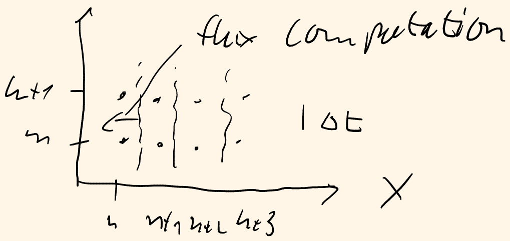
1.7.1. Gudanov method

$$\partial_t u + \partial_x F(u) = 0$$

let us assume that we have a finite differencing representation
we want to solve the Riemann problem at the cell interfaces

$$F^{n+1}_{n+\frac{1}{2}} = \frac{1}{\Delta t} \int_t^{t+\Delta t} F(u(x_{n+\frac{1}{2}}, t))\, dt$$

the latter will give you a constraint n the time step to ensure the neighbring do not intersect
then copute the numerical flux as



$$F^{n+1}_{n+\frac{1}{2}} \simeq F(u(x_{n+\frac{1}{2}}, t))$$

for small Δt and this will be only a first approximation

Gudanov

$$F_{n+\frac{1}{2}} = \begin{cases} \min(F(u)), & u_L \le u \le u_R \text{ if } u_L < u_R \\ \max(F(u)), & u_L \ge u \ge u_R \text{ if } u_L > u_R \end{cases}$$

$$F = F_H - (1-\phi)(F_H - F_L)$$

high order flux, weight factor, low-order flux

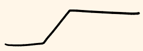
to extend this to higher orders, one can write

$$0 \le \phi \le 1$$

discontinuity, smooth problems

$$\phi(\vartheta) = \operatorname{minmod}(1, \vartheta) = \begin{cases} 0, & \vartheta \le 0 \\ \vartheta, & 0 \le \vartheta \le 1 \\ 1, & \vartheta > 1 \end{cases}$$

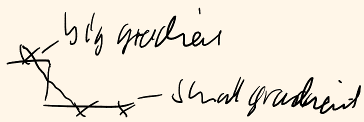
Min Mod - limiter



$$\phi(\vartheta) = \max(0, \min(1, 2\vartheta), \min(\vartheta, 2)) \qquad \vartheta = \frac{u_n - u_{n-1}}{u_{n+1} - u_n}$$

- gradient

Superbee - limiter

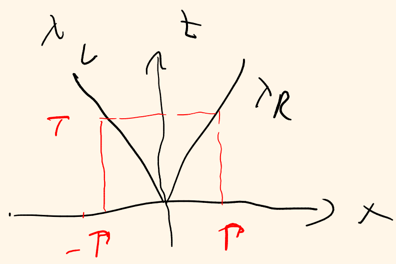


1.7.2. other approximate Riemann solvers

$$\partial_t u + \partial_x F = 0$$

HLLE - Solver (Hartenet al 1983, Einsfeld et al 1988)

assumption after decaying of the initial discontinuity of the local Riemann problem, only two waves are propagating in different directions



$$u(x,t) = \begin{cases} u_L, & \frac{x}{t} < \lambda_L \\ u_{HLLE} & \\ u_R, & \frac{x}{t} < \lambda_R \end{cases}$$

$$\int_{-\Gamma}^{\Gamma} u(x,T)\,dx = \underbrace{\int_{-\Gamma}^{\Gamma} u(x,0)}_{\partial_x F \text{ integrated}} + \underbrace{\int_0^T F(u(-\Gamma,t))\,dt - \int_0^T F(u(\Gamma,t))\,dt}_{\partial_t u \text{ integrated}}$$

Insert the proposed solution from above

$$= \underbrace{\int_{-\Gamma}^{0} u_L\,dx + \int_0^{\Gamma} u_R\,dx}_{\Gamma(u_L+u_R)} + \underbrace{\int_0^T F(u(-\Gamma,t))\,dt - \int_0^T F(u(\Gamma,t))\,dt}_{T(F_L - F_R)}$$

$$\int_{-T}^{T} u(x,T)\,dx = \underbrace{\int_{-\Gamma}^{\lambda_L T} u_L\,dx}_{\text{constant}} + \int_{T\lambda_L}^{T\lambda_R} u_{HLLE}\,dx + \int_{T\lambda_R}^{\Gamma} u_R\,dx$$

$$= u_L(T\lambda_L + \Gamma) + u_{HLLE}(\lambda_R - \lambda_L) + u_R(\Gamma - T\lambda_R)$$

$$u_{HLLE} = \frac{\lambda_R u_R - \lambda_L u_L + F_L - F_R}{\lambda_R - \lambda_L}$$

$$F = \begin{cases} F_L \\ \frac{\lambda_R F_L - \lambda_L F_R + \lambda_L \lambda_R (u_R - u_L)}{\lambda_R - \lambda_L} \\ F_R \end{cases}$$

where $\lambda_L = \min(0, \lambda_-(u_L), \lambda_-(u_R))$ minimum speed of the left moving fields

$\lambda_R = \max(0, \lambda_+(u_L), \lambda_+(u_R))$

$$F_{LLF} = \frac{1}{2}(F_L + F_R) - \frac{1}{2}(u_R - u_L)$$

a simplified version of this flux local lax Friedrichs flux