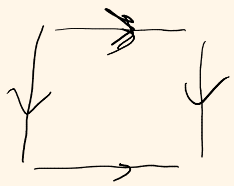
Lecture "symmetry" in QM

2023-11-08



arrow is time evolution

group basics:
books: Wigner (group theory) 1959, group theory in a nut shell Zee (2019) for physis
for math people elementary introduction 2015 Hall

definition: $G \ni a, b$
like "+" or "*" $a, b \mapsto a+b,\ a*b$ or $ab \in G$

G-set

rules: neutral element e with $eg = g \ \forall g \in G$

inverse g^{-1} with $g^{-1}g = e$

associative $(ab)g = a(bg)$

$eg = g$

$(g^{-1}g)g = g^{-1}(gg) \stackrel{?}{=} g$

groups could be transformation, so the same rules for them
in general non commutaive, $ab \neq ba$ "non-Abelian"

define "commutator" $C_{ab} = a^{-1}b^{-1}ab$

because $e = (ba)^{-1}ba$

$\stackrel{?}{=} a^{-1}b^{-1}ba = a^{-1}ea = e$

set of commutators, maybe it is a subset and it has some structures
commutator(derived) group

$C_G = \{C_{ab} \mid a, b \in G\}$ $\quad e \in C_G$ for a, a^{-1}

subgroup

product of numbers we use for the commutators

if $c, c' \in C_G$, then also $cc' \in C_G$ and $c^{-1} \in C_G$ $\Big\} \Leftrightarrow c^{-1}c' \in C_G$

$[\underbrace{(ba)^{-1}}_{C}ab]^{-1}(b'a')^{-1}a'b'$

$= (ab)^{-1}(ba)(b'a')^{-1}a'b'$ .... $p^{-1}q^{-1}pq$

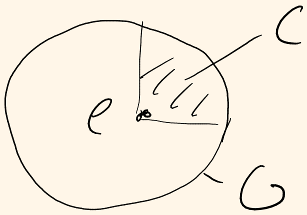
$= b^{-1}a^{-1}ba\,a'^{-1}b'^{-1}a'b'$

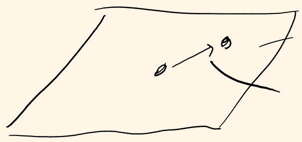
$a' = a^{-1}$

$b' = a$

simple groups are
groups with trivial subgroups

$= b^{-1}(a^{-1}b)a$





plane

vector some kind
of transformation

just one discrete subgroup

or quantize just one
multiplication in some numbers

subgroup in "transformation groups" G acting on M

be $s \in$ Object (being transformed by G)

$G_s = \{g \in G \mid g \cdot s = s\}$ ▷ origin   is a subgroup

$g, g' \in G_s$, $(g^{-1}g')s = g^{-1}\cdot(g'\cdot s) = g^{-1}\cdot s = s$

▷ G: all motions of space, rotation and translations

fixed point $O \in M$ is kept fixed $\Leftrightarrow$ G_O = all rotations

G = all matrices $d \times d$ be M a set of $d \times d$ matrices $\det(\cdot) \neq 0$

"general linear group" $GL(d)$ (anything)

$g \in G$, $m \in M \mapsto g m g^{-1}$ take $m = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ is invariant under G

matrices $= \mathbb{1}_d$ if $g g^{-1} = \mathbb{1}_d$

$\mapsto g m g^T$ orthogonal $O(d)$

$m = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$ is invariant under G $\quad g m g^T = m$ $\quad O(1,3)$

orthogonal group

contugation $\boxed{g m g^{-1} = m'}$

if G act on space $v \in \mathbb{R}^d$

sets numbers

$g \cdot \vec{v} = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} = \vec{v}'$ "bilinear form $\vec{v}, \vec{u} \mapsto \vec{u} \cdot \vec{v}$ "scalar product"

$(\vec{u}, \vec{v})_m = \vec{u}^T m \vec{v}$ Gewichtung weight (eg $\vec{u} = \vec{v} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$)

$g \downarrow$ transformation invariant $g^T m g = m$ $\quad t^2 - x^2 - y^2 - z^2 = (\vec{u}, \vec{u})_m$

$(g \cdot \vec{u}, g \cdot \vec{v})_m = (g \vec{u})^T m (g \vec{v}) = \vec{u}^T \underbrace{(g^T m g)}_{m} \vec{v} = \vec{u}^T m \vec{v} = (\vec{u}, \vec{v})_m$

conserved under transformation

Continuous version

another subgroup

1-parameter subgroup

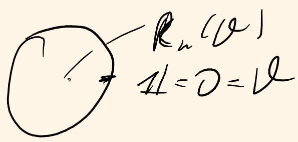
$e, g, gg = g^2, g^3, \ldots, g^{-1}, g^{-2}, \ldots$

rotation by axis $\hat{n}$, angles ϑ unit vector

"cyclic group generated by g"

$R_n(\vartheta) R_n(\vartheta') = R_n(\vartheta + \vartheta' \bmod 2\pi)$

$g^n g^m = \qquad g^{-n} g^m = g^{m-n}$



has same topological structure

boosts along axis $\hat{x}$,

rapidity q (or v, β, γ)

$B_{\hat{x}}(q) B_{\hat{x}}(q') = B_{\hat{x}}(q + q')$

$\beta = \frac{v}{c} = \tanh q$

$B_{\hat{x}}(q) = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}$

$t, x \qquad t', x'$

$x(0) \xrightarrow{q} x(q)$

derivative in space time
can define the infinitesimal operation
on the group

$= \exp\left(\frac{q}{c} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$