Fundamental Concepts of Ring Theory

(Lesson 9)

1. Let A be a ring with unity. Show that if u is invertible in A then so is $-u$.

2. Prove that $\mathbb{Z}_p$ is a integral domian iff p is prime.

3. The *direct product* of the rings A and B is the cartesian product $A \times B$ endowed with the operations defined by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),$$
$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2, y_1 y_2).$$

(i) Prove that $A \times B$ is a ring with these operations.

(ii) Prove that if A and B are Abelian, then so is its direct product.

(iii) Describe the divisors of zero of $A \times B$.

(iv) Describe the invertible elements of $A \times B$.

(v) Assume that A and B are non trivial rings. Explain why $A \times B$ can not be an integral domain by providing a counterexample.

4. Let $(A, +, \cdot)$ be a ring.

(i) Prove that for any $a, b, c \in A$,

$$a(b - c) = ab - ac, \quad (b - c)a = ba - ca.$$

(ii) Assume that $a, b \in A$ are such that $ab = -ba$. Prove that

$$(a + b)^2 = (a - b)^2 = a^2 + b^2.$$

(iii) Assume that A is an integral domain. Prove that for all $a, b \in A$, if $a^2 = b^2$ then $a = b$ or $a = -b$.

(iv) If A is an integral domain, for any $a \in A$, if $a = a^{-1}$, then $x \in \{-1, 1\}$.

(v) Prove that if $(A, +)$ is a cyclic group, then A is a commutative ring.

5. Let $(A, +, \cdot)$ be a nontrivial ring with unity and $a, b, c \in A$.

(i) Prove that if a is invertible, then

$$ab = ac \;\Rightarrow\; b = c,$$

and that a has only one multiplicative inverse.

(ii) Prove that if $a^2 = 0$, then $a+1$ and $a-1$ are invertible.

(iii) Prove that if a and b are invertible, then ab is invertible.

(iv) Prove that $(A^{\times}, \cdot)$ is a group.

6. Let $(F, +, \cdot)$ be a field with $|F| = m \in \mathbb{N}$. Prove that

$$\forall x \in F \setminus \{0\} : \quad x^{m-1} = 1. \tag{1}$$

7. Let A be a commutative ring and $a, b \in A$. Prove that if ab is invertible, then a and b are both invertible.

8. Let $(A, +, \cdot)$ be a nontrivial ring and $a, b, c \in A$.

1. Prove that if $a \notin \{-1, 1\}$ and $a^2 = 1$, then $a+1$ and $a-1$ are zero divisors
2. Prove that if ab is a divisor of zero, then either a or b is a zero divisor.

9. Prove that in a nontrivial commutative ring with unity, a zero divisor cannot be invertible.

10. Consider $A = (\mathbb{Z}, \oplus, \odot)$ where

$$a \oplus b = a + b - 1, \quad a \odot b = ab - (a+b) + 2.$$

1. Prove that A is a commutative ring with unity. Indicate the zero element, the unit, and the negative of an arbitrary a.
2. Is A an integral domain?

11. Consider $A = (\mathbb{Q} \times \mathbb{Q}, \oplus, \odot)$ where

$$(a,b) \oplus (c,d) = (a+c, b+d), \quad (a,b) \odot (c,d) = (ac - bd, ad + bc).$$

(i) Prove that A is a commutative ring with unity. Indicate the zero element, the unit element, and the negative of an arbitrary a.

(ii) Prove that A is a field and indicate the multiplicative inverse of an arbitrary nonzero element.

12. Explain why $\mathbb{R}^X$ is neither a field nor an integral domain. Describe its zero divisors. Here X denotes an arbitary set.

13. Let $\Omega \neq \varnothing$ be a set and consider $A = (\mathscr{P}(\Omega), \Delta, \cap)$.

(i) Prove that A is a commutative ring with unity.

(ii) Describe the zero divisors and the invertible elements of A.

(iii) Explain why A is not an integral domain.

(iv) Give the tables for additiona and multiplication of A for $\Omega = \{a, b, c\}$.

14. Let G be an additive Abelian group. An endomorphism on G is a homomorphism from G into G. Prove that $\operatorname{End}(G)$, the set of endomorphisms on G becomes a ring with unity when it's endowed with addition and the composition product.

15. Let $(A, +, \cdot)$ be a ring. An element $a \in A$ is said to be *nilpotent* if

$$\exists n \in \mathbb{N} : \quad a^n = 0.$$

(i) Prove that if A has a unit element and $a \in A$ is nilpotent, then both $a + 1$ and $a - 1$ are invertible.

(ii) Prove that if A is commutative and $a \in A$ is nilpotent, then xa is nilpotent, for all $x \in A$.

(iii) Prove that if A is commutative and $a, b \in A$ are nilpotent, then $a + b$ is nilpotent.

16. Show that a ring that is cyclic under addition is commutative.

17. Let n be an integer greater than 1 . In a ring in which $x^n = x$ for all x, show that $ab = 0$ implies $ba = 0$.

18. Let a belong to a ring R. Let $S = \{x \in R \mid ax = 0\}$. Show that S is a subring of R.

19. Suppose that there is a positive even integer n such that $a^n = a$ for all elements a of some ring. Show that $-a = a$ for all a in the ring.

20. Let R be a ring. The center of R is the set $\{x \in R \mid ax = xa \text{ for all } a \text{ in } R\}$. Prove that the center of a ring is a subring.

21. Suppose that R is a ring and that $a^2 = a$ for all a in R. Show that R is commutative.

22. Show that $4x^2 + 6x + 3$ is a unit in $Z_8[x]$.

23. Let R be a commutative ring with more than one element. Prove that if for every nonzero element a of R we have $aR = R$, then R has a unity and every nonzero element has an inverse.

24. Suppose that R is a ring with no zero divisors and that R contains a nonzero element x such that $x^2 = x$. Show that x is the unity of R.

25. Let R be a ring. Prove that $a^2 - b^2 = (a + b)(a - b)$ for all $a, b \in R$ if and only if R is commutative.