# Exercises on Gravitational Wave Data Analysis

## Jürgen Ehlers Spring School on Gravitational Physics

**Exercise 1:** We have seen in the lecture that the noise is uncorrelated in Fourier domain, i.e, the value of the noise at a particular frequency does not depend on the value at other frequencies, which is translated into the following expression:

$$\langle \tilde{n}^*(f)\tilde{n}(f)\rangle = \frac{1}{2}\delta(f-f')S_n(f). \tag{1}$$

Let's prove this expression.

**Hints:**

- Express $\tilde{n}(f)$ and $\tilde{n}(f')$ in terms of the Fourier transform of $n(t)$: $n(t) = \int_{-\infty}^{\infty} = \tilde{n}(f)e^{-i2\pi ft}$. (Remember to use a different time variable for $\tilde{n}(f)$ and $\tilde{n}(f')$, for example $t$ and $t'$, or $t+\tau$).
- Work the expression until you can use the Wiener-Khinchin theorem, which relates the autocorrelation function of a stationary process to its power spectral density (PSD) in the Fourier domain. We have seen the expression in the lecture: $\langle n(t)n(t+\tau)\rangle = R(\tau) = \int_{-\infty}^{\infty}(1/2)S_n(f)e^{-i2\pi f\tau}df$

**Exercise 2:** We have seen in the lecture the match filter as the optimal detection statistics for stationary Gaussian noise:

$$\hat{s} \equiv (d|h) = 4\mathrm{Re}\int_0^{\infty} df\,\frac{\tilde{d}(f)\tilde{h}(f)^*}{S_n(f)} \tag{2}$$

In this exercise we are going to obtain it as the linear filter that maximizes the expected value of the signal-to-noise ratio (SNR). We define

$$\hat{s} = \int_{-\infty}^{\infty} dt d(t)K(t) \tag{3}$$

where $d(t) = h(t) + n(t)$ and $K(t)$ is a linear filter. The SNR can be defined as $S/N$, where $S = \langle\hat{s}\rangle$ (the expected value of the filtered data) and $N = \sqrt{\langle\hat{s}^2\rangle_{h=0} + \langle\hat{s}\rangle^2_{h=0}}$ (the rms of the filtered data assuming that there is no signal). Find the linear filter $K(t)$ that maximises S/N.

**Hints:**

- Remember that we can always work with $\langle n(t)\rangle = 0$.

- After computing $S = \langle \hat{s} \rangle$, it should not depend on $n(t)$.
- After working $N^2$, your will encounter $\langle \tilde{n}^*(f)\tilde{n}(f) \rangle$ in your expression. Use the result from the previous exercise to simplify the expression.
- Once $S$ and $N$ are computed, express $S/N$ in terms of the inner product that we have seen in the lectures:
  $(a|b) \equiv \mathrm{Re} \int_{-\infty}^{\infty} df \, \frac{\tilde{a}(f)\tilde{b}(f)^*}{(1/2)S_n(f)} = 4\mathrm{Re} \int_0^{\infty} df \, \frac{\tilde{a}(f)\tilde{b}(f)^*}{S_n(f)}$.
  Note: it might be useful to define a variable $\tilde{u}(f) = \frac{1}{2} S_n(f)\tilde{K}(f)$.
- Show that the (Fourier transform of the) filter that maximises the inner product is proportial to $\tilde{h}(f)/S_n(f)$

**Exercise 3:** We have seen in the lecture that we can employ the maximum likelihood estimation of the template parameters to define the maximum likelihood detection statistic. Remember that the likelihood ratio for the hypothesis of a signal described by $h(\lambda)$ is (in log for commodity):

$$\ln \Lambda(\mathcal{H}_\lambda, d) = (d|h(\lambda)) - \frac{1}{2}(h(\lambda)|h(\lambda)) \tag{4}$$

Consider a signal $h = Ag(x)$ with known for $g(x)$ and unknown amplitude $A$. Find the value of A that maximises the log-likelihood ratio $\ln \Lambda(\mathcal{H}_A, d)$ and compute the maximum likelihood detection statistics $\ln \Lambda(\mathcal{H}_{A_{max}}, d)$.

## GWOSC Tutorials

If you want to learn more about gravitational wave data analysis, you can check the useful tutorials from the Gravitational Wave Open Science Center (GWOSC) at the following link: https://gwosc.org/tutorials/