

1. Homomorphisms and isomorphisms

When studying an algebraic structure, we are not only interested in the objects that possess this structure, but also in the morphisms between objects of the same type that preserve such structure. This is true through out all branches of mathematics.

In the case of group theory, the objects are groups and the structure preserving maps are called group-homomorphisms.

Let (G, \cdot) and $(H, *)$ be groups.

Definition 1.1. A group-homomorphism from G to H is a map $\varphi: G \rightarrow H$ such that

$$\varphi(a \cdot b) = \varphi(a) * \varphi(b)$$

for all $a, b \in G$.

Remark 1.1.1. From now and on, we omit the adjective “group” in “group-homomorphism”, since we will be working only on the realm of group theory (**Grp**).

Note that φ transforms a product $a \cdot b$ (using the operation of G) into the product $\varphi(a) * \varphi(b)$ (using the operation of H). This is why we say φ preserves the structure: it takes a product in G and maps it to a product in H . In this case (of groups) there are no more structure to be preserved.

However, the definition above just guarantees that the structure is preserved in one direction only: from G to H . If, in addition, it is possible to preserve the structure the other way around, we obtain a more interesting type of map: an *isomorphism*.

Definition 1.2. An **isomorphism** is a bijective homomorphism.

Thus, an isomorphism $\varphi: G \rightarrow H$ is just a homomorphism that has inverse $\varphi^{-1}: H \rightarrow G$. In this case, G and H are said to be of the same isomorphism type, or *isomorphic* for short. We write $G \cong H$. You may ask *isn't it necessary that the inverse be a homomorphism also (so that the structure is preserved the other way around)?* And it turns out that *no*, because it follows straight from the definition. (Why?)

Exercise 1. Prove φ^{-1} is a homomorphism if φ is an isomorphism.

Exercise 2. Prove \cong is an equivalence relation (over which set?)

Lemma 1.3. If $\varphi: G \rightarrow H$ is an isomorphism, then

$$(i) \quad |G| = |H|$$

- (ii) G is Abelian if and only if H is Abelian
- (iii) φ preserves the order of elements, that is, $|x| = |\varphi(x)|$.

Example 1. (i) Let us prove $(\mathbb{R}, +) \cong (\mathbb{R}^+, \cdot)$. Define $\psi: \mathbb{R} \rightarrow \mathbb{R}^+ : x \mapsto \exp(x)$. Notice

$$\psi(x + y) = \exp(x + y) = \exp(x) \cdot \exp(y) = \psi(x) \cdot \psi(y).$$

Since ψ is injective and surjective (facts known from elementary calculus), ψ is an isomorphism. (Are there any other maps that show $(\mathbb{R}, +) \cong (\mathbb{R}^+, \cdot)$?)

- (ii) If X and Y have the same cardinality, then $S_X \cong S_Y$. Indeed, if $\omega: X \rightarrow Y$ is a bijection (recall the definition of cardinality), and given any $\alpha \in S_X$, the map $\omega \circ \alpha \circ \omega^{-1}: Y \rightarrow Y$ is also a bijection. Thus, define $\varphi: S_X \rightarrow S_Y$ by $\varphi(\alpha) = \omega \circ \alpha \circ \omega^{-1}$ for every $\alpha \in S_X$. Diagrammatically, we have

$$X \xrightarrow{\alpha} X \xrightarrow{\omega} Y \xrightarrow{\omega \circ \alpha \circ \omega^{-1}} Y.$$

Since

$$\varphi(\alpha \circ \beta) = \omega \circ (\alpha \circ \beta) \circ \omega^{-1} = (\omega \circ \alpha \circ \omega^{-1}) \circ (\omega \circ \beta \circ \omega^{-1}) = \varphi(\alpha) \circ \varphi(\beta),$$

φ is a homomorphism. The reader must easily verify that φ is an isomorphism.

- (iii) The groups S_3 and $\mathbb{Z}/6\mathbb{Z}$ are not isomorphic as one is Abelian and the other is not. This follows from Theorem 1.3.
- (iv) $(\mathbb{R} \setminus \{0\}, \cdot)$ and $(\mathbb{R}, +)$ are not isomorphic. The former has an element of order 2 and the latter does not have any element of order 2.

Exercise 3. (i) Provide an example of a group with only one element.

- (ii) Prove S_3 is the unique nonabelian group of order 6 up to isomorphism.
- (iii) Prove $A \times B \cong B \times A$ if A and B are any groups.

2. Subgroups

The objective is not to learn this concepts only, but mainly to learn to use them to study the principal objects of group theory.