Dawson Beatty October 6, 2018

Trilateration Equation and Error

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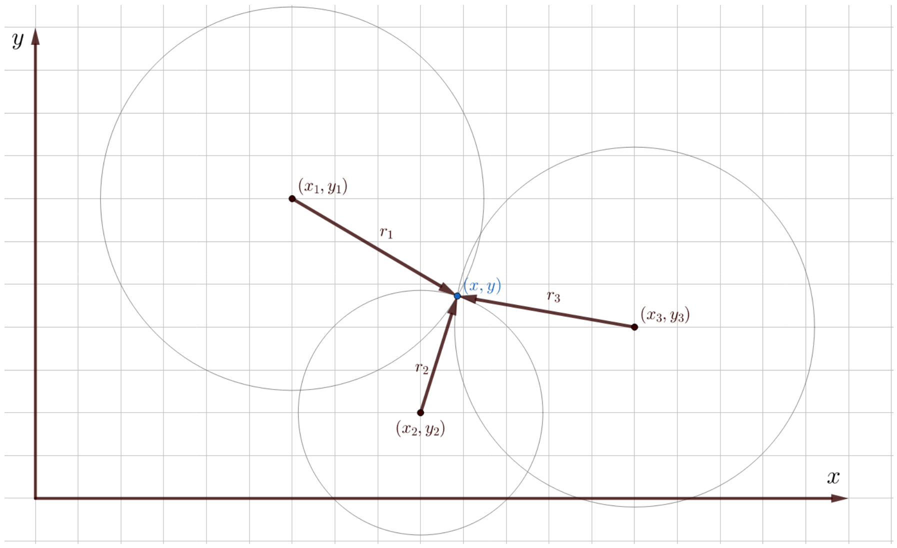
1 Basic Trilateration

Premise of trilateration is that there are three beacons from which the distance can be measured. Need at least three beacons for unique intersection.



Need to find point (x, y) which satisfies all of these equations:

$$(x - x_1)^2 + (y - y_1)^2 = r_1^2$$
$$(x - x_2)^2 + (y - y_2)^2 = r_2^2$$
$$(x - x_3)^2 + (y - y_3)^2 = r_3^2$$

Subtract 2 from 1, 3 from 2. Forms system of equations which can be solved.

$$-2\,x(x_1 - x_2) - 2\,y(y_1 - y_2) = r_1^2 - r_2^2 - x_1^2 + x_2^2 - y_1^2 + y_2^2$$
$$-2\,x(x_2 - x_3) - 2\,y(y_2 - y_3) = r_2^2 - r_3^2 - x_2^2 + x_3^2 - y_2^2 + y_3^2$$

$$x = \frac{(y_1 - y_3)y_2^2 - r_1^2 y_3 + r_2^2 y_3 + x_1^2 y_3 - x_2^2 y_3 + y_1^2 y_3 - \left(r_2^2 - r_3^2 - x_2^2 + x_3^2 + y_3^2\right) y_1 + \left(r_1^2 - r_3^2 - x_1^2 + x_3^2 - y_1^2 + y_3^2\right) y_2}{2\left((x_2 - x_3)y_1 - (x_1 - x_3)y_2 + x_1 y_3 - x_2 y_3\right)}$$
$$y = -\frac{(x_1 - x_3)x_2^2 - r_1^2 x_3 + r_2^2 x_3 + x_1^2 x_3 - (x_2 - x_3)y_1^2 + (x_1 - x_3)y_2^2 - \left(r_2^2 - r_3^2 + x_3^2 + y_3^2\right)x_1 + \left(r_1^2 - r_3^2 - x_1^2 + x_3^2 + y_3^2\right)x_2}{2\left((x_2 - x_3)y_1 - (x_1 - x_3)y_2 + x_1 y_3 - x_2 y_3\right)}$$

If we plug in some values:

$$\begin{aligned} x_1 &= 12 & y_1 &= 14 & r_1 &= 9 \\ x_2 &= 18 & y_2 &= 4 & r_2 &= 6.4 \\ x_3 &= 28 & y_3 &= 8 & r_3 &= 8.2 \end{aligned}$$

$$x = 20.07$$
$$y = 10.04$$

2 Error in Distance

Generally, uncertainty in a function f is given as:

$$\sigma_f = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 + \cdots}$$

We now consider some variance in the reported distances, which is given by the RF module:

$$\begin{aligned} r_1 &= r_1 \pm \sigma_{r1} \\ r_2 &= r_2 \pm \sigma_{r2} \\ r_3 &= r_3 \pm \sigma_{r3} \end{aligned}$$

The values for x and y above are given as a function of $x_1, x_2, x_3, y_1, y_2, y_3, r_1, r_2, r_3$: we'll call them g, h (same functions as above):

$$\begin{aligned} x_{\text{rover}} &= g(x_1, x_2, x_3, y_1, y_2, y_3, r_1, r_2, r_3) \\ y_{\text{rover}} &= h(x_1, x_2, x_3, y_1, y_2, y_3, r_1, r_2, r_3) \end{aligned}$$

The propogated errors in x and y are given:

$$\sigma_x = \sqrt{\left(\frac{\partial g}{\partial r_1}\right)^2 \sigma_{r1}^2 + \left(\frac{\partial g}{\partial r_2}\right)^2 \sigma_{r2}^2 + \left(\frac{\partial g}{\partial r_3}\right)^2 \sigma_{r3}^2}$$

Right now we only consider the range error, so all other σ values are zero.

$$\begin{aligned} \frac{\partial g}{\partial r_1} &= \frac{r_1 y_2 - r_1 y_3}{(x_2 - x_3)y_1 - (x_1 - x_3)y_2 + x_1 y_3 - x_2 y_3} \\ \frac{\partial g}{\partial r_2} &= -\frac{r_2 y_1 - r_2 y_3}{(x_2 - x_3)y_1 - (x_1 - x_3)y_2 + x_1 y_3 - x_2 y_3} \\ \frac{\partial g}{\partial r_3} &= \frac{r_3 y_1 - r_3 y_2}{(x_2 - x_3)y_1 - (x_1 - x_3)y_2 + x_1 y_3 - x_2 y_3} \end{aligned}$$

By applying the example values:

$$\begin{array}{ccc} x_1 = 12 & y_1 = 14 & r_1 = 9 \\ x_2 = 18 & y_2 = 4 & r_2 = 6.4 \\ x_3 = 28 & y_3 = 8 & r_3 = 8.2 \end{array}$$

$$\begin{aligned} \frac{\partial g}{\partial r_1} &= 0.2903 \\ \frac{\partial g}{\partial r_2} &= 0.3097 \\ \frac{\partial g}{\partial r_3} &= -0.6613 \end{aligned}$$

This value defines the confidence we will have in our positioning just from trilateration. If we have $\sigma_x = 1\text{m}$, we can be 67% that our position lies within one meter of the mean. If $2\sigma_x = 1\text{m}$ then 95% confident, $3\sigma_x = 1\text{m}$ means 99.7%.

Let's test some values:

$$\sigma_{r1} = .45 \qquad \sigma_{r2} = .41 \qquad \sigma_{r3} = .32$$

$$\sigma_x = 0.6337 \qquad \sigma_y = 0.9010$$

3 Pod Placement Error

It gets worse! There will be some error in the placement of the pods as well:

$$\begin{aligned} \sigma_{x1} &= 0.5 & \sigma_{y1} &= 0.5 \\ \sigma_{x2} &= 0.5 & \sigma_{y2} &= 0.5 \\ \sigma_{x3} &= 0.5 & \sigma_{y3} &= 0.5 \end{aligned}$$

Each of the above (somewhat arbitrary values) means that we're 68% confident that we can get the pod within 0.5 meters of the desired location in each direction, or 95% confident that it's within a meter.

The partials here are really long so I won't print them, but now we're considering $\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3}, \frac{\partial g}{\partial y_1}, \frac{\partial g}{\partial y_2}, \frac{\partial g}{\partial y_3}$ in addition to the trilateration errors.

$$\sigma_x = \sqrt{\left(\frac{\partial g}{\partial r_1}\right)^2 \sigma_{r1}^2 + \left(\frac{\partial g}{\partial r_2}\right)^2 \sigma_{r2}^2 + \left(\frac{\partial g}{\partial r_3}\right)^2 \sigma_{r3}^2 + \left(\frac{\partial g}{\partial x_1}\right)^2 \sigma_{x1}^2 + \left(\frac{\partial g}{\partial x_2}\right)^2 \sigma_{x2}^2 + \left(\frac{\partial g}{\partial x_3}\right)^2 \sigma_{x3}^2 + \left(\frac{\partial g}{\partial y_1}\right)^2 \sigma_{y1}^2 + \left(\frac{\partial g}{\partial y_2}\right)^2 \sigma_{y2}^2 + \left(\frac{\partial g}{\partial y_3}\right)^2 \sigma_{y3}^2}$$

$$\sigma_x = 0.7453 \qquad \sigma_y = 1.073$$