lecture - 2024-02-28 pherical stars

T^α_α

field equations: $R_{\mu\nu} = -\kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$

↳ $\partial \Gamma + \Gamma\Gamma$ non linear partial differential equation

↳ ∂g

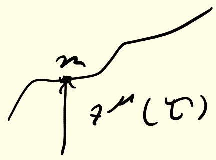
analytical / simple solution?

ideal fluid $T_{\mu\nu} = \left(\rho + \frac{p}{c^2} \right) u_\mu u_\nu - p g_{\mu\nu}$ not friction

vacuum $\rho = 0 = p$

no point particle
because δ-function not easy to solve

point particle $T^{\mu\nu}(x) = \int d\tau \, m u^\mu u^\nu \delta(x^\mu - z^\mu(\tau))$



assume symmetries

ansatz for line element:

- static -> independent of $t = x^0$ except dt
- spherical symmetry / isotropic
  - -> depends on spatial coordinates only via

$$r^2 = \|\vec{x}^2\| \qquad \vec{x} \qquad d\vec{x}$$

$$\text{"}d\vec{x}^2\text{"} = dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta \, d\varphi^2$$

$$\text{"}\vec{x} \cdot d\vec{x}\text{"} = r\,dr$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = A(r)\,dt^2 - B(r)\,dr^2 - C(r)\, r^2 \left(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2 \right) - \underbrace{D(r)\,dr\,dt}_{\text{remove it}}$$

dt comes from the Minkowski metric, it is static and we want in local coordinates this metric by

$=1$ by changing / gauge-fixing r

(redefine) shifting t
$dt^2 = (dt + \ldots dr)^2$
$t \to t(t', r)$

Calculate $g_{\mu\nu} \to \Gamma^\rho_{\mu\nu} \to R^\alpha{}_{\nu\alpha\beta} \to R_{\mu\nu}$ $\quad A' = \frac{dA}{dr}$, $A'' = \frac{d^2A}{dr^2}$

$$R_{tt} = -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B} \right) - \frac{A'}{rB}$$

$$R_{rr} = \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B} \right) - \frac{B'}{rB} \qquad R_{\varphi\varphi} = R_{\vartheta\vartheta} \cdot \sin^2\vartheta$$

$$R_{\vartheta\vartheta} = \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B} \right)$$

all other zero

insert them into the Einstein equations

$$R_{\mu\nu} = -\kappa \left(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T \right)$$

where $T = T^{\mu}_{\ \mu} = (\rho + \frac{p}{c^2}) \underbrace{u^{\mu} u_{\mu}}_{c^2} - p \underbrace{\delta^{\mu}_{\mu}}_{4} = \rho c^2 - 3p$

$= g^{\mu\nu} T_{\mu\nu}$

$$R_{\mu\nu} = -\kappa \left((\rho + \tfrac{p}{c^2}) u_{\mu} u_{\nu} - \tfrac{1}{2}(\rho c^2 - p)\, g_{\mu\nu} \right)$$

$c^2 = g^{\mu\nu} u_{\mu} u_{\nu}$ $\quad u^{i} = 0$ static $\quad c^2 = \frac{1}{A} u_t^2$, $\quad [u_{\mu}] = c\sqrt{A}(1,0,0,0)$

$R_{tt} = -\frac{1}{2}\kappa(\rho c^2 + 3p) A$

$R_{rr} = -\frac{1}{2}\kappa(\rho c^2 - p) B$

$R_{\vartheta\vartheta} = -\frac{1}{2}\kappa(\rho c^2 - p) r^2$

$R_{\varphi\varphi} = \sin^2\vartheta \, R_{\vartheta\vartheta}$

combining equations

$$\frac{R_{tt}}{A} + \frac{R_{rr}}{B} + \frac{2R_{\vartheta\vartheta}}{r^2} = -2\kappa\rho c^2$$

$$\hookrightarrow \left(1 - \frac{1}{B}\right) + r\frac{B'}{B^2} = \kappa r^2 \rho c^2$$

$$\frac{d}{dr}\Big(\underbrace{r(1 - \tfrac{1}{B})}_{2Gc^2 \, m(r)}\Big) = \kappa r^2 \rho c^2$$

definition

not the integrated density in spacetime $\longrightarrow$ $m'(r) = 4\pi r^2 \rho(r)$

$$m(r) = 4\pi \int_0^r \bar{r}^2 \rho(\bar{r})\, d\bar{r} \qquad (\mathrm{I})$$

gravitational energy

if flat space

$= \int d^3x\, \rho$

we $D_\nu T^{\mu\nu} = 0$ recall follows from field equations

$= D_\nu \frac{G^{\mu\nu}}{8\pi} = 0$

$\mu = 0 \rightarrow$ continuity equation

$\mu = i \rightarrow$ Euler equation

case $\mu = r$ hydrostatic equilibrium equation

$$p'(r) = -(\rho c^2 + p)\frac{A'}{2A} \qquad \text{relativistic version} \qquad (\mathrm{II})$$

from $R_{\mu\nu}$ with (I) we get $B = \left(1 - \frac{2Gm}{c^2 r}\right)^{-1}$
left and right sides

$$\frac{A'}{2A} = \frac{1}{r^2}\left[\frac{4\pi G}{c^4} p r^3 + \frac{Gm(r)}{c^2}\right]\left[1 - \frac{2Gm(r)}{rc^2}\right]^{-1} \qquad (\mathrm{III})$$

Remark:
recall for Newtonian limit $A \simeq 1 + 2U(r)$
(Newtonian potential

3 equations: (I), (II), (III) Talman-Oppenheimer-Valkoff equations (TOV-equations)

but 4 functions: m, A, p, ρ

so you need to specify relation between pressure and density $p \leftrightarrow \rho$

equation of state EOS

simple examples:
- -baxotropic EOS $p = p(\rho)$ you have to modell your equation of state
- -polytropic EOS $p = k\rho^{\gamma}$ $n = (\gamma - 1)^{-1}$
- -piecewise polytropic EOS

example neutron star - polytropic EOS

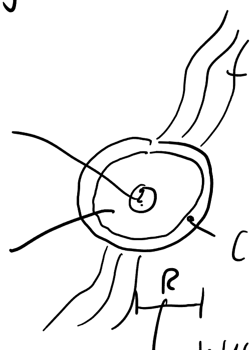
$n \simeq 1$, $k \simeq 100$, $G = 1 = c$ Units: km
$\gamma = 2$

they have very complicated structure
proximate by piecewise polytrope, works fine

$p^+ + e^- \rightarrow n + \nu_e$

$10\,\mathrm{km} < R < 15\,\mathrm{km}$

masses $M = m(r=R) \sim 1..3 M_\odot$



mass-radius relations

$p(\rho)$ $\exists$ maximum mass
mass $= M_{max} \Rightarrow$ collapse - no static solution
$\hookrightarrow$ black hole

compact stars: very dense
more than normal star
(neutron stars, black hole, white dwarf

Analytic solution

exterior (vacuum) solution : $r > R$

$p = 0 = \rho \rightarrow m(r) = \text{const} = M$ \ total energy

try : $A = 1 - \frac{2GM}{c^2 r}$ $\rightarrow$ (III) fullfilled

line element $\rightarrow$ Schwarzschild metric :

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr - r^2 (d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)$$

Birkhoffs theorem: exterior solution of spherically symmetric star is Schwarzschild solution, asymtotically flat.

Interior solutions

fluid is incompressible $\rho = \text{const}$

a limit because $v_s \sim \infty$ $\Rightarrow$ infinit the speed of sound

solution : $r < R$: $m(r) = 4\pi \int_0^r \bar{r}^2 \underbrace{\rho(\bar{r})}_{\text{const}} d\bar{r}$

$$= \frac{4}{3} \pi r^3 \rho$$

$$p(r) = \rho c^2 \frac{\left(1 - \frac{2\mu r^2}{R^3}\right)^{1/2} - \left(1 - \frac{2\mu}{R}\right)^{1/2}}{3\left(1 - \frac{2\mu}{R}\right)^{1/2} - \left(1 - \frac{2\mu r^2}{R^3}\right)^{1/2}} \qquad \mu = \frac{GM}{c^2}$$

$$A(r) = \frac{c^2}{4}\left[3\sqrt{1 - \frac{3\mu}{R}} - \sqrt{1 - \frac{2\mu r^2}{R^3}}\right]^2$$

limiting case $p \rightarrow \infty$ at $r = 0$

$\hookrightarrow 3\left(1 - \frac{2\mu}{R}\right)^{1/2} - 1 = 0 \quad \rightarrow \underbrace{\frac{GM}{c^2 R}}_{} = \frac{4}{9}$ — C : compactness

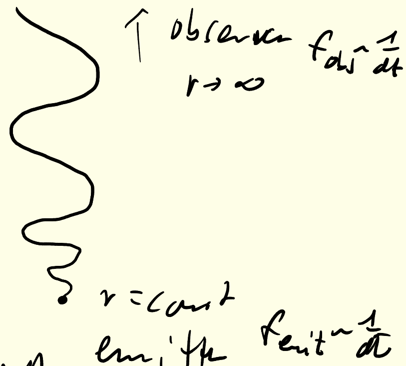
$\Rightarrow$ Buchdahl limit

$$C < \frac{4}{9}$$

"Clasical" test of General Relativity

- perihelion advance of Mercury
- light deflection by the Sun
- gravitational Redshift

Gravitational Redshift



clock at some position in a gravitational field in Schwarzschild metric

$r = const$

observing the clock far away position

proper time $$d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) \underbrace{dt^2}_{\text{time of observer}} + \dots + \underbrace{dr^2}_{=0 \text{ because } r = const}$$

at infinity it goes to Minkowski Metric

↳ redshift

$$z = \frac{f_{emit} - f_{obs}}{f_{obs}} = \underbrace{\frac{f_{emit}}{f_{obs}}}_{\sim \frac{dt}{d\tau}} - 1$$

$$1 + z = \frac{dt}{d\tau} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2}$$

approx $\quad z \approx \frac{GM}{c^2 r} + O(G^2)$

example:

- Pound-Rebka experiment
- Einstein Tower
- GPS etc
- optical atomic clocks

Motion in Schwarzschild metric

insert metric of Schwarzschild in geodesic equation $\quad \vartheta = \frac{\pi}{2} = \frac{\tau}{4}$

$$\left(\frac{dr}{d\lambda}\right)^2 = \underbrace{e^2}_{\text{energy}} - \left(1 - \frac{2GM}{c^2 r}\right)\left(\frac{\ell^2}{r^2} + \varepsilon\right) \quad \cdot \quad \frac{d\phi}{d\lambda} = \frac{\ell}{r^2}$$

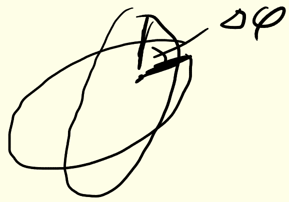
$$\varepsilon = \frac{u_\mu u^\mu}{c^2} = \begin{cases} 1, & \text{massive particles} \\ 0, & \text{massless} \ -\text{"}- \end{cases}$$

$\ell = const$ - Angular momentum

$e = const$ - energy

Perihel advance

approximate solution for this motion for $\varepsilon = 1$
bound orbits

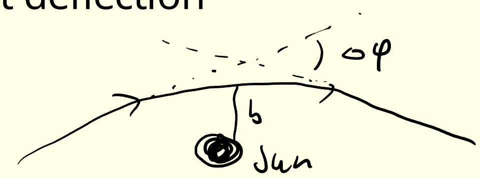


$$\Delta\varphi \simeq \frac{6\pi GM}{c^2 a(1-e^2)}$$

a: semi-major axis
e: eccentricity

for Mercury $\Delta\varphi \simeq 43''$ fits observations

light deflection



massless particle

$$\Delta\varphi \simeq \frac{4GM}{c^2 b}$$