JESS 2024 Cosmology Problem Set – With Solutions
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1 Sahara of stars

Compare the number of grains of sand on Earth with the number of stars in the visible universe.

For this estimate, assume that a grain of sand occupies 1 cubic millimetre and that sand covers 10% the Earth to a depth of 1 metre. Assume galaxies are 5 million light years apart and contain 200 billion stars, and that the visible universe has a radius of 15 billion light years.

Solution

Earth has a radius of 6400km. This gives a surface of $4\pi(6400000)^2$ square meters. Divide this by 10 and multiply by 1 to get volume of sand in cubic meters. Then convert to cubic millimeters by multiplying by 10^9, to obtain about 5×10^{22}.

The dimater of the causal region of the universe is about 30 billion light years, so there would be about 6000 galaxies in one direction. This gives $(6 \times 10^3)^3$ galaxies, each with 2×10^{11} stars, i.e. about 4×10^{22} stars.

Thus there are roughly as many stars in the visible universe as there are grains of sand on Earth. And a seed contains roughly that many atoms. By contrast, the human body consists of a bit more than 10^{27} atoms...

2 Interstellar travel

How long would it take to reach the centre of our galaxy, and come back, at constant Earth-like acceleration? Compare the subjective time elapsed for the astronaut with the time elapsed on Earth. What is the maximum velocity reached? How long (in terms of astronaut time) would it take to reach the Andromeda galaxy, about 2 million light years away?

Hints: recall your special relativity formulae. Express Earth acceleration in terms of light-years per years squared. The distance to the centre of the galaxy is about 26000 light years.

Solution

At constant acceleration, the spacetime trajectory is a hyperbola. A solution for an object moving in the x direction with acceleration g is

$$x = \frac{1}{g}\cosh(g\tau)\,, \qquad t = \frac{1}{g}\sinh(g\tau)\,, \tag{1}$$

where τ is the proper time. Check: the 4-acceleration is

$$\mathbf{a} = (\frac{d^2t}{d\tau^2}, \frac{d^2x}{d\tau^2}, 0, 0) = g^2(t, x, 0, 0) \quad \text{with} \quad \mathbf{a}\cdot\mathbf{a} = g^4(t^2 - x^2) = g^2. \tag{2}$$

Thus the velocity is $dx/dt = \tanh(g\tau)$.

Now note that one year is about $3 \cdot 10^7$ seconds, and one light year is $3 \cdot 10^7 \cdot 3 \cdot 10^8$ meters. Thus, surprisingly, standard Earth surface acceleration is

$$g \approx 10\, m/s^2 \approx 1\, (light - yr)/(yr)^2. \tag{3}$$

So if $\tau = 10$ years of subjective astronaut time, then the distance covered is about $x = \cosh(10) \approx 11000$ light years! (A more precise calculation gives a slightly higher value.) Thus, if the rocket accelerates for the first 10 years, then turns around and decelerates for another 10 years, then after 20 years one reaches the centre of the galaxy. Another 20 years are required for the return trip. But on Earth 44000 years (in fact, more precisely a little over 50000 years) will have passed. Relativity has consequences!

By the same calculation, Andromeda can be reached after about 15 years ($x = \cosh(15)$).

If you feel ambitious, calculate the energy required for such a trip. (The most efficient way to generate such energy would be from matter-antimatter annihilation. I find that the required mass is about that of an asteroid.)

3 Friedmann equations

Calculate the connections and Ricci tensor for a flat FLRW metric, with line element

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)\,. \tag{4}$$

Use these to derive the Friedmann equations in the presence of a perfect fluid.

Solution

Here are the expressions for general spatial curvature k (the flat case corresponds to $k = 0$ and $g_{(3)ij} = \delta_{ij}$):

$$\Gamma^0_{ij} = \tfrac{\dot a}{a} g_{ij} = a\dot a g_{(3)ij} \tag{5}$$
$$\Gamma^k_{0j} = \tfrac{\dot a}{a}\delta^k_j \tag{6}$$
$$\Gamma^k_{ij} = \Gamma^k_{(3)ij} = k g_{(3)ij} x^k \tag{7}$$
$$\Gamma^0_{00} = \Gamma^0_{0j} = \Gamma^k_{00} = 0 \tag{8}$$
$$R_{00} = -3\tfrac{\ddot a}{a} \tag{9}$$
$$R_{ij} = (\tfrac{\ddot a}{a} + 2\tfrac{\dot a^2}{a^2} + 2\tfrac{k}{a^2}) g_{ij} = (a\ddot a + 2\dot a^2 + 2k) g_{(3)ij} \tag{10}$$
$$R_{0j} = 0 \tag{11}$$
$$R = 6(\tfrac{\ddot a}{a} + \tfrac{\dot a^2}{a^2} + \tfrac{k}{a^2}). \tag{12}$$

From these it follows that the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ has the components

$$G_{00} = 3\tfrac{\dot a^2}{a^2} + 3\tfrac{k}{a^2} \tag{13}$$
$$G_{ij} = (\tfrac{-2\ddot a}{a} - \tfrac{\dot a^2}{a^2} - \tfrac{k}{a^2}) g_{ij} = (-2a\ddot a - \dot a^2 - k) g_{(3)ij} \tag{14}$$
$$G_{0j} = 0\,. \tag{15}$$

The Friedmann equations immediately follow.

4 Age of the universe

Calculate the age of the universe, with the current best-fit fractional energy densities inferred by data from the PLANCK satellite (with 1σ error bars):

$$\Omega_r = (9.15 \pm 0.34) \times 10^{-5}\,, \tag{16}$$
$$\Omega_m = 0.308 \pm 0.012\,, \tag{17}$$
$$\Omega_k = -0.005 \pm 0.017\,, \tag{18}$$
$$\Omega_\Lambda = 0.692 \pm 0.012\,, \tag{19}$$

and

$$H_0 = (67.8 \pm 0.9)\,\mathrm{km\,s^{-1}\,Mpc^{-1}}\,. \tag{20}$$

Repeat the calculation for a flat universe that contains only matter ($\Omega_m = 1, \Omega_\Lambda = \Omega_r = \Omega_k = 0$) and for a flat universe containing only matter and vacuum energy ($\Omega_{\rm m} = 0.308, \Omega_\Lambda = 1 - \Omega_{\rm m}$).

Solution

Recall that the Friedmann equation is given by

$$3H^2 = \frac{\rho_{r,0}}{a^4} + \frac{\rho_{\rm m,0}}{a^3} - \frac{3k}{a^2} + \Lambda. \tag{21}$$

If we choose units in which the current scale factor is $a_0 = 1$, then $\rho_{\rm r,0}$ and $\rho_{\rm m,0}$ denote the current energy densities in radiation and pressure free matter, while Λ denotes the contribution from a cosmological constant. A useful way to re-write Eq. (21) can be obtained by dividing through by the current critical density $3H_0^2$, to obtain

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_{\rm r}}{a^4} + \frac{\Omega_{\rm m}}{a^3} + \frac{\Omega_{\rm k}}{a^2} + \Omega_\Lambda, \tag{22}$$

where the current fractional energy densities are conventionally defined as

$$\Omega_{\rm r} = \frac{\rho_{r,0}}{3H_0^2}, \quad \Omega_{\rm m} = \frac{\rho_{m,0}}{3H_0^2}, \quad \Omega_{\rm k} = \frac{-k}{H_0^2}, \quad \Omega_\Lambda = \frac{\Lambda}{3H_0^2}. \tag{23}$$

The Friedmann equation can be recast as the differential

$$\mathrm{d}t = \frac{\mathrm{d}u}{H_0\sqrt{\Omega_{\rm r}u^{-2} + \Omega_{\rm m}u^{-1} + \Omega_{\rm k} + \Omega_\Lambda u^2}}. \tag{24}$$

Integrating from $u = 0$ to $u = 1$ (the integral needs to be done numerically) gives us the time since the big bang, namely

$$t_0 \approx 0.960\frac{1}{H_0} \approx (13.80 \pm 0.04) \times 10^9 \text{ years}. \tag{25}$$

Repeating the calculation with matter only, one finds

$$t_0 = \frac{2}{3H_0} \approx 9.6 \times 10^9 \text{ years}. \tag{26}$$

This would be in conflict with the inferred ages of the oldest stars and galaxies.

Repeating the calculation with matter and vacuum energy only, one gets

$$t_0 \approx 0.957 \frac{1}{H_0} \approx (13.76 \pm 0.04) \times 10^9 \text{ years.} \tag{27}$$

The radiation phase thus impacts the lifetime of the universe rather insignificantly.

5 Redshift

Consider the following thought experiment: two observers A and B are at rest in a static universe, and separated by a distance R. Observer A sends a pulse of light (of frequency ν) towards B. While the pulse is travelling, B moves away from A and comes to rest again at twice the initial separation. Then B receives the pulse of light - how is the frequency of the pulse shifted? Now consider a second experiment: C and D are at rest in an initially static universe. C emits a pulse of light towards D. While the pulse is travelling, the universe starts expanding. When it has expanded to twice its size, D receives the pulse of light - how is the frequency now altered?

Solution

In the first case, B moved and is at rest again, but there was no change to the spacetime. B observes the frequency ν. In the second case, the wave expands with the spacetime, and hence the wavelength becomes twice as large when D receives the signal. The frequency is thus halved, to $\nu/2$.

6 Common Misconceptions

Try to answer questions typically asked during popular science talks (of course these questions are based on false assumptions, but it is a good exercise to ?find the wrong assumption and to try to answer them correctly):

a) What is outside of the universe?

b) What does the universe expand into?

c) If the universe expands and everything becomes bigger, then how can we observe this, if our equipment and we ourselves are also becoming bigger?

d) If everything is moving away from us, then aren't we at the centre of the universe?

e) Where did the big bang happen?

Solution

Cf. the discussion in class, but briefly:

a) The question doesn't make sense, as the universe makes its own space when expanding. All of space belongs to (is inside) the universe.

b) Again, the universe makes its own space (microphysically, we don't know how to answer this question yet)

c) The metric is not everywhere FLRW, that's just an approximation on large scales. Inside galaxies, the metric is more Schwarzschild-like, and does not expand. So only the vast spaces between galaxies expand, not the galaxies themselves.

d) Think of the analogy of a balloon that is blown up, with raisins attached to model galaxies. All galaxies move away from each other, but there is no centre.

e) Everywhere, cf. the time reverse of the answer above.

7 A preview of the horizon problem

Some aspects of the big bang model are puzzling - here's a preview. Consider the presently known universe. For quick order-of-magnitude calculations, the time since the big bang and the size of the observable universe (on the order of 10 billion years and 10 billion light years) are approximately given by 10^{60} Planck times and 10^{60} Planck lengths respectively. The universe might very well be much larger, but this would only strengthen the argument. Let us also assume for simplicity that the universe is filled purely by radiation (adding matter and vacuum energy complicates the mathematics, but does not change the essence of the argument).

1. Then how does the scale factor evolve?

2. How large was the observable part of the universe at one nanosecond after the big bang, and at the Planck time? Compare the age and size of the universe at these times.

3. Assume for a second the popular idea that the universe started from a point. Then at the Planck time, and at one nanosecond after the big bang, how large would you expect the universe roughly to have been? Compare with your calculations above and discuss!

Solution

During the radiation dominated phase, the scale factor evolves as

$$a(t) = a_0(t - t_0)^{1/2}, \quad \text{here} \quad a(t) = 10^{60}\left(\frac{t}{10^{60}}\right)^{1/2}, \tag{28}$$

if we use Planck units.

Then at one Planck time of age, we have

$$t = 1, \qquad a = 10^{30}. \tag{29}$$

Thus, even though the universe is just one Planck time old, it already has a radius of 10^{30} Planck lengths! This seems vastly incompatible with the idea that the origin of the universe was in the form of a big bang singularity. It seems impossible to reconcile this with causality. (You can repeat a similar calculation for $t = 1ns$, where the results are similar but somewhat less extreme). This simple calculation already renders the popular idea of a singular beginning highly implausible. It is just one of many problems with the original hot big bang model. Proposed resolutions (all this largely hypothetical) involve an inflationary phase, cosmic bounces and/or quantum effects such as the universe tunnelling into existence at a non-zero classical size.