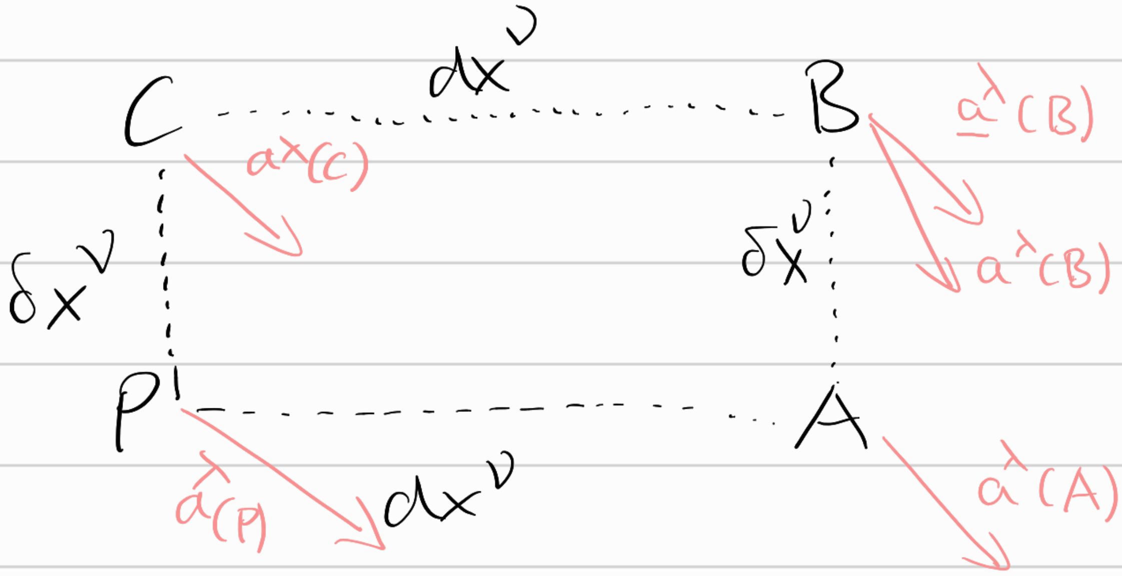
Curvature tensor

Let us study the following parallel transport of a contravariant vector a^λ.



From P to A to B (PAB)
From P to C to B (PCB)

We assume dx^ν and δx^ν are small, and use our earlier result for the parallel transport between two closeby points:

at A from P: $a^\lambda(A) = a^\lambda(P) - \Gamma^\lambda_{\mu\nu}(P) a^\mu(P) dx^\nu$

at B from A: $a^\lambda(B) = a^\lambda(A) - \Gamma^\lambda_{\rho\sigma}(A) a^\rho(A) \delta x^\sigma$

Inserting $a^\lambda(A)$ (as function at P) into $a^\lambda(B)$ yields:

$$a^\lambda(B) = a^\lambda(P) - \Gamma^\lambda_{\mu\nu}(P)\, a^\mu(P)\, dx^\nu$$
$$- \underline{\Gamma^\lambda_{\rho\sigma}(A)} \left[a^\rho(P) - \Gamma^\rho_{\beta\nu}(P)\, a^\beta(P)\, dx^\nu \right] \delta x^\sigma$$

To express $\Gamma^\lambda_{\rho\sigma}(A)$ at P we approximate it as:

$$\Gamma^\lambda_{\rho\sigma}(A) \simeq \Gamma^\lambda_{\rho\sigma}(P) + \Gamma^\lambda_{\rho\sigma,\tau}(P)\, dx^\tau$$

which introduces terms up to order dx^3, but we only keep terms up to order dx^2.

This yields for $a^\lambda(B)$ via PAB

$$a^\lambda(B) = a^\lambda \underbrace{- \Gamma^\lambda_{\mu\nu}\, a^\mu dx^\nu}_{} \underbrace{- \Gamma^\lambda_{\rho\sigma}\, a^\rho \delta x^\sigma}_{}$$
$$+ \Gamma^\lambda_{\rho\sigma} \Gamma^\rho_{\beta\nu}\, a^\beta dx^\nu \delta x^\sigma + \Gamma^\lambda_{\rho\sigma,\tau}\, a^\rho dx^\tau \delta x^\sigma + \text{order}(dx^3)$$

where we did not write out the explicit dependency on P, because all terms are at P.

Repeating the analysis via PCB yields

$$\underline{a}^\lambda(B) = a^\lambda - \Gamma^\lambda_{\mu\nu} a^\mu \delta x^\nu - \Gamma^\lambda_{\rho\sigma} a^\rho dx^\sigma + \Gamma^\lambda_{\rho\sigma}\Gamma^\rho_{\beta\nu} a^\beta \delta x^\nu dx^\sigma + \Gamma^\lambda_{\rho\sigma,\tau} a^\rho \delta x^\tau dx^\sigma$$

Note that the appearance of dx and δx is slightly different (indices)

The difference of $a^\lambda(B)$ and $\underline{a}^\lambda(B)$ can now be computed from properties at P. (using $\tau \to \nu$; $\rho \to \beta$ for dummy indices)

$$\delta a^\lambda = a^\lambda(B) - \underline{a}^\lambda(B) = a^\beta (dx^\nu \delta x^\sigma - dx^\sigma \delta x^\nu)(\Gamma^\lambda_{\rho\sigma}\Gamma^\rho_{\beta\nu} + \Gamma^\lambda_{\beta\sigma,\nu})$$

Showing that only terms of order dx^2 remain.

Noting that ν and σ are dummy indices, we could also write

$$\delta a^\lambda = a^\beta (dx^\sigma \delta x^\nu - dx^\nu \delta x^\sigma)(\Gamma^\lambda_{\rho\nu}\Gamma^\rho_{\beta\sigma} + \Gamma^\lambda_{\beta\nu,\sigma})$$

Adding both versions of δa^λ with a factor $\frac{1}{2}$, then

$$\delta a^\lambda = -\frac{1}{2} a^\beta R^\lambda{}_{\beta\nu\sigma}(dx^\sigma \delta x^\nu - dx^\nu \delta x^\sigma)$$

with $$R^\lambda{}_{\beta\nu\sigma} = -\Gamma^\lambda_{\rho\nu}\Gamma^\rho_{\beta\sigma} + \Gamma^\lambda_{\rho\sigma}\Gamma^\rho_{\beta\nu} - \Gamma^\lambda_{\beta\nu,\sigma} + \Gamma^\lambda_{\beta\sigma,\nu}$$

which is called the curvature tensor or Riemann tensor.

Note: some refrences follow different conventions for the Christoffel symbols, and thus, have a different sign for the $\Gamma\Gamma$ terms compared to the $\partial\Gamma$ terms.

Substituting its definition and writing out all terms, shows that (by the contractions with $\delta x^\nu dx^\rho$ and $dx^\nu \delta x^\rho$, using again dummy indices) the result for a^λ is the same.

The interpretation obtained from this definition is that the parallel transport of vectors does in general depend on the path, if the curvature tensor does not vanish.

An alternative definition is

$$\nabla_\beta \nabla_\nu a_\lambda - \nabla_\nu \nabla_\beta a_\lambda = R^\lambda{}_{\beta\nu\sigma}\, a_\lambda$$