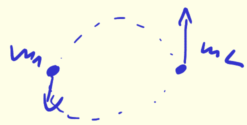
2024-04-06 lecture

Inspiral of binary systems

$$L_{GW} = -\frac{dE_{orb}}{dt}$$ releasing energy due to inspiral and emitting gravitational wave



now $l_0(t)$

kinetic energy

$$E_{orb} = E_k + U$$

$$E_k = \frac{1}{2} m_1 \omega_1^2 r_1^2 + \frac{1}{2} m_2 \omega_2^2 r_2^2 = \frac{1}{2}\frac{G\mu M}{l_0}$$ reduced mass $\mu = \frac{m_1 m_2}{M}$

$$r_1 = \frac{m_2 l_0}{M} \qquad r_2 = \frac{m_1 l_0}{M} \qquad \omega_k = \sqrt{\frac{GM}{l_0^3}}$$

$$E_k = \frac{1}{2}\mu l_0^2 \omega_k^2 \qquad U = -G\frac{m_1 m_2}{l_0} = -\frac{G\mu M}{l_0}$$

potential energy

$$\Rightarrow E_{orb} = -\frac{1}{2}\frac{G\mu M}{l_0}$$

$$\frac{dE_{orb}}{dt} = +\frac{1}{2}\frac{G\mu M}{l_0^2}\frac{dl_0}{dt} = -E_{orb}\frac{1}{l_0}\frac{dl_0}{dt}$$

L_{GW} insert

$$\frac{1}{l_0}\frac{dl_0}{dt} = -\frac{\frac{dE_{orb}}{dt}}{E_{orb}} = \frac{L_{GW}}{E_{orb}}$$

Luminosity L_{GW} is derived from the h metric

3rd time derivative

$$L_{GW} = \frac{G}{5c^5}\sum_{k,l,m=1}^{3} \dddot{Q}_{km}\left(t-\frac{r}{c}\right)\dddot{Q}^{km}\left(t-\frac{r}{c}\right) = \frac{32\mu^2 G^4 M^3}{5c^5 l_0^5}$$

quadrupole moments

$$\frac{1}{l_0}\frac{dl_0}{dt} = -\frac{32}{5}\frac{G^{\cancel{4}\,3}\mu^{\cancel{2}}M^{\cancel{3}\,2}}{c^5 l_0^{\cancel{5}\,4}}\frac{2\cancel{l_0}}{\cancel{G\mu M}} = -\frac{64}{5}\frac{G^3\mu M^2}{c^5}\frac{1}{l_0^4}$$

$$\frac{dl_0}{dt} = -\frac{64}{5}\frac{G^3\mu M^2}{c^5}\frac{1}{l_0^3}$$

$$\int_{l_0^{init}}^{l_0(t)} \bar{l}_0^{\,3}\, d\bar{l}_0 = \int_{t_{init}}^{t} -\frac{64}{5}\frac{G^3\mu M^2}{c^5}\, d\bar{t}$$

$$l_0^4 = (l_0^{init})^4 - \frac{64}{5} \frac{G^3}{c^5} \mu M^2 t$$

$$l_0(t) = \sqrt[4]{(l_0^{ini})^4 - \frac{256}{5} \frac{G^3}{c^5} \mu M^2 t}$$

$$l_0(t) = l_0^{init} \left(1 - \frac{t}{t_{coal}}\right)^{1/4} \quad t_{coal} = \frac{5}{256} \frac{c^5}{G^3} \frac{(l_0^{init})^4}{\mu M^2}$$

$$\omega_k(t) = \sqrt{\frac{GM}{l_0^3(t)}} \quad l_0 \text{ insert}$$

$$= \omega_k^{init} \left(1 - \frac{t}{t_{coal}}\right)^{-3/8} \quad \omega_k^{init} = \sqrt{\frac{GM}{(l_0^{init})^3}}$$

$$f_{GW}(t) = \frac{2\omega_k(t)}{2\pi} = f_{GW}^{init} \left(1 - \frac{t}{t_{coal}}\right)^{-3/8} \quad f_{GW}^{init} = \frac{1}{\pi} \sqrt{\frac{GM}{(l_0^{init})^3}}$$

$$f_{GW}(t \to t_{coal}) \to \infty$$

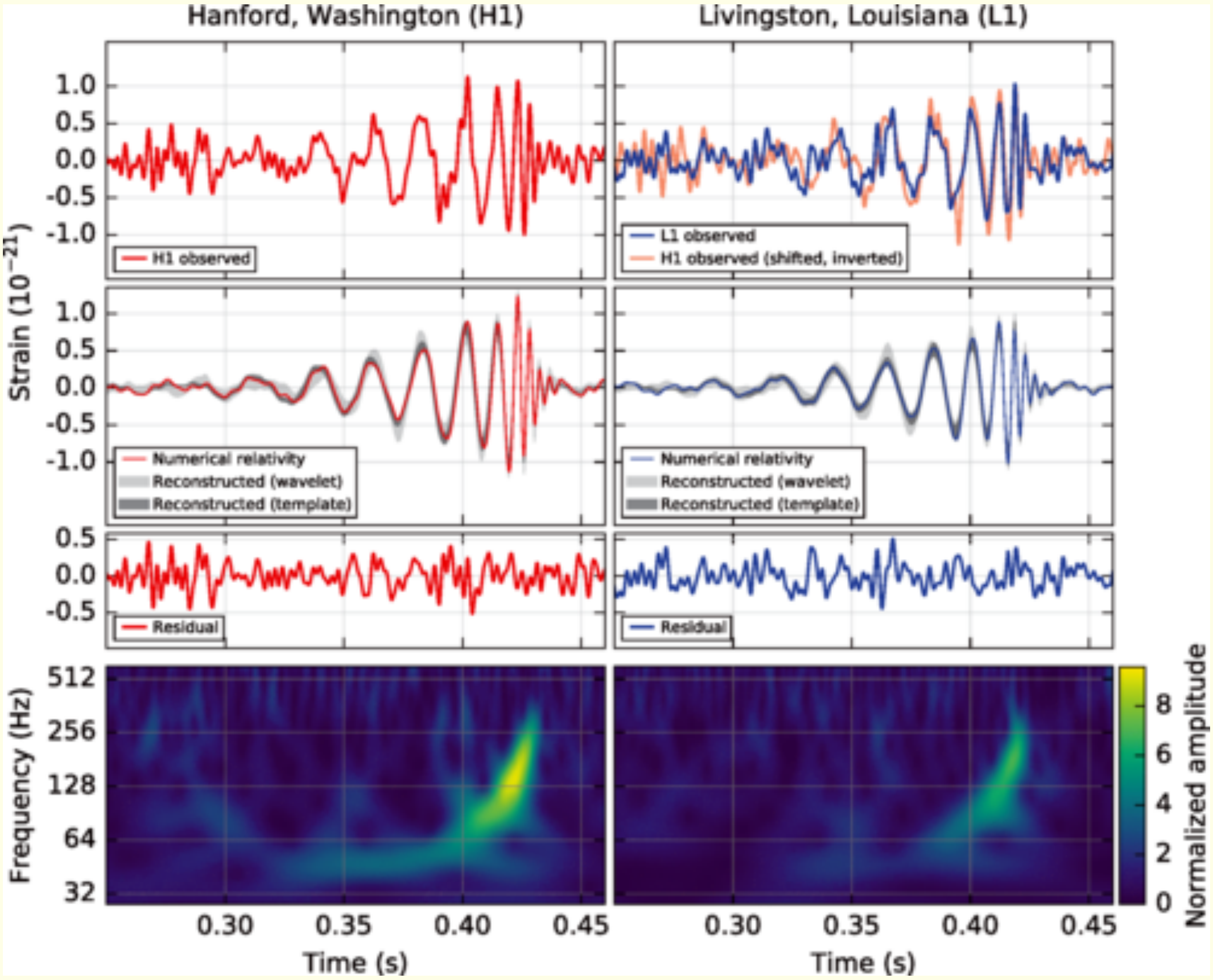
$$h_0(t) = \frac{4 \mu M G^2}{c^4 r \, l_0(t)} = \frac{4 \mu M G^2}{c^4 r} \frac{\omega_k(t)^{2/3}}{G^{1/3} M^{1/3}} = \frac{4 \mu M G^2}{c^4 r} \frac{f_{GW}(t)^{2/3} \pi^{2/3}}{G^{1/3} M^{1/3}}$$

chirp-mass $\mathcal{M}^{5/3} = \mu M^{2/3}$

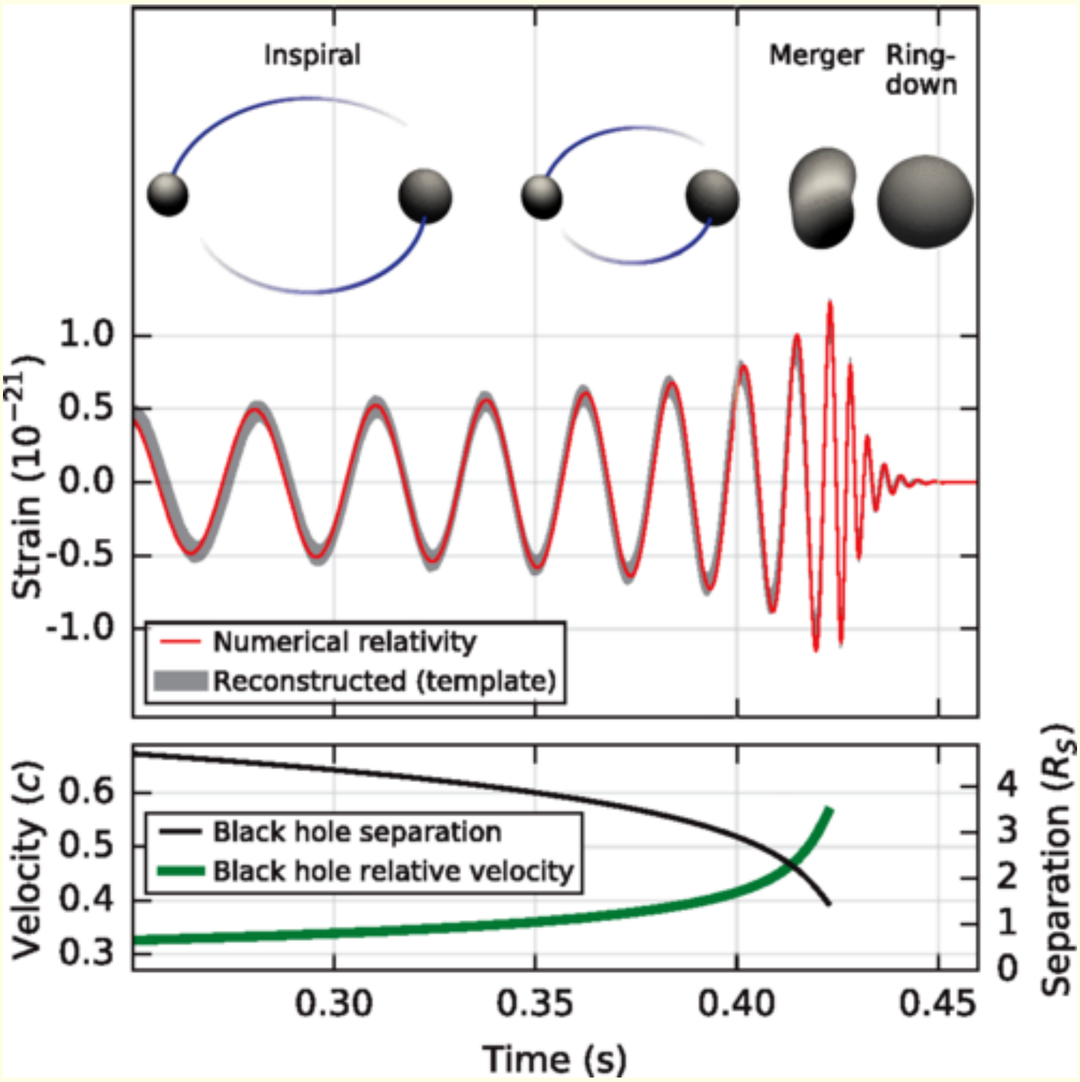
$$h_0(t) = \frac{4 \pi^{2/3} G^{5/3}}{c^4 r} f_{GW}(t)^{2/3} \mathcal{M}^{5/3}$$

$$h_0(t \to t_{coal}) \to \infty$$

frequency $f_{GW}\big|_{max} \simeq 150 \text{ Hz}$



Hanford, Washington (H1)
Livingston, Louisiana (L1)
Strain (10^{-21})
H1 observed
L1 observed
H1 observed (shifted, inverted)
Numerical relativity
Reconstructed (wavelet)
Reconstructed (template)
Residual
Frequency (Hz)
Normalized amplitude
Time (s)



Inspiral
Merger
Ring-down
Strain (10^{-21})
Numerical relativity
Reconstructed (template)
Velocity (c)
Separation (R_S)
Black hole separation
Black hole relative velocity
Time (s)

$$f_{GW}(t) = \frac{1}{\pi}\sqrt{\frac{GM}{(\ell_0^{ini})^3}}\left(1-\frac{t}{t_{coal}}\right)^{-3/8} \quad \ell_0^{ini} - \text{model}$$

$$f_{GW}^{-8/3}(t) = \frac{(8\pi)^{8/3}}{5}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3}(t_c - t)$$

$$\mathcal{M} = \frac{f_{GW}(t)^{-8/5}}{(t_c - t)^{3/5}}\,\frac{5^{3/5}}{(8\pi)^{8/5}}\,\frac{c^3}{G}$$

example: $f_{GW}(t) = 150\,\text{Hz}$ estimating chirp-mass

$t = 0.42\,\text{s}$ $\mathcal{M} = 30\,M_\odot$

$$R = \left(\frac{GM}{\omega_k^2}\right)^{1/3} = 350\,\text{km} \quad \text{distance}$$

$$\omega_k = 2\pi\frac{f_{GW}}{2}$$

Schwarzschildradius

$$R_S^1 = \frac{2G}{c^2}m_1 \qquad m_1 = m_2$$

$$\mathcal{M} = \mu^{3/5}M^{2/5} = \left(\frac{m_1 m_2}{M}\right)^{3/5}M^{2/5} = 2^{-1/5}m_1 \Rightarrow m_1 = 35\,M_\odot$$

$$R_S^1 \approx 103\,\text{km}$$

It has to be 2 black holes, because if there are two stars, the radii of the stars would be larger and they would have already collided, so we need the black holes

After inspiral of binary systems - remnant black hole

$$e^{i\omega_{GW}t} = \underbrace{e^{2\pi i f_{GW} t}}_{\text{wave}} \underbrace{e^{-t/\tau_{damp}}}_{\text{damping}}$$

$$\omega = \omega_R + i\omega_I \qquad f_{GW} = \frac{\omega_R}{2\pi} \qquad \tau_{damp} = \frac{1}{\omega_I}$$

Quasi-normal modes $\quad f_{GW} \simeq 260\,\mathrm{Hz} \quad \tau_{damp} \simeq 4\,\mathrm{ms}$

a test for GR

Using the same assumptions as before the inspiral

$$g_{\mu\nu} = \underbrace{g^0_{\mu\nu}}_{\text{background metric of outside of black hole}} + \underbrace{h_{\mu\nu}}_{\text{pertubation}}$$

no spinning black hole

$G = c = 1$

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)$$

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{0\lambda\varrho}(h_{\varrho\mu,\nu} + h_{\varrho\nu,\mu} - h_{\mu\nu,\varrho}) + \frac{1}{2} h^{\lambda\varrho}(g^0_{\varrho\mu,\nu} + g^0_{\varrho\nu,\mu} - g^0_{\mu\nu,\varrho}) + O(h)$$

$$\frac{\partial}{\partial x^\alpha}\Gamma^{\alpha}_{\mu\nu}(h) - \frac{\partial}{\partial x^\nu}\Gamma^{\alpha}_{\mu\alpha}(h)$$

pertubation only scalar field pertubation for the black hole

massless scalar field

$\Box\,\Phi = 0$ for the box derivative we use the Schwarzschild metric

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, g^{0\mu\nu}\partial_\nu \Phi\right) = 0 \qquad g = \det g_{\mu\nu}$$

$$\sqrt{-g} = r^2 \sin\vartheta$$

$$\Phi(t, r, \vartheta, \varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{\psi_{\ell m}(r)}{r} \underbrace{Y_{\ell m}(\vartheta, \varphi)}_{\substack{\text{spherical harmonics} \\ \text{angular functions}}} e^{-i\omega t}$$

$$Y_{\ell m} = P_\ell(\vartheta)\, e^{im\varphi} = \frac{1}{\sin^2\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\,\partial_\vartheta P_\ell) - \frac{m^2}{\sin^2\vartheta} P_\ell = -\ell(\ell+1) P_\ell$$

insert Y_{lm} in here and the metric

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu \Phi\right)=0$$

$$\left(1-\frac{2M}{r}\right)^2\frac{d^2\psi}{dr^2}+\frac{2M}{r}\left(1-\frac{2M}{r}\right)\frac{d\psi}{dr}+\Big(\omega^2-\underbrace{\left(1-\frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2}+\frac{2M}{r^3}\right)}_{V_{eff}(r)}\Big)\psi=0$$

$$\frac{dr}{dr_*}=1-\frac{2M}{r} \qquad r_*=r+2M\log\left(\frac{r}{2M}-1\right)$$

$$\frac{d\psi}{dr}=\frac{d\psi}{dr_*}\frac{dr_*}{dr}=\frac{d\psi}{dr_*}\frac{1}{\left(1-\frac{2M}{r}\right)}$$

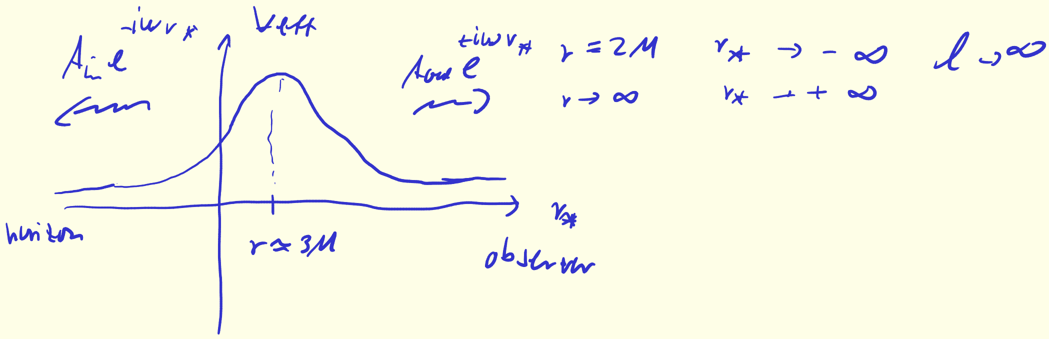
$$\frac{d^2\psi}{dr^2}=\frac{d}{dr}\left(\frac{d\psi}{dr_*}\frac{1}{\left(1-\frac{2M}{r}\right)}\right)=\frac{d^2\psi}{dr_*^2}\frac{dr_*}{dr}\frac{1}{1-\frac{2M}{r}}+\frac{d\psi}{dr_*}\frac{d}{dr}\left(\frac{1}{1-\frac{2M}{r}}\right)$$

rewrite in the new coordinate r_*

$$\frac{d^2}{dr_*^2}\psi+\left(\omega^2-V_{eff}(r)\right)\psi=0$$

$V'_{eff}(r)=0$ for max and min

$r\simeq 3M$ – circular orbit for photons



fundamental mode at $l=m=2$

search for eigenvalues of the this equation, for that we need 2 boundary conditions

$$\frac{d^2}{dr_*^2}\psi+\left(\omega^2-V_{eff}(r)\right)\psi=0$$

eigenvalue

$$\psi(r)=A_{in}e^{-i\omega r_*}+A_{out}e^{i\omega r_*}$$

$r\to 2M$ $\quad \psi(r)\sim A_{in}e^{-i\omega r_*}$ $\quad$ asymptotic region

$r\to\infty$ $\quad \psi(r)\sim A_{out}e^{i\omega r_*}$ $\quad$ integrating numerically to get the ω

$l = 2 \qquad f = \frac{\omega_R}{2\pi} \sim 12\ kHz\ \frac{M_\odot}{M} \quad$ using $60\ M_\odot$

$\tau = \frac{1}{\omega_I} \sim 0.035\ ms\ \frac{M}{M_\odot}$

V_{eff} for many interactions

$$V_{eff}(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} + (1-s^2)\frac{2M}{r^3}\right]$$

$s = 0$ scalar
$s = -1$ em
$s = -2$ grav.

for spinning $\chi = 0.7 \quad s = -2,\ l = m = 2$

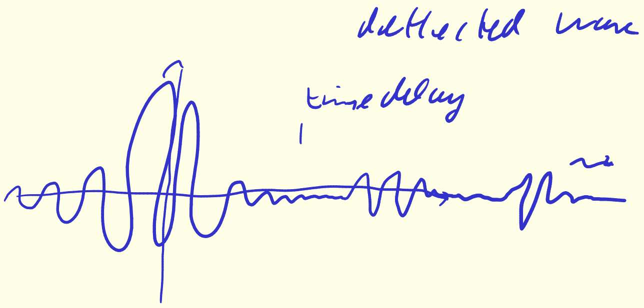
$M\omega_R = 0.5198 \qquad M\omega_I = 0.0818$

$f = \frac{M\omega_R}{2\pi M} \approx 250\ kHz$

$\tau = \left(\frac{(M\omega_I)}{M}\right)^{-1} = 4\ ms$

modified solution to have also
contribution from the black hole
not only the outward solution wave

Lecturer:
Research in ring down of the black holes
allowing patial reflection of the wave at the black hole



testing GR at the horizon
with gravitational waves
signals after the ringdown
some kind of echo

maybe quantum reflection

maybe without a horizon