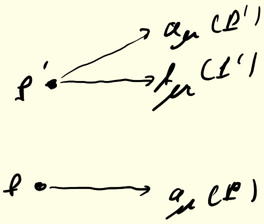
lecture-2024-02-27

## Parallel transport



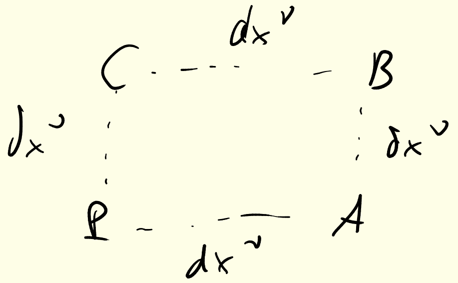

$$a_\mu(P') - A_\mu(P') = (a_{\mu,\nu} - \Gamma^\lambda_{\mu\nu} a_\lambda)\, dx^\nu$$

$$\delta a_\mu = \Gamma^\lambda_{\mu\nu} a_\lambda\, dx^\lambda \quad \text{covariant}$$

$$\delta a^\mu = -\Gamma^\mu_{\lambda\nu} a^\lambda dx^\nu \quad \text{contravariant}$$

where $A_\mu(P') = a_\mu(P) + \delta a_\mu(P)$

## Curvature




P to A to B (PAB)

P to C to B (PCB)

$$a^\lambda(A) = a^\lambda(P) - \Gamma^\lambda_{\mu\nu}(P)\, a^\mu(P)\, dx^\nu$$

$$a^\lambda(B) = a^\lambda(A) - \Gamma^\lambda_{\rho\sigma} a^\rho(A)\, \delta x^\sigma$$

if $dx^\nu$ small, so Taylor expansion around A

$$\Gamma^\lambda_{\rho\sigma}(A) = \Gamma^\lambda_{\rho\sigma}(P) + \Gamma^\lambda_{\rho\sigma,\mu}(P)\, dx^\mu$$

all at P

$$a^\lambda(B) = a^\lambda - \Gamma^\lambda_{\mu\nu} a^\mu dx^\nu - \Gamma^\lambda_{\rho\sigma} a^\rho \delta x^\sigma + \Gamma^\lambda_{\rho\sigma}\Gamma^\rho_{\beta\nu} a^\beta dx^\nu \delta x^\sigma + \Gamma^\lambda_{\rho\sigma,\tau} a^\rho dx^\tau \delta x^\sigma + O(dx^3)$$

via PCB $\underline{a}^\lambda(B) = a^\lambda - \Gamma^\lambda_{\mu\nu} a^\mu \delta x^\nu - \Gamma^\lambda_{\rho\sigma} a^\rho dx^\sigma + \Gamma^\lambda_{\rho\sigma}\Gamma^\rho_{\beta\nu} a^\beta \delta x^\nu dx^\sigma + \Gamma^\lambda_{\rho\sigma,\tau} a^\rho \delta x^\tau dx^\sigma$

$$\delta a^\lambda = a^\lambda(B) - \underline{a}^\lambda(B)$$
$$= a^\beta (dx^\nu \delta x^\sigma - dx^\sigma \delta x^\nu)(\Gamma^\lambda_{\rho\sigma}\Gamma^\rho_{\beta\nu} + \Gamma^\lambda_{\beta\sigma,\nu})$$

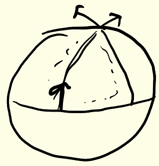
exchange $\nu \to \sigma$
$\sigma \to \nu$

$$\delta a^\lambda = a^\beta (dx^\sigma \delta x^\nu - dx^\nu \delta x^\sigma)(\underline{\Gamma^\lambda_{\rho\nu}\Gamma^\rho_{\beta\sigma}} + \Gamma^\lambda_{\beta\nu,\sigma})$$

$$\delta a^\lambda = -\frac{1}{2} a^\beta R^\lambda_{\beta\nu\sigma}(dx^\sigma \delta x^\nu - dx^\nu \delta x^\sigma)$$

$$R^\lambda_{\beta\nu\sigma} = -\Gamma^\lambda_{\beta\nu,\sigma} + \Gamma^\lambda_{\beta\sigma,\nu}$$

Another definition



$$\nabla_\gamma \nabla_\beta V_\alpha - \nabla_\beta \nabla_\gamma V_\alpha = R^{\sigma}{}_{\alpha\beta\gamma} V_\sigma$$

remarks:

-in flat space $R^{\sigma}{}_{\alpha\beta\gamma} = 0$

- in 4D 256 components
for easier computation you can use symmetries

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} \qquad R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$$

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} \qquad R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0$$

number of independent components

| | | | |
|---|---|---|---|
| dimension | 2 | 3 | 4 |
| components | 1 | 6 | 20 |

Bianchi identity $\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0$

$$\nabla_{[e} R_{ab]cd}$$

Ricci tensor: $R_{ab} = R^{c}{}_{abc}$

Ricci scalar: $R = R_a{}^a$

$$\nabla_e R_{bc} - \nabla_c R_{be} + \nabla_a R^{a}{}_{bec} = 0$$

$$\nabla_b R^{b}{}_{c} - \nabla_c R + \nabla_a R^{ab}{}_{bc} = 0$$

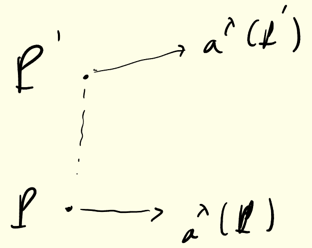
$$\nabla_a R^{ab}{}_{bc} = \nabla_a R^{ba}{}_{cb} = \nabla_a R^{a}{}_{c}$$

$$\Rightarrow 2\nabla_b R^{b}{}_{c} - \nabla_c R = \nabla_b \left(2R_c{}^b - \delta_c{}^b R\right) = 0$$

$$\Rightarrow \nabla_b \left(R^{bc} - \tfrac{1}{2} g^{bc}\right) = 0$$

$G^{ab} = R^{ab} - \frac{1}{2} g^{ab} R$ Einstein tensor

## Christoffel symbols




lenghts of $a^\lambda(P)$

$$g_{\mu\nu} a^\mu a^\nu \big|_P = g_{\rho\nu} a^\rho a^\nu \big|_{P'}$$

$$g_{\mu\nu}(P') \simeq g_{\mu\nu}(P) + g_{\mu\nu,\rho}(P)\, dx^\rho$$

$$a^\mu(P') = a^\mu(P) - \Gamma^\mu_{\sigma\rho}(P)\, a^\sigma(P)\, dx^\rho$$

$$\Rightarrow g_{\mu\nu} a^\mu a^\nu \big|_{P'} = (g + g\, dx)(a + a\, dx)(a + a\, dx)$$

$$\Rightarrow \underbrace{(g_{\mu\nu,\rho} - g_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - g_{\sigma\nu}\Gamma^\sigma_{\mu\rho})}_{\nabla_\rho g_{\mu\nu}} a^\mu a^\nu dx^\rho = 0$$

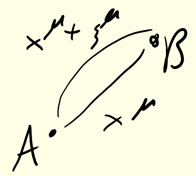
$$\underbrace{\nabla_\rho g_{\mu\nu}}_{=0} \; a^\mu a^\nu dx^\rho = 0$$

for the same lenght of the vector, the covariant derivative must be vanish

$$\Gamma^\alpha_{\mu\rho} = \frac{1}{2} g^{\alpha\nu} (g_{\mu\nu,\rho} + g_{\nu\rho,\mu} - g_{\rho\mu,\nu})$$

## Geodesics

$$S = \int_A^B ds = \int_A^B \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda$$




$$\bar{x}^\mu(\lambda) = x^\mu(\lambda) + \varepsilon \xi^\mu(\lambda)$$

$$\bar{S} = \int_A^B \Big( \underbrace{g_{\mu\nu}(\bar{x}^\alpha) \frac{d\bar{x}^\mu}{d\lambda} \frac{d\bar{x}^\nu}{d\lambda}}_{f(\bar{x}^\alpha, \bar{u}^\alpha)} \Big)^{1/2} d\lambda$$

$$u^\mu = \dot{x}^\mu = \frac{dx^\mu}{d\lambda} \qquad \bar{u}^\mu = \dot{\bar{x}}^\mu = \dot{x}^\mu + \varepsilon \dot{\xi}^\mu = u^\mu + \varepsilon \dot{\xi}^\mu$$

Taylor expansion

(1) $$f(\bar{x}^\alpha, \bar{u}^\alpha) = f(x^\alpha, u^\alpha) + \varepsilon \Big( \xi^\alpha \frac{\partial f}{\partial x^\alpha} + \dot{\xi}^\alpha \frac{\partial f}{\partial u^\alpha} \Big) + O(\varepsilon^2)$$

(2) $$\frac{d}{d\lambda}\Big( \frac{\partial f}{\partial u^\alpha} \xi^\alpha \Big) = \frac{\partial f}{\partial u^\alpha} \dot{\xi}^\alpha + \frac{d}{d\lambda}\Big( \frac{\partial f}{\partial u^\alpha} \Big) \xi^\alpha$$

$$f(\bar{x}^\alpha, \bar{u}^\alpha) - f(x^\alpha, u^\alpha) = \varepsilon\left(\frac{\partial f}{\partial x^\alpha} - \frac{d}{d\lambda}\frac{\partial f}{\partial u^\alpha}\right)\xi^\alpha + \varepsilon\frac{d}{d\lambda}\left(\frac{\partial f}{\partial u^\alpha}\xi^\alpha\right)$$

$$\delta S = \bar{S} - S = \int_A^B \delta f\, ds = \int_A^B (f(\bar{x}^\alpha, \bar{u}^\alpha) - f(x^\alpha, u^\alpha))\, d\lambda$$

$$= \varepsilon\int_A^B \left[\frac{\partial f}{\partial x^\alpha} - \frac{d}{d\lambda}\left(\frac{\partial f}{\partial u^\alpha}\right)\right]\xi^\alpha d\lambda + \underbrace{\varepsilon\int_A^B \frac{d}{d\lambda}\left(\frac{\partial f}{\partial u^\alpha}\xi^\alpha\right)d\lambda}_{= \varepsilon \frac{\partial f}{\partial u^\alpha}\xi^\alpha\Big|_A^B = 0}$$

if $\xi(A) = \xi(B) = 0$

Extremum for $S$

so $\delta S = 0 \Rightarrow \frac{d}{d\lambda}\left(\frac{\partial f}{\partial u^\alpha}\right) - \frac{\partial f}{\partial x^\alpha} = 0$

Lagrange function of particle (squared Lagrangian)

$$L = g_{\mu\nu} u^\mu u^\nu = f^2$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial u^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} = 0 \qquad \frac{\partial L}{\partial u^\alpha} = 2 g_{\mu\alpha} u^\mu$$

$$\frac{\partial L}{\partial x^\alpha} = g_{\mu\nu,\alpha} u^\mu u^\nu$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial u^\alpha}\right) = 2\frac{d g_{\mu\alpha}}{ds} u^\mu + 2 g_{\mu\alpha}\frac{du^\mu}{ds}$$

$$= 2 g_{\mu\alpha,\nu} u^\nu u^\mu + 2 g_{\mu\alpha}\frac{du^\mu}{ds}$$

$$g_{\mu\alpha,\nu} u^\nu u^\mu = g_{\nu\alpha,\mu} u^\mu u^\nu$$

$$= g_{\mu\alpha,\nu} u^\nu u^\mu + g_{\nu\alpha,\mu} u^\mu u^\nu + 2 g_{\mu\alpha}\frac{du^\mu}{ds}$$

using Euler Lagrange $\frac{d}{ds}\frac{\partial L}{\partial u^\alpha} = \frac{\partial L}{\partial x^\alpha}$

$$2 g_{\mu\alpha}\frac{du^\mu}{ds} + g_{\mu\alpha,\nu} u^\nu u^\mu + g_{\nu\alpha,\nu} u^\nu u^\mu + g_{\nu\alpha,\mu} u^\mu u^\nu = g_{\mu\nu,\alpha} u^\mu u^\nu \quad | \cdot g^{\alpha\rho}$$

$$\Rightarrow \frac{du^\rho}{ds} + \underbrace{\frac{1}{2} g^{\alpha\rho}\left[g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right]}_{\Gamma^\rho_{\mu\nu}} u^\mu u^\nu = 0$$

$$\frac{du^\rho}{ds} + \Gamma^\rho_{\mu\nu} u^\mu u^\nu = 0$$

**Geodesic equation**

# General Relativity

Newton $\vec{F} = m_I \vec{g} = -m_G \nabla\phi$

Poisson equation $\nabla^2\phi = 4\pi G\rho$

$$\frac{d^2\vec{x}}{dt^2} = -\frac{m_g}{m_I}\nabla\phi$$

Newtons equivalence principle (EP) $m_G = m_I$

$$\vec{\nabla}^2 \to \Box^2 = \nabla_\alpha\nabla^\alpha = \eta_{\mu\nu}\nabla^\mu\nabla^\nu \hat{=} \partial_t^2 - \nabla^2$$

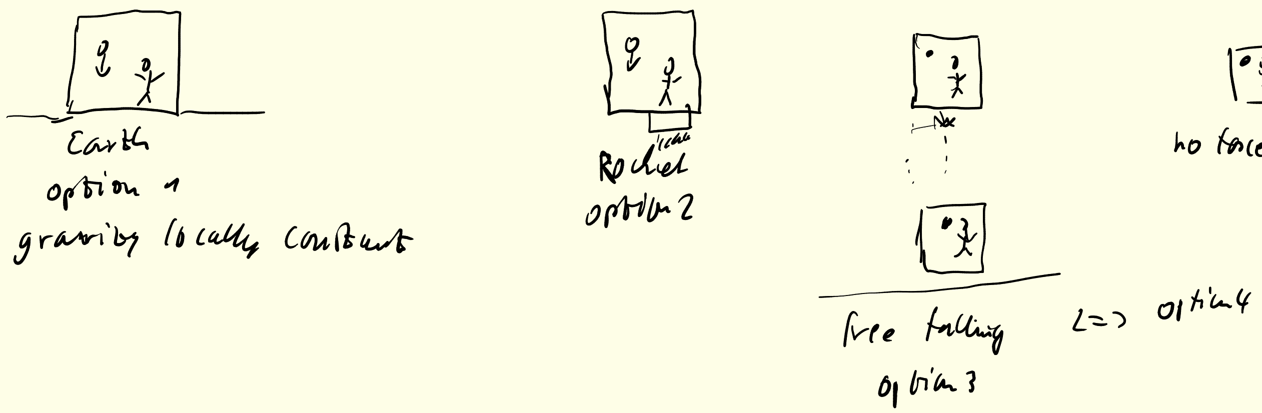
$\rho_0$ in general not the rest mass density, Volume changes in special Relativity

$$\rho = \gamma^2\rho_0$$

with these assumptions we can not rewrite the equations in general relativity way

## More on EP

example person in elevator or box in different situations




## weak EP - universality of free fall

$m_G = m_I$

- trajectories of freely falling test particle
are independent of the structure and composition

## Strong EP

Review by Clifford Will

- weak EP is valid and for gravitating particles
- outcome of experiment independent of velocity of apparatus
- independent where and when

energy momentum tensor

$$T^{\mu\nu} = \rho u^\mu u^\nu = \rho_0 \gamma^2 u^\mu u^\nu$$ tensor rank 2 symmetric

$$T^{00} = \rho u^0 u^0 = \gamma^2 \rho c^2$$

$$T^{0i} = T^{i0} = \gamma^2 \rho c u^i$$ energyflux $\frac{1}{c}$ in t direction

$$T^{ij} = \gamma^2 \rho u^i u^j$$

Minkowski $\partial_\mu T^{\mu\nu} = 0$ why

Assume a perfect fluid

$$T^{\mu\nu} = \left(p + \frac{\rho}{c^2}\right) u^\mu u^\nu - p g^{\mu\nu}$$

$$\partial_\mu (c^2) = \partial_\mu (u_\nu u^\nu) = \partial_\mu u_\nu u^\nu + u^\nu \partial_\mu u_\nu = 2 (\partial_\mu u^\nu) u_\nu$$

$$u^\nu (\partial_\mu u_\nu) = \eta^{\nu\alpha} u_\alpha (\partial_\mu u_\nu) = u_\alpha (\partial_\mu \eta^{\nu\alpha} u_\nu) = u_\alpha (\partial_\mu u^\alpha) = u_\nu (\partial_\mu u^\nu)$$

$$\partial_t \rho + \nabla (\rho \vec{u}) = 0$$ continuity equation

$$\rho (\partial_t + \vec{u} \cdot \nabla) = \vec{\nabla} \cdot p$$ Euler equation

## Einstein field equations

earlier $\nabla_\beta \left(R^{\beta\sigma} - \frac{1}{2} g^{\beta\sigma} R\right) = 0$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -k T_{\mu\nu}$$

$$k = \frac{8\pi G}{c^4}$$

alternative

$$\nabla_\beta \left(R^{\beta\sigma} - \frac{1}{2} g^{\beta\sigma} R + \Lambda g^{\beta\sigma}\right)$$

$$\nabla_\alpha g_{\mu\nu} = 0$$

## Connection to Newton

$$R_{\mu\nu} = -k \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right)$$

$|\vec{u}| < c$ look at $T_{00}$ weak fields $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ Minkowski small $h_{\mu\nu} \ll \eta_{\mu\nu}$ in some coordinates

$\rho \gg p$ - pressure

$$\Gamma^\sigma_{\mu\nu} \approx \frac{1}{2} \eta^{\sigma\kappa} \left(h_{\nu\kappa,\mu} + h_{\mu\kappa,\nu} - h_{\mu\nu,\kappa}\right) + O(h^2)$$

$$R_{00} = \Gamma^{\mu}_{0\mu,0} - \Gamma^{\mu}_{00,\mu} + \underbrace{\Gamma^{\mu}_{0\mu}\Gamma^{\nu}_{\nu 0}}_{\sim h^2} - \underbrace{\Gamma^{\nu}_{00}\Gamma^{\mu}_{\nu\mu}}_{\sim h^2} \approx \Gamma^{\mu}_{0\mu,0} - \Gamma^{\mu}_{00,\mu} + \mathcal{O}(h^2)$$

$$\Rightarrow \frac{1}{2}\delta^{ij}\frac{\partial^2 h_{00}}{\partial x^i \partial x^j} = k\left(\rho c^2 \gamma - \frac{1}{2}\rho c^2 \gamma g_{00}\right) \simeq \frac{1}{2} k \rho c^2$$

$$\frac{1}{2}\nabla^2 h_{00} = \frac{1}{2} k \rho c^2 \qquad \nabla^2 \phi = 4\pi \rho G = 2\pi \rho G$$

$$g_{00} = 1 + \frac{2\phi}{c^2} \qquad g_{00} = 1 + h_{00}$$

$$h_{00} = \frac{2\phi}{c^2}$$

$$\nabla^2 \frac{2\phi}{c^2} = k \rho c^2$$

$$\nabla^2 \phi = \frac{k \rho c^4}{2} \Rightarrow k = \frac{8\pi G}{c^4}$$