lecture 2023-12-20

Lie Algebra / Kommuatoren

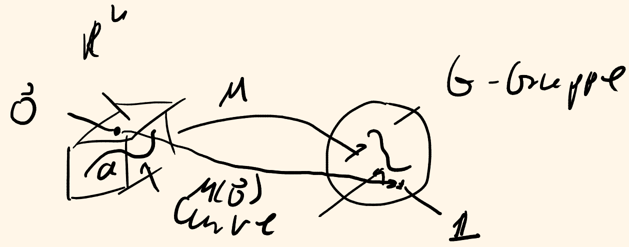
nicht kommutierende Operatoren, z.b. $[\hat{x}, \hat{p}] = i\hbar$

diese entstehen aus Symmetrien, kommt aus der Darstellung und den Symmetrien im Quantensystem

represent "classical" transformations the operators

starting point for creating Lie Algebra

parametrise a group with real numbers



stetig (homeomorph)

topologie erhalten

e.g. matrix $M = M(\vec{a})$

fix $M(\vec{0}) = \mathbb{1}$

consider "small parameters" $\delta\vec{a}$

matrix not in the group

$$M(\delta\vec{a}) = \mathbb{1} + \sum_i \delta a^i X_i + \dots \qquad O(|\delta a|^2)$$

has properties for Taylor expansion

generators

rotation in 2D

$\vartheta = 0 \Rightarrow M(0) = \mathbb{1}$

$$M(\vartheta, \dots) = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}$$

$\Downarrow$

$\det M(\vartheta) = 1$

expand the functions

$$M(\vartheta) = \mathbb{1} + \delta\vartheta \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{X_\vartheta} + \dots$$

trace X_ϑ

antisymmetric

boost (Lorentz boost) in t, x "plane"

$$\Lambda(v_x) = \begin{pmatrix} \gamma & v_x \gamma / c^2 \\ v_x \gamma & \gamma \end{pmatrix}$$

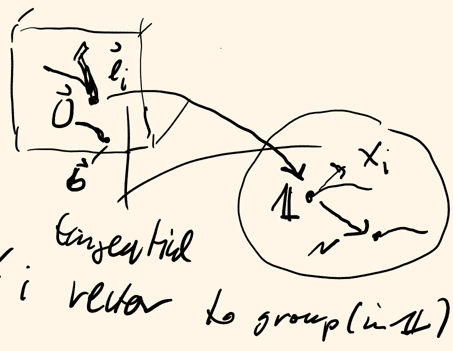
expand

$$\approx \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{v_x}{c} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{X_{v_x}} + \dots$$

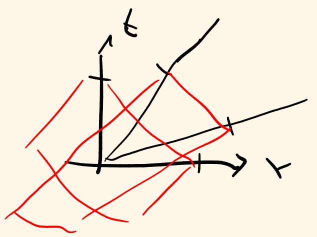
symmetric

$\det X_{v_x} = -1$

$\vec{e}_i, X_i$ tangential vector to group (in $\mathbb{1}$)



displace curve to the new parameter b and the curve is displaced to the group element N



diagonal shear

Lie-Algebra: $\mathcal{L}$ vector space

differenzierbare Gruppen

elements in a vector space(real or complex scalars) unitary trick

inner "product", anti-symmetric Lie-Product bilinear

$[X, Y] \in \mathcal{L} \quad X, Y \in \mathcal{L}$ not a scalar product

for matrices $[X, Y] = XY - YX$ with Jacobi-identity (see tabelsheets)

"structure constants"

be $\{X_i\}$ a basis of $\mathcal{L}$ only linear independent
n vectors

scalar product for matrices "Hilbert-Schmidt-Frobenius"

$$\mathrm{tr}\{A^+ B\} = \mathrm{tr}\{B^+ A\}$$

$$\mathrm{tr}\{\underbrace{X_{vx}^+ X_{vx}}_{=\mathbb{1}}\} = 2$$

$$A = (a_{ij})$$

$$\mathrm{tr}\{\underbrace{X_{v}^+ X_{v}}_{=\mathbb{1}}\} = 2$$

$$\mathrm{tr}\{A^+ B\} = \sum_{j,i} b_{ij}^* a_{ij}$$

$$(A^+)_{ji} a_{ij}$$

$$\mathrm{tr}\{A^+ A\} = \sum_{j,i} a_{ij}^* a_{ij} = \sum_{ij} |a_{ij}|^2$$

with killing form we can define some "scalar product" but could be not positiv definit

$$\underbrace{[X_i, X_j]}_{\text{is element of group}} = \sum_k \underbrace{c_{ij}^k}_{\text{structure constants}} X_k$$

∇ area-preserving 2D maps SL(2) (special linear maps group) if with R or C it means matrices are real or complex

Lie algebra contains $X_{v_1} \; X_{vx}$

third "Pauli" matrix

$$\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} = -2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

trace of commutators are always zero.

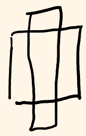
$tr[X_{v_1}, X_{v_2}] = 0$

$[\mathcal{L}, \mathcal{L}] \subseteq \mathcal{L}$ a sub algebra

$J = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ generator rotation

$k = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ "boost" or

$L = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ "squeeze shear" $\begin{pmatrix} 1+w & 0 \\ 0 & 1-w \end{pmatrix} \approx \exp(wL) = \begin{pmatrix} e^{w} & 0 \\ 0 & e^{-w} \end{pmatrix}$



$[J, k] = -2L$

$[J, L] = +2k$

$[k, L] = +2J$

	J	k	L
J	0	$-2L$	$+2k$
k	$+2L$	0	$2J$
L	$-2k$	$-2J$	0

structure constants of $sl(2)$

small letters for Lie-algebra

not cyclic

compare Pauli matrices

	σ_1	σ_2	σ_3
σ_1	0	$2i\sigma_3$	$-2i\sigma_2$
σ_2	$-2i\sigma_3$	0	$2i\sigma_1$
σ_3	$2i\sigma_2$	$-2i\sigma_1$	0

structure constants $+\varepsilon_{jkl}$

factor i .. $SU(2)$

group element $u = \exp(i\vartheta\sigma_1)$

$i\sigma_1$ is generator (not hermitian)

	$i\sigma_1$	$i\sigma_2$	$i\sigma_3$
$i\sigma_1$	0	$-2i\sigma_3$	$+2i\sigma_2$
$i\sigma_2$	$+2i\sigma_3$	0	$-2i\sigma_1$
$i\sigma_3$	$-2i\sigma_2$	$+2i\sigma_1$	0

algebra $su(2)$

have cyclic $[i\sigma_j, i\sigma_k] = 2\varepsilon_{jkl} i\sigma_l$

structure constants

Levi-Civitá

unitary matrices