CVE154 Exam 2, Part 2

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1 Blank space

Consider the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ of N equations and N unknowns.

1. If $\boldsymbol{A}$ is upper-triangular, $\boldsymbol{x}$ is directly obtained using backward substitution.
2. The augmented (coefficient) matrix is formed by appending $\boldsymbol{b}$ as a new column to $\boldsymbol{A}$.
3. The system has a unique solution if the determinant of $\boldsymbol{A}$ is not zero.
4. If $\boldsymbol{A}$ is lower-triangular, $\boldsymbol{x}$ is directly obtained using forward substitution.
5. If the inverse of the coefficient matrix, $\boldsymbol{A}^{-1}$, exists, then $\boldsymbol{x}$ is computed as $\boldsymbol{A}^{-1}\boldsymbol{b}$. *See p. 387 of the main reference.*
6. $\boldsymbol{A}$ is well-conditioned if its condition number of $\boldsymbol{A}$ is not significantly greater than unity. *See Section 2.1, Subsection "Ill Conditioning" of Numerical Methods in Engineering with Python 3.*

Consider direct methods for solving the above linear system.

7. Partial pivoting interchanges rows of the augmented matrix to find the pivot element. *See p. 373 - 374 of the main reference.*
8. LU decomposition involves factoring $\boldsymbol{A}$ into a product of a lower- and an upper-triangular matrix, with the requirement that $\boldsymbol{A}$ is nonsingular.
9. The Gauss-Jordan method extends Gaussian elimination by transforming the system into an equivalent form $\boldsymbol{Dx} = \boldsymbol{d}$ where $\boldsymbol{D}$ is diagonal. *See Item 12 of Exercise Set 6.1 (p. 370) of the main reference.*
10. In the Doolittle variant of LU decomposition, the lower-triangular factor of $\boldsymbol{A}$ has 1s on the diagonal. *See p. 405 of the main reference.*
11. Complete pivoting looks for the pivot element across rows and columns of the augmented matrix. *See p. 379 of the main reference.*
12. In the Crout variant of LU decomposition, the diagonal elements of the upper-triangular factor of $\boldsymbol{A}$ are 1s. *See p. 405 of the main reference.*
13. Cholesky decomposition involves factoring $\boldsymbol{A}$ as a product of a lower- and an upper-triangular matrix, each being a transpose of the other. *See p. 405 of the main reference.*

2 Million reasons

3 pt. In the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ of N equations and N unknowns, the coefficient matrix has a determinant of -1709. Is the system solvable? Why do you think so?

> That $\boldsymbol{A}$ has a nonzero determinant implies that it is nonsingular, and so the linear system has a solution.

4 pt. Suppose for a physical problem you derived two equivalent linear system representations: $\boldsymbol{Ax} = \boldsymbol{a}$ and $\boldsymbol{By} = \boldsymbol{b}$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are the vectors containing unknown quantities. $\boldsymbol{A}$ and $\boldsymbol{B}$ have condition numbers of 2.024 and 2024, respectively. Which linear system should you use?

> $\boldsymbol{Ax} = \boldsymbol{a}$ is preferable, on the following basis. The further from unity the condition number of a coefficient matrix is, the less well-conditioned it is, and, consequently, the less reliable the solution is of the linear system.

5 pt. Discuss how Gaussian elimination can be used to compute the inverse of a square, nonsingular matrix $\boldsymbol{A}$.

> See pp. 388 - 389 of the main reference.

3 Look at those cavemen go

10 pt. Say you have a robot equipped with distance sensors so that you can collect Cartesian coordinates of N points — *i.e.*, (x_1, y_1, z_1), (x_2, y_2, z_2), …, (x_N, y_N, z_N) — on the roof of a cave. Hoping to simplify the estimation of the cave's dimensions, you hypothesize that the cave is adequately modelled as a portion of a sphere $(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=\rho^2$. Here, the problem is determining from collected data the parameters α, β, γ, and ρ. Show that this problem can be modelled as a system of N linear equations in four unknowns.

> Each of the N collected points must satisfy the sphere equation, *i.e.*,
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> $$(x_i-\alpha)^2+(y_i-\beta)^2+(z_i-\gamma)^2=\rho^2,$$
>
> for $i=1,2,\ldots,N$. Expanding the terms and rearranging, one can arrive at
>
> $$\begin{aligned}&2x_i\alpha+2y_i\beta+2z_i\gamma\\&+\rho^2-\alpha^2-\beta^2-\gamma^2\\&\qquad\qquad=x_i^2+y_i^2+z_i^2.\end{aligned}$$
>
> The subscripted terms, being collected data, are known, and so is the quantity $b_i=x_i^2+y_i^2+z_i^2$. Since α, β, γ, and ρ are unknown, then the quantity $\theta=\rho^2-\alpha^2-\beta^2-\gamma^2$ is also unknown. Therefore, each of the N collected data points must satisfy
>
> $$2x_i\alpha+2y_i\beta+2z_i\gamma+\theta=b_i,$$
>
> whence the linear system:
>
> $$\begin{bmatrix}2x_1 & 2y_1 & 2z_1 & 1\\ 2x_2 & 2y_2 & 2z_2 & 1\\ \vdots & \vdots & \vdots & \vdots\\ 2x_N & 2y_N & 2z_N & 1\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\\ \gamma\\ \theta\end{bmatrix}=\begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_N\end{bmatrix}.$$

Bonus

3 pt. Estimate the number of pedestrian crossings inside the campus. Explain your thought process.