Continuous Symmetry Groups in Quantum Physics

Winter Term 2023/24
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Problem Set No 2 Hand out: 01 November 2023
Hand in: 08 and 15 November

Problem 2.1 – General Concepts in Group Theory (3 points)

(a) A subgroup is a subset of a group that is “closed” under the group operations. Show that H is a subgroup of G if and only if for all $h_1, h_2 \in H$, we have $h_1^{-1} h_2 \in H$.

(b) The multiplication table of a group lists the products one may form from all pairs g_1, g_2 of group elements. Imagine you have a group with nine elements, and you label them “1”, “2” …“9”. Obviously, each line of the multiplication table is then a permutation of the nine elements (Cayley theorem). For each column, this is also true, although column no. 3 and line no. 3 will in general differ (non-commutative group). This looks like the properties of a Sudoku puzzle. Find out why most of the Sudokus do not give a proper multiplication table. Find a group with nine elements and check whether its table is a proper Sudoku.

(c) The “character” of an element in a matrix group is defined by the trace of its matrix,

$$\chi(g) = \operatorname{tr} M(g) \tag{2.1}$$

It is invariant under a basis change (aka similarity transformation) where $M(g) \mapsto M'(g) = S^{-1} M(g)\, S$. Show that for rotation matrices in 3D, the character only depends on the rotation angle, but not on the axis of rotation.

Problem 2.2 – Discrete Groups and Characters (1 point)

Consider the symmetry group of a two-dimensional square and of the three-dimensional tetrahedron. (You can list the corners of a tetrahedron as $\{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}$.) Compute the character of a few symmetry elements and comment.

Problem 2.3 – Subgroups and boosts (3 points)

Remember the Lorentz transformation along the x-axis,

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \Lambda(v) \begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \tag{2.2}$$

with the dimensionless velocity $\beta = v/c$ and the famous Lorentz factor γ. (a) Show that for a small velocity, we have

$$\Lambda(v) \approx \mathbb{1} + \beta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.3}$$

(b) Compute the exponential

$$\exp\left[\frac{q}{c}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \tag{2.4}$$

and link the parameter q, called "rapidity" to the boost velocity v. For small v, obviously $v \approx q$, right?

(c) Since exponentials are easy to multiply, two boosts along the x-direction sum up in their rapidities: $\Lambda(v')\Lambda(v) = \Lambda(v'')$ with $q'' = q' + q$. Verify that this corresponds to the "relativistic addition of velocities".