Continuous Symmetry Groups in Quantum Physics

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Problem 1.1 – Invariance and Symmetry Groups (5 points)

We see in the lecture that a symmetry transformation can be defined with respect to an "invariant property", e.g.: the transformed trajectory $\boldsymbol{r}'(t)$ solves the same equations of motion as $\boldsymbol{r}(t)$.

(a) Show that such a property naturally implies a group structure on the set of symmetry transformations.

(b) Consider the set of $d \times d$ matrices R that leave the scalar product between two real vectors invariant, i.e., $(R\,\mathbf{x}) \cdot (R\,\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$. Show that they form the group $\mathrm{O}(d)$.

(c) In $2d$-dimensional phase space, a transformation $(x, p) \mapsto (x', p') = T(x, p)$ is called "canonical", if the new coordinates x', p' are "canonically conjugate", i.e. if their Poisson bracket is $\{x', p'\} = 1$ (if $2d > 2$, read: $\{x_i', p_j'\} = \delta_{ij}$ and $\{x_i', x_j'\} = 0$ etc.). Consider the case that x', p' are the solutions to Hamilton's equations. Take the time derivative of their Poisson bracket and show that it is zero: "Time evolution is a canonical transformation". Give an example for a nonlinear transformation T.

Solution (point c). First question: How to understand the Poisson bracket $\{x', p'\}$ when x' and p' are the solutions to the equations of motion (aka Hamilton's equations)? Let us write them as $(x', p') = (x(t), p(t)) = T(x, p; t)$ as a function of initial coordinates x, p and adopt the rule that the Poisson brackets are always computed with respect to these x, p.

Second: To work out the time derivative, remember that in the Poisson bracket, x' and p' enter as a product, so the usual rule for derivatives applies:

$$\frac{\mathrm{d}}{\mathrm{d}t}\{x', p'\} = \left\{\frac{\mathrm{d}x'}{\mathrm{d}t}, p'\right\} + \left\{x', \frac{\mathrm{d}p'}{\mathrm{d}t}\right\}$$

Now, the equations of motion in Hamiltonian mechanics are provided by the Poisson brackets with H:

$$\ldots = \{\{x', H\}, p'\} + \{x', \{p', H\}\} = \{\{x', p'\}, H\}$$

using the Jacobi identity in the second step. At least at the initial time, the Poisson bracket between x' and p' is 1, a constant that gives zero in the Poisson bracket with H. The constant $\{x', p'\} = 1$ satisfies this differential equation and it coincides with the initial conditions. By the uniqueness of the solution, it must be the solution. As an alternative proof, iterate and compute higher derivatives. They involve higher Poisson brackets (always with H in the left entry) and are all zero at the initial time. Hence in the Taylor series of $\{x', p'\}$, all terms vanish except for the zeroth-order (constant) one.

(d) Consider the vector space of linear functions on phase space, e.g.

$$Q = Q(x, p) = a\,x + b\,p \tag{1.1}$$

Show that the Poisson bracket $\{Q_1, Q_2\}$ is the area spanned by the vectors $(a_1, b_1)^\mathsf{T}$ and $(a_2, b_2)^\mathsf{T}$. Check the sign of this area and generalise to a $2d$-dimensional phase space.

(e) If Hamilton's equations are linear, you have learned in your analysis lecture that the time-evolved coordinates $x'(t), p'(t)$ depend linearly on the initial conditions. Writing this transformation in block form

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = \begin{pmatrix} R & A \\ B & S \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} \tag{1.2}$$

construct a few simple solutions to the invariance conditions

$$R\,S^\mathsf{T} - A\,B^\mathsf{T} = 1\,, \quad R\,A^\mathsf{T} - A\,R^\mathsf{T} = 0\,, \quad \ldots \tag{1.3}$$

Note: for $2d = 2$ or 4, there are at most 3 or 10 "independent" transformations. They generate the so-called "symplectic groups" Sp(2), Sp(4).

Problem 1.2 – Rotations, intuitively (4 points)

In problem 2.1, we have already encountered the matrices of the orthogonal group O(d). We specialise here to the case of SO(3) and construct the Rodriguez formula, an explicit expression for a rotation matrix.

(a) Check the following count of independent parameters in SO(3): Its matrices R are orthogonal and have unit determinant. The first condition is equivalent to $R^\mathsf{T} R = \mathbb{1}$ which is an equation between symmetric matrices, with 6 independent entries. Hence of the 9 parameters in R, only 3 are left free. For the determinant, we get $1 = \det \mathbb{1} = (\det R^\mathsf{T})\,(\det R) = (\det R)^2$, giving the solutions $\det R = \pm 1$. Only one of the solution is allowed, but this does not restrict the number of real parameters. (This removes a "disconnected component" from the group. For the general case of SO(3), a similar count leads to $\frac{1}{2}d(d-1)$ parameters or "dimensions".)

Solution (point a). At face value, a real 3×3 matrix R has nine independent real parameters. The matrix equation $R^\mathsf{T} R = \mathbb{1}$ involves, however, a symmetric matrix, so it contains only three (on the diagonal) and three (below the diagonal, say) independent parameters. So three parameters remain for the group O(3) ("orthogonal matrices"). The determinant only fixes a sign and singles out one "component" of the group (which has the unit matrix $\mathbb{1}$ in the "middle"). The other component with the negative unit matrix $-\mathbb{1}$ (space inversion) is disjoint from the first one. Extending this to SO(4), we get $16 - 4 - 6 = 6$ free parameters.

A similar calculation can be done for the unitary groups, e.g., SU(2). These 2×2 complex matrices contain eight independent real parameters. The condition for unitarity, $U^\dagger U = \mathbb{1}$, is an equation between Hermitean matrices, hence it contains two real parameters on the diagonal and two for the complex entry below the diagonal, say. This makes four, leaving still four parameters. But now the determinant is constrained by unitarity only to have $1 = [\det(U)]^* \det(U) = |\det(U)|^2$, leaving a real phase parameter free. This one is fixed in the "special" group SU(2), leaving only three free parameters.

(b) A rotation $R = R_{\boldsymbol{u}}(\theta)$ can be specified by a unit vector $\boldsymbol{u}$ (rotation axis) and an angle θ. Make a count of parameters that this captures all dimensions of SO(3). Rotations about the same axis commute, right? This implies $R_{\boldsymbol{u}}(\theta_1 + \theta_2) = R_{\boldsymbol{u}}(\theta_1)\, R_{\boldsymbol{u}}(\theta_2)$, right?

(c) Make a sketch of the action of such a rotation on a vector $\boldsymbol{r}$. Adopt the "right-hand rule" for the sense of the rotation. Show that for a small angle $\delta\theta$, we have

$$\delta\theta \to 0: \quad R_{\boldsymbol{u}}(\delta\theta)\, \boldsymbol{r} = \boldsymbol{r} + \delta\theta\, (\boldsymbol{u} \times \boldsymbol{r}) + \ldots \tag{1.4}$$

We say that "the vector product generates a rotation".

(d) For an arbitrary angle, check the following differential equation

$$R_{\boldsymbol{u}}(\theta + \mathrm{d}\theta) = R_{\boldsymbol{u}}(\mathrm{d}\theta)\, R_{\boldsymbol{u}}(\theta) = R_{\boldsymbol{u}}(\theta) + \mathrm{d}\theta\, \boldsymbol{u} \times R_{\boldsymbol{u}}(\theta) + \ldots \tag{1.5}$$

whose formal solution is, of course, $R_{\boldsymbol{u}}(\theta) = \exp[\theta\, (\boldsymbol{u} \times \_)]$ where $\boldsymbol{u} \times \_$ is shorthand for the linear operation "form the vector product with $\boldsymbol{u}$ from the left".

(e) Compute recursively the powers of $\boldsymbol{u} \times \_$: even (nonzero) powers turn out proportional to $\mathbb{1} - \boldsymbol{u}\, (\boldsymbol{u} \cdot \_)$ (a projector orthogonal to $\boldsymbol{u}$) and odd powers proportional to $\boldsymbol{u} \times \_$. Collect the even and odd powers in the Taylor series of $\exp[\theta\, (\boldsymbol{u} \times \_)]$ and find the Rodrigues formula:

$$\exp[\varphi\, \boldsymbol{u} \times \_] = \mathbb{1} + \sin(\varphi)\, \boldsymbol{u} \times \_ + (\cos\varphi - 1)\left[\mathbb{1} - \boldsymbol{u}\, (\boldsymbol{u} \cdot \_)\right] \tag{1.6}$$