lecture 2023-01-10 Lie Algebra for space-time transformations

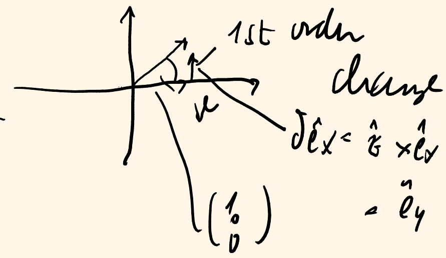
Galilei group
Lorentz group

{ rotations
switch between reference frames (inertial observers)

typical parametrisations

rotations $\hat n$, ϑ $\quad R_n(\vartheta) \approx \mathbb{1} + \vartheta \hat n \times \_ + \dots = \mathbb{1} + \vartheta X_{\hat n}$ generates rotations around n-axis

z-axis $\quad \begin{pmatrix} c & -s & 0 \\ +s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \approx \mathbb{1} + \begin{pmatrix} 1 & -\vartheta & 0 \\ \vartheta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \dots$



n axis vector has 3 components so we have 3 generators

$$= \mathbb{1} + \hat z \times \_ \qquad X_3 = X_{\hat z} = \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}$$

structure constants

$$X_i = X_1 = X_2 = X_3 \qquad X_n = \sum_i n_i X_i$$

$[X_i, X_j] = ?$

choose $n = i$, $m = j$ $\qquad + m \times (\_ \times n)$ $\qquad b(ca) - c(ba)$

$[X_{\hat n}, X_{\hat m}] = \hat n \times (\hat m \times \_) - m \times (n \times \_)$

Jacobi identity or bac-cab

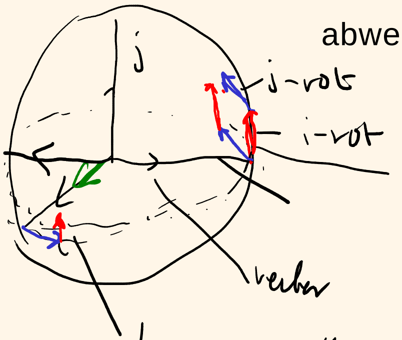
$$= m(n \cdot \_) - \cancel{\_(n \cdot m)} - n(m \cdot \_) + \cancel{\_(m \cdot n)}$$

$$= \_ \times (m \times n) = - \_ \times (n \times m)$$
$$= (n \times m) \times \_ = X_{\hat n \times \hat m}$$

$\hat n = e_i \quad \hat m = e_j \quad \hat n \times \hat m = e_i \times e_j = \varepsilon_{ijk} e_k$

hence $[X_i, X_j] = \varepsilon_{ijk} X_k$ $\quad$ structure constants $C_{ij}^k = \varepsilon_{ijk}$
("pure geometry") $\quad$ for $SO(3)$ rotations

abweichung ist structure constant



parallelogram (= commutator)
closes only up to 1st order

line not closed - rotation around k-axis

Hermitian generators

$J = iX$

$[J_k, J_l] = i\varepsilon_{klm} J_m$

same as angular momentum operators
why? angular momentum also elements of a lie algebra (basis of Lie Algebra)

$\vec{J}'$

to Lie group of quantum mechanis transformations
which is a representation of rotations

$\hat{T}_n(\varphi) \sim \hat{T}(\vec{a})$

if $R(\vec{a})R(\vec{b}) = R(\vec{c})$ the T's must satisfy

$T(\vec{a})T(\vec{b}) = T(\vec{c})$

a parametrised form of

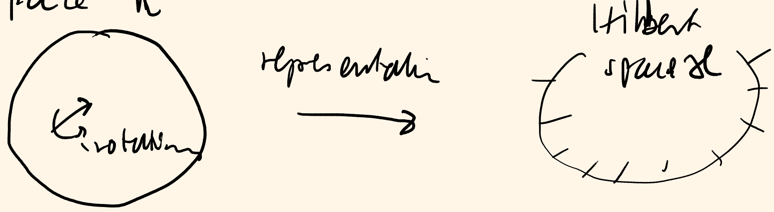
$$T(R)\,T(R') = T(RR')$$

there Lie algebras have the same structure constants

T "represents" R

space R^3

SO(3)



unitary trafo group

Wigner theorem:
a group of transformations that leaves scalar products invariant (propabilites are the same in all systems) $|\langle T(\psi)|T(\chi)\rangle| = |\langle\psi|\chi\rangle|$

is a group of unitaries or of anti-unitaries linear trafos.

symmetries in quantum mechanics with Wigner theorem

in quantum mechanis Hilbert spaces / systems

Quantum mechanic systems are representations("live in") of the classical symmetry groups mathematics of Lie algebras - representations are well understood and " just a few"

example rotations So(3) or SU(2) which unitary representation — representation — spin

for each dimension $2j+1 = 1, 2, \ldots$ — dim $= 1$ "scalar" (invariant) $j = 0$

$= 2$ "spinor" $j = 1/2$

exactly one irruducible representation — $= 3$ "vector" $j = 1$

Quanten zahl

(Quanten Zahl)

supergesetz: muss so sein