Star trek

The Planar Circular Restricted Three-Body Problem: equations of motion

[after Wang Sang Koon, Martin W. Lo, Jerrold E. Marsden, Shane D. Ross, Chaos, vol. 10, n. 2, pp. 4227-469 (2000)]

Two main bodies, such as the Earth and the Moon, have masses $m_E = 1-\mu$ and $m_M = \mu$, so that the total mass is normalized to one. These bodies rotate in the plane counter-clockwise about their common center of mass (placed at the origin) and with the angular velocity normalized to one. The third body, the spacecraft, has a small mass that does not affect the primary motion, and is free to move in the plane. One chooses a rotating coordinate system so that the origin is at the center of mass and the Earth and the Moon are fixed on the x-axis at $(-\mu, 0)$ and $(1-\mu, 0)$, respectively. Let (x, y) be the position of the spacecraft in the plane (so these are the position coordinates relative to the positions of the Earth and the Moon, not relative to an inertial frame). The equations of motion of the spacecraft are

$$\ddot{x} - 2\dot{y} = \Omega_x$$
$$\ddot{y} + 2\dot{x} = \Omega_y,$$
$$\text{where } \Omega = \frac{x^2+y^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{\mu(1-\mu)}{2}$$

Here Ω_x, Ω_y are the partial derivatives of Ω and $r_1 = \sqrt{(x+\mu)^2 + y^2}$, $r_2 = \sqrt{(x-1+\mu)^2 + y^2}$ are the distances of the comet from the main bodies.

For the following tasks fix $\mu = 0.05$.

Task 1: Show that the equations of motion follow from the Hamiltonian

$$\frac{(p_x+y)^2}{2} + \frac{(p_y-x)^2}{2} - \Omega(x, y)$$

and show that the "Energy" (Jacobi integral) $E = \Omega - \frac{\dot{x}^2+\dot{y}^2}{2}$ is conserved.

Equilibrium points and the Hill's region

Equations of motion have 5 equilibrium points (Lagrange points) L_{1-5}, see "Lagrangian points" in Wikipedia.

Task 2: Find the coordinates of Lagrange points $L_{1,2,3}$ lying on the symmetry line $y = 0$ numerically, and calculate the values of $\Omega_{1,2}$ at points $L_{1,2}$.

The set of points x, y for which $\Omega(x, y) \geq E$ defines a region available for a spacecraft with energy E. It is called Hill's region.

Task 3: Using contour plots in gnuplot or other drawing program, draw several Hill's regions. Draw Hill regions around $E = \Omega_2$. For which E a "neck" allowing to go from outside the Moon to the inside region (and vice verse) appears?

Symplectic integration

We split the Hamiltonian in two parts

$$H = H_1 + H_2 \qquad H_1 = \frac{(p_x + y)^2}{2} + \frac{(p_y - x)^2}{2} \qquad H_2 = -\Omega$$

Task 4: Write explicitly evolution operators $\Psi_{1,2}(\Delta t)$ over time step Δt for Hamiltonians H_1 and H_2. Combine these operators in the Störmer-Verlet integration step [see sec. 4.2.2 of Leimkuhler, Reich, *Simulating Hamiltonian Dynamics*.]

$$\Psi_2(\frac{\Delta t}{2})\Psi_1(\Delta t)\Psi_2(\frac{\Delta t}{2})$$

Test accuracy of energy conservation (for a long enough sattelite trajectory) in depenendence on the time step.

Finding interesting treks

Suppose that the radii of bodies are $R_1 = 0.2$, $R_2 = 0.01$.

Task 5: Fix a position on the Earth surface, and an initial energy such that the spacecraft cannot escape the Earth-Moon system, but can reach the Moon. By changing the direction of the initial velocity, find a trajectory which at least 4 times rotates around the moon and then returns to the Earth. What maximal number of rotations can you find?

Task 6: Fix a position on the Earth surface, and an initial energy such that the spacecraft can barely escape the Earth-Moon system. By changing the direction of the initial velocity, find a trajectory which at least 4 times rotates around the moon and then escapes the Earth-Moon system. What maximal number of rotations can you find?

Task 7: Find a trajectory that comes from the outer space, makes several rotations around the Moon, then several rotations around the Earth, and then escapes to the outer space.