CVE154 Exam 1

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This exam is primarily aimed to assess your ability to mathematically model an engineering problem as a root-finding task. The value of such computational thinking is that it allows us to address the (near-)realistic former by solving the abstract latter using established mathematical and computing tools.

While the use of AI tools to answer this exam is not prohibited, it is nevertheless of ethical interest to disclose such use. This is in line with the MSU Policy on the Fair and Ethical Use of AI and Its Applications. As such, please include a brief statement of which AI tools did you use in answering this exam.

Write your answers in an A4-sized document saved as a PDF file with a filename following the pattern `EXM-01_<Class number>_<ID number>`. (For example, if your ID number is 2013-0024 and you are enrolled to the W45M456 class, then your answers should be in the file `EXM-01_W45M456_2013-0024.pdf`). To submit your answers, first create a Google Drive folder named `CVE154_<ID number>`, owned by your G.MSUIIT account. Then upload the PDF file to the folder, making sure that the instructor's G.MSUIIT account has editor access to the folder. Lastly, send a link to the uploaded PDF file via email with the subject `CVE154 Exam 1`.

1 First things first

The following scenarios require you to retrieve information from your arsenal of first principles, and, from such information, synthesize reasoned answers.

5 marks In trying to find some root of a function $f(x)$ over the interval $[m, n]$, you notice that $f(m) > 0$ and $f(n) > 0$. Reason whether or not a root exists in the said interval.

> If $f(x)$ is continuous over $[m, n]$, one of the following can happen: (i) the interval does not contain a root; (ii) the interval has one root (*e.g.*, a parabola opening upwards from a vertex on the x-axis); (iii) the interval has an odd number of roots (*e.g.*, a sinusoid shifted upwards by its amplitude); and (iv) the interval contains an even number of roots (*e.g.*, a standard sinusoid). Moreover, these possibilities also hold if $f(x)$ is not continuous over $[m, n]$.

3 marks Suppose running bisection method converges to and outputs $\hat{p}$. How do you check if $\hat{p}$ is a root of some function $g(x)$?

> One can check if $g(\hat{p})$ is practically close enough to 0.

7 marks Say you have "blackbox" access to a function $f(x)$ and its derivative $f'(x)$, *i.e.*, you know the values of $f(x)$ and $f'(x)$ for every x you provide, but you do not know how such values are computed. You notice that at $x = 17$, the derivative is approximately close to 0. Come up with a strategy for finding one or more roots of $f(x)$.

Since $f'(17) \approx 0$, then it is reasonable to expect that $f(x)$, assuming continuous, has a (local) extremum near $x = 17$. Intuitively, there is a "peak" or a "dip" at $x = 17$, the left and right sides of which have slopes of opposite signs. A sensible strategy then is to use Newton-Raphson at initial estimates $x_0 < 17$ and $x_0 > 17$. Of course, this strategy is but a heuristic, and further information on $f(x)$ and/or $f'(x)$ are needed to at least imply multiplicity of roots.

An alternative strategy is as follows. First, check the sign of $f(17)$. Then, randomly select some $y > 17$ and use the `zbrac` and/or `zbrak` routines (see Section 9.1 of *Numerical Recipes, 3rd ed.*) to find an interval satisfying the conditions of the intermediate value theorem. If such an interval is found, one can use bisection method to find a root. Next, randomly select some $y < 17$ and use the same approach.

10 marks Derive Newton-Raphson iteration schemes[1] to approximate $\sqrt[n]{A}$ and $\frac{1}{\sqrt{A}}$, where A is positive.

Observe that $\sqrt[n]{A}$ is the root of $f(x) = x^n - A$, whose derivative is $f'(x) = nx^{n-1}$. Then, the Newton-Raphson iteration is

$$x_{k+1} \leftarrow x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^n - A}{nx_k^{n-1}} = \left(1 - \frac{1}{n}\right) x_k + \frac{Ax_k}{nx_k^n}. \tag{1}$$

Along the same thought, $\frac{1}{\sqrt{A}}$ is the root of $g(x) = x^2 - \frac{1}{A}$, whose derivative is $g'(x) = 2x$. Then, the Newton-Raphson iteration is

$$x_{k+1} \leftarrow x_k - \frac{g(x_k)}{g'(x_k)} = x_k - \frac{x_k^2 - \frac{1}{A}}{2x_k} = \frac{x_k}{2} + \frac{1}{2Ax_k}. \tag{2}$$

10 marks You are to find a point on the graph of $y = 3\cosh\left(\frac{x}{3}\right)$ that is closest to $(9, 17)$. Cast the original problem as finding the root of some function $f(x)$. Also derive a Newton-Raphson iteration scheme to approximate said root.

The distance between the point $(9, 17)$ and any point (x, y) on the graph of $y = 3\cosh\left(\frac{x}{3}\right)$ can be expressed as a function w of x:

$$w(x) = \sqrt{(x-9)^2 + \left(3\cosh\left(\frac{x}{3}\right) - 17\right)^2}. \tag{3}$$

The desired point is one whose ordinate x minimizes $w(x)$. Moroever, $w(x)$ is minimized with respect to x when its derivative with respect to x

$$w'(x) = \frac{x + \left(3\cosh\left(\frac{x}{3}\right) - 17\right)\sinh\left(\frac{x}{3}\right) - 9}{\sqrt{(x-9)^2 + \left(3\cosh\left(\frac{x}{3}\right) - 17\right)^2}} \tag{4}$$

is zero. Thus, the equivalent root-finding task is to find x for which $f(x) = w'(x) = 0$. The Newton-Raphson iteration scheme is therefore:

$$x_{k+1} \leftarrow x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{w'(x_k)}{w''(x_k)}, \tag{5}$$

where $w''(x_k)$ can be easily obtained using freely available computer algebra systems (*e.g.*, David Scherfgen's Derivative Calculator).

[1] Recall that an iteration scheme takes the form of $w_{k+1} \leftarrow$ some function of w_k, where w_k represents the k-th iteration estimate.

The above approach is okay, except for the square root. This calls for a pro move. Observed[2] that a point $\left(x, 3\cosh\frac{x}{3}\right)$ with the least distance from $(9,17)$ is the same point with the *least squared distance* from $(9,17)$. So, we have

$$v(x) = w^2(x) = (x-9)^2 + \left(3\cosh\left(\frac{x}{3}\right) - 17\right)^2 \tag{6}$$

to minimize; ergo, finding the root of

$$f(x) = v'(x) = 2x + \left(6\cosh\left(\frac{x}{3}\right) - 34\right)\sinh\left(\frac{x}{3}\right) - 18, \tag{7}$$

whose derivative is a bit more manageable:

$$f'(x) = v''(x) = 2\sinh^2\left(\frac{x}{3}\right) + \frac{\cosh\left(\frac{x}{3}\right)\left(6\cosh\left(\frac{x}{3}\right) - 34\right)}{3} + 2. \tag{8}$$

2 Hang in there

A cable supported at its ends and hanging due a uniform gravitational force, is often considered to take the form of a *catenary*, assuming that the cable has a uniform density ρ and a fixed cross-sectional area A_0 throughout its length. Static equilibrium implies that the horizontal component of the tension at any point along the wire is a constant quantity T_0. In Cartesian coordinates arranged so that the y-axis passes through the vertex, the elevation y and the horizontal displacement x of any point along the cable are related as follows:

$$y(x) = \alpha\cosh\frac{x}{\alpha}, \qquad \text{where} \quad \alpha = \frac{T_0}{\rho g A_0}. \tag{9}$$

One may thus treat α as a parameter whose value dictates the shape of the catenary.

Epstein is tasked to design a cable whose ends are supported at the top of two H_{sup}-ft high pylons situated X_{span} ft apart. The maximum sag is Y_{sag} ft from the support points. Epstein wants to find the value of α corresponding to $H_{\mathsf{sup}} = 40$, $X_{\mathsf{span}} = 100$, and $Y_{\mathsf{sag}} = 5.5$.

10 marks Using Equation (9), derive a root-finding problem describing Epstein's task.

Observe that α is the elevation of the lowest point, *i.e.*, at $x = 0$ where the maximum sag occurs. Then, by construction and by symmetry, the elevation of any supported end (*i.e.*, at $x = -50$ or at $x = 50$) is the sum of α and the maximum sag[3]:

$$\alpha\cosh\frac{50}{\alpha} = \alpha + 5.5 \quad\Longrightarrow\quad \underbrace{\alpha\cosh\frac{50}{\alpha} - \alpha - 5.5}_{f(\alpha)} = 0, \tag{10}$$

thus arriving at the desired root-finding formulation of Epstein's task.

7 marks Provide a suitable initial interval $[\underline{\alpha}_0, \overline{\alpha}_0]$ for finding the desired α via bisection. Then, provide an iteration budget (*i.e.*, the maximum number of iterations) if you wish to find the desired α to within 10^{-5} ft. Justify your choices.

For $[\underline{\alpha}_0, \overline{\alpha}_0]$ to be suitable, $f(\underline{\alpha}_0)$ and $f(\overline{\alpha}_0)$ must have opposing signs. Then, the iteration budget N can be set by

$$\frac{\overline{\alpha}_0 - \underline{\alpha}_0}{2^N} = 10^{-5} \quad\Longrightarrow\quad N = \log_2\frac{\overline{\alpha}_0 - \underline{\alpha}_0}{10^{-5}}. \tag{11}$$

[2] You passed the tense check.

[3] What about $f(\alpha) = \alpha + Y_{\mathsf{sag}} - H_{\mathsf{sup}}$?.

5 marks Using Equation (9), express Epstein's task as a fixed-point problem.

It follows directly from Equation (10) that:

$$\alpha = \underbrace{5.5 - \alpha \cosh \frac{50}{\alpha}}_{g(\alpha)}, \tag{12}$$

which is a fixed-point form. There are, of course, many other alternative equivalent expressions.

5 marks Derive a Newton-Raphson iteration scheme to numerically compute the desired α.

From Equation (10),

$$f'(\alpha) = \cosh\left(\frac{50}{\alpha}\right) - \frac{50}{x}\sinh\left(\frac{50}{\alpha}\right) - 1. \tag{13}$$

Then, the Newton-Raphson iteration would be

$$\alpha_{k+1} \leftarrow \alpha_k - \frac{f(\alpha_k)}{f'(\alpha_k)} = \alpha_k - \frac{\alpha \cosh \frac{50}{\alpha} - \alpha - 5.5}{\cosh\left(\frac{50}{\alpha}\right) - \frac{50}{x}\sinh\left(\frac{50}{\alpha}\right) - 1}. \tag{14}$$

3 Going with the flow

In hydraulics, the Manning formula (also known as Gauckler–Manning formula) empirically describes uniform liquid flow in an open channel[4] (*i.e.*, a conduit that does not completely enclose the liquid). For an open channel, the Manning formula relates the *volumetric flow rate*[5] q to the slope m_{channel}, the *flow cross-sectional area* a, and the *hydraulic radius* r_{hyd} as follows:

$$q = \frac{\sqrt{m_{\mathsf{channel}}}}{\mu_{\mathsf{Man}}} a r_{\mathsf{hyd}}^{\frac{2}{3}}. \tag{15}$$

The *Manning coefficient* (also known as *Gauckler–Manning coefficient*) μ_{Man} is empirically determined and is dependent on factors like roughness and sinuosity of the channel.

(Perhaps it is best to undestand how to compute flow cross-sectional area and the hydraulic radius by means of an example. Say you have a pipe with radius r, wherethrough some liquid flows up to midway of the height of the pipe. The flow cross-sectional area is half the pipe cross-sectional area, *i.e.*, $0.5\pi r^2$. The hydraulic radius is the ratio of the flow cross-sectional area and the pipe's wetted perimeter (*i.e.*, that portion of the pipe's cross-sectional perimeter that is "wetted" in the sense of being in contact with the liquid). In this case, the wetted perimeter is just half the circumference of the pipe cross-section, *i.e.*, πr. Thus, the hydraulic radius is $0.5r$. If the pipe were to fully conduct the liquid, then the flow cross-sectional area is πr^2, and the hydraulic radius is $0.5r$.)

In a bid to be be branded as the spitting image of a cutting-edge waterworks design firm, Hawk Tuah Enterprises assigned Haliey to design of a rectangular open channel that would transmit water from the outlet of an upper reservoir (at 174.07 m above sea level) to the inlet of a lower reservoir (at 7.1 m above sea level). The outlet of the upper reservoir discharges water at a rate of 17.03 cms (cubic meters per second). Haliey, upon checking the map coordinates of the terminal points of the channel, notes that the outlet of the upper reservoir and the inlet of the lower reservoir are separated by 98.04 km. Moreover, it is required to match the volumetric flow rate in the channel to that of the discharge from the upper reservoir. Adapting an existing rectangular channel design with a width of 4.75 m and a Manning coefficient of 0.0017, Haliey must then determine the height of the water flowing through the channel.

[4] An *open channel* is a conduit that does not completely enclose the liquid passing passing through it.

[5] As the name suggests, it is the volume of liquid flowing through a cross-section of the conduit per unit time.

10 marks Assuming uniform flow conditions, use Equation (15) to derive a root-finding problem describing Haliey's task.

From the given, $q = 17.03$, $\mu_{\mathsf{Man}} = 0.0017$,

$$m_{\mathsf{channel}} = \frac{174.07 - 7.1}{98040} \approxeq 0.00170308, \quad \text{and} \quad a = 4.75h,$$

where h is the desired water flow height. The wetted perimeter is $4.75 + 2h$, so that the hydraulic radius is

$$r_{\mathsf{hyd}} = \frac{a}{4.75 + 2h} = \frac{4.75h}{2h + 4.75}.$$

Therefore, from Equation (15),

$$\underbrace{\frac{\sqrt{m_{\mathsf{channel}}}}{\mu_{\mathsf{Man}}} \frac{4.75^{\frac{5}{3}} h^{\frac{5}{3}}}{(2h + 4.75)^{\frac{2}{3}}} - 17.03}_{f(h)} = 0, \tag{16}$$

which is the desired root-finding formulation of Haliey's task.

5 marks Assuming uniform flow conditions, use Equation (15) to express Haliey's task as a fixed-point problem.

A straightforward reformulation is to rearrange Equation (16) into:

$$h = \underbrace{\frac{(17.03\mu_{\mathsf{Man}})^{\frac{3}{5}}}{4.75 m_{\mathsf{channel}}^{\frac{3}{10}}} (2h + 4.75)^{\frac{2}{5}}}_{g(h)}. \tag{17}$$

Note that there are many other alternative equivalent expressions.

5 marks Derive a Newton-Raphson iteration scheme for numerically computing the height of water flow.

This is just

$$h_{k+1} \leftarrow h_k - \frac{f(h_k)}{f'(h_k)}, \tag{18}$$

where the derivative may be determined by a computer algebra system.

4 Never give up

7 marks Be like Rick.