Classical Gravity from QFT

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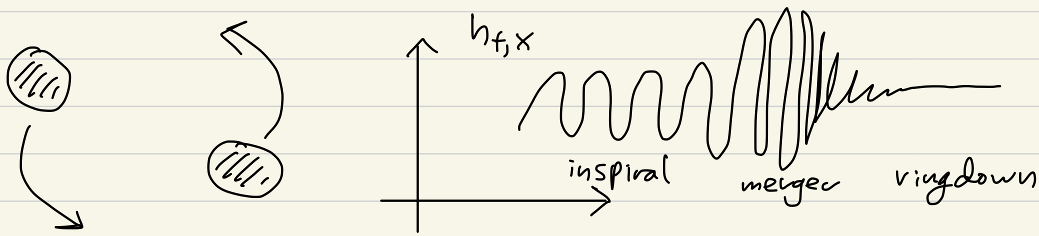
Recent reviews – Rafael Porto 1601.04914
– Michèle Levi 1807.01699

What does QFT have to do with classical physics? Actually, quite a lot!! But first, let's review the context:

① Introduction – the Binary Inspiral



$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad \Big\} \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

We can't solve Einstein's equations exactly, so approximate! There are two options:

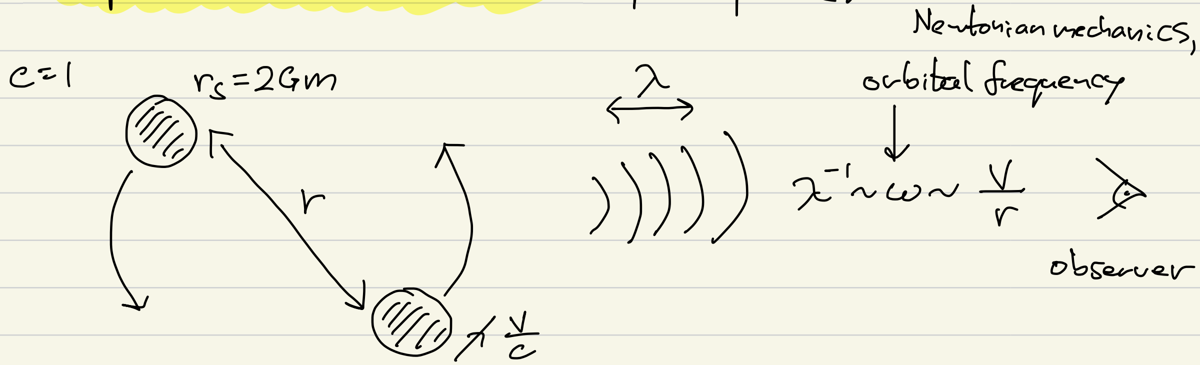
1) Numerical Relativity (NR) – good for the merger
2) Perturbation Theory – good for the inspiral

Inspiral is the longest stage, with LISA we'll see more of it!

QFT provides a convenient framework for applying perturbation theory

More specifically, we will use an EFT framework:

② **Separation of Scales** (inspiral phase)



The problem contains three length scales, organized as:

$$r_s \ll r \ll \lambda$$

Internal Zone : finite-size effects,
Near Zone : orbital scale
Far Zone : gravitational wave scale

This hierarchy of scales is known as the EFT tower. We will focus on the near zone.

Here, we can assume BHs (or NSs) are point particles, with suitable corrections to describe finite size, tides, spin, etc. This is the bread and butter of QFT!

$\langle T \rangle = -\frac{\langle V \rangle}{2}$ $\left\{ \frac{mv^2}{r} \sim \frac{GmM}{r^2} \right\}$

centrifugal balances gravitational force!

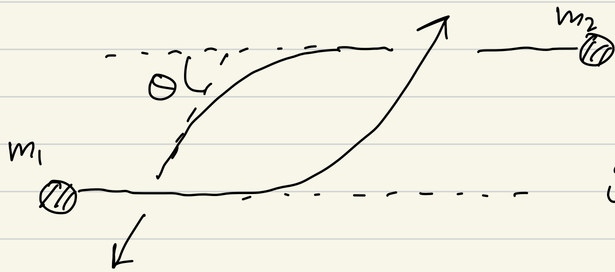
2.1 Post-Newtonian (PN) Regime

$$\varepsilon = \frac{Gm}{r} \sim v^2 \text{ (virial th}^{\underline{m}}\text{)}$$ } applies to bound orbits

- The scheme most applicable to bound orbits.
- Relies on the virial theorem, comparable scales.
- Starting point 0PN = Newtonian physics!
- Focus on e.g. 2-body Hamiltonian

2.2 Post-Minkowskian (PM) regime

If we have scattering bodies, or elliptic orbits!

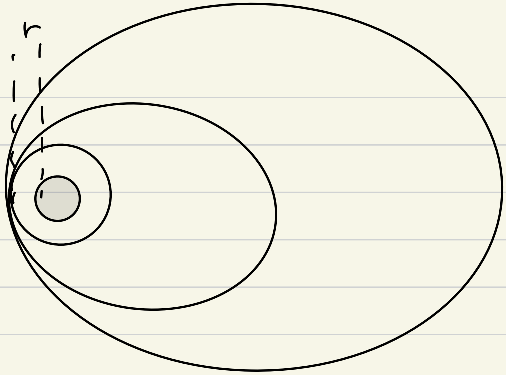


$$\varepsilon = \frac{GM}{r}$$ } no virial th$^{\underline{m}}$

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{G}\, h_{\mu\nu}$$

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$$

- Assume weak fields, but fast velocities!
- No separation between near zone & far zone
- Infinitely high orders in the PN expansion!
- Focus here on calculating asymptotic quantities, e.g. scattering angle θ.
- Encode physics in gauge-invariant quantity, unlike e.g. Hamiltonian
- Relevant for highly elliptic bound orbits:

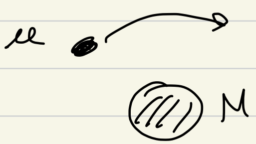


With fixed r at closest approach, we have faster velocities at the same point... no more virial theorem.

Eventually, the orbit goes elliptic $\rightarrow$ hyperbolic ... a scattering encounter!

2.3 Gravitational Self-Force (GSF)

Extreme-mass-ratio Inspirals (EMRIs)



$$\varepsilon = \frac{\mu}{M} \ll 1$$

- GSF is the probe limit, which we can solve! Solution to geodesic equation is known analytically.
- Corrections in GSF describe infinitely high orders in both PM & PN regimes!

Ultimately: a major goal of future GW development is to combine info from the different regimes, to get a more complete picture!

③ Single-Particle EFT

Let's describe a single particle in a GR background, using field theory. The particle can be described by a worldline action:

$$S_{pp} = -m\int ds$$ } trajectory defined by extremising the proper time s

$$= -m\int\sqrt{g_{\mu\nu}dx^\mu dx^\nu} = -m\int d\tau\,\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$$ } $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$

Pro tip: we can also use $S_{pp} = -\frac{m}{2}\int d\tau\left(e^{-1}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + e\right)$

$$S_{pp} = -\frac{m}{2}\int d\tau\,L(\tau) = -\frac{m}{2}\int d\tau\,g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

Variation of this action gives rise to the geodesic equation in a curved background:

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial\dot{x}^\mu}\right) = \frac{\partial L}{\partial x^\mu} \qquad \frac{\partial L}{\partial x^\mu} = g_{\nu\rho,\mu}\dot{x}^\nu\dot{x}^\rho$$

$$\frac{\partial L}{\partial\dot{x}^\mu} = 2g_{\mu\nu}\dot{x}^\nu \Rightarrow \frac{d}{d\tau}\left(\frac{\partial L}{\partial\dot{x}^\mu}\right) = 2g_{\mu\nu}\ddot{x}^\nu + 2g_{\mu\nu,\rho}\dot{x}^\nu\dot{x}^\rho$$

$$\Rightarrow g_{\mu\nu}\ddot{x}^\nu = \frac{1}{2}\left(g_{\nu\rho,\mu} - g_{\mu\nu,\rho} - g_{\mu\rho,\nu}\right)\dot{x}^\nu\dot{x}^\rho$$

$$\ddot{x}^\mu(\tau) = -\Gamma^\mu{}_{\nu\rho}\dot{x}^\nu\dot{x}^\rho$$ } $\Gamma^\mu{}_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma}\right)$

However, this ignores the internal structure of our compact object. We need to modify the worldline action to incorporate these!

We can ignore $R_{\mu\nu}=0$, as it vanishes on support of the Einstein equations, so the key thing required is:

$$E_{\mu\nu} = R_{\mu\alpha\nu\beta}\dot{x}^\alpha \dot{x}^\beta \qquad B_{\mu\nu} = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\mu} R^{\alpha\beta}{}_{\delta\nu}\dot{x}^\gamma \dot{x}^\delta$$

The point-particle action now takes the form

$$S_{pp} = m\int d\tau \left[\frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + C_{E^2} E_{\mu\nu}^2 + C_{B^2} B_{\mu\nu}^2 + \dots\right]$$

The Wilson coeffs "Love numbers" C_{E^2/B^2} describe response to adiabatic tides. They show up at $O(r_s^4)$!

This point-particle description works in PM & PN contexts!

④ Two-Body Dynamics

Now we seek to describe two objects interacting via GR. Our overall action (in the near zone) is:

$$S_{tot}[g_{\mu\nu}, x_1^\mu(\tau_1), x_2^\mu(\tau_2)] = S_{EH}[g_{\mu\nu}] + S_{pp}^{(1)}[g_{\mu\nu}, x_1^\mu(\tau_1)] + S_{pp}^{(2)}[g_{\mu\nu}, x_2^\mu(\tau_2)]$$

The Einstein-Hilbert action is

$$S_{EH}[g_{\mu\nu}] = -\frac{1}{16\pi G}\int d^4x\,\sqrt{\det g}\;R[g_{\mu\nu}]$$

When varied, this gives rise to the Einstein field equations. In this case, matter sourced by the two worldlines:

$$\frac{\delta S_{tot}}{\delta g_{\mu\nu}} = 0 \Rightarrow G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

$$T_{\mu\nu}(x) = \sum_{a=1,2} m_a \int d\tau_a\, \dot{x}_\mu \dot{x}_\nu\, \delta^4(x - x_a(\tau))$$

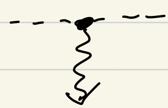
We could solve these equations perturbatively! Let's expand our fields:

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \qquad \kappa = \sqrt{32\pi G}$$

$$\Rightarrow \Box h_{\mu\nu} = -\frac{\kappa}{2} P_{\mu\nu;\rho\sigma} T^{\rho\sigma} + O(\kappa^2) \qquad P_{\mu\nu;\rho\sigma} = \eta_{\mu(\rho}\eta_{\sigma)\nu} - \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma}$$

$$\int_k e^{-ik\cdot x}\,\Box h_{\mu\nu} = -\frac{\kappa}{2} P_{\mu\nu;\rho\sigma} \int_k e^{-ik\cdot x}\, T^{\rho\sigma}$$

$$\Rightarrow h_{\mu\nu}(k) = \frac{\kappa}{2}\,\frac{P_{\mu\nu;\rho\sigma}}{k^2}\, T^{\rho\sigma}(k)$$



where, $T^{\mu\nu}(k) = \sum_a m_a \int d\tau_a\, e^{ik\cdot x_a(\tau_a)}\, \dot{x}_a^\mu \dot{x}_a^\nu$

In PN, $\dot{x}_a^\mu = (1,0) + O(v)$, so focus on T^{00} //

⑤ Non-Relativistic GR (NRGR)

Split the metric into internal/near/far zones:

$$g_{\mu\nu} = \eta_{\mu\nu} + \cancel{h^{\text{internal}}_{\mu\nu}}^{\,0} + h^{\text{near}}_{\mu\nu} + h^{\text{far}}_{\mu\nu}$$

Working in the near zone, our EFT already accounts for $h^{\text{internal}}_{\mu\nu}$. The others have scalings:

$$\left.\partial_t h^{\text{near}}_{\mu\nu} \sim \frac{v}{r} h^{\text{near}}_{\mu\nu}, \quad \partial_i h^{\text{near}}_{\mu\nu} \sim \frac{1}{r} h^{\text{near}}_{\mu\nu},\right\} p^{\text{pot}}_{\mu} \sim \left(\frac{v}{r}, \frac{1}{r}\right)$$

$$\left.\partial_t h^{\text{far}}_{\mu\nu} \sim \frac{v}{r} h^{\text{far}}_{\mu\nu}, \quad \partial_i h^{\text{far}}_{\mu\nu} \sim \frac{v}{r} h^{\text{far}}_{\mu\nu}\right\} p^{\text{rad}}_{\mu} \sim \left(\frac{v}{r}, \frac{v}{r}\right)$$

To go from near → far zone, we "integrate out" the near-zone gravitons. Formally, in QFT language:

$$e^{iW[x^a_\mu, h^{\text{far}}_{\mu\nu}]} = \int \mathcal{D}h^{\text{near}}_{\mu\nu} \exp\left\{ i S_{\text{tot}}[g_{\mu\nu}, x_1^\mu(\tau), x_2^\nu(\tau)] \right\}$$

What does this mean?? Formally, it is an instruction to sum over all possible values of $h^{\text{near}}_{\mu\nu}$, "field configurations". Then, plug back into the action

In practice, the solution is dominated by solutions to the classical EoMs, with other possibilities suppressed by powers of $\hbar$, Planck's constant. This is known as the "saddle-point approximation".

6.1 Effective Potential

$$S_{int} = \frac{K}{2}\int d^4x\, T^{\mu\nu}(x)\, h_{\mu\nu}(x) \quad \Big\}\text{ because } T^{\mu\nu} = 2\frac{\delta S}{\delta g_{\mu\nu}}$$

$$= \frac{K}{2}\int d^4x \int_h e^{-ih\cdot x}\, T^{\mu\nu}(h) \int_q e^{-iq\cdot x}\, h_{\mu\nu}(q)$$

$$= \frac{K}{2}\int_{h,q} (2\pi)^4\, \delta^4(h+q)\, T^{\mu\nu}(h)\, h_{\mu\nu}(q)$$

$$= \frac{K}{2}\int_q T^{\mu\nu}(-q)\, h_{\mu\nu}(q)$$

$\downarrow$ plug in leading-order solution

$$= \frac{K^2}{4}\int_q T_{\mu\nu}(-q)\, \frac{P^{\mu\nu;\rho\sigma}}{q^2}\, T_{\rho\sigma}(q) \quad \Big\}\ P^{00;00} = \frac{1}{2}$$

$$\approx 4\pi G \int_{q^0,\vec{q}} T_{00}(-q)\left(\frac{1}{-(q^0)^2 + \vec{q}^{\,2}}\right) T_{00}(q) \quad \Big\}\ q^0 \text{ suppressed}$$

$$\approx 4\pi G \cdot m_1 m_2 \int_{q^0,\vec{q}} \int_{\tau_1,\tau_2} e^{iq\cdot(x_2-x_1)}\left(+\frac{1}{\vec{q}^{\,2}} + \cdots\right)$$

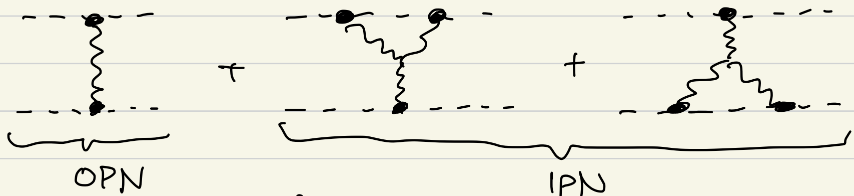
$$= 4\pi G\, m_1 m_2 \int_{\tau_1,\tau_2} \delta(\tau_1-\tau_2) \int_{\vec{q}} e^{-i\vec{q}\cdot(\vec{x}_2-\vec{x}_1)}\left(\frac{1}{+\vec{q}^{\,2}}\right)$$

$$\boxed{\int_{\vec{q}} \frac{1}{\vec{q}^{\,2}}\, e^{-i\vec{q}\cdot\vec{r}} = \frac{1}{4\pi r}} \qquad t \approx \tau_1 \approx \tau_2$$

$$= +\int_t \frac{G m_1 m_2}{r} \quad \Big\}\ r = |\vec{x}_1 - \vec{x}_2| \quad \Big\}\ \text{The Newtonian Potential!!}$$

Working order-by-order, the solution appears as a sum of Feynman diagrams:

$$e^{iW[\vec{x}_a]} =$$



$$+ \cdots$$

Doing this calculation (in detail!) we learn that

$$\boxed{W = \int dt \left[L^{0PN} + L^{1PN} + \cdots \right]}$$

$$L^{0PN} = \frac{1}{2} m_1 \vec{v}_1^{\,2} + \frac{1}{2} m_2 \vec{v}_2^{\,2} + \frac{G m_1 m_2}{r} \quad \Big\} \; r = |\vec{x}_1 - \vec{x}_2|$$

$$L^{1PN} = \frac{1}{8} m_1 \vec{v}_1^{\,4} + \frac{1}{8} m_2 \vec{v}_2^{\,4} - \frac{G^2 m_1 m_2 (m_1 + m_2)}{2r^2}$$

$$+ \frac{G m_1 m_2}{2r} \left[3(\vec{v}_1^{\,2} + \vec{v}_2^{\,2}) - 7 \vec{v}_1 \cdot \vec{v}_2 - \frac{(\vec{v}_1 \cdot \vec{r})(\vec{v}_2 \cdot \vec{r})}{r^2} \right]$$

Known as the Einstein-Infeld-Hoffmann Lagrangian, the first relativistic correction to Newton's law of gravitation!

Current state-of-the-art is 4PN order!