Lecture 2024-02-29
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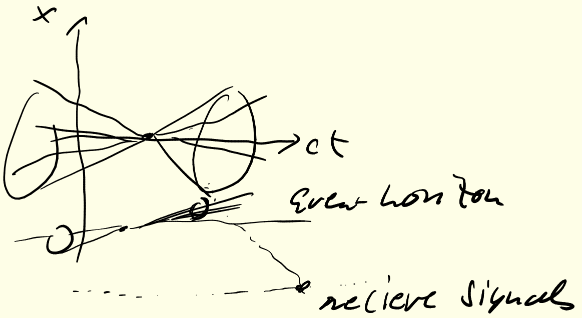
Blackholes (BH)

Proper definiton of blackholes
typical properties, special causal structure
Do BH come in families, or are they all different from each other
conformal completion, Killing vectors, Penrose diagrams, explici solution Schwarzschild solution
Birkhoff & Israel theorems, Kerr metric, Penrose process, Lense Thirring effect

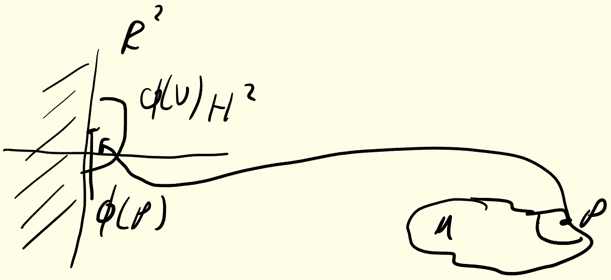
Definiton:

Region of spacetime from which no signal(causal worldline) can escape to infinity.



Proper definition of spacetime infinity:
conformal completion - concept of manifold with boundary $\mathcal{M}$

$\forall p \in \mathcal{M}$, $\exists$ an hommomorphism ϕ from a vicinity of p to an open subset of the halfplane $H^n = \{(x^0, \dots, x^{n-1}) \in \mathbb{R}^n, x^{n-1} \geq 0\}$



A point p is a boundary of the manifold:

iff $\exists$ a homeomorphism ϕ such that $\phi(p) \in \partial H^n$

$\partial H^n = \{(x^0, \dots, x^{n-1}) \in H^n, x^{n-1} = 0\}$

spacetime is manifold & metric. We need to extend the metric to boundary of H (∂H)

spacetime $(\mathcal{M}, g)$ admits a conformal completion iff

$\exists$ a Lorentzian manifold with boundary $(\hat{\mathcal{M}}, \hat{g})$

with a smooth scalar field $\Omega : \hat{\mathcal{M}} \to \mathbb{R}^+$ such that

1. $\hat{\mathcal{M}} = \mathcal{M} \cup \mathcal{J}$, where $\mathcal{J} = \partial \hat{\mathcal{M}}$

"scri"

2. on $\mathcal{M}$, $\hat{g} = \Omega^2 g$

3. on $\mathcal{J}$, $\Omega = 0$

4. on $\mathcal{J}$, $d\Omega \neq 0$

$\hat{g}$ - conformal metric   preserve the causal structure of the spacetime

$\mathcal{J}$ - conformal boundary of $(\mathcal{M}, g)$
within the conformal completion $(\hat{\mathcal{M}}, \hat{g})$

Condition 2 and 3 garantee that J is at infinite distance from all the points of M.

Distances measured by $\hat{g}$ are given by the "line element"

$$d\hat{s}^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu = \Omega^2 g_{\mu\nu} dx^\mu dx^\nu = \Omega^2 ds^2$$

$$\Rightarrow ds^2 = \frac{d\hat{s}^2}{\Omega^2} \to \infty \quad \text{at } \mathcal{J}, \Omega = 0 \qquad \Delta s = \int ds = \int \frac{d\hat{s}}{\Omega} = \lim_{\Omega \to 0} \int \frac{d\hat{s}}{\Omega} \to +\infty$$

Condition 4 ensures that J is regular hypersurface. - (smooth (a C^2) submanifold)

The black hole region of $(\hat{\mathcal{M}}, \hat{g})$ is the region from which no future-directed causal worldline reaches $\mathcal{J}$.

Event horizon - the white hole

Event horizon is the topological boundary of BH

The wordlines of white hole region only goes away from the white hole, not gong in.
No past-directed causal worldlines reaches J.

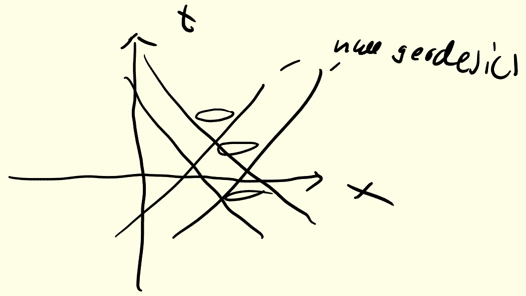
The domain of outer communications $\langle\langle \mathcal{M} \rangle\rangle$ is the set of points that do not lie in the holes.

Visualization of the whole causal structure of spacetime at infinity with conformal completion.
Penrose diagrams with conformal completion.

example Minkowski metric $\quad c=1$

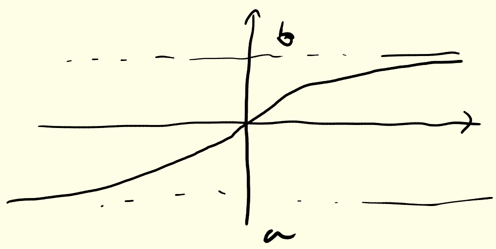
$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

with $t \in \mathbb{R},\ r \in \mathbb{R}^+,$



change coordinates to bring $\mathbb{R}$ and $\mathbb{R}^+$ to a finite intervall.

this type of function



$\rightarrow$ arctan

change the coordinates

$$u = t - r \qquad t = \frac{v+u}{2}$$

$$v = t + r \qquad r = \frac{v-u}{2}$$

This wil preserve the aspects of lightcones the null geodesics will such that , iff be the 45 degrees inched lines.

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = d\bar{s}^2$$

$$dt = \frac{dv}{2} + \frac{du}{2} \qquad ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$$dr = \frac{dv}{2} - \frac{du}{2} \qquad = -\left(\frac{dv+du}{2}\right)^2 + \left(\frac{dv-du}{2}\right)^2 + \left(\frac{v-u}{2}\right)^2 (d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$$= -\frac{dv^2}{4} - \frac{dv\,du}{2} - \cancel{\frac{du^2}{4}} + \cancel{\frac{dv^2}{4}} - \frac{dv\,du}{2} + \cancel{\frac{du^2}{4}} + (\ldots)(\ldots)$$

$$ds^2 = -dv\,du + \left(\frac{v-u}{2}\right)^2 (d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$$\hat{g}_{\mu\nu} = \begin{pmatrix} 0 & -1 & & \\ -1 & 0 & & \\ & & \left(\frac{v-u}{2}\right)^2 & \\ & & & \left(\frac{v-u}{2}\right)^2 \sin^2\theta \end{pmatrix}$$

The ranges of coordinates u and v is such that

$$-\infty < u < v < +\infty$$

now use arctan fnction to change to

$$\underline{u} = \arctan u$$

$$\underline{v} = \arctan v$$

$-\frac{\pi}{2} < \mathcal{U} < \mathcal{V} < \frac{\pi}{2}$

$-\frac{\pi}{4} < \mathcal{U} < \mathcal{V} < \frac{\pi}{4}$

$u = \tan \mathcal{U}$

$v = \tan \mathcal{V}$

change to $(\mathcal{U}, \mathcal{V}, \theta, \varphi)$

$$du = d(\tan \mathcal{U}) = \frac{d\mathcal{U}}{\cos^2 \mathcal{U}} \qquad dv = \frac{d\mathcal{V}}{\cos^2 \mathcal{V}}$$

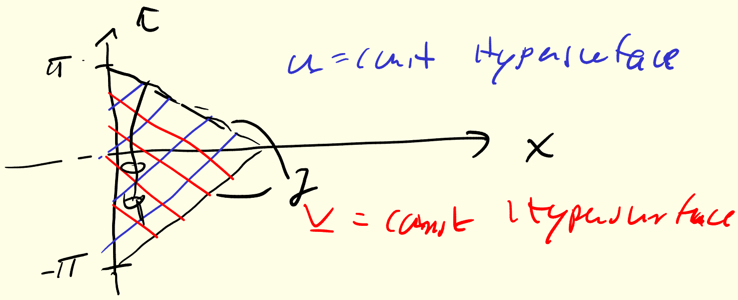
$$\Rightarrow \quad ds^2 = \frac{1}{4\cos^2\mathcal{V}\cos^2\mathcal{U}}\left(-4\, d\mathcal{U}\, d\mathcal{V} + \sin^2(\mathcal{V}-\mathcal{U})(d\theta^2 + \sin^2\theta\, d\varphi^2)\right)$$

final trafo $\quad \tau = \mathcal{V} + \mathcal{U} \qquad \mathcal{U} = \frac{\tau - \chi}{2}$

$\chi = \mathcal{V} - \mathcal{U} \qquad \mathcal{V} = \frac{\tau + \chi}{2}$

with $\quad 0 < \chi < \pi \quad , \quad \chi - \pi < \tau < \pi - \chi$

this defines the following domain

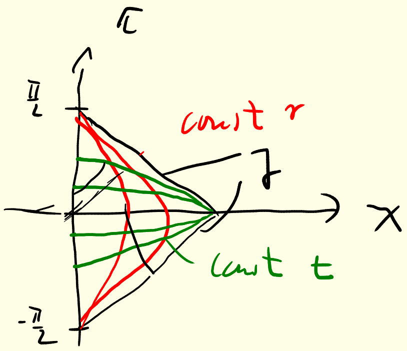


$$\tau = \arctan(t - r)$$
$$\chi = \arctan(t + r)$$

$$t = \frac{\sin\tau}{\cos\tau + \cos\chi} \qquad r = \frac{\sin\chi}{\cos\tau + \cos\chi}$$

all possible path of the photons or null geodesics

$\mathcal{U} = \text{const} \qquad \tau = \chi + \text{const}$



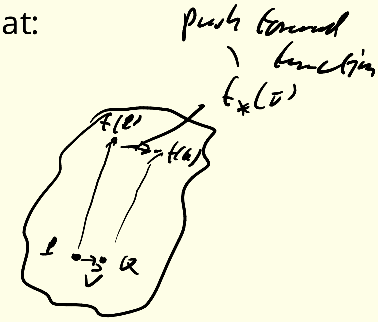
We can never prove the exitence of a BH somewhere. We can never be on the boundary. Some typical signatures for specific BH allows to derive physical consequences of their presence. (the acting on surrounding things, interactions with BH)
no direct measurements
stars around supermassive BH -> timelike geodesic (orbits)
photons travelling close to a BH (gravitational lensing)
photons from accretion disks close to BH (Event horizon telescope) -> null geodesics

-> study geodesic of BH spacetimes
we need explicit motions of BH, solve Einstein equation $\quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$

for solving, try at first Vacuum $T_{\mu\nu} = 0$, $\Lambda = 0$ no cosmic constant

symmetric spacetime with Lie group G (Group that is also a manifold)

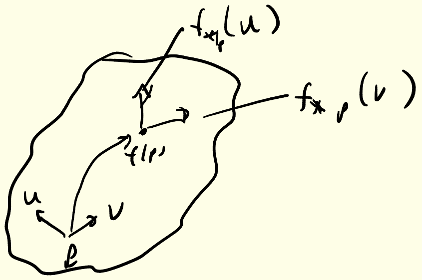
A symmetry group a spacetime (M,g) iff there is an action of G on M such that:



any $f \in G$ defines an isometry of (M,g)

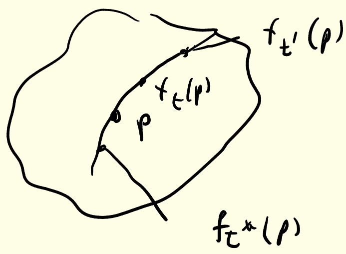
$\forall p \in M$, $\forall (u,v) \in T_p(M)^2$ $\quad g_{f(p)}(f_* u, f_* v) = g_p(u,v)$

at p



Consider a 1-d Lie Group, all its elements can be refered by a parameter t:

$G = \{ f_t, t \in I \}$ some interval



The killing vector $\vec{\xi}$ of G at p is defined as $\frac{df_t(p)}{dt}$ generators

The killing vector gives the direction and speed by which G moves p.

For a killing vector ξ, one has $D_\mu \xi_\nu + D_\nu \xi_\mu = 0$

$D_{(\mu} \xi_{\nu)} = 0$

In coordinates $(x^0, \dots, x^{n-1})$ adopted to ξ, in example such that $\xi = \partial_0$.

then G is a isometry iff $\partial_0 g_{\alpha\beta} = 0$

x_0 is then called ignorable coordinate.

time invariance - stationary

(M,g) is stationary, iff

1. translation group $(\mathbb{R},+)$ is an isometry group
2. orbits are everywhere timelike $\Leftrightarrow \exists$ killing vector is everywhere timelike
   $(f_t(p))$

2.'' (pseudo stationary) the orbits are timelike in the vicinity of the conformal boundary $\mathcal{J}$ of a conformal completion of (M,g)
$\Leftrightarrow$ $\exists$ killing vector timelike vector in a vicinity of $\mathcal{J}$

nothing moves at all :

(M,g) is pseudo (resp. strictly) static iff

1. it is pseudo stationary
2. $\exists$ killing vector is orthogonal to a family of hypersurfaces

Lemaite synchronous coordinate

$$ds^2 = -d\tau^2 + a(\tau,x)^2 dx^2 + r(\tau,x)^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)$$

solve the null radial geodesic equation

p the momentum vectors of particle $p^\mu \partial_\mu p = 0$

Show that for any Killing vector k, are has $p^\mu \partial_\mu (k \cdot p) = 0$

$E = -\partial_t p$ is conserved along the geodesic

$L = \partial_\varphi \cdot p$

energy angular momentum

particle falling radially $p = p^t \partial_t + p^r \partial_r$

denoting λ an affine parameter along particle trajectory $p^\mu = \frac{dx^\mu}{d\lambda}$

$$p^t = \frac{dt}{d\lambda} \qquad p^r = \frac{dr}{d\lambda}$$

Using $p^2 = 0$ (particle is massless)

$$r(\lambda) = r_0 - E\lambda$$

tortoise

$$t(\lambda) = \text{const} - r_*(\lambda) \quad \text{where } r_* = r + 2m \ln\left|\frac{r}{2m} - 1\right|$$

Any ingoing radial null geodesic can be labelled by a constant v

such that $t(\lambda) = v - r_*(\lambda)$

Lets use this relation to define a new coordinate system $(v, r, \vartheta, \varphi)$

$$v = t + r_*(x)$$
$$r = r$$
$$\vartheta = \vartheta$$
$$\varphi = \varphi$$

In these so-called null Eddington-Finkelstein coordinates

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dv^2 + 2\, dv\, dr + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)$$

Divergence at r_s disappeared: it was just a pathology of schwarzschild-Droste coordinates, called coordinate singularity

$r=0$ divergence can be showed by Kretschmann scalar is shows you the curvature of the spacetime

$$K = R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} = \frac{48 m^2}{r^6} \xrightarrow{r\to 0} \infty$$

This reflects a physical pathology called a "curvature singularity"

Checking that this is a null

$$v(\lambda) = v_0$$
$$r(\lambda) = -\lambda \qquad \lambda \in (\lambda_0, 2m, \dots)$$
$$\theta = \theta_0$$
$$\varphi = \varphi_0$$

ingoing geodesic (from M_I to M_{II}) in NIEF

null Eddington-Finkelstein

This follows a photon through r_s where tangent vector gives the future direction in M_{II}

In M_{II}, r_s has to decrease towards the future along any causal wordline

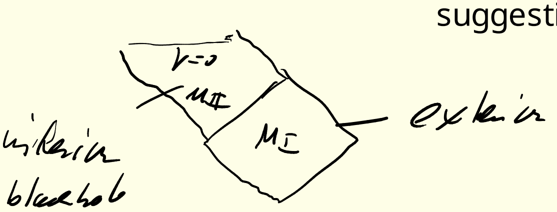
$r=0$ is an unavoidablle end of time (not a place in time)

This property help conclude that M_{II} in the BH region of Schwarzschild spacetime

Penrose diagram of Schwarzschild spacetime

(before maximal extension)

suggestion: timelike circular geodesics to



Kerr Metric solution with less symmetries for $R_{\mu\nu}=0$

1963, Roy Kerr

$$ds^2 = \left(-1 - \frac{2mr}{\ell^2}\right)dt^2 - \frac{4am\, r\sin^2\theta}{\ell^2}\, dt\, d\varphi + \frac{\ell^2}{\Delta}dr^2 + \ell^2 d\theta^2 +$$
$$+ \left(\ell^2 + a^2 + \frac{2a^2 m\, r \sin^2\theta}{\ell^2}\right)\sin^2\theta\, d\varphi^2$$

with $m>0$ $\quad 0 < \underset{\text{spin parameter}}{a} < m$

$$\ell^2 = r^2 + a^2\cos^2\theta \qquad \Delta = r^2 - 2mr + a^2 = (r-r_+)(r-r_-) \qquad r_\pm = m \pm \sqrt{m^2-a^2}$$

Kerr is pseudo stationary and asymmetric (Killing vector $\xi = \partial_t$
Killing vector $\eta = \partial_\varphi$

Kerr metric in astrophysics

For rotating stars and rotating black holes

Carter-Robinson-theorem

Let (M,g) be a 4d asymptotically flat spacetime containing a BH with a connected event horizon H. If

(M,g) is stationary & axisymmetric

g solves $R_{\mu\nu} = 0$

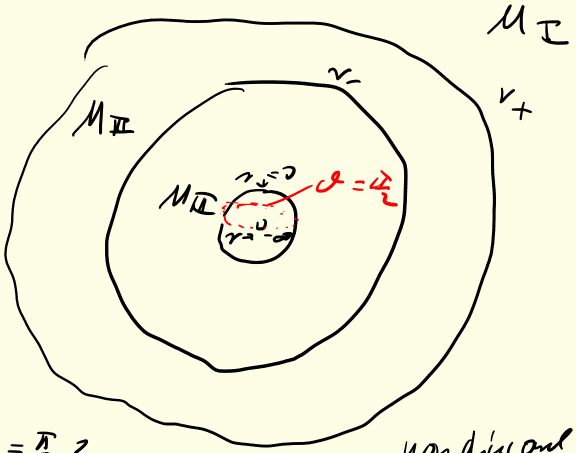
there is no closed causal curve in $\langle\langle M \rangle\rangle$ (no time travel)

then $\langle\langle M \rangle\rangle = \langle\langle \text{Kerr} \rangle\rangle$

problems with the metric $g_{\mu\nu}$

$$\Delta = 0 \;\Rightarrow\; r = r_\pm \;, \quad \rho^2 = 0 \;\Rightarrow\; r = 0 \text{ and } \vartheta = \frac{\pi}{2}$$

Define the subregion

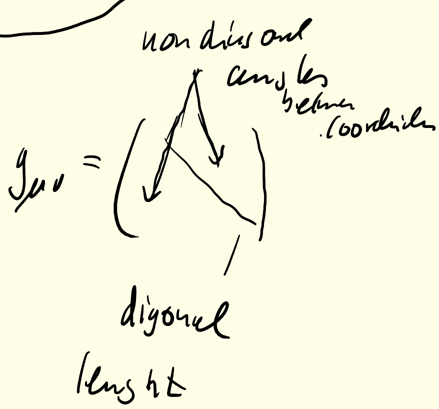


$$M_I = \mathbb{R} \times (r_+, \infty) \times S^2$$

$$M_{II} = \mathbb{R} \times (r_-, r_+) \times S^2$$

$$M_{III} = \mathbb{R} \times (-\infty, r_-) \times S^2$$

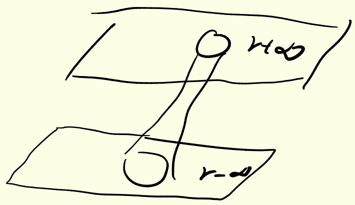
Ring singularity $R = \{ p \in M, r = 0, \vartheta = \frac{\pi}{2} \}$



$r \to \infty$ some wormhole

$r \to -\infty$

$a \to 0$



Ring singularity is a curvature singularity

$$\text{Kretschmann} \xrightarrow[\substack{r \to 0 \\ \vartheta = \frac{\pi}{2}}]{} \pm\infty$$

$$K \sim \frac{1}{\rho}$$

So it can not be curved. On the contrary, the divergence at $r_\pm$ are mere coordinate singularities that disappears, when switching to some null coordinates

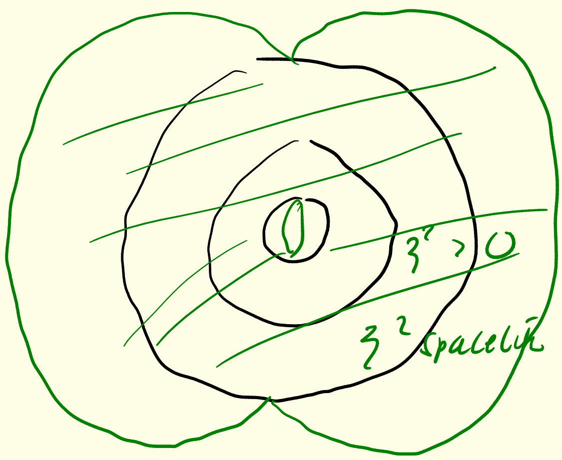
$\xi^2 = \partial_t^2 = g_{tt}$ for investigating the stationary of the blackhole

– strictly stationary

otherwise pseudo-stationary

$\iff r < r_{\mathcal{E}-}(\vartheta)$ or $r > r_{\mathcal{E}+}(\vartheta)$

where $r_{\mathcal{E}\pm} = m \pm \sqrt{m^2 - a^2\cos^2\vartheta}$



The region where ξ is spacelike is called the ergoregion. And because it is part of $\langle\langle \text{kerr} \rangle\rangle$ it is possible to extend energy from the BH by sending particles into the ergoregion according to the so-called Penrose process.