lecture 2023-12-07

Problem set 2 until 2nd February 2024

4.1. GRHD in 3+1 form

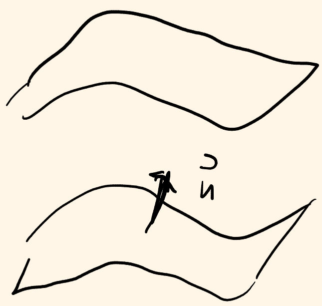
mass conservation I $\partial_t(\sqrt{\gamma}\tilde{D}) + \partial_k(\sqrt{\gamma}\hat{D}(\alpha v^k - \beta^k)) = 0$

momentum conservation II $\partial_t(\sqrt{\gamma}S_i) + \partial_k(S_i(\alpha v^k - \beta^k) + \alpha p\,\delta^k_i) = \alpha\sqrt{\gamma}\,T^{\mu}{}_{\nu}\,\Gamma^{\nu}{}_{i\mu}$

energy conservation III $\partial_t(\sqrt{\gamma}\tau) + \partial_k(\sqrt{\gamma}(\tau(\alpha v^k - \beta^k) + \alpha p v^k)) = \alpha^2\sqrt{\gamma}\,T^{\nu\mu}$

$\partial_\mu(\ln\alpha) - \Gamma^{0}{}_{\mu\nu}$

rewrite this system in 3+1 form



γ_{ij}, K_{ij}

helpers:

(covariant derivative wrt γ_{ij})

$$\frac{1}{\sqrt{\gamma}}\partial_k(\sqrt{\gamma}\,\omega^k) = D_k\omega^k$$

$\hat{D}$ - rest mass density

use first equation for $\tilde{D}$:

$$\frac{1}{\sqrt{\gamma}}\partial_k(\sqrt{\gamma}\tilde{D}(\alpha v^k - \beta^k)) = D_k(\tilde{D}(\alpha v^k - \beta^k))$$

$$= D_k(\hat{D}\alpha v^k) - \beta^k\partial_k\hat{D} - D_k\beta^k\tilde{D}$$

also note $\frac{1}{\sqrt{\gamma}}\partial_t(\sqrt{\gamma}\tilde{D}) = \partial_t\tilde{D} + \frac{\hat{D}}{2}\partial_t\ln\gamma$

$$\underbrace{\partial_t\ln\gamma = -2\alpha K + 2D_k\beta^k}_{\text{ADM equations}}$$

$$\partial_t\hat{D} + \frac{\hat{D}}{2}[-2\alpha K + 2D_k\beta^k] + D_k(\hat{D}\alpha v^k) - \beta^k\partial_k\hat{D} - D_k\beta^k\tilde{D} = 0$$

$$\boxed{\Rightarrow \partial_t\tilde{D} - \beta^k D_k\hat{D} + D_k(\hat{D}\alpha v^k) = \alpha\hat{D}K}$$

equation for τ

$$T_{\mu\nu} = \rho_0 h u_\mu u_\nu + p g_{\mu\nu}$$

Lorentz factor

$$v^i = \frac{u^i}{\alpha u^0} + \frac{\beta^i}{\alpha}, \quad u^i = \left(v^i - \frac{\beta^i}{\alpha}\right) W \quad u^0 = \frac{W}{\alpha} \quad W = \frac{1}{\sqrt{1-v^2}}$$

$$g_{\alpha\beta} = \begin{pmatrix} -\alpha^2 + \beta_k \beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \quad g^{\alpha\beta} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i \beta^j}{\alpha^2} \end{pmatrix}$$

$$\Gamma^0_{00} = \left(\partial_t \alpha + \beta^m \partial_m \alpha - \beta^m \beta^n K_{mn}\right) \frac{1}{\alpha}$$

$$\Gamma^0_{0i} = \frac{\partial_i \alpha - \beta^n K_{in}}{\alpha}$$

$$\Gamma^0_{ij} = -\frac{K_{ij}}{\alpha}$$

$$\partial_t(\sqrt{\gamma}\tau) + \partial_k\left(\sqrt{\gamma}\tau(\alpha v^k - \beta^k) + \alpha p v^k\right) = \alpha^2 \sqrt{\gamma}\left(T^{0\mu}\partial_\mu \ln\alpha - \Gamma^0_{\mu\nu} T^{\mu\nu}\right)$$

$$T^{00}\partial_0 \ln\alpha = (\rho_0 h u^0 u^0 + p g^{00})\partial_0 \ln\alpha = \frac{\rho_0 h W^2}{\alpha^2}\frac{\partial_t \alpha}{\alpha} - \frac{p}{\alpha^2}\frac{\partial_t \alpha}{\alpha}$$

$$T^{0i}\partial_i \ln\alpha = (\rho_0 h u^i u^0 + p g^{0i})\partial_i \ln\alpha = \left(\rho_0 h\left(v^i - \frac{\beta^i}{\alpha}\right) W \frac{W}{\alpha} + p\frac{\beta^i}{\alpha^2}\right)\frac{\partial_i \alpha}{\alpha}$$

$$= \left(\rho_0 h v^i \frac{W^2}{\alpha} - \rho_0 \frac{\beta^i}{\alpha^2} h W^2 + \frac{p\beta^i}{\alpha^2}\right)\frac{\partial_i \alpha}{\alpha}$$

$$\Gamma^0_{00} T^{00} = \left(\partial_t \alpha + \beta^m \partial_m \alpha - \beta^m \beta^n K_{mn}\right)\frac{1}{\alpha}\cdot\left(\rho_0 \cdot h u^0 u^0 + p g^{00}\right)$$

$$= \left(\partial_t \alpha + \beta^m \partial_m \alpha - \beta^m \beta^n K_{mn}\right)\frac{1}{\alpha}\cdot\left(\rho_0 \cdot h \frac{W}{\alpha}\frac{W}{\alpha} + p \cdot -\frac{1}{\alpha^2}\right)$$

$$\Gamma^0_{ij} T^{ij} =$$

write down the right side ithout terms of the 0. components

$$\alpha^2 \sqrt{\gamma}^{-1} \left(T^{0\mu} \partial_\mu \ln\alpha - \Gamma^0_{\mu\nu} T^{\mu\nu} \right)$$

$\Rightarrow$

$$\partial_t \alpha : \quad \frac{\rho_0 h w^2 \partial_t \alpha}{\alpha^3} - \frac{p}{\alpha^3}\partial_t \alpha - \frac{\partial_t \alpha}{\alpha}\frac{\rho_0 h w^2}{\alpha^2} + \frac{\partial_t \alpha\, p}{\alpha^3} = 0$$

$$v^i \beta^m K_{mn} : \quad -\beta^m K_{in} \rho_0 h \frac{w^2}{\alpha} v^i \cdot 2 + 2 \cdot \beta^m K_{in} \frac{\rho_0 h w^2 v^i}{\alpha^3} = 0$$

because $\Gamma^0_{0i} T^{0i} = \Gamma^0_{i0} T^{i0}$

if collect all terms

$$\alpha^2 \left(T^{0\mu} \ln\alpha - \Gamma^0_{\mu\nu} T^{\mu\nu} \right) = -\rho_0 h w^2 \partial_j \alpha\, v^j + \alpha p K + \alpha K_{ij} \rho_0 h w^2 v^i v^j$$

if we again use the same steps for the equation for D

$$\partial_t \tau - \beta^k \partial_k \tau + D_k \left(\alpha v^k (\tau + p) \right) = (\tau + p + \hat{D}) \left(\alpha v^m v^n K_{mn} - v^n \partial_n \alpha \right) +$$
$$+ \alpha K (\tau + p)$$

let us look at a fluid comoving with Eulerian observers

change in internal energy

$$\partial_t \tau = \alpha K (\tau + p) \qquad \alpha K = -\partial_t \ln \gamma$$

due to change in volume element

and due to pressure

$$\Rightarrow \alpha K p = -p \partial_t \ln \gamma$$

first law of thermodynamics

and, we avoid deriving the last equations:

$$\partial_t S^i - \mathcal{L}_{\vec{\beta}} S^i + D_k \left[\alpha \left(S^i v^k - \gamma^{ik} p \right) \right] = -(\tau + \hat{D}) D^i \alpha + \alpha K S^i$$

$$\mathcal{L}_{\vec{\beta}} S^i = \beta^n \partial_n S^i - S^k \partial_k \beta^i$$

4.2. the equation of state

some special cases:

dust: $p = 0$ non interacting particles, so they can not create pressure

ideal gas: $\overset{\text{heat}}{dQ} = \underbrace{dU}_{\text{internal energy}} + p\,dV$

heat capacity at constant volume $c_V = \frac{1}{M}\left(\frac{dQ}{dT}\right)_V$ $\quad c_p = \frac{1}{M}\left(\frac{dQ}{dT}\right)_p$

heat capacity at constant pressure

adiabatic index $\gamma = \frac{c_p}{c_V}$

$$pV = nkT$$

$$dQ = \underbrace{dU + p\,dV = M c_V\,dT}_{\text{constant } V} + nk\,dT$$

$$c_p = \left(\frac{M c_V\,dT + nk\,dT}{M\,dT}\right)_p = c_V + \frac{k}{m} = c_V + \frac{nk}{M} \rightarrow n\cdot m = M \quad \text{particle mass} \quad m = \frac{M}{n}$$

$$\gamma = 1 + \frac{k}{m c_V}$$

$$nT = \frac{U}{m c_V} \rightarrow pV = \frac{kU}{m c_V} = (\gamma - 1)\overset{\text{total energy}}{U}$$

$$\rho_0 = \frac{nm}{V}$$

$$U = n\cdot m\cdot \varepsilon \qquad p(\rho_0, \varepsilon) = (\gamma - 1)\rho_0 \varepsilon$$

foradiabtic processes (no heat processes)

$$dQ = 0 \rightarrow 0 = d\varepsilon + p\,d\left(\frac{1}{\rho_0}\right)$$

$$= \frac{1}{\gamma - 1} d\left(\frac{p}{\rho_0}\right) + p\,d\left(\frac{1}{\rho_0}\right)$$

$$\frac{dp}{p} = \gamma \frac{d\rho_0}{\rho_0} \qquad p(\rho_0) = \underbrace{C}_{\text{integration constant}}\rho_0^{\gamma} \quad \text{polytropic equation of state}$$

$$\varepsilon = \frac{C}{\gamma - 1} \rho_0^{\gamma - 1}$$

in practise, people tabulated equation of states(EOS)

$$p = p(\rho_0, T, Y_e)$$

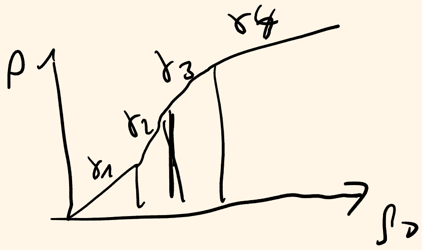
this ist potentially a complicated function dependency
on the density, temprature and composition

people simplify tabulated EOS, by first assuming
o-temperature and a fixed composition

$$p(\rho_0, T, Y_e) \mapsto p(\rho_0)$$

approximated by piecewise polytypes

$$p_i = C_i (\rho_0)^{\gamma_i}$$



where p is continuous between different pieces or thes generalized
piecewise polytropics
to make p one time differentiable to have a continuous speed of sound

$$p_i = C_i \rho_0^{\gamma_i} + \Delta_i$$

to still include heat people assume an additional pressure component
that is similar to an ideal gas

$$p = p_{cold} + p_{th}$$

piecewise polytrope

thermal pressure $p_{th} = \rho_0 \varepsilon_{th} (\gamma - 1)$

$\varepsilon - \varepsilon_{cold}$