2024-03-05 lecture

Slow motion approximation - inspiral of compact binaries

$|x_i| < \varepsilon$ (localisation) $\quad T^{part}_{\mu\nu} \neq 0$

$\lambda_{GW} = \frac{2\pi c}{\omega}$ $\qquad \omega \varepsilon \lll c$

Assumption: $\lambda_{GW} \gg \varepsilon \qquad \frac{2\pi c}{\omega} \gg \varepsilon \qquad v_{typical} \ll c$

approximation:

$$\Box_F \bar{h}_{\mu\nu} = \frac{16\pi G}{c^4} T^{part}_{\mu\nu} \qquad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

$$\partial_\mu \bar{h}^\mu_\nu = 0 \qquad \Box_F = -\frac{\partial^2}{c^2\partial t^2} + \nabla^2$$

$$\bar{h}_{\mu\nu}(t, x^j) = \int_{-\infty}^{\infty} \bar{h}_{\mu\nu}(\omega, x^j) e^{-i\omega t} d\omega \qquad j = 1,2,3$$

$$T_{\mu\nu}(t, x^j) = \int_{-\infty}^{\infty} T_{\mu\nu}(\omega, x^j) e^{-i\omega t} d\omega$$

$$\left(\nabla^2 - \frac{1}{c^2}(-i\omega)^2\right)\bar{h}_{\mu\nu}(\omega, x^j) = -\frac{16\pi G}{c^4} T_{\mu\nu}(\omega, x^j)$$

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\bar{h}_{\mu\nu}(\omega, x^j) = -\frac{16\pi G}{c^4} T_{\mu\nu}(\omega, x^j)$$

Exterior part

$$T_{\mu\nu} = 0 \qquad \left(\nabla^2 + \frac{\omega^2}{c^2}\right)\bar{h}_{\mu\nu}(\omega, x^j) = 0$$

move to polar coordinates $(x, y, z) \to (r, \vartheta, \varphi)$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta \frac{\partial}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}$$

wave equation as solution assumption

$$\bar{h}_{\mu\nu}(\omega, r) = \underbrace{\frac{A_{\mu\nu}(\omega)}{r} e^{i\frac{\omega}{c}r}}_{\text{outgoing solution}} + \frac{Z_{\mu\nu}(\omega)}{r} e^{-i\frac{\omega}{c}r}$$

only outgoing solution so $\frac{Z_{\mu\nu}(\omega)}{r} e^{-i\frac{\omega}{c}r} \to 0$

Interior solution

$$\kappa = \frac{16\pi G}{c^4}$$

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\bar{h}_{\mu\nu}(\omega, x^j) = -\kappa T_{\mu\nu}(\omega, x^j)$$

(0)

$$\int_V \left(\underset{(1)}{\nabla^2} + \underset{(2)}{\frac{\omega^2}{c^2}}\right)\bar{h}_{\mu\nu}(\omega, x^j)\, d^3x = \int_V -\kappa T_{\mu\nu}(\omega, x^j)\, d^3x$$

$$\nabla^2 = \nabla \cdot \nabla$$

(1):

$$\int_V \nabla^2 \bar{h}_{\mu\nu}(\omega, x^j)\, d^3x = \int_V \nabla \cdot \left(\nabla \bar{h}_{\mu\nu}(\omega, x^j)\right) d^3x$$

Gauss's theorem

approximation on a sphere

$$= \int_A (\nabla h_{\mu\nu})^k\, dS_k \simeq 4\pi\varepsilon^2 \frac{\partial}{\partial r}\left(\frac{A_{\mu\nu}(\omega)}{r} e^{i\frac{\omega}{c}r}\right)\Bigg|_{r=\varepsilon}$$

on $\int T_{\mu\nu} = 0$

$$= 4\pi\varepsilon^2\left[-\frac{A_{\mu\nu}}{r^2} e^{+i\frac{\omega}{c}r} + i\frac{\omega}{c}\frac{A_{\mu\nu}}{r} e^{i\frac{\omega}{c}r}\right]\Bigg|_{r=\varepsilon}$$

$$\simeq 4\pi\varepsilon^2\left[-\frac{A_{\mu\nu}}{\varepsilon^2} e^{i\frac{\omega}{c}\varepsilon} + i\frac{\omega}{c}\frac{A_{\mu\nu}}{\varepsilon} e^{i\frac{\omega}{c}\varepsilon}\right]$$

(1) Taylor expansion

$$\simeq -4\pi A_{\mu\nu}(\omega) + O(\varepsilon)$$

(2):

$$\int_V \frac{\omega^2}{c^2}\bar{h}_{\mu\nu}(\omega, x^j)\, d^3x = \frac{\omega^2}{c^2}\int_V \bar{h}_{\mu\nu}(\omega, x^j)\, d^3x \lesssim \frac{\omega^2}{c^2}\left|\bar{h}_{\mu\nu}\right|_{\max\, \mathrm{in}\, V} \frac{4}{3}\pi\varepsilon^3$$

Because higher term of ε, we have ε^3, we neglect this contribution

$$(2) \to 0$$

(1) and (2) in (0)

$$-4\pi A_{\mu\nu}(\omega) = -\kappa \int_V T^{\mathrm{pert}}_{\mu\nu}(\omega, x^j)\, d^3x$$

$$A_{\mu\nu}(\omega) = \frac{4G}{c^4}\int_V T^{\mathrm{pert}}_{\mu\nu}(\omega, x^j)\, d^3x$$

$$\bar{h}_{\mu\nu}(\omega, r) = \frac{A_{\mu\nu}}{r} e^{i\frac{\omega}{c}r} = \frac{4G}{c^4}\frac{e^{i\omega r}}{r}\int_V T^{\mathrm{pert}}_{\mu\nu}(\omega, x^j)\, d^3x$$

Inverse Fourier trafo

$$\bar{h}_{\mu\nu}(t, r) = \frac{\bar{A}_{\mu\nu}}{r} e^{i\frac{\omega}{c} r} = \frac{4G}{c^4} \frac{1}{r} \underbrace{\int_V T^{\mu\nu}_{\mu\nu}\left(t - \frac{r}{c}, x^i\right) d^3x}$$

we need to take assumptions and laws, what restrict our solution for $T^{\mu\nu}_{\mu\nu}$, which are allowed

conservation law: $\partial_\nu T^{\mu\nu} = 0$

$$\frac{1}{c}\frac{\partial}{\partial t} T^{\mu 0} = -\frac{\partial}{\partial x^k} T^{\mu k}$$

Gaussian law or $T_{\mu\nu} = 0$

$$\frac{1}{c}\frac{\partial}{\partial t} \int_V T^{\mu 0} d^3x = -\int_V \frac{\partial}{\partial x^k} T^{\mu k} d^3x = -\int_S T^{\mu k} dS_k = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t} \int_V T^{\mu 0} = 0$$

$$\frac{1}{c}\int_V T^{\mu 0} = \text{const} \rightarrow \bar{h}^{\mu 0} = \text{const} = 0$$

Tensor-virial theorem

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int_V \underbrace{T^{00} x^k x^m d^3x}_{\text{this is a quadropole}} = 2\int_V T^{nk} d^3k$$

like dipole in electrostatic

$\vec{d}_{el} = \sum_i q_i \vec{r}_i$ dipole so in case of quadropole you have more terms of $\vec{r}_i$

in gravitation $\vec{d}_G = \sum_i m_i \vec{r}_i$ $\quad \frac{d}{dt}\vec{d}_G = 0$ – discrete space

$$q^{km}(t) = \frac{1}{c^2}\int_V T^{00} x^k x^m d^3x$$ quadropole moment tensor – continuous space

using tensor-virial theorem and last equation of quadropole

$$\int T^{nk} d^3x = \frac{1}{2} \frac{d^2}{dt^2} q^{km}(t)$$

amplitude

$$\Rightarrow \bar{h}^{ik} = \frac{2G}{rc^4} \frac{d^2}{dt^2} q^{ik}\left(t - \frac{r}{c}\right)$$

it $\bar{h}$ like a potential, so we have $\frac{1}{r}$ potential for wave

for gravitional waves we need a kind of asymmtry, symmetric events or objects does not produce gravitational waves.

Why is it? Is it because of the theory, it has natural symmetries, or is it the nature itself, that it is symmetric the space time.

amplitude

$$\frac{G}{c^4} = 10^{-44} \frac{s^2}{km}$$ very small amplitude

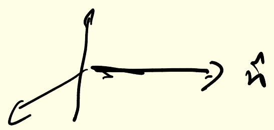
TT gauge

$$P_{jk} = \delta_{jk} - n_j n_k$$ 3d-euclidian

projector

$$n^\alpha = (0, n^i) \qquad n^i = \frac{x^i}{r}$$ propagation vector

$$(P_{jk} v^k) n^j = 0 \qquad P_{jk} v^k \perp n_j$$



Newton gravtitional waves why not there, do we doing that

transverse $n^j P_{jk} = 0$

traceless

$$P_{jkmn} = P_{jm} P_{kn} - \frac{1}{2} P_{jk} P_{mn}$$

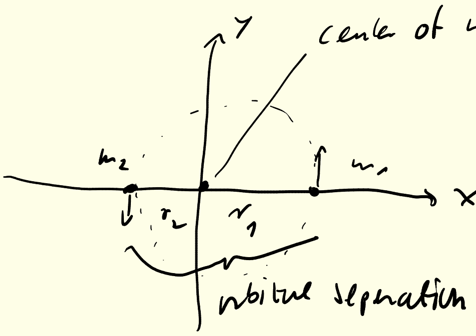
$$\delta^{jk} P_{jkmn} = \delta^{mn} P_{jkmn} = 0$$

$$h^{TT}_{ij} = P_{ijmn} \bar{h}^{mn} = P_{ijmn} h^{mn}$$

$$h^{TT}_{\mu 0} = 0$$

$$h^{TT}_{jk}(t, r) = \frac{2G}{c^4 r} \frac{d^2}{dt^2} \overbrace{Q^{TT}_{jk}\left(t - \frac{r}{c}\right)}^{P_{jkmn} q^{mn}}$$

Quadropole tensor



$m_1 = (x_1, y_1)$

$m_2 = (x_2, y_2)$

$M = m_1 + m_2$

reduced mass $\mu = \frac{m_1 m_2}{M}$

$$X_{cm} = \frac{r_1 m_1 - r_2 m_2}{M} = 0 \qquad r_1 = \frac{r_2 m_2}{m_1} \qquad r_1 = \frac{m_2}{M} l_0$$

$$l_0 = r_1 + r_2 = r_2\left(1 + \frac{m_2}{m_1}\right) = \frac{r_2 M}{m} \qquad r_2 = \frac{m_1}{M} l_0$$

keplers law

$$\frac{G m_1 m_2}{l_0^2} = m_1 \omega_k^2 r_1^2 = \frac{m_1 m_2}{M} l_0 \omega_k^2$$

$$\frac{G m_1 m_2}{l_0^2} = \frac{m_1 m_2}{M} l_0 \omega_k^2$$

$$\omega_k^2 = \frac{GM}{l_0^3} \quad \text{keplerian frequency}$$

$$x_1 = r_1 \cos(\omega_k t) = \frac{m_2}{M} l_0 \cos(\omega_k t) \qquad x_2 = -r_2 \cos(\omega_k t) = -\frac{m_1 l_0}{M} \cos(\omega_k t)$$

$$y_1 = r_1 \sin(\omega_k t) = \frac{m_2}{M} l_0 \sin(\omega_k t) \qquad y_2 = -r_2 \sin(\omega_k t) = -\frac{m_1 l_0}{M} \sin(\omega_k t)$$

$$T^{00} = c^2 \sum_{n=1}^{2} m_n \delta(x - x_n) \delta(y - y_n) \delta z$$

$$q^{xx} = m_1 \int_V \delta(x - x_1) \delta(y - y_1) \delta(z) x^2 \, dx\, dy\, dz + m_2 \int_V \delta(x - x_2) \delta(y - y_2) \delta z \, x^2 \, dx\, dy\, dz$$

$$q^{xx} = m_1 x_1^2 + m_2 x_2^2 = m_1 \frac{m_2^2}{M^2} l_0^2 \cos^2(\omega_k t) + m_2 \frac{m_1^2}{M} l_0^2 \cos^2(\omega_k t)$$

use reduced mass

$$= \mu l_0^2 \cos^2(\omega_k t) = \frac{\mu l_0^2}{2} \cos(2\omega_k t) + \text{const}$$

$$\cos(2\alpha) = 2\cos^2\alpha - 1$$

the same steps as before

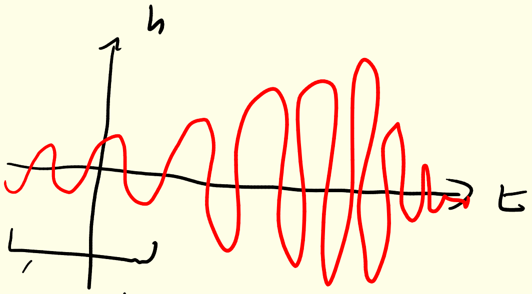
$$q^{yy} = m_1 y_1^2 + m_2 y_2^2 = \dots = \mu l_0^2 \sin^2(\omega_k t) = -\frac{\mu l_0^2}{2} \cos(2\omega_k t) + \text{const}$$

$$\sin^2\alpha = \frac{1}{2} - \frac{1}{2}\cos(2\alpha)$$

$$q^{yy} = -q^{xx}$$

$$q^{xy} = \mu l_0^2 \cos(\omega_K t)\sin(\omega_K t) = \frac{\mu l_0^2}{2}\sin(2\omega_K t)$$

quadrupole moment frequency is the double of the keplerian frequency



quadrupole frequency

could be ~ms

~s

$$\bar{h}^{TT}_{ij}(t,r) = \frac{2G}{c^4 r}\frac{d^2}{dt^2} q_{ij}(t-\frac{r}{c})$$

$$\partial_t q^{xx} = -\frac{\mu}{2} l_0^2 (2\omega_K)\sin(2\omega_K t)$$

$$\partial_t^2 q^{xx} = -2\mu l_0^2 \omega_K^2 \cos(2\omega_K t) = -2\mu \cancel{l_0} \frac{GM}{l_0^{\cancel{2}}}\cos(2\omega_K t)$$

$$= -\frac{2\mu M G}{l_0}\cos(2\omega_K t)$$

$$\partial_t^2 q^{yy} = -\partial_t^2 q^{xx}$$

$$\partial_t^2 q^{xy} = -\frac{2\mu M G}{l_0}\sin(2\omega_K t)$$

$$\bar{h}^{TT}_{ij} = -\underbrace{\frac{2G}{c^4 r}\frac{2\mu G}{l_0}}_{h_0\text{-amplitude}} \Lambda_{ijkl} q^{kl}(t-\frac{r}{c})$$

$$q_{ij} = \begin{array}{c|cc} & x & y \\ \hline x & \cos(2\omega_K t) & \sin(2\omega_K t) \\ y & \sin(2\omega_K t) & -\cos(2\omega_K t) \end{array}$$

$$h_0 = \frac{4G^2 \mu M}{l_0 c^4 r}$$ → one can measure the l_0, M radii and mass of the objects

example PSR 1913+16

$m_1 \sim m_2 \sim 1.4 M_\odot$

$r = 5\,\mathrm{kpc}$

$h_0 \simeq 10^{-23}$ for

binary neutron stars and black holes

frequency range of 10–1000 Hz

LISA instrument range $10^{-4} - 10^{-1}$ Hz   supermassive - black hole
binaries