Continuous Symmetry Groups in Quantum Physics

Winter Term 2023/24

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Problem Set No 5 Hand out: 21 December 2023

Discussion: 10 and 17 January 2024

Problem 5.1 – From one commutator to the other (1 point)

In a Lie group represented by matrices $M(\mathbf{a})$, expand the group commutator that we defined as $C(\delta\mathbf{a}, \delta\mathbf{b}) = [M(\delta\mathbf{a})]^{-1}\,[M(\delta\mathbf{b})]^{-1}\,M(\delta\mathbf{a})\,M(\delta\mathbf{b})$ for small parameters $\delta\mathbf{a}$, $\delta\mathbf{b}$ around zero (so that $M(\delta\mathbf{a}) = \mathbb{1} + \sum_i \delta a^i X_i + \ldots$ with a set of generators $\{X_i\}$). As a first step, justify that the result is of the form

$$C(\delta\mathbf{a}, \delta\mathbf{b}) = \mathbb{1} + \sum_{ij} \delta a^i \delta b^j G_{ij} \tag{5.1}$$

with no linear terms. (Consider the special cases $\delta\mathbf{a} = \mathbf{0}$ or $\delta\mathbf{b} = \mathbf{0}$.)

Problem 5.2 – Differential operators and the Lie bracket (5 points)

In this exercise, you play with differential operators of a more or less general form with the aim to find “closed Lie algebras” or “invariant vector spaces”.

(a) Consider the components of the orbital angular momentum operator like

$$L_3 = -\mathrm{i}\hbar\left(x\partial_y - y\partial_x\right) \tag{5.2}$$

Compute its action on quadratic expressions in the three variables x, y, z. Check that these “binomials” form a 6-dimensional vector space. Identify an invariant sub-space (of dimension one) where invariance means that L_3 maps the corresponding polynomial to zero. Compute a matrix representation of L_3 in the “orthogonal” five-dimensional subspace. Think of a “natural” choice for the basis in this space.

(b) Check how to construct a Lie algebra with differential operators of the form $D^{n,m} = q^{(n)}(x_i)\,\partial_x^m$ taking the usual commutator (in the operator sense) for the Lie bracket. Here $q^{(n)}$ are n’th order polynomials in the 3D coordinates x_i, and ∂^m is a m’th order differential operator. You may start playing with the special value $m = 1$ (first order derivatives) and check what values of n give a (“closed”) Lie algebra. “Interesting” cases would be where the algebra is not infinite-dimensional, i.e., not all values $n, m = 0, 1, 2\ldots$ are allowed.

(c) Evaluate the Lie bracket (commutator) for covariant derivatives that appear for charged particles, e.g.

$$\left[\hat{p}_i - \mathrm{i}\, e A_i\, , \hat{p}_j - \mathrm{i}\, e A_j\right] \tag{5.3}$$

with the standard momentum operator $\hat{\mathbf{p}} = -\mathrm{i}\hbar\nabla$. The nonzero result you get may be called a "curvature" because the vector potential $\mathbf{A}$ prescribes how the complex amplitude (phase) of a wave function has to be "parallel transported" from one position to a neighboring one. Repeat the game with the components of the gradient operator in spherical coordinates, $\nabla_r = \partial/\partial r$, $\nabla_\theta = (1/r)\partial/\partial\theta$, $\nabla_\varphi = 1/(r\sin\theta)\partial/\partial\varphi$. By choosing these curvilinear coordinates, one also gets a "curvature". It may be called an artefact of this choice of coordinates, since it does not appear in Cartesian coordinates. The final variation is to use a "non-Abelian" covariant derivative

$$\left[\hat{p}_i - \mathrm{i}\, e\, \sigma_a A_i^a\, , \hat{p}_j - \mathrm{i}\, e\, \sigma_b A_j^b\right] \tag{5.4}$$

where the σ_a are the Pauli matrices and there are actually three vector potentials $\mathbf{A}^a$ with $a = 1, 2, 3$ (summation over a, b in Eq. (5.4)). This kind of "parallel transport" appears for wave functions that have two components representing, for example spin or isospin states. It plays a key role in the theory of weak interactions.

Problem 5.3 – Rotations and their Lie group (3 points)

(a) We have seen how geometric transformations like translations and rotations can be implemented as acting on functions, e.g.

$$\psi(x) \mapsto \psi(x - b)\, , \qquad \psi(x, y) \mapsto \psi(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta) \tag{5.5}$$

Consider infinitesimal parameters δb and $\delta\theta$ and compute the generators of these transformations (that are differential operators). Check that you get up to a factor $\mathrm{i}\hbar$, the familiar momentum and angular momentum operators of quantum mechanics.

(b) In texts on differential geometry, one often identifies a vector field with a differential operator. This may be illustrated with the concept of a Killing vector field. Look up its definition. Consider again the orbital angular momentum operator (5.2) and construct a vector field by letting it act on the position vector $\mathbf{r}$ (which is, of course, also a vector field):

$$\mathbf{v}(\mathbf{r}) = L_3\, \mathbf{r} \tag{5.6}$$

What you get should be the Killing vector field corresponding to rotations around the z-axis (useful for a system with the corresponding symmetry). Play the game also for other components of the angular momentum operator and formulate a rule how the components of the Killing vector field (at a given position $\mathbf{r}$) are related to the coefficients of the angular momentum, understood as a differential operator.