Les Houches Lectures on 2D Gravity and Random Matrix Models

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Abstract

(This is a draft! Also, bibliography to come!) These four lectures ~~were~~ (are to be) given at the Les Houches 2024 Summer School on Quantum Geometry, 5th–9th August 2024. The material begins with early motivations for studying 2D quantum gravity: as a route to understanding the path integral definition of string theory, but then moves on to the effective 2D quantum gravity that arises when studying near extreme charged black holes. Both applications emphasize the two primary (and intersecting) interpretations of the random matrix model: As a 't Hooftian means of defining the sum over surfaces through tesselations, and as a Wignerian means of statistically characterising the spectrum of the (holographic dual) Hamiltonian of the theory. Perturbative and non-perturbative aspects are uncovered in detail.

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1 Lecture 1

Before diving into the details of quantizing 2D gravity, it is worth reminding ourselves why we want to do such a thing at all. Of course, one simple reason is that it is a simple enough theory that we might learn useful lessons for quantizing gravity in other dimensions, but there are (at least) two major areas where issues of quantum gravity in higher dimensions already rely on our understanding of quantum gravity in 2D. The first is string theory. A common approach to it involves formulating the dynamics of strings moving in some background in terms of a two dimensional "polyakov" action for the worldsheet of the string. The spacetime cooordinates resemble fields in some 2D spacetime, but the dynamics also involves summing over all possible geometries and topologies of the worksheet, which is a form of quantum gravity. The second motivation is the physics of higher dimensional black holes.

1.1 Motivations from string theory

Outside of the famous critical dimensions of string theory, the effective scalar coming from Weyl rescaling of the 2D metric does not decouple from the theory, and must be included for consistency. What is typically done is to choose a reference metric $\hat{g}_{ab}$ and write the physical metric as conformal to that: $g_{ab} = e^{2\phi}\hat{g}_{ab}$, where ϕ is a scalar for which there is a a special 2D "Liouville" conformal field theory action. There's also additional fields (we'll refer to as X generically) with their own CFT action (for example they could be interpreted as additional spacetime coordinates) and of course there are the Fadeev-Popov ghosts from gauge fixing, which have their own CFT. These theories are all coupled together implicitly through the requirement that the total central charge of the theory should vanish. Ignoring the ghost sector henceforth, the problem of Liouville+matter is an important 2D quantum gravity theory with features that will inspire us later, so we should talk about it.

1.1.1 Liouville theory plus matter

The matter+Liouville sector has partition function:

$$Z = \int D\varphi DX e^{-S_{\rm tot}} , \tag{1}$$

where $S_{\rm tot}$ includes $S_{\rm L}(\varphi)$ and $S_{\rm matter}(X)$, which will be taken to be some conformal field theory representing the matter sector such that

$$c_{\rm L} + c = 26 , \tag{2}$$

cancelling the -26 coming from the ghost sector. These days the Liouville action is usually written for some $\varphi = \phi/b$ in the following form:

$$S_{\rm L} = \frac{1}{4\pi}\int d^2z\sqrt{\hat{g}}\left\{\hat{g}^{ab}\partial_a\varphi\partial_b\varphi + Q\hat{R}\varphi + \mu e^{2b\varphi}\right\} , \tag{3}$$

where the Q term modifies the stress tensor for a scalar to (writing *e.g.* the holomorphic part)

$$T(z) = -\partial_z\varphi\partial_z\varphi + Q\partial_z^2\varphi , \tag{4}$$

with which (in computing the usual OPE) for a vertex operator $\exp(ik\cdot\varphi)$ yields the following (Q-shifted) conformal dimension: $\Delta = -\left(\frac{1}{4}k^2 + \frac{i}{2}Qk\right)$. The last term in the action looks like the insertion of a background involving such a vertex operator with $ik = 2b$, and since conformal invariance requires that $\Delta = 1$ we obtain the condition:

$$Q = b + b^{-1} . \tag{5}$$

The $T(z)T(z)$ OPE yields the central charge $c_L = 1+6Q^2$ for the Liouville sector, and from (2):

$$Q = \sqrt{\frac{25-c}{6}}\,, \quad \text{with} \quad b = \frac{Q}{2} - \frac{\sqrt{Q^2-4}}{2} = \frac{1}{2}\left[\sqrt{\frac{25-c}{6}} - \sqrt{\frac{1-c}{6}}\right]\,, \tag{6}$$

where the sign of the root was chosen to match to the classical limit, which is $b \to 0$ or $Q \to \infty$. We will discuss this limit soon.

Recalling that in this action $\hat{g}_{ab}$ is a reference metric and that it is $\hat{g}_{ab}\mathrm{e}^{2b\varphi}$ that is the physical metric, we see that the last term (times $\sqrt{\hat{g}}$) is really the determinant of the physical metric, and hence its integral gives the two dimensional area $A = \int d^2z\sqrt{\hat{g}}\mathrm{e}^{2b\varphi}$. Then μ has the interpretation as a cosmological constant.

1.1.2 KPZ scaling and a critical point

It is natural to wonder about the partition function computed by integrating over surfaces of fixed area, which we can denote as:

$$Z|_A = \int D\varphi DX \mathrm{e}^{-S_{\rm tot}}\delta\left(\int d^2z\sqrt{\hat{g}}\mathrm{e}^{2b\varphi} - A\right)\,, \tag{7}$$

and so the total partition function is

$$Z = \int_0^\infty dA \mathrm{e}^{-\mu A} Z_A\,, \tag{8}$$

a Boltzman factor with an entropic weight. It is interesting to ask what the large A behaviour of Z_A is, and the result (of refs. [?,?] is that

$$Z|_A \sim A^{(\gamma^{(0)}_{\rm str}-2)\frac{\chi}{2}-1} = A^{\gamma_{\rm str}-3} \tag{9}$$

where $\gamma^{(0)}_{\rm str}$ is often called the (genus zero) string susceptibility, while $\gamma_{\rm str} = 2-(2-\gamma^{(0)}_{\rm str})\chi/2$ is the string susceptibility at genus h, and $\chi = 2-2h$ is the Euler number of the surface. Using the form of the Liouville action, the dependence of $\Gamma_{\rm str}$ on the parameters of the theory can be deduced from a scaling argument. Shifting $\varphi \to \varphi + \rho/2b$, the Liouville term linear in φ shifts the action by $Q\rho\chi/2b$, and within the δ function the integral gets scaled by e^ρ. Since $\delta(ax) = \delta(x)/a$, we can write:

$$Z|_A = \mathrm{e}^{-\frac{Q\rho\chi}{2b}-\rho} Z|_{\mathrm{e}^{-\rho}A}\,, \tag{10}$$

whereupon setting $\mathrm{e}^\rho = A$ yields:

$$\gamma^{(0)}_{\rm str} = 2 - \frac{Q}{b} = \frac{1}{12}\left[(c-1) - \sqrt{(c-1)(c-25)}\right]\,, \quad \text{or} \tag{11}$$

$$\gamma_{\rm str} = 2 - \left(\frac{1-h}{12}\right)\left[25 - c + \sqrt{(c-1)(c-25)}\right]\,. \tag{12}$$

(To summarize, we've effectively used the ability to shift φ to rescale area A.) Notice that the asymptotic behaviour (9) allows integral (8) to be done, giving:

$$Z \sim \mu^{2-\gamma_{\rm str}}\,. \tag{13}$$

This is a characteristic behaviour that we'll seek for later. For example for "pure gravty", *i.e.* $c = 0$, on the sphere $g = 0$, $\gamma^{(0)}_{\rm str} = -\frac{1}{2}$, giving $Z \sim \mu^{\frac{5}{2}}$. Notice also that this behaviour also means that the expectation value of the area, $\langle A\rangle = -\partial \ln Z/\partial\mu$, diverges as $\mu \to 0$, which makes sense, and $\gamma_{\rm str}$ is a measure of the rate.

1.1.3 A special classical limit

It's interesting to rescale the Liouville field according to $\varphi = \phi/b$, in which case the action can be wrriten:

$$S_{\rm L} = \frac{1}{4\pi b^2}\int d^2z\sqrt{\hat g}\left\{\hat g^{ab}\partial_a\phi\partial_b\phi + (1+b^2)\hat R\phi + b^2\mu e^{2\phi}\right\} , \tag{14}$$

with equations of motion:

$$2\nabla^2_{\hat g}\phi - \hat R = b^2(2\mu e^{2\phi} + \hat R) . \tag{15}$$

Here, the parameter b^2 is playing the role of $\hbar$, and there's an analogue of a classical limit, where $b \to 0$ and hence $Q \to \infty$. Note that the central charges diverge: $c_L \to +\infty$ and $c \to -\infty$. Writing the above in terms of the physical metric $g_{ab} = e^{2\phi}\hat g_{ab}$ and using the identity relating 2D Ricci scalars:

$$R(e^{2\phi}\hat g) = e^{-2\phi}\hat g(R(\hat g) - 2\nabla^2_{\hat g}\phi) , \tag{16}$$

the classical equation of motion is simply

$$R = -2\mu b^2 , \tag{17}$$

which is Liouville's equation, telling us here (if we hold constant $\mu b^2 = 1$) that the physical metric has constant negative curvature: $R = -2$. This will likely remind you of things (to be) seen in the lectures of Turiaci on JT gravity. Let's have a glance at some of that story.

1.2 Motivations from black holes in $D > 2$.

1.2.1 Reissner-Nordström black hole in $D = 4$

Quick (maybe to be expanded) description of (say) $D = 4$ Reissner-Nordström black holes follows: The extremal (T=0) limit of the metric is AdS$_2 \times S^2$, with the cosmological constant of AdS$_2$ and radius of the S^2 set by the charge $\bar Q$. There is an $SL(2,\mathbb{R})$ conformal symmetry due to the AdS$_2$ factor. The black hole has an entropy $S_0 = A/4G_N$, where A is the area of the horizon.

1.2.2 The near-horizon, low-temperature limit

The low-temperature geometry is "nearly" AdS$_2$ and the sphere's size can fluctuate away from that set by $\bar Q$. There is an effective 2D theory of gravity describing the dynamics, with a special coupling to a field Φ that represents the fluctuations of the sphere's radius, resulting in an action that schematically takes the (Euclidean) form:

$$S_{\rm JT} = -\frac{1}{2}\int_M\sqrt{g}\,\Phi(R+2) - \int_{\partial M}\sqrt{h}\,\Phi_b(K-1) - S_0\left(\frac{1}{4\pi}\int_M\sqrt{g}R + \frac{1}{2\pi}\int_{\partial M}\sqrt{h}K\right) , \tag{18}$$

where R is the Ricci scalar of metric g_{ab}, and for the boundary (∂M) terms, Φ_b is the value of Φ there, h is the induced metric and K is the extrinsic curvature. Euclidean signature has been chosen, such that Euclidean time has period $\beta = 1/T$.

Note that S_0 multiplies $\chi = 2 - 2h - b$, the Euler number of M, where h is the number of handles and b the number of boundaries. (Not to be confused with Liouville parameter!!) This will result in a factor of $e^{\chi S_0}$ in computations using this action. For example, as we will see shortly, for the partition function $Z_{JT}(\beta) = \int D\Phi Dg \exp(-S_{\rm JT})$, the leading (disc) order will have a factor e^{S_0} since there $h = 0$ and $b = 1$.

The equation of motion for Φ dynamically sets $R = -2$, and the dynamics reduces to a careful treatment of the boundary dynamics. The latter is a special theory of quantum mechanics where the boundary can fluctuate according to a Schwarzian action, while keeping its length fixed to be β. Turiaci's lectures discuss this in detail, and a key exhibit of this Schwarzian dynamics is the result for the disc partition function:

$$Z_0(\beta) = \frac{e^{S_0}\gamma^{3/2}}{\sqrt{2\pi}}\frac{1}{\beta^{3/2}}e^{\frac{2\pi^2\gamma}{\beta}} , \tag{19}$$

and $\gamma = \Phi_r$, a (renormalized) value of the scalar on the asymptotic boundary where the Schwarzian is defined. (We'll likely set it to $1/2$ later on in these lectures.)

1.2.3 Jackiw-Teitelboim (JT) gravity and the Euclidean GPI

(Probably will say a bit about the general case involving multiple insertions of $Z(\beta)$, and any number of handles.... decomposition into gluing trumpets onto Weil-Petersson volumes. Compute or show a few examples maybe.)

1.3 Performing the Euclidean Path Integral

Let us now step back and look at the problem we've been trying to get to grips with. The core point is that there is some two dimensional gravity theory of g_{ab}, which is largely topological, made interesting by either having a "quantum" deformation (going to non-zero b for Liouville) or by virtue of how it is coupled to an additional sector (dynamical Φ for Jackiw-Teitelboim) or both. The gravitational path integral approach to either story involves summing over all metrics of all topologies. In the case of JT gravity, we were lucky that the dynamics froze the bulk dynamics to be constant curvature metrics, in which case the topological sum over metrics boiled down to enumeration of the properties of the volume of moduli space of hyperbolic metrics, with some needlework to do at the geodesic boundaries in order to connect them to the Schwarzian dynamics living at the asymptotic boundaries where Φ_b lives.

How do we do the sum over metrics and topologies in more general cases? Moreover, what lies beyond the (asymptotic) expansion in the topological expansion parameter? What do we learn about the whole business of defining quantum gravity as a path integral over all geometries and topologies? It's entirely an educated guess/analogy inspired by successes with (non-gravitational) theories, but it isn't guaranteed to be a complete definition. Can we learn in 2D whether it is or not?

To begin to answer some of these questions, let's return to the simplest model we can think of, and start approaching the problem from scratch.

1.3.1 Hermitian random matrix model, tessellations and topology

A possible way to handle the sum over metrics and topologies is to make the problem more manageable by breaking the surface up into pieces. Imagine (see figure XX) using little quadrilaterals ("squares") of some fixed area to build up curved surfaces.

- We can make discrete surfaces of any topology by gluing them together to make something with v vertices, e edges, and n faces (the number of squares). The Euler number of a surface made this way is $\chi = v - e + n$.

- Local positive or negative curvature at some point can be approximated by packing in more or less than four squares meeting.

The partition function representing the sum over all closed surfaces would be of the form:

$$Z_{\text{discrete}} = \sum_{h=0}^{\infty} \nu_B^{2h-2} \sum_{n=0}^{\infty} \mathrm{e}^{-\mu_B n} Z_{h,n} \ , \tag{20}$$

Where all possible topologies are being summed with some weight $\nu_B^{-\chi}$ and area (total number of squares) are being Boltzman weighted with some bare cosmological constant μ_B.

The question is then how to evaluate the $Z_{h,n}$ "entropic" factors? In this simple case of "pure gravity" where there's no coupling to some other sector to contend with (*i.e.,* $c = 0$ in the language of section 1.1.1), $Z_{h,n}$ is simply the number of tesselations at given genus and area.

Here's a way to count these, (following the classic work of Brezin, Itzykson, Parisi, and Zuber '78). Decorate the squares in one of our diagrams drawn above such that there's a point at the centre of each square, and the lines joining them by crossing the edge connecting adjacent squares. Counting all possible such "dual" diagrams is equivalent to counting our tessellations. But these look like Feynman graphs for some theory of quartic interactions. To be precise, consider the (zero-dimensional) "field theory":

$$Z(N,g) = \mathrm{e}^{-E(N,g)} = \int dM \exp\left\{-N\mathrm{Tr}\left(\frac{M^2}{2} + g M^4\right)\right\} , \tag{21}$$

Where M is an $N \times N$ Hermitian matrix and the measure is $dM = \prod_i M_{ii} \prod_{i<j} dM_{ij} dM^*_{ij}$, the Haar measure over matrix elements.

Propagators in this theory are represented by double lines that can be thought of as each propagating an index of the matrix, and the Feynman rules give a factor of $\frac{1}{N}$ for each propagator and gN for each vertex. (See figure XX))

Let's focus on the factors of N that would result form a computation of a diagram. A connected vacuum diagram with P propagators, V vertices and L loops would have a factor $g^V N^{-P} N^V N^L$ (since every closed loop has a free index running around in it that can take any of N values). So this means that the partition function of our theory (for closed diagrams) can be written as (taking the log for closed sector):

$$\log(Z(N,g)) = \sum_{h=0}^{\infty} \left(\frac{1}{N}\right)^{2h-2} \sum_{n=1}^{\infty} g^n Z_{h,n} \ , \tag{22}$$

since by the dual construction $V = n$, $P = e$, and $L = v$ and $\chi = v - e - n = 2 - 2h$. So comparing to equation (20) we can identify that the matrix model has just the right sort of parameters we need for our discrete model, *i.e.,* we have a map $\nu_B \longleftrightarrow 1/N$ and $\mathrm{e}^{-\mu_B} \longleftrightarrow g$.

Our job now is to evaluate $Z_{h,n}$. Let's start by working at leading order, at sphere topology, which is the leading order in the large N limit.

1.3.2 Leading order in large N, the Dyson gas, eigenvalue repulsion

Since the action of our model involves only the trace of powers of M, it is prudent to work in terms of the eigenvalues of M. The trace gives a large $U(N)$ invariance under $M \to UMU^\dagger$, where $U \in U(N)$, and we can "gauge fix" by writing $M = U\Lambda U^\dagger$ where Λ is the diagonal matrix $\Lambda = \mathbf{diag}\{\lambda_1, \lambda_2, \cdots \lambda_N\}$. The integral over matrix elements of M now becomes an integral over the N eigenvalues λ_i and the volume of the unitary group, which will appear as an overall factor (which we can drop). The result is

$$Z(N,g) = \int \prod_i d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 \exp\left\{-N \sum_i \left(\frac{\lambda_i^2}{2} + g\lambda_i^4\right)\right\} . \tag{23}$$

There is a Jacobian $\Delta^2(\lambda)=\prod_{i<j}(\lambda_i-\lambda_j)^2$ appearing as a result of this procedure. $\Delta(\lambda)$ is the Vandermonde determinant. The Jacobian can be computed directly by conputing the determinant of the metric obtained by writing out $(\delta M)^2$ in the decomposition of M into U and Λ, or by using a Fedeev-Popov type procedure, or by reasoning as follows. The change of variables to the eigenvalue basis makes sense for generic λ_i, but it is going to fail whenever any two are identical. (The change of variables can't distinguish between the two, so it is singular there. There's an enhanced symmetry $U(2)$ in this case.) This means the Jacobian J should vanish whenever this happens. This is like the change of variables from rectangular to spherical polars. There $J=r^2\sin\theta d\theta d\phi dr$. But at $\theta=0$ and π there is an ambiguity under rotations of x into y, the change of variables is singular, and $J=0$. So this determines that $J=\prod_{i<j}(\lambda_i-\lambda_j)^\beta$. Dimensional analysis finishes the job. On the one hand, $dM\sim\lambda^{N^2}$, while on the other, $\prod_i d\lambda_i J\sim\lambda^N\lambda^{\beta N(N-1)/2}$, and hence β must be 2.

A suggestive way of writing our eigenvalue problem is as a "Dyson gas":

$$Z(N,g)=\int\prod_i d\lambda_i\exp\left\{-N\sum_i\left(\frac{\lambda_i^2}{2}+g\lambda_i^4\right)+2\sum_{i<j}\log|\lambda_i-\lambda_j|\right\}\,, \tag{24}$$

where there are N particles with positions at λ_i, in a potential $NV(\lambda_i)=N(\lambda_i^2/2+g\lambda_i^4)$ with a logarithmic (1D Coulomb) interparticle repulsion. Let's look at the $N\to\infty$ limit. Intuitively, we should expect a smooth saddle point solution to our system (at least for some range of g) represented by a droplet of eigenvalues formed by the balance between the attraction from the potential dragging them to the origin and the logarithmic repulsion coming from $\sim N$ other elements of the droplet forcing it to spread out.

To find it, replace λ_i by a smooth parameter $\lambda(X)$, where $X=i/N$ runs from 0 to 1, and sums become integrals according to $\frac{1}{N}\sum_i=\int_0^1 dX$. The model is then $Z(N,g)=\mathrm{e}^{-N^2E(g)_{\rm sph}}$, where:

$$E(g)_{\rm sph}=\lim_{N\to\infty}\left\{\int_0^1 dX\left(\frac{\lambda(X)^2}{2}+g\lambda(X)^4\right)+\int_0^1\int_0^1 dXdY\log|\lambda(X)-\lambda(Y)|\right\}\,, \tag{25}$$

and $\lambda(X)$ is determined by the equations of motion $\delta E(g)_{\rm sph}/\delta\lambda(X)=0$, which is:

$$\lambda(X)+g4\lambda(X)^3-2\mathrm{P}\int_0^1\frac{dY}{\lambda(X)-\lambda(Y)}\,, \tag{26}$$

where P denotes the principal part of the integral.

Solving this can be carried out as follows. Introduce a density of eigenvalues $\rho_0(\lambda)$ defined by $dX=\rho_0(\lambda)d\lambda$, and since our potential is even, our solution is going to lie on some symmetric interval that we can parameterize as $(-2a,2a)$. We should normalize the density according to $\int_{-2a}^{2a}d\lambda\rho_0(\lambda)=1$. Our stationarity condition can then be written as

$$\frac{V'(\lambda)}{2}=\mathrm{P}\int_{-2a}^{2a}d\mu\frac{\rho(\mu)}{\lambda-\mu}\,, \tag{27}$$

where a prime denotes a λ-derivative (and μ is a dummy variable, hopefully not to be confused with μ seen earlier).

Extending to the complex λ plane, consider the function

$$F(\lambda)=\int d\mu\frac{\rho_0(\mu)}{\lambda-\mu} \tag{28}$$

with the following properties:

- It is analytic on the plane, with a cut on the interval $(-2a, 2a)$;
- As $|\lambda| \to \infty$, $F(\lambda) \to 1/\lambda$ (following from the normalization condition);
- It is real for λ real outside of the cut;
- It has a discontinuity across the cut:

$$F(\lambda \pm i\epsilon) = \frac{1}{2}V'(\lambda) \mp i\pi\rho_0(\lambda) \,. \tag{29}$$

The above properties fix $F(\lambda)$ to be of the form:

$$F(\lambda) = \frac{1}{2}V'(\lambda) - \frac{P(\lambda)}{2}\sqrt{(\lambda-2a)(\lambda+2a)} \,, \tag{30}$$

where $P(\lambda)$ is quadratic in λ. (As we wil see later, this is a special case of something more general, where when $V(\lambda)$ is of order p, the polynomial is of order $p-2$.) The density is then extracted as:

$$\rho_0(\lambda) = \frac{1}{2\pi}P(\lambda)\sqrt{(4a^2-\lambda^2)} \,. \tag{31}$$

Some experimentation (expand the square root in for large λ) shows that with $P(\lambda) = A\lambda^2 + B\lambda + C$, one gets $A = 4g$, $B = 0$ and $C = 1 + 8ga^2$ (from simply matching to the terms in $V'(\lambda)/2$) and setting the coefficient of λ^{-1} to unity results in the following equation results for a:

$$12ga^4 + a^2 - 1 = 0 \,. \tag{32}$$

How do we use this solution? Well, first note that rewriting the integrals in equation (25) as λ integrals (with the aid of our explicit ρ_0 and also equation (26) allows $E(g)_{\rm sph}$ to be written as:

$$E(g)_{\rm sph} - E(0)_{\rm sph} = \frac{1}{24}(a^2-1)(9-a^2)) - \frac{1}{2}\log a^2 \,. \tag{33}$$

Consider working perturbatively around $g = 0$. One can write:

$$a^2 = \frac{1}{24g}\left[(1+48g)^{\frac{1}{2}} - 1\right] \tag{34}$$

$$= 1 - 12g + 288g^2 - g^3 + 290304g^4 - 10450944g^5 + \cdots \tag{35}$$

From here one can solve for $E(g)_{\rm sph}$ perturbatively, getting

$$E(g)_{\rm sph} - E(0)_{\rm sph} = 2g - 18g^2 + 288g^3 - 6048g^4 + \frac{746496}{5}g^5 + \cdots \,, \tag{36}$$

and each of these numbers can be associated to the number of ways of drawing a diagram at genus 0 (a planar diagram) with 1, 2, 3, 4, and 5 vertices. See figure XX. Try computing a few and checking! This is precisely the sort of entropic factors $Z_{h,n}$ we sought, for $h = 0$. You might be worried about the alternating signs here, but it is ok. Throw in an overall minus sign to match to the $\log Z(N, g)$ we're really interested in, and then the odd powers of g have minus signs, but this matches the fact that the sign of g in the original matrix model is such that it is really $(-g)^n$ that should multiply diagrams with n vertices (tessellations with n faces).

Well, overall this is all very nice, but it is not the theory we are looking for. Spheres made out of a handful of tesselated squares are not very good approximations to smooth geometry! We need to take a continuum limit.

1.3.3 A critical point and a continuum limit: Double scaling

In fact, we need to make the number of squares large, and we find that regime by going to the edge of the radius of convergence of the small g expansion, where the perturbative treatment begins to break down. This is at $g_c = -\frac{1}{48}$. Following our noses and expanding in $g - g_c$ small, we find that the leading non-trivial behaviour is:

$$N^2 E(g)_{\text{sph}} \simeq N^2 (g_c - g)^{\frac{5}{2}} + \cdots \tag{37}$$

(there are higher order terms that won't survive the scaling limit to be taken shortly), which translates to the following behaviour for the connected diagrams in terms of the bare parameters we wrote previously:

$$\log(Z_{\text{sph}}(N, g)) \simeq \frac{1}{\nu_B^2} (\mu_B - \mu_c)^{\frac{5}{2}} + \cdots \tag{38}$$

So now comes the famous double scaling limit. While taking the large N limit, let's also approach this special point, and correlate the rate at which we take large N with the approach to the critical point. Write (Brezin-Kazakov, Gross-Migdal, Douglas-Shenker):

$$\mu_B - \mu_c = \mu \delta^4 \,, \quad \nu_B \equiv \frac{1}{N} = \hbar \delta^5 \,, \tag{39}$$

where $\delta \to 0$ in the limit, and we are being playful with the notation $\hbar$ for the renormalized topological expansion parameter $1/N$, but this will fit with modern conventions you'll see elsewhere. Think of δ as setting a length scale, like the size of a square in the tessellation. For a given surface of fixed physical size we are using more and more squares to approximate it (as we are making them smaller) and so the continuum is being approached, and we see that our gravity partition function computed by our methods at this order is:

$$Z_{\text{sph}} \simeq \frac{1}{\hbar^2} \mu^{\frac{5}{2}} \,, \tag{40}$$

which is just the behaviour we hoped to see from the KPZ scaling discussion around equation (13).

1.3.4 Living on the Edge

In fact, we could have done this analysis starting with a cubic matrix model. This would correspond to tessellation with triangles. The detailed expressions one gets turn out to be quite different. V' is quadratic now, so $P(\lambda)$ must be linear, for example. The spectral density is not symmetric any more. But still it will turn out that you can tune the coupling in the potential to find the scaling behaviour (40), suggesting that there is some universality going on.

Stepping back, where does this interesting universal behaviour come from? How can we get more of it? The core point is that the *edge* of the spectral density is controlling the behaviour in the double-scaling limit. We shall develop techniques for speaking directly to that in Lecture 2, but for now from this lecture we have the language to state it clearly:

At generic g, the edge of the density is just a square root fall-off. This is already interesting and well-studied in fact, and there are communities of statistical physicists who live at this square root edge and learn all kinds of universal physics from it. They even scale into it, in a way similar to what we do with gravity! For an all-too-brief moment there, our community and those communities ran along in parallel (early to mid-1990s), glancing over at each other, in an interesting bit of history. (I'll give some references later, but you can get a head start by googling the Tracy-Widom distribution for example, and note the date on that paper.)

However, at $g = g_c$, notice that $a^2 = 2$, and the $P(\lambda)$ we computed above becomes $\frac{1}{12}(8 - \lambda^2)$, resulting in the spectral density:

$$\rho_0(\lambda) = \frac{1}{24\pi}(8 - \lambda^2)^{\frac{3}{2}} . \tag{41}$$

The fall-off has changed to a $3/2$ power at each edge. This is at the root of the change in universality.

The same change in universality can come about with triangles (cubic potentials) since a linear $P(\lambda)$ is enough to change one or the other edge (but not both, like in the symmetric case) from $(\lambda - \lambda_0)^{\frac{1}{2}}$ to $(\lambda - \lambda_0)^{\frac{3}{2}}$ behaviour.

From the point of view of the gravity theory we find in the continuum limit it is saying that there's universal physics that does not care about the details of the tessellation (triangles or squares) which is precisely as it should be!

1.3.5 Multicritical points and double scaling

First a pause for a change of convention. What will follow will fit better if in the example above we send $\lambda \to \sqrt{2}\lambda$, with the result that $V(\lambda) = \lambda^2 + g\lambda^4$ and then $g_c = -\frac{1}{12}$. The resulting spectral density is on the interval $(-2, 2)$:

$$\rho_0(\lambda) = \frac{P(\lambda)}{2\pi}\sqrt{4 - \lambda^2} . \tag{42}$$

It's clear what to do next (Kazakov '89). Including higher order potentials, say $V(\lambda)$ of even order $p = 2k$, will result in a $P(\lambda)$ of order $p - 2$. That allows for tuning parameters to give $P(\lambda) \sim (4 - \lambda^2)^{\frac{p}{2}-1}$ and hence a spectral density with edges that fall off as $(\lambda - 2)^{k-\frac{1}{2}}$. In fact, one can derive these "multicritical" potentials under just such considerations and they are:

$$V_{2k}(\lambda) = \sum_{m=1}^{k}(-1)^{m-1}\frac{k!(m-1)!}{(k-m)!(2m)!}\lambda^{2m} , \tag{43}$$

(*e.g.* the familiar $V_4 = \lambda^2 - \frac{1}{12}\lambda^4$, and $V_6 = \frac{3}{2}\lambda^2 - \frac{1}{4}\lambda^4 + \frac{1}{60}\lambda^6$) and the spectral densities are:

$$\rho_0(\lambda) = \frac{1}{\pi}\frac{(k!)^2}{(2k)!}(4 - \lambda^2)^{k-\frac{1}{2}} . \tag{44}$$

Playing a bit more will yield that the generalization of (40) that results from this is:

$$Z_{\rm sph} \simeq \frac{1}{\hbar^2}\mu^{k+\frac{1}{2}} , \tag{45}$$

where the case we did upstairs was $k = 2$. Looking back at (13) suggests that there could exist a sensible continuum theory for every k such that $\gamma_{\rm str}$ on the sphere is:

$$\gamma_{\rm str}^{(0)} = 2 - \left(k + \frac{1}{2}\right) = 1 - \frac{2k-1}{2} . \tag{46}$$

1.3.6 A special family of CFTS

As mentioned above, for the X sector, we're going to focus on some $c \leq 1$ CFTs, the (p, q) minimal models. The integers p and q are mutually prime (and when $|p - q| = 1$, the model is unitary). The central charge is:

$$c = 1 - \frac{6(p-q)^2}{pq} . \tag{47}$$

Some examples: The $(3,2)$ model has $c=0$ and is simply the trivial theory, containing only the vacuum, while the $(4,3)$ theory is a celebrity: It has $c=\frac{1}{2}$ and is the critical Ising model. The $(5,2)$ theory is also of great interest in some circles, with central charge $c=-\frac{22}{5}$ it is non-unitary, and is the critical Lee-Yang model, which can be thought of as the Ising model in a background magnetic field tuned to special imaginary values. Finally, the $(1,2)$ model is also interesting. It has $c=-2$ and pertains to a certain topological model once gravity is included, as we'll mention (I hope) later.

It is fun to work out the Liouville parameters Q and b in equation (6) for this family, with the nice results:

$$Q=\frac{(p+q)}{\sqrt{pq}}\,,\qquad b=\sqrt{\frac{q}{p}}\,, \tag{48}$$

and the genus zero string susceptibility is simply:

$$\gamma_{\rm str}^{(0)}=2-\frac{Q}{b}=1-\frac{p}{q}\,. \tag{49}$$

Note that the $(1,2)$, $(3,2)$ and $(5,2)$ models are early entries in a special series, which we can write as the $(2k-1,2)$ series. They are $k=1,2,3$ cases, and yes this is the same k of the previous subsection! We see that the $\gamma_{\rm str}^{(0)}$ for these is precisely that predicted below equation (45). That's pretty nice, I hope you agree.

1.4 Looking ahead

Before moving forward, it is time to develop some more powerful methods for tackling all this, and through which some of the statements made above about edges and universality will become much more straightforward. And that's just the start of what we can do with those methods! They are the orthogonal polynomial methods developed originally in this context in the classic Bessis, Itzykson and Zuber ('80) paper.