



## Cyclic Groups, Normality, Quotients and the Isomorphism Theorems

(Lessons 6, 7, and 8)

1. Prove that if  $A$  is a subset of  $B$  then  $\langle A \rangle \leq \langle B \rangle$ . Give an example where  $A \subseteq B$  with  $A \neq B$  but  $\langle A \rangle = \langle B \rangle$ .
2. Prove that the subgroup generated by any two distinct elements of order 2 in  $S_3$  is all of  $S_3$ .
3. Prove that the subgroup of  $S_4$  generated by  $(12)$  and  $(12)(3\ 4)$  is a noncyclic group of order 4.
4. A group  $H$  is called finitely generated if there is a finite set  $A$  such that  $H = \langle A \rangle$ .
  - (i) Prove that every finite group is finitely generated.
  - (ii) Prove that  $\mathbb{Z}$  is finitely generated.
  - (iii) Prove that every finitely generated subgroup of the additive group  $\mathbb{Q}$  is cyclic. (If  $H$  is a finitely generated subgroup of  $\mathbb{Q}$ , show that  $H \leq \langle \frac{1}{k} \rangle$ , where  $k$  is the product of all the denominators which appear in a set of generators for  $H$ .)
  - (iv) Prove that  $\mathbb{Q}$  is not finitely generated.
5. Let  $\varphi : G \rightarrow H$  be a homomorphism and let  $E$  be a subgroup of  $H$ . Prove that  $\varphi^{-1}(E) \leq G$  (i.e., the pullback of a subgroup under a homomorphism is a subgroup). If  $E \trianglelefteq H$  prove that  $\varphi^{-1}(E) \trianglelefteq G$ . Deduce that  $\ker \varphi \trianglelefteq G$ .
6. Prove that if  $N \trianglelefteq G$  and  $H$  is any subgroup of  $G$  then  $N \cap H \trianglelefteq H$ .
7. Let  $N$  be a *finite* subgroup of a group  $G$  and assume  $N = \langle S \rangle$  for some subset  $S$  of  $G$ . Prove that an element  $g \in G$  normalizes  $N$  if and only if  $gSg^{-1} \subseteq N$ .
8. Prove that if  $G/Z(G)$  is cyclic then  $G$  is abelian. (Hint: If  $G/Z(G)$  is cyclic with generator  $xZ(G)$ , show that every element of  $G$  can be written in the form  $x^a z$  for some integer  $a \in \mathbb{Z}$  and some element  $z \in Z(G)$ .)
9. Let  $A$  and  $B$  be groups. Show that  $\{(a, 1) \mid a \in A\}$  is a normal subgroup of  $A \times B$  and the quotient of  $A \times B$  by this subgroup is isomorphic to  $B$ .
10. Let  $A$  be an abelian group and let  $D$  be the (diagonal) subgroup  $\{(a, a) \mid a \in A\}$  of  $A \times A$ . Prove that  $D$  is a normal subgroup of  $A \times A$  and  $(A \times A)/D \cong A$ .
11. Suppose  $A$  is the non-abelian group  $S_3$  and  $D$  is the diagonal subgroup  $\{(a, a) \mid a \in A\}$  of  $A \times A$ . Prove that  $D$  is not normal in  $A \times A$ .

12. Let  $G$  be a group, let  $N$  be a normal subgroup of  $G$  and let  $\overline{G} = G/N$ . Prove that  $\bar{x}$  and  $\bar{y}$  commute in  $\overline{G}$  if and only if  $x^{-1}y^{-1}xy \in N$ . (The element  $x^{-1}y^{-1}xy$  is called the *commutator* of  $x$  and  $y$  and is denoted by  $[x, y]$ .)
13. Let  $G$  be a group. Prove that  $N = \langle x^{-1}y^{-1}xy \mid x, y \in G \rangle$  is a normal subgroup of  $G$  and  $G/N$  is abelian ( $N$  is called the *commutator subgroup* of  $G$ ).
14. Show that if  $|G| = pq$  for some primes  $p$  and  $q$  (not necessarily distinct) then either  $G$  is abelian or  $Z(G) = 1$ .
15. Prove that if  $H$  and  $K$  are finite subgroups of  $G$  whose orders are relatively prime then  $H \cap K = 1$ .
16. Suppose  $H$  and  $K$  are subgroups of finite index in the (possibly infinite) group  $G$  with  $|G : H| = m$  and  $|G : K| = n$ . Prove that  $\text{l.c.m.}(m, n) \leq |G : H \cap K| \leq mn$ . Deduce that if  $m$  and  $n$  are relatively prime then  $|G : H \cap K| = |G : H| \cdot |G : K|$ .
17. Use Lagrange's Theorem in the multiplicative group  $(\mathbb{Z}/p\mathbb{Z})^\times$  to prove *Fermat's Little Theorem*: if  $p$  is a prime then  $a^p \equiv a \pmod{p}$  for all  $a \in \mathbb{Z}$ .
18. Let  $p$  be a prime and let  $n$  be a positive integer. Find the order of  $\bar{p}$  in  $(\mathbb{Z}/(p^n - 1)\mathbb{Z})^\times$  and deduce that  $n \mid \varphi(p^n - 1)$  (here  $\varphi$  is Euler's function).
19. Let  $G$  be a finite group, let  $H$  be a subgroup of  $G$  and let  $N \trianglelefteq G$ . Prove that if  $|H|$  and  $|G : N|$  are relatively prime then  $H \leq N$ .
20. Prove that if  $N$  is a normal subgroup of the finite group  $G$  and  $(|N|, |G : N|) = 1$  then  $N$  is the unique subgroup of  $G$  of order  $|N|$ .
21. If  $A$  is an abelian group with  $A \trianglelefteq G$  and  $B$  is any subgroup of  $G$  prove that  $A \cap B \trianglelefteq AB$ .
22. Use Lagrange's Theorem in the multiplicative group  $(\mathbb{Z}/n\mathbb{Z})^\times$  to prove *Euler's Theorem*:  $a^{\varphi(n)} \equiv 1 \pmod{n}$  for every integer  $a$  relatively prime to  $n$ , where  $\varphi$  denotes Euler's  $\varphi$ -function.
23. Prove that if  $H$  is a normal subgroup of  $G$  of prime index  $p$  then for all  $K \leq G$  either
  - (i)  $K \leq H$  or
  - (ii)  $G = HK$  and  $|K : K \cap H| = p$ .
24. Let  $C$  be a normal subgroup of the group  $A$  and let  $D$  be a normal subgroup of the group  $B$ . Prove that  $(C \times D) \trianglelefteq (A \times B)$  and  $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$ .
25. Let  $M$  and  $N$  be normal subgroups of  $G$  such that  $G = MN$ . Prove that  $G/(M \cap N) \cong (G/M) \times (G/N)$ .
26. Let  $p$  be a prime and let  $G$  be a group of order  $p^a m$ , where  $p$  does not divide  $m$ . Assume  $P$  is a subgroup of  $G$  of order  $p^a$  and  $N$  is a normal subgroup of  $G$  of order  $p^b n$ , where  $p$  does not divide  $n$ . Prove that  $|P \cap N| = p^b$  and  $|PN/N| = p^{a-b}$ . (The subgroup  $P$  of  $G$  is called a Sylow  $p$ -subgroup of  $G$ . This exercise shows that the intersection of any Sylow  $p$ -subgroup of  $G$  with a normal subgroup  $N$  is a Sylow  $p$ -subgroup of  $N$ .)

27. Prove that there are only two distinct groups of order 4 (up to isomorphism), namely  $\mathbb{Z}_4$  and  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ . (*Hint:* By Lagrange's Theorem, a group of order 4 that is not cyclic must consist of an identity and three elements of order 2.)
28. Let  $H, K$  be subgroups of a group  $G$ . Then  $HK$  is a subgroup of  $G$  if and only if  $HK = KH$ .
29. Let  $H \leq G$ . Prove  $aHa^{-1}$  is a subgroup for each  $a \in G$ , and  $H \cong aHa^{-1}$ .
30. Let  $G$  be a finite group and  $H$  a subgroup of  $G$  of order  $n$ . If  $H$  is the only subgroup of  $G$  of order  $n$ , then  $H$  is normal in  $G$ .
31. If  $H$  is a cyclic subgroup of a group  $G$  and  $H$  is normal in  $G$ , then every subgroup of  $H$  is normal in  $G$ .
32. If  $H$  is a normal subgroup of a group  $G$  such that  $H$  and  $G/H$  are finitely generated, then so is  $G$ .
33. Let  $N \trianglelefteq G$  and  $K \trianglelefteq G$ . If  $N \cap K = \langle e \rangle$  and  $N \vee K = G$ , then  $G/N \cong K$ .
34. If  $f : G \rightarrow H$  is a homomorphism,  $H$  is abelian and  $N$  is a subgroup of  $G$  containing  $\text{Ker } f$ , then  $N$  is normal in  $G$ .
35. If  $N \trianglelefteq G$ ,  $[G : N]$  finite,  $H \leq G$ ,  $|H|$  finite, and  $[G : N]$  and  $|H|$  are relatively prime, then  $H \leq N$ .