Continuous Symmetry Groups in Quantum Physics

Winter Term 2023/24
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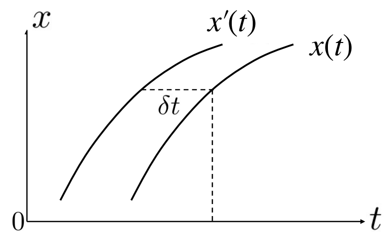
Problem Set No 0 Discussion: 18 October 2023 + X

Problem 0.1 – Shifting Time (3 points)

The figure on the right illustrates two trajectories that are "shifted in time" relative to another.
(a) Describe in words how the shifted trajectory $x'(t)$ differs from $x(t)$. Try to use the words "shift forward in time".
(b) Give an equation that relates the two trajectories.



(c) Expand this equation to first order in δt. The mathematical operation that provides the difference $x'(t) - x(t)$ (for small δt) is called the "generator" of the transformation (here: of time translation).

Problem 0.2 – Shifting Space – globally (4 points)

Imagine a system of particles that interact via a potential $V(x_1, \ldots, x_n)$. Consider the situation that the potential only depends on the differences $x_i - x_j$ of the particle coordinates.

(1) Show that such a potential is *globally translation invariant*, i.e. for all b, we have

$$V(x_1 + b, \ldots, x_n + b) = V(x_1, \ldots, x_n)\,. \tag{0.1}$$

Expand this equation for small b to first order.

(2) Imagine that the particles perform small-amplitude oscillations around their equilibrium positions, $x_i = x_{0i} + u_i$. Justify that the equations of motion can be written in the form

$$m_i \ddot{u}_i + \sum_j K_{ij} u_j = 0 \tag{0.2}$$

with a symmetric matrix K_{ij}.

(3) Show that the "center-of-mass mode" $u_i = u$ provides a solution to Eq. (0.2) with $\ddot{u}_i = 0$.

(4) In molecular and solid-state physics, it is customary to introduce the "dynamical matrix" $D_{ij} = K_{ij}/\sqrt{m_i m_j}$. Show that it is symmetric and that its eigenvectors provide the "normal (vibrational) modes" of the system.

Problem 0.3 – Runge-Lenz Vector (3 points)

In Lagrangian mechanics (and in the lecture), a key result is the theorem of Emmy Noether that links the generator of a symmetry transformation to a conservation law. Recall its statement: We consider a transformation $\mathcal{T}$ of the generalised coordinates q_i of the form

$$\mathcal{T} \,:\, q_i \mapsto q_i + \delta\varepsilon f_i(q, \dot{q}; t)\,, \tag{0.3}$$

where the function $f_i(q, \dot{q}; t)$ depends on all the coordinates $q = \{q_i\}$, their time derivatives and maybe on time, and $\delta\varepsilon$ is a "small parameter" (infinitesimal transformation). If the change ("variation") in the Lagrangian can be written as a total time derivative,

$$\delta L = L(\mathcal{T} q, \mathcal{T} \dot{q}; t) - L(q, \dot{q}; t) = \delta\varepsilon \frac{\mathrm{d}K}{\mathrm{d}t}\,, \tag{0.4}$$

then the quantity

$$F = \sum_i f_i \frac{\partial L}{\partial \dot{q}_i} - K \tag{0.5}$$

is conserved, i.e., $\mathrm{d}F/\mathrm{d}t = 0$.

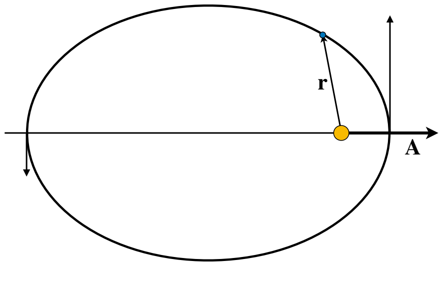
(1) Consider a point mass m in a spherically symmetric potential $V = -\alpha/r^n$ ($n = 1, 2, \ldots$). Be $\boldsymbol{p} = m\dot{\boldsymbol{r}}$ its momentum and $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ its angular momentum relative to the force centre $\boldsymbol{r} = \boldsymbol{0}$. We apply the transformation

$$\boldsymbol{r} \mapsto \boldsymbol{r} + \delta\boldsymbol{\varepsilon} \times \boldsymbol{L} \tag{0.6}$$

with an infinitesimally small vector $\delta\boldsymbol{\varepsilon}$. Assuming the variation of the Lagrangian to take the form

$$\delta L = \frac{n\alpha}{r^{n+2}} \left[r^2 \left(\delta\boldsymbol{\varepsilon} \cdot \boldsymbol{p}\right) - (\delta\boldsymbol{\varepsilon} \cdot \boldsymbol{r})(\boldsymbol{r} \cdot \boldsymbol{p}) \right] = \frac{\mathrm{d}}{\mathrm{d}t}\left(m\alpha \frac{\delta\varepsilon \cdot \boldsymbol{r}}{r} \right) \tag{0.7}$$

(the last expression only holds for $n = 1$, Kepler problem and Coulomb potential), show that the so-called *Runge-Lenz vector* is conserved (check carefully all signs)



$$\boldsymbol{A} = \boldsymbol{p} \times \boldsymbol{L} - m\alpha \frac{\boldsymbol{r}}{r}\,. \tag{0.8}$$

(2) Construct $\boldsymbol{A}$ graphically as shown in the figure and check that its direction is parallel to the major axis of the Kepler ellipse.

(3) Final checks: – Prove Eq. (0.7). – The excentricity of the ellipse is $|\boldsymbol{A}|/m\alpha$.

Problem 0.4 – Variations on Jacobi (4 points)

We are going to use extensively commutators $[\cdot, \cdot]$ in the lecture because they provide a way to implement and check symmetries in quantum physics. With the usual definition, the commutator satisfies the *Jacobi identity*:

$$[A\,,[B\,,C]] + [B\,,[C\,,A]] + [C\,,[A\,,B]] = 0 \tag{0.9}$$

(This is so trivial and tedious that you are not required to check it.)

(1) Re-order the terms in Eq. (0.9) to find answers for questions à la "When I first commute C with B (from the left) and then with A (from the left), and compare to the other way round, I get …" and "This funny commutator is a 'product', but it is not 'associative'. Its 'defect' of being associative is given by …"

(2) Show that the Jacobi identity holds for double vector products $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$ between ordinary 3D-vectors.

(3) Show that Eq. (0.9) holds for the Poisson brackets $\{\cdot, \cdot\}$ between functions on phase space with the definition

$$\{f, g\} = \sum_i \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \sum_i \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x_i} \tag{0.10}$$

Prove the Poisson theorem: "The Poisson bracket of two conserved quantities is again conserved." (A quantity f is called "conserved" when its Poisson bracket with the Hamiltonian H vanishes.)

(4) Consider the (huge!) set of first-order differential operators $F = \sum_i f_i \partial/\partial x_i$ acting on smooth functions of variables x_i ($i = 1, 2 \ldots$). Show that the commutator $[F, G]$ defined by its action on a function φ

$$[F, G]\varphi = F[G\varphi] - G[F\varphi] \tag{0.11}$$

is again a differential operator of the same type. And again, the Jacobi identity holds …but is there any need to check that explicitly?

Problem 0.5 – Trivial up to a Phase and Unit Operators (2 points)

In quantum physics, you know that one cannot differentiate two states (ket vectors) that only differ by a global phase factor. (In mathematical jargon, the physical space of kets is a *projective Hilbert space*.)

Prove the following theorem in this context: If C is a (linear) operator whose action on any ket is a multiplication with a complex number, $C|\psi\rangle = z(\psi)|\psi\rangle$, then C is proportional to the unit operator. (In other words, the number z does *not* depend on ψ.)