# Game Theory Notes (Week 2)

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December 2018

## 1 Mixed Strategies and Nash Equilibrium (I)

Say you have a game where the UN sets up it's bomb-checking stations on a particular road every hour. Then say there's $n$-number of roads for the UN to choose to set up and $n$-number of roads for the Somalian bomber to go down. Then if the UN gave a clear structure on which road they were checking and when they were checking that road, the bomber could easily bypass the security stations and smuggle the bomb through for explosion. Thus, the UN uses a mixed strategy: randomization of the checkpoints. So the UN will randomly (every hour) change it's checkpoint location to another road. This eliminates the ability for any bomber to know exactly where the UN station(s) will be. And this gives the UN the opportunity to have a higher chance at catching a bomber if there's an individual planning to bomb the local area via one of the streets that the UN *could* be on.

## 2 Mixed Strategies and Nash Equilibrium (II)

*Mixed Strategies*

Recall that it would be a pretty bad idea to play any deterministic strategy in the Matching Pennies Game. Because say you choose heads, (and you win when both choose heads, or both choose tails) then your opponent will want to play tails. In which case you would want to change your guess to tails, and in which case your opponent will want to change their guess to heads. Then you keep going around and around in this circle of everyone changing their answer. Thus, an idea emerges: confuse the opponent by playing *randomly*.

We must also now reform the original definition that we had for a strategy. Define a *strategy* $s_i$ for agent $i$ as any probability distribution over the actions $A_i$. So the definition of the various strategies are:

- **pure strategy**: only one action is played with positive probability.
- **mixed strategy**: more than one action is played with positive probability. These actions are called the *support* of the mixed strategy.

So let the set of *all strategies* for $i$ be $S_i$, and let the set of *all strategy profiles* be $S = S_1 \times ... \times S_n$.

*Utility under Mixed Strategies*

What is your *payoff* if all the players follow mixed strategy profile $s \in S$? Then we we can't just read this number from the game matrix anymore, We won't always end up in the same cell. Instead, use the idea of **expected utility** from decision-theory:

$$u_i(s) = \sum_{a \in A} u_i(a) Pr(a|s) \tag{1}$$

s.t.

$$Pr(a|s) = \prod_{j \in N} s_j(a_j). \tag{2}$$

So say that you have a 2x2 Standard Matrix, then suppose that the weighted probability of choosing either action 1 or action 2 is 50%/50%. Then say that player 2 has the same mixed strategy of choosing at random between action 1 or action 2. Thus, within any given utility value/outcome, the result is a 25% chance of that utility being chosen.

*Best Response and Nash Equilibrium*

Our definitions of best response and Nash equilibrium generalize from actions to strategies:

$s_i^* \in BR(s_{-i})$ iff $\forall s_i \in S_i, u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$.
*Definition (Best Response)*

$s = \langle s_1, ..., s_n \rangle$ is a **Nash equilibrium** iff $\forall i, s_i \in BR(s_{-i})$.
*Definition (Nash Equilibrium)*

Thus, emerges one of the most important concepts in Game Theory:

**Every finite game has a Nash equilibrium**.
*Theorem (Nash, 1950)*

In our current definition of Nash equilibrium in terms of strategy, and adjust this to be in terms of action ($a$), then this becomes the *Pure Strategy Nash*. The issue with this, is when we put the Nash equilibrium in terms of actions, the theorem above *no longer is true*. The theorem holds, explicitly, in terms of strategy.

Moving back to our Matching Pennies example/scenario, we find the following probabilities for the pennies landing in what each player *wants*:

| | | 0.5 | 0.5 |
|---|---|---|---|
| | | Heads | Tails |
| 0.5 | Heads | $1, -1$ | $-1, 1$ |
| 0.5 | Tails | $-1, 1$ | $1, -1$ |

*Figure 1*

Thus, there is a 50/50 chance of being correct, and it's best to randomize your *own* choice because then in *best* scenario, you have a 50/50 shot of being correct when accounting for the chance that your opponent is *also* randomizing their answer.

In terms of the coordination game (involving pure cooperation), then we find something "unusual", in similarity in the game of pure competition:

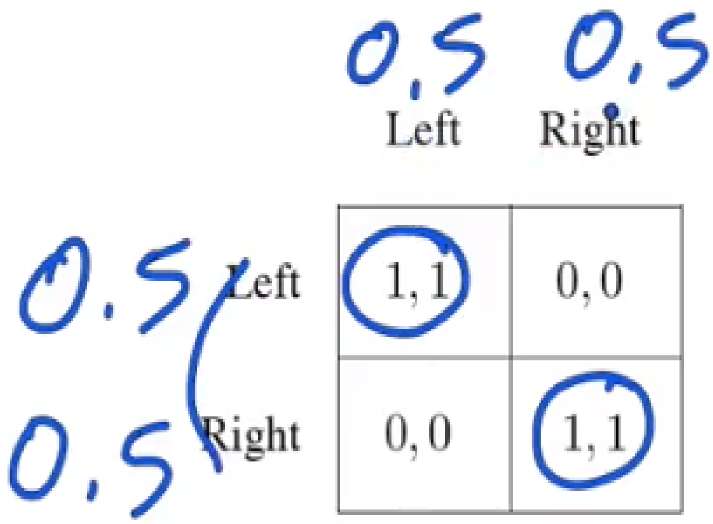


*Figure 2*

In this game, you account for randomness (via mixed strategy), by actually having only a 50/50 shot of choosing the right answer when accounting for the "opponents" randomness. Thus, we find that the with mixed strategy, you have a 50/50 of being correct - and this is potentially the best strategy to take.

Lastly, when we incorporate the mixed strategy idea into the Prisoner's Dilemma, there's actually only one Nash equilibrium, and the following box is circled:

| | $C$ | $D$ |
|---|---|---|
| $C$ | $-1, -1$ | $-4, 0$ |
| $D$ | $0, -4$ | $-3, -3$ |

*Figure 3*

Because the tendency (over time) is to take the action of being disruptive rather than cooperating. Thus, the strategy of tendency, no matter what your "opponent" were to choose, is to choose the action of being disruptive - when accounting also for mixed strategy.

## 3 Computing Mixed Nash Equilibrium

Reverting back to our idea of the game "Date Night" where we have the two movies to choose from, and the given standard matrix:

| | B | F |
|---|---|---|
| B | 2,1 | 0,0 |
| F | 0,0 | 1,2 |

*Figure 4*

In general, it's hard to compute Nash equilibria, but it's easy when you can guess the *support* of a player's action/decision (thus giving us intuition to their strategy). For this date night example, let's look for an equilibrium where all actions are part of the support (via mixed strategies). Now let's introduce some math/statistics concepts in order to determine people's actions. Let player 2 play $B$, with $p$, $F$ with $1-p$ intuitively (because when choosing a single action, the probability of choosing the other action would be one minus the probability of choosing the first action). Thus arises the concept: **"If player 1 best-responds with a mixed strategy, player 2 must make them indifferent between $F$ and $B$"**. So why? Because in order for player 1 to truly best-respond to the situation using a *mixed strategy*, then player 2 must be making player 1 feel indifferent to either of the presented options (even though player 1 *prefers* a specific action!) Following the mathematical concepts:

$$u_1(B|(p,1-p)) = u_1(F|(p,1-p)) \tag{3}$$

$$2p + 0(1-p) = 0p + 1(1-p) \tag{4}$$

$$\therefore p = \frac{1}{3} \tag{5}$$

Likewise, player 1 must randomize to make player 2 indifferent (why is player 1 willing to randomize? - Answer this question on your own, but probably due to simply wanting to be with their date). Now, let player 1 play $B$ with $q$, $F$ with

$1-q$. Then the mathematical concept arises:

$$u_2(B|(q,1-q)) = u_2(F|(q,1-q)) \tag{6}$$

$$q + 0(1-q) = 0q + 2(1-q) \tag{7}$$

$$\therefore q = \frac{2}{3} \tag{8}$$

Thus the mixed strategies

$$(\frac{2}{3}, \frac{1}{3})$$

,

$$(\frac{1}{3}, \frac{2}{3})$$

are the Nash equilibrium of this game. If you're confused at this point - don't worry. This is a very difficult concept to grasp, and read through a couple more times to make sure that you truly understand. The following can, conceptually, help you also understand.

*Interpreting Mixed Strategy Equilibria*

What does it mean to play a mixed strategy? Well there's actually many different interpretations, but here are some of the most used ideas:

- Randomize to **confuse** your opponent (think back to the Matching Pennies Game).

- Randomize when **uncertain** about the other player's action (think back to the Date Night Game).

- Mixed strategies are a concise description of what might happen in **repeated play** (where the equilibrium is the probability, over time, of a certain action being chosen): count of pure strategies in the limit.

- Mixed strategies describe **population dynamics**: 2 agents chosen from a population, all having deterministic strategies. *Mixed strategy* gives the probability of getting each *pure strategy.*

Hence, these four primary ideas are the reasons for mixed strategies to either be used, or to better understand the mixed strategy-equilibria that emerges in various games.

# 4 Hardness Beyond 2x2 Games - Basic

*Algorithms*

Two example algorithms for finding the Nash equilibrium (NE) would be the Linear Complementarity formulation (LCP) [Lemke-Howson '64], and also Support Enumeration Method [Porter et al. '04]. Both of these algorithms can be explored more in depth by looking up the actual research papers, but basically the former is a very "smart" way of setting up to find the NE in mathematical terms, whereas the ladder fixes the supports among players to better understand underlying actions, finding the NE.

*From Algorithms to Complexity Analysis*

Still, finding even a single Nash equilibrium seems hard; how do we capture that "hardness"? Now, enter PPAD ("Polynomial Parity Arguments on Directed Graphs") [Papadimitriou '94]. At a high level:

- FNP problems are constructive versions of NP problems (F = "Functional").

- TFNP is a subclass of FNP for problems for which a solution is guaranteed to exist (T = "Total").

- PPAD is a subclass of TFNP where the proofs are based on parity arguments in directed graphs.

So in the hierarchy of hardness problems, where is PPAD? Taking the following diagram of where Polynomial time problems (P) exist, non-Polynomial (NP), and non-Polynomial-complete (NP-complete), we can better see the sheer difficulty/complexity of solving certain problems:

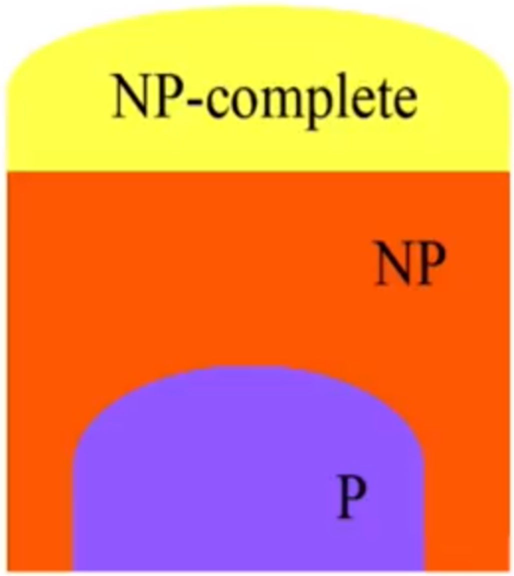


*Figure 5*

Now we actually don't know *exactly* where PPAD lies in this graph, but with much research: the most educated guess places PPAD within NP as displayed below. But truly, we don't know whether the entire class doesn't all collapse and become one in the same (it's widely believed that it does not). But we do *know* that PPAD lies somewhere between P and NP:

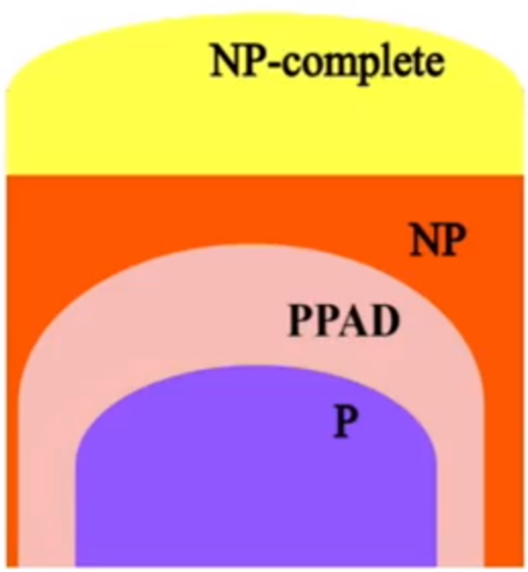


*Figure 6*

So what does this have to do with the complexity of solving Nash equilibria? That's where the following comes into play. The Theorem of Complexity of the Nash Equilibrium:

Computing a Nash equilibrium is PPAD-complete...

- for games with $\geq 4$ players [Daskalakis, Goldberg, Papadimitriou '05].
- for games with 3 players [Chen, Deng '05] and [Daskalakis, Papadimitriou '05].
- for games with 2 players [Chen, Deng '06].

Again, very important concept, and read research journals/articles if you don't truly understand, but basically these problems become very hard to solve, but the completeness guarantees that there must exist an supremum (as in Real Analysis definition of the Reals being complete). Thus, there exists at least a single solution of a Nash equilibrium at all of these games.

# 5 Hardness Beyond 2x2 Games - Advanced

*Early History*

In **1928 von Neumann** found/proved the existence of Equilibrium in 2-player, zero-sum games. This proof uses Brouwer's fixed point theorem, which in turn led directly to algorithms: Danzig '57 - equivalent to LP duality, and Khachiyan '79 - polynomial time solvable. Then in **1950 Nash** found/proved the existence of Equilibrium in multiplayer, general-sum games. His proof also uses Brouwer's fixed point theorem; and led to intense effort on equilibrium algorithms: Kuhn '61, Mangasarian '64, *Lemke-Howson '64*, Rosenmuller '71, Wilson '71, Scarf '67, Eaves '72, Laan-Talman '79, *Porter et al. '04*, ... And all of these are exponentially difficult in the worst case scenario.

*The Lemke-Howson Algorithm*

The following is intense mathematical formulation, and will be explained in the following paragraph:

$$\sum_{k \in A_2} u_1(a_1^j, a_2^k) \cdot s_2^k + r_1^j = U_1^*, \forall j \in A_1 \tag{9}$$

$$\sum_{j \in A_1} u_2(a_1^j, a_2^k) \cdot s_1^j + r_2^k = U_2^*, \forall k \in A_2 \tag{10}$$

$$\sum_{j \in A_1} s_1^j = 1, \sum_{k \in A_2} s_2^k = 1 \tag{11}$$

$$s_1^j \geq 0, s_2^k \geq 0, \forall j \in A_1, \forall k \in A_2 \tag{12}$$

$$r_1^j \geq 0, r_2^k \geq 0, \forall j \in A_1, \forall k \in A_2 \tag{13}$$

$$r_1^j \cdot s_1^j = 0, r_2^k \cdot s_2^k = 0, \forall j \in A_1, \forall k \in A_2 \tag{14}$$

To better understand this math, we must first understand both $s$ and $r$. The $s$'s denote the mixed strategy used by the two players. For example, referring to equation 9, $s_2$ denotes the probability with which player 2 will decide to play action $k$ in their mixed strategy. The $r$'s denote the **slack** variables in these equations. To better understand, we refer to equation 9 again, this equation encompasses for any action of player 1 ($\forall j \in A_1$). For any action of player 1, we analyze the value that it gives with respect to the strategy of the other player. Thus, we look at all the action available for player 2 ($\sum_{k \in A_2}$), and we look at the *pay-off* to player 1 for playing a particular action $j$ ($u_1(a_1^j, a_2^k)$). Then we normalize the equation by the statistical probability of a strategy $a_2^j$ being enacted via $s_2^k$.

So far, we have the expected pay-off for play 1 for this particular strategy s.t. player 2 is playing a certain mixed strategy $s_2$. So we will have various pay-offs based on the assumptions of possibilities for all possible strategies, but what we want is *payer 1 to best respond to that strategy of player 2* (because in equilibrium, every player is best responding to the other player. So call $U_1^*$ the pay-off to player 1 in the Nash equilibrium. Typically, the best-response pay-off will be the supremum, whereas the summation encapsulates the pay-off of any action j. Introduce the slack variable $r$, and this is how much player 1 is missing relative to their best response when they're playing strategy $j$.

The conditions on each $s$ and $r$, make this the linear complementarity problem because each of the choices must be greater than or equal to 0 (and if the slack variable is 0, then we are in Nash equilibrium, *except* when strategies that are actually played with zero probability are played by the player). By this, if the probability of a certain strategy is greater than 0, then the slack variable must be 0, and when they're playing with a zero probability, you don't care what the slack variable is because they're not playing that strategy at all. These concepts also work the same way for player 2.

Along comes Lemke-Howson who suggest to find the solution in a manner (this is only a brief overview - not an in-depth explanation) where you initialize the $s$'s and $r$'s in a particular way (in fact, to artificially initialize them all to 0). Then, one-by-one, alternating between taking either $s$ or $r$ and removing them from the set that has the current value, finally replacing it with a complementary variable - until you reach an equilibrium. This is also known as the pivoting procedure.

*Support Enumeration Method: Porter et al. 2004*

**Step 1: Finding a Nash equilibrium with a specific support**

$$\sum_{a_{-1} \in \sigma_{-1}} p(a_{-i}) u_i(a_i, a_{-i}) = v_i, \forall i \in \{1, 2\}, a_i \in \sigma_i \tag{15}$$

$$\sum_{a_{-1} \in \sigma_{-1}} p(a_{-i}) u_i(a_i, a_{-i}) \leq v_i, \forall i \in \{1, 2\}, a_i \notin \sigma_i \tag{16}$$

$$p_i(a_i) \geq 0, \forall i \in \{1, 2\}, a_i \in \sigma_i \tag{17}$$

$$p_i(a_i) = 0, \forall i \in \{1, 2\}, a_i \notin \sigma_i \tag{18}$$

$$\sum_{a_i \in \sigma_i} p_i(a_i) = 1, \forall i \in \{1, 2\} \tag{19}$$

With these equations, we're looking for a mixed strategy that gives one strategy for each of the players that will give a Nash equilibrium (with the best-response which is denoted by $v_i$ for a particular action $i$). The other summation/boundaries indicate an action that will be no greater than the best response. Notice that here, we're fixing the support (we only use it in the assumption, but never in the equation itself). So we play with $p_i$'s of positive probability because if we use the remaining strategy with zero probability, it doesn't matter if we're best responding (this is a natural and hidden assumption of this).

**Step 2: Smart heuristic search through all sets of support**.

In this idea, we assume that the players have the same number in actions, s.t. randomization of either player choosing a single action is the same. Then the next step is after equating supports, we then start with small supports and then gradually work up to larger supports for certain actions. The complexity of the problems as we deal with more people, more actions, and more pay-offs, grows exponentially into a "hard" problem (explained in section 4 and the basics of NP-hardness).

# 6 Example: Mixed Strategy Nash

Take an example of mixed strategy being a soccer penalty kick. Then in this game, the mixed strategy is evidently used in order to throw off that player's opponent. Mixed strategies are heavily used in sports and competitive games because the player wants to be unpredictable. This raises the question: *how do equilibrium strategies adjust to skills?* And in the case of this specific game, should a kicker who kicks penalty kicks worse to their right than left, kick more often to the left more than the right? Given is the correlated standard matrix:

| $Kicker/Goalie$ | $Left$ | $Right$ |
|---|---|---|
| $Left$ | $0,1$ | $1,0$ |
| $Right$ | $1,0$ | $0,1$ |

*Figure 7*

Thus, clearly in mixed strategy, the best response is 50/50 for both players. But suppose the kicker is worse at kicking to the left comparative to the right s.t. the following standard matrix forms:

| $Kicker/Goalie$ | $Left$ | $Right$ |
|---|---|---|
| $Left$ | $0,1$ | $1,0$ |
| $Right$ | $.75,.25$ | $0,1$ |

*Figure 8*

Now mathematically set this up s.t. $p$ correlates with the kicker kicking to their left, and $1-p$ correlates with the kicker kicking to their right. Then by solving for $p$ (given in the equations above), we find that $p = \frac{4}{7}$ so that the player has a higher probability at making the goal if they kick to their left. If we set this up s.t. $q$ correlates with the goalie diving to the left, and $1-q$ correlates with the goalie diving to the right. Then, by again using the equations above, we find that $q = \frac{4}{7}$, where the goalie has a higher probability at saving the goal if the player kicks to the right. So the Nash equilibrium is $(\frac{4}{7}, \frac{3}{7})$ for the kicker, and $(\frac{4}{7}, \frac{3}{7})$ for the goalie, because the goalie is more likely to catch the ball if they're prepared for being kicked to the right.

In a mixed equilibrium, the goalie's strategy must have the kicker indifferent. The goalie then goes right more than left, and kicker still goes left and right with equal probability (due to what was explained above about the Nash equilibrium of this problem). Hence, *the goalie's strategy adjusts*, and the kicker actually adjusts to kick more to their weak side!

The goalie now has a slight advantage, and wins $\frac{4}{7}$ of the time. If the goalie still played equal probability, then the kicker could always go left and win $\frac{1}{2}$ the time instead of $\frac{3}{7}$. By adjusting the strategy to keep the kicker indifferent, the goalie takes advantage of the kicker's weak right kick and wins more often (interesting comparative stats)!

## 7 Data: Professional Sports and Mixed Strategies

Refer to video on Coursera for more explicit detail, because I will be keeping this part brief. Basically, a guy named Ignacio Palacios-Heurta in 2003 recorded 1417 penalty kicks from FIFA (soccer) games. He measured many more variables, but for sake of simplicity, we'll be working with only kicker/goalie moving right or left.

| Kicker/Goalie | Left | Right |
|---|---|---|
| Left | .58, .42 | .95, .05 |
| Right | .93, .07 | .70, .30 |

*Figure 9*

And from here we can actually compute the Nash equilibrium of mixed strategy, and we find that:

| | Goalie Left | Goalie Right | Kicker Left | Kicker Right |
|---|---|---|---|---|
| Nash Freq. | .42 | .58 | .38 | .62 |
| Actual Freq. | .42 | .58 | .40 | .60 |

*Figure 10*

Thus, we find that players randomize pretty well over time. There's many other factors of pressure, or whether or not players become predictable over time. When involving mixed strategy equilibria, we do see randomization making it significantly harder to predict, but this is evident in sports, in nature (which way a squirrel runs to get out of your way), and in society/business interactions (audits by tax authorities).