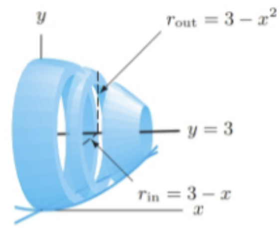


Figure 8.23: One slice (a disk-with-a-hole)

Known Cross-Section Base

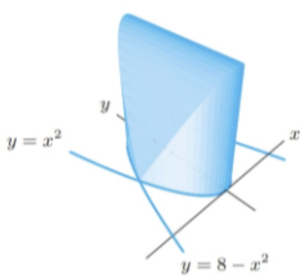


Figure 8.25: The solid for Example 4

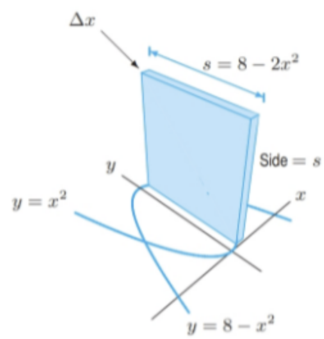


Figure 8.26: A slice of the solid for Example 4

We view the solid as a loaf of bread sitting on the xy-plane and made up of square slices. A typical slice of thickness Δx is shown in Figure 8.26. The side length, s, of the square is the distance (in the y direction) between the two curves, so $s = (8 - x^2) - x^2 = 8 - 2x^2$, giving

$$\text{Volume of slice} \approx s^2\,\Delta x = (8 - 2x^2)^2\,\Delta x.$$

Thus

$$\text{Total volume} = V \approx \sum s^2\,\Delta x = \sum (8 - 2x^2)^2\,\Delta x.$$

As the thickness Δx of each slice tends to zero, the sum becomes a definite integral. Since the curves $y = x^2$ and $y = 8 - x^2$ intersect at $x = -2$ and $x = 2$, these are the limits of integration. We have

$$\begin{aligned} V = \int_{-2}^{2} (8 - 2x^2)^2\,dx &= \int_{-2}^{2} (64 - 32x^2 + 4x^4)\,dx \\ &= \left(64x - \frac{32}{3}x^3 + \frac{4}{5}x^5\right)\Bigg|_{-2}^{2} = \frac{2048}{15} \approx 136.5. \end{aligned}$$

Arc Length

1. derivative of f(x) = change in y over change in x, thus change in y equals

derviative of f(x) times change in x

2. take increasingly smaller triangles along the perimteter of the arc
3. determine formula for hypotenuse of triangle using sides delta-x and delta-y
4. determine riemann summ of permieter distances
5. compute indefinite intrgral across domain

$$\text{Length } \approx \sqrt{(\Delta x)^2 + (\Delta y)^2} \approx \sqrt{(\Delta x)^2 + \left(f'(x)\,\Delta x\right)^2} = \sqrt{1 + (f'(x))^2}\,\Delta x.$$

Thus, the arc length of the entire curve is approximated by a Riemann sum:

$$\text{Arc length} \approx \sum \sqrt{1 + \left(f'(x)\right)^2}\,\Delta x.$$

Since x varies between a and b, as we let Δx tend to zero, the sum becomes the definite integral:

For $a < b$, the arc length of the curve $y = f(x)$ from $x = a$ to $x = b$ is given by

$$\text{Arc length} = \int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

Parametric Arc Length

Arc Length of a Parametric Curve

A particle moving along a curve in the plane given by the parametric equations $x = f(t)$, $y = g(t)$, where t is time, has speed given by:

$$v(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

We can find the distance traveled by a particle along a curve between $t = a$ and $t = b$ by integrating its speed. Thus,

$$\text{Distance traveled} = \int_a^b v(t)\,dt.$$

If the particle never stops or reverses its direction as it moves along the curve, the distance it travels is the same as the length of the curve. This suggests the following formula:

If a curve is given parametrically for $a \le t \le b$ by differentiable functions and if the velocity $v(t)$ is not 0 for $a < t < b$, then

$$\text{Arc length of curve} = \int_a^b v(t)\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

- the difference in x and the difference in y for each new sectino of the arc length represents two sides of a right triangle
- the hypotenuse of that trianlge represents the distance traveled / the permiter of the arc over that domain
- that value is also equivalent to the distance traveled given that e are finding the integral of velocity

- the indefinite integral of the formula for the hypotneuse over the given domain represents the total arc length over that domain

8.4 Density and Mass

To find total quantity from density, divide the region into small pieces in such a way that the density is approximately constant on each piece, and add the contributions of the pieces

- if density and volume are given, the total mass can be computed by finding the integral of dneisty times volume, etc.
- determine formula for slices over the domain for volume
- determine formula for slices of density over the domain
- determeine indefinite integral of their product

Center of Mass

`moment of mass about pivote = mass * displacement from pivot`

- the seesaw balances if the total moment is zero
- thus, the center of mass is the point about which the total moment is zero

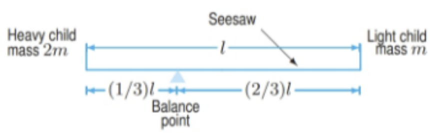


Figure 8.57: Children on seesaw

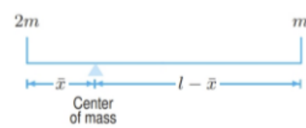


Figure 8.58: Center of mass of point masses

Suppose the center of mass in Figure 8.58 is at a distance of $\bar{x}$ from the left end. The moment of the left mass about the center of mass is $-2m\bar{x}$ (it is negative since it is to the left of the center of mass); the moment of the right mass about the center of mass is $m(l-\bar{x})$. The system balances if

$$-2m\bar{x}+m(l-\bar{x})=0 \quad \text{or} \quad ml-3m\bar{x}=0 \quad \text{so} \quad \bar{x}=\frac{1}{3}l.$$

Thus, the center of mass is $l/3$ from the left end.

We use the same method to calculate the center of mass, $\bar{x}$, of the system in Figure 8.59. The sum of the moments of the three masses about $\bar{x}$ is 0, so

$$m_1(x_1-\bar{x})+m_2(x_2-\bar{x})+m_3(x_3-\bar{x})=0.$$

Solving for $\bar{x}$, we get

$$m_1\bar{x}+m_2\bar{x}+m_3\bar{x}=m_1x_1+m_2x_2+m_3x_3$$
$$\bar{x}=\frac{m_1x_1+m_2x_2+m_3x_3}{m_1+m_2+m_3}=\frac{\sum_{i=1}^{3}m_ix_i}{\sum_{i=1}^{3}m_i}.$$

Generalizing leads to the following formula:

The **center of mass** of a system of n point masses $m_1, m_2, \ldots, m_n$ located at positions $x_1, x_2, \ldots, x_n$ along the x-axis is given by

$$\overline{x}=\frac{\sum x_im_i}{\sum m_i}.$$

The numerator is the sum of the moments of the masses about the origin; the denominator is the total mass of the system.

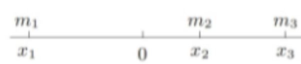


Figure 8.59: Discrete masses m_1, m_2, m_3

Continuous Mass Density

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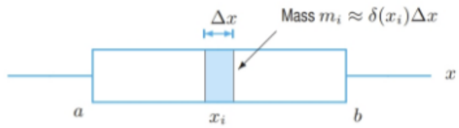

Figure 8.60: Calculating the center of mass of an object of variable density, $\delta(x)$

Δx. On each piece, the density is nearly constant, so the mass of the piece is given by density times length. See Figure 8.60. Thus, if x_i is a point in the i^{th} piece,

$$\text{Mass of the } i^{\text{th}} \text{ piece, } m_i \approx \delta(x_i)\Delta x.$$

Then the formula for the center of mass, $\bar{x} = \sum x_i m_i / \sum m_i$, applied to the n pieces of the object gives

$$\bar{x} = \frac{\sum x_i \delta(x_i)\Delta x}{\sum \delta(x_i)\Delta x}.$$

In the limit as $n \to \infty$ we have the following formula:

The **center of mass** $\overline{x}$ of an object lying along the x-axis between $x = a$ and $x = b$ is

$$\overline{x} = \frac{\int_a^b x\delta(x)\,dx}{\int_a^b \delta(x)\,dx},$$

where $\delta(x)$ is the density (mass per unit length) of the object.

As in the discrete case, the denominator is the total mass of the object.

Two- and Three-Dimensional Regions

For a system of masses that lies in the plane, the center of mass is a point with coordinates $(\bar{x}, \bar{y})$. In three dimensions, the center of mass is a point with coordinates $(\bar{x}, \bar{y}, \bar{z})$. To compute the center of mass in three dimensions, we use the following formulas in which $A_x(x)$ is the area of a slice perpendicular to the x-axis at x, and $A_y(y)$ and $A_z(z)$ are defined similarly. In two dimensions, we use the same formulas for $\bar{x}$ and $\bar{y}$, but we interpret $A_x(x)$ and $A_y(y)$ as the lengths of strips perpendicular to the x- and y-axes, respectively.

For a region of constant density δ, the center of mass is given by

$$\bar{x} = \frac{\int x\delta A_x(x)\,dx}{\text{Mass}} \qquad \bar{y} = \frac{\int y\delta A_y(y)\,dy}{\text{Mass}} \qquad \bar{z} = \frac{\int z\delta A_z(z)\,dz}{\text{Mass}}.$$

The expression $\delta A_x(x)\Delta x$ is the moment of a slice perpendicular to the x-axis. Thus, these formulas are extensions of that on page 444. In the two- and three-dimensional case, we are assuming that the density δ is constant. If the density is not constant, finding the center of mass may require a double or triple integral from multivariable calculus.