Abstract Algebra 2024–I
Homework 2

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1. Determine which of the following binary operations are associative.

(a) the operation $\star$ on $\mathbb{Z}$ defined by $a \star b = a - b$

(b) the operation $\star$ on $\mathbb{R}$ defined by $a \star b = a + b + ab$

(c) the operation $\star$ on $\mathbb{Q}$ defined by $a \star b = \dfrac{a+b}{5}$

(d) the operation $\star$ on $\mathbb{Z} \times \mathbb{Z}$ defined by $(a, b) \star (c, d) = (ad + bc, bd)$

(e) the operation $\star$ on $\mathbb{Q} \setminus \{0\}$ defined by $a \star b = \dfrac{a}{b}$

2. Prove that addition of residue classes in $\mathbb{Z}/n\mathbb{Z}$ is associative. (Assume it is well defined.)

3. Determine which of the following sets are groups under addition:

(a) the set of rational numbers (including $0 = 0/1$) in lowest terms whose denominators are odd

(b) the set of rational numbers (including $0 = 0/1$) in lowest terms whose denominators are even

(c) the set of rational numbers of absolute value < 1

(d) the set of rational numbers of absolute value ≥ 1 together with 0

(e) the set of rational numbers with denominators equal to 1 or 2

(f) the set of rational numbers with denominators equal to 1, 2 or 3

4. Let $G = \{z \in \mathbb{C} \mid z^n = 1 \text{ for some } n \in \mathbb{Z}^+\}$.

(a) Prove that G is a group under multiplication (called the group of *roots of unity* in $\mathbb{C}$).

(b) Prove that G is not a group under addition.

5. Let $G = \{a + b\sqrt{2} \in \mathbb{R} \mid a, b \in \mathbb{Q}\}$.

(a) Prove that G is a group under addition.

(b) Prove that the nonzero elements of G are a group under multiplication. ("Rationalize the denominators" to find multiplicative inverses.)

6. Find the orders of each element of the additive group $\mathbb{Z}/12\mathbb{Z}$.

7. Find the orders of the following elements of the multiplicative group $(\mathbb{Z}/12\mathbb{Z})^{\times}$:

$$\overline{1}, \overline{-1}, \overline{5}, \overline{7}, \overline{-7}, \overline{13}.$$

8. Find the orders of the following elements of the additive group $\mathbb{Z}/36\mathbb{Z}$:

$$\overline{1}, \overline{2}, \overline{6}, \overline{9}, \overline{10}, \overline{12}, \overline{-1}, \overline{-10}, \overline{-18}.$$

9. Let x be an element of G. Prove that $x^2 = 1$ if and only if $|x|$ is either 1 or 2.

10. Let x be an element of G. Prove that if $|x| = n$ for some positive integer n then $x^{-1} = x^{n-1}$.

11. Let x and y be elements of G. Prove that $xy = yx$ if and only if $y^{-1}xy = x$ if and only if $x^{-1}y^{-1}xy = 1$.

12. Let $x \in G$ and let $a, b \in \mathbb{Z}^+$.

(a) Prove that $x^{a+b} = x^a x^b$ and $(x^a)^b = x^{ab}$.

(b) Prove that $(x^a)^{-1} = x^{-a}$.

(c) Establish part (a) for arbitrary integers a and b (positive, negative or zero).

13. For x an element in G show that x and x^{-1} have the same order.

14. If x and g are elements of the group G, prove that $|x| = |g^{-1}xg|$. Deduce that $|ab| = |ba|$ for all $a, b \in G$.

15. Prove that if $x^2 = 1$ for all $x \in G$, then G is abelian.

16. Assume H is a nonempty subset of $(G, \star)$ which is closed under the binary operation on G and is closed under inverses, i.e., for all h and k elements of H it holds $hk, h^{-1} \in H$. Prove that H is a group under the operation $\star$ restricted to H (such a subset H is called a subgroup of G).

17. Prove that if x is an element of the group G then $\{x^n \mid n \in \mathbb{Z}\}$ is a subgroup (cf. the preceding exercise) of G (called the cyclic subgroup of G generated by x).

18. Compute the order of each of the elements in (a) D_6, (b) D_8, and (c) D_{10}.

19. Let σ be the permutation

$$1 \mapsto 3 \quad 2 \mapsto 4 \quad 3 \mapsto 5 \quad 4 \mapsto 2 \quad 5 \mapsto 1$$

and let τ be the permutation

$$1 \mapsto 5 \quad 2 \mapsto 3 \quad 3 \mapsto 2 \quad 4 \mapsto 4 \quad 5 \mapsto 1.$$

Find the cycle decompositions of each of the following permutations: $\sigma, \tau, \sigma^2, \sigma\tau, \tau\sigma$, and $\tau^2\sigma$.

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