Game Theory Notes (Week 3)

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1 Beyond the Nash Equilibrium

All you really need to know about this section is there are deeper ideas that lie behind the numbers in the Nash equilibrium (which if you're following, should make complete sense). Then there's also various other scenarios of games dealing with Nash equilibrium. This is kind of a warm-up to solution concepts in next videos to prepare you for ideas outside of just Nash equilibrium.

2 Strictly Dominated Strategies & Iterative Removal

"Rationality"

A basic premise is that players maximize their payoffs. This isn't necessarily due to greed, etc. but maybe someone is altruistic, or believes in a different idea/concept. Then this person is going to have a higher valuation in the Nash equilibrium which actually makes them more predictable. What if all players know this? And they know that other players know it? And they know that other players know that they know it? Etc.

Strictly Dominated Strategies

A strictly dominated strategy can never be a best reply. By this knowledge, let us remove these various actions associated with the dominated strategy because clearly it won't be played. Assuming that all players are rational, then all players know this - so let us iterate. We'll iterate, removing actions that can be strictly dominated by other actions. Running this process to termination is called **iterated removal of strictly dominated strategies**. Thus, a strategy $a_i \in A_i$ is strictly dominated by $a_i' \in A_i$ if

$$u_i(a_i, a_{-i}) < u_i(a_i', a_{-i}), \forall a_{-i} \in A_{-i} \tag{1}$$

Now we'll dive into an example to better understand the nuances of how using this method works. Say our standard matrix for some game is:

	L	C	R
U	$3,0$	$2,1$	$0,0$
M	$1,1$	$1,1$	$5,0$
D	$0,1$	$4,2$	$0,1$

Figure 1

If we look carefully, we can see that R is strictly dominated by C s.t. the payoff for the player choosing C will always be better than choosing R for that specific player. So we can now eliminate R:

	L	C
U	$3,0$	$2,1$
M	$1,1$	$1,1$
D	$0,1$	$4,2$

Figure 2

Iterating by the logic from above, for player 2 M is strictly dominated by U, giving us:

	L	C
U	3, 0	2, 1
D	0, 1	4, 2

Figure 3

Again, we see that L is strictly dominated by C, resulting with:

	C
U	2, 1
D	4, 2

Figure 4

Finally, it's clear here that for both players, U is strictly dominated by D, giving:

	L	C	R
U	3, 0	2, 1	0, 0
M	1, 1	1, 1	5, 0
D	0, 1	4, 2	0, 1

Figure 5

So by using this iterative process, we can determine (by our given theorems) that we conclusively have a unique Nash equilibrium, which is also the best response for both players. This process identifies a unique predicted play which coincides with the only (unique) Nash equilibrium.

Looking at another (near identical) example, we're given the following standard matrix:

	L	C	R
U	$3,1$	$0,1$	$0,0$
M	$1,1$	$1,1$	$5,0$
D	$0,1$	$4,1$	$0,0$

Figure 6

Notice that we can remove R once again using the IRoSDS method, but then we're caught up by

	L	C
U	$3,1$	$0,1$
M	$1,1$	$1,1$
D	$0,1$	$4,1$

Figure 7

Circled in red are the characteristics that will be used to eliminate the next option. If players are willing to randomize, then say we assign U with a 50%

chance of choosing, and D with 50% as well. Then if we calculate the payoff of of choosing only either U or D, we're given the expected payoff of $(1.5, 1)$ if player 1 chooses L, and $(2, 1)$ if player 1 chooses C. Either way, this dominates M. So we say that M is dominated by the mixed strategy that selects U and D with equal probability. Giving:

	L	C
U	$3, 1$	$0, 1$
D	$0, 1$	$4, 1$

Figure 8

So at this point there are no other strategies that are strictly dominated. The Nash equilibria of this specific game would be an infinite number of solutions because this correlates with the matching pennies problem shown before.

This process *preserves Nash equilibria*. This can, most definitely, be used as a **preprocessing step** before computing an equilibrium. Some games are solvable using this technique - those games are then called **dominance solvable**. The *order of removal* when there are multiple strictly dominated strategies **does not matter** (think about this for a while, because it's not necessarily intuitive)!

Weakly Dominated Strategies

A strategy $a_i \in A_i$ is **weakly dominated** by $a'_i \in A_i$ if:

$$u_i(a_i, a_{-i}) \leq u_i(a'_i, a_{-i}), \forall a_{-i} \in A_{-i}, \text{ and} \tag{2}$$

$$u_i(a_i, a_{-i}) < u_i(a'_i, a_{-i}), \text{ for some } a_{-i} \in A_{-i} \tag{3}$$

We can also remove this iteratively too, **but**:

- They can be best replies,
- Order of removal *can* matter, and
- At least one equilibrium preserved.

Recall the Keynes Beauty Contest Game, could you solve it via iterative elimination of Weakly Dominated Strategies? Nope!

Summary

- Players maximize their payoffs (they don't play *strictly* dominated strategies, nor do they play *strictly* dominated strategies, given what remains.
- Nash equilibria are a subset of what remains.
- Do we see such behavior in reality?

3 Dominated Strategies & Iterative Removal: An Application

Feeding Behavior among Pigs and Iterative Strict Dominance

For the next example, we'll be analyzing a feeding behavior pigs, given the following situation. An experiment by B.A. Baldwin and G.B. Meese in 1979 from "Social Behavior in Pigs Studied by Means of Operant Conditioning" *Animal Behavior*, Vol 27, pp 947-957. (See also J. Harrington (2011) *Games, Strategies and Decision Making*, Worth Publishers).

In this study, two pigs are in a cage, one is larger which we'll call the "dominant" pig for sake of simplicity. In order to get the food, a single pig would have to go press lever to get food to arrive to the cage. The food and lever are on opposite sides of the cage for this study. So the way it would work is one pig would run to press the lever while the other pig would be the first to get the food (for better visual/audio explanation, visit the course on Coursera: video 3-3).

Say 10 units (or pellets) of food arrive per 1 pressed-lever. The typical split of food consumption is as follows:

- If large pig gets to food first then 1,9 split (1 pellet/unit for small, 9 for large).
- If small pig gets to food first then 4,6 split.
- If both pigs get to the food at the same time then 3,7 split.
- Pressing the lever (for food) costs 2 units of food in energy.

Then by condensing our qualitative data to quantitative data in terms of the Standard Matrix, we see:

$Small/Large$	$Press$	$Wait$
$Press$	$1,5$	$-1,9$
$Wait$	$4,4$	$0,0$

Figure 9

By the process of IRoSDS, we find that the Nash equilibrium is when the smaller pig waits, and the larger pig presses the lever:

$Small/Large$	$Press$	$Wait$
~~$Press$~~	~~$1,5$~~	~~$-1,9$~~
$Wait$	$4,4$	$0,0$

Figure 10

Pig Behavior: Frequency of pushing the lever per 15 minutes, after ten tests (learning...) Baldwin and Meese (1979)

Let's now take a look at how the pigs *actually* behaved during this study:

	$Alone$	$Together$
$LargePigs$	75	105
$SmallPigs$	70	5

Figure 11

The pigs performed just as we expected! Through the use of IRoSDS we can simplify the problem in order to find the Nash equilibrium, and it's clear that over a highly iterative process (such as this study), the pigs performed to how we expected: the larger pig would push the lever, while the smaller pig would wait and only eat a smaller portion of the food).

Iterative Strict Dominance

Are pigs rational? Well, we assume not. Do they know game theory? Wild guess, again probably not. But we do notice that the pigs tend to seem to learn and respond to incentives. The pigs were clearly learning due to their behavior over many iterated processes when analyzing their behavior, thus the pigs were learning to not play a strictly dominated strategy. Furthermore, the pigs were learning to not play strictly dominated strategies out of what remains. Learning, evolution, and survival of the fittest are some of the most power game theory tools that we can use to predict behavior.

4 Maxmin Strategies

Maxmin Strategies

Player i's **maxmin strategy** is a strategy that maximizes i's worst-case payoff, in the situation where all the other players (whom we denote $-i$) happen to play the strategies which cause the greatest harm to i. The **maxmin value** (or **safety level**) of the game for player i is that minimum payoff guaranteed by a maxmin strategy. We can the formulate the formal definition for maxmin as:

The **maxmin strategy** for player i is $\arg\max_{s_i} \min_{s_{-i}} u_i(s_1, s_2)$, and the **maxmin value** for player i is $\max_{s_i} \min_{s_{-i}} u_i(s_1, s_2)$.

*Definition **(Maxmin)***

Why would i want to play a maxmin strategy? It could be due to many reasons, but just for a couple examples:

- A conservative agent maximizing worst-case payoff.
- A paranoid agent who believes everyone is out to get him.

Minmax Strategies

Player i's **minmax strategy** against player $-i$ in a 2-player game is a strategy that minimizes $-i$'s best-case payoff, and the **minmax value** for i against $-i$ is payoff. We can, similarly, formulate the formal definition for minmax as:

In a two-player game, the **minmax strategy** for a player i against player $-i$ is $\arg\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$, and player i's **minmax value** is $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$.

*Definition **(Minmax)***

Why would i want to play a minmax strategy? Again, there's many different potential scenarios, but an example would be to punish the other agent as much as possible.

Minmax Theorem

According to von Neumann (1928), the Minmax theorem was; *in any finite, two-player, zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and his minmax value.* From this, we can take away three major points:

1. Each player's maxmin value is equal to his minmax value. The maxmin value for player 1 is called the **value of the game**.
2. For both players, the set of maxmin strategies coincides with a set of minmax strategies.
3. Any maxmin strategy profile (or, equivalently, minmax strategy profile) is a Nash equilibrium. Furthermore, these are all the Nash equilibria. Consequently, all Nash equilibria have the same payoff vector (namely, those in which player 1 gets the value of the game).

Saddle Point: Matching Pennies

Moving back to our example of the matching pennies game; graphically, we gain better intuition as to the payoffs, with the corresponding strategy spaces:

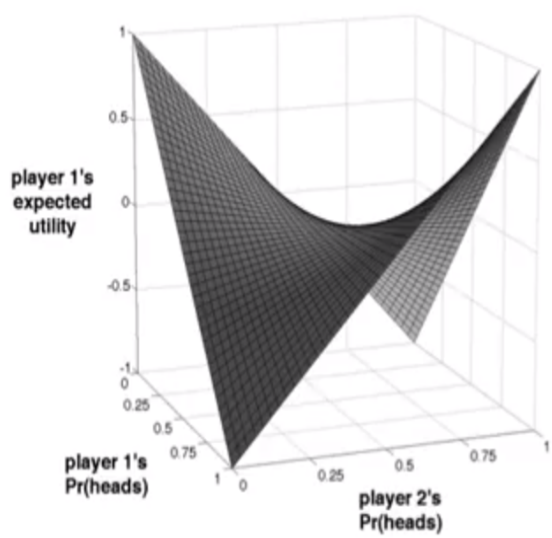

Figure 12

When we further analyze the point in which we reach Nash equilibrium, we take a slice at the point in which neither party is moving in the x nor the y axes. When looking at the 3-dimenional structure, this is clearly at 0.5 on the x-axis, 0.5 on the y-axis, and 0 on the z-axis:

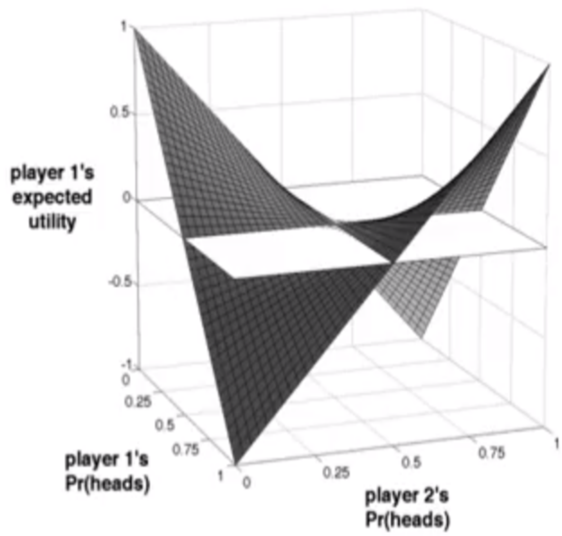

Figure 13

This is also called the saddle point in mathematical terms.

Computing Minmax

For 2 players, minmax is solvable with LP (Linear Programming); so the goal is to minimize U_1^*, which is subject to the following (don't worry, heavy math but there's an explanation at the end):

$$\sum_{k \in A_2} u_1(a_1^j, a_2^k) \cdot s_2^k \leq U_1^*, \forall j \in A_1 \tag{4}$$

$$\sum_{k \in A_2} s_2^k = 1 \tag{5}$$

$$s_2^k \geq 0, \forall k \in A_2 \tag{6}$$

U_1^* is the value of the game (i.e. the payoff to player 1 in equilibrium). In the equations above, we're specifying from player 2's point of view (this works either way). So for all actions that player 1 might consider, player 1 will look at all player 2's strategies for the optimal mixed strategy s_2. Thus, player 1's best response strategy will never exceed the value of the game (U_1^*) because we're trying to minimize this value. Thus, we're going to find the lowest value of U_1^* that player 1 doesn't have a preference as to which option they choose from their array of pure strategies A_1 (i.e. 50/50). Then we finally multiply the probability of player 2's mixed strategy play s_2^k. Computing minmax is, in fact, a linear program, which implies that this is efficiently solvable (explained in Week 2).

5 Maxmin Strategies - Advanced

2x2 Zero-Sum Games

Minmax or maxmin produces the same result as method for finding the Nash equilibrium in general 2x2 games; let's check this against the penalty kick game. Say we have the following Standard Matrix now for the penalty kick game:

		Goalie	
		L	R
Kicker	L	$0.6, 0.4$	$0.8, 0.2$
	R	$0.9, 0.1$	$0.7, 0.3$

Figure 14

Thus, we formulate the question: how does the Kicker maximize his minimum? And by using the equation that we derived directly above, we can apply this concept:

$$\max_{s_1} \min_{s_2} [s_1(L)s_2(L) \cdot 0.6 + s_1(L)s_2(R) \cdot 0.8 + s_1(R)s_2(L) \cdot 0.9 + ...] \tag{7}$$

Where the 0.6 comes from the Kicker's probability of making the goal when kicking to the left, while the Goalie dives to the left. The 0.8 is when the Kicker's probability of making the goal when kicking to the left, while the Goalie dives to the right, so on and so forth. Of course, this isn't a complete answer though, so: what is his minimum? Well, take

$$\min_{s_2} [s_1(L)s_2(L) \cdot 0.6 + s_1(L)s_2(R) \cdot 0.8 + s_1(R)s_2(L) \cdot 0.9 + ...] \tag{8}$$

And by simply rearranging the equation to be entirely in terms of kicking to the left, we see

$$= \min_{s_2} [s_1(L)s_2(L) \cdot 0.6 + s_1(L)(1 - s_2(L)) \cdot 0.8 + (1 - s_1(L))s_2(L) \cdot 0.9 + ...] \tag{9}$$

$$= \min_{s_2}[(0.2 - s_1(L) \cdot 0.4) \cdot s_2(L) + (0.7 + s_1(L) \cdot 0.1)] \tag{10}$$

Since this is in terms of s_2 where we want it in terms of s_1. First, take derivative of this function with respect to s_1 (to find s_1's minimum)

$$\implies 0.2 - s_1(L) \cdot 0.4 = 0 \tag{11}$$

$$\implies s_1(L) = \frac{1}{2}, s_1(R) = \frac{1}{2} \tag{12}$$

So now that we know the Kicker's minimum, how does the Goalie minimize the Kicker's maximum? Refer to equation 7, because we will use the exact same set-up since this is a game of pure competition. Then we can find what the Kicker's maximum is through the following,

$$\max_{s_1}[s_1(L)s_2(L) \cdot 0.6 + s_1(L)s_2(R) \cdot 0.8 + s_1(R)s_2(L) \cdot 0.9 + ...] \tag{13}$$

$$= \max_{s_1}[s_1(L)s_2(L) \cdot 0.6 + s_1(L)(1 - s_2(L)) \cdot 0.8 + (1 - s_1(L))s_2(L) \cdot 0.9 + ...] \tag{14}$$

$$= \max_{s_1}[(0.1 - s_2(L) \cdot 0.4) \cdot s_1(L) + (0.7 + s_2(L) \cdot 0.2)] \tag{15}$$

Once again, this in terms of both s_1 and s_2, where we only want it in terms of s_2. Thus, take the first derivative of this function with respect to s_2 (to find s_2's maximum).

$$\implies 0.1 - s_2(L) \cdot 0.4 = 0 \tag{16}$$

$$\implies s_2(L) = \frac{1}{4}, s_2(R) = \frac{3}{4} \tag{17}$$

This means that the randomization in equilibrium for s_2 is $\frac{1}{4}$ if the Goalie goes left, or $\frac{3}{4}$ if the Goalie goes right. This is definitely not intuitive, so review this concept extensively until you understand.

6 Correlated Equilibrium: Intuition

Examples

Consider again, the Date Night Game/Example. Intuitively, the best outcome seems a 50/50 split between (F, F), (B, B) (because typically couples are still going to want to go see the same movie together, rather than getting the payoff of 0, where they go to 2 different movies). But there's no way to achieve this, so either someone loses out (unfair) or both players often miscoordinate. If we take a different example, where there's different outcomes, let's create the Traffic Game.

Say two cars approach an intersection at the exact same time. Then each car has the decision to either go or to wait for the other player/car to go. If both go at the same time - there's a crash which is the worst outcome, and if both wait at the same time - there's a mild irritation for the other car not going. The Standard Form would be set up as

	go	wait
go	$-10, -10$	$1, 0$
wait	$0, 1$	$-1, -1$

Figure 15

Of course, the Nash equilibrium is for one player to go, and one player to wait. But, there's also opportunity for mixed strategy as well.

Intuition

What is the natural solution here? In today's world, it's a *traffic light*: a fair randomizing device that tells one of the agents to go and the other to wait. There's also evident benefits that come from this fair, randomizing device:

- The negative payoff outcomes are completely avoided.
- Fairness is achieved.
- The sum of social welfare can exceed that of any Nash equilibrium.

We could use this same idea to achieve the fair outcome in the Date Night Game, e.g. You flip a coin and predetermine what heads and tails represent in terms of action, but you'll both be going to a single movie together. This idea is known as a **Correlated Equilibrium** (informally) defined as a randomized assignment of (potentially correlated) action recommendations to agents, such that nobody wants to deviate. It's not as though you physically cannot go when at a stop light, it's just a very, *very* good recommendation to stop at the light s.t. you would not want to deviate. When flipping a coin, the coin isn't forcing you to go to a movie, but it's a randomized action recommendation s.t. neither you nor your partner would want to deviate.