1. Ring Homomorphisms and the Fundamental Theorems of Isomorphism

1.1. Ring Homomorphisms

To understand better rings and their properties we have to look at maps between ring that preserve the structure, namely that preserve both addition and multiplication. This maps are called ring-homomorphisms.

Definition 1.1. Let A and B be rings. A map $\psi\colon A \to B$ is a **ring-homomorphism** if, for all $a, b \in A$,

(i) $\varphi(a+b) = \varphi(a) + \varphi(b)$, and

(ii) $\psi(a \cdot b) = \psi(a)\psi(b)$.

A bijective ring-homomorphism is called a **ring-isomorphism**. If there is an isomorphism between A and B, we say these rings are isomorphic, which is denoted by $A \cong B$.

Note that condition (i) means ψ is a group-homomorphism from $(A, +_A)$ to $(B, +_B)$. This implies that $\psi(0_A) = 0_B$ since $\psi(0_A) = \psi(0_A + 0_A) = \psi(0_A) + \psi(0_A)$ whence $\psi(0_A) = 0_B$ by substraction. A ring-homomorphism from A to A is called an **endomorphism** of A.

Remark 1.1.1. If both A and B are rings with unity, a ring-homomorphism $\psi\colon A \to B$ must in addition satisfy $\psi(1_A) = 1_B$.

Given a homomorphism of rings $\psi\colon A \to B$, the kernel of ψ is

$$\operatorname{Ker}\psi = \{x \in A \mid \varphi(x) = 0\}.$$

The image of ψ is $\operatorname{Im}\psi = \psi(A)$.

Exercise 1. Prove $\operatorname{Ker}\psi$ is an ideal of A and $\operatorname{Im}\psi$ is a subring of B. Why is it not necessarily true that $\operatorname{Im}\psi$ is an ideal of B?

Example 1. (i) If B is a subring of a ring A, the canonical injection $\iota\colon B \to A\colon b \mapsto b$ is a ring-homomorphism. In particular, the identity $\mathrm{Id}_A\colon A \to A$ is an isomorphism.

(ii) If A is a ring and $I \trianglelefteq A$ an ideal, the canonical projection $\pi\colon A \to A/I$ is a surjective homomorphism of rings. Indeed, we have

$$\begin{aligned} \pi(a_1) + \pi(a_2) &= (a_1 + I) + (a_2 + I) = (a_1 + a_2) + I = \pi(a_1 + a_2), \quad \text{and} \\ \pi(a_1)\pi(a_2) &= (a_1 + I)(a_2 + I) = (a_1 a_2) + I = \pi(a_1 a_2). \end{aligned}$$

In particular, the projection $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is a ring homomorphism, called **reduction modulo** n.

(iii) If $(A_\lambda)_{\lambda\in\Lambda}$ is a family of rings, the αth canonical projection $p_\alpha\colon \prod_{\lambda\in\Lambda} A_\lambda \to A_\alpha$ is a surjective ring-homomorphism.

(iv) The rings $2\mathbb{Z}$ and $3\mathbb{Z}$ are not isomorphic. If there were an isomorphism $\psi : 2\mathbb{Z} \to 3\mathbb{Z}$ such that $\psi(2) = 3k$ for some integer $k \neq 0$, then

$$\psi(4) = \psi(2+2) = \psi(2) + \psi(2) = 6k,$$

but

$$\psi(4) = \psi(2\cdot 2) = \psi(2)\cdot\psi(2) = 9k^2.$$

Thus $6k = 9k^2$, and since $k \neq 0$, then $2 = 3k$, which is impossible. Therefore $2\mathbb{Z} \ncong 3\mathbb{Z}$.

(v) Let A be a commutative ring with unity, then the map

$$\psi : A[x] \to A \quad : \quad p(x) \mapsto p(0)$$

is a ring homomorphism that maps a polynomial to its constant term. In general, the evaluation map $\psi_\alpha : A[x] \to A$ defined by

$$\psi_\alpha\left(a_0 + a_1x + \cdots + a_nx^n\right) = a_0 + a_1\alpha + \cdots + a_n\alpha^n,$$

where $\alpha \in A$ is fixed, is a homomorphism of $A[x]$ onto A.

(vi) There is no ring homomorphism $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}$ for any $n \in \mathbb{Z}^+$.

1.2. Properties of Ring Homomorphisms

Proposition 1.2. *Let $\psi\colon A \to B$ be a morphism of rings. Then*

(i) $f(-a) = -f(a)$ *for all* $a \in A$

(ii) $f(na) = nf(a)$ *for all* $a \in A$ *and* $n \in \mathbb{Z}$

(iii) $f(a^n) = f(a)^n$ *for all* $a \in A$ *and* $n \in \mathbb{Z}^+$

(iv) $\operatorname{Ker}\psi$ *is an ideal of* A

(v) $\operatorname{Im}\psi$ *is a subring of* B

(vi) ψ *is injective if and only of* $\operatorname{Ker}\psi = 0$

(vii) if $a \in A$ *is invertible, so is* $\psi(a)$

(viii) if $\mathfrak{p}$ *is an ideal of* B*, then* $\psi^{-1}(\mathfrak{p})$ *is an ideal of* A.

(ix) if $\mathfrak{p}$ *is a maximal (prime) ideal of* B*, then* $\psi^{-1}(\mathfrak{p})$ *is a maximal (prime) ideal of* A.

(x) the composition of ring homomorphisms is a ring homomorphism

Proof. Straight from the definitions and left as an exercise to the student. □

1.3. Fundamental Theorems of Isomorphism

caracterizacion primos y maximales con cocientes

The proofs of the following results are left as exercises to the reader. The diagrams should be used as mnemoteniques.

Theorem 1.3 (Fundamental Theorem on Ring-Homomorphisms)**.** *Let $\psi : A \to B$ be a ring-homomorphism and I an ideal of A with $I \subseteq \operatorname{Ker}\psi$. Then there is a unique ring-homomorphism $\overline{\psi}\colon A/I \to B$ such that $\psi = \overline{\psi} \circ \pi$, i.e.,*

$$\overline{\psi}(a+I) = \psi(a) \quad \textit{for all } a \in A.$$

In other words, such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\pi"'] & B \\ A/I \arrow[ur, dashed, "\overline{\psi}"'] & \end{tikzcd}$$

Moreover

(i) $\operatorname{Im}\overline{\psi} = \operatorname{Im}\psi$*,*

(ii) $\operatorname{Ker}\overline{\psi} = (\operatorname{Ker}\psi)/I$

(iii) $\overline{\psi}$ *is an isomorphism if and only if* ψ *is an epimorphism and* $I = \operatorname{Ker}\psi$*.*

Proof (Sketch). Define $\overline{\psi}\colon A/I \to B$ by $\overline{\psi}(a+I) = \psi(a)$. By using the fact that $I \subseteq \operatorname{Ker}\psi$, we show $\overline{\psi}$ is well-defined. This shows the existence. If $\varphi\colon A/I \to B$ is another ring-homomorphism such that $\psi = \varphi \circ \pi$, then $\overline{\psi} \circ \pi = \varphi \circ \pi$. Since π is surjective, it has a right inverse and thus $\overline{\psi} = \varphi$. This shows the uniqueness. On the other hand, (i) $\psi(a) \in \operatorname{Im}\psi$ if and only if $\psi(a) = \overline{\psi}(a+I) \in \operatorname{Im}\overline{\psi}$. Hence $\operatorname{Im}\psi = \operatorname{Im}\overline{\psi}$. Note (ii)

$$\begin{aligned} \operatorname{Ker}\overline{\psi} &= \{a+I \in A/I \mid \overline{\psi}(a+I) = 0\} \\ &= \{a+I \in A/I \mid \psi(a) = 0\} \\ &= \{a+I \in A/I \mid a \in \operatorname{Ker}\psi\} \\ &= \operatorname{Ker}\psi/I \end{aligned}$$

Finally, (i) shows $\overline{\psi}$ is surjective if and only if ψ is, and (ii) implies $\overline{\psi}$ is injective if $\operatorname{Ker}\psi = I$ since $\operatorname{Ker}\overline{\psi} = I/I = I$. These facts imply (iii). □

The following corollaries are known as the fundamental theorems of ring-homomorphisms. Their proof follows from the fundamental theorem of ring-homomorphisms and thus we leave the details to the reader.

Corollary 1.4 (Second Isomorphism Theorem)**.** *Let I and J be ideals in a ring A. Then*

$$\frac{I}{I \cap J} \cong \frac{I+J}{J}.$$

Corollary 1.5 (Third Isomorphism Theorem)**.** *Let I and J be ideals in a ring A with $J \subseteq I$. Then*

$$\frac{A/J}{I/J} \cong \frac{A}{I}.$$