MODULI SPACES OF RIEMANN SURFACES – EXERCISES

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Lecture 1: moduli spaces of Riemann surfaces and their stratification

Exercise 1.

(1) *Consider a genus* 0 *curve with three marked points* $(\mathbb{P}^1, p_1, p_2, p_3)$*. Find the (unique)* $g \in \mathrm{PSL}(2, \mathbb{C})$ *that maps* $(\mathbb{P}^1, p_1, p_2, p_3)$ *to* $(\mathbb{P}^1, 0, 1, \infty)$*.*

(2) *Consider a genus* 0 *curve with four marked points* $(\mathbb{P}^1, p_1, p_2, p_3, p_4)$*. The element* $g \in \mathrm{PSL}(2, \mathbb{C})$ *found in part (1) maps* $(\mathbb{P}^1, p_1, p_2, p_3, p_4)$ *to* $(\mathbb{P}^1, 0, 1, \infty, t)$*. Find an expression for* t *as a function of* p_1, p_2, p_3, p_4*.*

Exercise 2. *For the reader familiar with Riemann–Roch and Riemann–Hurwitz, convince yourself that the complex dimension of* $\mathcal{M}_g = \mathcal{M}_{g,0}$ *is* $3g - 3$*. To this end, consider the moduli space of pairs* (Σ, f)*, where* Σ *is a genus* g *Riemann surface and* f *is a degree* d *holomorphic map from* Σ *to* $\mathbb{P}^1$ *(i.e. a meromorphic function on* X*). Such a space is sometimes referred to as a Hurwitz space, denoted* $\mathcal{H}_{g,d}$*. Compute its dimension in two different ways.*

- *The dimension of* $\mathcal{H}_{g,d}$ *equals the dimension of* $\mathcal{M}_g$*, counting the "number of deformation parameters" of the Riemann surface* Σ*, plus the "number of deformation parameters" of the function* f*. Compute the latter via Riemann–Roch.*
- *Directly compute the dimension of* $\mathcal{H}_{g,d}$ *using Riemann–Hurwitz.*

Conclude that $\dim \mathcal{M}_g = 3g - 3$*.*

Exercise 3. *The Euler characteristic of an orbifold* X *is defined as*

$$\chi(X) = \sum_G \frac{\chi(X_G)}{|G|}\,, \tag{0.1}$$

where X_G *is the locus of points with automorphism group* G*. Prove that* $\chi(\mathcal{M}_{1,1}) = -\frac{1}{12}$*.*

Exercise 4.

(1) *List all strata of* $\overline{\mathcal{M}}_{2,1}$*.*

(2) *Consider a stable graph* Γ *of type* (g, n)*. Show that the dimension of the stratum is* $\dim(\mathcal{M}_\Gamma) = \dim(\overline{\mathcal{M}}_{g,n}) - |E_\Gamma|$*.*

Lecture 2: Witten's conjecture

Exercise 5. *Employ the geometric string and dilaton equations, together with the projection formula and the expression $[\Gamma] = \frac{1}{|\mathrm{Aut}(\Gamma)|}\xi_{\Gamma,*}\mathbf{1}$ for the Poincaré dual of boundary strata, to prove the following equations satisfied by Witten's correlators.*

- ***String equation.*** *Integrals over $\overline{\mathcal{M}}_{g,n+1}$ with no ψ_{n+1} are reduced to integrals over $\overline{\mathcal{M}}_{g,n}$:*
$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{d_1} \cdots \psi_n^{d_n} = \sum_{i=1}^{n} \int_{\overline{\mathcal{M}}_{g,n}} \left(\prod_{j \neq i} \psi_i^{d_i} \right) \psi_i^{d_i - 1} . \tag{0.2}$$
In Witten's notation, the string equation amounts to the removal of a τ_0:
$$\langle \tau_{d_1} \cdots \tau_{d_n} \tau_0 \rangle_g = \sum_{i=1}^{n} \langle \tau_{d_1} \cdots \tau_{d_i - 1} \cdots \tau_{d_n} \rangle_g . \tag{0.3}$$
- ***Dilaton equation.*** *Integrals over $\overline{\mathcal{M}}_{g,n+1}$ with a single power of ψ_{n+1} are reduced to integrals over $\overline{\mathcal{M}}_{g,n}$:*
$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{d_1} \cdots \psi_n^{d_n} \psi_{n+1} = (2g - 2 + n) \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} . \tag{0.4}$$
In Witten's notation, the string equation amounts to the removal of a τ_1:
$$\langle \tau_{d_1} \cdots \tau_{d_n} \tau_1 \rangle_g = (2g - 2 + n) \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g . \tag{0.5}$$

Exercise 6. *Knowing the string equation and the integral $\int_{\overline{\mathcal{M}}_{0,3}} \mathbf{1} = \langle \tau_0^3 \rangle_0 = 1$, show that all genus 0, ψ-class intersection numbers are determined. Can you prove the following closed formula:*
$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_0 = \binom{n-3}{d_1, \ldots, d_n} , \tag{0.6}$$
where $\binom{D}{d_1,\ldots,d_n} = \frac{D!}{d_1! \cdots d_n!}$ is the multinomial coefficient?

Exercise 7. *Knowing the string equation, the dilaton equation, and the integral $\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \langle \tau_1 \rangle_1 = \frac{1}{24}$, show that all genus 1, ψ-class intersection numbers are determined. Can you prove the following closed formula:*
$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_1 = \frac{1}{24} \left(\binom{n}{d_1, \ldots, d_n} - \sum_{\epsilon_1, \ldots, \epsilon_n \in \{0,1\}} \binom{n - |\epsilon|}{d_1 - \epsilon_1, \ldots, d_n - \epsilon_n} (|\epsilon| - 2)! \right) , \tag{0.7}$$
where $|\epsilon| = \epsilon_1 + \cdots + \epsilon_n$?

Exercise 8. *Prove that $\langle \tau_1 \rangle_1 = \frac{1}{24}$ using the following facts.*

(1) The following identity holds for arbitrary line bundle $\mathcal{L}$: $c_1(\mathcal{L}) = \frac{1}{k} c_1(\mathcal{L}^{\otimes k})$.
(2) For an arbitrary line bundle $\mathcal{L}$, we have $c_1(\mathcal{L}) = [Z - P]$, where Z and P are the divisors of zeros and poles of a generic meromorphic section of $\mathcal{L}$ and $[\,\cdot\,]$ denotes the Poincaré dual[1].
(3) Consider the cotangent line bundle $\mathcal{L}_1^{\otimes k} \to \overline{\mathcal{M}}_{1,1}$. There is a canonical identification of the vector space of holomorphic sections of $\mathcal{L}_1^{\otimes k}$ and the vector space of modular forms of weight k.

[1]Poincaré duality for orbifolds involves the automorphism group. More precisely, if Z is a sub-orbifold of X with underlying topological space $\hat{Z}$, then $[Z] = \frac{1}{|G|}[\hat{Z}]$, where G is the automorphism group of a generic point in $\hat{Z}$.

(4) The following (combination of) Eisenstein series

$$
\begin{aligned}
G_4(\tau) &= \sum_{\lambda \in (\mathbb{Z}+\tau\mathbb{Z})\setminus\{0\}} \frac{1}{\lambda^4}\,, \\
G_6(\tau) &= \sum_{\lambda \in (\mathbb{Z}+\tau\mathbb{Z})\setminus\{0\}} \frac{1}{\lambda^6}\,, \\
\tilde{G}_{12}(\tau) &= \left(\frac{G_4(\tau)}{2\zeta(4)}\right)^3 - \left(\frac{G_6(\tau)}{2\zeta(6)}\right)^2,
\end{aligned}
\tag{0.8}
$$

are modular forms of weight 4*,* 6*, and* 12 *respectively. Besides, they have a unique simple zero at* $\tau = \frac{1+\mathrm{i}\sqrt{3}}{2}$*,* $\tau = \mathrm{i}$*, and* $\tau = +\mathrm{i}\infty$ *respectively.*

Exercise 9. *Define the differential operators*

$$
L_{-1} = \hbar \frac{\partial}{\partial t_0} - \hbar^2 \left(\sum_{k \geq 1} t_k \frac{\partial}{\partial t_{k-1}} + \frac{t_0^2}{2} \right), \tag{0.9}
$$

$$
L_0 = \hbar \frac{\partial}{\partial t_1} - \hbar^2 \left(\sum_{k \geq 0} \frac{2k+1}{3}\, t_k \frac{\partial}{\partial t_k} + \frac{1}{24} \right). \tag{0.10}
$$

Prove the following:

- *The string equation and* $\langle \tau_0^3 \rangle_0$ *are equivalent to the equation* $L_{-1}\, Z = 0$*.*
- *The dilaton equation and* $\langle \tau_1 \rangle_1 = \frac{1}{24}$ *are equivalent to the equation* $L_0\, Z = 0$*.*

Exercise 10. *Prove that the collection* $(L_n)_{n \geq -1}$ *of differential operators defined by equation (0.9), (0.10), and*

$$
L_n = \hbar \frac{\partial}{\partial t_{n+1}} - \hbar^2 \left(\sum_{k \geq 0} \frac{(2n+2k+1)!!}{(2n+3)!!(2k-1)!!}\, t_k \frac{\partial}{\partial t_{k+n}} + \frac{1}{2} \sum_{\substack{a,b \geq 0 \\ a+b=n-1}} \frac{(2a+1)!!(2b+1)!!}{(2n+3)!!} \frac{\partial^2}{\partial t_a \partial t_b} \right) \tag{0.11}
$$

for $n \geq 1$ *is indeed a representation of the Virasoro algebra:* $[L_m, L_n] = \hbar^2 (m-n) L_{m+n}$*. This, together with the form (0.11) of the operators, proves that* $(L_n)_{n \geq -1}$ *form an Airy ideal (see Vincent's lectures).*

Exercise 11 (☠). *Show that the Virasoro constraints are equivalent to the following* topological recursion *for Witten's correlators:*

$$
\begin{aligned}
\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g &= \sum_{m=2}^{n} \frac{(2d_1 + 2d_m - 1)!!}{(2d_1+1)!!\,(2d_m-1)!!} \langle \tau_{d_1+d_m-1} \tau_{d_2} \cdots \widehat{\tau_{d_m}} \cdots \tau_{d_n} \rangle_g \\
&\quad + \frac{1}{2} \sum_{a+b=d_1-2} \frac{(2a+1)!!\,(2b+1)!!}{(2d_1+1)!!} \Bigg(\langle \tau_a \tau_b \tau_{d_2} \cdots \tau_{d_n} \rangle_{g-1} \\
&\qquad + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{ d_2,\ldots,d_n \}}} \langle \tau_a \tau_{I_1} \rangle_{g_1} \langle \tau_b \tau_{I_2} \rangle_{g_2} \Bigg).
\end{aligned}
\tag{0.12}
$$

Prove that the above recursion is equivalent to the Eynard–Orantin topological recursion formula (see Vincent's lectures) on the Airy spectral curve $(\mathbb{P}^1,\ x(z) = \frac{z^2}{2},\ y(z) = z,\ B(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2})$*:*

$$\omega_{g,n}(z_1, \ldots, z_n) = (-1)^n \sum_{\substack{d_1, \ldots, d_n \geq 0 \\ d_1 + \cdots + d_n = 3g-3+n}} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \prod_{i=1}^{n} \frac{(2d_i + 1)!!}{z_i^{2d_i+2}} \mathrm{d}z_i \,. \tag{0.13}$$

Lecture 3: cohomological field theories and topological recursion

Exercise 12. *Let (V, η, e, Ω) be a CohFT with unit. Prove that $(V, \eta, e, \star)$ forms a Frobenius algebra, that is, it satisfies*

$$\eta(v_1 \star v_2, v_3) = \eta(v_1, v_2 \star v_3)\,. \tag{0.14}$$

A Frobenius algebra is equivalent to a 2D topological field theory $\mathcal{Z}$ via the following assignments: $\mathcal{Z}(S^1) = V$ for the Hilbert space of states on the circle and

$$\begin{aligned} \mathcal{Z}\big(\quad\big) &= \eta \colon V \otimes V \to \mathbb{Q}\,, \\ \mathcal{Z}\big(\quad\big) &= e \colon \mathbb{Q} \to V\,, \\ \mathcal{Z}\big(\quad\big) &= \star \colon V \otimes V \to V\,, \end{aligned} \tag{0.15}$$

for the morphisms. The partition function $\mathcal{Z}(\Sigma_{g,n,m})$ of any genus g surfaces connecting n initial states to m final states can be reconstructed from the above values using the TFT properties.

Exercise 13. *Prove that $\exp(2\pi^2\kappa_1)$ is the CohFT obtained from the trivial one under the action of the following translation:*

$$T(u) = \sum_{k \geq 1} \frac{(-2\pi^2)^k}{k!} u^{k+1} = u\big(1 - e^{-2\pi^2 u}\big)\,. \tag{0.16}$$

Exercise 14. *Prove, using Mumford's formula, that $\Lambda(t)\Lambda(-t) = 1$. This is sometimes referred to as Mumford's relation. Deduce the relations $\lambda_g^2 = 0$.*

Exercise 15 (). *Show that the CohFT associated to the following spectral curve*

$$\left(\mathbb{P}^1,\ x(z) = -f\log(z) - \log(1-z),\ y(z) = -\log(z),\ B(z_1,z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}\right)\,. \tag{0.17}$$

is the triple Hodge $\Lambda(1)\Lambda(f)\Lambda(-f-1)$. This is the CohFT underlying the (framed) topological vertex. The large framing limit recovers the so-called Lambert curve computing Hurwitz numbers.

Exercise 16. *Consider the spectral curve*

$$\left(\mathbb{P}^1,\ x(z) = \frac{z^2}{2},\ y(z) = \frac{\sin(2\pi z)}{2\pi z},\ B(z_1,z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}\right)\,. \tag{0.18}$$

Using the CohFT/topological recursion correspondence and the expression for the Weil–Petersson form $\exp(2\pi^2\kappa_1)$ in terms of Givental's action (exercise 13), show that the topological recursion correlators associated to the above spectral curve compute the differential of the Laplace transform of the Weil–Petersson volumes:

$$\omega_{g,n}(z_1,\dots,z_n) = \mathrm{d}_{z_1} \cdots \mathrm{d}_{z_n} \left(\prod_{i=1}^{n} \int_0^\infty \mathrm{d}L_i\, e^{-z_i L_i}\right) V_{g,n}^{\mathrm{WP}}(L_1,\dots,L_n)\,. \tag{0.19}$$