1.5. Parallelisation

why do we need this in our work?

Let us consider an example of 5 refinement levels and 100^3 points per level.
We want simulate approx 10^5 timestep
this leads to the following number of operations

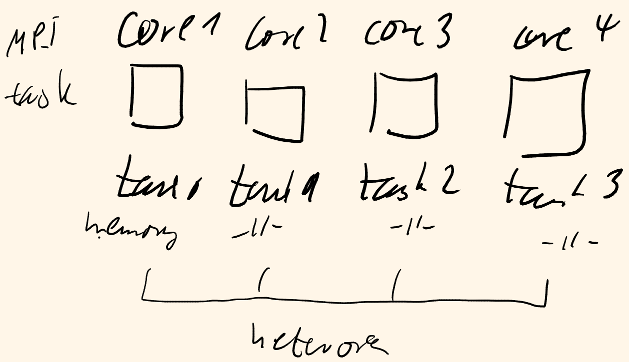
$$\left(\underbrace{10^5}_{\text{timestep}}\underbrace{(1+2+4+8+16)}_{10^7}\right) 10^6 \cdot 10^4 = 10^{17} \text{ operations}$$

with one processor 1GHz this would take one year

to avoid this, we want to parallelize the code where possible and there are 2 main strategies

MPI message passing interface

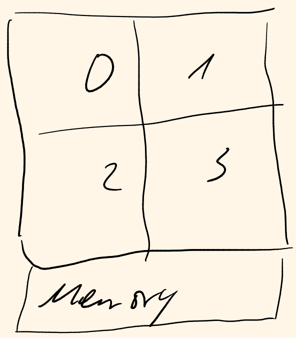
you start multiple task and every task has its own memory



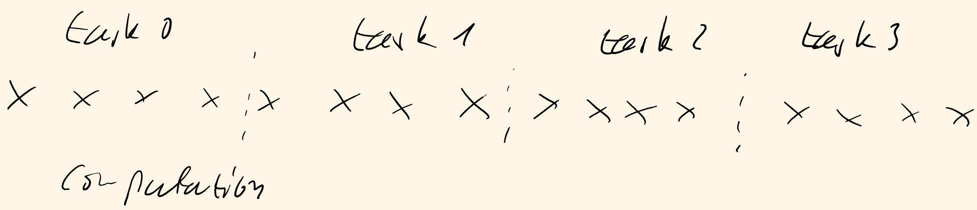
+ scales to larger systems
- sensitive to your network connection
(Ethernet 10Mbits to 1Gbits
Infiniband/Omnipath 10Gbits to 50Gbits

OpenMPI (opne multi-processing)

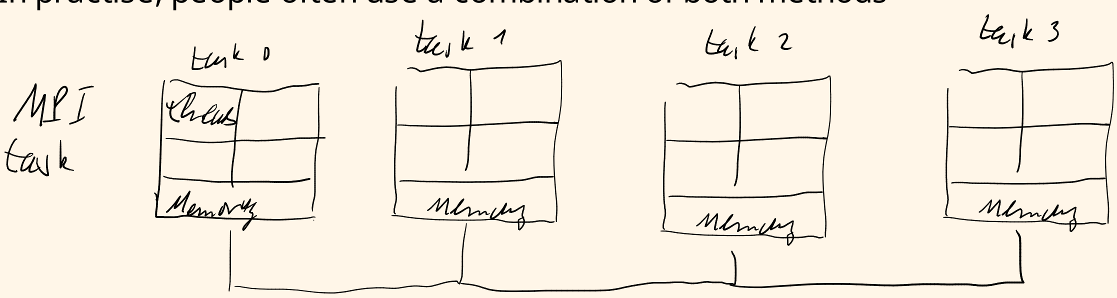
shared memory for different threas



- limited to a single processor
+ simpler to implement



In practise, people often use a combination of both methods

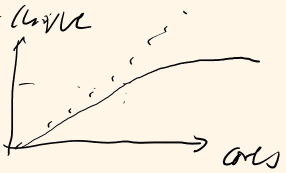


to measure your code performance, you test how your code scales

strong scaling

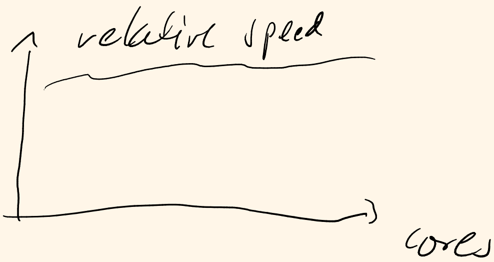
you use the same problem size and increase the number of used cores

$$\frac{T(N_{cores})}{T(1\,core)} = speedup$$



weak scaling

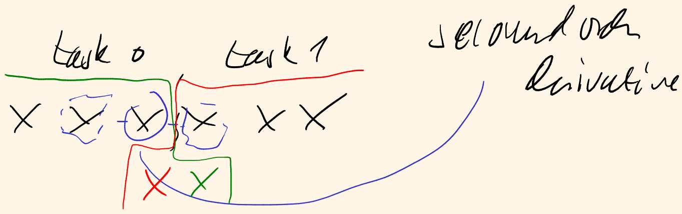
you adjust your problem to the number f used cores



Minimum of cores to start the simulation depends on the memory reserved for a single core, normally 2Gbit for one core

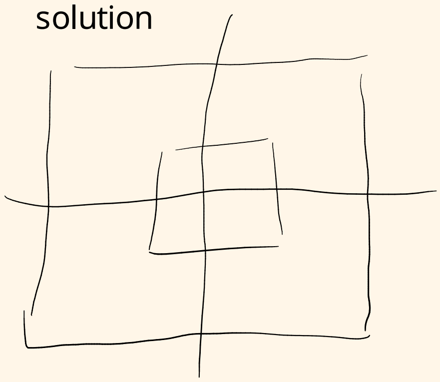
example for parallezing simple option

MPI

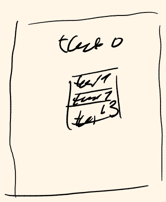


so instead of the simple decomposition with equal number of points per MPI task, you can also parallelize task-based where depending on your work the points are distributed

solution



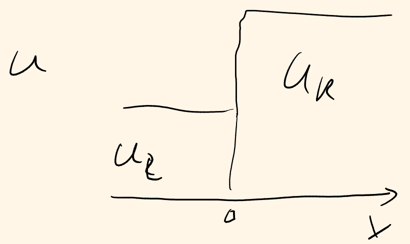
every task get the same amount of points and the same levels to compute



my distribution of tasks

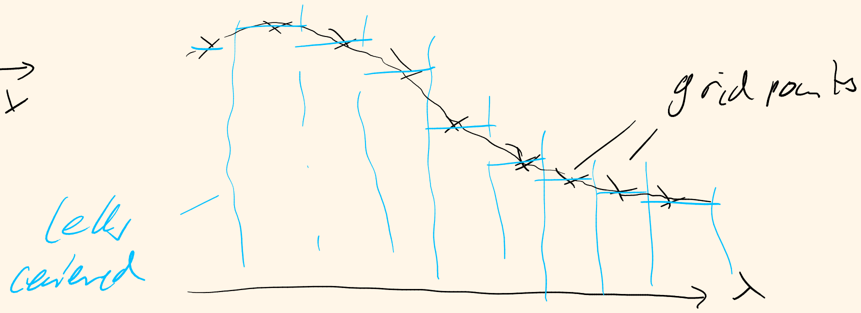
1.6. Riemann Problem

handle discontiuity problems in our simulations



$$u = \begin{cases} u_L, & x<0 \\ u_R, & x>0 \end{cases}$$

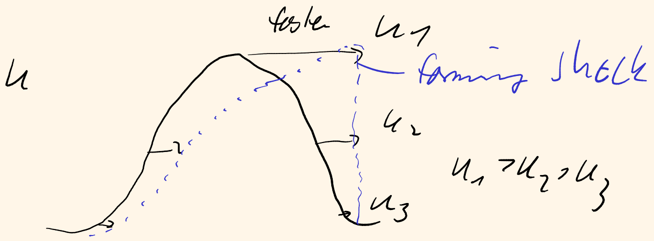
in our simulations every cell surface basically is a Riemann Problem



in hydrodynamic simluations in addition to the understanding that you have discontinuities on the cell interfaces, you can form "real" shocks

One example is Burges Equation

$$\partial_t u + u \partial_x u = 0$$



$$u_1 > u_2 > u_3$$

start with an example linearized gas equation

$$\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + \frac{a^2}{\rho_0} \frac{\partial \rho}{\partial x} = 0$$

initial data

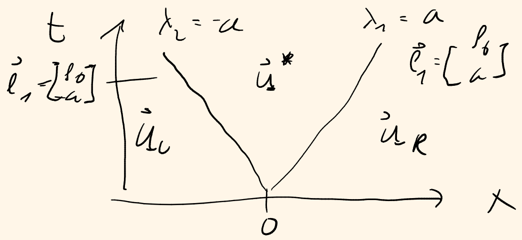
$$\begin{bmatrix} \rho \\ u \end{bmatrix} = \begin{cases} \begin{bmatrix} \rho_L \\ u_L \end{bmatrix}; & x \leq 0 \\ \begin{bmatrix} \rho_R \\ u_R \end{bmatrix}; & x > 0 \end{cases}$$

we can write the equation in conservative form

$$\frac{\partial \vec{u}}{\partial t} + A \frac{\partial \vec{u}}{\partial x} = 0, \quad \vec{u} = \begin{bmatrix} \rho \\ u \end{bmatrix}, \quad A = \begin{pmatrix} 0 & \rho_0 \\ \frac{a^2}{\rho_0} & 0 \end{pmatrix}$$

eigenvalues are here $\lambda_{1,2} = \pm a$

eigenvector $\vec{e}_1 = \begin{bmatrix} \rho_0 \\ -a \end{bmatrix}$ $\quad \vec{e}_2 = \begin{bmatrix} \rho_0 \\ a \end{bmatrix}$



we can express $\vec{U}_L, \vec{U}_R$ by using the eigenvectors:

$$\vec{U}_L = \alpha_1 \vec{e}_1 + \alpha_2 \vec{e}_2 \;; \qquad \alpha_1 = \frac{a\rho_L - \rho_0 u_L}{2a\rho_0} \qquad \vec{U}_R = \beta_1 \vec{e}_1 + \beta_2 \vec{e}_2$$

$$\alpha_2 = \frac{a\rho_L + \rho_0 u_L}{2a\rho_0}$$

$$\vec{U}^* = \alpha_1 \vec{e}_1 + \beta_2 \vec{e}_2$$

constant solution, where we take into account that the flux on the right is moving left and the flux from left is moving right

lets look at the generic conservation law

$$\partial_t u + \partial_x F(u) = 0$$

to get a weak solution for this equation, we write it in integral formwith the help of smooth and compact test functions $\Phi(x,t)$

$$\Phi(t,\infty) = \Phi(\infty,x) = \Phi(-\infty,x) = 0$$

$$\int_0^\infty \int_{-\infty}^\infty (\partial_t u + \partial_x F(u)) \, \Phi(x,t) \, dx\, dt = 0$$

productrule

$$= \int_0^\infty \int_{-\infty}^\infty \Big(\underbrace{\partial_t(\Phi u)}_{-\int_{-\infty}^{\infty} \Phi(x,0)\, u(x,0)\, dx} - u\partial_t \Phi + \underbrace{\partial_x(F\Phi)}_{0} - uF\partial_x \Phi \Big) \, dx\, dt$$

$$\rightarrow \int_0^\infty \int_{-\infty}^\infty u(\partial_t \Phi + F \partial_x \Phi) \, dx\, dt = -\int_{-\infty}^{\infty} \Phi(x,0)\, u(x,0)\, dx$$

then is u a weak solution from $\quad \partial_t u + \partial_x F(u) = 0$

example Burger equation

$$u(x,0) = \begin{cases} u_L, & x<0 \\ u_R, & x>0 \end{cases} \qquad F(u) = \frac{u^2}{2}$$

$$\frac{d}{dt}\int_{x_L}^{x_R} u\, dx = \int_{x_L}^{x_R} \partial_t u\, dx = -\int_{x_L}^{x_R} \partial_x F\, dx = F(u_L) - F(u_R)$$

speed of the jump itself

$$= \frac{d}{dt}\Big[(x_R - x_L) u_R + \overset{\text{speed of discontinuity}}{c} t (u_L - u_R) \Big] = c \cdot (u_R - u_L)$$

Rankine Hugoint jump condition

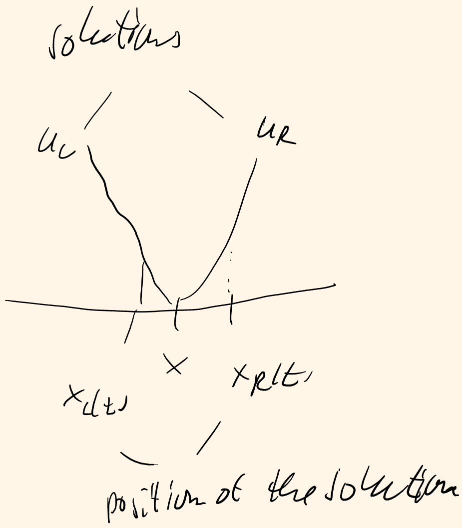
$$c = \frac{F(u_R) - F(u_L)}{u_R - u_L} \simeq \frac{u_R^2 - u_L^2}{2(u_R - u_L)} = \frac{u_R + u_L}{2}$$

For the Burgers equation this can be used to get the following weak solution

$$u(x,t) = \begin{cases} u_L, & x \leq u_L t = x_L \\ \frac{x}{t}, & u_L t \leq x \leq u_R t \\ u_R, & x > u_R t = x_R \end{cases}$$

if $u_R > u_L$ rarefaction wave

$u_L > u_R$ shock wave



Burger equation is an example for
real shock waves.

the Riemann Problem gives an impression of not wanted
shack waves in yours solution, where is none of them. So you have to try
to avoid them. Next time we see ways to solve that.

We will see next time how the flux comes into play and how to solve it with that.