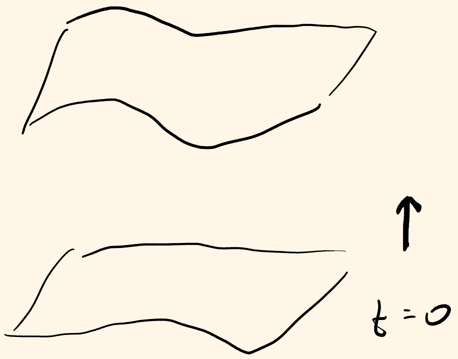
lecture 2023-12-14

5. Binary neutron Star initial data

the formalism that we want to use is the extended conformal Thin Sandwich approach



$\mathcal{H} = 0$

$\mathcal{M} = 0$

constraints: energy and momentum conservation

in Maxwell Equations we have

$\nabla \cdot B = 0 \qquad \nabla \cdot E \sim \rho$

how to compute data on this first slice

equations that folow from the 3+1 decomposition of the field equations and that correspond to energy and momentum conservation

$$\mathcal{H} = {}^{(3)}R + K^2 - K_{\mu\nu} K^{\mu\nu} - 16\pi \rho_{ADM} = 0$$

Hamiltonian constraint

$$\mathcal{M}^i = D_j \left(K^{ij} - \gamma^{ij} K \right) - 8\pi j^i_{ADM} = 0$$

Momentum constraint

we start with the conformal transformation of the metric

$$\bar{\gamma}_{ij} = \psi^{-4} \gamma_{ij}$$

and we define

$$\bar{u}_{ij} = \partial_t \gamma_{ij}$$

and we will demand that the volume element of the confrmal metrix stay momententanly constant

$$\partial_t \bar{\gamma} = 0$$

since we introduced ψ , we hace one more degree of freedom and can enforce

$$\bar{\gamma} = 1 \rightarrow \partial_t \ln \psi = \partial_t \ln \gamma^{\frac{1}{12}}$$

$$u_{ij} = \partial_t(\psi^{-4}\gamma_{ij}) = \psi^{-4}(\partial_t \gamma_{ij} - 4\gamma_{ij}\partial_t \ln\psi)$$

$$= \psi^{-4}(\partial_t \gamma_{ij} - \tfrac{1}{3}\gamma_{ij}\partial_t \ln\gamma) = \psi^{-4} u_{ij}$$

if we now use the trace-freepart of the evolution equation of γ_{ij} then we get

$$u_{ij} = \partial_t \gamma_{ij}(\gamma^{mn}\partial_t \gamma_{mn}) = -2\alpha A_{ij} + (\underline{\underline{L}}\beta)_{ij}$$

with the conformal killing form

$$(\underline{\underline{L}}\beta)_{ij} = D_i\beta_j + D_j\beta_i - \tfrac{2}{3}\gamma_{ij} D_k\beta^k$$

$$K_{ij} = \tfrac{1}{2\alpha}(\partial_t \gamma_{ij} - D_i\beta_j - D_j\beta_i)$$

$$K^{ij} = \underbrace{A^{ij}}_{\text{trace-free part of the curvature}} + \tfrac{1}{3}\gamma^{ij}\underbrace{K}_{\text{trace}} \rightsquigarrow A^{ij} = \tfrac{1}{2\alpha}[(\underline{\underline{L}}\beta)^{ij} - u^{ij}]$$

$$= \frac{\psi^{-4}}{2\alpha}[(\overline{\underline{\underline{L}}\beta})^{ij} - \bar{u}^{ij}]$$

finally we also get a conformal trace free part of the extrinsic curvature

$$\bar{A}^{ij} = \psi^{10}A^{ij} = \tfrac{1}{2\bar{\alpha}}[(\overline{\underline{\underline{L}}\beta})^{ij} - \bar{u}^{ij}]$$

$$\text{with } \bar{\alpha} = \psi^{-6}\alpha$$

so we obtain finally the CTS: equation.

$$\mathcal{H} \rightarrow 8\bar{D}^2\psi - \bar{R}\psi + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} - \tfrac{2}{3}\psi^5 K + 16\pi\psi^5\rho = 0$$

$$\mathcal{M} \rightarrow \bar{D}_j\left[\tfrac{1}{2\bar{\alpha}}(\overline{\underline{\underline{L}}\beta})^{ij}\right] - \bar{D}_j\left[\tfrac{1}{2\bar{\alpha}}\bar{u}^{ij}\right] - \tfrac{2}{3}\psi^6\bar{D}^i K - 8\pi\psi^{10} j^i = 0$$

solve $\mathcal{H}$ for ψ $\quad$ solve $\mathcal{M}^i \rightarrow \beta^i$

assumptions: conformal derivative

conformal metric
an example

$$\bar{\gamma}_{ij} = \delta_{ij} \Rightarrow \bar{R} = 0 \qquad \bar{D}^2 = \Delta$$

$\bar{D}^2 \rightarrow$ normal partiall Derivative

$$\bar{u}_{ij} = 0 \qquad K = 0$$

$\bar{D}_j \rightarrow$ no Γ

$\bar{D}_j = \partial_j$

α can also be choosen

ρ, j^i will determine your solution and depend on the fluid properties

assume you know ρ, j^i you can solve for ψ, β^i and get the full space time

$$\gamma_{ij} = \psi^{+4} \bar{\gamma}_{ij}$$

$$K^{ij} = \psi^{-10} \bar{A}^{ij} + \frac{1}{3}\gamma^{ij} K$$

$$\bar{A}^{ij} = \frac{1}{2\alpha}\left[(\bar{\mathbb{L}}\beta)^{ij} - \bar{u}^{ij} \right]$$

the approach can be extended by chosen the time derivative of
the trace of the extrinsic curvature $\partial_t K$

$$\partial_t K = -D^2 \alpha + \alpha\left(K_{ij} K^{ij} + 4\pi(\rho + \underset{\sim}{S}) \right) + \beta^i D_i K$$

$$S_{ij} S^{ij}$$

$$\Rightarrow \bar{D}^2(\alpha\psi) = \alpha\psi\left(\frac{7}{8}\psi^{-8}\bar{A}_{ij}\bar{A}^{ij} + \frac{5}{12}\psi^4 K^2 + \frac{1}{8}\bar{R} + \right.$$
$$\left. + 2\pi\psi^4(\rho + S) \right) - \psi^5 \partial_t K + \psi^5 \beta^i \bar{D}_i K$$

equation for α

to solve this set of equations we need bundary conditions, and often people use

$$\lim_{r\to\infty} \alpha = 1 \qquad \lim_{r\to\infty} \beta^i = 0 \text{ or } \Omega\varphi \qquad \lim_{r\to\infty} \gamma_{ij} = \delta_{ij}$$

$$\lim_{r\to\infty} \bar{\alpha} = 1 \qquad \lim_{r\to\infty} K^{ij} = 0$$

often people condructing binaries

$$\beta^i = B^i + \Omega \phi^i$$

corotating frame $\lim_{r\to\infty} B^i = 0$

5.2. deriving the matter equations

$$\nabla_\mu T^{\mu\nu} = 0 \rightsquigarrow$$ conservation if energy momentum with respect to the fluid

$$\nabla_\mu (\rho_0 u^\nu) = 0$$ continuity equation

if we pick again

$$T^{\mu\nu} = (\rho_0 (1+\varepsilon) + p) u^\mu u^\nu + p g^{\mu\nu}$$

$$h = 1 + \frac{p}{\rho_0} + \varepsilon$$ then we can rewrite the equations and get:

$$u^\nu \nabla_\nu (h u_\mu) + D_\mu h = 0$$

as next step we can split the fluids 4-velocity

$$u^\mu = u^0 (\xi^\mu + V^\mu)$$

time like, $\xi^0 = 1$ through normalization

purely spatial, $n_\mu V^\mu = 0$, normal vector to the hypersurface

$$D_j\left(\frac{h}{u^t} + \hat{u}_j V^j\right) + V^j (D_j \hat{u}_i - D_i \hat{u}_j) = 0$$

$$D_i (\alpha u^0 \rho_0 V^i) = 0$$

$$\hat{u}_i = \gamma^\mu_i h u_\mu$$

psi is the killing vector

ξ

5.2.1. example corotating Binaries

like the earth and moon system, the same side of the one object is directed to the rotating frame

$v^i = 0$ velocity of fluid is zero

$\hookrightarrow D_j\left(\frac{h}{u^0}\right) = 0 \Rightarrow \frac{h}{u^0} = \text{const}$ constant can be found $u_\mu u^\mu = -1$

5.2.2 example irrotational Binaries

the stars moving around each other

in general relativity you can define irrotational motion through the definition of vorticity tensor

$$\omega_{\mu\nu} = \gamma^\gamma_\mu \gamma^\delta_\nu (\nabla_\delta (h u_\gamma) - \nabla_\gamma (h u_\delta))$$

if $\omega_{\mu\nu} = 0 \Rightarrow h u_\alpha = \nabla_\alpha \Phi$ gradient of a scalar potential

solve elliptical equation for the velocity potential Φ - gives you fluid flow $\Rightarrow h$

5.2.3. example generic systems

for spining stars: you split the spatial velocity v^i into a irrotational part that comes from the gradient of a scalar field and an additional rotational component

for describing generic orbital otion, you need to modify ξ^μ to account e.g. for ecentricity -> allowing for eccentric motion or

introduce some small radial velocity to account for gravitational wave losses.