CSE 15: Discrete Mathematics
Final Exam – Fall 2020

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1. **Logical Equivalences**

p	q	r	$\neg p$	$\neg q$	$\neg r$	$(p \to q)$	$(\neg r \to \neg q)$	$(p \to q) \land (\neg r \to \neg q)$	$\neg r \to \neg p$
F	F	F	T	T	T	T	T	T	T
F	F	T	T	T	F	T	T	T	T
F	T	F	T	F	T	T	F	F	T
F	T	T	T	F	F	T	T	T	T
T	F	F	F	T	T	F	T	F	F
T	F	T	F	T	F	F	T	F	T
T	T	F	F	F	T	T	F	F	F
T	T	T	F	F	F	T	T	T	T

$(p \to q) \land (\neg r \to \neg q)$	$\neg r \to \neg p$	$(p \to q) \land (\neg r \to \neg q) \equiv \neg r \to \neg p$
T	T	T
T	T	T
F	T	F
T	T	T
F	F	T
F	T	F
F	F	T
T	T	T

- According to the truth tables, because there are two spots that are not equal (rows 3 and 6). Hence, $(p \to q) \land (\neg r \to \neg q)$ is not equivalent to $\neg r \to \neg p$.

2. **Quantifiers**

   a) $\forall x(x^2 - 1 > 0) : 0^2 - 1 > 0 \Rightarrow -1 > 0$
   This statement is **FALSE** because there exists a x-value that makes the statement false, i.e plugging 0 in for x, which results in $-1 > 0$.

   b) $\exists x(x^3 - 1 = 0) : 1^3 - 1 = 0 \Rightarrow 0 = 0$
   This statement is **TRUE** because there exists a x-value that makes the statement true, i.e plugging in 1 in for x, which would result in $0 = 0$.

   c) $\forall x \exists y(x^2 + y^2 = 1) : 1^2 + 2^2 = 1 \Rightarrow 5 \neq 1$
   This statement is **FALSE** because there does not exist any number that makes this statement true, i.e plugging in 1 for x and 2 for y, the result is $5 \neq 1$.

   d) $\exists x \exists y(x^4 + y^4 = 0)$
   This statement is **TRUE** because there exists both a x-value and a y-value that makes the statement true, i.e plugging in 0 for both x and y, which results in $0 = 0$.

3. **Negation of Complex Sentences**

   a) $\neg \exists y \exists x P(x, y) \equiv \forall y \neg \exists x P(x, y) \equiv \forall y \forall x \neg P(x, y)$
   Using De Morgan's Law for quantifiers, we change the quantifiers to $\forall$ and move the negation sign to the predicate to get $\forall y \forall x \neg P(x, y)$ which is equivalent to the original statement of $\neg \exists y \exists x P(x, y)$.

   b) $\neg \forall x \exists y(P(x, y) \wedge Q(x, y) \equiv \exists x \neg \exists y(P(x, y) \vee Q(x, y)) \equiv \exists x \forall y(\neg P(x, y) \vee \neg Q(x, y))$
   By using De Morgan's Law for quantifiers, we move the negation symbol to $\exists y$ which we then change to $\forall$ and move the negation symbol to the predicates. We then distribute the negation symbol between $P(x, y)$ and $Q(x, y)$ and change the $\wedge$ to $\vee$. As a result, we are left with $\exists x \forall y(\neg P(x, y) \vee \neg Q(x, y))$ which is equivalent to $\neg \forall x \exists y(P(x, y) \wedge Q(x, y)$.

   c) $\neg \exists y(\exists x R(x, y) \vee \forall x S(x, y)) \equiv \forall y \neg(\exists x R(x, y) \vee \forall x S(x, y)) \equiv \forall y(\neg \exists x R(x, y) \wedge \neg \forall x S(x, y)) \equiv \forall y(\forall x \neg R(x, y) \wedge \exists x \neg S(x, y))$
   Through the use of De Morgan's Law for quantifiers, we move the negation symbol down the statement. Both the quantifiers and logic operators change to their counterpart, such as $\forall$ changes to $\exists$ and $\vee$ changes to $\wedge$ and the negation symbol reaches the predicates. As such, when following this process, the result is $\forall y(\forall x \neg R(x, y) \wedge \exists x \neg S(x, y))$ which is equivalent to $\neg \exists y(\exists x R(x, y) \vee \forall x S(x, y))$.

4. **Cartesian Products**

   a) $A \times C : \{(a, True), (b, True), (c, True), (a, False), (b, False), (c, False)\}$

   b) There are 54 elements in $A \times B \times C \times A$. This is the case where the cardinality of A = 3, B = 3, and C = 2. Following the Cartesian product, $3 \times 3 \times 2 \times 3 = 54$ elements in $A \times B \times C \times A$.

   c) $A \times B \times C \times A : \{(a, dog, True, a)\}$

   d) $A \times B$ and $B \times A$ are not equal unless A = B or A or B is an empty set. This is the case as the orders of the sets can differ based on the order of the pairs.

   e) $(True, True, dog)$ is **not** an element of $C \times B \times C$ as it does not follow the order in which the parts of the elements appear.

5. **Arguments**

   a) p : Enrolled in CSE015.
   q : Has access to computer Lab 2.
   $\neg q$: Jennifer does not have access to computer Lab 2.
   $\neg p$: Jennifer is not enrolled in CSE015.

   By setting the predicates as p for being enrolled in CSE015, q as every student having access to computer Lab 2, and $\neg p$ as Jennifer not having access to computer Lab 2, we can use Universal Modus Tollens to clarify whether this argument is correct or not. As such, this statement is **correct** by Universal Modus Tollens.

   b) This argument is not a valid argument. This argument does not follow any of the rules of inference that would thus state whether it is a correct argument or not. The information that was given does not justify that it is correct. Therefore, this argument is **not correct**.

6. **Functions**

a) $f(x, y) = x + y$

$f(1, 0) = 1 + 0 = 1$
$f(0, 1) = 0 + 1 = 1$

$f(x, y) = x + y$ is surjective. This function is surjective because any ordered pair substituted would output a image that corresponds to the first variable. Such as (1,0), if these variables were to be substituted, there would be an image of 1. Therefore, $f(x, y) = x + y$ is **surjective**.

b) $f(x, y) = |x + y|$

$f(-1, 0) = |-1 + 0| = 1$

$f(x, y) = |x + y|$ is none of the former. This function is none of the former because with any ordered pair, you are unable to get a negative number as an image. Therefore, $f(x, y) = |x + y|$ is **none of the former**.

c) $f(x, y) = x^2 - y$

$f(2, 1) = 2^2 - 1 = 3$

$f(x, y) = x^2 - y$ is surjective. This function is surjective because when substituting any ordered pair, there is an image that corresponds to said ordered pair. For example, if (2, 1) were substituted, the image corresponding to it would be 3. Therefore, $f(x, y) = x^2 - y$ is **surjective**.

7. **Order of Growth**

a) $f_1(n) = 4n^2 + \log n$ $\quad f_2(n) = 21n + \sqrt{n}$
$f_1(n)$ is big-O of $f_2(n)$

$f_1(n)$ is big-O of $f_2(n)$ because, if we were to ignore all the constants and ignore all the lower order terms, n grows faster than that of n^2. Therefore, $f_1(n)$ is big-O of $f_2(n)$.

b) $f_1(n) = 3n^2 + 2^n$ $\quad f_2(n) = 4n^2 + 45n$
$f_1(n)$ is big-O of $f_2(n)$

$f_1(n)$ is big-O of $f_2(n)$ because, after ignoring all constants and ignoring all the lower order terms, n^2 grows faster than that of 2^n as n^2 grows faster. Therefore, $f_1(n)$ is big-O of $f_2(n)$.

c) $f_1(n) = 2n^3 + 4n \log n \qquad f_2(n) = n \log n + 3n^2 + 12n^3$
Both functions are big-O's of each other.

Both functions are big-O's of each other because, after ignoring all constants and ignoring all the lower order terms, $f_1(n)$ has n^3 and $f_2(n)$ also has n^3. Since both grow at the same rate, we can conclude that both of the functions are big-O's of one another.

d) $f_1(n) = 4n^2 \log n^2 + 21n \qquad f_2(n) = 0.002n^3 + n$
$f_1(n)$ is big-O of $f_2(n)$

$f_1(n)$ is big-O of $f_2(n)$ because, when we ignore all constants and all lower order terms, $n^2 \log n^2$ grows faster than n^3. Therefore, $f_1(n)$ is big-O of $f_2(n)$.

8. **Induction**

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$$\sum_{i=1}^{n} 2^{i-1} \cdot i = 2^n(n-1) + 1$$

Basis Step:

$P(1) = 2^1(1-1) + 1$
$P(1) = 1$

Inductive Step:
Let n = k:

$P(k) = 2^k(k-1) + 1$
$P(k) = 2^k \cdot k - 2^k + 1$

If n = k = k+1, let n = k+1:

$P(k+1) = 2^{k+1}((k+1) - 1) + 1$
$P(k+1) = 2^{k+1}(k+1) - 2^{k+1} + 1$

Since $k \to k+1$, and k = k+1, this formula is **true** for each n $\geq$ 1

9. **Modular Arithmetic** $m = 13$

a) $5 +_m 8$
$5 +_{13} 8$
13 mod 13 = 0

b) 9(modulo$m$)
9(modulo13) = 0
$0 \times 13 = 0$
$9 - 0 = 9$

$9 + (-9) = 0$

Therefore, the additive inverse of 9(modulo13) is -9.

c) Is $4 \equiv 38(\bmod m)$?

$38(\bmod 13) = 12$

Because $38(\bmod 13) = 12$, it is not equivalent to 4.

d) $-4 \bmod m$
$-4 \bmod 13 = 4$
As 13 is unable to go into -4, we are left with 0 and a remainder of 4.

$13 - 4 = 9$
$9 = -4 \bmod 13$
Division Algorithm:
$-4 \bmod 13 = -4 - 13(\frac{-4}{13})$
Through simplification, the result is 0.

10. **Cryptography**

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
U	V	W	X	Y	Z															
21	22	23	24	25	26															

Ciphertext:
HMWGVIXIDQEXL
8 13 23 7 22 9 24 9 4 17 5 24 12

Plaintext:
4 9 19 3 18 5 20 5 0 13 1 20 8
DISCRETE MATH