2024-03-04 lecture

Gravitational Waves

$$g_{\mu\nu} = \underbrace{g^0_{\mu\nu}}_{\text{background metric}} + \overset{\text{perturbation}}{h_{\mu\nu}} \qquad |h_{\mu\nu}| \ll g^0_{\mu\nu}$$

solution of $G_{\mu\nu}$

external source related. the original space time

here we take $g^0_{\mu\nu} = \eta_{\mu\nu} = -c^2dt^2 + dx^2 + dy^2 + dz^2$

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$$

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu} \quad | g^{0\mu\nu}$$

$$g_{\mu\nu}g^{0\mu\nu} = \delta_\nu^\nu + h_{\mu\nu}g^{0\mu\nu}$$

$$T_{\mu\nu} = \underbrace{T^0_{\mu\nu}}_{\text{for background solution}} + T^{pert}_{\mu\nu} \quad \text{- for perturbation}$$

$$|T^{pert}_{\mu\nu}| \ll |T^0_{\mu\nu}|$$

$$R_{\mu\nu} = \frac{\partial}{\partial x^\alpha}\Gamma^\alpha_{\mu\nu} - \frac{\partial}{\partial x^\nu}\Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\sigma\alpha}\Gamma^\sigma_{\mu\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\mu\alpha}$$

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}) \qquad {}_{,\mu} = \frac{\partial}{\partial x^\mu}$$

$$= \Gamma^\lambda_{\mu\nu}(\eta) + \underbrace{\frac{1}{2}\eta^{\lambda\rho}(h_{\rho\mu,\nu} + h_{\rho\nu,\mu} - h_{\mu\nu,\rho}) - \frac{1}{2}\eta^{\lambda\rho}(\overset{0}{\eta_{\rho\mu,\nu}} + \overset{0}{\eta_{\rho\nu,\mu}} - \overset{0}{\eta_{\mu\nu,\rho}})}_{\Gamma^\lambda_{\mu\nu}(h)} + O(h^2)$$

$$g^{\mu\nu} = g^{\mu\nu 0} - h^{\mu\nu}$$

$$R_{\mu\nu} = \overset{\to 0}{R_{\mu\nu}(\eta)} + \Gamma^\alpha_{\mu\nu,\alpha}(h) - \Gamma^\alpha_{\mu\alpha,\nu}(h) + \overset{\to 0}{\Gamma^\alpha_{\sigma\alpha}(\eta)}\Gamma^\sigma_{\mu\nu}(h) + \Gamma^\alpha_{\sigma\alpha}(h)\overset{\to 0}{\Gamma^\sigma_{\mu\nu}(\eta)}$$
$$- \Gamma^\alpha_{\sigma\nu}(\eta)\Gamma^\sigma_{\mu\nu}(h) - \Gamma^\alpha_{\sigma\nu}(\eta)\Gamma^\sigma_{\mu\alpha}(h) - \Gamma^\alpha_{\sigma\nu}(h)\underset{\to 0}{\Gamma^\sigma_{\mu\alpha}(\eta)} + O(h^2)$$

$$\underset{\to 0}{R_{\mu\nu}(\eta)} = \frac{8\pi G}{c^4}\left(T^0_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^0\right) \to 0$$

$$R_{\mu\nu}(h) = \frac{8\pi G}{c^4}\left(T^{\text{part}}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{\text{part}}\right) + O(h^2)$$

the term $h_{\mu\nu}T^0$ neglected

$$\Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} = \frac{1}{2}\eta^{\alpha\rho}\left(\frac{\partial^2 h_{\rho\nu}}{\partial x^\alpha\partial x^\nu} + \frac{\partial^2 h_{\rho\nu}}{\partial x^\alpha\partial x^\mu} - \frac{\partial^2 h_{\mu\nu}}{\partial x^\alpha\partial x^\rho}\right)$$

$$- \frac{1}{2}\eta^{\alpha\rho}\left(\frac{\partial^2 h_{\rho\mu}}{\partial x^\nu\partial x^\alpha} + \frac{\partial^2 h_{\rho\alpha}}{\partial x^\nu\partial x^\mu} - \frac{\partial^2 h_{\mu\sigma}}{\partial x^\nu\partial x^\rho}\right)$$

$$= \frac{1}{2}\left(\frac{\partial^2 h^\alpha_\mu}{\partial x^\alpha\partial x^\nu} + \frac{\partial^2 h^\alpha_\nu}{\partial x^\alpha\partial x^\mu} \underbrace{- \eta^{\alpha\rho}\frac{\partial^2 h_{\mu\nu}}{\partial x^\alpha\partial x^\rho}}_{\Box_F h_{\mu\nu}} - \frac{\partial^2 h^\alpha_\mu}{\partial x^\nu\partial x^\alpha} \underbrace{- \frac{\partial^2 h^\rho_\alpha}{\partial x^\nu\partial x^\mu}}_{\text{the same thing}} + \frac{\partial^2 h^\rho_\mu}{\partial x^\nu\partial x^\rho}\right)$$

$$\Box_F = \eta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta} = -\frac{\partial^2}{c^2\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$R_{\mu\nu} = \frac{1}{2}\left(\frac{\partial^2 h^\alpha_\mu}{\partial x^\alpha\partial x^\nu} + \frac{\partial^2 h^\alpha_\nu}{\partial x^\alpha\partial x^\mu} - \Box_F h_{\mu\nu} - \frac{\partial^2 h^\alpha_\alpha}{\partial x^\nu\partial x^\mu}\right)$$

$$\Box_F h_{\mu\nu} - \underbrace{\left(\frac{\partial^2 h^\alpha_\mu}{\partial x^\alpha\partial x^\nu} + \frac{\partial^2 h^\alpha_\nu}{\partial x^\alpha\partial x^\mu} - \frac{\partial^2 h}{\partial x^\mu\partial x^\nu}\right)}_{\to 0} = -\frac{16\pi G}{c^4}\left(T^{\text{part}}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{\text{part}}\right)$$

$x^{\mu'} = x^\mu + \varepsilon^\mu$ infinitesimal small

$|h'_{\mu\nu}| \ll 1$

harmonic gauge

$$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0$$

$$\frac{\partial x^{\alpha'}}{\partial x^\mu} = \delta^\alpha_\mu + \frac{\partial\varepsilon^\alpha}{\partial x^\mu}$$

$$g_{\mu\nu} = g'_{\alpha\beta}\frac{\partial x^{\alpha'}}{\partial x^\mu}\frac{\partial x^{\beta'}}{\partial x^\nu}$$

$$= (\eta_{\alpha\beta} + h'_{\alpha\beta})\left(\delta^\alpha_\mu + \frac{\partial\varepsilon^\alpha}{\partial x^\mu}\right)\left(\delta^\beta_\nu + \frac{\partial\varepsilon^\beta}{\partial x^\nu}\right)$$

$$g_{\mu\nu} = \eta_{\alpha\beta} + h'_{\alpha\beta} + \frac{\partial\varepsilon_\nu}{\partial x^\mu} + \frac{\partial\varepsilon_\mu}{\partial x^\nu} + O(h^2) = \eta_{\mu\nu} + h_{\mu\nu}$$

$$\Rightarrow \quad h'_{\mu\nu} = h_{\mu\nu} - \varepsilon_{\nu,\mu} - \varepsilon_{\mu,\nu} + O(h^2)$$

$$g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0 \rightarrow \frac{1}{2}\eta^{\mu\nu}\eta^{\lambda\rho}\left(\frac{\partial h_{\rho\mu}}{\partial x^\nu} + \frac{\partial h_{\rho\nu}}{\partial x^\mu} - \frac{\partial h_{\mu\nu}}{\partial x^\rho}\right)$$

$$\eta^{\mu\nu} - h^{\mu\nu} \qquad = \frac{1}{2}\eta^{\lambda\rho}\left(\frac{\partial h^{\nu}_{\rho}}{\partial x^\nu} + \frac{\partial h^{\mu}_{\rho}}{\partial x^\mu} - \frac{\partial h}{\partial x^\rho}\right)$$

$$= \frac{1}{2}\eta^{\lambda\rho}\left(2\frac{\partial h^{\mu}_{\rho}}{\partial x^\mu} - \frac{\partial h}{\partial x^\rho}\right) = 0$$

$$\frac{\partial}{\partial x^\mu} h^{\mu}_{\nu} = \frac{1}{2}\frac{\partial h}{\partial x^\nu}$$

$$\frac{\partial}{\partial x^\rho}\frac{\partial}{\partial x^\mu} h^{\mu}_{\nu} = \frac{1}{2}\frac{\partial}{\partial x^\rho}\frac{\partial h}{\partial x^\nu}$$

then the term goes to $\underbrace{\left(\frac{\partial^2 h^{\alpha}_{\mu}}{\partial x^\alpha \partial x^\nu} + \frac{\partial^2 h^{\alpha}_{\nu}}{\partial x^\alpha \partial x^\mu} - \frac{\partial^2 h}{\partial x^\mu \partial x^\nu}\right)}_{0}$

$$\Box_F h_{\mu\nu} = -\frac{16\pi G}{c^4}\left(T^{pert}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{pert}\right)$$

$$\frac{\partial}{\partial x^\mu} h^{\mu}_{\nu} = \frac{1}{2}\frac{\partial h}{\partial x^\mu} \qquad\qquad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$$

$$\Box_F \bar{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\Box_F h = -\frac{16\pi G}{c^4}\left(T^{pert}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{pert}\right)$$

$$\eta^{\mu\nu}\left(\Box_F h_{\mu\nu}\right) = \eta^{\mu\nu}\left(\frac{16\pi G}{c^4}\left(T^{pert}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{pert}\right)\right)$$

$$\Box_F h = -\frac{16\pi G}{c^4}\left(T^{pert} - \frac{1}{2}\underbrace{\eta_{\mu\nu}\eta^{\mu\nu}}_{4}T^{pert}\right)$$

$$\Box_F h = +\frac{16\pi G}{c^4}T^{pert}$$

$$\Box_F \bar{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\Box_F h = -\frac{16\pi G}{c^4}\left(T^{pert}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{pert}\right)$$

$$\Box_F \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}^{part}$$

$$\partial_\mu \bar{h}^\mu_\nu = 0$$

outside of source $\Box_F \bar{h}_{\mu\nu} = 0$ $\Box_F = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$

wave equation metric is of the form of wave

$$\frac{\partial}{\partial x^\mu} \bar{h}^\mu_\nu + \underbrace{\frac{1}{2}\eta^\mu{}_\nu h}_{} = \underbrace{\frac{1}{2}\frac{\partial}{\partial x^\mu} h}_{}$$

$$\frac{\partial}{\partial x^\mu} \bar{h}^\mu_\nu = 0$$

Gamma-ray burst GRB170817A both signals at the same time detected

Gravitational waves GW170817

Grav. detection

$$-3\cdot 10^{-15} < \frac{v_{GW} - v_{EM}}{v_{EM}} < 1\cdot 10^{-16}$$

Wave solution

$$\bar{h}_{\mu\nu} = Re\{\underbrace{A_{\mu\nu}}_{\text{amplitude}} e^{i\overbrace{k_\alpha}^{\text{wave vector}} x^\alpha}\} \qquad \vec{k} = (\overbrace{\tfrac{\omega}{c}}^{\text{frequency}}, \underline{k})$$

① $\vec{k}$ is null vector

$$\Box_F \bar{h}_{\mu\nu} = 0 = \eta^{\alpha\beta} \underbrace{\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta} e^{ik_\gamma x^\gamma}}_{} A_{\mu\nu} = A_{\mu\nu}\eta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\left(ik_\beta e^{ik_\gamma x^\gamma}\right)$$

$$ik_\gamma \underbrace{\frac{\partial x^\gamma}{\partial x^\beta}}_{\delta^\gamma_\beta} e^{ik_\gamma x^\gamma} = ik_\beta e^{ik_\gamma x^\gamma}$$

$$= -A_{\mu\nu}\eta^{\alpha\beta} k_\beta k_\gamma \frac{\partial x^\gamma}{\partial x^\alpha} e^{ik_\gamma x^\gamma} = -A_{\mu\nu} \underbrace{\eta^{\alpha\beta} k_\alpha k_\beta}_{=0\ =\ \eta^{\alpha\beta}k_\alpha k_\beta} e^{ik_\gamma x^\gamma} = 0$$

$\rightarrow \vec{k}$ is null vector

② $\vec{k} \perp A_{\mu\nu}$

use harmonic gauge property

$$\partial_\mu \bar{h}^\mu{}_\nu = 0 = \eta^{\mu\alpha} \partial_\mu \bar{h}_{\alpha\nu} = \eta^{\mu\alpha} \partial_\mu \left(A_{\alpha\nu} e^{ik_\gamma x^\gamma} \right)$$

$$= \eta^{\mu\alpha} A_{\alpha\nu}\, ik_\mu\, e^{ik_\gamma x^\gamma} = i \underbrace{A^\mu{}_\nu k_\mu}\, e^{ik_\gamma x^\gamma} = 0$$

$$A^\mu{}_\nu k_\mu = 0$$

propagation along x - axes

$$\longrightarrow x^1 = x \qquad \bar{h}^\mu{}_\nu = \bar{h}^\mu{}_\nu(\chi(t,x)) \qquad \chi = t - \frac{x}{c}$$

$$0 = \partial_\mu \bar{h}^\mu{}_\nu = -\frac{1}{c}\partial_t \bar{h}^t{}_\nu + \partial_x \bar{h}^x{}_\nu = \frac{1}{c}\partial_\chi \big(\underbrace{\bar{h}^t{}_\nu - \bar{h}^x{}_\nu}_{\text{const}} \big) = 0$$

$$\bar{h}^t{}_t = \bar{h}^x{}_t, \quad \bar{h}^t{}_x = \bar{h}^x{}_x \quad \bar{h}^t{}_y = \bar{h}^x{}_y \quad \bar{h}^t{}_z = \bar{h}^x{}_z \quad \text{4 conditions}$$

$$d\chi = dt - \frac{dx}{c}$$

$$d\chi + \frac{dx}{c} = dt \qquad dx = c(dt - d\chi)$$

$$= -\frac{1}{c}\left(\frac{\partial}{\partial \chi} + c\frac{\partial}{\partial x} \right) \bar{h}^t{}_\nu + \frac{\partial}{\partial x} \bar{h}^x{}_\nu$$

$$x^{\mu'} = x^\mu + \varepsilon^\mu \qquad h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu + O(h^2)$$

with this trafo we can reduce the degrees of freedom about 4

trace use

$$\bar{h}'_{\mu\nu} = h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h' = h_{\mu\nu} - \partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu - \frac{1}{2}\eta_{\mu\nu} h'$$

$$h' = \eta^{\mu\nu} h'_{\mu\nu} = h'^\nu{}_\nu = h - 2\partial_\alpha \varepsilon^\alpha \qquad (h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h) = \bar{h}_{\mu\nu}$$

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu + \eta_{\mu\nu} \partial_\alpha \varepsilon^\alpha$$

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \underbrace{\partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu}\partial_\alpha \xi^\alpha}_{=0} = \bar{h}_{\mu\nu}$$

$$= 0 = \Box \xi^\mu \quad 4 \text{ Conditions} \quad \text{not physical Conditions}$$

TT gauge transverse $\bar{h}^t{}_x = \bar{h}^t{}_y = \bar{h}^t{}_z = 0$

traceless $\bar{h}_{yy} = -\bar{h}_{zz} \rightarrow \bar{h}^\mu{}_\mu = 0$

harmonic gauge transverse

$$\bar{h}_{\mu\nu} = \begin{array}{c|cccc} & t & x & y & z \\ \hline t & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 \\ y & 0 & 0 & \bar{h}_{yy} & \bar{h}_{yz} \\ z & 0 & 0 & \bar{h}_{yz} & -\bar{h}_{yy} \end{array} \qquad 4\times 4$$

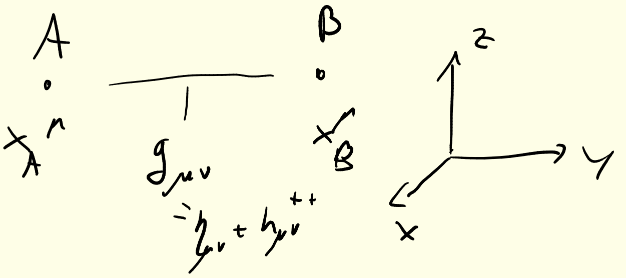
$\Rightarrow \bar{h}_{yy}, \bar{h}_{yz}$

traceless

symmetry

$$\bar{h}^\mu{}_\mu = 0 = h^\mu{}_\mu - \frac{1}{2}\eta^\mu{}_\mu h^\lambda{}_\lambda = -h^\mu{}_\mu$$

Distance change between to points



$t<0$ at rest

$t=0$ a plane GW along $\vec{x}$

$$\delta x^\mu = \delta x^\mu_B - \delta x^\mu_A$$

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = (\eta_{\mu\nu} + h^{TT}_{\mu\nu})\, dx^\mu dx^\nu$$

$$\{x^{\mu'}_A\} : ds^2 = \eta'_{\alpha\beta}\, dx^{\alpha'} dx^{\beta'} + O(h^2) \qquad t_A = \frac{\tau}{c} \qquad g_{\mu'\nu'}|_A = \eta_{\mu'\nu'}$$

proper time

$$\delta x^{i'} = \delta x^{i'}_B \qquad \left.\frac{dx^{\mu'}}{d\tau}\right|_A = (1,0,0,0)$$

$$\frac{D^2 \delta x^\mu}{D\tau^2} = R^\mu{}_{\alpha\beta\gamma} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\delta x^\gamma \quad \text{Geodesic deviation}$$

$$\frac{d^2 \delta x^i}{dt^2} = R^i_{00j}\, \delta x^j \quad \text{spatial distance}$$

$$R_{\alpha\kappa\lambda\mu} = \frac{1}{2}\left(\frac{\partial^2 g_{\alpha\mu}}{\partial x^\kappa \partial x^\lambda} + \frac{\partial^2 g_{\kappa\lambda}}{\partial x^\alpha \partial x^\mu} - \frac{\partial^2 g_{\alpha\lambda}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\kappa\mu}}{\partial x^\alpha \partial x^\lambda} \right)$$

$$+ g_{\nu\sigma}\left(\Gamma^\nu_{\kappa\lambda} \Gamma^\sigma_{\alpha\mu} - \Gamma^\nu_{\kappa\mu} \Gamma^\sigma_{\alpha\lambda} \right)$$

$$= \frac{1}{2}\left(\frac{\partial^2 h_{\alpha\mu}}{\partial x^\kappa \partial x^\lambda} + \underbrace{\frac{\partial^2 h_{\kappa\lambda}}{\partial x^\alpha \partial x^\mu}}_{\partial^2 h_{00} \to 0} - \underbrace{\frac{\partial^2 h_{\alpha\lambda}}{\partial x^\kappa \partial x^\mu}}_{h_{i0} \to 0} - \underbrace{\frac{\partial^2 h_{\kappa\mu}}{\partial x^\alpha \partial x^\lambda}}_{h_{0j} \to 0} + O(h^2) \right)$$

$$R_{i00j} = \frac{1}{2} \frac{\partial^2 h^{TT}_{ij}}{\partial x^0 \partial x^0}$$

$$\delta x^i = \delta x^i_0 + \delta x^i_1$$

$$R^\lambda_{00n} = \eta^{\lambda i} R_{i00n} = \frac{1}{2} \eta^{\lambda i} \frac{\partial^2 h^{TT}_{in}}{c^2 \partial t^2}$$

integration of $\dfrac{d^2 \delta x^1}{c\,dt^2} = \dfrac{1}{2} \eta^{\lambda i} \dfrac{\partial^2 h^{TT}_{ij}}{\partial x^0 \partial x^0}$

with δx^i $\quad \delta x^i = \delta x^i_0 + \frac{1}{2} \eta^{ji} h^{TT}_{ik} \delta^k_0$

$$\delta x^1 = \delta x^1_0 + \underbrace{\frac{1}{2} \eta^{11} h^{TT}_{11} \delta x^1_0}_{=0} = \delta x^1_0$$

$$\delta x^2 = \delta x^2_0 + \frac{1}{2} \underbrace{\eta^{22}}_{=1} \left(h^{TT}_{22} \delta x^2_0 + h^{TT}_{23} \delta x^3_0 \right)$$

$$\delta x^3 = \delta x^3_0 + \frac{1}{2} \underbrace{\eta^{33}}_{=1} \left(h^{TT}_{32} \delta x^2_0 + h^{TT}_{33} \delta x^3_0 \right)$$

$$= \delta x^3_0 + \frac{1}{2}\left(h^{TT}_{32} \delta x^2_0 - h^{TT}_{22} \delta x^3_0 \right)$$