Continuous Symmetry Groups in Quantum Physics

Winter Term 2023/24
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Problem Set No 6 Hand out: 19 January 2024
Discussion: 24 and 31 January 2024

Problem 6.1 – Translating boosts (3 points)

(a) A Galilei transformation can be represented by the linear map

$$\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{v}\, t\,, \quad t' = t \tag{6.1}$$

Check that this commutes with spatial translations $\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{b}$. This is a bit strange because in quantum mechanics, Galilei transformations are generated by the centre-of-mass operator (multiplied by $-M$), while translations are generated by the momentum operator.

(b) In the language of Poisson brackets $\{f\,, g\} = \partial_x f\, \partial_p g - \partial_p f\, \partial_x g$ for Hamilton mechanics, a Galilei transformation is generated by

$$\boldsymbol{r}' = \boldsymbol{r} + \{\boldsymbol{r}\,, K(\boldsymbol{v})\}\,, \quad \boldsymbol{p}' = \boldsymbol{p} + \{\boldsymbol{p}\,, K(\boldsymbol{v})\} \quad \text{with} \quad K(\boldsymbol{v}) = \boldsymbol{v}\cdot(-M\boldsymbol{r} + \boldsymbol{p}\, t) \tag{6.2}$$

The same scheme generates translations via $T(\boldsymbol{b}) = \boldsymbol{p}\cdot\boldsymbol{b}$. Check that the Poisson bracket $\{K(\boldsymbol{v})\,, T(\boldsymbol{b})\}$ between the generators is nonzero, although simply a constant in phase space. We may say that the Poisson algebra of the generators for Poincaré symmetry extends by a constant the set of generators of coordinate transformations.

(c) In the relativistic setting, a Galilei transformation becomes a Lorentz boost

$$\boldsymbol{r}' = \gamma(\boldsymbol{r} + \boldsymbol{v}\, t)\,, \quad t' = \gamma(t + \boldsymbol{r}\cdot\boldsymbol{v}/c^2) \tag{6.3}$$

Consider two infinitesimal boosts with velocities $\boldsymbol{u}, \boldsymbol{v}$ and compute their commutator. Show that you get a term of order $1/c^2$ which has the structure of a rotation around the axis $\boldsymbol{u}\times\boldsymbol{v}$.

Problem 6.2 – Scalars and vectors (3 points)

(a) A *vector* operator is defined in quantum mechanics as a set of three operators A_i ($i = 1, 2, 3$) whose commutator with the angular momentum operator $\boldsymbol{J}$ has the form (Einstein summation)

$$[J_i\,, A_j] = \mathrm{i}\hbar\, \epsilon_{ijk} A_k \tag{6.4}$$

A *scalar* operator commutes with $\boldsymbol{J}$. Show that the scalar (the vector) product of two vector operators is a scalar (vector) operator.

(b) A set of four operators (A^μ) is a (contravariant) 4-*vector* provided the following commutators hold. Split into temporal and spatial components $(A^\mu) = (A_0, \boldsymbol{A})$: A_0 is scalar, and $\boldsymbol{A}$ is vectorial [Eq. (6.4)]. With respect to the generators $\boldsymbol{K}$ of Lorentz boosts, we have

$$[\boldsymbol{K}, A_0] = -\mathrm{i}\frac{\hbar}{c}\boldsymbol{A}, \quad [K_i, A_j] = -\mathrm{i}\frac{\hbar}{c}\delta_{ij}A_0 \tag{6.5}$$

Show that the Minkowski product $A^\mu V_\mu = A_0 V_0 - \boldsymbol{A}\cdot\boldsymbol{V}$ of two 4-vectors is a Lorentz scalar, i.e., it commutes with $\boldsymbol{J}$ *and* $\boldsymbol{K}$.

(c) The momentum operator $(P^\mu) = (H/c, \boldsymbol{P})$ is a 4-vector. Consider the commutators $[P_i, K_j]$ in the non-relativistic limit and check how to recover the standard non-relativistic commutation relations between momentum and centre-of-mass coordinate $\boldsymbol{K} = -M\boldsymbol{R}$.

Problem 6.3 – The Lorentz algebra and its generators (1 point)

(a) The commutators for the generators $\boldsymbol{J}$ and $\boldsymbol{K}$ of the Lorentz group are

$$[J_i, J_k] = \mathrm{i}\hbar\,\epsilon_{ikl}J_l, \qquad [J_i, K_j] = \mathrm{i}\hbar\,\epsilon_{ijl}K_l, \qquad [K_i, K_j] = -\mathrm{i}\frac{\hbar}{c^2}\,\epsilon_{ijk}J_k, \tag{6.6}$$

We set $\hbar = c = 1$ in the following. Show that the complex linear combinations

$$\boldsymbol{A} = \boldsymbol{J} + \mathrm{i}\,\boldsymbol{K}, \qquad \boldsymbol{B} = \boldsymbol{J} - \mathrm{i}\,\boldsymbol{K} \tag{6.7}$$

commute and satisfy separately SU(2)-commutators,

$$[\boldsymbol{A}, \boldsymbol{B}] = 0, \qquad [A_i, A_j] = 2\mathrm{i}\,\epsilon_{ijk}A_k, \qquad [B_i, B_j] = 2\mathrm{i}\,\epsilon_{ijk}B_k \tag{6.8}$$

At the level of the algebra, the Lorentz group is thus (complex) isomorphic to two SU(2) algebras, in jargon: lor $\simeq$ su(2) $\oplus$ su(2) (lowercase notation for Lie algebras)

(b) The group is not really a tensor product SU(2) $\otimes$ SU(2) because of the algebra complexification. After all, this tensor product is a compact group, and indeed, the Lorentz group is not compact. (Give an argument why.) To understand the relation between the group and the algebra, consider an irreducible representation of the A- and B-algebras with the quantum numbers $j = 0$ and $j' = 1/2$. By equating $\boldsymbol{A} = \boldsymbol{0}$ and $\boldsymbol{B} = \boldsymbol{\sigma}$ (why?), give the representation of the operators $\boldsymbol{J}$ and $\boldsymbol{K}$. Compute the exponentials $R(\boldsymbol{n}, \theta) = \exp(-\mathrm{i}\theta\boldsymbol{n}\cdot\boldsymbol{J})$ (for rotations) and $S(\boldsymbol{q}) = \exp(-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{K})$ (for boosts) and apply them to a "spin up" state (eigenvector of J_3 with eigenvalue $+\frac{1}{2}$). The particles corresponding to this representation are called "left-handed Weyl fermions". In the standard model, they correspond to neutrinos, electrons, and quarks (all massless) before the symmetry breaking due to the Higgs mechanism.