Continuous Symmetry Groups in Quantum Physics

Winter Term 2023/24
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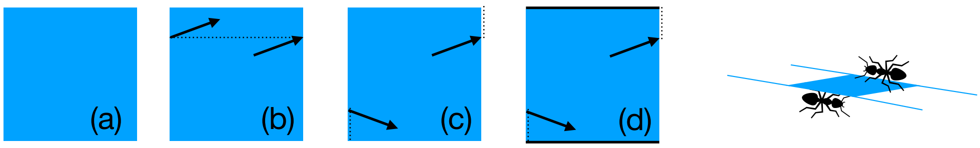
Problem Set No 4 Hand out: 29 November 2023
Discussion: 06 and 13 December

Problem 4.1 – Topology (4 points)

Consider a square, as shown in the figure.



In case (a), all four edges are distinct. Show that this square is (path) connected and simply connected. – Case (b): opposite points on the left and right edge are identified. Show that with this topology, the square is infinitely path-connected. Its homotopy group is isomorphic to $\mathbb{Z}$. – Case (c): points on the left and right edge are "crosswise" identified. You get a Möbius strip: embedded in three dimensions, you can go continuously from "one side" to the other side of the surface. Find M. C. Escher's famous illustration of this with crawling ants. Now ants have a head and a tail: when you meet a fellow ant on the opposite side of the Möbius strip, do you see her "face to face" or "upside down" (as shown on the right)? – Show that the border of the Möbius strip is a single closed line. Think about applying Stokes' theorem: How to evaluate the integral over the Möbius surface that is clearly subtended by its border? – Case (d): the top and bottom edges (that were already a single line in case(c)) are identified as a single point. What we get is a closed surface that has, in three dimensions, no "inner" or "outer" face. Find discussions on the Web whether this can be used as a bottle (order one from Klein & co).

Problem 4.2 – Points, crystals, and space groups (4 points)

(© Martin Wilkens) The symmetry group of a solid object can be introduced as a subgroup of the Euclidean group E(3) that is formed, as you recall, by the rotation group $\mathcal{R}(3)$ and the translation group $\mathcal{T}(3)$ put together (in a "semi-direct product" $\mathcal{R}(3) \ltimes \mathcal{T}(3)$). The rotations may be called the *point group* of three-dimensional space because they keep one point fixed. The whole of E(3) is the *space group* of Euclidean space in three dimensions.

We now specialise to a solid object made from a set of points Γ that are distinguished by some property (e.g., "red" and "blue" atoms). Let us write $\Gamma = (\Gamma_1, \ldots, \Gamma_N)$ where Γ_j collects all positions of atoms of type j. The *space group* $\mathcal{S}(\Gamma)$ of this solid body contains those Euclidean transformations that map all sets of points into themselves, i.e., any blue atom is mapped to a blue one etc.:

$$\mathcal{S}(\Gamma) := \left\{ g \in \mathrm{E}(3) \quad \text{with} \quad g(\Gamma_1) = \Gamma_1, \ldots, g(\Gamma_N) = \Gamma_N \right\} \tag{4.1}$$

Important subgroups of the space group are the *point group* of Γ:

$$P(\Gamma) = \mathcal{R}(3) \cap \mathcal{S}(\Gamma) \tag{4.2}$$

and the *translation group* of Γ:

$$\mathcal{T}(\Gamma) = \mathcal{T}(3) \cap \mathcal{S}(\Gamma) \tag{4.3}$$

(a) Show that $\mathcal{T}(\Gamma)$ is an invariant subgroup of $\mathcal{S}(\Gamma)$. Recall that this means that for any $h \in \mathcal{T}(\Gamma)$ and any $g \in \mathcal{S}(\Gamma)$, we have $g\, h\, g^{-1} \in \mathcal{T}(\Gamma)$.

(b) Is it always true that the factor group $\mathcal{S}(\Gamma)/\mathcal{T}(\Gamma) \cong P(\Gamma)$? Recall that the factor group G/H is formed by the so-called left cosets $g\,H = \{g\,h \mid h \in H\}$ with binary operation (*Verknüpfung*) $(g\,H)\circ(g'\,H) = (g\,g')H$.

(c) We call the collection of atoms Γ a "crystal" if there are three linearly independent vectors $\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3 \in \mathbb{R}^3$ such that $(\mathbb{1}, \boldsymbol{t}_k) \in \mathcal{S}(\Gamma)$ for $k = 1, 2, 3$. (Notation $(\boldsymbol{R}, \boldsymbol{b})$ with rotation matrix $\boldsymbol{R}$ and translation vector $\boldsymbol{b}$ for an element of E(3).)

Show that there are three vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3 \in \mathbb{R}^3$ so that

$$\mathcal{T}(\Gamma) = \left\{ (\mathbb{1}, \sum_j n^j \boldsymbol{a}_j) \quad \text{with} \quad n^1, n^2, n^3 \in \mathbb{Z} \right\} \tag{4.4}$$

The vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ are called the *primitive lattice vectors* of the crystal; they are not unique. The image of the origin (an arbitrary reference point) under the translation group, $\mathcal{T}(\Gamma)(\mathbf{0})$ (aka "orbit" of $\mathbf{0}$), is called the *Bravais lattice* of the crystal.