Planar Quads in Architecture Surfaces

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Abstract

In this paper I will discuss how to deal with planar quads in architectural free-form surfaces. The first chapter will cover the problems behind planar quads for constructability, there benifits and the mathematical measures and the constraints goals for optimization. The second chapter will embed several pre-processing algorithms that generates a candidate mesh for being a planar quads mesh. Therefore, in the third chapter, I will optimize those candidate meshes according to some conditions that qualify them to be PQ meshes. Those conditions are based on scientifique papers. And finally by combining the chapter two and chapter three, iteratively a subdivision method algorithm and a quad planarization will be done in order to have a PQ mesh.

1 Construction

PQ meshes are found everywhere in architectural design. They must show different results from the mere geometry since the planarity of faces should obbey the goals in order to fulfil the basis properties of the Planar Quad meshes (Zadravec, Schiftner, and Wallner 2010).

1.1 PQ Geometric Properties

Since a triangular face in space is always planar, then they are easier to deal with when the curvature is very high. The latter leads the quad panels to bend and the four vertices will want to move from the plane. Such constraint is a disadvantage for PQ meshes over triangular meshes. If the warping height exceeds a certain limit while measuring it, the four vertices on each of the quad panels should be independant from its neighbouring vertex see fig. 1. This is the only solution to have flat panels for constructability.

In theory two parallel vectors in space enclosed at each point by two other vectors not necesseraly parallel form a planar face, for more information refer to the author (Glymph et al. 2004). Consider each row of faces $f_{i,j}$ is a PQ strip.

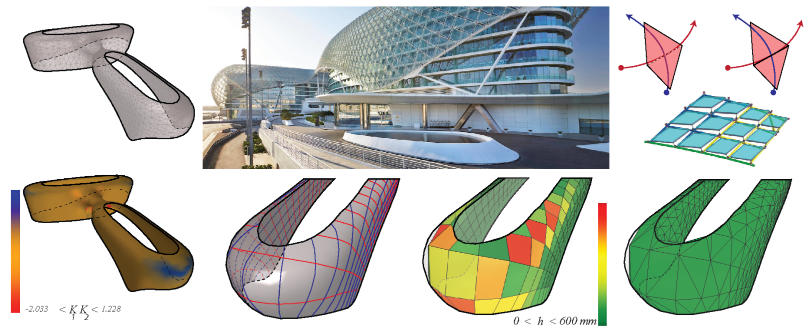


Figure 1: The hinge is affected by the high Gaussian curvature on surface of the Yas Island Hotel By Zaha Hadid. The difference between PQ meshes and triangular meshes.

PQ meshes are composed by vertices $\mathbf{v}_{i,j}$ with a valence $\pm k/4 (k \in Z)$ where along each vertex a curve of family A and a curve of family B intersect. N-geons can appear with a valence $k \neq 4$ so called singularities.

1.2 Benefits

Planar quads are prefered over triangular meshes for there aesthetically and constructabilty reason. The advantages of planar quads over other meshes for constructability are that:

- PQ meshes are less heavy.
- Triangle meshes are a waste of material, quad panels are full panels.
- Less energy consumption during the fabrication.

1.2.1 Conical Meshes

- Offset properties
  - Proof:
- Planar 100%
  - Proof:
- Discretize the curvature directions
  - Proof:
- Good for glass structures:
  - Proof:

1.2.2 Conjugate Networks

- Admit free torsion node
  - Proof:
- Discretize the curvature directions
  - Proof:
- Planar X%:
  - Proof:
- The advantage of designing a conjugate direction field is that the user's pocess a free control on the PQ mesh layout.
- Good for glass structures:
  - Proof:

1.3 Metrics-Measures(Quality)

To have planar quads, several measures are mentioned below. For a better quality, the mathematical measures and the conditions are classified by face and by meshfig. 2. Thus some conditions will be translated to some custom goals that will make the quality of the PQ mesh better.

The measurements and conditions applied to the mesh itself are:

The measurements and conditions applied to the elements of the mesh are:

2 Algorithmic Strategies

2.1 Several Pre-Processing Techniques

2.1.1 Translational Surfaces

Translational surfaces are: * Limited and easy to generate

- Planar 100%:
  - Proof: As describe in the planarity measure, the geometric principale of translational surfaces allows to succeed an ultimate planarity result on the panels.
- Homogeneous:
  - Proof: Another translational surfaces geometric principale is where the two parallel vectors enclosed by two others having the same length, reduces the variances between the panels.

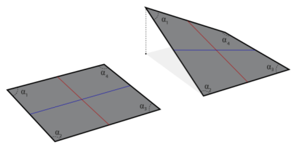
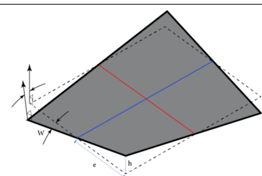
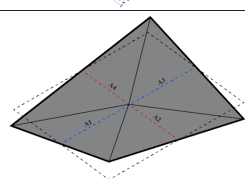
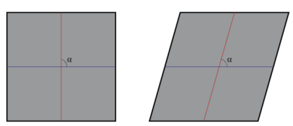
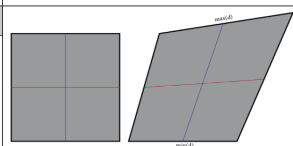
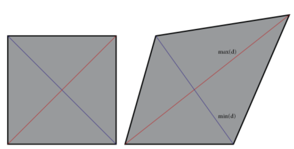
Function Name	Technical Drawings	Mathematical Function	Acceptable Range	Ideal Value
Planarity and Convexity: A panel is flat and convex if and only the difference between the sum of the internal angles and 2π should be zero.		$\delta_{PQ} = \sum \alpha_i$ - 2	0 to 0.1	0.0
Warping Angles Ratio: The measure of a quadrilateral element from being planar. Max element corner normal angular deviation from normal of mean plane.		$PQ_{warping} = arc\ sin\ (\frac{h}{e})$	0.9 to 1.0	0.0
Taper Ratio: Maximum ratio of lengths derived from opposite edges.		$PQ_{taper} = 4\ (\frac{A_{min}}{\sum A_i})$	0.3 to 1.0	1.0
Skew Ratio: Maximum |cos α| where α is the angle between edges at quad center.		$PQ_{skew} = 1 - (\frac{\lvert\frac{\pi}{2} - \min(\alpha)\rvert}{\frac{\pi}{2}})$	0 to 0.5	0.0
Element Area: The area on each quad is divided by two.		$PQ_{area} = \frac{max(d) \times min(d)}{2}$	NONE	NONE
Diagonals Aspect Ratio: Maximum distance between diagonals of the quad face divided by the minimum distance of diagonals.		$\eta_{PQ} = \frac{\max(d)}{\min(d)}$	1 to 5.0	1.0

Figure 2: Table showing the main different measures of PQ meshes see (Gokhale (2008), guide (2018) and Geometry and Generator (2016))

- Good for glass structures:
  - Proof:

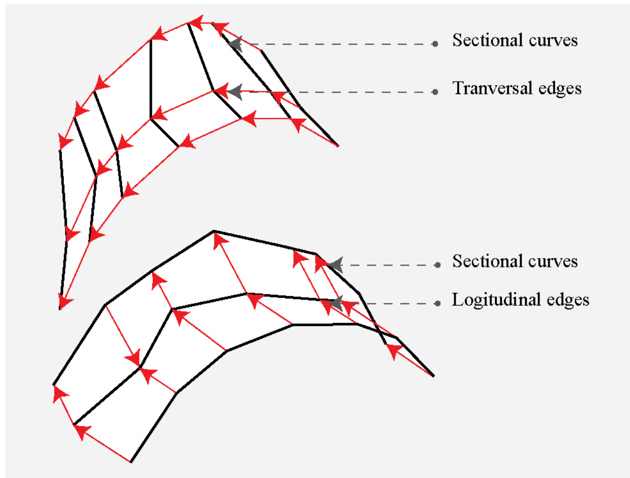


Figure 3: Translational surfaces principals

Translational surfaces admit a huge variety of shapes for a gridshell of quadrangular planar mesh see (Glymph et al. 2004). If the sectional curves are plane and the vectors are parallel having the same length the result will respond to the design principale of a translation surface. Assuming that one direction of the quad mesh net to be the sectional curve, two design principales can appear see fig. 3:

- The row of longitudinal sectional curves form parallel vectors see fig. 3 left top.
- The row of lateral sectional curves form parallel vectors see fig. 3 left down.

2.1.1.1 Row of sectional curves translated over a set of parallel vectors

The family of sectional curves C_i translated over a set of parrallel vectors is generated as follow: A random spatial curve called generatrix is translated against another random spatial curve called directrix as seen in fig. 4. Thus considering that translation by equal length gives homogeneous results of the planar quads.

Several gometrical shapes have been developped in architecture during the history using translation surfaces. The elliptical paraboloid is the most familiar shape found. It is generated using the same principal, translating one parabolic curve against another.

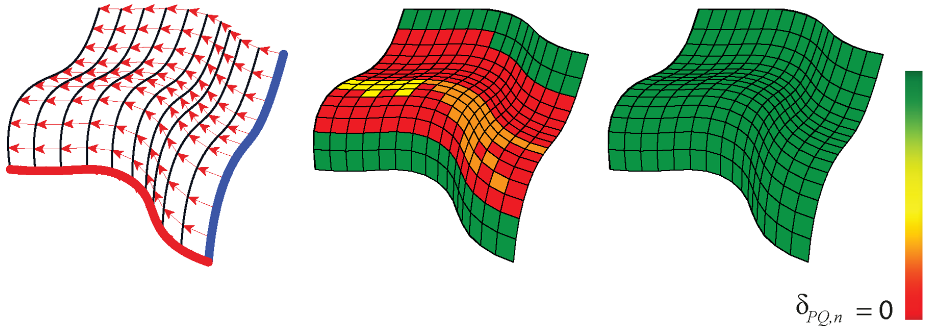
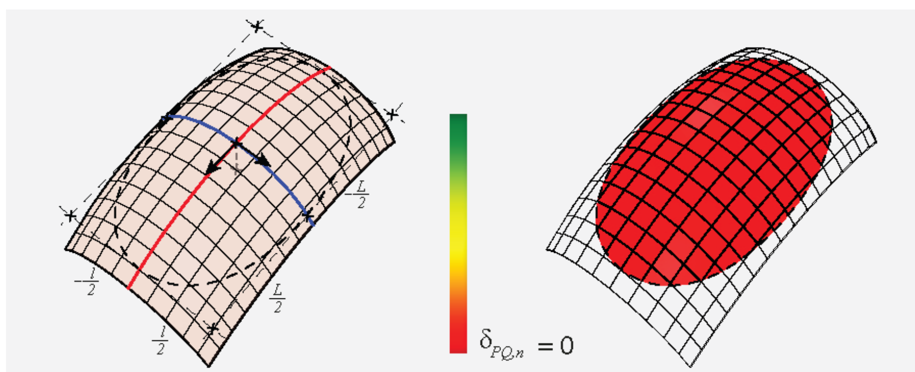


Figure 4: Geometric principale for translation surfaces and planarity measure fulfilled.



In translational surfaces, some geometrical shapes admit boolean and joining operations, for example, the hyperbolic paraboloid is a type of translational surfaces that acknowledge such operations. By translating a parabolic curve over a hyperbolic the result is as seen in fig. 5

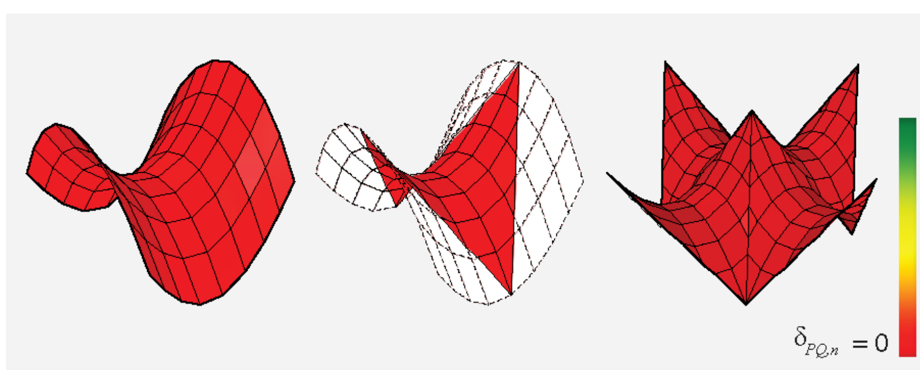


Figure 5: Translated hyberbolic paraboloid and joining possibilities

2.1.1.2 Scale-Translation Surfaces

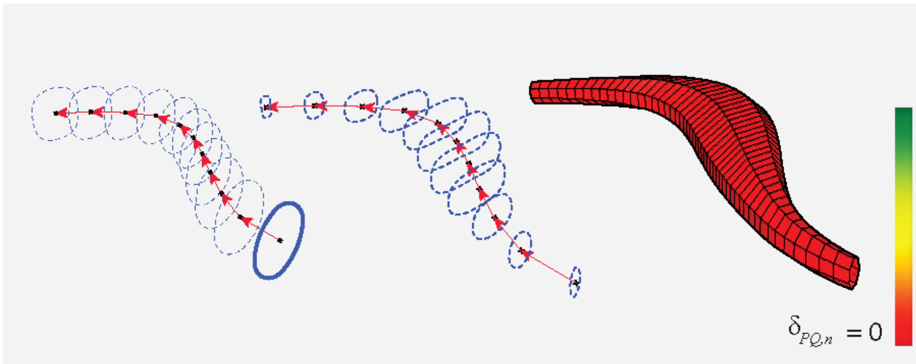


Figure 6: Centric scale-translation expansion

Scale Translational surfaces are generating by adding a scale parameter to the output curvesC_n. After translating the sectional curve v_0 on each point v_i equally distant at the directrix curve, the output curves can be scaled uniformly or non-uniformly controlled by the user. The centrical expansion of any curve gives a new curve having parallel edges. The center of expansion can be chosen randomly(Glymph et al. 2004) but in this technique centric expansion has been chosen. The edges that enclose the quads are connected from each curve C_{i+1} at the vertex v_i. The resulting algorithm gives planar quad meshes see fig. 6.

2.1.2 Conjugate Networks

2.1.2.1 What is a CDF on a Triangular Mesh.(Liu et al. 2011)

On a smooth surface $S \subset \mathbb{R}^3$, the tangent vectors $\mathbf{v}_p, \mathbf{w}_p$ are conjugate if and only if they are treated as two vectors in $\mathbb{R}^3$. The CDF is a tool for non-photorealistic rendering in order to visualize the surface topology. Therefore it is a powerful tool for surface remeshing and alignement control. On a triangular face f_i as seen in fig. 7 of a triangular mesh $\mathbb{R}^3 = (V, E, F)$ a CDF is:

- Four vectors $\{\mathbf{v_i}, \mathbf{w_i}.{-}\mathbf{v_i}{-}\mathbf{w_i}\}$
- Two scalar parameters $\{\theta_i, \alpha_i\}$:
  - θ_i oriented angle between $e_{i,1}$ and $\mathbf{v_i}$
  - α_i oriented angle between $\mathbf{v_i}$ and $\mathbf{w_i}$
  - They define the following: $\mathbf{v_i} = (\mathbf{cos}\theta_\mathbf{i}, \mathbf{sin}\theta_\mathbf{i})^T$ and $\mathbf{w_i} = (\mathbf{cos}(\theta_\mathbf{i} + \alpha_\mathbf{i}), \mathbf{sin}(\theta_\mathbf{i} + \alpha_\mathbf{i})^T$

2.1.2.2 The relation between PQ meshes and Conjugate Networks.(Liu et al. 2006)

Conjugate curve networks are families of curves $A, B \subset \Phi$: For each $p \in \Phi$ unique curves of both family A, B should appear. Since T_1, T_2 are conjugate then

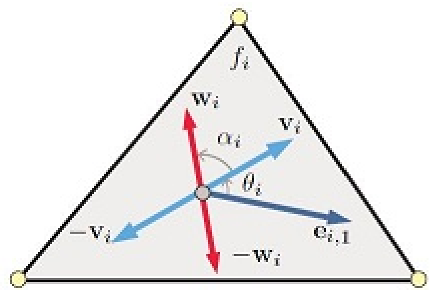


Figure 7: A CDF on a Triangular face

they predefine A and get B by *integrating the vector field directions conjugate of family A*

Examples of Conjugate Curve Networks on Surfaces suited for PQ meshes: (Liu et al. 2006) * *The network of principale curvature lines* * In a translational surface of the form $p(u, v) = \mathbf{q}(u) + \mathbf{q}(v)$ where its directrix curve $\mathbf{q}(u)$ is translated along a another curve generatrix $\mathbf{q}(v)$ and vice versa see (fig. 8 left). * Not suited for PQ meshes: * *Epipolar curves*: The translation of a point p along a line l and the intersection of the planes threw the points $p(i)$ with that surfaceΦ generate asymptotic curves that are not suited for such meshing see (fig. 8 center). * *Isophotic curves are conjugate to the system of the steepest descent curves with respect to the z-axis* see (fig. 8 right).

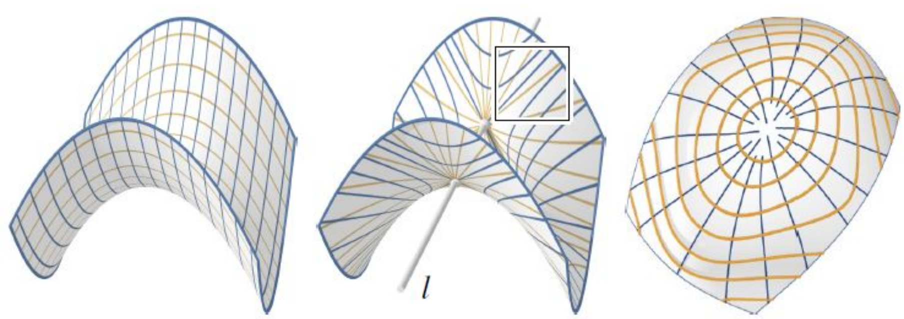


Figure 8: Various Conjugate Networks

2.1.2.3 Generating a Quad-Dominant mesh by a CDF without singularities.

The input mesh should not be random, it is preferable to use a remeshing tool such as mesh machine to control the edges length and the fixed boundaries or others, for more info see (Piker, n.d.).

2.1.2.3.1 Alignment with curvature (Mueller and Adriaenssens 2018).

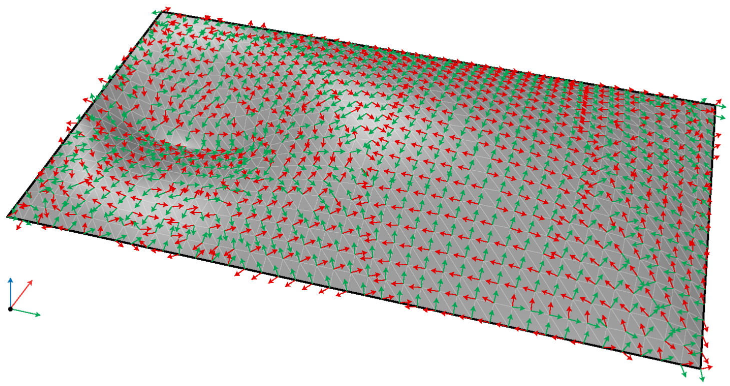


Figure 9: Surface tangency by extracting the maximum and minimum principal directions in red and green.

The quality of the mesh is always better when the panels are aligned with the curvature or the stress lines. In order to induce orthogonality, I will introduce in this section the alignement by extracting the principal directions $e_{p,1}$ and $e_{p,2}$ as surface tangents responding to the second fondamental form.

2.1.2.3.2 Rigid Umbilics

Rigid umbilics are usually found where the curvature is the same.

2.1.2.3.3 Interpolating vector field with ***N-PolyVector Field***

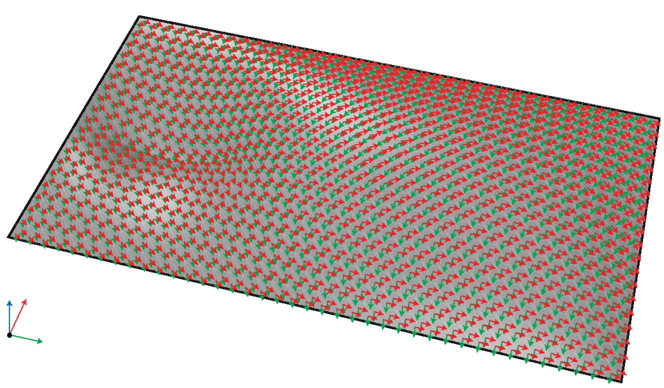


Figure 10: Smoothed vector field using n-polyVector field algorithm

In order to find a smoothed and well aligned vector field,in this section I will use (Greco 2018) as a plugin developed by (Mueller and Adriaenssens 2018). The algorithm is based on finding the trade-off between neighboring faces so that the parallel transport succeed. It uses the novel method proposed by (Diamanti et al. 2014) called *N-Poly Vector Field*. While selecting a subset of points, the vector field is able to be generated smoothly and continuously. It finds the smoothest vector field by smoothing and continuously interpolate the two vectors parallely. This method is different from the one used in (Liu et al. 2011) where it uses a signed permutation technique in order to find the correct vector's relation between neighbouring vertices. In fig. 10, it is well clear how the smoothed vector field and the parallel transport have been well generated.

2.1.2.3.4 Conjugate Direction Field

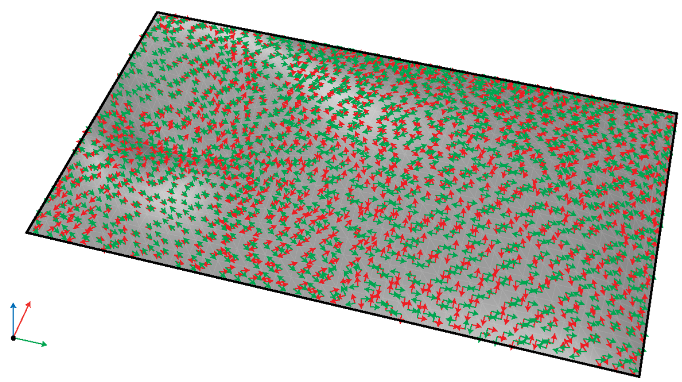


Figure 11: Conjugate field after smoothing previously the vector field

After smoothing the vector field in the previous step, a quad mesh can be generated after defining the conjugate networks, (Liu et al. 2011) took further this topic for more information refer to the reading. From the previous step I define a conjugate vector field with (Greco 2018) developed by (Mueller and Adriaenssens 2018) using an algorithm provided in (LibDirectional 2018).

2.1.2.3.5 Global Parametrization

This method is based on the *global parametrization with frame fields* it is generated using a custom component in developped by the author of (Greco 2018). It is used to shape the new mesh in a different typology, the latter has to be aligned with some given vetors by interest of the user. The Mixed-integer quadrangulation by (Bommes, Zimmer, and Kobbelt 2009) is one way to do that and another way is using *Anisotropic remeshing to concentrate the elements in the regions with more details*(Mueller and Adriaenssens 2018) published by (Diamanti et al. 2014) an open source library can do that see (Libgil 2017).

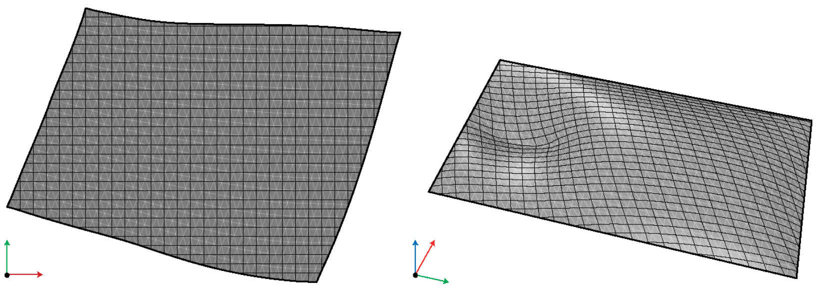


Figure 12: Left: Global Parametrization mapping with frame fields. Right: Conjugate curve network tracing on the mesh.

2.1.2.3.6 Tracing Streamlines

The streamlines are traced after integrating the Vector field. They are generated using a custom component in (Greco 2018) that is developed using the 4^{th} order Runge-Kutta for more information refer to (Mueller and Adriaenssens 2018).

2.1.3 Conical Meshes

- Conical meshes are planar quad meshes which discretize principal curvature lines, posses an offset at a constant distance as well as planar connecting elements. See fig. 14,
- A conical mesh is conical if and only if all of its vertices $\mathbf{v}_i$ are conical.

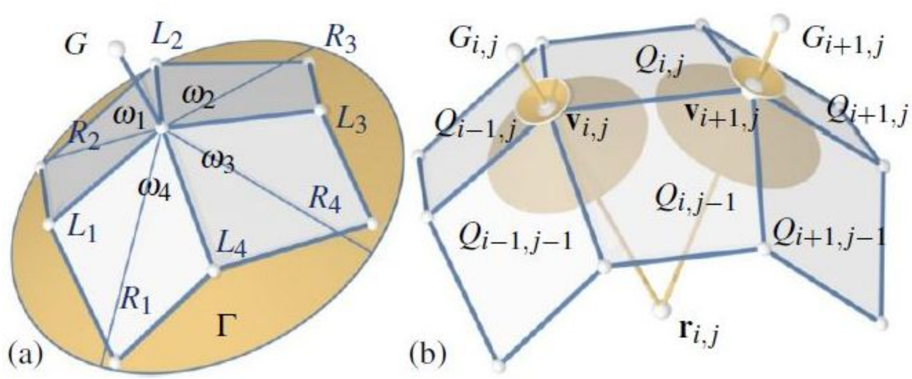


Figure 13: Faces Configuration of a Conical Ceshes

- $\mathbf{v}$ is a conical vertex if and only if the four face planes meeting at $\mathbf{v}$ are tangent to a common sphere (Liu et al. 2006) see fig. 13 (a) and (b)

2.1.3.1 The Angle Criterion of a Conical Mesh

- The sum of the opposite angles on a vertex **v** should always be equals to zero so:
- **v** is a conical vertex if and only if the charaterization of a conical mesh interior angles should respond to this function:

$$\omega_1 + \omega_3 = \omega_2 + \omega_4$$

2.1.3.2 The Offset Properties,

- Triangular meshes are missing the the offset property at a constant distance.
- However conical meshes have this property while generating conical meshes at the offset.
- The fact that the faces of a conical mesh are incident to a common vertex $\mathbf{v}_{i,j}$ and tangent to a cone with an axis $Q_{i,j}$. After offseting the axis remains the same and the faces are still tangent to the cone.

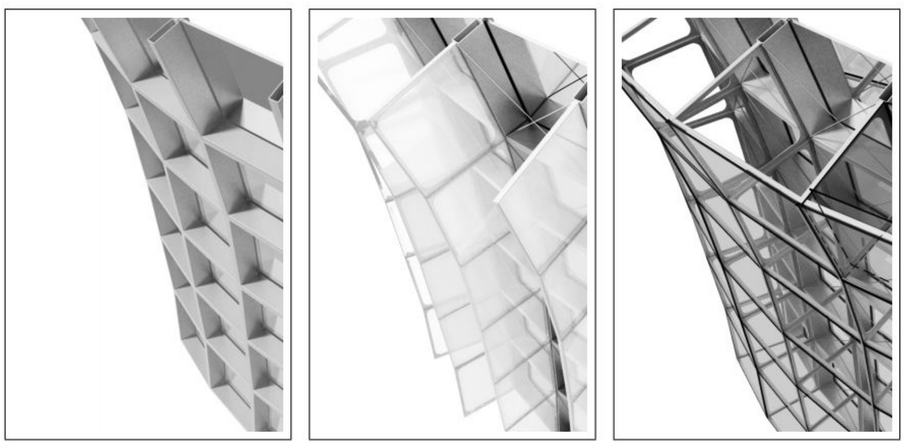


Figure 14: Constant offset of a Conical Mesh see (Pottmann and Wallner 2008)

The Languerre transformation (Liu et al. 2006) contains one of the instances for offsetting planes by a fixed distance along their normal vector. The Languerre transformation preserves the conical meshes at the offset.

2.1.3.3 The Normals,

- *The spherical image* is a fact where the vertice $\mathbf{n}_{ij}$ of a PQ mesh built on a unit sphere are converted to the normal vectors of $Q_{i,j}$.
- As the four faces incident to a common vertice $\mathbf{n}_{ij}$ tangent to the same cone Γ_{ij}, the normal vectors $\overrightarrow{n_{ij}}$ on each of the four faces share the same angle with the cone's axis $Q_{i,j}$.
- Consequently the spherical image of the principale curvature network returns an orthogonal curve network on a sphere.

2.1.3.4 Planar Quads Perturbation

- Quad mesh as input with vertices $\mathbf{v}_{i,j}$
- minimally perturb the vertices into a new positions while maintaining the PQ meshes properties.
- Penalty Linear fonctunality combination:

1. $Q_{i,j}$ is convex and planar if and only if the sum of angles is equals to 2π:

$$c_{pq} = \phi^1{}_{i,j} + ... + \phi^4{}_{i,j} - 2\pi = 0. \tag{1}$$

   Another Planarity Constraints on all the $\mathbf{v}i, j$, while computing the unit vectors along the egdes of each face consider the determinant on each edge equals to zero: $\lambda^T{}_{det} c_{det} = 0$.

2. Two energies terms:
   - Fairness (which simplifies the bending energies in the rows and columns on each polygon of the mesh):

$$f_{fair} = \sum_{i,j} [(\mathbf{v}_{i+1,j} - 2\mathbf{v}_{i,j} + \mathbf{v}_{i-1j})^2 + (\mathbf{v}_{i,j+1} - 2\mathbf{v}_{i,j} + \mathbf{v}_{i,j-1})^2].$$

   - Closeness:

     While the polygons are defined as squares, this fonction minimizes the distance between the original surface Φ and the vertices $\mathbf{v}i, j$ of the perturbed mesh, where $\mathbf{y}i, j$. are the closest points to the mesh, otherwise the undifined squares are set to zero :

$$f_{close} = \sum_{i,j} ||\mathbf{v}_{i,j} - \mathbf{y}_{i,j}||^2.$$

3. SQP(Sequential Quadratic Programming)

   As subject to the constraints above the Langrangian functions is written as follow:

$$f_{PQ} = w_1 f_{fair} + w_2 f_{close} + \lambda^T{}_{pq} c_{pq} + \lambda^T{}_{det} c_{det} \tag{2}$$

   The SQP minimizes the fairness and closeness subject to the planarity constraints of f_{PQ}. While w_1 and w_2 are user spicified constants to control fairness and closeness.

   SQP works only for up to 1000 vertices per mesh

4. Therefore another function by combining the constraints in {eq. 1} by summing up the angles on all the polygons. In addition to that a final function is added to minimize the objectives in {eq. 2}.

2.1.3.5 Generating Conical Meshes

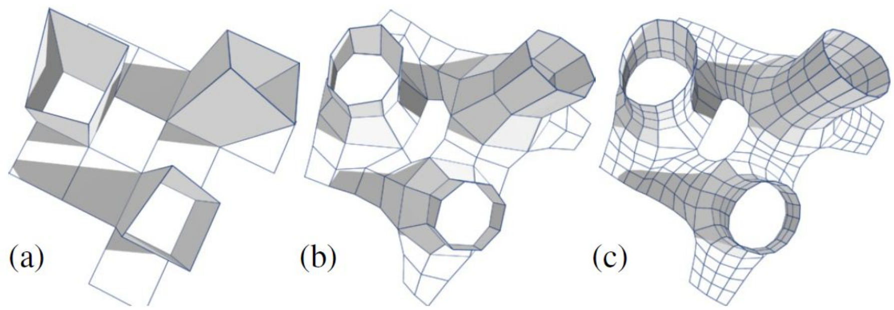


Figure 15: Coarse Mesh Subdivision and PQ perturbation sequence

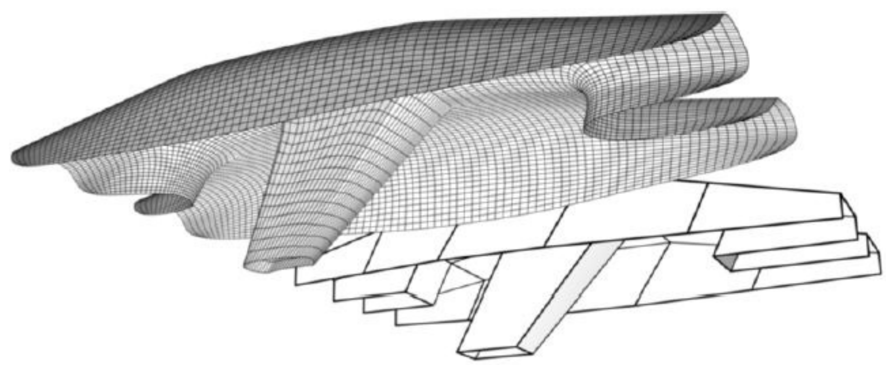


Figure 16: Conical mesh after combining a Catmull-Clark subdivision and a conical optimization

$$\omega_1 + \omega_3 - \omega_2 - \omega_4 = 0. \tag{3}$$

$$\phi^1{}_{i,j} + \phi^3{}_{i,j} - \pi = 0, \phi^2{}_{i,j} + \phi^4{}_{i,j} - \pi = 0. \tag{4}$$

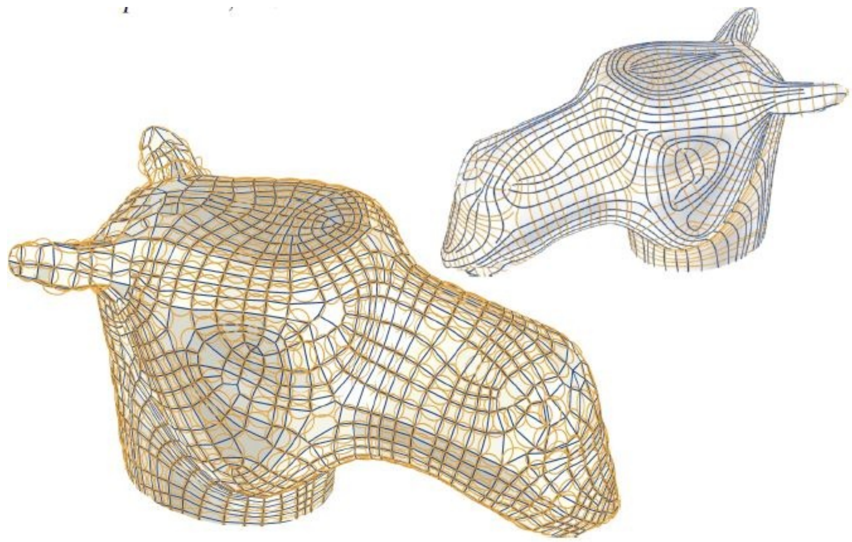


Figure 17: Circular Mesh generated after optimizing a conical mesh generated from principal curvature

2.2 Optimization (K2)goals

2.2.1 Quad-Dominant Mesh(Optimize to a PQ mesh by imposing some properties)

Subdivision Strategy (Starting with a Coarse Quad-Dominant mesh)

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