Mandatory assignment 2
MEK4300

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Exercise 1

We are to solve the dimensionless non-linear Falkner-Skan equation

$$f''' + ff'' + \beta\left(1 - f'^2\right) = 0 \tag{1}$$

where $f = f(\eta)$, $\eta = y\sqrt{\frac{U(1+m)}{2\nu x}}$ and $\beta = \frac{2m}{1+m}$. m is is called the Falkner-Skan power-law parameter.

To solve this I introduce a new function $H = f'$. Inserted, this gives me two equations

$$\begin{aligned} H'' + fH + \beta\left(1 - H^2\right) &= 0 \\ H - f' &= 0 \end{aligned} \tag{2}$$

I then introduce two test functions v, q and rewrite the equation set on variational form

$$\begin{aligned} -\int H'v'dx + \int fHvdx + \int \beta vdx - \int \beta H^2 vdx &= 0 \\ \int Hqdx - \int f'qdx &= 0 \end{aligned} \tag{3}$$

This can be implemented directly into FEniCS.

Part a

Velocity profiles and shear-stress profiles for selected β values can be seen in figure 1.

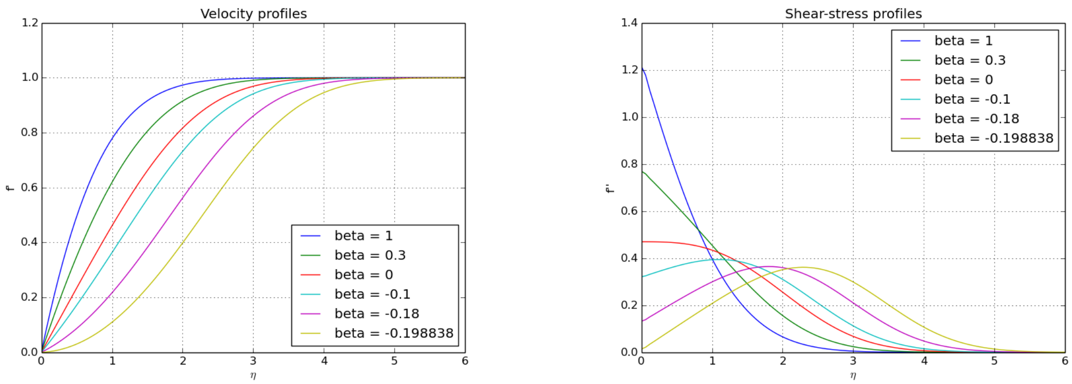


Figure 1: Plots of the velocity profiles and the shear-stress profiles for selected values of β.

Part b

When we have $\beta = -0.198838$ we see from figure 1 that the shear-stress is very close to 0 at $x = 0$. When $\beta = -0.19884$ we have that the shear-stress is exactly 0. Two different solutions for the shear-stress when $\beta = -0.1$ can be seen in figure 2

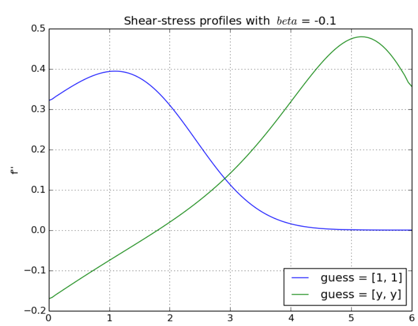


Figure 2: Plots of the shear-stress profile for two different initial guesses.

Exercise 2

Case 2D-1, steady

Solving the steady case was done by a straight forward implementation of the steady Navier-Stokes equation using FEniCS. It was solved for three different meshes.

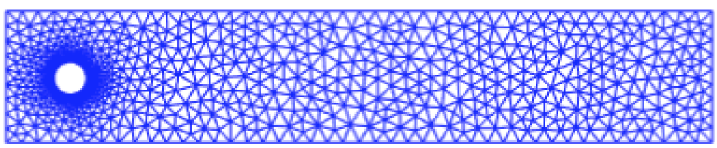
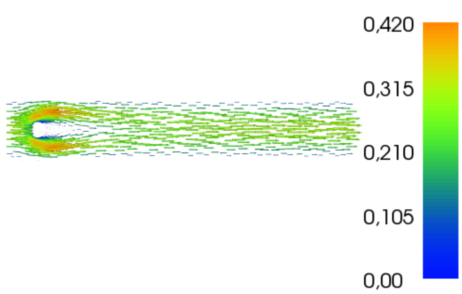


Figure 3: The figure displays the coarsest mesh used, yielding 18155 unknowns.



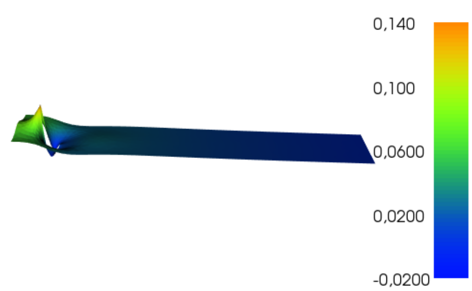


Figure 4: Plots of the velocity profile and the pressure profile for the coarsest mesh.

In table 1 we see that all the calculated values are within the lower and

Table 1: Calculated values for three different meshes for the steady state.

Unknowns	C_D	C_L	L_a	ΔP
18155	5.5758	0.0109	0.0847	0.1174
43417	5.5774	0.0105	0.0846	0.1175
177541	5.5773	0.0106	0.0846	0.1175
lower bound	5.5700	0.0104	0.0842	0.1172
upper bound	5.5900	0.0110	0.0852	0.1176

upper bound.

Case 2D-2, unsteady

The Navier-Stokes equation is solved using a implicit pressure correction scheme, IPCS. This is done by first calculating a tentative velocity, u^*, then calculating the pressure with the tentative velocity, for so to calculate the correct velocity.

Table 2: Calculated values for the unsteady state.

Unknowns	C_{Dmax}	C_{Lmax}	St	ΔP
18155	3.2290	1.0542	0.2967	2.4853
lower bound	3.2200	0.9900	0.2950	2.4600
upper bound	3.2400	1.0100	0.3050	2.5000

We see the C_{Lmax} is a little of.

Exercise 3

We are to use a mixing length method to implement a solver for a plane turbulent channel flow. with the following momentum equation

$$0 = \nu \frac{\partial^2 \bar{u}}{\partial y^2} - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} - \frac{\partial \overline{u'v'}}{\partial y} \tag{4}$$

where $\overline{u'v'} = -l^2 \left|\frac{\partial \bar{u}}{\partial x}\right| \frac{\partial \bar{u}}{\partial x}$.

Part a

We are given a hint that the pressure gradient is constant and we are given a value for $v^* = \sqrt{\nu \frac{\partial \bar{u}}{\partial x}} = 0.05$. Integrating the momentum equation across the channel we get

$$\begin{aligned} \frac{1}{\rho}\int_0^H \frac{\partial \bar{p}}{\partial x} dy &= \int_0^H \nu \frac{\partial^2 \bar{u}}{\partial y^2} dy + \int_0^H l^2 \frac{\partial \overline{u'v'}}{\partial y} dy \\ &= \nu \frac{\partial \bar{u}}{\partial y}\bigg|_0^H + l^2 \overline{u'v'}\bigg|_0^H \end{aligned} \tag{5}$$

By the wall we must have that the mean fluctuating velocities must be zero. So we then end up with the first term on the left hand side. Rearranging the equation and inserting values gives

$$\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} = \frac{2(v^*)^2}{H} \tag{6}$$

Close to the wall we must have that

$$y^+ = \frac{y v^*}{\nu} < 1 \tag{7}$$

For a non skewed mesh we must have that the first inner node must be located at y-coordinate

$$y \leq \frac{\nu}{v^*} = \frac{1}{Re} = \frac{1}{1000} \tag{8}$$

For a channel height of 1 meter we must have then have 1001 nodes in order to ensure that the first inner node is located at this coordinate.

Calculating for half a channel height of 1m with FEniCS gives a velocity profile seen in figure 5.

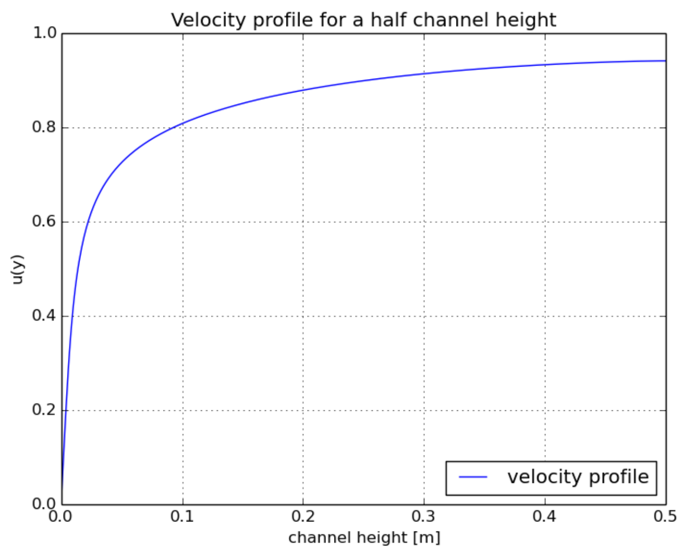


Figure 5: Figure displays velocity profile in the lower half of a channel with a height of 1 meter using a mixing length model.

Part b

Using the values given in the exercise I get result seen in figure 6.

We see that $u^+ = y^+$ is a good match for $y^+ < 5$, but for larger values we don't have a match. This can be achieved somewhat by adjusting the constants given in the exercise. Two examples can be seen in figure 7.

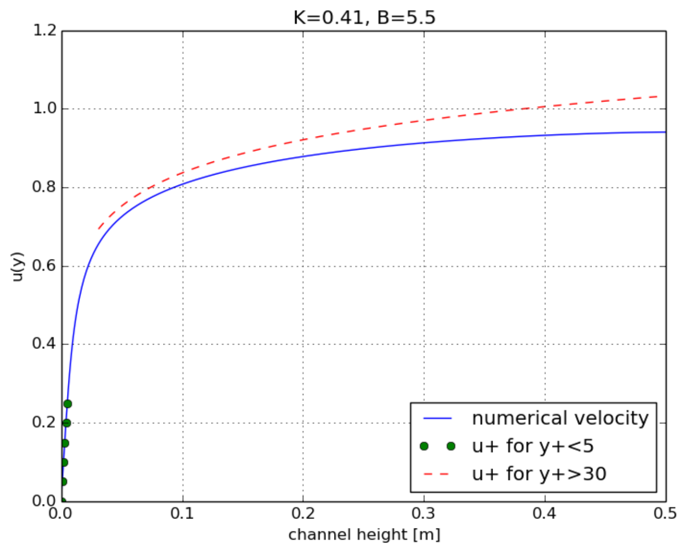


Figure 6: Velocity profile and u^+ for different channel areas with the values given in the exercise.

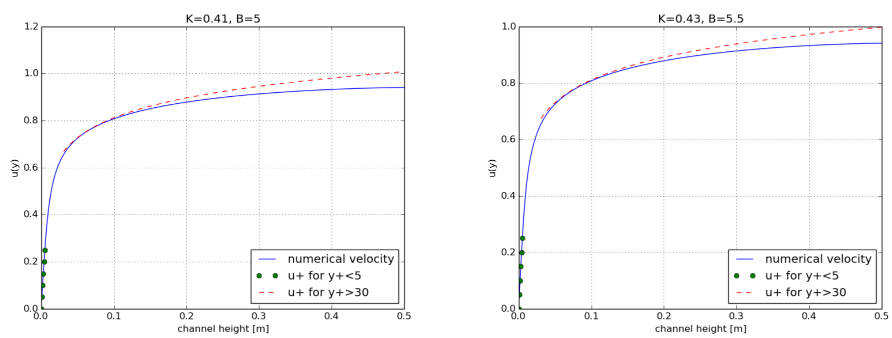


Figure 7: Velocity profile and u^+ for different channel areas with alternative constants.

Exercise 4

The Navier-Stokes equation (NS) for incompressible, Newtonian fluids, with the continuity equation reads

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j} + f_V \tag{9}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{10}$$

where the volume forces, f_V, is so much smaller than the viscous forces so we can ignore it.

RANS

We consider the parameters as a combination of one time averaged term and one fluctuating term

$$\begin{aligned} u &= \bar{u} + u' \\ p &= \bar{p} + p' \end{aligned} \tag{11}$$

First we prove that the continuity equation is true for all parameters. The time averaged is simple

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0 \tag{12}$$

If we now subtract this equation from the original continuity equation, we end up with

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \tag{13}$$

Now we insert this into NS and get

$$\frac{\partial}{\partial t}\left(\bar{u_i}+u_i'\right)+\left(\bar{u_j}+u_j'\right)\frac{\partial}{\partial x_j}\left(\bar{u_i}+u_i'\right)=-\frac{1}{\rho}\frac{\partial}{\partial x_i}\left(\bar{p}+p'\right)+\nu\frac{\partial^2}{\partial x_j^2}\left(\bar{u_i}^2+p_i'^2\right) \tag{14}$$

We now time average the entire equation and use the similarities

$$\begin{aligned}\bar{f}' &= 0\\ \overline{\frac{\partial f}{\partial s}} &= \frac{\partial \bar{f}}{\partial s}\\ \bar{\bar{f}} &= \bar{f}\\ \overline{f+g} &= \bar{f}+\bar{g}\end{aligned} \tag{15}$$

Then we end up with

$$\frac{\partial \bar{u_i}}{\partial t}+\bar{u_j}\frac{\partial \bar{u_i}}{\partial x_j}+\bar{u_j'}\frac{\partial \bar{u_i'}}{\partial x_j}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_j^2} \tag{16}$$

The third term on the left hand side can be written, by using the product rule, as

$$\bar{u_j'}\frac{\partial \bar{u_i'}}{\partial x_j}=\frac{\partial \overline{u_i' u_j'}}{\partial x_j}-\bar{u_i'}\frac{\partial \bar{u_j'}}{\partial x_j} \tag{17}$$

where the last term is zero as shown in the continuity equation. If we move this term to the right hand side and multiply with ρ

$$\rho\frac{\partial \bar{u_i}}{\partial t}+\rho\bar{u_j}\frac{\partial \bar{u_i}}{\partial x_j}=-\frac{\partial \bar{p}}{\partial x_i}+\mu\frac{\partial^2 \bar{u_i}}{\partial x_j^2}-\frac{\partial}{\partial x_j}\overline{u_i' u_j'} \tag{18}$$

The left hand side is the $\rho\frac{D\bar{u_i}}{Dt}$ from the book.

TKE

To obtain an equation for the turbulent kinetic energy we start of with subtracting NS from RANS. For most of the terms this is an easy operation but on the convective term we can, because of continuity, use:

$$\begin{aligned}u_j\frac{\partial u_i}{\partial x_j}-\overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} &= \frac{\partial u_i u_j}{\partial x_j}-\frac{\partial \overline{u_i u_j}}{\partial x_j}\\ &= \frac{\partial}{\partial x_j}\left(u_i u_j-\overline{u_i u_j}\right)\\ &= \frac{\partial}{\partial x_j}\left(\overline{u_i}u_j'+u_i'\overline{u_j}+u_i'u_j'\right)\end{aligned} \tag{19}$$

This gives us

$$\frac{\partial u_i'}{\partial t}+\overline{u_j}\frac{\partial u_i'}{\partial x_j}=-\frac{1}{\rho}\frac{\partial p'}{\partial x_i}+\nu\frac{\partial^2 u_i'}{\partial x_j^2}+\frac{\partial}{\partial x_j}\overline{u_i'u_j'}-\frac{\partial}{\partial x_j}\overline{u_i}u_j'-\frac{\partial}{\partial x_j}u_i'u_j' \tag{20}$$

Now we dot this equation with u_i' and then we time average the entire equation

$$\overline{u_i'\left(\frac{\partial u_i'}{\partial t}+\overline{u_j}\frac{\partial u_i'}{\partial x_j}=-\frac{1}{\rho}\frac{\partial p'}{\partial x_i}+\nu\frac{\partial^2 u_i'}{\partial x_j^2}+\frac{\partial}{\partial x_j}\overline{u_i'u_j'}-\frac{\partial}{\partial x_j}\overline{u_i}u_j'-\frac{\partial}{\partial x_j}u_i'u_j'\right)} \tag{21}$$

Now we write out in terms and arrange using product rule and continuity. The first term:

$$\overline{u_i'\frac{\partial u_i'}{\partial t}}=\frac{1}{2}\frac{\partial\overline{u_i'u_i'}}{\partial t} \tag{22}$$

The second term:

$$\overline{u_i'\overline{u_j}\frac{\partial u_i'}{\partial x_j}}=\frac{1}{2}\overline{u_j}\frac{\partial\overline{u_i'u_i'}}{\partial x_j} \tag{23}$$

Our left hand side has now turned into

$$\frac{\partial}{\partial t}\frac{1}{2}\overline{u_i'u_i'}+\overline{u_j}\frac{\partial}{\partial x_j}\frac{1}{2}\overline{u_i'u_i'}=\frac{DK}{Dt} \tag{24}$$

The three last terms on our right hand side can be written as

$$\overline{u_i'\frac{\partial}{\partial x_j}\overline{u_i'u_j'}}-\overline{u_i'\frac{\partial}{\partial x_j}\overline{u_i}u_j}-\overline{u_i'\frac{\partial}{\partial x_j}u_i'u_j'}=0-\overline{u_i'u_j'}\frac{\partial\overline{u_i}}{\partial x_j}-\frac{1}{2}\frac{\partial}{\partial x_j}\overline{u_j'u_i'u_i'} \tag{25}$$

Then our full equation can be written as

$$\frac{DK}{Dt}=-\frac{\partial}{\partial x_j}\left(u_j'\left(\frac{1}{2}u_i'u_i+\frac{p'}{\rho}\right)\right)-\overline{u_i'u_j'}\frac{\partial\overline{u_i}}{\partial x_j}+\nu\overline{u_i'}\frac{\partial^2\overline{u_i'}}{\partial x_j^2} \tag{26}$$

Our last term is the same term as the result from calculating the last two terms in White (IV and V).

$$\begin{aligned}IV&=\frac{\partial}{\partial x_j}\left[\nu u_i'\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial x_j'}{\partial x_i}\right)\right]\\&=\nu\left[\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}+u_i'\frac{\partial^2 u_i'}{\partial x_j^2}+\frac{\partial u_i'}{\partial x_j}\frac{\partial u_j'}{\partial x_i}+u_i'\frac{\partial^2 u_j'}{\partial x_i\partial x_j}\right]\end{aligned} \tag{27}$$

Subtracting V from IV leaves us with $u_i' \frac{\partial^2 u_i'}{\partial x_j^2}$ and our equations match.

Reynold Stress Equation

We take the RANS equation and subtract the NS again. We then dot it with a variable u_k'. This gives us

$$u_k' \left(\frac{\partial u_i'}{\partial t} + \overline{u_j} \frac{\partial u_i'}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u_i'}{\partial x_j^2} + \frac{\partial}{\partial x_j}\overline{u_i' u_j'} - \frac{\partial}{\partial x_j}\overline{u_i} u_j' - \frac{\partial}{\partial x_j} u_i' u_j' \right) \tag{28}$$

We also dot the k'th RANS-NS equation with a variable u_i' which gives

$$u_i' \left(\frac{\partial u_k'}{\partial t} + \overline{u_j} \frac{\partial u_k'}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_k} + \nu \frac{\partial^2 u_k'}{\partial x_j^2} + \frac{\partial}{\partial x_j}\overline{u_k' u_j'} - \frac{\partial}{\partial x_j}\overline{u_k} u_j' - \frac{\partial}{\partial x_j} u_k' u_j' \right) \tag{29}$$

Now we add the two equations together. The left hand side can be written as

$$u_k' \frac{\partial u_i'}{\partial t} + u_i' \frac{\partial u_k'}{\partial t} + u_k' \overline{u_j} \frac{\partial u_i'}{\partial x_j} + u_i' \overline{u_j} \frac{\partial u_k'}{\partial x_j} = \frac{D u_k' u_i'}{Dt} \tag{30}$$

The first term on the right hand side can be written as

$$\begin{aligned} \frac{1}{\rho}\left[u_k' \frac{\partial p'}{\partial x_i} + u_i' \frac{\partial p'}{\partial x_k} \right] &= \frac{1}{\rho}\left[\frac{\partial (u_k' p')}{\partial x_i} - p' \frac{\partial u_k'}{\partial x_i} + \frac{\partial (u_i' p')}{\partial x_k} - p' \frac{\partial u_i'}{\partial x_k} \right] \\ &= -\frac{p'}{\rho}\left(\frac{\partial u_k'}{\partial x_i} + \frac{\partial u_i'}{\partial x_k} \right) + \frac{\partial}{\partial x_j}\frac{p'}{\rho}\left(u_k' \delta_{ij} + u_i \delta_{kj} \right) \end{aligned} \tag{31}$$

The second term

$$u_k' \frac{\partial^2 u_i'}{\partial x_j^2} + u_i' \frac{\partial^2 u_k'}{\partial x_j^2} = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j} u_i' u_k' - 2 \frac{\partial u_k'}{\partial x_i}\frac{\partial u_i'}{\partial x_k} \tag{32}$$

The third term vanishes when we time average.

The fourth term

$$u_k' \frac{\partial}{\partial x_j}\overline{u_i} u_j' + u_i' \frac{\partial}{\partial x_j}\overline{u_k} u_j' = u_k' u_j' \frac{\partial \overline{u_i}}{\partial x_j} + u_i' u_j' \frac{\partial \overline{u_k}}{\partial x_j} \tag{33}$$

The fifth term

$$
\begin{aligned}
u'_k \frac{\partial}{\partial x_j} u'_i u'_j + u'_i \frac{\partial}{\partial x_j} u'_k u'_j &= u'_k u'_j \frac{\partial}{\partial x_j} u'_i + u'_i u'_j \frac{\partial}{\partial x_j} u'_k \\
&= \frac{\partial}{\partial x_j} u'_i u'_k u'_j
\end{aligned}
\tag{34}
$$

This leaves us with all nine right hand side terms found in equation (6-18) in White.