

1. Ring Homomorphisms and the Fundamental Theorems of Isomorphism

1.1. Ring Homomorphisms

To understand better rings and their properties we have to look at maps between ring that preserve the structure, namely that preserve both addition and multiplication. This maps are called ring-homomorphisms.

Definition 1.1. Let A and B be rings. A map $\psi: A \rightarrow B$ is a **ring-homomorphism** if, for all $a, b \in A$,

$$(i) \quad \psi(a + b) = \psi(a) + \psi(b), \text{ and}$$

$$(ii) \quad \psi(a \cdot b) = \psi(a)\psi(b).$$

A bijective ring-homomorphism is called a **ring-isomorphism**. If there is an isomorphism between A and B , we say these rings are isomorphic, which is denoted by $A \cong B$.

Note that condition (i) means ψ is a group-homomorphism from $(A, +_A)$ to $(B, +_B)$. This implies that $\psi(0_A) = 0_B$ since $\psi(0_A) = \psi(0_A + 0_A) = \psi(0_A) + \psi(0_A)$ whence $\psi(0_A) = 0_B$ by subtraction. A ring-homomorphism from A to A is called an **endomorphism** of A .

Remark 1.1.1. If both A and B are rings with unity, a ring-homomorphism $\psi: A \rightarrow B$ must in addition satisfy $\psi(1_A) = 1_B$.

Given a homomorphism of rings $\psi: A \rightarrow B$, the kernel of ψ is

$$\text{Ker } \psi = \{x \in A \mid \psi(x) = 0\}.$$

The image of ψ is $\text{Im } \psi = \psi(A)$.

Exercise 1. Prove $\text{Ker } \psi$ is an ideal of A and $\text{Im } \psi$ is a subring of B . Why is it not necessarily true that $\text{Im } \psi$ is an ideal of B ?

Example 1. (i) If B is a subring of a ring A , the canonical injection $\iota: B \rightarrow A: b \mapsto b$ is a ring-homomorphism. In particular, the identity $\text{Id}_A: A \rightarrow A$ is an isomorphism.

(ii) If A is a ring and $I \trianglelefteq A$ an ideal, the canonical projection $\pi: A \rightarrow A/I$ is a surjective homomorphism of rings. Indeed, we have

$$\begin{aligned} \pi(a_1) + \pi(a_2) &= (a_1 + I) + (a_2 + I) = (a_1 + a_2) + I = \pi(a_1 + a_2), \quad \text{and} \\ \pi(a_1)\pi(a_2) &= (a_1 + I)(a_2 + I) = (a_1a_2) + I = \pi(a_1a_2). \end{aligned}$$

In particular, the projection $\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$ is a ring homomorphism, called **reduction modulo n** .

- (iii) If $(A_\lambda)_{\lambda \in \Lambda}$ is a family of rings, the α th canonical projection $p_\alpha: \prod_{\lambda \in \Lambda} A_\lambda \rightarrow A_\alpha$ is a surjective ring-homomorphism.
- (iv) The rings $2\mathbb{Z}$ and $3\mathbb{Z}$ are not isomorphic. If there were an isomorphism $\psi: 2\mathbb{Z} \rightarrow 3\mathbb{Z}$ such that $\psi(2) = 3k$ for some integer $k \neq 0$, then

$$\psi(4) = \psi(2 + 2) = \psi(2) + \psi(2) = 6k,$$

but

$$\psi(4) = \psi(2 \cdot 2) = \psi(2) \cdot \psi(2) = 9k^2.$$

Thus $6k = 9k^2$, and since $k \neq 0$, then $2 = 3k$, which is impossible. Therefore $2\mathbb{Z} \not\cong 3\mathbb{Z}$.

- (v) Let A be a commutative ring with unity, then the map

$$\psi: A[x] \rightarrow A \quad : \quad p(x) \mapsto p(0)$$

is a ring homomorphism that maps a polynomial to its constant term. In general, the evaluation map $\psi_\alpha: A[x] \rightarrow A$ defined by

$$\psi_\alpha(a_0 + a_1x + \cdots + a_nx^n) = a_0 + a_1\alpha + \cdots + a_n\alpha^n,$$

where $\alpha \in A$ is fixed, is a homomorphism of $A[x]$ onto A .

- (vi) There is no ring homomorphism $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}$ for any $n \in \mathbb{Z}^+$.

1.2. Properties of Ring Homomorphisms

Proposition 1.2. *Let $\psi: A \rightarrow B$ be a morphism of rings. Then*

- (i) $f(-a) = -f(a)$ for all $a \in A$
- (ii) $f(na) = nf(a)$ for all $a \in A$ and $n \in \mathbb{Z}$
- (iii) $f(a^n) = f(a)^n$ for all $a \in A$ and $n \in \mathbb{Z}^+$
- (iv) $\text{Ker } \psi$ is an ideal of A
- (v) $\text{Im } \psi$ is a subring of B
- (vi) ψ is injective if and only if $\text{Ker } \psi = 0$
- (vii) if $a \in A$ is invertible, so is $\psi(a)$
- (viii) if \mathfrak{p} is an ideal of B , then $\psi^{-1}(\mathfrak{p})$ is an ideal of A .
- (ix) if \mathfrak{p} is a maximal (prime) ideal of B , then $\psi^{-1}(\mathfrak{p})$ is a maximal (prime) ideal of A .
- (x) the composition of ring homomorphisms is a ring homomorphism

Proof. Straight from the definitions and left as an exercise to the student. □

1.3. Fundamental Theorems of Isomorphism

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