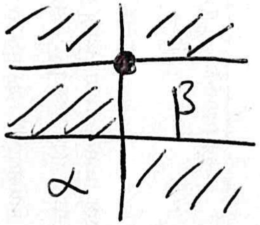
Av(231)

A non empty perm looks like

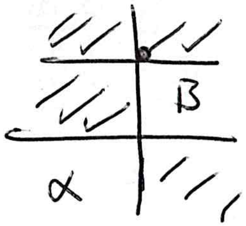


where $\alpha, \beta \in Av(231)$

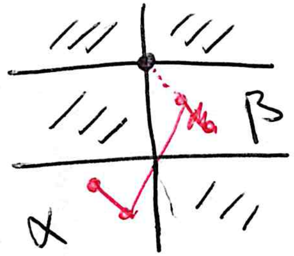
Consider a finitely based subclass of $Av(231)$.

[In fact, they all are finitely based but this uses facts I haven't shown yet]

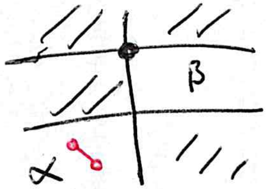
Ex $Av(231, 2143)$



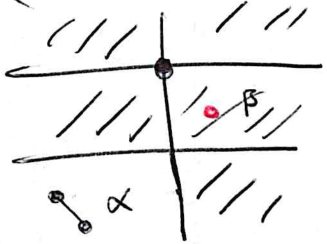
$\alpha, \beta \in Av(231, 2143)$ but additionally need $\alpha \cap \beta$ to avoid 2143.



Can consider whether the bottom left is increasing or contains 21.



so $\alpha \in Av(21)$
$\beta \in Av(231, 2143)$



so $\alpha \in Av(231, 2143) \cap Co(21)$
$\beta = \varepsilon$.

Let F be g.f for $Av(231, 2143)$

$$F = 1 + \frac{x}{1-x} \cdot F + x \cdot \left(F - \frac{1}{1-x} \right)$$

solving gives

$$F = \frac{1-2x}{1-3x-x^2}$$

This case analysis can be done for an arbitrary set of patterns.

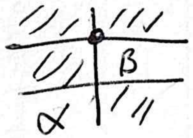
We just need to consider all ways to write $\sigma = \alpha_1 \oplus \alpha_1$ — sum

If $\pi \in Av(231)$ is $\pi = \alpha_1 \ominus \alpha_2$

then $\alpha_1 \in Av(21)$

Moreover π starts with n.

These are enough to show that ~~the~~ an proper subclass~~es~~ C are rational, since each



must have either α, β in a proper subclass of C.

For one pattern Mansour and Vainshtein 2001/02.

Then by induction all equations of form

- ~~…~~ $R_1 \circ R_2$ where R_1, R_2 are rational

or

- $R_1 \circ F$

(Base case $Av(1)$ g.f is 1 which is rational...)

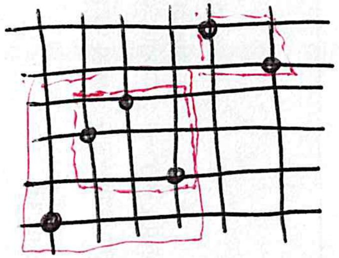
This kind of case analysis is going to be at the heart of the algorithms we discuss today.

Simple permutations

An *interval* in a permutation is a set of consecutive indices with consecutive values.

Ex. 134265

Geometrically, can draw a $k\times k$ box around the points.



Every perm of length n contains the *trivial* length 1 and n intervals. Every other we call *proper*.

~~A perm of length ≥ 2 is simple if it contains~~

A perm is *simple* if it contains no proper intervals.

Ex. $\{1, 12, 21, 3142, 2143, \ldots\}$

Count: $1, 2, 0, 2, 46, 338, \ldots$

$\sim \frac{n!}{e^2}$ (Albert, Atkinson and Klazar)

Not P-recursive

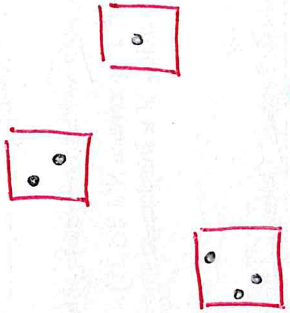
(no linear rec with poly coeffs)

[We will from now on only consider perms simple if length ≥ 2]

The inflation of $\pi[\alpha_1, \ldots, \alpha_k]$ is the perm obtained by replacing $\pi(i)$ with α_i.

Example

$2413[12, 1, 312, 21]$



$= 45\ 8\ 312\ 76$.

We ~~write~~ treat inflations of length 2 specially:

The sum $\alpha_1 \oplus \alpha_2 = 12[\alpha_1, \alpha_2]$

if it is a sum, we say it is sum-decomposable [else sum-indec]

For every perm

$\pi = \alpha_1 \oplus \alpha_2 \oplus \ldots \oplus \alpha_n$ for unique sum indecomposable perms.

(the sum components)

(similar for skew  $\pi_1 \ominus \pi_2$)

Thm (Albert and Atkinson)

Every perm π length ≥ 2 can be written as inflation

$$\pi = \sigma[\alpha_1, \ldots, \alpha_n]$$

where σ is unique.

If length $\pi \geq 4$ α_i unique

if length $\pi = 2$ ~~then~~ and α_i sum/skew indecom ($\overset{12}{\text{sum}}$/$\overset{21}{\text{skew}}$)

then it is unique.

Proof (skip?)

If A and B are maximal proper intervals with non-emty intersection then union must be whole perm.

So index/values of A and B not interior of $[1, \ldots, n]$

Therefore either $12[\alpha_1, \alpha_2]$ or $21[\alpha_1, \alpha_2]$.

If α_i indecomposable then unique.

Other case, maximal proper intervals are disjoint.

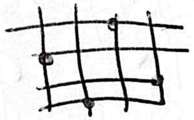
Maximality $\Rightarrow$ pattern is simple, and uniquely determined

Thm (Albert and Atkinson/Schmerl and Trotter)

Every simple π can delete a single point to get simple or it is exceptional

Here exceptional means

2 4 6 ... 1 3 5 ...



and so on.

and its symmetries.

A consequence is that

Corollary

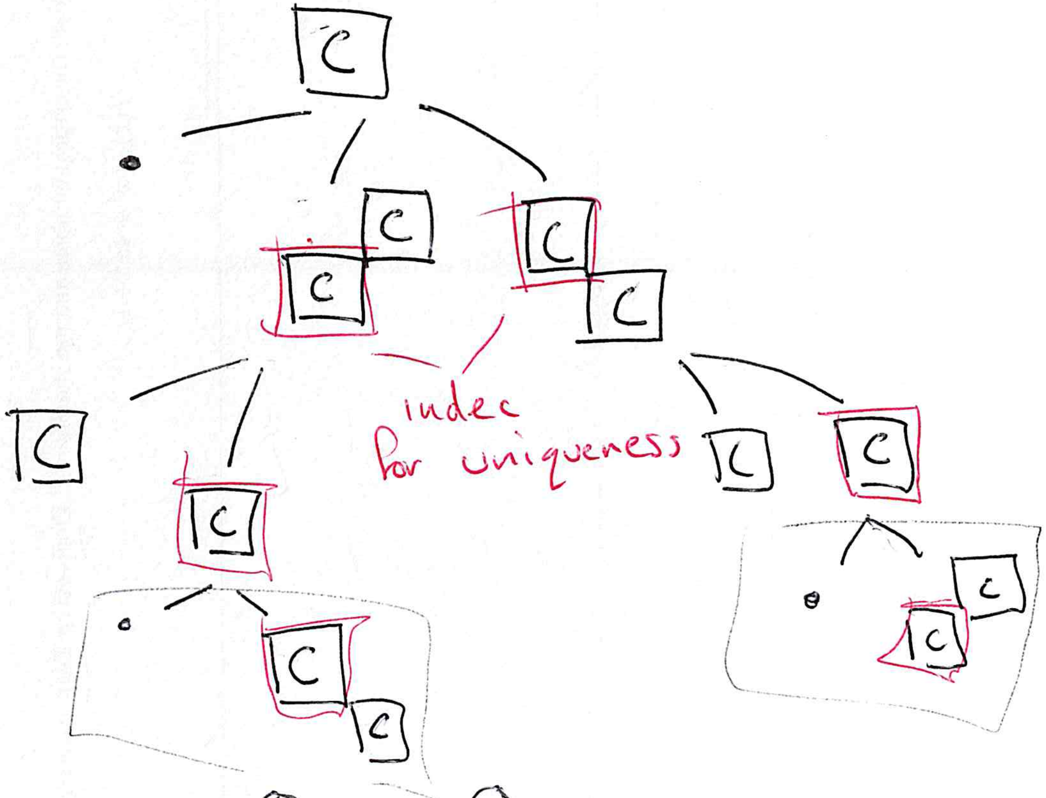
Every simple perm contains a subperm that is simple of length $n-1$ or $n-2$.

$\Rightarrow$ Every simple $n \geq 4$ contains one of 3142 or 2413.

Consider $Av($~~[illegible]~~$, 2413, 3142)$

which contains only 12 and 21 that are simple.

Then it is either 1, sum dec or skew dec.



$$F = x + F^{\oplus} + F^{\ominus}$$

$$F^{\oplus} = F^{\not\oplus} \cdot F, \quad F^{\ominus} = F^{\not\ominus} \cdot F$$

$$F^{\not\oplus} = x + F^{\ominus}, \quad F^{\not\ominus} = x + F^{\oplus}$$

Using this result, and following from a theorem of Higman, we get that

Theorem (Murphy 2002 PhD Thesis, Albert and Atkinson 2005)

Any class with finitely many simple perms is finitely based.

~~This result also gives an effective algorithm, fully described by~~

This also gives an algorithm for computing the simple perms if it is finite.

If you generate them brute force and ever have 0 of length n and $n+1$, you know you have them all.

(3.5)

A much more effective procedure exists!

Thm (Pierrot and Rossin)

There is an algorithm that computes the simple perms which is polynomial in the number of simples in the class.

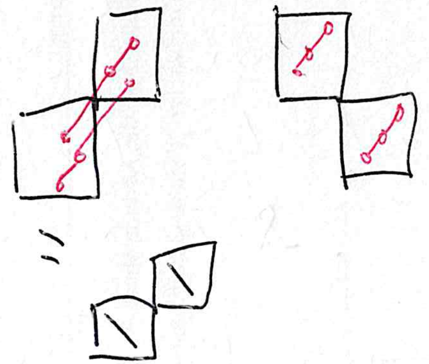
Solving gives

$$F = \frac{1 - x - \sqrt{1-6x+x^2}}{2}$$

which genf for Large Schröder numbers

If we avoided an extra pattern we could do a similar case analysis as before

Ex $Av(123, 2413, 3142)$



It follows that

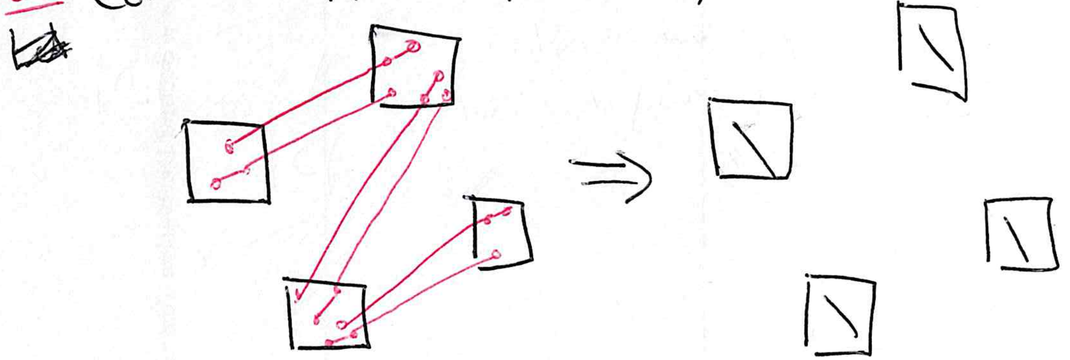
Thm

Any class with no simples has an algebraic generating function.

[brushing some details under the carpet]

What if we contain 3142, then may contain inflations

Ex Consider $Av(123, 2413, 531642)$



By earlier thm simples ar 12, 21 and 3142
can write down eqs

$$F = x + F^{\oplus} + F^{\ominus} + F^{S}, \quad F^{S} = \left(\frac{x}{1-x}\right)^4, \quad F^{\oplus} = \left(\frac{x}{1-x}\right)^2 \quad F^{\ominus} = F^{\oplus} \circ F$$

$$F^{\ominus} = x + F^{\oplus} + F_{S}.$$

solving gives

$$F = \frac{x - 3x + 4x^2 - 2x^3 + x^5}{1 - 5x + 9x^2 - 8x^3 + 3x^4 - x^5}$$

This idea can be used to give an algorithm for enumerating any class with finitely many simples.

Thm (Albert and Atkinson) (2005)
The generating function of every class with finitely many simples is algebraic.

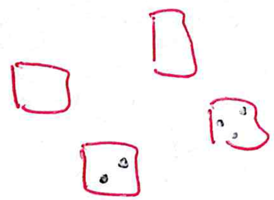
This algorithm I am about to present was implemented by Bassino, Bouvel, Pierrot, Pivoteau and Rossin, 2017

but the one I show is ambiguous, as I will use the inc-exclusion

They gave a non-ambiguous version which I'll discuss briefly at the end - and is your homework to implement!

To do this we use embeddings, which are essentially inflation but we allow the empty permutation.

~~e.g.~~ Ex $3142[\varepsilon, 12, \varepsilon, 213] =$



we say this is an embedding of 12435 in 3142.

6

To compute the inflations of σ in $Av(B)$ we do the following

① Compute the set of all embeddings of each $\pi \in B$ in σ.

② If all embeddings have at most one no empty perm go to step 4. else

③ Pick an embedding of in σ say $(\pi_1, \ldots, \pi_k)$ with at least two non-empty, let π_i be one.

Either we avoid $(\ldots, \pi_{i-1})$ ~~or~~ so we remove it
or we contain it, so we avoid $(\ldots, \pi_{i-1}, \varepsilon, \pi_{i+1}, \ldots)$

④ All have at most one (with containment conditions in cells) so we are cartesian product of what is in cells.

To compute we need to find gen f of $Av(B) \cap Co(\pi_1) \cap Co(\pi_2) \cap \ldots$

So do

$Av(B) \cap Co(\pi_2) \cap \ldots \setminus Av(B \cup \{\pi_1\}) \cap Co(\pi_2)$

etc

The classes in these cells are either subclasses or the class itself, so we get an ~~system~~ equation $F = x + p(\underbrace{F, F_1, \ldots, F_m}_{})$ which we solve for alg g.f. (g.f of subclasses) ← strong from subpattern so induc

If we want a non-ambiguous version we need to enumerate

$$Av(B) \cap Co(\pi_1) \cap \ldots \cap Co(\pi_k)$$

We still have to avoid ~~[illegible]~~ embeddings of B, but must contain at least one embedding of π_1 in σ, of π_2 in σ.

To do this you need a data structure of form

{ set of embeddings to avoid
and set of sets to contain.

Can still do case analysis until all embeddings are local

nothing

if you contain

$\{\pi_1, \ldots, \pi_k\}$ then contain each $\{\pi_1, \varepsilon, \ldots\}$, $\{\varepsilon, \pi_2, \varepsilon, \ldots\}$ etc

(and many other details)

The obvious question, how do I know if a class has finitely many simples?

Luckily

Theorem (Brignall, Ruškuc and Vatter, 2008)

It is decidable if a finitely based class contains infinitely many simple permutations.

The proof relies on the following

Theorem (Brignall, Huczynska, Vatter, 2008)

Every sufficiently long simple contains one of , $<|\cdot$, $V|\cdot$ or a proper pin sequence.

and syms

these are perm classes

I won't discuss the pin sequences, it is where the complexity comes in.

Fortunately, Bassino, Bouvel, Pierrot and Rossin give an algorithm:

Thm (2015)

For a finite set B, there is an algorithm that decides if $Av(B)$ has a finite number of simples in

they give it!

$$O(\underbrace{n}_{\sum_{\pi \in B}|\pi|} \log n + \underbrace{s}_{\text{maximal size of pin perm}}^{\overbrace{2k}^{\text{number of pin perms}}}) \quad \left[\text{or } O(n \log n + \underbrace{p}_{p = \prod_{\pi \text{ is a pin perm}} |\pi|}^{2})\right]$$

polynomial time.