Lecture 1 2023-10-19

90 minutes lecture and on other weeks we have a seminar with the problems

block I - numerical methods

books
structured like Introduction to 3+1 numerical relativity

numerical relativity

1.1: method of lines

idea: rewriting your set of PDEs as a set of ODEs

example: the wave equation

$$\partial_t^2 u = c^2 \partial_x^2 u$$

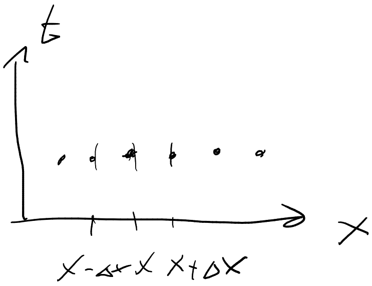
get ride of all space derivatives

so the ride side is zero

rewrite / approximate our derivative as

$$\partial_x^2 u(t,x) \approx \frac{u(t,x+\Delta x) - 2u(t,x) + u(t,x-\Delta x)}{\Delta x^2}$$

$u(t,x)$ should be continous function and the $\Delta x <<< 1$



$$\partial_t^2 u(t,x) \approx c^2 \left(\frac{u(t,x+\Delta x) - 2u(t,x) + u(t,x-\Delta x)}{\Delta x^2} \right)$$

$$\partial_t u(t,x) = v(t,x)$$

$$\partial_t v = c^2 \left(\frac{u(t,x+\Delta x) - 2u(t,x) + u(t,x-\Delta x)}{\Delta x^2} \right)$$

$$\Rightarrow \quad \partial_t \vec{y} = f(\vec{y}) \qquad \vec{y} = \begin{pmatrix} u \\ v \end{pmatrix} \qquad f = \begin{pmatrix} v \\ c^2(\ldots) \end{pmatrix}$$

1.2: Courant-Friedrichs-Lewy condition (CFL)

the CFL condition expresses the timestep to ensure stability of your simulation

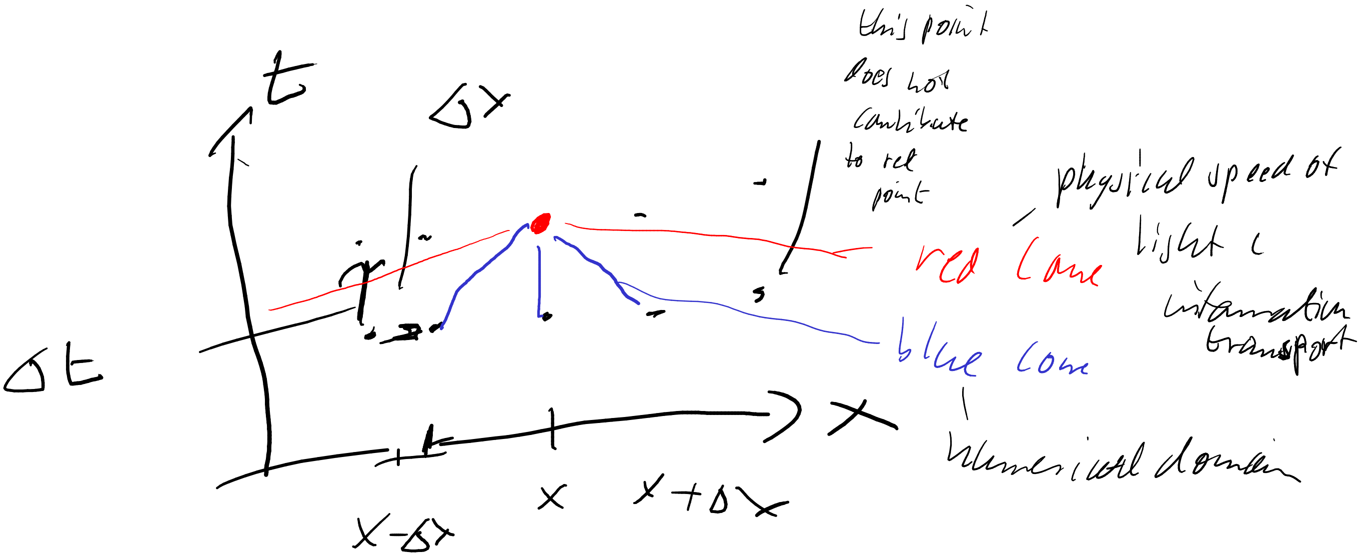
example: $$\partial_t u = -c\,\partial_x u = -c\left(\frac{u(t,x+\Delta x) - u(t,x-\Delta x)}{2\Delta x}\right)$$

Euler upwind

spatial derivative 2nd order

$$\partial_t u \stackrel{!}{=} \frac{u(t+\Delta t,x) - u(t,x)}{\Delta t}$$

$$\Rightarrow u(t+\Delta t, x) = u(t,x) - \frac{c\Delta t}{2\Delta x}\left(u(t,x+\Delta x) - u(t,x-\Delta x)\right)$$



to ensure that the physical domain of dependence is covered by the numerical domain of dependence we can decrease Δt

shift the data points in the red cone

alternativly you can also increase your order of differentation

we would need to make sure, that $\boxed{\left|\frac{\Delta t}{\Delta x}\right| \le \left|\frac{1}{c}\right|}$

CFL condition

$\lambda \le \frac{1}{2}$

$\lambda = \frac{c\Delta t}{2\Delta x} = C_{CFL}$

$\frac{c}{2} = u$

$\frac{c}{2}\frac{\Delta t}{\Delta x} \le \frac{1}{2}$

Courant-number

$C_{CFL} = \frac{u\Delta t}{\Delta x}$ $\quad C_{CFL} < 1$

CFL number

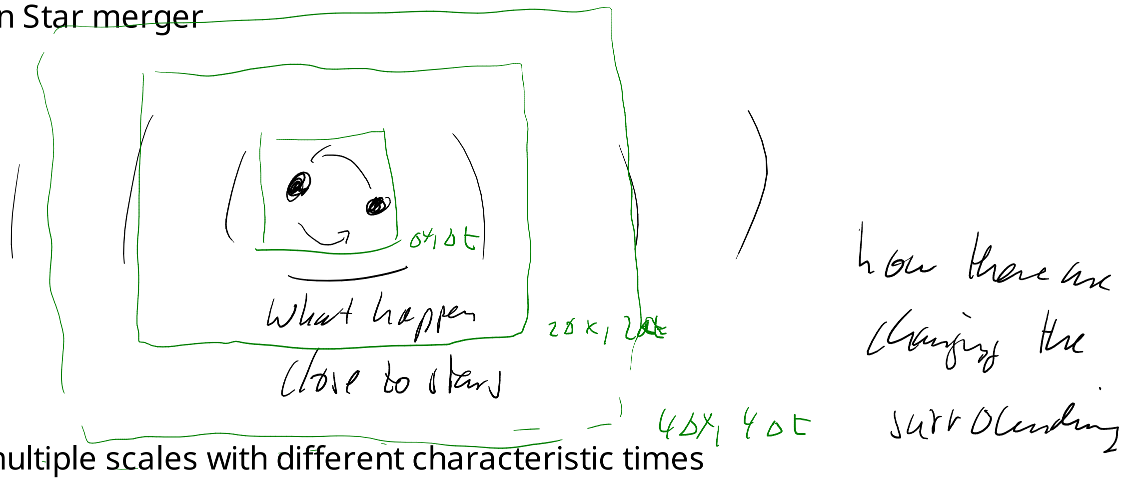
or absolute values

here $u = c$ $\quad C_{CFL} = \frac{c\Delta t}{\Delta x} < 1$ or $\frac{\Delta t}{\Delta x} < \frac{1}{c}$

https://en.wikipedia.org/wiki/Courant%E2%80%93Friedrichs%E2%80%93Lewy_condition

1.3. adaptive mesh refinement

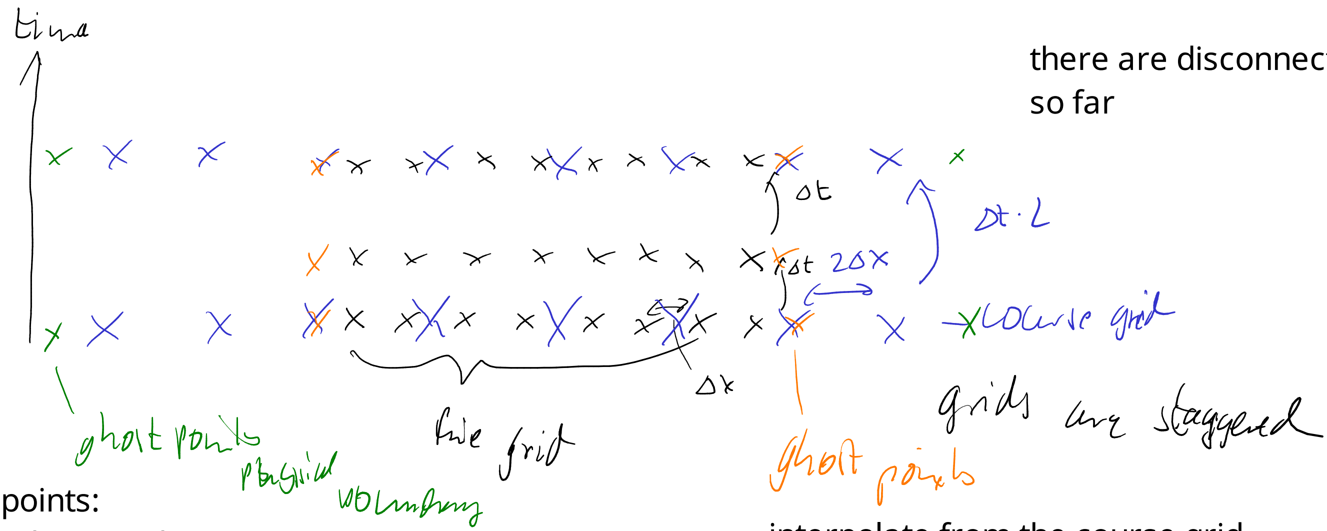
example: Neutron Star merger



want resolve multiple scales with different characteristic times

now in 1D

there are disconnected
so far



out points:
boundary conditions
or ghost point with physical condtions

interpolate from the course grid
for the fine grid ghost points in space and time
It is called prolongation

connection between the grids
if your levels are aligned in time, you restrict your information to the course grid,
which means that you rewrite your values on the course grid via the interpolation of the fine one

this approach is one of the simplest refinement strategies and
was introduced by Berger & Oliger

you can extend this e.g. by enforce also that fluxes accross refinement boundaries are conserved
by introducing an correction step, which goes back to Berger & Colella

1.4. Explicit vs implicit schemes

Les as assume: $\frac{d}{dt}\vec{q} = A(t)\,\vec{q}$; $A(t) = \begin{pmatrix} -1 & 0 \\ 0 & -100 \end{pmatrix}$

apply the Euler scheme would lead to

$$\begin{pmatrix} q_1(t+\Delta t) \\ q_2(t+\Delta t) \end{pmatrix} = \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} + \Delta t \cdot \begin{pmatrix} -1 & 0 \\ 0 & -100 \end{pmatrix} \cdot \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}$$

Following the CFL condition, whee would have a necessary time stepping wrt:

q_1 $\Delta t \sim 1$ and for q_2 $\Delta t \sim \frac{1}{100}$

when use explicit schemes and the same timestep for different variables, this gets computation vey expensive.

alternative: implicit scheme

$$\begin{pmatrix} q_1(t+\Delta t) \\ q_2(t+\Delta t) \end{pmatrix} = \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -100 \end{pmatrix} \begin{pmatrix} q_1(t+\Delta t) \\ q_2(t+\Delta t) \end{pmatrix} \Delta t$$

$$\Rightarrow \begin{pmatrix} 1+\Delta t & 0 \\ 0 & 1+100\,\Delta t \end{pmatrix} \begin{pmatrix} q_1(t+\Delta t) \\ q_2(t+\Delta t) \end{pmatrix} = \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}$$

$$q_1(t+\Delta t) = \frac{1}{1+\Delta t}\, q_1(t)$$

$$q_2(t+\Delta t) = \frac{1}{1+100\,\Delta t}\, q_2(t)$$

example 2 $\frac{\partial}{\partial t} q - \frac{\partial}{\partial x}\left(D \frac{\partial}{\partial x} q\right) = b$

for an explicit schem you would end up with

$$q(t+\Delta t) = q(t) + \underbrace{\frac{\Delta t\, D}{\Delta x^2}}\big(q(t, x+\Delta x) - 2q(t,x) + q(t, x-\Delta x)\big) + \Delta t\, b$$

CFL condition $\Delta t \leq \frac{1}{2} \frac{\Delta x^2}{D}$

$\lambda \leq \frac{1}{2}$

$\lambda = \frac{\Delta t}{\Delta x^2} D$

$C_{CFL} = u \cdot \frac{\Delta t}{\Delta x} \leq \frac{1}{2}$

$u = \frac{D}{\Delta x}$ $\quad \frac{D}{\Delta x} \frac{\Delta t}{\Delta x} \leq \frac{1}{2}$

$\Delta t \leq \frac{1}{2} \frac{\Delta x^2}{D}$