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Problem Set 3

Cyclic Groups, Normality, Quotients and the Isomorphism Theorems

(Lessons 6, 7, and 8)

1. Prove that if A is a subset of B then $\langle A\rangle \leq \langle B\rangle$. Give an example where $A \subseteq B$ with $A \neq B$ but $\langle A\rangle = \langle B\rangle$.

2. Prove that the subgroup generated by any two distinct elements of order 2 in S_3 is all of S_3.

3. Prove that the subgroup of S_4 generated by (12) and (12)(3 4) is a noncyclic group of order 4.

4. A group H is called finitely generated if there is a finite set A such that $H = \langle A\rangle$.

(i) Prove that every finite group is finitely generated.

(ii) Prove that $\mathbb{Z}$ is finitely generated.

(iii) Prove that every finitely generated subgroup of the additive group $\mathbb{Q}$ is cyclic. (If H is a finitely generated subgroup of $\mathbb{Q}$, show that $H \leq \langle \frac{1}{k}\rangle$, where k is the product of all the denominators which appear in a set of generators for H.)

(iv) Prove that $\mathbb{Q}$ is not finitely generated.

5. Let $\varphi : G \to H$ be a homomorphism and let E be a subgroup of H. Prove that $\varphi^{-1}(E) \leq G$ (i.e., the pullback of a subgroup under a homomorphism is a subgroup). If $E \trianglelefteq H$ prove that $\varphi^{-1}(E) \trianglelefteq G$. Deduce that $\ker \varphi \trianglelefteq G$.

6. Prove that if $N \trianglelefteq G$ and H is any subgroup of G then $N \cap H \trianglelefteq H$.

7. Let N be a *finite* subgroup of a group G and assume $N = \langle S\rangle$ for some subset S of G. Prove that an element $g \in G$ normalizes N if and only if $gSg^{-1} \subseteq N$.

8. Prove that if $G/Z(G)$ is cyclic then G is abelian. (Hint: If $G/Z(G)$ is cyclic with generator $xZ(G)$, show that every element of G can be written in the form $x^a z$ for some integer $a \in \mathbb{Z}$ and some element $z \in Z(G)$.)

9. Let A and B be groups. Show that $\{(a, 1) \mid a \in A\}$ is a normal subgroup of $A \times B$ and the quotient of $A \times B$ by this subgroup is isomorphic to B.

10. Let A be an abelian group and let D be the (diagonal) subgroup $\{(a, a) \mid a \in A\}$ of $A \times A$. Prove that D is a normal subgroup of $A \times A$ and $(A \times A)/D \cong A$.

11. Suppose A is the non-abelian group S_3 and D is the diagonal subgroup $\{(a,a) \mid a \in A\}$ of $A \times A$. Prove that D is not normal in $A \times A$.

12. Let G be a group, let N be a normal subgroup of G and let $\overline{G} = G/N$. Prove that $\overline{x}$ and $\overline{y}$ commute in $\overline{G}$ if and only if $x^{-1}y^{-1}xy \in N$. (The element $x^{-1}y^{-1}xy$ is called the *commutator* of x and y and is denoted by $[x,y]$.)

13. Let G be a group. Prove that $N = \langle x^{-1}y^{-1}xy \mid x, y \in G\rangle$ is a normal subgroup of G and G/N is abelian (N is called the *commutator subgroup* of G).

14. Show that if $|G| = pq$ for some primes p and q (not necessarily distinct) then either G is abelian or $Z(G) = 1$.

15. Prove that if H and K are finite subgroups of G whose orders are relatively prime then $H \cap K = 1$.

16. Suppose H and K are subgroups of finite index in the (possibly infinite) group G with $|G : H| = m$ and $|G : K| = n$. Prove that l.c.m.$(m,n) \leq |G : H \cap K| \leq mn$. Deduce that if m and n are relatively prime then $|G : H \cap K| = |G : H| \cdot |G : K|$.

17. Use Lagrange's Theorem in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$ to prove *Fermat's Little Theorem*: if p is a prime then $a^p \equiv a \pmod p$ for all $a \in \mathbb{Z}$.

18. Let p be a prime and let n be a positive integer. Find the order of $\overline{p}$ in $(\mathbb{Z}/(p^n-1)\mathbb{Z})^\times$ and deduce that $n \mid \varphi(p^n - 1)$ (here φ is Euler's function).

19. Let G be a finite group, let H be a subgroup of G and let $N \trianglelefteq G$. Prove that if $|H|$ and $|G : N|$ are relatively prime then $H \leq N$.

20. Prove that if N is a normal subgroup of the finite group G and $(|N|, |G : N|) = 1$ then N is the unique subgroup of G of order $|N|$.

21. If A is an abelian group with $A \trianglelefteq G$ and B is any subgroup of G prove that $A \cap B \trianglelefteq AB$.

22. Use Lagrange's Theorem in the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$ to prove *Euler's Theorem*: $a^{\varphi(n)} \equiv 1 \pmod n$ for every integer a relatively prime to n, where φ denotes Euler's φ-function.

23. Prove that if H is a normal subgroup of G of prime index p then for all $K \leq G$ either

(i) $K \leq H$ or

(ii) $G = HK$ and $|K : K \cap H| = p$.

24. Let C be a normal subgroup of the group A and let D be a normal subgroup of the group B. Prove that $(C \times D) \trianglelefteq (A \times B)$ and $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$.

25. Let M and N be normal subgroups of G such that $G = MN$. Prove that $G/(M \cap N) \cong (G/M) \times (G/N)$.

26. Let p be a prime and let G be a group of order $p^a m$, where p does not divide m. Assume P is a subgroup of G of order p^a and N is a normal subgroup of G of order $p^b n$, where p does not divide n. Prove that $|P \cap N| = p^b$ and $|PN/N| = p^{a-b}$. (The subgroup P of G is called a Sylow p-subgroup of G. This exercise shows that the intersection of any Sylow p-subgroup of G with a normal subgroup N is a Sylow p-subgroup of N.)

27. Prove that there are only two distinct groups of order 4 (up to isomorphism), namely $\mathbb{Z}_4$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. (*Hint*: By Lagrange's Theorem, a group of order 4 that is not cyclic must consist of an identity and three elements of order 2.)

28. Let H, K be subgroups of a group G. Then HK is a subgroup of G if and only if $HK = KH$.

29. Let $H \leq G$. Prove aHa^{-1} is a subgroup for each $a \in G$, and $H \cong aHa^{-1}$.

30. Let G be a finite group and H a subgroup of G of order n. If H is the only subgroup of G of order n, then H is normal in G.

31. If H is a cyclic subgroup of a group G and H is normal in G, then every subgroup of H is normal in G.

32. If H is a normal subgroup of a group G such that H and G/H are finitely generated, then so is G.

33. Let $N \trianglelefteq G$ and $K \trianglelefteq G$. If $N \cap K = \langle e \rangle$ and $N \vee K = G$, then $G/N \cong K$.

34. If $f : G \to H$ is a homomorphism, H is abelian and N is a subgroup of G containing $\operatorname{Ker} f$, then N is normal in G.

35. If $N \trianglelefteq G$, $[G : N]$ finite, $H \leq G$, $|H|$ finite, and $[G : N]$ and $|H|$ are relatively prime, then $H \leq N$.