Problem Set 1

Wave Equation

due date: November 23, 2022 NR_Hydro

A Solving the wave equation

In this problem set, we will consider a simple non-relativistic example and will investigate the one-dimensional wave equation for the scalar field ϕ

$$\frac{\partial^2 \phi(x,t)}{\partial t^2} = c^2 \frac{\partial^2 \phi(x,t)}{\partial x^2}, \tag{1}$$

with $c = const$.

(A.1) Fully first order formulation (10 Marks)

As a first task, rewrite the wave equation (Eq. (1)) in a fully first order form by using the auxiliary variables $\eta = \partial_t \phi$ and $\chi = \partial_x \phi$. Collect them into the state vector $\mathbf{u} = (\phi, \eta, \chi)$ and show that the vector equation can be written as

$$\partial_t \mathbf{u} + \mathbf{A} \partial_x \mathbf{u} = \mathbf{S}, \tag{2}$$

where $\mathbf{A}$ is a matrix and $\mathbf{S}$ is the vector of source terms. Analyze matrix $\mathbf{A}$ by evaluating its eigenvalues and eigenvectors.

(A.2) Solving the wave equation

In the next step, please consider the following initial condition

$$\phi(x,0) = e^{\sin^2\left(\frac{\pi x}{L}\right)} - 1, \quad 0 \leq x \leq L, \tag{3}$$

with periodic boundary condition

$$\phi(x,t) = \phi(x \pm L, t). \tag{4}$$

For simplicity, you should choose $L = 1 = c$.

Write a code to solve the system 2 numerically using

1. **(15 Marks)** second-order spatial finite differencing stencils and the forward Euler method, and
2. **(25 Marks)** forth-order spatial finite difference and the forth-order Runge-Kutta method.

For the spatial grid setting, you can discretize the equation in space using an N-points grid in cell center:

$$x_i = \left(i - \frac{1}{2}\right) \Delta x, \quad i = 1, 2, \ldots, N, \tag{5}$$

with $\Delta x = L/N$ and $\mathbf{u}_i = \mathbf{u}(x_i)$. Note that the time step for this hyperbolic system has to follow the Courant-Friedrich-Levy condition

$$\max |\lambda_i| \frac{\Delta t}{\Delta x} \leq C_{max}, \tag{6}$$

where λ_i is the characteristic speed of the system and C_{max} is the maximum allowed Courant number depending (where the exact number will depend on the exact discretization scheme; typically $C_{max} = 1$). You can choose $\Delta t = C \frac{\Delta x}{\max|\lambda_i|}$ for Courant number $C \leq 1$ in this problem. Perform convergence test by varying the resolution Δx and Δt to verify your result.

(A.3) Parallelization

In the following we will extend the programm by employing different parallelization schemes, please parallelize your code using

1. **(25 Marks)** Open MP, and
2. **(25 Marks)** MPI

Show that your parallelized code gives the same result as the serial version and measure the performance of your code by performing a strong scaling test (i.e. fix your problem size and measure the execution time with different number of processors). You may need to use a large number of grid points $N > 1000$ for the test.

– Please submit solutions to peter.nee@aei.mpg.de. –

$\partial_t \vec{u} = F(\vec{u})$ $\quad \partial_t^2 \varphi = c^2 \partial_x^2 \varphi$ $\quad \partial_x \varphi = \chi$

$\partial_x \chi = \partial_x^2 \varphi$

$\partial_t^2 u = \partial_x u$

$v = \partial_t u$

$\partial_t u = v$

$\partial_t v = \partial_t^2 u = \partial_x u$

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix}$$

A.2.

initial condition $\varphi(x,0) = e^{\sin^2\left(\frac{\pi x}{2}\right)} - 1$

$\partial_t \varphi = \eta$

$\partial_x \varphi = \chi = \varphi_{0,x}$

$$\eta = \begin{cases} 0 \\ \varphi_{it} = \pm c \dfrac{\partial \varphi}{\partial x} = \pm \varphi_{0x} \end{cases}$$

choose your own condition for η

A.2.1.

(1) $f(x)$

(2) $f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) + O(\Delta x^3)$

(3) $f(x-\Delta x) = f(x) - \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) + O(\Delta x^3)$

(2)-(3)

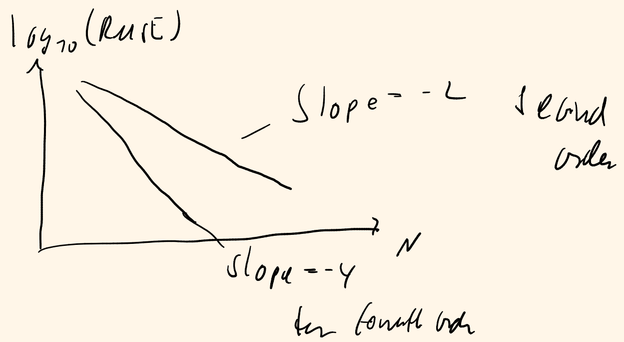
$f(x+\Delta x) - f(x-\Delta x) = 2\Delta x f'(x) + O((\Delta x)^3)$

$$f'(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$$

(4) $f(x+2\Delta x) = \ldots$ Taylor expansion up to fourth order

(5) $f(x+2\Delta x) = \ldots$

$$RMSE = \frac{1}{N} \sqrt{\sum_{i=0} \left(\varphi'_{num}(x_i) - \varphi'_{analytic}(x_i) \right)^2}$$



$$\partial_t^2 \varphi = c^2 \partial_x^2 \varphi$$

$$\partial_t^2 u = \partial_x u$$

$$v = \partial_t u$$

$$\partial_t u = v$$

$$\partial_t v = \partial_t^2 u = \partial_x u$$

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix}$$

$$\partial_x \varphi = \chi$$

$$\partial_t \varphi = \eta$$

$$\partial_x \chi = \partial_x^2 \varphi = \frac{1}{c^2} \partial_t^2 \varphi$$

$$= \frac{1}{c^2} \partial_t \eta$$

$$\partial_t \eta = \partial_t^2 \varphi = c^2 \partial_x^2 \varphi$$

$$= c^2 \partial_x \chi$$

$$\partial_t \chi = \partial_t \partial_x \varphi$$

$$= \partial_x \partial_t \varphi$$

$$= \partial_x \eta$$

$$\partial_x \chi = \frac{1}{c^2} \partial_t \eta$$

$$\vec{u} = \begin{pmatrix} \varphi \\ \eta \\ \chi \end{pmatrix}$$

$$\partial_t \varphi = \eta$$

$$\partial_t \eta = c^2 \partial_x \chi$$

$$\partial_t \chi = \partial_x \eta$$

$$\chi = \partial_x \varphi$$

$$\partial_x \chi = \frac{1}{c^2} \partial_t \eta$$

$$\partial_t \vec{u} + A \vec{u} = \vec{S}$$

$$\begin{pmatrix} \partial_t \varphi \\ \partial_t \eta \\ \partial_t \chi \end{pmatrix} + A \cdot \begin{pmatrix} \varphi \\ \eta \\ \chi \end{pmatrix} = \vec{S}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\vec{S} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\partial_t \varphi + a_{11} \varphi + a_{12} \eta + a_{13} \chi = 0$$

$$\partial_t \eta + a_{21} \varphi + a_{22} \eta + a_{23} \chi = 0$$

$$\partial_t \chi + a_{31} \varphi + a_{32} \eta + a_{33} \chi = 0$$

$$\chi - \partial_x \varphi = 0 \qquad a_{13} = 1 \quad a_{11} = -\partial_x$$

$$\partial_t \varphi - \eta = 0 \qquad a_{12} = -1$$

$$\partial_t \eta - c^2 \partial_x \chi = 0 \qquad a_{23} = -c^2 \partial_x$$

$$\partial_t \chi - \partial_x \eta = 0 \qquad a_{32} = -\partial_x$$

$$A = \begin{pmatrix} -\partial_x & -1 & 1 \\ 0 & 0 & -c^2 \partial_x \\ 0 & -\partial_x & 0 \end{pmatrix}$$

$$\partial_t \varphi + \chi - \partial_x \varphi - \eta = 0$$

$$\partial_t \varphi = \eta + \partial_x \varphi - \chi$$

$$\partial_t \eta = + c^2 \partial_x \chi$$

$$\partial_t \chi = \partial_x \eta$$

$$\partial_x^2 \varphi = c^2 \partial_t^2 \varphi$$

$$\partial_x \varphi = \eta$$

$$\partial_x^2 \varphi = \partial_x \eta = c^2 \partial_t^2 \varphi$$

$$\partial_x \eta = c^2 \partial_t \chi$$

$$\partial_t \varphi = \chi$$

$$\partial_t^2 \varphi = \partial_t \chi = \frac{1}{c^2} \partial_x^2 \varphi$$

$$= \frac{1}{c^2} \partial_x \eta$$

$$\partial_t \eta = \partial_t (\partial_x \varphi)$$

$$= \partial_x (\partial_t \varphi)$$

$$= \partial_x \chi$$

$$\partial_t \varphi = \chi$$

$$\partial_t \chi = \frac{1}{c^2} \partial_x \eta$$

$$\partial_t \eta = \partial_x \chi$$

$\partial_t \vec{u} + A \partial_x \vec{u} = \vec{S}$

$\partial_x \varphi = \chi$ $\qquad \partial_t^2 \varphi = c^2 \partial_x^2 \varphi$

$\underline{\partial_t \varphi = \eta}$ $\qquad \partial_t^2 \varphi = \partial_t \eta$

$\partial_x^2 \varphi = \partial_x \chi$ $\qquad \underline{\partial_t \eta = c^2 \partial_x \chi}$

$\vec{u} = \begin{pmatrix} \varphi \\ \eta \\ \chi \end{pmatrix}$

$\partial_t \begin{pmatrix} \varphi \\ \eta \\ \chi \end{pmatrix} + A \partial_x \begin{pmatrix} \varphi \\ \eta \\ \chi \end{pmatrix} = S$

$\partial_t \varphi + a_{11} \partial_x \varphi + a_{12} \partial_x \eta + a_{13} \partial_x \chi = S_1$

$\partial_t \eta + a_{21} \partial_x \varphi + a_{22} \partial_x \eta + a_{23} \partial_x \chi = S_2$

$\partial_t \chi + a_{31} \partial_x \varphi + a_{32} \partial_x \eta + a_{33} \partial_x \chi = S_3$

$\partial_x \varphi = \chi$ $\qquad$ I $\partial_t \varphi = \eta \rightarrow a_{11} = a_{12} = a_{13} = 0$ $\qquad$ linear propagation speed

$\partial_t \eta = c^2 \partial_x \chi$ $\qquad S_1 = \eta$

$\partial_t \chi = \partial_t \partial_x \varphi$

$= \partial_x (\partial_t \varphi)$ $\qquad \partial_t \varphi = \eta$

$\partial_t \chi = \underline{\partial_x \eta}$

I $\partial_t \varphi = \eta$ $\qquad a_{11} = a_{12} = a_{13} = 0$

$\boxed{\partial_x \varphi = \chi}$ $\qquad \partial_t \eta = c^2 \partial_x \chi$

$\partial_t \eta - c^2 \partial_x \chi = 0$

II $\partial_t \eta - c^2 \partial_x \chi + \partial_x \varphi = \chi$ $\qquad a_{21} = 1 \quad a_{23} = -c^2 \quad a_{22} = 0 \quad S_2 = \chi$

$\vec{u} = \begin{pmatrix} \varphi \\ \eta \\ \chi \end{pmatrix}$ $\qquad$ III $\partial_t \chi - \partial_x \eta = 0$

$a_{32} = -1 \quad a_{31} = a_{33} = 0$

$S = \begin{pmatrix} \eta \\ \chi \\ 0 \end{pmatrix}$ $\qquad S_3 = 0$

$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \underline{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -c^2 \\ 0 & -1 & 0 \end{pmatrix}}$

$\partial_t \vec{u} + A \partial_x \vec{u} = \vec{S}$

$\underline{\partial_t \begin{pmatrix} \varphi \\ \eta \\ \chi \end{pmatrix}} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -c^2 \\ 0 & -1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} \varphi \\ \eta \\ \chi \end{pmatrix} = \begin{pmatrix} \eta \\ \chi \\ 0 \end{pmatrix}$

$$\partial_t \begin{pmatrix}\varphi\\ \eta\\ \chi\end{pmatrix} + \begin{pmatrix}0&0&0\\0&0&-c^2\\0&-1&0\end{pmatrix}\partial_x\begin{pmatrix}\varphi\\ \eta\\ \chi\end{pmatrix} = \begin{pmatrix}\eta\\0\\0\end{pmatrix}$$

I $\partial_t \varphi = \eta$

$\partial_x \varphi = \chi$ $\quad \partial_t \eta = c^2 \partial_x \chi$

$\underline{\mathrm{II}}$ $\partial_t \eta - c^2 \partial_x \chi = 0$

$\underline{\mathrm{III}}$ $\partial_t \chi - \partial_x \eta = 0$

$\varphi(0,x) = e^{\sin^2(\frac{\pi}{L}x)} - 1 \quad 0 \le x \le L$

$\partial_x \varphi = \chi \qquad \partial_x \chi = e^{\sin^2(\frac{\pi}{L}x)} \cdot (2\sin\frac{\pi}{L}x) \cdot \frac{\pi}{L}$

$\partial_t \varphi = \eta$

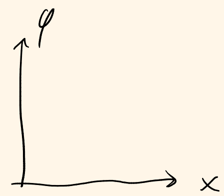
$\eta = 0$

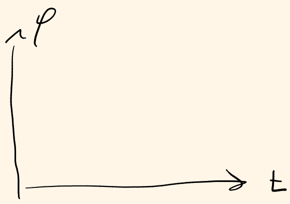
you can choose for eta a arbitary initial condition
experimenting with some
with equal 0
with equal constant
with equal chi

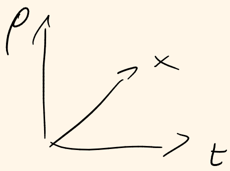
$\partial_t \varphi = \eta$

$\partial_t \eta = +c^2 \partial_x \chi$

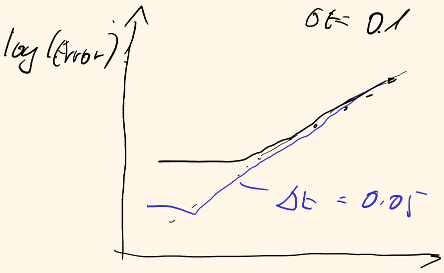
$\partial_t \chi = \partial_x \eta$







Konvergence test



① $\Delta x = 0.1$
② $\Delta x = 0.05$
③ $\Delta x = 0.025$

$\rightarrow \Delta\Delta x_{12}$

$\Delta\Delta x_{23}$

$$\frac{u_{i+1} - u_{i-1}}{2h} + O(\Delta x^2)$$

$u(x) = x$

not too simple choosing

do not change delta t