Spherical stars

field eqs: complicated

$$R_{\mu\nu} = -\kappa\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right), \quad \kappa = 8\pi G$$

analytic/"simple" solutions?

↳ ideal fluid $T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u_\mu u_\nu - p g_{\mu\nu}$

OR vacuum $\rho = p = 0$

↳ symmetries

ansatz for line element:

- static ⇔ indep. of $t = x^0$ (except dt)
- spherically symmetric / isotropic
  ↳ depends on spatial coord. only via

$$r^2 = "\vec{x}^2"$$
$$"d\vec{x}^2" = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$
$$"\vec{x}\cdot d\vec{x}" = r\,dr$$

$$\hookrightarrow ds^2 = g_{\mu\nu} dx^\mu dx^\nu = A(r)dt^2 - B(r)dr^2 - \underbrace{C(r)}_{\substack{1 \text{ by changing/} \\ \text{gauge-fixing } r}} r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) - \underbrace{D(r)\,dr\,dt}_{\substack{\text{remove by} \\ \text{shifting } t}}$$

calculate: $g_{\mu\nu} \to \Gamma^{\rho}_{\mu\nu} \to R^{\mu}{}_{\nu\alpha\beta} \to R_{\mu\nu}$ (exercise?)

$$R_{tt} = -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB} \qquad A' \equiv \frac{dA}{dr} \text{ etc}$$

$$R_{rr} = \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB}$$

$$R_{\theta\theta} = \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right)$$

$$R_{\phi\phi} = R_{\theta\theta} \sin^2\theta$$

all other zero

insert in:

$$R_{\mu\nu} = -\kappa\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right) = -\kappa\left[\left(\rho + \frac{p}{c^2}\right) u_\mu u_\nu - \frac{1}{2}(\rho c^2 - p) g_{\mu\nu}\right]$$

where $T = \left(\rho + \frac{p}{c^2}\right)\underbrace{u^\mu u_\mu}_{c^2} - p \underbrace{\delta^\mu_\mu}_{4} = \rho c^2 - 3p$

$$\left.\begin{array}{l} c^2 = g^{\mu\nu} u_\mu u_\nu \\ u^i = 0 \text{ (static)} \end{array}\right\} \rightarrow c^2 = \frac{1}{A}(u_0)^2 \quad \curvearrowright [u_\mu] = c\sqrt{A}\,(1,0,0,0)$$

then:

$$R_{tt} = -\tfrac{1}{2}\kappa(\rho c^2 + 3p)A$$
$$R_{rr} = -\tfrac{1}{2}\kappa(\rho c^2 - p)B$$
$$R_{\theta\theta} = -\tfrac{1}{2}\kappa(\rho c^2 - p)r^2 \qquad (*)$$

eliminate p:

$$\frac{R_{tt}}{A} + \frac{R_{rr}}{B} + \frac{2R_{\theta\theta}}{r^2} = -2\kappa\rho c^2$$

$$\hookrightarrow \underbrace{\left(1-\frac{1}{B}\right) + \frac{rB'}{B^2}}_{\frac{d}{dr}\left[r\left(1-\frac{1}{B}\right)\right]} = \kappa r^2 \rho c^2$$

$$\frac{d}{dr}\Big[\underbrace{r\left(1-\frac{1}{B}\right)}_{2Gm(r)c^2\text{ def.!}}\Big] \quad \curvearrowright \quad \boxed{m'(r) = 4\pi r^2 \rho(r)} \qquad (\mathrm{I})$$

then: $m(r) = 4\pi \int_0^r d\bar{r}\,\bar{r}^2 \rho(\bar{r}) = \int \overset{\text{flat}}{d V}\,\rho$

<u>not</u> the integrated density ρ

$$\tilde{m}(r) = \int \underset{\text{curved}}{dV}\,\rho = \int_0^r 4\pi\bar{r}^2\sqrt{B(\bar{r})}\,d\bar{r}\cdot\rho(\bar{r})$$

use $\nabla_\nu T^{\mu\nu} = 0$ Recall: follows from field eqs.

$\mu = 0$: continuity eq.

$\mu =$ spatial: Euler eq. (momentum conservation)

case $\mu = r$ $\curvearrowright$ eq. of hydrostatic equilibrium

$$\hookrightarrow \boxed{p'(r) = -(\rho c^2 + p)\frac{A'}{2A}} \qquad (\mathrm{II})$$

from $(*)$ with $B = \left(1 - \frac{2Gm}{c^2 r}\right)^{-1}$

$$\boxed{\frac{A'}{2A} = \frac{1}{r^2}\left[\frac{4\pi G}{c^4}pr^3 + \frac{Gm}{c^2}\right]\left[1 - \frac{2Gm}{rc^2}\right]^{-1}} \qquad (\mathrm{III})$$

Recall: Newtonian limit $A \approx 1 + 2V(r)$ ...

3 equations: (I), (II), (III) ↯ Tolman-Oppenheimer-Volkoff (TOV) eqs.

but 4 functions: m, A, p, ρ

need to specify relation $p \leftrightarrow \rho$

↳ equation of state (EOS)

simple cases:

- barotropic EOS $p = p(\rho)$ (from data table)
- polytropic EOS $p = K \cdot \rho^{\gamma}$ OR: $\rho = \left(\frac{p}{K}\right)^{n/1+n} + np$ sometimes dropped

  polytropic index: $n = (\gamma - 1)^{-1}$
- piecewise polytropic EOS

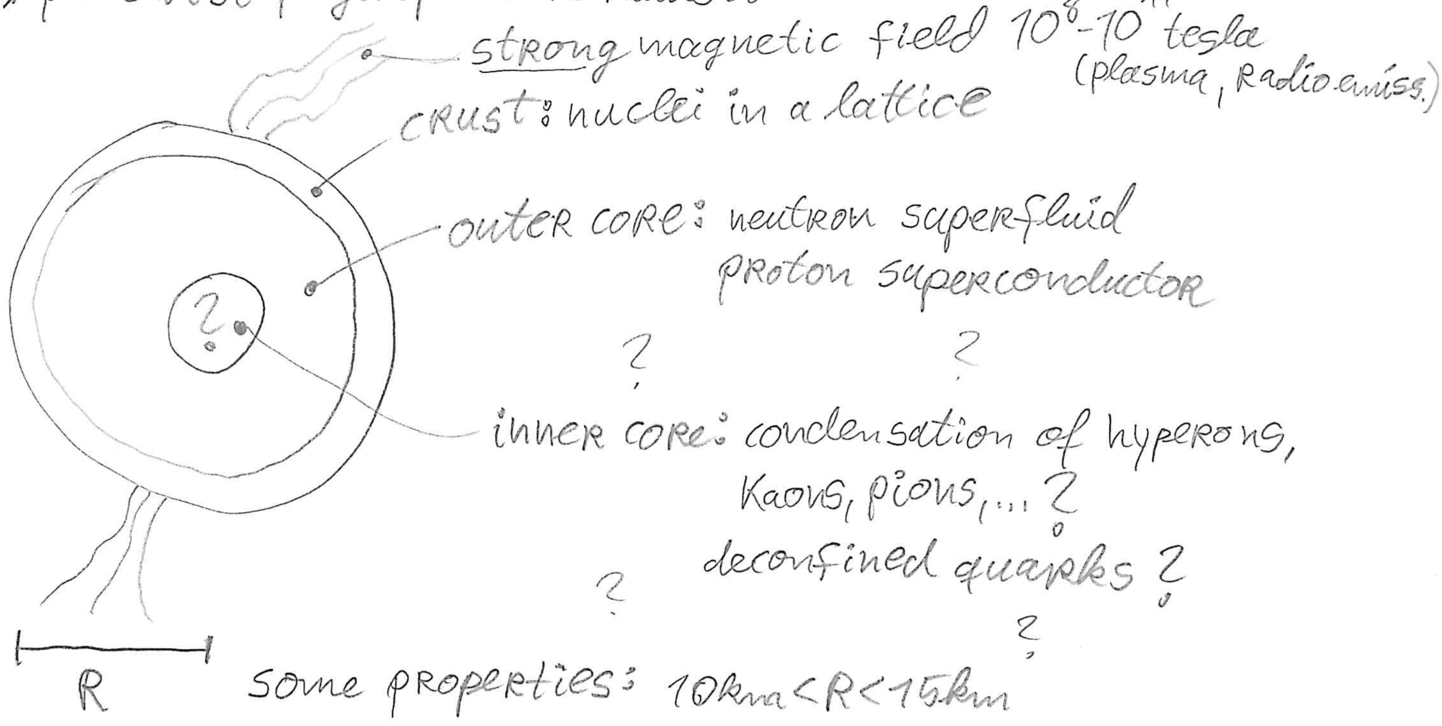
example: neutron star ≈ polytropic EOS $p^+ + e^- \longrightarrow n + \nu_e$

$n \approx 1$, $K \approx 100$, $G = 1 = c$, units: km

exercise: numerically integrate TOV + EOS

neutron stars have complicated structure

↳ piecewise polytrope more realistic



some properties: $10\,\text{km} < R < 15\,\text{km}$

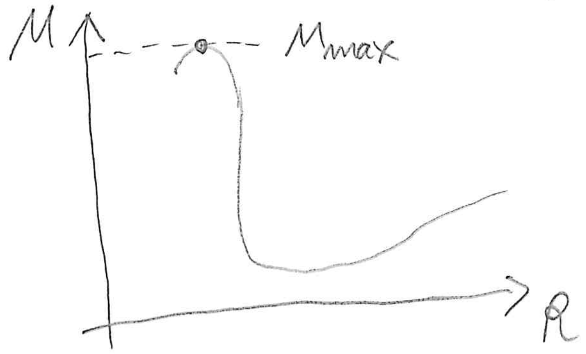
mass $M = m(r=R) \sim 1 \ldots 3 M_\odot$

(sun compressed to size of a city)

need more observations to constrain EOS !

mass-Radius Relation

↳ from numerical integration of TOV + EOS



$\exists$ maximum mass!
mass $> M_{max} \Rightarrow$ collapse!
white dwarf: $M_{max} \sim 1.5 M_\odot$
neutron star: $M_{max} \sim 2 \ldots 3 M_\odot$
black hole: no M_{max}

collapse

analytic solutions

exterior (vacuum) solution: $r > R$

↳ $p = 0 = \rho \rightsquigarrow m(r) = M = \text{const}$

try: $A = 1 - \frac{2GM}{c^2 r} \rightsquigarrow$ (III) fulfilled!

line element $\rightsquigarrow$ Schwarzschild metric:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 - \frac{2GM}{c^2 r}\right) dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

Birkhoff's theorem: exterior spherically symmetric solution = Schwarzschild metric

interior solution: for $\rho = \text{const} \rightsquigarrow$ incompressible

↳ unphysical, since sound speed $\sim \sqrt{\frac{dp}{d\rho}} = \infty$

Solution for $r < R$:

$$m(r) = 4\pi \int_0^r d\bar{r} \cdot \bar{r}^2 \cdot \rho = \frac{4}{3} \pi \rho r^3$$

$$p(r) = \rho c^2 \frac{(1 - 2\mu r^2/R^3)^{1/2} - (1 - 2\mu/R)^{1/2}}{3(1 - 2\mu/R)^{1/2} - (1 - 2\mu r^2/R^3)^{1/2}} \qquad \mu \equiv \frac{GM}{c^2}$$

$$A(r) = \frac{c^2}{4} \left[3\sqrt{1 - \frac{2\mu}{R}} - \sqrt{1 - \frac{2\mu r^2}{R^3}} \right]^2$$

limiting case: $p \to \infty$ at $r = 0$

↳ $3\left(1 - \frac{2\mu}{R}\right)^{1/2} - 1 = 0 \rightsquigarrow \underbrace{\frac{GM}{c^2 R}}_{C:\text{ compactness}} = \frac{4}{9}$

$\rightsquigarrow$ Buchdahl limit: $C < \frac{4}{9}$

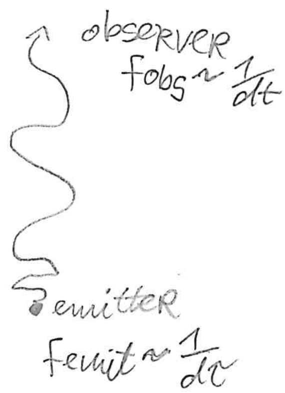
"classical" Tests of General Relativity

- perihelion advance of Mercury
- light deflection by Sun (grav. lensing)
- grav. Redshift

grav. Redshift

clock at $r = \text{const}$ in Schwarzschild metric:

↳ proper time: $d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) dt^2$ ↑ time of observer at $r \to \infty$



↳ redshift z: $z = \frac{f_{emit} - f_{obs}}{f_{obs}} = \underbrace{\frac{f_{emit}}{f_{obs}}}_{\frac{dt}{d\tau}} - 1$

$$\boxed{1 + z = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2}}$$

approx.: $z \approx \frac{GM}{c^2 r} + O(G^2)$

examples:
- Pound-Rebka experiment
- Einstein Tower
- GPS etc
- optical atomic clocks

...

Motion in Schwarzschild metric

insert $g_{\mu\nu}$ into geodesic eq.:

$$0 = \frac{dU^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu} U^\mu U^\nu, \quad U^\mu = \frac{dx^\mu}{d\tau}$$

Re-parametrize $\tau \to \lambda$, so it works for photons

$$\hookrightarrow \left(\frac{dr}{d\lambda}\right)^2 = e^2 - \left(1 - \frac{2GM}{c^2 r}\right) \cdot \left(\frac{\ell^2}{r^2} + \varepsilon\right), \quad \frac{d\phi}{d\lambda} = \frac{\ell}{r^2} \qquad (*)$$

$\varepsilon = \frac{U_\mu U^\mu}{c^2} = \begin{cases} 1 & \text{massive particle} \\ 0 & \text{massless "} \end{cases}$

$\ell = \text{const} \sim$ angular momentum

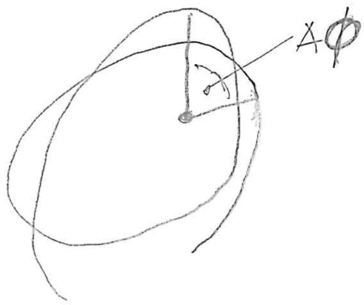
$e = \text{const} \sim$ energy

perihelion advance

solve (*) for $\varepsilon=1$, bound orbit

approximate solution: (weak-field)

0th order: Newtonian ellipse with semi-major axis: a and eccentricity: e



1st correction:

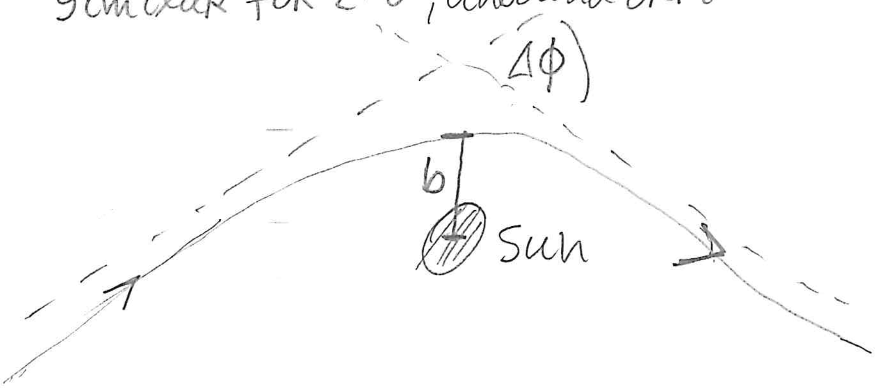
$$\boxed{\Delta\phi \approx \frac{6\pi GM}{c^2 a(1-e^2)}}$$

⇝ ellipse not closed!

for Mercury: $\Delta\phi \approx 43''$ ⇝ fits observations!

light deflection

similar for $\varepsilon=0$, unbound orbit



$$\boxed{\Delta\phi \approx \frac{4GM}{c^2 b}}$$