Continuous Symmetry Groups in Quantum Physics

Winter Term 2023/24
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Problem Set No 3 Hand out: 14 November 2023
Hand in: 22 and 29 November

Problem 3.1 – Symmetric and anti-symmetric products (3 points)

If we have a matrix representation of a group, then we may consider its tensor product with itself. This means that a group element g is mapped to the matrix

$$T(g) = M(g) \otimes M(g) \tag{3.1}$$

If $\mathscr{V}$ is a vector space on which $M(g)$ acts as a linear (invertible) transformation, then $T(g)$ acts on the tensor product space spanned by vectors $|u,\, v\rangle = |u\rangle \otimes |v\rangle \in \mathscr{V} \otimes \mathscr{V}$. Consider the "exchange operator" P_{ex} defined by

$$P_{\text{ex}}|u,\, v\rangle = |v,\, u\rangle \tag{3.2}$$

and extended by linearity to $\mathscr{V} \otimes \mathscr{V}$.

(a) Show that P_{ex} is unitary and commutes with all $T(g)$.

(b) Imagine you construct a basis of $\mathscr{V} \otimes \mathscr{V}$ made from eigenvectors of P_{ex}. What are the possible eigenvalues? Give examples of eigenvectors.

(c) Show that each eigenspace in (b) is invariant under all $T(g)$ and that the product representation splits in two sub-spaces of dimension $\frac{1}{2}n(n+1)$ and $\frac{1}{2}n(n-1)$ where $n = \dim \mathscr{V}$.

Problem 3.2 – Reducible or irreducible (2 points)

A "reducible representation" M of a group G is defined as follows: There exists a basis transformation S such that for all $g \in G$, we have

$$S\, M(g)\, S^{-1} = M_1(g) \oplus M_2(g) \tag{3.3}$$

where $\oplus$ denotes a "direct sum" such that M_1 and M_2 are block-matrices on the "diagonal". If this is not the case, a representation is called "irreducible".

(a) Think whether the following statement is a correct logical formulation of the concept of being "irreducible": For all vectors $|u\rangle, |v\rangle \in \mathscr{V}$ (the space on which the $M(g)$ are acting), there is a group element g such that

$$|v\rangle = M(g)|u\rangle \tag{3.4}$$

(If needed, we require the $M(g)$'s to be unitary and $|u\rangle$, $|v\rangle$ to have the same norm.)

(b) Show that the following statement is equivalent to M being irreducible: If $\mathscr{W} \subseteq \mathscr{V}$ is a subspace that is invariant under all group matrices, i.e., $M(g)|w\rangle \in \mathscr{W}$ for all $|w\rangle \in \mathscr{W}$, then we either have $\mathscr{W} = \mathscr{V}$ or $\mathscr{W} = \{0\}$ (both cases are called "trivial").

Problem 3.3 – Conjugate anything (5 points)

In the lecture, we tried to show that the *group commutators*

$$C[a, b] = a^{-1} b^{-1} a\, b =: \bar{a}\, \bar{b}\, a\, b\,, \quad a, b \in G \tag{3.5}$$

form a sub-group. This is actually wrong: the product of two C's is *not* a commutator. However, one may consider the sub-group Z that is *generated* when we consider all possible products of the C's. In the exercise, you complete the list of properties of this "commutator subgroup".

(a) The inverse of a (group) commutator is a commutator, $(C[a, b])^{-1} = C[b, a]$.

(b) The conjugation of a commutator with any group element g is a commutator, $g\, C[a, b]\, \bar{g} = C[a', b']$ (find the suitable elements a', b').

(c) Easy generalisation of (b): The conjugation of a product of commutators with any $g \in G$ is a product of commutators. This makes Z a so-called "invariant (or normal) sub-group" of G.

Now consider commutators in the algebra of a Lie group. We are going to learn that for a given basis set $\{G_i | i = 1, 2 \ldots\}$, the commutator can be written as a linear combination

$$[G_i\,, G_j] = C^k_{ij}\, G_k \qquad \text{(Einstein summation)} \tag{3.6}$$

where the numbers C^k_{ij} are called the "structure constants" of the algebra.

(d) Imagine that the G_i's are conjugated by an invertible operator S, i.e., $K_i = S\, G_i\, S^{-1}$. Show that when the commutators among the K_i's are computed, they satisfy the same structure constants as in Eq. (3.6).

(e) Finally, consider an operator that is generated by an element L of the Lie algebra: $S = \exp(\mathrm{i}\, \theta L)$. Compute the conjugation $S\, G_i\, S^{-1}$ up to first order in the parameter θ and use the structure constants to re-write this *action of the group on its algebra* as a linear map with matrix elements L_i^j:

$$S\, G_i\, S^{-1} = G_i + \theta\, L_i^j\, G_j + \ldots \tag{3.7}$$

This construction is called the *adjoint* representation: it associates each algebra element L to its matrix (L_i^j).