Exercises

1. Derive the addition of velocities using Lorentz transformations of the coordinates. Assume two inertial frames are at relative velocity v in Cartesian x-direction.
2. Compute the components $g_{\mu\nu}$ of the metric tensor in polar coordinates ($\mu \in r, \theta$)
$$x = r\cos(\theta)\,, \qquad y = r\sin(\theta)\,, \tag{157}$$
by using the invariance of the line element $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ when changing from Cartesian to polar coordinates.
3. By following a similar calculation as done for the contravariant vector, show that the covariant derivative of a covector is given as
$$\nabla_\mu A_\lambda = \partial_\mu A_\lambda - \gamma^\rho{}_{\mu\lambda} A_\rho \tag{158}$$
and determine $\gamma^\rho{}_{\mu\lambda}$
4. Show that $\nabla_\alpha g_{\mu\nu} = 0$ by using the definition of a covariant derivative of a covariant tensor of rank 2
$$\nabla_\nu T_{\lambda\mu} = \partial_\nu T_{\lambda\mu} - \Gamma^\alpha{}_{\lambda\nu} T_{\mu\alpha} - \Gamma^\alpha{}_{\mu\nu} T_{\lambda\alpha} \tag{159}$$
and assume
$$\Gamma^\alpha{}_{\mu\rho} = \frac{1}{2} g^{\alpha\nu} \left(\partial_\rho g_{\mu\nu} + \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu}\right) \tag{160}$$
and that $g_{\mu\nu} = g_{\nu\mu}$.
5. Show that the contraction of a tensor $T^\alpha{}_{\beta\gamma}$ results in a covariant vector (use $\beta = \alpha$ or $\gamma = \alpha$).
6. Show that $T^r U_{ir}$ is a tensor if T^i and U_{ij} are each tensors.