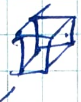
3 Relativistic Hydrodynamics

→ How to evolve Matter field.

3.1 Special Relativistic Hydrodynamics

 We will model small fluid elements:

in any fluid element, we consider to have large Number of particles

→ rest mass density ρ
→ energy density ε
→ Velocity v^i
→ pressure P

→ We can describe these systems typically either by moving with the fluid (Lagrangian Frame) or in the Lab frame (Eulerian frame)

In the following, we consider ideal fluids
(No Viscosity, heat conduction)

$$T_{\mu\nu} = (\rho + \varepsilon)\, u_\mu u_\nu + P\, \eta_{\mu\nu}$$

$\eta_{\mu\nu}$ → Minkowski, ∵ We are doing SRHD

ρ → density
↳ composed of : $\rho = \underbrace{\rho_0}_{\text{Rest Mass density}} (1+\varepsilon)$ ↳ Energy density

$$\rho_0 = \underbrace{n}_{\text{Number density}} . \underbrace{m}_{\text{particle's Mass}}$$

You can also define specific entalpy $h = 1 + \varepsilon + \frac{P}{\rho_0}$, and with this the energy momentum tensor can be written as:

$$T_{\mu\nu} = \rho_0 h\, u_\mu u_\nu + P\, \eta_{\mu\nu}$$

Note that $\rho \neq \rho_0 \neq \rho_{ADM}$
where you get ρ_{ADM} → by contracting $T_{\mu\nu}$ with the normal of the Hyper surface.

$$\rho_{ADM} = n^\mu n^\nu T_{\mu\nu} = n^\mu n^\nu (\rho_0 h u_\mu u_\nu + P g_{\mu\nu})$$

$$= \rho_0 h (u_\mu n^\mu)^2 - P$$

$$= \rho_0 h \, W^2 - P$$

$\hookrightarrow$ Lorentz factor

$$W = -u^\mu n_\mu = u^0 = \sqrt{1 - u_i u^i}$$ which follows from:

$$v^i = \frac{u^i}{u_0} = \sqrt{1-v^2} \, u^i$$

For example, if we move with the fluid ($W = 1$), then we get

$$\rho_{ADM} = \rho_0 h - P = \rho_0 (1 + \varepsilon) = \rho$$

Interesting to note that $\rightarrow$ any time and point our fluid is described by $\boxed{\rho_0, \varepsilon, v^i, p}$ (6 degrees of freedom)

$\rightarrow$ we need to find 6 equations. $\rightarrow$ Primitive Variables

$\rightarrow$ The 1^{st} equation comes from the conservation of particles

$$\partial_\mu (\rho_0 u^\mu) = 0$$

$\rightarrow$ The rest from Energy-Momentum Conservation.

$$\partial_\mu (T^{\mu\nu}) = 0$$

$\rightarrow$ The 6^{th} eqn to close the system is the Equation of State that connects the pressure to the density or other fluid variables.

$$P = P(\rho, \varepsilon)$$

* Typically we do not evolve the primitive variables, but conserved variables:

$D = \rho_0 W$ $\rightarrow$ Rest mass density

$S^\mu = \rho_0 h W u^\mu$ $\rightarrow$ Momentum Density

$E = \rho_0 \varepsilon W$ $\rightarrow$ internal Energy density

$$\partial_\mu (\rho_0 u^\mu) = \partial_\mu \left(\frac{D}{W} u^\mu\right) = \partial_0 \left(\frac{D u^0}{W}\right) + \partial_i \left(\frac{D u^i}{W}\right)$$

$$= \boxed{\partial_t D + \nabla_i (D v^i) = 0}$$

To derive the next Equation, we start from

$$T^{\mu}_{\nu} = \frac{S_{\nu} u^{\mu}}{W} + p\,\delta^{\mu}_{\nu} \quad , \text{ so that we get}$$

$$\partial_\mu \left(\frac{S_i u^\mu}{W} + p\,\delta^\mu_i \right) = \partial_t S_i + \partial_k (S_i V^k) + \partial_i p = 0$$

For our Equation for E, we start from

$$\partial_\mu (u_\nu T^{\mu\nu}) = T^{\mu\nu} \partial_\mu u_\nu \qquad (u^\mu u_\mu = -1)$$

Inserting the Energy momentum tensor leads to

$$\partial_\mu (u_\nu T^{\mu\nu}) = p\,\partial_\mu u^\mu$$

$$u_\mu T^{\mu\nu} = -\rho_0 (1+\varepsilon) u^\nu$$

$$\Rightarrow \partial_\mu (u_\nu T^{\mu\nu}) = -\partial_\mu (\rho_0 (1+\varepsilon) u^\mu) = p\,\partial_\mu u^\mu$$

$$= -\partial_\mu (\rho_0 \varepsilon u^\mu) = p\,\partial_\mu u^\mu$$

$$\partial_\mu (\rho_0 \varepsilon u^\mu) + p\,\partial_\mu u^\mu = 0$$

$$\boxed{\partial_t E + \partial_k (E V^k) + p\left(\partial_t W + \partial_k (W V^k)\right) = 0}$$

Unfortunately the last equation contains the derivative of the Lorentz factor, which we want to overcome

$$\tau = \tau_{ADM} - \rho_0 W = \rho_0 h W^2 - p - D$$

$$\Rightarrow \partial_t \tau + \partial_k \left((\tau + p) V^k \right) = 0$$

(3.2.1) Conservative to Primitive reconstruction

* So while we are simulating the conserved variables (D, S_i, τ) we always need to know the primitives to compute the fluxes. This conversion is non-trivial and general cases has to be done numerically.

A typical procedure is that you start with an initial ~~guess~~ guess p^*

④ General Relativistic Hydrodynamics

We start from the Energy-Momentum Tensor.

$$T_{\mu\nu} = \rho_0 h \, u_\mu u_\nu + P g_{\mu\nu}$$

regarding conservation laws:

$$\nabla_\mu (\rho_0 u^\mu) = 0 \qquad \partial_\mu \to \nabla_\mu$$

$$\nabla_\mu T^{\mu\nu} = 0$$

We have again 5 equations and need to ~~use~~ use the EoS as in the SRHD case.

Same hints for deriving / ~~extending~~ extending previously derived SRHD equations to GRHD

$$\nabla_\mu \xi^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, \xi^\mu\right)$$

$$\hookrightarrow \partial_\mu \left(\sqrt{-g}\, \rho_0 u^\mu\right) = 0$$

$$\partial_\mu \left(\sqrt{-g}\, T^\mu_{\ \nu}\right) = \sqrt{-g}\, \Gamma^\alpha_{\mu\nu} T^\mu_{\ \alpha}$$

$$g = -\alpha^2 \gamma$$

$$W = -u^\mu n_\mu = \alpha u^0$$

$$v^i = \frac{u^i}{\alpha u^0} + \frac{\beta^i}{\alpha}$$

So if you use these expression you end up with

$$\partial_t (\sqrt{\gamma}\, D) + \partial_k \left(\sqrt{\gamma}\, D \,(\alpha v^k - \beta^k)\right) = 0$$

$$\partial_t (\sqrt{\gamma}\, S_i) - \partial_k \left(\sqrt{\gamma}\, S_i (\alpha v^k - \beta^k) + \alpha P \delta^k_i\right) = \alpha \sqrt{\gamma}\, \Gamma^\mu_{\nu i} T^\nu_{\ \mu}$$

$$\partial_t (\sqrt{\gamma}\, \tau) + \partial_k \left(\sqrt{\gamma}\, \left(\tau (\alpha v^k - \beta^k) + \alpha P v^k\right)\right) = \alpha^2 \sqrt{\gamma} \left(T^{\nu\mu} \partial_\mu \ln\alpha - \Gamma^\nu_{\mu\nu} T^{\mu\nu}\right)$$