# Problem Set 1
Wave Equation

C++ programming language using

due date: November 23, 2023 NR_Hydro

## A Solving the wave equation

In this problem set, we will consider a simple non-relativistic example and will investigate the one-dimensional wave equation for the scalar field $\phi$

$$\frac{\partial^2 \phi(x,t)}{\partial t^2} = c^2 \frac{\partial^2 \phi(x,t)}{\partial x^2}, \tag{1}$$

with $c = const.$

### (A.1) Fully first order formulation (10 Marks)

As a first task, rewrite the wave equation (Eq. (1)) in a fully first order form by using the auxiliary variables $\eta = \partial_t \phi$ and $\chi = \partial_x \phi$. Collect them into the state vector $\mathbf{u} = (\phi, \eta, \chi)$ and show that the vector equation can be written as

$$\partial_t \mathbf{u} + \mathbf{A} \partial_x \mathbf{u} = \mathbf{S}, \tag{2}$$

where $\mathbf{A}$ is a matrix and $\mathbf{S}$ is the vector of source terms. Analyze matrix $\mathbf{A}$ by evaluating its eigenvalues and eigenvectors.

### (A.2) Solving the wave equation

In the next step, please consider the following initial condition

$$\phi(x,0) = e^{\sin^2\left(\frac{\pi x}{L}\right)} - 1, \quad 0 \le x \le L, \tag{3}$$

with periodic boundary condition

$$\phi(x,t) = \phi(x \pm L, t). \tag{4}$$

For simplicity, you should choose $L = 1 = c$.

Write a code to solve the system 2 numerically using

1. **(15 Marks)** second-order spatial finite differencing stencils and the forward Euler method, and
2. **(25 Marks)** forth-order spatial finite difference and the forth-order Runge-Kutta method.

For the spatial grid setting, you can discretize the equation in space using an N-points grid in cell center:

$$x_i = \left(i - \frac{1}{2}\right) \Delta x, \quad i = 1, 2, \ldots, N, \tag{5}$$

with $\Delta x = L/N$ and $\mathbf{u}_i = \mathbf{u}(x_i)$. Note that the time step for this hyperbolic system has to follow the Courant-Friedrich-Levy condition

$$\max |\lambda_i| \frac{\Delta t}{\Delta x} \le C_{max}, \tag{6}$$

where $\lambda_i$ is the characteristic speed of the system and $C_{max}$ is the maximum allowed Courant number depending (where the exact number will depend on the exact discretization scheme; typically $C_{max} = 1$). You can choose $\Delta t = C\frac{\Delta x}{\max|\lambda_i|}$ for Courant number $C \leq 1$ in this problem. Perform convergence test by varying the resolution $\Delta x$ and $\Delta t$ to verify your result.

## (A.3) Parallelization C++

In the following we will extend the programm by employing different parallelization schemes, please parallelize your code using

1. **(25 Marks)** Open MP, and
2. **(25 Marks)** MPI

Show that your parallelized code gives the same result as the serial version and measure the performance of your code by performing a strong scaling test (i.e. fix your problem size and measure the execution time with different number of processors). You may need to use a large number of grid points $N > 1000$ for the test.

– Please submit solutions to peter.nee@aei.mpg.de. –