CVE154 Exam 2

prepared by Christian Cahig for classes of A.Y. 2024-2025 S1

2024-11-04

This exam is primarily aimed to assess your ability to model an engineering problem as solving for N unknowns from M linear equations, *i.e.*,

$$\underbrace{\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & \cdots & A_{1,N} \\ A_{2,1} & A_{2,2} & A_{2,3} & \cdots & A_{2,N} \\ A_{3,1} & A_{3,2} & A_{3,3} & \cdots & A_{3,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{M,1} & A_{M,2} & A_{M,3} & \cdots & A_{M,N} \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_M \end{bmatrix}}_{\boldsymbol{b}},$$

where $\boldsymbol{x}$ are the unknowns. Intuitively, the *impulses* $\boldsymbol{b}$ applied to a physical system cause some *responses* $\boldsymbol{x}$ from the system, according to the *design parameters* $\boldsymbol{A}$ of the system. We typically want to solve for $\boldsymbol{x}$ given $\boldsymbol{b}$ and $\boldsymbol{A}$. It will suffice us for now to deal with real numbers, and to leave complex-valued representations to tomorrow's imaginations.

While the use of AI tools to answer this exam is not prohibited, it is nevertheless of ethical interest to disclose such use. This is in line with the MSU Policy on the Fair and Ethical Use of AI and Its Applications. As such, please include a brief statement indicating the AI tools you used and how you used them in answering this exam.

Write your answers in an A4-sized document saved as a PDF file with a filename following the pattern `EXM-02_<Class number>_<ID number>`. (For example, if your ID number is 2013-0024 and you are enrolled to the W45M456 class, then your answers should be in the file `EXM-02_W45M456_2013-0024.pdf`). To submit your answers, upload the PDF file to the Google Drive folder you created in Exam 1. Lastly, send a link to the uploaded PDF file via email with the subject `CVE154 Exam 2`.

The deadline for submitting your answers is 4 November 2024 at 1159 hours. You may, of course, submit at a later time, but your overall score will be computed as $0.80^t r$, where r is your raw score (*i.e.*, assuming you submitted no later than the deadline), and t is the continuous-valued number of hours elapsed since the deadline.

1 Truss issues

Consider the 1D truss system shown in Figure 1. The nominal dimensions of the members are summarized in Table 1.

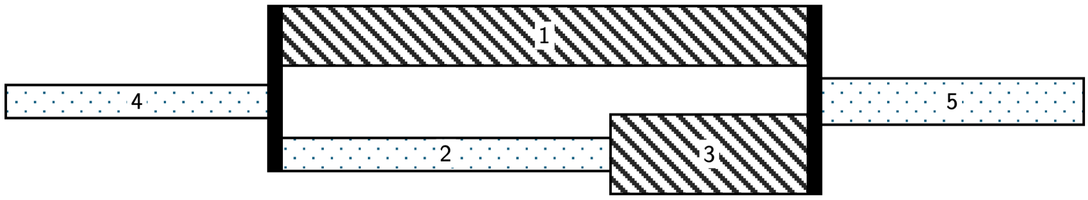

Figure 1 Member numbers are as shown. Dot-coloured members are made of a material having a 9-GPa modulus of elasticity, while the other members are made of a material with 900 MPa.

Table 1 Nominal dimensions for the members in the 1D truss system of Figure 1.

Member	Length (m)	Cross-sectional area (mm^2)
1	8.0	90.0
2	5.0	50.0
3	3.0	120.0
4	4.0	50.0
5	4.0	70.0

10 marks Assuming that the members are made of linearly elastic materials, and that their masses are negligible, show that the net external forces applied at the junctions (p_i's) are linearly related to the axial deformations of the junctions (δ_i's) as follows:

$$\underbrace{\begin{bmatrix} K_{1,1} & K_{1,2} & K_{1,3} & K_{1,4} & K_{1,5} \\ K_{2,1} & K_{2,2} & K_{2,3} & K_{2,4} & K_{2,5} \\ K_{3,1} & K_{3,2} & K_{3,3} & K_{3,4} & K_{3,5} \\ K_{4,1} & K_{4,2} & K_{4,3} & K_{4,4} & K_{4,5} \\ K_{5,1} & K_{5,2} & K_{5,3} & K_{5,4} & K_{5,5} \end{bmatrix}}_{\boldsymbol{K}} \underbrace{\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \end{bmatrix}}_{\boldsymbol{\delta}} = \underbrace{\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{bmatrix}}_{\boldsymbol{p}}, \tag{1}$$

where $\boldsymbol{K}$ is the *(global) stiffness matrix*. Indicate the junction numbering you used.

Recall from elementary mechanics that an axially loaded (prismatic) member made of a linearly elastic material behaves like a spring in that the axial loading (analogous to the spring force) is directly proportional to the axial deformation (analogous to the spring elongation), where the proportionality constant (analogous to the spring stiffness) is given by the modulus of elasticity E of the material and the design length ℓ and cross-sectional area a like so:

$$\text{axial load} = \frac{Ea}{\ell} \times \text{axial deformation}.$$

Here, the proportionality constant is referred to as the *stiffness* of the member. Therefore, in the present problem, it is straightforward to compute the stiffnesses of the members; these are summarized in Table 2. Moreover, the junction numbering is arbitrary, and so we shall use that which is shown in Figure 2.

One can eventually arrive at Equation (1) by first establishing how, for each member, the forces at the ends relate to the deformations at the ends, and then stitching these per-member relationships. As this answer key cannot possibly outweigh existing and battle-tested online

Table 2 Member stiffnesses for the 1D truss system in Figure 1.

Member	Stiffness (kN/m)
1	10.125
2	90.000
3	36.000
4	112.50
5	157.50

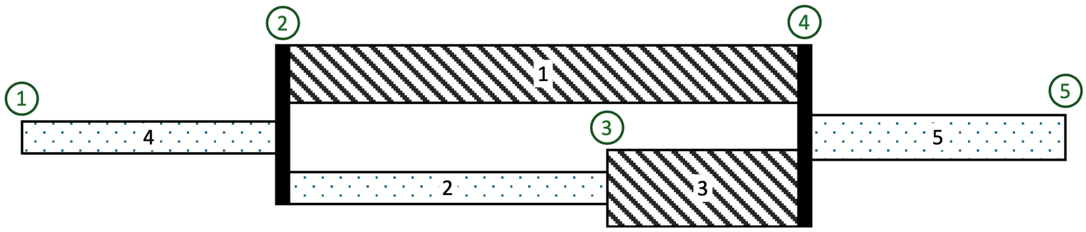

Figure 2 The encircled numbers represent the junction (or, node) indices for the 1D truss system in Figure 1.

pedagogical contents, the student is referred to StilHOT's CORNER video on stiffness of bars[1] and Seán Carroll's article, *Truss Analysis using the Direct Stiffness Method*[2] for (less improper) guides on how to derive Equation (1). One will inevitably arrive at the following rules for constructing $\boldsymbol{K}$. A diagonal element $K_{i,i}$ is the sum of the stiffnesses of all members connected to junction i. An off-diagonal element $K_{i,j}$ is negative of the stiffness of the member connecting junctions i and j, but is simply zero in the absence of such a member. The latter also implies that $\boldsymbol{K}$ is symmetric. Thus,

$$\underbrace{\begin{bmatrix} 112.5 & -112.5 & 0 & 0 & 0 \\ -112.5 & 212.625 & -90 & -10.125 & 0 \\ 0 & -90 & 126 & -36 & 0 \\ 0 & -10.125 & -36 & 203.625 & -157.5 \\ 0 & 0 & 0 & -157.5 & 157.5 \end{bmatrix}}_{\boldsymbol{K}} \underbrace{\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \end{bmatrix}}_{\boldsymbol{\delta}} = \underbrace{\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{bmatrix}}_{\boldsymbol{p}}. \tag{2}$$

Note that $\boldsymbol{p}$ contain quantities in kilonewtons and $\boldsymbol{\delta}$ contain quantities in meters. Looking at the pattern of which of its elements are zero, $\boldsymbol{K}$ is said to have a *banded* structure.

5 marks You are considering a case study wherein:

- the left end of Member 4 is fixed to a rigid support;
- a 350-N leftward force is applied to the junction connected to the right end of Member 1;
- a 100-N leftward force is applied to the junction connecting Members 2 and 3;
- a 400-N rightward force is applied to the junction connected to the left end of Member 5;
- a 700-N rightward force is applied to the right end of Member 5; and
- a 325-N rightward force acts on the junction connecting Members 2 and 3.

[1] https://youtu.be/UrZSoq7k4r0
[2] https://www.engineeringskills.com/posts/direct-stiffness-method

Show and reason how Equation (1) must be modified. Identify the unknown quantities and describe how to solve for these.

The case study conditions imply that $\delta_1 = 0$ m, $p_2 = 0$ kN, $p_3 = -0.1 + 0.325 = 0.225$ kN, $p_4 = -0.35 + 0.4 = 0.05$ kN, and $p_5 = 0.7$ kN. As such, Equation (2) becomes

$$\underbrace{\begin{bmatrix} 112.5 & -112.5 & 0 & 0 & 0 \\ -112.5 & 212.625 & -90 & -10.125 & 0 \\ 0 & -90 & 126 & -36 & 0 \\ 0 & -10.125 & -36 & 203.625 & -157.5 \\ 0 & 0 & 0 & -157.5 & 157.5 \end{bmatrix}}_{\boldsymbol{K}} \underbrace{\begin{bmatrix} 0 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \end{bmatrix}}_{\boldsymbol{\delta}} = \underbrace{\begin{bmatrix} p_1 \\ 0 \\ 0.225 \\ 0.05 \\ 0.7 \end{bmatrix}}_{\boldsymbol{p}}. \tag{3}$$

We are now to determine δ_2, δ_3, δ_4, δ_5, and p_1. (Understand that p_1 corresponds to a reaction force.) First, we solve for the remaining deformations from

$$\begin{bmatrix} 212.625 & -90 & -10.125 & 0 \\ -90 & 126 & -36 & 0 \\ -10.125 & -36 & 203.625 & -157.5 \\ 0 & 0 & -157.5 & 157.5 \end{bmatrix} \begin{bmatrix} \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.225 \\ 0.05 \\ 0.7 \end{bmatrix}, \tag{4}$$

which is a "subsystem" of Equation (3) formed by removing the first row and the first column of the augmented matrix. Then, p_1 is computed as:

$$p_1 = \begin{bmatrix} 112.5 & -112.5 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \end{bmatrix} = -112.5\delta_2. \tag{5}$$

5 marks You now modify the above case study by removing the force applied to the right end of Member 5, and fixing that end to a rigid support. Discuss what becomes of Equation (1), and how to solve for the resulting unknown quantities.

Now we have $\delta_1 = 0$ m, $\delta_5 = 0$ m, $p_2 = 0$ kN, $p_3 = 0.225$ kN, and $p_4 = 0.05$ kN, thus rendering Equation (2) as

$$\underbrace{\begin{bmatrix} 112.5 & -112.5 & 0 & 0 & 0 \\ -112.5 & 212.625 & -90 & -10.125 & 0 \\ 0 & -90 & 126 & -36 & 0 \\ 0 & -10.125 & -36 & 203.625 & -157.5 \\ 0 & 0 & 0 & -157.5 & 157.5 \end{bmatrix}}_{\boldsymbol{K}} \underbrace{\begin{bmatrix} 0 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ 0 \end{bmatrix}}_{\boldsymbol{\delta}} = \underbrace{\begin{bmatrix} p_1 \\ 0 \\ 0.225 \\ 0.05 \\ p_5 \end{bmatrix}}_{\boldsymbol{p}}. \tag{6}$$

The unknowns now are δ_2, δ_3, δ_4, p_1, and p_5, with the latter two corresponding to reaction forces of a statically indeterminate case. First, we solve for the remaining deformations from

$$\begin{bmatrix} 212.625 & -90 & -10.125 \\ -90 & 126 & -36 \\ -10.125 & -36 & 203.625 \end{bmatrix} \begin{bmatrix} \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.225 \\ 0.05 \end{bmatrix}, \tag{7}$$

which is a "subsystem" of Equation (6) formed by removing the first row, the fifth row, the first column, and the fifth column of the augmented matrix. Then, p_1 and p_5 are computed as:

$$\begin{bmatrix} p_1 \\ p_5 \end{bmatrix} = \begin{bmatrix} 112.5 & -112.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -157.5 & 157.5 \end{bmatrix} \begin{bmatrix} 0 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ 0 \end{bmatrix} = \begin{bmatrix} -112.5\delta_2 \\ -157.5\delta_4 \end{bmatrix}. \tag{8}$$

2 Mix and match

A production facility owned by White Party Industries has a network of five mixing vessels and connecting pipes. Each vessel is equipped with at least one inlet, at least one outlet, and agitating means, all configured to maintain a steady concentration of baby oil. Table 3 summarizes the designed flow rates to and fro the vessels.

Table 3 Nominal flow rates to and fro the mixing vessels of Section 2.

From	To	Flow rate (m^3/s)
Vessel 1	Vessel 2	7.00
Vessel 2	Vessel 3	6.00
Vessel 3	Vessel 4	3.00
Vessel 4	Vessel 5	2.00
Vessel 5	Vessel 4	4.00
Vessel 4	Vessel 3	5.00
Vessel 3	Vessel 2	2.00
Vessel 2	Vessel 1	4.00
External 20-mg/m^3 source	Vessel 1	4.00
External 15-mg/m^3 source	Vessel 5	2.00
Vessel 1	Vessel 3	1.00
Vessel 3	External	6.00

10 marks Assuming that steady-state mass balance holds, express the task of determining the vessel concentrations from the nominal flow rates as solving for $\boldsymbol{\kappa}$ in the linear system

$$\boldsymbol{M}\boldsymbol{\kappa} = \boldsymbol{i}. \tag{9}$$

The system is schematically schown in Figure 3.

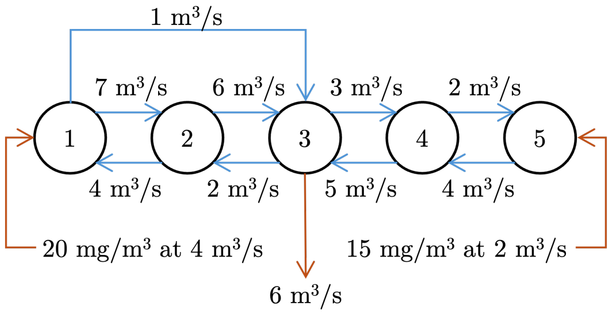

Figure 3 Vessels are represented by circles. Nominal flow rates of Table 3 are indicated by numbers besides arrows. Denote by κ_i the concentration in vessel i.

From the mass balance assumption,

$$-8\kappa_1 + 4\kappa_2 = 80, \tag{10a}$$
$$7\kappa_1 - 10\kappa_2 + 2\kappa_3 = 0, \tag{10b}$$
$$\kappa_1 + 6\kappa_2 - 11\kappa_3 + 5\kappa_4 = 0, \tag{10c}$$
$$3\kappa_3 - 7\kappa_4 + 4\kappa_5 = 0, \tag{10d}$$
$$2\kappa_4 - 4\kappa_5 = 30, \tag{10e}$$

or, equivalently,

$$\underbrace{\begin{bmatrix} -8 & 4 & 0 & 0 & 0 \\ 7 & -10 & 2 & 0 & 0 \\ 1 & 6 & -11 & 5 & 0 \\ 0 & 0 & 3 & -7 & 4 \\ 0 & 0 & 0 & 2 & -4 \end{bmatrix}}_{\boldsymbol{M}} \underbrace{\begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \\ \kappa_4 \\ \kappa_5 \end{bmatrix}}_{\boldsymbol{\kappa}} = \underbrace{\begin{bmatrix} 80 \\ 0 \\ 0 \\ 0 \\ 30 \end{bmatrix}}_{\boldsymbol{i}}. \tag{11}$$

5 marks Industrial control systems typically allow monitored quantities to deviate within acceptable ranges from their nominal values. Suppose that the flow rates have tolerances of 3 % from their nominal values. Set up $\boldsymbol{M}$ and $\boldsymbol{i}$ for when it is desired to determine the vessel concentrations corresponding to the minimum allowable flow rates.

The minmum allowable flow rates would be 97 % of the values in Table 3. Upon invoking mass balance, one will realize that the equations to be solved are just Equation (10) with both sides multiplied by 0.97. Therefore, $\boldsymbol{M}$ and $\boldsymbol{i}$ are as in Equation (11).

10 marks During an emergency where the flow rates peaked their allowable values, D. Dee made a judgment call to close the valves at both ends of the pipe connecting Vessels 1 and 3. Set up the linear system that must be solved to estimate the vessel concentrations, and comment on the quality of the solution based on the condition number of the coefficient matrix.

Here, there is no direct fluid communication between Vessels 1 and 3, and the remaining flow rates are at 103 % of their nominal values as reported in Table 3. The system is described shown in Figure 4.

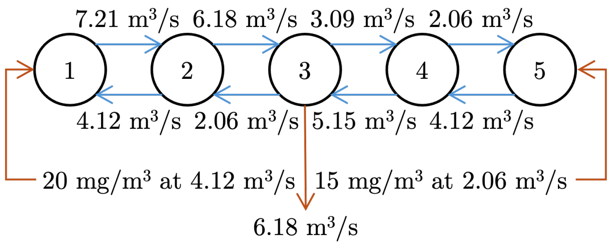

Figure 4 The system of Figure 3 but without the pipe connecting Vessels 1 and 3, and the remaining flow rates at 103 % of the nominal values in Table 3.

Invoking mass balance,

$$1.03\,(-7\kappa_1 + 4\kappa_2) = 1.03\,(80)\,, \tag{12a}$$
$$1.03\,(7\kappa_1 - 10\kappa_2 + 2\kappa_3) = 0, \tag{12b}$$
$$1.03\,(6\kappa_2 - 11\kappa_3 + 5\kappa_4) = 0, \tag{12c}$$
$$1.03\,(3\kappa_3 - 7\kappa_4 + 4\kappa_5) = 0, \tag{12d}$$
$$1.03\,(2\kappa_4 - 4\kappa_5) = 1.03\,(30)\,, \tag{12e}$$

or, equivalently,

$$\underbrace{\begin{bmatrix} -7 & 4 & 0 & 0 & 0 \\ 7 & -10 & 2 & 0 & 0 \\ 0 & 6 & -11 & 5 & 0 \\ 0 & 0 & 3 & -7 & 4 \\ 0 & 0 & 0 & 2 & -4 \end{bmatrix}}_{\boldsymbol{M}} \underbrace{\begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \\ \kappa_4 \\ \kappa_5 \end{bmatrix}}_{\boldsymbol{\kappa}} = \underbrace{\begin{bmatrix} 80 \\ 0 \\ 0 \\ 0 \\ 30 \end{bmatrix}}_{\boldsymbol{i}}. \tag{13}$$

It is evident from Equation (12) that $\boldsymbol{M}$ has a banded structure. Lastly, it is straightforward to verify using linear algebra tools that $\boldsymbol{M}$ has a small condition number (*e.g.*, as low as 0.078 and as high as about 19.8, according to the supported modes in NumPy's `numpy.linalg.cond`). In other words, we can expect that the concentrations determined by solving Equation (13) are fairly reliable in that $\boldsymbol{\kappa}$ will not dramatically change with slight deviations in the flow rates.

3 Bonus

3 marks In your stay in the university, what do you consider to be the most profound thing you've heard or learned (so far) from a teacher?

In an otherwise uneventful afternoon in 2023, I found myself in a conversation with two teachers: one I consider my Big Brother, the other we call Dad. At some point Big Brother asked Dad, "What keeps you going?", making reference to the latter's abundance of experience. Dad paused, as if searching for a response befitting his notoriety as a wordsmith. And befitting indeed it was: "The opportunity to help, and the opportunity to learn."