CSE 015: Discrete Mathematics
Fall 2020
Homework #6
Solution

Christian Duan
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1. **Recursively Defined Functions**

   (a) $f(n+1) = -2f(n)$

   $$a_n = \begin{cases} 3 & \text{for } n = 0 \\ -2f(n-1) & \text{for } n > 0 \end{cases}$$

   $f(5) = f(4+1) = -2f(4) = -2 * 48 = -96$
   $f(4) = f(3+1) = -2f(3) = -2 * -24 = 48$
   $f(3) = f(2+1) = -2f(2) = -2 * 12 = -24$
   $f(2) = f(1+1) = -2f(1) = -2 * -6 = 12$
   $f(1) = f(0+1) = -2f(0) = -2 * 3 = -6$
   $f(0) = 3$

   (b) $f(n+1) = 3f(n) + 7$

   $$a_n = \begin{cases} 3 & \text{for } n = 0 \\ 3f(n+1) + 7 & \text{for } n > 0 \end{cases}$$

   $f(5) = f(4+1) = 3f(4) + 7 = 3(523) + 7 = 1576$
   $f(4) = f(3+1) = 3f(3) + 7 = 3(172) + 7 = 523$
   $f(3) = f(2+1) = 3f(2) + 7 = 3(55) + 7 = 172$
   $f(2) = f(1+1) = 3f(1) + 7 = 3(16) + 7 = 55$
   $f(1) = f(0+1) = 3f(0) + 7 = 3(3) + 7 = 16$
   $f(0) = 3$

(c) $f(n+1) = f(n)^2 - 2f(n) - 2$

$$a_n = \begin{cases} 3 & \text{for } n = 0 \\ f(n-1)^2 - 2f(n-1) - 2 & \text{for } n > 0 \end{cases}$$

$f(5) = f(4+1) = f(4)^2 - 2f(4) - 2 = 19597$
$f(4) = f(3+1) = f(3)^2 - 2f(3) - 2 = 141$
$f(3) = f(2+1) = f(2)^2 - 2f(2) - 2 = 13$
$f(2) = f(1+1) = f(1)^2 - 2f(1) - 2 = -3$
$f(1) = f(0+1) = f(0)^2 - 2f(0) - 2 = 1$
$f(0) = 3$

(d) $f(n+1) = 3^{f(n)/3}$

$$a_n = \begin{cases} 3 & \text{for } n = 0 \\ 3^{f(n-1)/3} & \text{for } n > 0 \end{cases}$$

$f(5) = f(4+1) = 3^{f(4)/3} = 3$
$f(4) = f(3+1) = 3^{f(3)/3} = 3$
$f(3) = f(2+1) = 3^{f(2)/3} = 3$
$f(2) = f(1+1) = 3^{f(1)/3} = 3$
$f(1) = f(0+1) = 3^{f(0)/3} = 3$
$f(0) = 3$

2. **Recursively Defined Sequences**

(a) $a_n = 4n - 2$

$$a_n = \begin{cases} -2 & \text{for } n = 0 \\ a_{n-1} + 4 & \text{for } n > 0 \end{cases}$$

Basis step:
$a_0 = 4(0) - 2$
$a_0 = 0 - 2 = -2$

Induction step:
$a_k \to a_{k+1}$

Let n = k:
$a_k = 4k - 2$

Let n = k + 1:
$a_{k+1} = 4(k+1) - 2$

Recursive function:
$a_{k+1} = a_k + 4$

$a_k + 4 = 4k + 2$
$(4k - 2) + 4 = 4k + 2$

$\therefore a_n$ is true since a_{k+1} is $4k + 2$ and a_{k+1} from the recursion function also equates to $4k + 2$.

(b) $a_n = 1 + (-1)^n$

$$a_n = \begin{cases} 2 & \text{for } n = 0 \\ 2 - a_{n-1} & \text{for } n > 0 \end{cases}$$

Basis step:
$a_0 = 1 + (-1)^0$
$a_0 = 1 + 1 = 2$

Induction Step:
$a_k \to a_{k+1}$

Let n = k:
$a_k = 1 + (-1)^k$

Let n = k + 1:
$a_{k+1} = 1 + (-1)^{k+1}$

Recursive function:
$a_{k+1} = 2 - a_k$
$a_{k+1} = 2 - (1 + (-1)^k) = 1 + (-1)^{1+k}$

$\therefore a_n$ is true since a_{k+1} is $1 + (-1)^{1+k}$ and a_{k+1} from the recursion function also equates to $1 + (-1)^{1+k}$.

(c) $a_n = n(n - 1)$

$$a_n = \begin{cases} 0 & \text{for } n = 0 \\ a_{n-1} + 2(n-1) & \text{for } n > 0 \end{cases}$$

Basis step:
$a_0 = 0(0 - 1)$
$a_0 = 0(-1) = 0$

Induction Step:
$a_k \to a_{k+1}$

Let n = k:
$a_k = k(k - 1)$

Let n = k + 1:
$a_{k+1} = (k+1)((k+1)-1)$
$a_{k+1} = (k+1)(k)$
$a_{k+1} = (k+2-1)(k)$
$a_{k+1} = k^2 + 2k - k$
$a_{k+1} = k(k-1) + 2k$
$a_{k+1} = a_k + 2k$

Recursive function:
$a_{k+1} = a_{(k+1)-1} + 2(k-1)$
$a_{k+1} = a_k + 2kk$

$\therefore a_n$ is true since a_{k+1} is $a_k + 2k$ and a_{k+1} from the recursion function also equates to $a_k + 2k$.

(d) $a_n = n^2$

$$a_n = \begin{cases} 0 & \text{for } n = 0 \\ a_{n-1} + 2n - 1 & \text{for } n > 0 \end{cases}$$

Basis step:
$a_0 = 0^2$
$a_0 = 0$

Induction Step:
$a_k \to a_{k+1}$

Let n = k:
$a_k = k^2$

Let n = k + 1:
$a_{k+1} = (k+1)^2$
$a_{k+1} = k^2 + 2k + 1$
$a_{k+1} = a_k + 2k + 1$

Recursive function:
$a_{k+1} = a_{(k+1-1)} + 2(k+1) - 1$
$a_{k+1} = a_k + 2k + 1$

$\therefore a_n$ is true since a_{k+1} is $a_k + 2k + 1$ and a_{k+1} from the recursion function also equates to $a_k + 2k + 1$.

3. **Recursively Defined Sets**

(a) **Basis step**: The empty string in S , i.e., $\epsilon \in S$
Induction step: If $x \in S$ is a string in S, then $0x1$ is a string in S, i.e., $0x1 \in S$

S can be:
$\{\{\epsilon\}, \{x\}, \{0x1\}, \{000111\}\}$

The set of S contains strings that are comprised of the empty set or 0's and 1's. The strings in the set of S also have an equal number of 0's and 1's, where the string begins with 0. The strings in the set of S can also be infinitely long given that it has an equal number of 0's and 1's and begins with 0.