lecture 2023-11-29

representation(Darstellung) with matrix

many representations for the same group or operations

irreducible ("Building block") wigner/weyl: smallest units, description of elementary operations in quantum systems irrep (s) of symmetry groups

for every particle exist a representation

viele Dinge sind häßlich aber wahr und Dinge die schön sind sind meistens falsch, sagte ein Mathematiker

space of functions

Theorem: any unitary irrep of an Abelian group is 1D
(groups acts as phase factors)

$$\left(u^{-1} = u^{*} \qquad |u| = 1 \right)$$

▽ translations $T(d) \ni \vec{b}$ parameter of translation

representation $T(\vec{b}): \psi(\vec{x}) = e^{i\vec{k}\cdot\vec{x}} \mapsto \psi(\vec{x} - \vec{b}) = e^{-i\vec{k}\cdot\vec{b}}\,\psi(\vec{x})$

$$|u| = 1 \qquad u = e^{-i\vec{k}\cdot\vec{b}}$$

"plane waves, for each k , represent translations irrducible"

$$T(\vec{b})\,|k\rangle = e^{-i\vec{k}\cdot\vec{b}}\,|k\rangle$$

condition for representation $T(\vec{b})\,T(\vec{b}') = T(\vec{b} + \vec{b}')$

Σ be $|u\rangle, |v\rangle$ basis vectors of a representation $\mathcal{U}: g \mapsto \mathcal{U}(g)$

pick one $\mathcal{U}(g) \neq \mathbb{1}$ construct eigenvectors of $\mathcal{U}(g)\,|u\rangle = \alpha\,|u\rangle$

$$\mathcal{U}(g)\,|v\rangle = \beta\,|v\rangle$$

if $\alpha \neq \beta$, then $\langle v|u\rangle = 0$

$$\langle v|u\rangle = \langle v|\,\mathcal{U}^{\dagger}\mathcal{U}\,|u\rangle = \beta^{*}\langle v|u\rangle\alpha$$

$$0 = \langle v|u\rangle\left(1 - \frac{\alpha}{\beta}\right)$$

if $\alpha = \beta$
$\Rightarrow 0$
or $v \perp u \Rightarrow = 0$

$u(g) = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$ then $\underline{u}(g')\,|u\rangle = u(g')\,\frac{u(g)}{\alpha}\,|u\rangle$

$= \frac{u(g'g)}{\alpha}\,|u\rangle$ abelian group

$= \frac{1}{\alpha}\,u(g)\,\underbrace{\underline{u}(g')\,|u\rangle}_{|u\rangle \text{ also eigenvector}}$

also $\langle v|\underline{u}(g')|u\rangle = 0$

hence $u(g')\,|v\rangle = \alpha'\,|v\rangle$

subspace span $\{|u\rangle\}$ invariant under all $\underline{u}(g') \Rightarrow$ reducible representation

note $u(g')\,\underline{u}(g) = \underline{u}(g)\,\underline{u}(g') \;\Rightarrow\; [\underline{u}(g), \underline{u}(g')] = 0$

be diagonalized simultaneously

$\underline{u}$ $d \times d$ matrix

S_2 $d \times 2$ matrix $\qquad S_2 \qquad S_2^{+}\,\underline{u}(g')\,S_2$: projection into subspace

$\square$

chapter III Lie (continous)groups

matrix depend on continous (real) parameters

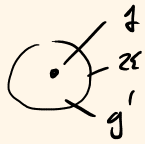
rotations, translations

formally continous group: "topological space"

there is a "metric" (distance) between group elements g, g'

$\mu(g,g') \in \mathbb{R}_+$ $\quad \mu(g,g') = 0 \Leftrightarrow g = g'$

to g, neighborhood $D_{g,\varepsilon} = \{ g' \mid \mu(g,g') < \varepsilon \}$



translations

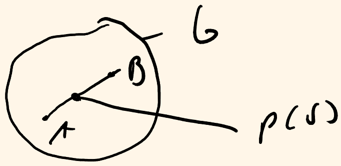
$\mu(R,R') = \max\limits_{\text{all } |u|=1} (|Ru - R'u|) = 2|\sin(\frac{\varphi-\varphi'}{2})|$

in 2D $R = R(\varphi)$

vectors are the same, then the rotations are the same and the representations are the same

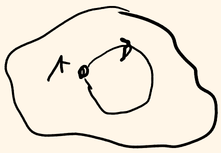
continuous functions $p: [0,1] \to G, s \mapsto p(s)$

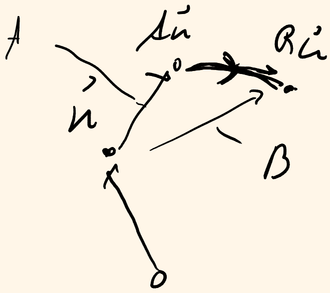
defines "path through group"



closed path = loop

$p(0) = p(1)$



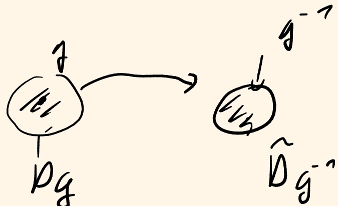


Liegroup (Hamels

Topological space, where

"take inverse" $g \mapsto g^{-1}$ and

"multiply" $g, g' \mapsto g \cdot g'$ are continuous differentiable maps. – tangent vectors



be G a transformation group

$p: [0,1] \to \mathbb{R}^d, \; s \mapsto \pi(p(s)) \cdot \vec{u}$ $\in \mathbb{R}^d$ – invariant and topology where is differentiable

representation as matrix