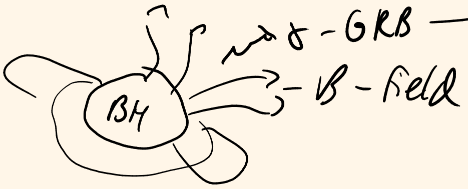
lecture 2024-01-25

2.7. general relativitics Magnetohydrodynamics

2.7.1. Introduction:

- magnetic breaking effects
- in the insprial of two neutron stars the effects are small,
maybe it leads to some electromagnet effects due to crast breaking
- in the merger, magnet field strength can be increased by orders of magnitude
    -> more mass ejection from the remnant due to magnet driven winds
    -> you can produce relativistic jets and launch short Gamma Ray Bursts



2.7.2 Deriving the general relativistics Magnetohydrodynamic Equations(GRMHD):

we need to solve Maxwell equations together with GR

$$\left.\begin{aligned} \nabla \cdot \vec{B} &= 0 \\ \nabla \cdot \vec{E} &= \rho_e \\ \nabla \times \vec{E} &= -\partial_t \vec{B} \\ \nabla \times \vec{B} &= \partial_t \vec{E} + \vec{j} \end{aligned}\right\}$$

$$E^\alpha n_\alpha = 0$$

$$B^\alpha n_\alpha = 0$$

$$E_a = \gamma_{ab} E^b$$

$$B_a = \gamma_{ab} B^b$$

$$j = \rho_e n^a + j^\alpha$$

some hyper- with $\vec{n}$
surface n_a

from special relativity, we know that we can introduce the Faraday tensor

$$F^{ab} = n^a E^b - n^b E^a + \varepsilon^{abc} B_c$$

$$E^a = F^{ab} n_b \; ; \quad B^a = \frac{1}{2} \varepsilon^{abc} F_{bc}$$

this allows us to write Maxwell equations as:

$$\nabla_\mu F^{\mu\nu} = 4\pi j^\nu$$

$$\nabla_{[\alpha} F_{\mu\sigma]} = 0$$

the stress-energy tensor connected to $F_{\alpha\beta}$ is

$$T_{ab}^{EM} = \frac{1}{4\pi}\left(F_{ac} F_b^{\ c} - \frac{1}{4} g_{ab} F_{cd} F^{cd} \right)$$

-> the total energy-momentum tensor is simply

$$T_{ab} = T_{ab}^{Hydro} + T_{ab}^{EM}$$

$$\rho_{ADM} = T_{ab} n^a n^b$$

$$= \rho_0 h W - p + \underbrace{\frac{1}{8\pi}(E^2 + B^2)}$$

$$j_a^{ADM} = -T_{bc} n^b \gamma_a^{\ c}$$

$$= \rho_0 h W u_a + \underbrace{\frac{1}{4\pi} \epsilon_{abc} E^b B^c}$$

$$S_{ab}^{ADM} = \rho_0 h u_a u_b + p \gamma_{ab} + \underbrace{\frac{1}{4\pi}\left[-E_a E_b - B_a B_b + \frac{1}{2} \gamma_{ab} (E^2 + B^2) \right]}$$

need of number of charges and electric current
the evolution equations for electric and magnet field:

$$\partial_t E^i - \mathcal{L}_{\vec\beta} E^i = \alpha K E^i - \overbrace{D_k (\alpha \epsilon^{kij} B_j)}^{\nabla \times B} - 4\pi \alpha j^i$$

source term

$$\partial_t B^i - \mathcal{L}_{\vec\beta} B^i = \alpha K B^i - D_k (\alpha \epsilon^{kij} E_j)$$

shift in ADM formalism, derivative operator, curved space time

$$D_a E^a = 4\pi \rho_e$$

$$D_a B = 0$$

generally to simplify the equations, one introduces

$$\tilde{E}^a = \sqrt{\gamma} E^a, \quad \tilde{B}^a = \sqrt{\gamma} B^a, \quad Q = \sqrt{\gamma} \rho_e, \quad J^a = \alpha \sqrt{\gamma} j^a$$

$$\partial_t Q + \partial_k (J^k - Q \beta^k) = 0$$

and Ohm's law gives you

$$J^k = (v^k + \beta^k) Q + \sigma_e \left[-(v^k + \beta^k) \overbrace{\tilde{E}^i v_i}^{\vec v \vec B} + \alpha \left(W \tilde{E}^k + \overbrace{\epsilon^{kij} v_i \tilde{B}_j}^{\vec v \times \vec B} \right) \right]$$

to solve this equation often it is used to make some assumptions about the fluid

2.7.2.1 ideal GRMHD
only works inside the Neutron star

only evolving the electric field not the B field

assumptions:
- conductivity is very large $\sigma \to \infty$

$$F^{ab} u_b = 0 \quad \Rightarrow \quad E^i = -\frac{1}{W} \epsilon^{ijk} u_j B_k$$

This also implies

$$E_a B^a = 0$$

and the evolution equation for B-field gets

$$\partial_t \hat{B}^i = \partial_k (v^i B^k - v^k \hat{B}^i)$$

the energy-momentum tensor simplifies

$$T^{EM}_{ab} = \frac{1}{4\pi} \left(b^2 u_a u_b + \frac{b^2}{2} g_{ab} - b_a b_b \right)$$

$$b^2 = b_a b^a \; ; \; b_a = \frac{1}{W} \left(B_a + u_a (B^k u_k) \right)$$

-> if you compare it together to the ideal fluid energy-momentum tensor, you see that $\frac{b^2}{4\pi}$

von Neumann stability for ODGL equations

is somehow related to the magnetic energy density
and $\frac{b^2}{8\pi}$ is the magnetic pressure

Following steps that we outlined for deriving the GRHD equations, one can alos write ideal GRMHD in conservative form

$$\partial_t \vec{q}_\alpha + \partial_k \vec{F}^k_\alpha = \vec{S}_\alpha$$

$$q_\alpha = (D, S_i, \tau, \hat{B}^i)$$

$$F^k_\alpha = \Big(D_v^k, S_i v^k + p \alpha \sqrt{\gamma} \delta^k_i - \frac{\alpha\sqrt{\gamma}}{4\pi W^2} B^k (B_i + u_i (B^j u_j)), \tau v^k + (p + \frac{b^2}{8\pi})(v^k + \beta^k)\sqrt{\gamma} + - \frac{\alpha\sqrt{\gamma}}{4\pi W^2} B^k (B^j u_j), \sqrt{\gamma} (v^k B^i - B^k v^i) \Big)$$

$$S_\alpha = \Big(0, -\tau \partial_i \alpha + \tau \partial_i \beta^k - \frac{1}{2} \alpha \sqrt{\gamma} S_{jk} \partial_i \gamma^{jk}, \alpha\sqrt{\gamma} S_{ij} K^{ij} - S^i \partial_i \alpha, 0 \Big)$$

2.7.2.2 Force-free GRMHD

$$T^{Hydro}_{\mu\nu} << T^{EM}_{\mu\nu}$$

this leads to $\nabla^\mu T_{\mu\nu} = 0$

$$D_\mu T_{\mu\nu} \approx D^\mu T^{EM}_{\mu\nu}$$

$$F_{\mu\nu} j^\mu = 0$$

-> this assumptions also decouples the evolution equations

2.7.2.3.) resistive GRMHD

-> is more realistic, as it allows to describe the interiorof the stars and
its exterior, ut the equations become longer and stiff
-> you need a smaller timestep or an impilicit scheme

2.7.3.) Implementation

2.7.3.1.) Divergence Cleaning

kappa - parameter determining the damping speed

$$\nabla_\mu ({}^*F^{\mu\nu} + g^{\mu\nu} \zeta) = \kappa n^\nu \zeta$$

some quantum mechanis

zeta

depends on div B

you get an evolution equation for ζ
that leads a change in the evolution of B, damping possible monopoles

2.7.3.2.) Vector potential evolution

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$A_\mu = n_\mu \Phi + A_\mu$$

$$\partial_t A_i = -E_i - \partial_i (\alpha \Phi - \beta^j A_j)$$

with

$\Phi = \frac{1}{\alpha} (\beta^j A_j)$ algebraic gauge

$\rightarrow \partial_t A_i = -E_i$

2.7.3.3.) constraint transport

$$E^{\hat{i}} = -\frac{1}{w}\,\epsilon^{ijk}\,u_j B_k$$

Hamiltonian physics

$$\rightarrow \partial_t \vec{\tilde{B}} + \nabla \times \vec{E} = 0$$