lecture 2024-02-26

Einstein telescope in 2030 around

Review of special relativity 1905

Minkowski

$$[\eta_{\mu\nu}] = \mathrm{diag}(1,-1,-1,-1)$$

Notation: Einstein sum convention

$$\sum_{\mu=0}^{3} x_\mu x^\mu = x_\mu x^\mu = x_\alpha x^\alpha \neq x_\alpha x_\alpha$$

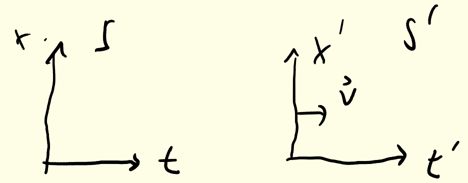
scalar product in Euclidean space

Euclidean $\vec{x}\cdot\vec{x} = (x_i \hat{e}^i)\cdot(x^j \hat{e}_j) = x_i x^j \hat{e}^i\cdot\hat{e}_j = x_i x^j \delta^i_j = x_i x^i = x_j x^j$

inertial frames:
same form where every law have the same form

$$\frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} + \frac{d^2z}{dt^2} = 0 \qquad x \to x' \text{ if inertial systems}$$

Systems S, S'



general ansatz

$$t' = At + Bx \qquad y' = y$$
$$x' = Dt + Ex \qquad z' = z$$

Galilean tansformations

$$t' = t \qquad A = 1 \quad B = 0$$
$$x' = x - vt \qquad E = 1 \quad D = -v$$

$$u'_x = \frac{dx'}{dt'} = \frac{d}{dx}(x - vt) = \frac{dx}{dt} - v = u_x - v$$
$$a'_x = \frac{du'_x}{dt'} = \frac{d}{dt}(u_x - v) = \frac{du_x}{dt} = a_x$$

Events

$$A \mathrel{\hat{=}} (t_A, x_A, y_A, z_A) \qquad \Delta t = t_B - t_A$$
$$B \mathrel{\hat{=}} (t_B, x_B, y_B, z_B) \qquad \Delta t' = t'_B - t'_A = t_B - t_A = \Delta t$$

$$\Delta r'^2 = (x'_B - x'_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2$$
$$\Delta x'^2 = (x'_B - x'_A)^2 = (x_B - vt - x_A + vt)^2 = (x_B - x_A)^2 = \Delta x^2$$

Lorentz-transformation

$$ct' = \gamma(ct - \beta x) \qquad y' = y \qquad \beta = \frac{v}{c}$$

$$x' = \gamma(x - \beta c t) \qquad z' = z \qquad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}}$$

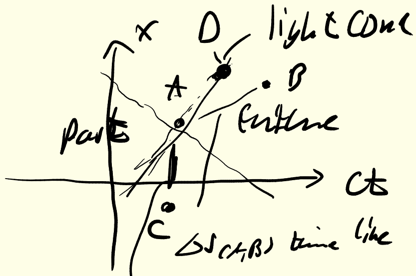
$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2 = 0$$

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = \Delta s'^2$$

$\Delta s^2 > 0$ timelike interval

$\Delta s^2 = 0$ lightlike or null -"-

$\Delta s^2 < 0$ space like -"-

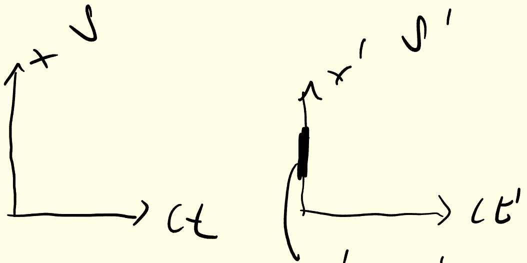


$\Delta s(A,B)$ time like

$\Delta s(A,C)$ space like

$\Delta s(A,D)$ light like

Length contraction



$$L' = x_B' - x_A'$$

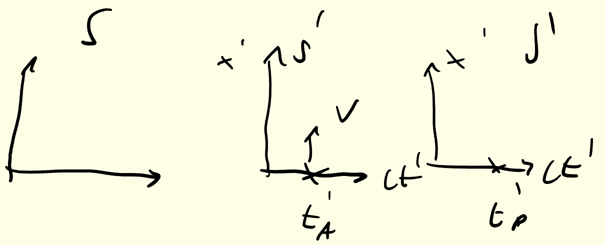
$$L' = \gamma(x_B - vt_B) - \gamma(x_A - vt_A)$$

$$t_A = t_B \qquad = \gamma(x_B - x_A) = \gamma L$$

$$L = \frac{L'}{\gamma}$$

Time dilation

T_0 - clock tick



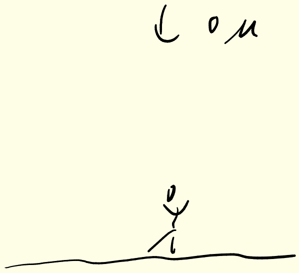
$$t_A = \gamma\left(t_A' + v\frac{x_A'}{c^2}\right)$$

$$t_B = \gamma\left(t_A' + T_0' + v\frac{x_B'}{c^2}\right)$$

$$t_B' = t_A' + T_0'$$

$$x_A' = x_B' \Rightarrow \qquad T = t_B - t_A = \gamma T_0'$$

Muon experiment: as example



decaying time is with higher velocities smaller

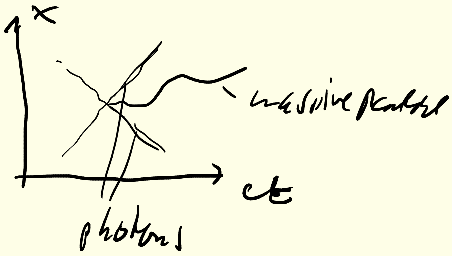
Minkowski line element

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$ Cartesian coordinates

$$ds'^2 = ds^2$$

$$\Delta s = \int_A^B ds$$

$ds^2 > 0$ timelike
$ds^2 = 0$ null
$ds^2 < 0$ spacelike



worldline described by $t(\lambda), x(\lambda), y(\lambda), z(\lambda)$ a parameter

$$c^2 d\tau^2 = ds^2 - dx^2 - dy^2 - dz^2$$

$$d\tau = \frac{dt}{\gamma}$$

proper time

$$\Delta\tau = \int_A^B d\tau = \int_A^B \left(1 - \frac{v(t)^2}{c^2}\right)^{1/2} dt$$

concept of four vectors

these objects are transform with lorentz trafos

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu$$

$$[\Lambda^\mu{}_\nu] = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ Lorentz trafo for x-trafos

$$= \left[\frac{\partial x'^\mu}{\partial x^\nu}\right]$$

$$\Lambda_\mu{}^\nu = \eta_{\mu\rho}\, \eta^{\nu\sigma}\, \Lambda^\rho{}_\sigma$$

four velocity

$$[u^\mu] = \gamma\left(c, \frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt}\right) = \gamma(c, \vec{u})$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \qquad x^0 = ct$$

four momentum

$$E = \gamma m_0 c^2$$

$$[p^\mu] = \underset{\text{Rest mass}}{m_0}[u^\mu] = \left(\frac{E}{c}, \vec{p}\right)$$

$$[\eta_{\mu\nu}] = \mathrm{diag}(1, -1, -1, -1)$$

$$p_\mu p^\mu = \eta_{\mu\nu} p^\mu p^\nu = \frac{E^2}{c^2} - \vec{p}\cdot\vec{p} = m_0^2 c^2$$

comoving frame

scalars are all the same in all reference systems, not changing

four force

$$f^\mu = \frac{dp^\mu}{d\tau} \qquad [f^\mu] = \gamma \frac{d}{dt}\left(\frac{E}{c}, \vec{p}\right)$$

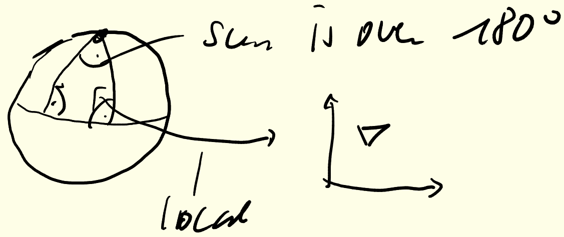
free particle, no forces acting $\frac{dp^\mu}{d\tau} = 0$

Introduction to tensors:

$\mathcal{M}$

Manifold: N-dimensional manifold is some space be locally mapped to the N-dimensional Euclidean space.
Points on the manifold $\mathcal{M}$ is described by N-points in Euclidean space.
There is a one-to-one correspondence between coordinates and points

example a sphere



$$x^\mu = x^\mu(\bar{x}^0, \bar{x}^1, \bar{x}^2, \dots, \bar{x}^n) \to \bar{x}^\mu = \bar{x}^\mu(x^0, x^1, \dots, x^n)$$

$$A^\mu_\nu = \frac{\partial x^\mu}{\partial \bar{x}^\nu} \text{ and } B^\nu_\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu}$$

exists and are invertible

scalars, vector, tensors in manifold

how they transforms

example $|\vec{v}|$

transformation to comoving system then $|\vec{v}'| = 0$

scalar

$\phi(x^\mu) = \phi(\bar{x}^\mu)$ object is not changed

here the scalar product is not changed under the transformations in the manifold

vectors

$$\bar{a}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu} a^\nu \quad \text{or} \quad \bar{b}_\mu = \frac{\partial x^\nu}{\partial \bar{x}^\mu} b_\nu$$

contravariant vector

covariant vector

compare with Lorentz trafo

$$\bar{x}^\mu = \Lambda^\mu{}_\nu x^\nu \qquad [\Lambda^\mu{}_\nu] = \frac{\partial \bar{x}^\mu}{\partial x^\nu}$$

linear transformation

bold four vector

$$\mathbf{a} = (a_\mu e^\mu) = (a^\mu e_\mu)$$

$$\mathbf{a}\cdot\mathbf{b} = (a_\mu e^\mu)\cdot(b^\nu e_\nu) = a_\mu b^\nu e^\mu e_\nu$$

Scalar products

orthogonal

$$a_\mu = \mathbf{a}\cdot e_\mu = (a_\nu \underbrace{e^\nu)\cdot e_\mu}_{\delta^\nu_\mu} = a_\mu \qquad a^\mu = \mathbf{a}\cdot e^\mu = (a^\nu \underbrace{e_\nu) e^\mu}_{\delta^\mu_\nu} = a^\mu$$

$$\mathbf{a}\cdot\mathbf{b} = (a^\mu e_\mu)\cdot(b^\nu e_\nu) = e_\mu e_\nu a^\mu b^\nu$$
$$= g_{\mu\nu} a^\mu b^\nu$$

raise an index

$$a^\mu g_{\mu\nu} = a^\mu e_\mu e^\nu = \mathbf{a}\, e_\nu = a_\nu \qquad g^{\mu\nu} = e^\mu e^\nu$$
$$g^\mu_{\ \nu} = e^\mu e_\nu$$

line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

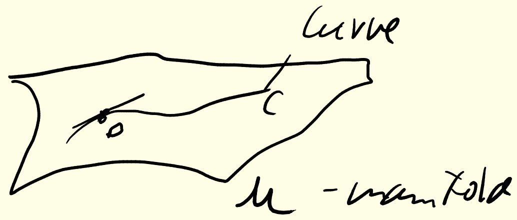
$$ds^2 = \bar{g}_{\alpha\beta} d\bar{x}^\alpha d\bar{x}^\beta \overset{!}{=} g_{\mu\nu} dx^\mu dx^\nu$$

$$= \bar{g}_{\alpha\beta} \frac{\partial \bar{x}^\alpha}{\partial x^\nu} dx^\nu \frac{d\bar{x}^\beta}{dx^\mu} dx^\mu$$

$$= \underbrace{\frac{\partial \bar{x}^\alpha}{\partial x^\nu} \frac{\partial \bar{x}^\beta}{\partial x^\mu}}\bar{g}_{\alpha\beta}\, dx^\nu dx^\mu$$

$$g_{\mu\nu} = \frac{\partial \bar{x}^\alpha}{\partial x^\nu} \frac{\partial \bar{x}^\beta}{\partial^\mu} g_{\alpha\beta}$$

tangential vector



$$x^\mu(\lambda)$$

$$t = \lim_{\delta\lambda \to 0} \frac{\delta s}{d\lambda}, \qquad \delta s = s(\lambda + \delta\lambda) - s(\lambda)$$

tangential vectors

in coordinates

$$t^\mu = \lim_{\delta\lambda \to 0} \frac{x^\mu(\lambda + \delta\lambda) - x^\mu(\lambda)}{\delta\lambda} = \frac{dx^\mu}{d\lambda}$$

coordinate change

instead of $x^\mu(\lambda)$ look at $\bar{x}^\mu(\lambda)$

$$\bar{x}^\mu(\lambda) = f^\mu(x^\mu(\lambda))$$

tangential vector

$$d\bar{x}^\mu = \frac{\partial f^\mu}{\partial x^\nu} dx^\nu \qquad \rightarrow \bar{t}^\mu = \frac{d\bar{x}^\mu}{d\lambda} = \frac{\partial \bar{x}^\mu}{\partial x^\nu} \frac{dx^\nu}{d\lambda} = \frac{\partial \bar{x}^\mu}{\partial x^\nu} t^\nu$$

tangential vectorcomponents transforms like a tensor
and they are contravariant

$$\frac{\partial}{\partial x^\mu} \phi(x^\mu) = \frac{\partial}{\partial x^\mu} \phi(\bar{x}^\mu) = \frac{\partial \bar{x}^\nu}{\partial x^\mu} \frac{\partial \phi(\bar{x}^\mu)}{\partial \bar{x}^\nu}$$

transforms like a covariant vector

scalar product transformation

$$\bar{a}^\mu \cdot \bar{b}_\mu = \left(\frac{\partial \bar{x}^\mu}{\partial x^\alpha} a^\alpha\right) \cdot \left(\frac{\partial x^\beta}{\partial \bar{x}^\mu} b_\beta\right)$$

$$= \frac{\partial \cancel{\bar{x}^\mu}}{\partial x^\alpha} \frac{\partial x^\beta}{\partial \cancel{\bar{x}^\mu}} a^\alpha b_\beta = \frac{\partial x^\beta}{\partial x^\alpha} a^\alpha b_\beta = \delta^\beta_\alpha a^\alpha b_\beta$$

$$= a^\alpha b_\alpha$$

Tensors

$$\bar{T}^{\alpha\beta} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{\mu}} \frac{\partial \bar{x}^{\beta}}{\partial x^{\nu}} T^{\mu\nu}$$ contravariant tensor of rank 2

$$\bar{T}_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial \bar{x}^{\alpha}} \frac{\partial x^{\nu}}{\partial \bar{x}^{\beta}} T_{\mu\nu}$$ covariant tensor of rank 2

$$\bar{T}^{\alpha}{}_{\beta} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial \bar{x}^{\beta}} T^{\mu}{}_{\nu}$$ mixed tensor of rank 2

$$\bar{T}^{\overbrace{\alpha \cdots \gamma}^{p}}{}_{\underbrace{\sigma \cdots \lambda}_{q}} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{\beta}} \cdots \frac{\partial x^{\zeta}}{\partial \bar{x}^{\lambda}} T^{\beta \cdots \delta}{}_{\rho \cdots \zeta}$$ rank (p, q)

Tensor algebra

addition:

$$\bar{a}^{\nu} + \bar{b}^{\nu} = \frac{\partial \bar{x}^{\nu}}{\partial x^{\alpha}} a^{\alpha} + \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} b^{\beta} = \frac{\partial \bar{x}^{\nu}}{\partial x^{\alpha}} (a^{\alpha} + b^{\alpha})$$

rank is the same

multiplication:

$$\bar{a}^{\alpha} \bar{b}^{\beta} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{\mu}} \frac{\partial \bar{x}^{\beta}}{\partial x^{\nu}} a^{\mu} b^{\nu}$$ changes the rank

contraction: $(p, q) \to (p-1, q-1)$

$$T^{\lambda\mu\nu}{}_{\lambda\alpha} = T^{\mu\nu}{}_{\alpha}$$

Trace: $T = T^{\alpha}{}_{\alpha}$

Symmetric $T_{\alpha\beta} = T_{\beta\alpha} \Leftrightarrow T_{(\alpha\beta)}$

$$T_{\alpha\beta\gamma} = T_{\alpha\gamma\beta} \Leftrightarrow T_{\alpha(\beta\gamma)}$$

Antisymmetric

$$T_{\alpha\beta} = -T_{\beta\alpha} \Leftrightarrow T_{[\alpha\beta]} \qquad T_{\alpha\beta\gamma} = -T_{\alpha\gamma\beta} \Leftrightarrow T_{\alpha[\beta\gamma]}$$

Number of independent components:

Symmetric : $\frac{n(n+1)}{2}$ | rank 2

antisymmetric : $\frac{n(n-1)}{2}$ | rank 2

Differentiation of tensors:

Are derivatives of tensors still tensors?

For scalars yes: $\bar{\partial}_\mu \phi = \frac{\partial x^\nu}{\partial \bar{x}^\mu} \partial_\nu \phi$

For higher tensors, not, so we need a new derivative: covariant derivative

derivative of a contravariant vector

$$\frac{\partial a^\mu}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha}\left(\frac{\partial x^\mu}{\partial \bar{x}^\nu}\bar{a}^\nu\right) = \frac{\partial \bar{x}^\rho}{\partial x^\alpha}\frac{\partial}{\partial \bar{x}^\rho}\left(\frac{\partial x^\mu}{\partial \bar{x}^\nu}\bar{a}^\nu\right)$$

$$= \underbrace{\frac{\partial \bar{x}^\rho}{\partial x^\alpha}\frac{\partial^2 x^\mu}{\partial \bar{x}^\rho \partial \bar{x}^\nu}\bar{a}^\nu}_{} + \frac{\partial \bar{x}^\rho}{\partial x^\alpha}\frac{\partial x^\mu}{\partial \bar{x}^\nu}\frac{\partial \bar{a}^\nu}{\partial \bar{x}^\rho}$$

$$\frac{\partial a^\mu}{\partial x^\alpha} - \frac{\partial^2 x^\mu}{\partial \bar{x}^\rho \partial \bar{x}^\nu}\frac{\partial \bar{x}^\rho}{\partial x^\alpha}\bar{a}^\nu = \frac{\partial a^\mu}{\partial x^\alpha} \underbrace{- \frac{\partial^2 x^\mu}{\partial \bar{x}^\nu \partial \bar{x}^\rho}\frac{\partial \bar{x}^\rho}{\partial x^\alpha}\frac{\partial \bar{x}^\nu}{\partial x^\kappa}}_{(-\Gamma^\mu{}_{\alpha\kappa})} a^\kappa = \frac{\partial \bar{x}^\rho}{\partial x^\alpha}\frac{\partial x^\mu}{\partial \bar{x}^\nu}\frac{\partial \bar{a}^\nu}{\partial \bar{x}^\rho}$$

$$\frac{\partial a^\mu}{\partial x^\alpha} + \Gamma^\mu{}_{\alpha\kappa} a^\kappa = \frac{\partial \bar{x}^\rho}{\partial x^\alpha}\frac{\partial x^\mu}{\partial \bar{x}^\nu}\frac{\partial \bar{a}^\nu}{\partial \bar{x}^\rho}$$

$$x'^\mu \to \bar{x}^\mu$$

$$\frac{\partial a'^\lambda}{\partial x^\sigma} - \frac{\partial^2 x'^\lambda}{\partial \bar{x}'^\nu \partial \bar{x}^\rho}\frac{\partial \bar{x}^\rho}{\partial x'^\sigma}\bar{a}^\nu = \frac{\partial a'^\lambda}{\partial x^\sigma} + \Gamma'^\lambda{}_{\sigma\rho} a'^\rho = \frac{\partial x'^\lambda}{\partial \bar{x}^\nu}\frac{\partial \bar{x}^\rho}{\partial x'^\sigma}\frac{\partial \bar{a}^\nu}{\partial \bar{x}^\rho}$$

$$\underbrace{\frac{\partial a^\mu}{\partial x^\alpha} + \Gamma^\mu_{\alpha\kappa} a^\kappa} = \frac{\partial x^\mu}{\partial x'^\lambda}\frac{\partial x'^\sigma}{\partial x^\alpha}\underbrace{\left(\frac{\partial a'^\lambda}{\partial x'^\sigma} + \Gamma'^\lambda_{\sigma\beta} a'^\beta\right)}$$

$$\nabla_\alpha a^\mu = \partial_\alpha a^\mu + \Gamma^\mu_{\alpha\kappa} a^\kappa$$

$$a^\mu_{;\alpha} = a^\mu_{,\alpha} + \Gamma^\mu_{\alpha\kappa} a^\kappa$$

$$\nabla_\nu T^{\lambda\mu} = \partial_\nu T^{\lambda\mu} + \Gamma^\lambda_{\alpha\nu} T^{\alpha\mu} + \Gamma^\mu_{\alpha\nu} T^{\lambda\alpha}$$

$$\nabla_\nu T_{\lambda\mu} = \partial_\nu T_{\lambda\mu} - \Gamma^\alpha_{\lambda\nu} T_{\mu\alpha} - \Gamma^\alpha_{\mu\nu} T_{\lambda\alpha}$$

Parallel transport

transport a vector in a manifold

a_μ covariant vector field

P': $a_\mu(P')$, $A_\mu(P') = a_\mu(P') + \delta a_\mu$

P: $a_\mu(P)$

P', P are close

(1) $a_\mu(P') = a_\mu(P) + a_{\mu,\nu}(P)\, dx^\nu$

(2) $A_\mu(P') = a_\mu(P) + \delta a_\mu(P)$

ansatz $\delta a_\mu = C^\lambda_{\mu\nu} a_\lambda dx^\nu$

$$a_\mu(P') - A_\mu(P') = \cancel{a_\mu(P)} + \underbrace{a_{\mu,\nu}(P)\, dx^\nu}_{\Delta a_\mu} - \cancel{a_\mu(P)} - \delta a_\mu(P)$$

$$= \Delta a_\mu - \delta a_\mu$$

$$= (a_{\mu,\nu} - \underbrace{C^\lambda_{\mu\nu}}_{\Gamma^\lambda_{\mu\nu}} a_\lambda)\, dx^\nu$$

$\Gamma^\lambda_{\alpha\beta} - \Gamma^\lambda_{\beta\alpha} = 0$

in GR it is symmetric in α, β

$$\delta a_\mu = \Gamma^\lambda_{\mu\nu} a_\lambda dx^\nu$$

$$\delta a^\mu = -\Gamma^\mu_{\lambda\nu} a^\lambda dx^\nu$$