UNIVERSIDAD YACHAY TECH

School of Mathematical and Computational Sciences
Abstract Algebra

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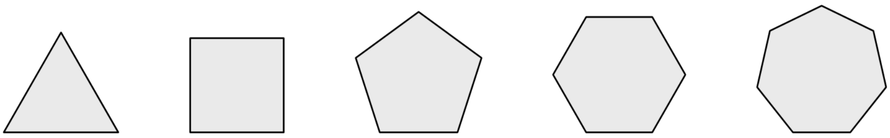
Lesson 4

1. The dihedral group

Objects have bilateral symmetry if they look the same when flipped over (usually in a specific direction, say vertically). The easiest geometric examples of objects with both rotational and bilateral symmetry are regular polygons.

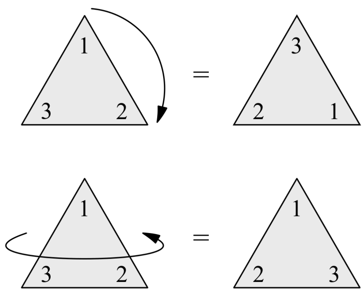


Observe that is not necessary to track the movement of all the points that make up a polygon in order to describe a rigid motion. Instead, we can simply keep track of their vertices. This motivates an important example of groups.

Some definitions first. A permutation of a set X is a bijection from X onto X. A regular polygon with n sides is called an n-gon. A symmetry of the n-gon is any rigid motion of of the n-gon. To be precise, a symmetry of the n-gon is just a permutation of $\{1, \dots, n\}$. From now and on, we will denote $[n] = \{1, \dots, n\}$.

It turns out, we can associate a group to the n-gon to study its rigid motions, that is, its symmetries. It is called the dihedral group of order $2n$ and denoted D_{2n}. Dihedral groups describe objects that have both rotational and bilateral symmetry.

Example 1. Both rotations and horizontal flips respect the shape of an equilateral triangle.



The first symmetry is decribed by the permutation $\sigma \colon [3] \to [3]$ defined by

$$\sigma(1) = 2, \quad \sigma(2) = 3, \quad \sigma(3) = 1.$$

The second one is described by $\tau \colon [3] \to [3]$ where

$$\tau(1) = 1, \quad \tau(2) = 3, \quad \tau(3) = 2.$$

Remark 1.0.1. We choose the notation D_{2n} because there are $2n$ symmetries associated to a n-gon. Namely, n rotations and n reflections.

Theorem 1.1. *(i)* $\{1, r, r^2, \ldots, r^{n-1}\}$ *are all distinct. Also* $r^n = 1$ *and* $|r| = n$.

(ii) $|s| = 2$

(iii) $s \neq r^i$ *for all* $i \in [n]_0$.

(iv) $sr^i = sr^j$ *for all* $i, j \in [n-1]$ *with* $i \neq j$.

(v) $rs = sr^{-1}$

(vi) $r^i s = sr^{-i}$ *for all* $0 \leq i \leq n$

Exercise 1. Prove or disprove $(sr^9)(sr^6) = r^9$.

2. Permutations and the symmetric group

Remark 2.0.1. (i) Not all permutations are cycles.

(ii)

(iii) Algorithm to factor a permutation into the product of cycles.

(iv) Every permutation has a cycle decomposition.

Definition 2.1.

Lemma 2.2. *Disjoint permutations commute.*

Lemma 2.3. *Every permutation of* $[n]$ *is either a cycle or a product of disjoint cycles.*