Continuous Symmetry Groups in Quantum Physics

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Problem 1.1 – Invariance and Symmetry Groups (5 points)

We see in the lecture that a symmetry transformation can be defined with respect to an "invariant property", e.g.: the transformed trajectory $\boldsymbol{r}'(t)$ solves the same equations of motion as $\boldsymbol{r}(t)$.

(a) Show that such a property naturally implies a group structure on the set of symmetry transformations.

(b) Consider the set of $d \times d$ matrices R that leave the scalar product between two real vectors invariant, i.e., $(R\,\mathbf{x}) \cdot (R\,\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$. Show that they form the group $\mathrm{O}(d)$.

(c) In $2d$-dimensional phase space, a transformation $(x, p) \mapsto (x', p') = T(x, p)$ is called "canonical", if the new coordinates x', p' are "canonically conjugate", i.e. if their Poisson bracket is $\{x', p'\} = 1$ (if $2d > 2$, read: $\{x'_i, p'_j\} = \delta_{ij}$ and $\{x'_i, x'_j\} = 0$ etc.). Consider the case that x', p' are the solutions to Hamilton's equations. Take the time derivative of their Poisson bracket and show that it is zero: "Time evolution is a canonical transformation". Give an example for a nonlinear transformation T.

(d) Consider the vector space of linear functions on phase space, e.g.

$$Q = Q(x, p) = a\,x + b\,p \tag{1.1}$$

Show that the Poisson bracket $\{Q_1, Q_2\}$ is the area spanned by the vectors $(a_1, b_1)^\mathsf{T}$ and $(a_2, b_2)^\mathsf{T}$. Check the sign of this area and generalise to a $2d$-dimensional phase space.

(e) If Hamilton's equations are linear, you have learned in your analysis lecture that the time-evolved coordinates $x'(t), p'(t)$ depend linearly on the initial conditions. Writing this transformation in block form

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = \begin{pmatrix} R & A \\ B & S \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} \tag{1.2}$$

construct a few simple solutions to the invariance conditions

$$R\,S^\mathsf{T} - A\,B^\mathsf{T} = 1\,, \quad R\,A^\mathsf{T} - A\,R^\mathsf{T} = 0\,, \quad \ldots \tag{1.3}$$

Note: for $2d = 2$ or 4, there are at most 3 or 10 "independent" transformations. They generate the so-called "symplectic groups" Sp(2), Sp(4).

Problem 1.2 – Rotations, intuitively (4 points)

In problem 1.1, we have already encountered the matrices of the orthogonal group $\mathrm{O}(d)$. We specialise here to the case of SO(3) and construct the Rodriguez formula, an explicit expression for a rotation matrix.

(a) Check the following count of independent parameters in SO(3): Its matrices R are orthogonal and have unit determinant. The first condition is equivalent to $R^{\mathsf{T}} R = \mathbb{1}$ which is an equation between symmetric matrices, with 6 independent entries. Hence of the 9 parameters in R, only 3 are left free. For the determinant, we get $1 = \det \mathbb{1} = (\det R^{\mathsf{T}})\,(\det R) = (\det R)^2$, giving the solutions $\det R = \pm 1$. Only one of the solution is allowed, but this does not restrict the number of real parameters. (This removes a "disconnected component" from the group. For the general case of SO(3), a similar count leads to $\frac{1}{2}d(d-1)$ parameters or "dimensions".)

(b) A rotation $R = R_{\boldsymbol{u}}(\theta)$ can be specified by a unit vector $\boldsymbol{u}$ (rotation axis) and an angle θ. Make a count of parameters that this captures all dimensions of SO(3). Rotations about the same axis commute, right? This implies $R_{\boldsymbol{u}}(\theta_1 + \theta_2) = R_{\boldsymbol{u}}(\theta_1)\, R_{\boldsymbol{u}}(\theta_2)$, right?

(c) Make a sketch of the action of such a rotation on a vector $\boldsymbol{r}$. Adopt the "right-hand rule" for the sense of the rotation. Show that for a small angle $\delta\theta$, we have

$$\delta\theta \to 0 : \qquad R_{\boldsymbol{u}}(\delta\theta)\, \boldsymbol{r} = \boldsymbol{r} + \delta\theta\, (\boldsymbol{u} \times \boldsymbol{r}) + \ldots \tag{1.4}$$

We say that "the vector product generates a rotation".

(d) For an arbitrary angle, check the following differential equation

$$R_{\boldsymbol{u}}(\theta + \mathrm{d}\theta) = R_{\boldsymbol{u}}(\mathrm{d}\theta)\, R_{\boldsymbol{u}}(\theta) = R_{\boldsymbol{u}}(\theta) + \mathrm{d}\theta\, \boldsymbol{u} \times R_{\boldsymbol{u}}(\theta) + \ldots \tag{1.5}$$

whose formal solution is, of course, $R_{\boldsymbol{u}}(\theta) = \exp[\theta\, (\boldsymbol{u} \times \_)]$ where $\boldsymbol{u} \times \_$ is shorthand for the linear operation "form the vector product with $\boldsymbol{u}$ from the left".

(e) Compute recursively the powers of $\boldsymbol{u} \times \_$: even (nonzero) powers turn out proportional to $\mathbb{1} - \boldsymbol{u}\, (\boldsymbol{u} \cdot \_)$ (a projector orthogonal to $\boldsymbol{u}$) and odd powers proportional to $\boldsymbol{u} \times \_$. Collect the even and odd powers in the Taylor series of $\exp[\theta\, (\boldsymbol{u} \times \_)]$ and find the Rodrigues formula:

$$\exp[\varphi\, \boldsymbol{u} \times \_] = \mathbb{1} + \sin(\varphi)\, \boldsymbol{u} \times \_ + (\cos\varphi - 1)\left[\mathbb{1} - \boldsymbol{u}\, (\boldsymbol{u} \cdot \_)\right] \tag{1.6}$$