1. Ring of Fractions

Throughout this section, $A \neq \{0\}$ is a commutative ring. The aim of this section is to prove that the commutative ring A is always a subring of a larger ring Q in which every non-zero element that is not a zero divisor is invertible. We had already proved that if a is not a zero divisor nor zero, then $ab = ac$ in A implies that $b = c$. Then a non-zero divisor element has similar properties to invertible elements.

The construction of $\mathbb{Q}$ is usually based on the construction of $\mathbb{Q}$ based on $\mathbb{Z}$. For this, note that every rational number may be represented in many different ways as the quotient of two integers, i.e.,

$$\frac{a}{b} = \frac{c}{d} \quad \text{iff} \quad ad = bc.$$

The relation

$$(a, b) \sim (c, d) \quad \text{iff} \quad ad = bc,$$

is an equivalence relation over the set $\mathbb{Z} \times \mathbb{Z} \setminus \{0\}$. The fraction $\frac{a}{b}$ is the equivalence class under $\sim$. Then $\mathbb{Q}$ is defined as the set of all equivalence classes, and the operations of addition and multiplication are given by

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}, \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

These operations are independent of the choice of representatives of the equivalence classes and make $\mathbb{Q}$ a field. The following theorem is a generalization of the above construction for any commutative ring.

Theorem 1.1. *Let A be an integral domain. Then the set $D = A \setminus \{0\}$ is a non-empty subset of A that does not contain any zero divisors, and there is a field Q such that*

(i) A is embedded in Q as a subring.

(ii) Every element of D is invertible in Q.

(iii) (Uniqueness of Q) Up to isomorphism, Q is the smallest field containing A in which all the elements of D are invertible.

Proof. (i) Let $T = A \times D$ and define the equivalence relation (this is left as an exercise for the reader) $\sim$ on T by

$$(a, b) \sim (c, d) \iff ad = bc.$$

We denote the equivalence class of (a, b) by $\frac{a}{b}$, i.e.,

$$\frac{a}{b} = \{(c, d) \in T \mid (a, b) \sim (c, d)\}.$$

Let's define Q as the set of all the equivalence classes defined above, that is,

$$Q = \left\{ \frac{a}{b} \mid (a,b) \in T \right\}.$$

Then Q is a commutative ring with unit with the operations given by

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}, \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Note that the unit of Q is $\frac{e}{e}$, for any $e \in D$, and $\frac{b}{a}$ is the multiplicative inverse of $\frac{a}{b} \in Q$. Since we have already constructed Q, we need to embed A into Q. For this, we define the ring homomorphism

$$\eta : A \to Q, \quad a \mapsto \frac{ad}{d},$$

where d is any element of D (it's an exercise to prove that η is an injective ring homomorphism which is well-defined, i.e., η does not depend on the choice of $d \in D$).

(ii) Let's prove that each $b \in D$ has a multiplicative inverse (under the embedding ϕ) in Q. The element b is represented in Q under ϕ by $\frac{bd}{d}$ for any $d \in D$. Then its multiplicative inverse in Q is the fraction $\frac{d}{bd}$. This fact is easy to prove since A is commutative and

$$\frac{bd}{d} \cdot \frac{d}{bd} = \frac{bd \cdot d}{bd^2} = 1 \quad \text{in } Q.$$

(iii) Considering ii), to prove that Q is the smallest ring containing A in which all the elements of D become invertible, is equivalent to proving that, if R is any commutative ring with unit, and

$$\kappa : A \to R$$

is an injective homomorphism such that $\kappa(d)$ is invertible in R for any $d \in D$, then there is an injective homomorphism $\theta : Q \to R$ such that $\theta \circ \eta = \kappa$.

Let $\kappa : A \to R$ be any injective homomorphism such that $\kappa(d)$ is invertible in R for any $d \in D$. Extend κ to the well-defined injective ring homomorphism (this is left as an exercise):

$$\theta : Q \to R, \quad \frac{a}{b} \mapsto \kappa(a)(\kappa(b))^{-1}.$$

Then for any $a \in A$ and any $e \in D$:

$$\begin{aligned}
\theta \circ \eta(a) &= \theta(\eta(a)), \\
&= \theta\left(\frac{ae}{e}\right), \\
&= \kappa(ae)(\kappa(e))^{-1}, \\
&= \kappa(a)\kappa(e)(\kappa(e))^{-1}, \\
&= \kappa(a).
\end{aligned}$$

Therefore, $\theta \circ \eta = \kappa$, completing the proof.

□

The field Q is called the field of fractions of A.

Example 1 (9.4)**.** If A is a field, then its field of fractions is just A itself.

Example 2 (9.5)**.** $\mathbb{Z}$ is an integral domain whose field of fractions is $\mathbb{Q}$. The subring $2\mathbb{Z}$ of $\mathbb{Z}$ has no zero divisors but has no unit, and its field of fractions is also $\mathbb{Q}$.

This lecture needs to be reviewed