Rings and their Properties

(Lessons 10 and 11)

1. Let A be a ring. The center of A is defined as

$$C(A) = \{a \in A \mid \forall x \in A : ax = xa\}.$$

Prove that $C(A)$ is a subring of A.

2. We say that a proper ideal I of R is maximal if there exists no other proper ideal J of R properly containing I. Let $I = \langle n \rangle$ be a principal ideal in $\mathbb{Z}$. Show that I is maximal if and only if $n = p$ with p a prime.

3. Let I and J be ideals of a ring A. Prove that $I + J$ is the smallest ideal of A containing both I and J.

4. Let I and J be ideals of a ring A.

(i) Prove that IJ is an ideal contained in $I \cap J$.

(ii) Prove that if A is abelian and $I + J = A$, then $IJ = I \cap J$.

5. List all the homomorphisms from $\mathbb{Z}_2$ to $\mathbb{Z}_4$.

6. Let $R = M_2(\mathbb{Z})$. Prove that the elements

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

are nilpotents of R whose sum is not nilpotent. Conclude then that the set of nilpotents in the noncommutative ring $M_2(\mathbb{Z})$ is not an ideal.

7. Prove that any homomorphic image of a commutative ring is a commutative ring.

8. The radical of J is the set

$$\text{rad}(J) = \{a \in A \mid \exists n \in \mathbb{Z} : a^n \in J\}.$$

Prove that $\text{rad}(J)$ is an ideal of A.

9. Let A and B be rings and $\eta : A \to B$ a surjective homomorphism.

(i) Assume that J is an ideal of A such that $J \supseteq \text{Ker}(\eta)$. Prove that $\eta(J)$ is an ideal of B.

(ii) Assume that B is a field. Prove that $\text{Ker}(\eta)$ is a maximal ideal, i.e., it is a proper ideal which is not strictly contained in any strictly larger proper ideal.

10. Let $m, n \in \mathbb{Z}$. Prove that
$$m\mathbb{Z} + n\mathbb{Z} = \gcd(m, n)\mathbb{Z}.$$

11. Establish each of the following isomorphisms:

(i) $\mathbb{Z}_{20}/4\mathbb{Z}_{20} \cong \mathbb{Z}_4$.

(ii) $\mathbb{Z}_{12}/6\mathbb{Z}_{12} \cong \mathbb{Z}_6$.

(iii) $\mathbb{Z}_{72}/12\mathbb{Z}_{72} \cong \mathbb{Z}_{12}$.

(iv) $\mathbb{R}[x]/\langle x^2 + 1 \rangle \cong \mathbb{C}$.

12. Let $f : A \to B$ be a surjective homomorphism of rings, where B is a commutative ring. Let I be the ideal generated by elements of the form $aa' - a'a$, with $a, a' \in A$. Show that:

(i) $I \subseteq \operatorname{Ker} f$.

(ii) $I = \operatorname{Ker} f$ if and only if the canonical surjection:
$$A/I \to A/\operatorname{Ker} f \cong B$$
is an isomorphism.

13. Show that the ideal $\langle x^2 - 2 \rangle$ of $\mathbb{Z}[x]$ is prime but not maximal by proving that
$$\mathbb{Z}[x]/\langle x^2 - 2 \rangle \cong \mathbb{Z}[\sqrt{2}].$$

14. Let $C[0, 1]$ be the ring of continuous functions from $[0, 1]$ to $\mathbb{R}$ with the structure described in Exercise 10.3.3. Show that
$$I = \left\{ f \in C[0, 1] \;\middle|\; f\left(\frac{1}{2}\right) = 0 \right\}$$
is a maximal ideal.

15. Let A and B be rings. Show that an ideal of $A \times B$ can always be written in the form $I \times J$, with $I \trianglelefteq A$ and $J \trianglelefteq B$. Deduce that an ideal of $A \times B$ is maximal if and only if it can be written as one of the forms $I \times B$, with I maximal in A, or $A \times J$, with J maximal in B.

16. Let A be a commutative ring and I a prime ideal of A such that A/I is finite. Show that I is maximal.

17. Let A be a commutative ring and I a proper ideal of A. Show that I is prime if and only if $A \setminus I$ is closed under multiplication.