Group 9: Simulation Over Three Major Indexes

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Introduction

In 2021, 56 percent of American adults, which equals 145 million people, invested their assets in the stock market. As more and more people tend to start investing in the stock market, numerous risk variables have also arisen naturally. Due to high volume which led to high volatility, investors nowadays become more cautious in terms of estimating the risk and the return before they buy stocks and determining whether the particular stock is suitable for a short term or long term.

To maximize their returns and to minimize their risk or to determine if the investors are investing it for a short term or long term, there are numerous difficulties such as lack of data and the uncertainty of economic flows which are the two most concerning factors for investors. To overcome these obstacles, we will conduct Monte Carlo simulation, bootstrap, and permutation tests over different sample sizes.

The index funds we will be simulating data for are the S&P 500, Nasdaq, and Dow Jones: the best three gauges of large United States stocks, and even the entire equities market. Our parameters of interest for these distributions are the mean, median, and variance, as these correspond to measures of potential return and risk. We are also interested in the distributions of these indexes e.g. their skewness and outliers.

Generate Data

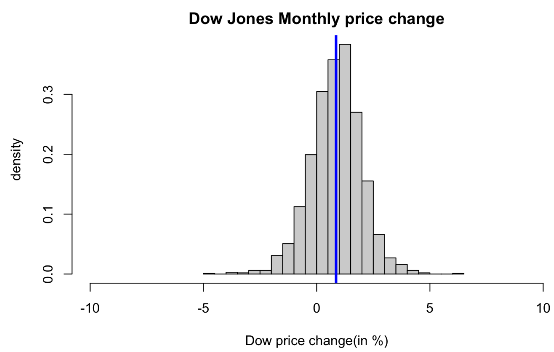
Parameters

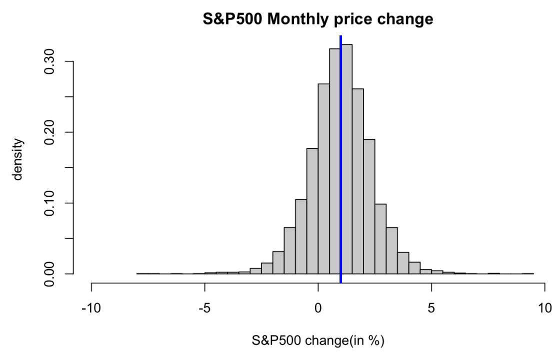
In order to create simulations of the monthly price changes of the stock indexes, we first have to understand a few key features of the indexes. First of all, the distribution is roughly symmetric, the probability of out-performing the average market performance is roughly the same as the probability of under-performing. Secondly, the stock market is full of surprises: the probability of having a price change of way above or below the mean could not be neglected. Thirdly, in the past few decades, the overall market return is positive, which means that the monthly price change should be centering at a slightly positive value. Based on those presumptions, we decided to use t distributions to simulation the three indexes. We used a t distribution with a df=7 instead of a standard normal distribution to generate the simulation because we believe that the tail of a t distribution fits better to the "relatively extreme" values of the real world monthly price change. After creating the t distributions, we apply the scaling factor s and the center m to adjust our distributions to make it more aligned with the real world data. The values of s and m are roughly calculated from the overall index performances.

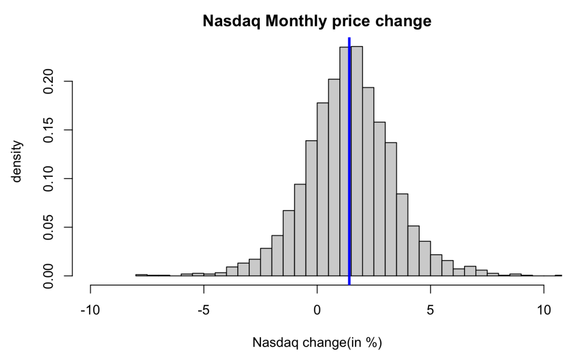
Indexes	m	s
Dow Jones	0.967	0.850
S&P500	1.153	1.000
Nasdaq	1.692	1.411

Histograms

With the parameters, we able to generate data of any data size. In order to see a better overall picture of the distributions, we used a large data size of 10000 to create the histograms for each indexes:







As shown in the histograms, we can see that the Dow Jones index has the smallest spread/variance in price changes; while the Nasdaq index has both the highest mean and variance among the three distributions. The means of all three distribution are all greater than zero, which indicates an average positive return in investment.

Choosing the weight parameter

When we are choosing our overall portfolio, it is unclear that what weight proportion we should be using for the three indexes. For this reason, we would be applying the Monte

Carlo method to find out the key parameters: mean, median, and standard deviations of the portfolio under different weight distributions. Since we want to diversify our portfolio, we would set a weight boundary on all individual index to be in between 10% and 60%. We selected six different combinations as the following:

Weight (Dow, S&P 500, Nasdaq)	Mean	Median	SD	Median-SD
(0.20, 0.20, 0.60)	1.233	1.215	1.749	-0.534
(0.20, 0.60, 0.20)	0.990	0.980	1.443	-0.463
(0.60, 0.20, 0.20)	0.996	0.954	1.416	-0.462
(0.33, 0.33, 0.34)	1.108	1.056	1.556	-0.500
(0.10, 0.40, 0.50)	1.165	1.118	1.682	-0.564
(0.40, 0.10, 0.50)	1.097	1.039	1.637	-0.599

Since the mean and the median represents the average ”return” of the month, and the standard deviation could be used to measure the fluctuations or ”risk” of the portfolio, we would want to find the portfolio that maximizes the return and minimizes the risk. However, this is impossible. From the table, we can see that there is a positive relationship between the mean and standard deviation. This could also be explained in the real world investment as higher profit comes with greater risk. Therefore, we would have to find our optimized balance between return and risk. Using a linear computation, we calculated the difference between median and standard deviation, and points out that the 60% Dow Jones, 20% S&P 500, and 20% Nasdaq combination is the linearly optimized portfolio. Therefore, this would be used as our combined t-mixture distribution.

Bootstrap

Suppose we are given a sample of monthly growth rates, how much uncertainty is there in the sample mean and sample variance? We will investigate this question using bootstrapping methods, where there is the assumption that we do not know the distribution of the sample data.

We will conduct bootstrapping for each distribution and compare the distribution and properties of their estimators at various sample sizes. Both empirical and non-parametric bootstrap methods will be used in order to compare the two.

Empirical Bootstrap

We first conduct the empirical bootstrap with $B = 10,000$ bootstrap samples. The samples to be used in the bootstrap are generated from the densities of each index.

With a small sample size ($n = 50$), we see that the distribution of sample means for each index is approximately normal, while the distribution of sample variance is right-skewed. We will also look at how these distributions change for $n = 500$ and $n = 1500$. For the sake of brevity, and due to the distributions being similar, only the histograms for Dow Jones will be shown.

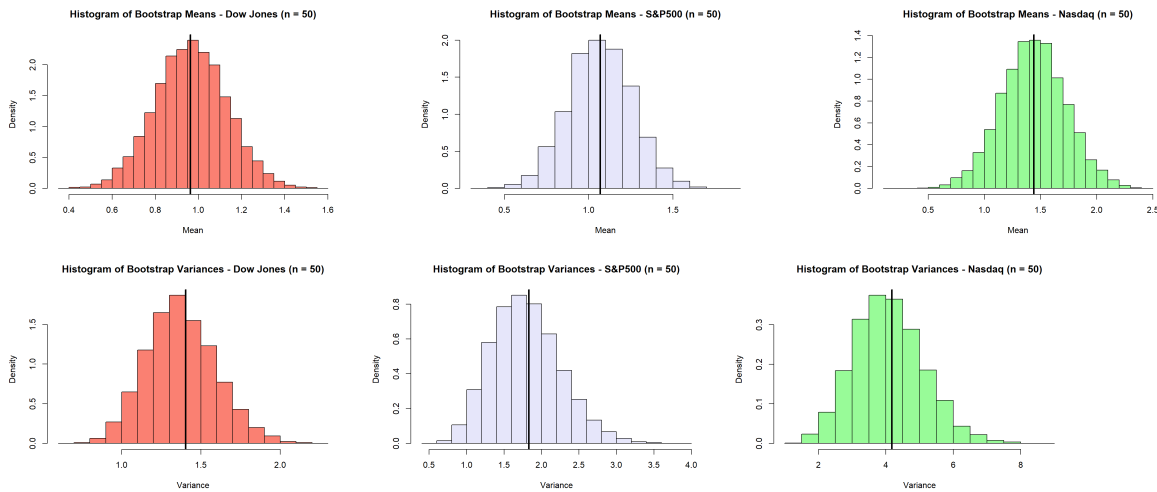


Figure 1: Histograms of the bootstrap sample means (top) and variances (bottom) for each distribution at $n = 50$. A vertical line is drawn at the location of the original sample's mean/variance.

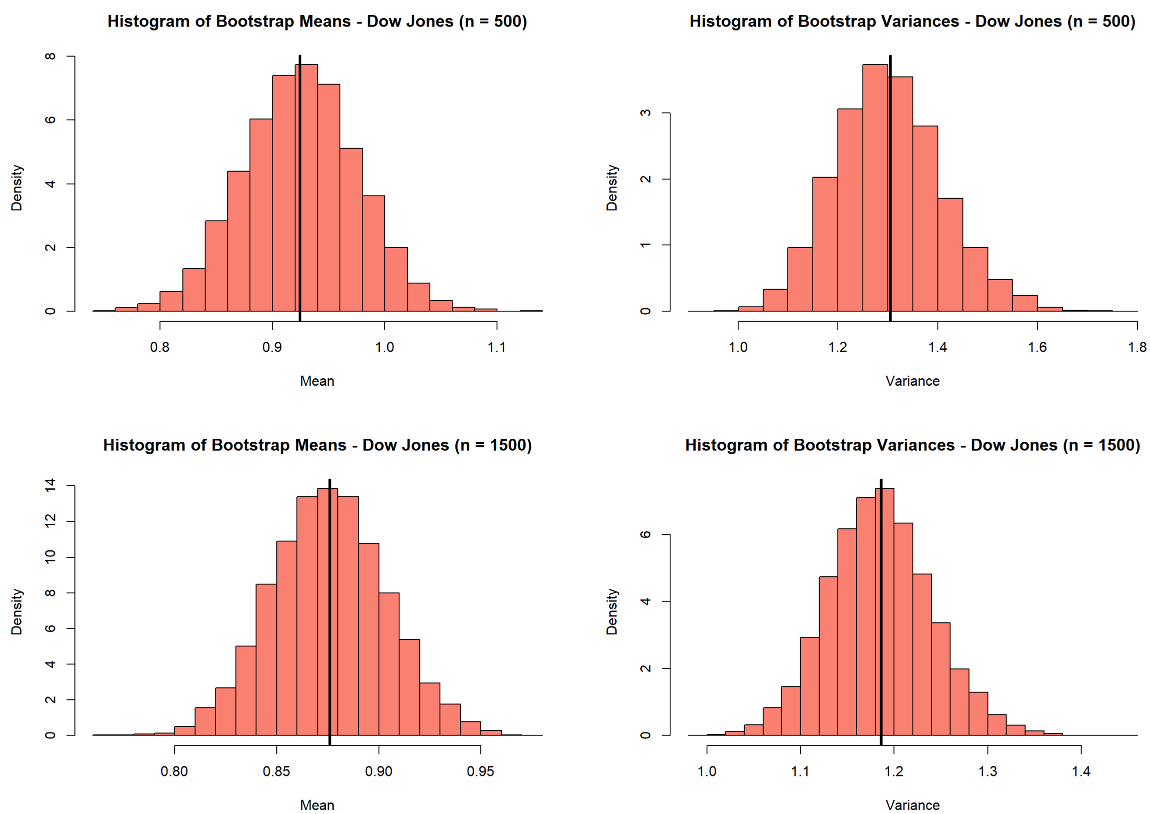


Figure 2: Histograms of bootstrap sample means (left) and variances (right) for Dow Jones, $n = 500, 1500$.

n	Index	CI	SE	Var	MSE	Bias	IQR
50	Dow Jones	[0.63809941583452, 1.29002468368498]	0.1659904	0.0275528	0.0275578	5.0e-06	0.2248776
	S&P 500	[0.702829859274412, 1.43739748761729]	0.1890382	0.0357354	0.0357321	-3.4e-06	0.2543077
	Nasdaq	[0.873593670971209, 2.01482019382601]	0.2876480	0.0827414	0.0827384	-2.9e-06	0.3860031
500	Dow Jones	[0.825740969259079, 1.02376882474241]	0.0508095	0.0025816	0.0025817	1e-07	0.0685148
	S&P 500	[0.79260424441335, 1.04416753035055]	0.0640358	0.0041006	0.0041004	-2e-07	0.0869131
	Nasdaq	[1.20061267538245, 1.53552516191457]	0.0851492	0.0072504	0.0072504	0e+00	0.1139676
1500	Dow Jones	[0.821269047751687, 0.931292716764252]	0.0279265	0.0007799	0.0007800	1e-07	0.0379749
	S&P 500	[0.895338034883729, 1.03214452130062]	0.0344494	0.0011868	0.0011867	0e+00	0.0462566
	Nasdaq	[1.27613103420144, 1.48108906474497]	0.0524637	0.0027524	0.0027525	1e-07	0.0711720

n	Index	CI	SE	Var	MSE	Bias	IQR
50	Dow Jones	[0.972111354335066, 1.82950536674124]	0.2188031	0.0478748	0.0486764	0.0008016	0.2971009
	S&P 500	[1.00192235090088, 2.77954018376947]	0.4561837	0.2081035	0.2093089	0.0012053	0.6266785
	Nasdaq	[2.24055974472008, 6.26202019663924]	1.0322192	1.0654766	1.0744286	0.0089520	1.3888589
500	Dow Jones	[1.10882656926564, 1.52897397125068]	0.1065772	0.0113587	0.0113660	7.30e-06	0.1421912
	S&P 500	[1.67552131043886, 2.4601300100902]	0.2006219	0.0402492	0.0402452	-4.00e-06	0.2681558
	Nasdaq	[3.23315587913212, 4.21447484318394]	0.2526306	0.0638222	0.0639138	9.16e-05	0.3414960
1500	Dow Jones	[1.07921929767641, 1.29818007534891]	0.0550448	0.0030299	0.0030317	1.8e-06	0.0726787
	S&P 500	[1.59563343516167, 1.9123238242635]	0.0813791	0.0066226	0.0066248	2.2e-06	0.1102052
	Nasdaq	[3.67439601519242, 4.55607555091896]	0.2251114	0.0506752	0.0506728	-2.4e-06	0.3042859

Figure 3: Tables containing properties of the distributions (empirical bootstrap) at different sample sizes , mean (left), variance (right). Confidence intervals computed using quantile method, $\alpha = 0.05$.

At greater sample sizes, we see that the distribution of means remains approximately normal. The distribution for the variance also appears to be close to normal. Looking at both tables, we see CI width, SE, bias, and IQR decrease with sample size, as expected.

Looking at the table for the mean, we see that there is little to no bias, even for the small sample. Nasdaq has the highest SE and Dow Jones has the lowest in all cases. This is also the same when it comes to IQR.

Looking at the table for the variance, we see there is a significant amount of bias in the small sample case, with Nasdaq having the most bias. Similar to the mean, Nasdaq has the highest SE and IQR in all cases. Interestingly, the SE of Nasdaq's variance does not change as much from $n = 500$ to $n = 1500$ as the other two indexes.

Parametric Bootstrap

For the parametric bootstrap, we again use $B = 10,000$ bootstrap samples. Since the densities are t-distributions, we thought it would be interesting to see the performance of generating the bootstrap samples instead from a normal distribution, with the parameters being the sample mean and sample standard deviation. We do this for the same range of sample sizes. For the sake of brevity, we will again only display the Dow Jones histograms.

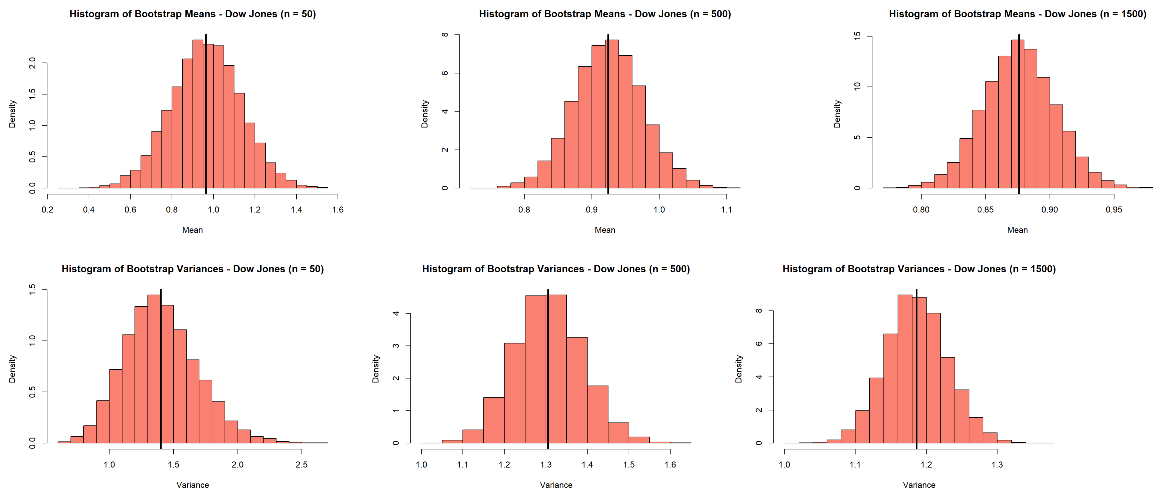


Figure 4: Histograms for bootstrap sample mean (top) and variance (bottom) for Dow Jones, $n = 50, 500, 1500$.

Similar to the empirical bootstrap distributions, the histograms for the mean are all roughly normal. For the variance, we again see they are right-skewed in the small sample case, but they approach a normal distribution for the larger sample sizes.

n	Index	CI	SE	Var	MSE	Bias	IQR
50	Dow Jones	[0.634452103602485, 1.2915009885559]	0.1672955	0.0279878	0.0279854	-2.4e-06	0.2246601
	S&P 500	[0.700757780286683, 1.54081218500419]	0.2138246	0.0457209	0.0457188	-2.1e-06	0.2845675
	Nasdaq	[0.550675660461829, 1.76871717485126]	0.3077572	0.0947145	0.0947068	-7.7e-06	0.4156484
500	Dow Jones	[0.826326880479681, 1.02600449970638]	0.0507567	0.0025762	0.0025760	-3e-07	0.0685016
	S&P 500	[1.00540205840054, 1.25121545810849]	0.0625964	0.0039183	0.0039181	-2e-07	0.0840964
	Nasdaq	[1.25964057503334, 1.59703731051389]	0.0854165	0.0072960	0.0072962	2e-07	0.1145346
1500	Dow Jones	[0.821919708471824, 0.931079577190969]	0.0279642	0.0007820	0.0007820	0e+00	0.0376386
	S&P 500	[0.904486697818722, 1.0435917144928]	0.0354870	0.0012593	0.0012593	0e+00	0.0477591
	Nasdaq	[1.34057240644964, 1.54350726339336]	0.0518846	0.0026920	0.0026926	6e-07	0.0703071

n	Index	CI	SE	Var	MSE	Bias	IQR
50	Dow Jones	[0.901618562009561, 2.01254766075715]	0.2842967	0.0808246	0.0808174	-7.20e-06	0.3780273
	S&P 500	[1.4773449083003, 3.27340663212661]	0.4613314	0.2128267	0.2128055	-2.12e-05	0.6140136
	Nasdaq	[2.9909812358446, 6.70967741594212]	0.9490558	0.9007068	0.9006168	-9.00e-05	1.2723220
500	Dow Jones	[1.15010743167645, 1.47375651560102]	0.0822221	0.0067605	0.0067611	6.00e-07	0.1103338
	S&P 500	[1.76079029073211, 2.25204518292492]	0.1254406	0.0157353	0.0157363	1.00e-06	0.1720991
	Nasdaq	[3.22431215255976, 4.13598051423201]	0.2305067	0.0531333	0.0531534	2.01e-05	0.3133102
1500	Dow Jones	[1.10218800575081, 1.27196697415485]	0.0433658	0.0018806	0.0018805	-1.0e-07	0.0574822
	S&P 500	[1.77547938705249, 2.05090199407315]	0.0698895	0.0048845	0.0048841	-4.0e-07	0.0934348
	Nasdaq	[3.74381665198165, 4.32643955220853]	0.1472991	0.0216970	0.0216951	-1.9e-06	0.1972613

Figure 5: Tables containing properties of the distributions (parametric bootstrap) at different sample sizes, mean (left), variance (right).

Both tables show some similar patterns as those for the empirical bootstrap. We again see that in all cases Nasdaq has the highest SE and IQR, and Dow Jones has the lowest.

Looking at the table for the mean, we see similar results to those obtained for the empirical bootstrap. Looking at the SE, bias, and IQR, the values obtained are almost the same. However, we can see that for $n = 500$, S&P 500's confidence interval covers a range of values significantly greater than those for the empirical bootstrap and does not contain the true mean of the t-distribution (though the lower bound is very close). All the confidence intervals for Nasdaq also contain slightly larger values than their empirical bootstrap counterparts, though they all still contain the true mean.

Looking at the table for the variance, one notable difference from the empirical bootstrap is that there is little to no bias in all cases, even for the small sample. We also see that most (but not all) SE and IQR values differ from those in the empirical bootstrap.The largest difference in these values occurs for Nasdaq when $n = 1500$, which may be due to Nasdaq having the "least normal" distribution of the three indexes. In addition, we see that most of the confidence intervals differ in the range of values covered, though only for the small sample would I deem the difference is significant. Despite this difference, they still contain the true variance, which could be partially attributed to the width of the confidence interval.

Permutation Test

Short-term

Does a certain index outperform for a short amount of time in the stock market? We were interested in comparing each index and to find out if it had a significant difference in mean, return. For the short term permutation test with the number of permutations of 10,000, we set the sample size as 63 days, which equals 3 months of trading days and the statistic referred to the absolute value of each mean difference.

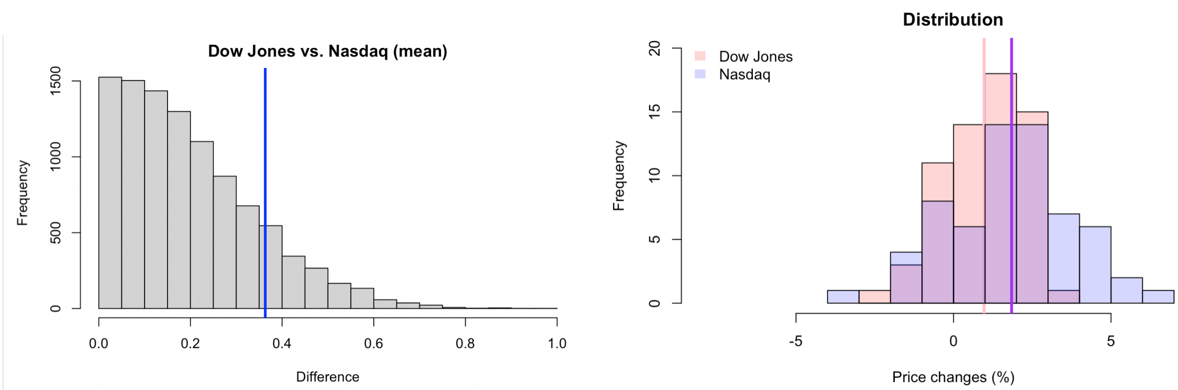


Figure 6: The above figures are the histogram of the test statistic of Dow Jones vs. Nasdaq. The p-value is 0.266, indicating a low significance. From this fact, we cannot reject the hypotheses due to p-value. Furthermore, looking at each distribution, it is most likely that Nasdaq will outperform the Dow Jones in terms of return.

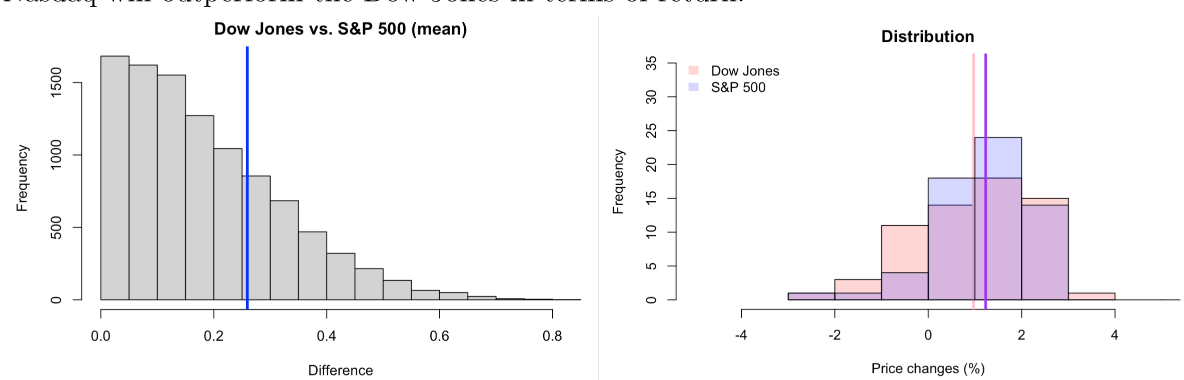


Figure 7: The above figures are the histogram of the test statistic of Dow Jones vs. S&P 500. The p-value is 0.004, indicating a high significance. This refers to the fact that two means were different from each other. Furthermore, the histograms show that it is extremely difficult to tell whether one index outperforms the other.

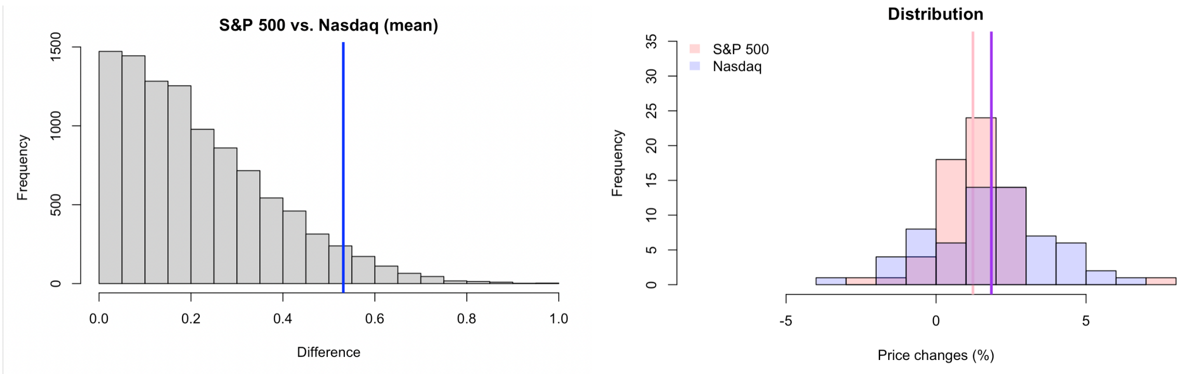


Figure 8: The above figures are the histogram of the test statistic of S&P 500 vs. Nasdaq. The p-value is 0.04, indicating a somewhat high significance. This refers to the fact that two means were likely different from each other. Furthermore, the histograms imply that Nasdaq will most likely outperform the S&P 500 in terms of return.

Long-term

Does a certain index outperform for a long amount of time in the stock market? We were also interested in comparing each index and to find out if it had a significant difference in mean return. For the long term permutation test with the number of permutations of 10,000, we set the sample size as 1300 days, which equals 5 years of trading days and the statistic referred to the absolute value of each mean difference.

As expected, all the p-value from permutation tests came out with extremely low, which indicates high significance. Furthermore, even in the histogram distributions, there were no noticeable differences between each index, which makes us difficult to tell which index outperformed in the long term.

However, this fact is understandable since these three major indexes are the best three gauges of large United States stocks, meaning they all do outperform in the entire equity market. Especially, in the long term, the stocks have time to recover from losses and any other uncertain factors. Furthermore, there are numerous stocks that are overlapping in these three indexes. For example, if your return is negative in S&P 500, then, returns in any other indexes are most likely to be negative as well. Therefore, it makes sense to have no differences.

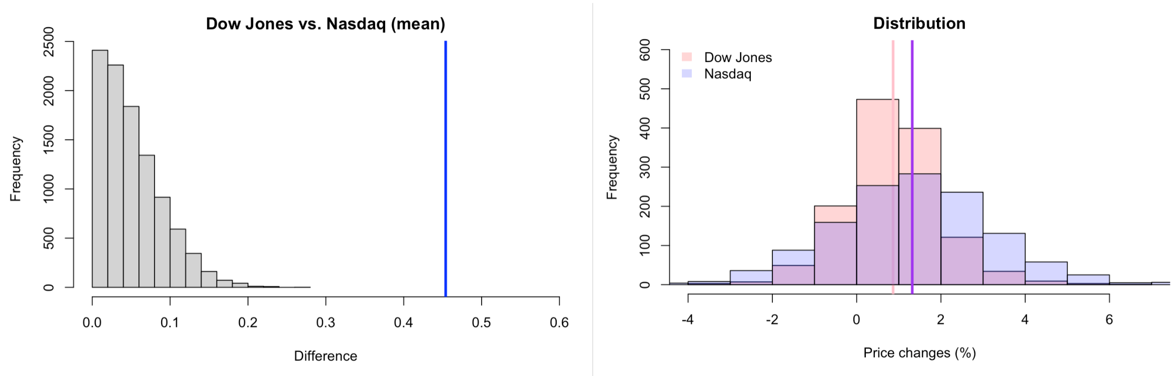


Figure 9: P-value: 0.000999

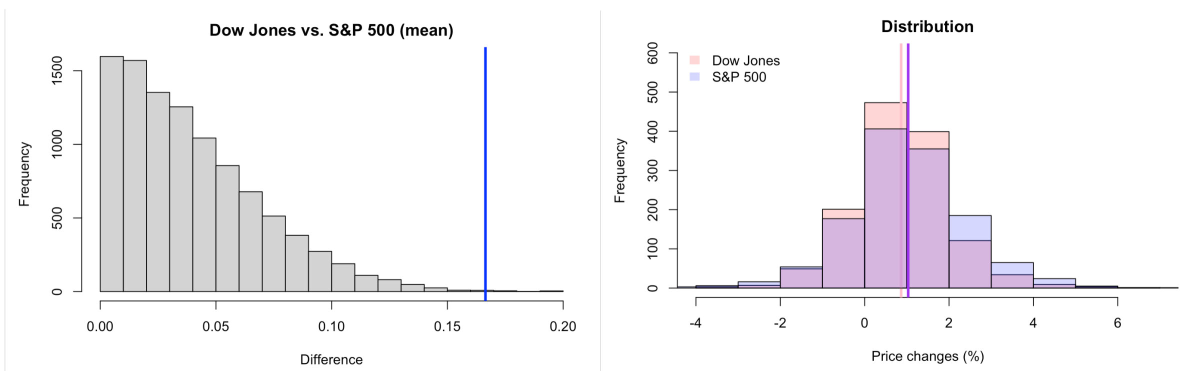


Figure 10: P-value: 0

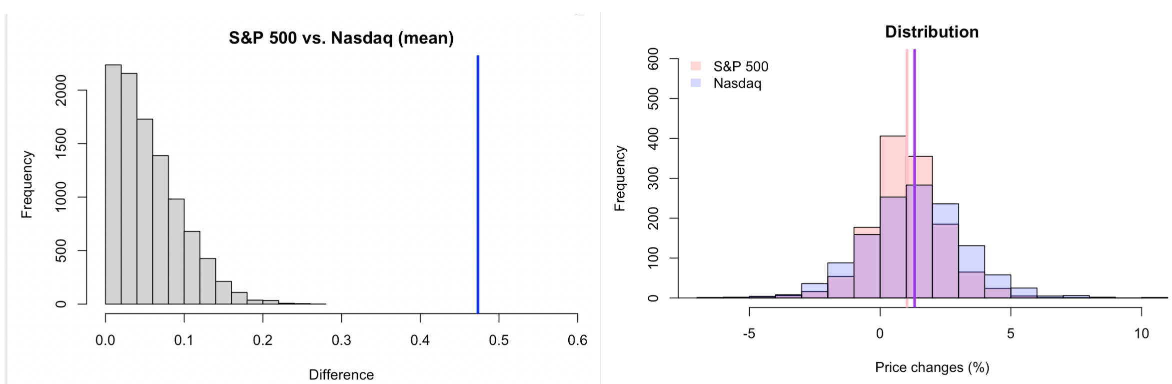


Figure 11: P-value: 0

Discussion & Conclusion

Given our three distributions, we investigated what proportion of each would create the most optimized portfolio, i.e. the one that best maximizes return and minimizes risk, using Monte Carlo simulation. Ultimately, we saw that there is a positive relationship between the mean and standard deviation, which can explained in real world investments as higher profit comes with greater risk. We found that the best combination of these distributions was 60% Dow Jones, 60% S&P 500, and 20% Nasdaq.

Using both the empirical and parametric bootstrap, we analyzed the distribution and properties for the sample mean and sample variance at various sample sizes. As stated before, these two statistics correspond to measures of the potential return and potential risk, respectively. Through our comparison of the empirical and parametric bootstrap, we come to the conclusion that the parametric bootstrap performs quite similarly to the empirical, at least when we use a normal distribution to generate bootstrap samples. This result was expected as the true densities are essentially normal distributions with heavier tails (t-distributions). Should we have chosen densities for the bootstrap samples more dissimilar from the t-distributions, I suspect we may have seen different results.

Based on the result from permutation test, it is extremely difficult to tell that a certain index outperforms the other for a long term investment. However, in a short term investment, Nasdaq will outperform the other two indexes in terms of return. Therefore, if investors are thinking about which index to invest in for a short-term, Nasdaq will be the outstanding choice. On the other hand, for a long term, all three indexes will be a great option to invest since there were no significant differences.

References

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