Les Houches Lectures on 2D Gravity and Random Matrix Models

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Abstract

(This is a draft! Also, bibliography to come!) These four lectures ~~were~~ (are to be) given at the Les Houches 2024 Summer School on Quantum Geometry, 5th–9th August 2024. The material begins with early motivations for studying 2D quantum gravity: as a route to understanding the path integral definition of string theory, but then moves on to the effective 2D quantum gravity that arises when studying near extreme charged black holes. Both applications emphasize the two primary (and intersecting) interpretations of the random matrix model: As a 't Hooftian means of defining the sum over surfaces through tesselations, and as a Wignerian means of statistically characterising the spectrum of the (holographic dual) Hamiltonian of the theory. Perturbative and non-perturbative aspects are uncovered in detail.

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1 Lecture 1

Before diving into the details of quantizing 2D gravity, it is worth reminding ourselves why we want to do such a thing at all. Of course, one simple reason is that it is a simple enough theory that we might learn useful lessons for quantizing gravity in other dimensions, but there are (at least) two major areas where issues of quantum gravity in higher dimensions already rely on our understanding of quantum gravity in 2D. The first is string theory. A common approach to it involves formulating the dynamics of strings moving in some background in terms of a two dimensional "Polyakov" action for the worldsheet of the string. The spacetime cooordinates resemble fields in some 2D spacetime, but the dynamics also involves summing over all possible geometries and topologies of the worksheet, which is a form of quantum gravity. The second motivation is the physics of higher dimensional black holes, where (for example) the low temperature physics of Reissner-Nordström black holes reduces to a study of the effective 2D theory that governs the geometry of the near-horizon region.

1.1 Motivations from string theory

Outside of the famous critical dimensions of string theory, the effective scalar coming from Weyl rescaling of the 2D metric does not decouple from the theory, and must be included for consistency. What is typically done is to choose a reference metric $\hat{g}_{ab}$ and write the physical metric as conformal to that: $g_{ab} = \mathrm{e}^{2\phi}\hat{g}_{ab}$, where ϕ is a scalar for which there is a a special 2D "Liouville" conformal field theory action. There's also additional fields (we'll refer to as X generically) with their own CFT action (for example they could be interpreted as additional spacetime coordinates) and of course there are the Fadeev-Popov ghosts from gauge fixing, which have their own CFT. These theories are all coupled together implicitly through the requirement that the total central charge of the theory should vanish. Ignoring the ghost sector henceforth, the problem of Liouville+matter is an important 2D quantum gravity theory with features that will inspire us later, so we should talk about it.

1.1.1 Liouville theory plus matter

The matter+Liouville sector has partition function:

$$Z = \int D\varphi DX \mathrm{e}^{-S_{\mathrm{tot}}} \,, \tag{1}$$

where $S_{\rm tot}$ includes $S_{\rm L}(\varphi)$ and $S_{\rm matter}(X)$, which will be taken to be some conformal field theory representing the matter sector such that

$$c_{\rm L} + c = 26 \,, \tag{2}$$

cancelling the -26 coming from the ghost sector. These days the Liouville action is usually written for some $\varphi = \phi/b$ in the following form:

$$S_{\rm L} = \frac{1}{4\pi}\int d^2z\sqrt{\hat{g}}\left\{\hat{g}^{ab}\partial_a\varphi\partial_b\varphi + Q\hat{R}\varphi + \mu e^{2b\varphi}\right\} \,, \tag{3}$$

where the Q term modifies the stress tensor for a scalar to (writing *e.g.* the holomorphic part)

$$T(z) = -\partial_z\varphi\partial_z\varphi + Q\partial_z^2\varphi \,, \tag{4}$$

with which (in computing the usual OPE) for a vertex operator $\exp(ik\cdot\varphi)$ yields the following (Q-shifted) conformal dimension: $\Delta = -\left(\frac{1}{4}k^2 + \frac{i}{2}Qk\right)$. The last term in the action looks like the insertion of a background involving such a vertex operator with $ik = 2b$, and since conformal invariance requires that $\Delta = 1$ we obtain the condition:

$$Q = b + b^{-1} \,. \tag{5}$$

The $T(z)T(z)$ OPE yields the central charge $c_L = 1+6Q^2$ for the Liouville sector, and from (2):

$$Q = \sqrt{\frac{25-c}{6}} \,, \quad \text{with} \quad b = \frac{Q}{2} - \frac{\sqrt{Q^2-4}}{2} = \frac{1}{2}\left[\sqrt{\frac{25-c}{6}} - \sqrt{\frac{1-c}{6}}\right] \,, \tag{6}$$

where the sign of the root was chosen to match to the classical limit, which is $b \to 0$ or $Q \to \infty$. We will discuss this limit soon.

Recalling that in this action $\hat{g}_{ab}$ is a reference metric and that it is $\hat{g}_{ab}e^{2b\varphi}$ that is the physical metric, we see that the last term (times $\sqrt{\hat{g}}$) is really the determinant of the physical metric, and hence its integral gives the two dimensional area $A = \int d^2z\sqrt{\hat{g}}\,e^{2b\varphi}$. Then μ has the interpretation as a cosmological constant.

1.1.2 KPZ scaling and a critical point

It is natural to wonder about the partition function computed by integrating over surfaces of fixed area, which we can denote as:

$$Z|_A = \int D\varphi DX e^{-S_{\rm tot}}\delta\left(\int d^2z\sqrt{\hat{g}}\,e^{2b\varphi} - A\right) \,, \tag{7}$$

and so the total partition function is

$$Z = \int_0^\infty dA e^{-\mu A} Z_A \,, \tag{8}$$

a Boltzman factor with an entropic weight. It is interesting to ask what the large A behaviour of Z_A is, and the result (of refs. [?, ?] is that

$$Z|_A \sim A^{(\gamma_{\rm str}^{(0)}-2)\frac{\chi}{2}-1} = A^{\gamma_{\rm str}-3} \tag{9}$$

where $\gamma_{\rm str}^{(0)}$ is often called the (genus zero) string susceptibility, while $\gamma_{\rm str} = 2-(2-\gamma_{\rm str}^{(0)})\chi/2$ is the string susceptibility at genus h, and $\chi = 2-2h$ is the Euler number of the surface. Using the form of the Liouville action, the dependence of $\Gamma_{\rm str}$ on the parameters of the theory can be deduced from a scaling argument. Shifting $\varphi \to \varphi + \rho/2b$, the Liouville term linear in φ shifts the action by $Q\rho\chi/2b$, and within the δ function the integral gets scaled by e^{ρ}. Since $\delta(\alpha x) = \delta(x)/\alpha$, we can write:

$$Z|_A = e^{-\frac{Q\rho\chi}{2b}-\rho} Z|_{e^{-\rho}A} \,, \tag{10}$$

whereupon setting $e^{\rho} = A$ yields:

$$\gamma^{(0)}_{\rm str} = 2 - \frac{Q}{b} = \frac{1}{12}\left[(c-1) - \sqrt{(c-1)(c-25)}\right] \,, \quad \text{or} \tag{11}$$

$$\gamma_{\rm str} = 2 - \left(\frac{1-h}{12}\right)\left[25 - c + \sqrt{(c-1)(c-25)}\right] \,. \tag{12}$$

(To summarize, we've effectively used the ability to shift φ to rescale area A.) Notice that the asymptotic behaviour (9) allows integral (8) to be done, giving:

$$Z \sim \mu^{2-\gamma_{\rm str}} \,. \tag{13}$$

This is a characteristic behaviour that we'll seek for later. For example for "pure gravty", *i.e.* $c = 0$, on the sphere $g = 0$, $\gamma^{(0)}_{\rm str} = -\frac{1}{2}$, giving $Z \sim \mu^{\frac{5}{2}}$. Notice also that this behaviour also means that the expectation value of the area, $\langle A\rangle = -\partial \ln Z/\partial\mu$, diverges as $\mu \to 0$, which makes sense, and $\gamma_{\rm str}$ is a measure of the rate.

1.1.3 A special classical limit

It's interesting to rescale the Liouville field according to $\varphi = \phi/b$, in which case the action can be wrriten:

$$S_{\rm L} = \frac{1}{4\pi b^2}\int d^2z\sqrt{\hat{g}}\left\{\hat{g}^{ab}\partial_a\phi\partial_b\phi + (1+b^2)\hat{R}\phi + b^2\mu e^{2\phi}\right\} \,, \tag{14}$$

with equations of motion:

$$2\nabla^2_{\hat{g}}\phi - \hat{R} = b^2(2\mu e^{2\phi} + \hat{R}) \,. \tag{15}$$

Here, the parameter b^2 is playing the role of $\hbar$, and there's an analogue of a classical limit, where $b \to 0$ and hence $Q \to \infty$. Note that the central charges diverge: $c_L \to +\infty$ and $c \to -\infty$. Writing the above in terms of the physical metric $g_{ab} = e^{2\phi}\hat{g}_{ab}$ and using the identity relating 2D Ricci scalars:

$$R(e^{2\phi}\hat{g}) = e^{-2\phi}\hat{g}(R(\hat{g}) - 2\nabla^2_{\hat{g}}\phi) \,, \tag{16}$$

the classical equation of motion is simply

$$R = -2\mu b^2 \,, \tag{17}$$

which is Liouville's equation, telling us here (if we hold constant $\mu b^2 = 1$) that the physical metric has constant negative curvature: $R = -2$. This will likely remind you of things (to be) seen in the lectures of Turiaci on JT gravity. Let's have a glance at some of that story.

1.2 Motivations from black holes in $D > 2$.

1.2.1 Reissner-Nordström black hole in $D = 4$

Quick (maybe to be expanded) description of (say) $D = 4$ Reissner-Nordström black holes follows: The extremal (T=0) limit of the metric is AdS$_2 \times S^2$, with the cosmological constant of AdS$_2$ and radius of the S^2 set by the charge $\bar{Q}$. There is an $SL(2,\mathbb{R})$ conformal symmetry due to the AdS$_2$ factor. The black hole has an entropy $S_0 = A/4G_N$, where A is the area of the horizon.

1.2.2 The near-horizon, low-temperature limit

The low-temperature geometry is "nearly" AdS_2 and the sphere's size can fluctuate away from that set by $\bar{Q}$. There is an effective 2D theory of gravity describing the dynamics, with a special coupling to a field Φ that represents the fluctuations of the sphere's radius, resulting in an action that schematically takes the (Euclidean) form:

$$S_{\mathrm{JT}} = -\frac{1}{2}\int_M \sqrt{g}\Phi(R+2) - \int_{\partial M}\sqrt{h}\Phi_b(K-1) - S_0\left(\frac{1}{4\pi}\int_M \sqrt{g}R + \frac{1}{2\pi}\int_{\partial M}\sqrt{h}K\right), \quad (18)$$

where R is the Ricci scalar of metric g_{ab}, and for the boundary (∂M) terms, Φ_b is the value of Φ there, h is the induced metric and K is the extrinsic curvature. Euclidean signature has been chosen, such that Euclidean time has period $\beta = 1/T$.

Note that S_0 multiplies $\chi = 2-2h-b$, the Euler number of M, where h is the number of handles and b the number of boundaries. (Not to be confused with Liouville parameter!!) This will result in a factor of $\mathrm{e}^{\chi S_0}$ in computations using this action. For example, as we will see shortly, for the partition function $Z_{JT}(\beta) = \int D\Phi Dg \exp(-S_{\mathrm{JT}})$, the leading (disc) order will have a factor e^{S_0} since there $h=0$ and $b=1$.

The equation of motion for Φ dynamically sets $R=-2$, and the dynamics reduces to a careful treatment of the boundary dynamics. The latter is a special theory of quantum mechanics where the boundary can fluctuate according to a Schwarzian action, while keeping its length fixed to be β. Turiaci's lectures discuss this in detail, and a key exhibit of this Schwarzian dynamics is the result for the disc partition function:

$$Z_0(\beta) = \frac{\mathrm{e}^{S_0}\gamma^{3/2}}{\sqrt{2\pi}}\frac{1}{\beta^{3/2}}\mathrm{e}^{\frac{2\pi^2\gamma}{\beta}}, \quad (19)$$

and $\gamma = \Phi_r$, a (renormalized) value of the scalar on the asymptotic boundary where the Schwarzian is defined. (We'll likely set it to $1/2$ later on in these lectures.)

1.2.3 Jackiw-Teitelboim (JT) gravity and the Euclidean GPI

(Probably will say a bit about the general case involving multiple insertions of $Z(\beta)$, and any number of handles.... decomposition into gluing trumpets onto Weil-Petersson volumes. Compute or show a few examples maybe.)

1.3 Performing the Euclidean Path Integral

Let us now step back and look at the problem we've been trying to get to grips with. The core point is that there is some two dimensional gravity theory of g_{ab}, which is largely topological, made interesting by either having a "quantum" deformation (going to non-zero b for Liouville) or by virtue of how it is coupled to an additional sector (dynamical Φ for Jackiw-Teitelboim) or both. The gravitational path integral approach to either story involves summing over all metrics of all topologies. In the case of JT gravity, we were lucky that the dynamics froze the bulk dynamics to be constant curvature metrics, in which case the topological sum over metrics boiled down to enumeration of the properties of the volume of moduli space of hyperbolic metrics, with some needlework to do at the geodesic boundaries in order to connect them to the Schwarzian dynamics living at the asymptotic boundaries where Φ_b lives.

How do we do the sum over metrics and topologies in more general cases? Moreover, what lies beyond the (asymptotic) expansion in the topological expansion parameter? What do we learn about the whole business of defining quantum gravity as a path integral over all geometries and topologies? It's entirely an educated guess/analogy inspired by successes with (non-gravitational) theories, but it isn't guaranteed to be a complete definition. Can we learn in 2D whether it is or not?

To begin to answer some of these questions, let's return to the simplest model we can think of, and start approaching the problem from scratch.

1.3.1 Hermitian random matrix model, tessellations and topology

A possible way to handle the sum over metrics and topologies is to make the problem more manageable by breaking the surface up into pieces. Imagine (see figure XX) using little quadrilaterals ("squares") of some fixed area to build up curved surfaces.

- We can make discrete surfaces of any topology by gluing them together to make something with v vertices, e edges, and n faces (the number of squares). The Euler number of a surface made this way is $\chi = v - e + n$.
- Local positive or negative curvature at some point can be approximated by packing in more or less than four squares meeting.

The partition function representing the sum over all closed surfaces would be of the form:

$$Z_{\text{discrete}} = \sum_{h=0}^{\infty} \nu_B^{2h-2} \sum_{n=0}^{\infty} \mathrm{e}^{-\mu_B n} Z_{h,n} , \tag{20}$$

Where all possible topologies are being summed with some weight $\nu_B^{-\chi}$ and area (total number of squares) are being Boltzman weighted with some bare cosmological constant μ_B.

The question is then how to evaluate the $Z_{h,n}$ "entropic" factors? In this simple case of "pure gravity" where there's no coupling to some other sector to contend with (*i.e.*, $c = 0$ in the language of section 1.1.1), $Z_{h,n}$ is simply the number of tesselations at given genus and area.

Here's a way to count these, (following the classic work of Brezin, Itzykson, Parisi, and Zuber '78). Decorate the squares in one of our diagrams drawn above such that there's a point at the centre of each square, and the lines joining them by crossing the edge connecting adjacent squares. Counting all possible such "dual" diagrams is equivalent to counting our tessellations. But these look like Feynman graphs for some theory of quartic interactions. To be precise, consider the (zero-dimensional) "field theory":

$$Z(N,g) = \mathrm{e}^{-E(N,g)} = \int dM \exp\left\{-N\mathrm{Tr}\left(\frac{M^2}{2} + gM^4\right)\right\} , \tag{21}$$

Where M is an $N \times N$ Hermitian matrix and the measure is $dM = \prod_i M_{ii} \prod_{i<j} dM_{ij} dM^*_{ij}$, the Haar measure over matrix elements.

Propagators in this theory are represented by double lines that can be thought of as each propagating an index of the matrix, and the Feynman rules give a factor of $\frac{1}{N}$ for each propagator and gN for each vertex. (See figure XX))

Let's focus on the factors of N that would result form a computation of a diagram. A connected vacuum diagram with P propagators, V vertices and L loops would have a factor $g^V N^{-P} N^V N^L$ (since every closed loop has a free index running around in it that can take any of N values). So this means that the partition function of our theory (for closed diagrams) can be written as (taking the log for closed sector):

$$\log(Z(N,g)) = \sum_{h=0}^{\infty} \left(\frac{1}{N}\right)^{2h-2} \sum_{n=1}^{\infty} g^n Z_{h,n} , \tag{22}$$

since by the dual construction $V = n$, $P = e$, and $L = v$ and $\chi = v - e - n = 2 - 2h$. So comparing to equation (20) we can identify that the matrix model has just the right sort of parameters we need for our discrete model, *i.e.*, we have a map $\nu_B \longleftrightarrow 1/N$ and $e^{-\mu_B} \longleftrightarrow g$.

Our job now is to evaluate $Z_{h,n}$. Let's start by working at leading order, at sphere topology, which is the leading order in the large N limit.

1.3.2 Leading order in large N, the Dyson gas, eigenvalue repulsion

Since the action of our model involves only the trace of powers of M, it is prudent to work in terms of the eigenvalues of M. The trace gives a large $U(N)$ invariance under $M \to UMU^\dagger$, where $U \in U(N)$, and we can "gauge fix" by writing $M = U\Lambda U^\dagger$ where Λ is the diagonal matrix $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots \lambda_N\}$. The integral over matrix elements of M now becomes an integral over the N eigenvalues λ_i and the volume of the unitary group, which will appear as an overall factor (which we can drop). The result is

$$Z(N,g) = \int \prod_i d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 \exp\left\{-N \sum_i \left(\frac{\lambda_i^2}{2} + g\lambda_i^4\right)\right\} . \tag{23}$$

There is a Jacobian $\Delta^2(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)^2$ appearing as a result of this procedure. $\Delta(\lambda)$ is the Vandermonde determinant. The Jacobian can be computed directly by conputing the determinant of the metric obtained by writing out $(\delta M)^2$ in the decomposition of M into U and Λ, or by using a Fedeev-Popov type procedure, or by reasoning as follows. The change of variables to the eignenvalue basis makes sense for generic λ_i, but it is going to fail whenever any two are identical. (The change of variables can't distinguish between the two, so it is singular there. There's an enhanced symmetry $U(2)$ in this case.) This means the Jacobian J should vanish whenever this happens. This is like the change of variables from rectangular to spherical polars. There $J = r^2 \sin\theta d\theta d\phi dr$. But at $\theta = 0$ and π there is an ambiguity under rotations of x into y, the change of variables is singular, and $J = 0$. So this determines that $J = \prod_{i<j}(\lambda_i - \lambda_j)^\beta$. Dimensional analysis finishes the job. On the one hand, $dM \sim \lambda^{N^2}$, while on the other, $\prod_i d\lambda_i J \sim \lambda^N \lambda^{\beta N(N-1)/2}$, and hence β must be 2.

A suggestive way of writing our eigenvalue problem is as a "Dyson gas":

$$Z(N,g) = \int \prod_i d\lambda_i \exp\left\{-N \sum_i \left(\frac{\lambda_i^2}{2} + g\lambda_i^4\right) + 2\sum_{i<j} \log|\lambda_i - \lambda_j|\right\} , \tag{24}$$

where there are N particles with positions at λ_i, in a potential $NV(\lambda_i) = N(\lambda_i^2/2 + g\lambda_i^4)$ with a logarithmic (1D Coulomb) interparticle repulsion. Let's look at the $N \to \infty$ limit. Intuitively, we should expect a smooth saddle point solution to our system (at least for some range of g) represented by a droplet of eigenvalues formed by the balance between the attraction from the potential dragging them to the origin and the logarithmic repulsion coming from $\sim N$ other elements of the droplet forcing it to spread out.

To find it, replace λ_i by a smooth parameter $\lambda(X)$, where $X = i/N$ runs from 0 to 1, and sums become integrals according to $\frac{1}{N}\sum_i = \int_0^1 dX$. The model is then $Z(N,g) = e^{-N^2 E(g)_{\rm sph}}$, where:

$$E(g)_{\rm sph} = \lim_{N\to\infty} \left\{ \int_0^1 dX \left(\frac{\lambda(X)^2}{2} + g\lambda(X)^4\right) + \int_0^1 \int_0^1 dX dY \log|\lambda(X) - \lambda(Y)| \right\} , \tag{25}$$

and $\lambda(X)$ is determined by the equations of motion $\delta E(g)_{\rm sph}/\delta\lambda(X) = 0$, which is:

$$\lambda(X) + g4\lambda(X)^3 - 2\mathrm{P}\int_0^1 \frac{dY}{\lambda(X) - \lambda(Y)} , \tag{26}$$

where **P** denotes the principal part of the integral.

Solving this can be carried out as follows. Introduce a density of eigenvalues $\rho_0(\lambda)$ defined by $dX = \rho_0(\lambda)d\lambda$, and since our potential is even, our solution is going to lie on some symmetric interval that we can parameterize as $(-2a, 2a)$. We should normalize the density according to $\int_{-2a}^{2a} d\lambda \rho_0(\lambda) = 1$. Our stationarity condition can then be written as

$$\frac{V'(\lambda)}{2} = \mathrm{P}\int_{-2a}^{2a} d\mu \frac{\rho(\mu)}{\lambda - \mu}\,, \tag{27}$$

where a prime denotes a λ-derivative (and μ is a dummy variable, hopefully not to be confused with μ seen earlier).

Extending to the complex λ plane, consider the function

$$F(\lambda) = \int d\mu \frac{\rho_0(\mu)}{\lambda - \mu} \tag{28}$$

with the following properties:

- It is analytic on the plane, with a cut on the interval $(-2a, 2a)$;
- As $|\lambda| \to \infty$, $F(\lambda) \to 1/\lambda$ (following from the normalization condition);
- It is real for λ real outside of the cut;
- It has a discontinuity across the cut:

$$F(\lambda \pm i\epsilon) = \frac{1}{2}V'(\lambda) \mp i\pi\rho_0(\lambda)\,. \tag{29}$$

The above properties fix $F(\lambda)$ to be of the form:

$$F(\lambda) = \frac{1}{2}V'(\lambda) - \frac{P(\lambda)}{2}\sqrt{(\lambda - 2a)(\lambda + 2a)}\,, \tag{30}$$

where $P(\lambda)$ is quadratic in λ. (As we wil see later, this is a special case of something more general, where when $V(\lambda)$ is of order p, the polynomial is of order $p-2$.) The density is then extracted as:

$$\rho_0(\lambda) = \frac{1}{2\pi}P(\lambda)\sqrt{(4a^2 - \lambda^2)}\,. \tag{31}$$

Some experimentation (expand the square root in for large λ) shows that with $P(\lambda) = A\lambda^2 + B\lambda + C$, one gets $A = 4g$, $B = 0$ and $C = 1 + 8ga^2$ (from simply matching to the terms in $V'(\lambda)/2$) and setting the coefficient of λ^{-1} to unity results in the following equation results for a:

$$12ga^4 + a^2 - 1 = 0\,. \tag{32}$$

How do we use this solution? Well, first note that rewriting the integrals in equation (25) as λ integrals (with the aid of our explicit ρ_0 and also equation (26) allows $E(g)_{\mathrm{sph}}$ to be written as:

$$E(g)_{\mathrm{sph}} - E(0)_{\mathrm{sph}} = \frac{1}{24}(a^2 - 1)(9 - a^2)) - \frac{1}{2}\log a^2\,. \tag{33}$$

Consider working perturbatively around $g = 0$. One can write:

$$a^2 = \frac{1}{24g}\left[(1 + 48g)^{\frac{1}{2}} - 1\right] \tag{34}$$

$$= 1 - 12g + 288g^2 - g^3 + 290304g^4 - 10450944g^5 + \cdots \tag{35}$$

From here one can solve for $E(g)_{\rm sph}$ perturbatively, getting

$$E(g)_{\rm sph} - E(0)_{\rm sph} = 2g - 18g^2 + 288g^3 - 6048g^4 + \frac{746496}{5}g^5 + \cdots\ , \tag{36}$$

and each of these numbers can be associated to the number of ways of drawing a diagram at genus 0 (a planar diagram) with 1, 2, 3, 4, and 5 vertices. See figure XX. Try computing a few and checking! This is precisely the sort of entropic factors $Z_{h,n}$ we sought, for $h=0$. You might be worried about the alternating signs here, but it is ok. Throw in an overall minus sign to match to the $\log Z(N,g)$ we're really interested in, and then the odd powers of g have minus signs, but this matches the fact that the sign of g in the original matrix model is such that it is really $(-g)^n$ that should multiply diagrams with n vertices (tessellations with n faces).

Well, overall this is all very nice, but it is not the theory we are looking for. Spheres made out of a handful of tesselated squares are not very good approximations to smooth geometry! We need to take a continuum limit.

1.3.3 A critical point and a continuum limit: Double scaling

In fact, we need to make the number of squares large, and we find that regime by going to the edge of the radius of convergence of the small g expansion, where the perturbative treatment begins to break down. This is at $g_c = -\frac{1}{48}$. Following our noses and expanding in $g-g_c$ small, we find that the leading non-trivial behaviour is:

$$N^2 E(g)_{\rm sph} \simeq N^2 (g_c - g)^{\frac{5}{2}} + \cdots \tag{37}$$

(there are higher order terms that won't survive the scaling limit to be taken shortly), which translates to the following behaviour for the connected diagrams in terms of the bare parameters we wrote previously:

$$\log(Z_{\rm sph}(N,g)) \simeq \frac{1}{\nu_B^2}(\mu_B - \mu_c)^{\frac{5}{2}} + \cdots \tag{38}$$

So now comes the famous double scaling limit. While taking the large N limit, let's also approach this special point, and correlate the rate at which we take large N with the approach to the critical point. Write (Brezin-Kazakov, Gross-Migdal, Douglas-Shenker):

$$\mu_B - \mu_c = \mu\delta^4\ , \quad \nu_B \equiv \frac{1}{N} = \hbar\delta^5\ , \tag{39}$$

where $\delta \to 0$ in the limit, and we are being playful with the notation $\hbar$ for the renormalized topological expansion parameter $1/N$, but this will fit with modern conventions you'll see elsewhere. Think of δ as setting a length scale, like the size of a square in the tessellation. For a given surface of fixed physical size we are using more and more squares to approximate it (as we are making them smaller) and so the continuum is being approached, and we see that our gravity partition function computed by our methods at this order is:

$$Z_{\rm sph} \simeq \frac{1}{\hbar^2}\mu^{\frac{5}{2}}\ , \tag{40}$$

which is just the behaviour we hoped to see from the KPZ scaling discussion around equation (13).

1.3.4 Living on the Edge

In fact, we could have done this analysis starting with a cubic matrix model. This would correspond to tessellation with triangles. The detailed expressions one gets turn out to be quite different. V' is quadratic now, so $P(\lambda)$ must be linear, for example. The spectral density is not symmetric any more. But still it will turn out that you can tune the coupling in the potential to find the scaling behaviour (40), suggesting that there is some universality going on.

Stepping back, where does this interesting universal behaviour come from? How can we get more of it? The core point is that the *edge* of the spectral density is controlling the behaviour in the double-scaling limit. We shall develop techniques for speaking directly to that in Lecture 2, but for now from this lecture we have the language to state it clearly:

At generic g, the edge of the density is just a square root fall-off. This is already interesting and well-studied in fact, and there are communities of statistical physicists who live at this square root edge and learn all kinds of universal physics from it. They even scale into it, in a way similar to what we do with gravity! For an all-too-brief moment there, our community and those communities ran along in parallel (early to mid-1990s), glancing over at each other, in an interesting bit of history. (I'll give some references later, but you can get a head start by googling the Tracy-Widom distribution for example, and note the date on that paper.)

However, at $g = g_c$, notice that $a^2 = 2$, and the $P(\lambda)$ we computed above becomes $\frac{1}{12}(8-\lambda^2)$, resulting in the spectral density:

$$\rho_0(\lambda) = \frac{1}{24\pi}(8-\lambda^2)^{\frac{3}{2}} . \tag{41}$$

The fall-off has changed to a $3/2$ power at each edge. This is at the root of the change in universality.

The same change in universality can come about with triangles (cubic potentials) since a linear $P(\lambda)$ is enough to change one or the other edge (but not both, like in the symmetric case) from $(\lambda-\lambda_0)^{\frac{1}{2}}$ to $(\lambda-\lambda_0)^{\frac{3}{2}}$ behaviour.

From the point of view of the gravity theory we find in the continuum limit it is saying that there's universal physics that does not care about the details of the tessellation (triangles or squares) which is precisely as it should be!

1.3.5 Multicritical points and double scaling

First a pause for a change of convention. What will follow will fit better if in the example above we send $\lambda \to \sqrt{2}\lambda$, with the result that $V(\lambda) = \lambda^2 + g\lambda^4$ and then $g_c = -\frac{1}{12}$. The resulting spectral density is on the interval $(-2, 2)$:

$$\rho_0(\lambda) = \frac{P(\lambda)}{2\pi}\sqrt{4-\lambda^2} . \tag{42}$$

It's clear what to do next (Kazakov '89). Including higher order potentials, say $V(\lambda)$ of even order $p = 2k$, will result in a $P(\lambda)$ of order $p-2$. That allows for tuning parameters to give $P(\lambda) \sim (4-\lambda^2)^{\frac{p}{2}-1}$ and hence a spectral density with edges that fall off as $(\lambda-2)^{k-\frac{1}{2}}$. In fact, one can derive these "multicritical" potentials under just such considerations and they are:

$$V_{2k}(\lambda) = \sum_{m=1}^{k}(-1)^{m-1}\frac{k!(m-1)!}{(k-m)!(2m)!}\lambda^{2m} , \tag{43}$$

(*e.g.* the familiar $V_4 = \lambda^2 - \frac{1}{12}\lambda^4$, and $V_6 = \frac{3}{2}\lambda^2 - \frac{1}{4}\lambda^4 + \frac{1}{60}\lambda^6$) and the spectral densities are:

$$\rho_0(\lambda) = \frac{1}{\pi}\frac{(k!)^2}{(2k)!}(4-\lambda^2)^{k-\frac{1}{2}} . \tag{44}$$

Playing a bit more will yield that the generalization of (40) that results from this is:

$$Z_{\mathrm{sph}} \simeq \frac{1}{\hbar^2}\mu^{2+\frac{1}{k}} , \tag{45}$$

where the case we did upstairs was $k = 2$. Looking back at (13) suggests that there could exist a sensible continuum theory for every k such that γ_{str} on the sphere is:

$$\gamma_{\mathrm{str}}^{(0)} = 2 - \left(2 + \frac{1}{k}\right) = -\frac{1}{k} . \tag{46}$$

1.3.6 A special family of CFTs

As mentioned above, for the X sector, we're going to focus on some $c \leq 1$ CFTs, the (p, q) minimal models. The integers p and q are mutually prime (and when $|p - q| = 1$, the model is unitary). The central charge is:

$$c = 1 - \frac{6(p-q)^2}{pq} . \tag{47}$$

Some examples: The $(3, 2)$ model has $c = 0$ and is simply the trivial theory, containing only the vacuum, while the $(4, 3)$ theory is a celebrity: It has $c = \frac{1}{2}$ and is the critical Ising model. The $(5, 2)$ theory is also of great interest in some circles, with central charge $c = -\frac{22}{5}$ it is non-unitary, and is the critical Lee-Yang model, which can be thought of as the Ising model in a background magnetic field tuned to special imaginary values. Finally, the $(1, 2)$ model is also interesting. It has $c = -2$ and pertains to a certain topological model once gravity is included, as we'll mention (I hope) later.

It is fun to work out the Liouville parameters Q and b in equation (6) for this family, with the nice results:

$$Q = \frac{(p+q)}{\sqrt{pq}} , \qquad b = \sqrt{\frac{q}{p}} . \tag{48}$$

The formula derived earlier for the genus zero string susceptibility is $\gamma_{\mathrm{str}}^{(0)} \stackrel{?}{=} 2 - \frac{Q}{b} = 1 - \frac{p}{q}$, but this must be used *only* when the model is unitary. It is only in such a case that the constant μ in the Liouville sector properly acts as a cosmological operator in theory. If the theory is unitary, one can write $p = k + 1$ and $q = k$, in which case

$$\gamma_{\mathrm{str}}^{(0)} = -\frac{1}{k} . \qquad \text{(unitary series)} \tag{49}$$

This matches the result we got above in equation (46), and initially in the literature it was thought this meant that the multicritical points corresponded to the unitary series of models, but this turned out to be an illusion. Taking into account the dimensions of the operators that appear in the model (we will discuss this later) yield a different formula for the critical exponent:

$$\gamma_{\mathrm{str}}^{(0)} = -\frac{2}{p+q-2} , \tag{50}$$

for which there is another series that fits the bill. Note that the $(1, 2)$, $(3, 2)$ and $(5, 2)$ models are early entries in a special series, which we can write as $(2k-1, 2)$ series. They are $k = 1, 2, 3$ cases, and yes *this is* the same k of the previous subsection! We see using (50) that we get

$$\gamma_{\mathrm{str}}^{(0)} = -\frac{1}{k} , \qquad \text{(non-unitary } (2k-1, 2) \text{ series)} \tag{51}$$

which matches the prediction below equation (45). That's pretty nice, I hope you agree.

1.4 Looking ahead

Before moving forward, it is time to develop some more powerful methods for tackling all this, and through which some of the statements made above about edges and universality will become much more straightforward. And that's just the start of what we can do with those methods! They are the orthogonal polynomial methods developed originally in this context in the classic Bessis, Itzykson and Zuber ('80) paper.

2 Lecture 2

2.1 Orthogonal polynomials

Let's start again, writing general even potential $V(\lambda)=\sum_i g_i\lambda^{2i}$, and so:

$$Z(N,g)=\int\prod_i d\lambda_i\prod_{i<j}(\lambda_i-\lambda_j)^2\exp\left\{-\frac{N}{\gamma}\sum_i V(\lambda_i))\right\}\ , \tag{52}$$

and a parameter γ has been introduced for later use. We will carry it along with us for a while, and its value will become apparent later.

Let us a define a family of orthogonal polynomials in λ, denoted $P_n(\lambda)=\lambda^n+\cdots$, that are orthogonal with respect to the measure $d\mu(\lambda)=d\lambda \mathrm{e}^{-\frac{N}{\gamma}V(\lambda)}$:

$$\int d\lambda \mathrm{e}^{-\frac{N}{\gamma}V(\lambda)}P_n(\lambda)P_m(\lambda)=h_n\delta_{nm}\ , \tag{53}$$

where h_n is a normalization, and $h_0=\int d\lambda \mathrm{e}^{-\frac{N}{\gamma}V(\lambda)}$. There is an infinite set of such polynomials $(n,m\in\mathbb{Z}^+)$, but N of them (counting from $n=0$) will be used to define the matrix model. The $P_n(\lambda)$ satisfy a recursion relation:

$$\lambda P_n(\lambda)=P_{n+1}(\lambda)+R_nP_{n-1}(\lambda)\ , \tag{54}$$

for even $V(\lambda)$. (We will discuss cases with non-even potentials later, when we look at another important class of matrix models). The fact that the recursion relation truncates follows from orthogonality (I'll insert a proof later).

Before going any further it is worth noting that the simplest example, a Gaussian model with potential $\frac{N}{\gamma}V(\lambda)\sim\lambda^2$, the P_n are Hermite polynomials $H_n(\lambda)$ This gives a nice intuition for how physics is described for the whole model in terms of the polynomials: For any particular one of the λ_i, the problem is simply the simple harmonic oscillator, for which the eigensolutions are the Hermite functions $\tilde{\psi}_n(\lambda_i)=\mathrm{e}^{-\frac{N}{\gamma}\frac{\lambda^2}{2}}H_n(\lambda_i)$ with energies that have a dependence $\sim(n+\frac{1}{2})$. Each of the N problems with coordinate λ_i can be in any of these quantum oscillator states.

In fact, the N different sectors are largely independent: The physics in one sector does not really affect the physics in another. This will mean that the collective description of the physics across the different sectors (a "many body" description) will be that of a *free* system. The logarithmic repulsion that keeps the λ_i's from coinciding can instead be re-interpreted as having prepared many-body states that are fermionic. We won't pursue that interpretation here, but it can be useful.

By definition, the first step of the following is true, but then the second step uses recursion (54) after multiplying by λ to the left, after which orthogonality is used again:

$$h_{n+1}=\int d\mu P_{n+1}\lambda P_n=\int d\mu[P_{n+2}+R_{n+1}P_n]P_n=R_{n+1}h_n\ , \tag{55}$$

(where the λ dependence is dropped to avoid clutter), resulting in

$$R_n = \frac{h_n}{h_{n-1}} \ . \tag{56}$$

The $P_n(\lambda)$ can be used to rewrite the matrix integral $Z(N, g_i)$ itself. Since (by taking linear combinations of rows or columns) the Vandermonde determinant can be written in terms of the first N of the $P_n(\lambda)$ as:

$$\Delta = \det\|\lambda_i^{j-1}\|_{i,j=1}^N = \det\|P_{j-i}(\lambda_i)\|_{i,j=1}^N \ , \tag{57}$$

it is easy to rewrite $Z(N, g_i)$ in terms of properties of the polynomials, by expanding out the determinant factors and using the orthogonality relation. The outcome will simply be a combinatorial factor multiplied by a factor of h_n from each of the N sectors, with result:

$$Z(N, g_i) = N! \prod_{n=0}^{N-1} h_n \ , \tag{58}$$

but we have another way or writing this given the product relation (56):

$$\begin{aligned} Z(N, g_i) &= N! h_0^N R_1^{N-1} \cdots R_{N-2}^2 R_{N-1} \\ &= N! h_0^N \prod_{n=1}^{N-1} R_n^{N-n} \\ &= N! h_0^N \exp\left(\sum_{n=1}^{N-1} (N-n) \log R_n\right) \\ &= N! h_0^N \exp\left(N^2 \cdot \frac{1}{N} \sum_{n=1}^{N-1} \left(1 - \frac{n}{N}\right) \log R_n\right) \ . \end{aligned} \tag{59}$$

We will take this expression further in a moment, but the key point is to realize that if we know the R_n recursion coefficients, we can compute the partition function outright.

2.2 Determining the polynomials by recursion

So the next step is to seek relations for the R_n, and in fact they follow from simple identities that the model satisfies by definition. Let us study the action of $d/d\lambda$ on the $P_n(\lambda)$. It is clear that

$$\int d\lambda \mathrm{e}^{-\frac{N}{\gamma}V(\lambda)} P_n(\lambda) \frac{d}{d\lambda} P_n(\lambda) = 0 \ , \tag{60}$$

but if the derivative results in two polynomials of the same order being integrated, we get, using (53):

$$\int d\lambda \mathrm{e}^{-\frac{N}{\gamma}V(\lambda)} P_{n-1}(\lambda) \frac{d}{d\lambda} P_n(\lambda) = n h_{n-1} \ . \tag{61}$$

Integrating by parts and using (53) again yields:

$$\frac{N}{\gamma} \int d\lambda \mathrm{e}^{-\frac{N}{\gamma}V(\lambda)} V'(\lambda) P_n(\lambda) P_{n-1}(\lambda) = n h_{n-1} \ . \tag{62}$$

This gives non-trivial recursion relations for R_n, because the $V'(\lambda)$ insertion can be simplified recursively using the relation (54), followed by use of orthogonality to do the resulting integrals.

For example in the Gaussian case, $V'=\lambda$, and so the left hand side is:

$$\frac{N}{\gamma}\int d\lambda e^{-\frac{N}{\gamma}V(\lambda)}(P_{n+1}(\lambda)+R_nP_{n-1}(\lambda))P_{n-1}(\lambda)=\frac{N}{\gamma}R_nh_{n-1}\ , \tag{63}$$

and comparing to the right hand side of (63) yields that $R_n=\gamma n/N$, which is the familiar recursion coefficient for the Hermite polynomials, up to the factor of γ/N.

When higher order potentials are used, there'll be more uses of the recursion relations, resulting in more terms, making the problem non-linear. Let's look at the quartic case we studied before, with $V=\lambda^2+g\lambda^4$, (note the choice $g_2=1$ for the quadratic term *vs* the $\frac{1}{2}$ in section **??**, so there will be slightly different algebra; also, g_4 is just written as g as before). After a bit of algebra the result is:

$$R_n\left[2+4g(R_{n-1}+R_n+R_{n+1})\right]=\frac{\gamma n}{N}\ . \tag{64}$$

We are now ready to make a swift connection to our earlier explorations at large N.

2.3 Large N physics

As before, at large N, the index n/N becomes a continuous coordinate X that runs from 0 to 1. The orthogonal polynomials become functions of X, $P_n(\lambda)\to P(X,\lambda)$, and so do the recursion coefficients: $R_n\to R(X)$. A single shift in the index n is a shift by $\epsilon\equiv\frac{1}{N}$. In these terms we can rewrite the equation (64) as:

$$R(X)\left[2+4g(R(X-\epsilon)+R(X)+R(X+\epsilon))\right]=\gamma X\ . \tag{65}$$

The leading large N behaviour comes from dropping the ϵ, giving:

$$12gR(X)^2+2R(X)-\gamma X=0\ , \tag{66}$$

which has solution:

$$R(X)=\frac{-1\pm\sqrt{1+12g\gamma X}}{12g}\ . \tag{67}$$

Note the similarity with, for example, equation (32), if one were to set $\gamma X=1$. This is not an accident, and we'll return to this later.

What to do with this solution? Well, let's return to our expression (59) for the partition function and write it in terms of continuous large N variables too. Let's take a logarithm to get the connected diagrams. After throwing away some uninteresting additive constants, we have the rather simple expression:

$$\log Z(N,g_i)=N^2\int_0^1(1-X)\log R(X)dX\ . \tag{68}$$

Now let's see how to efficiently get the physics we saw before! Using (67) we can expand $R(x)$ in small g to get:

$$R(X)=X\gamma-12X^2\gamma^2g+288X^3\gamma^3g^2-8640X^4\gamma^4g^3+290304X^5\gamma^5g^4+O\left(g^5\right)\ , \tag{69}$$

and its logarithm can similarly be expanded, and inserted into the integral (68). Dividing by N^2 and multiplying by a minus sign produces what was called $E_{\rm sph}(g)$ in Lecture 1, and indeed, carrying this all out gives:

$$E(g)_{\rm sph}-E(0)_{\rm sph}=\frac{1}{2}\gamma g-\frac{9}{8}\gamma^2g^2+\frac{9}{2}\gamma^3g^3-\frac{189}{8}\gamma^4g^4+\frac{729}{5}\gamma^5g^5+\frac{40095}{56}\gamma^6g^6+\cdots\ , \tag{70}$$

which, upon setting $\gamma=4$ to match Feynman rule conventions, gives our earlier result (36). Hopefully this has convinced you of the power of the methods we're using!

2.4 Critical physics at large N

Let's now look for critical behaviour. There is potentially interesting behaviour for the integral in the neighbourhood of $X = 1$, but there's also interesting behaviour if $R(X)$ goes to 1, which we might suspect is an interesting place given our earlier work on this same problem. How can this happen? Well, if X were to go to unity and *also* if $\gamma \to 1$, then we can see that $R(X) \to R_c = 1$ when $g \to g_c = -\frac{1}{12}$. Aha, this is (in this normalization) our old friend the critical potential for the quartic case! Indeed, this is where the series expansion of (67) diverges.

Let's set $g = g_c$ and $\gamma = 1$. Our equation for $R(X)$ is simply $-R^2 + 2R = \gamma X$ which is

$$1-(R-1)^2 = \gamma X \quad \Longrightarrow \quad R(x) = 1 + (1-\gamma X)^{\frac{1}{2}} \,. \tag{71}$$

This can be inserted into the integral to give, in the neighbourhood of $\gamma X = 1$:

$$\log Z \sim N^2 (1-\gamma X)^{\frac{5}{2}} \,, \tag{72}$$

where, putting in the scaling $\gamma X = 1 - \mu\delta^4$ and $1/N = \hbar\delta^5$ yields for the connected partition function:

$$Z_{\mathrm{sph}} \simeq \frac{1}{\hbar^2}\mu^{\frac{5}{2}} \,, \tag{73}$$

reproducing the KPZ scaling result just as we saw before in equation (40).

Note that the introduction of γ has been helpful here. Instead of tuning the potential to criticality in a scaled manner, we instead then inserted the critical value g_c of the coupling and kept γ away from 1. Then criticality is approached by sending γ (or the combination γX) to unity in a scaled way. The two procedures are equivalent and we will adopt the latter henceforth.

It is now easy to state what happens for higher multicritical points. The critical potentials ensure the following form for $R(X)$:

$$R(x) = 1 + (1-\gamma X)^{\frac{1}{k}} \,, \tag{74}$$

and the integral formula and double-scaling indeed yields the generalization $Z_{\mathrm{sph}} \simeq \frac{1}{\hbar^2}\mu^{2+\frac{1}{k}}$ that we guessed at the end of Lecture 1.

2.5 A useful representation

Let us turn now to the leading spectral density and see what's going on there in this orthogonal polynomial language. There is a very useful integral representation of the leading density in terms of the function $R(X)$:

$$\tilde{\rho}_0(\lambda) = \frac{1}{\pi}\int_0^1 \frac{dX}{\sqrt{4R(X)-\lambda^2}} \,, \tag{75}$$

where only contributions coming from where the square root is real are kept. A derivation may be found in ref. [?], **and I will review it in a later draft**. Let's try to understand it. Some intuition can be obtained by studying the simple Gaussian case, for which $R = X$ (setting $\gamma = 1$ for simplicity), when the formula readily yields the famous Wigner semi-circle law for the $k = 1$ case:

$$\rho_0(\lambda) = \frac{\sqrt{4-\lambda^2}}{2\pi} \,. \tag{76}$$

In this representation it is clear that the the droplet endpoint positions are at $\lambda_c = \pm 2\sqrt{R_c}$, where $R_c \equiv R(X = 1)$. This connects the tuning to critical behaviour to the endpoints of the distribution, as we suggested would emerge in Lecture 1. This will become even sharper below.

While we're at it, we can rapidly deduce that the k-generalization of the equation for $R(X)$, with $\gamma = 1$, can quickly give us the leading spectral densities for the k model. Try it by simply using $R(X) = 1 + (1-X)^{\frac{1}{k}}$ in the integral (75) and recover the expression (44).

2.6 Double Scaling in full

As we saw, in the neighbourhood of the critical points, scaled versions of all the key quantities survive to define the continuum physics. The full set of scalings for any k turn out to be:

$$\begin{aligned} \gamma X &= 1 + \mu\delta^{2k} \\ X &= 1+(x-\mu)\delta^{2k} \\ R(X) &= 1-u(x)\delta^2 \\ \frac{1}{N} &= \hbar\delta^{2k+1}\,, \end{aligned} \tag{77}$$

where as before the parameter $\delta \to 0$ in the limit. We have defined a scaled coordinate x which gives a refined probe of the region near $X = 1$. It will be allowed to range over the whole real line, but remember that the power of δ multiplying it means that the whole real line is zoomed out from the infinitessimal neighbourhood of $X = 1$!. (The region $-\infty \leq x \leq \mu$ will have special significance shortly.) The scaling function $u(x)$ is the scaling of $R(X)$ away from its critical value, in the neighbourhood of the endpoint of the spectral density as we've seen, and indeed $u(x)$ will probe the behaviour of the endpoint in the scaled region.

With these relations, in the limit $\delta \to 0$ the surviving physics for our gravity partition function (68) is:

$$Z = \frac{1}{\hbar^2}\int_{-\infty}^{\mu} (x-\mu)u(x)dx\ , \tag{78}$$

or alternatively

$$u(x) = \hbar^2 \left.\frac{\partial^2 \widetilde{F}}{\partial \mu^2}\right|_{\mu=x} . \tag{79}$$

In the Gaussian case, k=1, and the relation R_n=$\gamma n/N$, which became $R(X)$=γX, simply yields $u(x)$=$-x$, where μ has been set to zero, since the model has no parameter to which this quantity corresponds in the scaling limit.

We can also apply these scalings to the leading spectral density of (75). All the physics comes from the neighbourhood of the $X = 1$ limit and as discussed above, this is also the neighbourhood of an end λ_c of the spectral density. So generally the double-scaling limit ought to include scaling λ away from λ_c by a scaled amount:

$$\lambda = \lambda_c - E\delta^2\ . \tag{80}$$

The power of δ is fixed by the denominator of the integral representation. The expectation should be that $\tilde{\rho}_0(\lambda)d\lambda \to \rho_0(E)dE$, once the scaling $\lambda = \lambda_c - E\delta^2$ is used. A factor δ^{2k-1} results from putting in the rest of the scaling relations, combining to give a factor δ^{2k+1}. To get something finite at large N, a factor of N should be multiplied in here, which gives the finite combination already identified: $\hbar^{-1}$. Thus:

$$\rho_0(E) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\mu} \frac{dx}{\sqrt{E-u_0(x)}}\ , \tag{81}$$

where $u_0(x)$ is the leading part of $u(x)$. This is an extremely useful formula that maybe you've seen used in the literature before. IT may have seemed like magic. Now you see where it comes from, and what the parameter x and the function $u_0(x)$ really mean.

2.7 String equations and beyond the sphere

The recursion relations for R_n at higher k become a *differential equation* for $u(x)$, which is often (for historical reasons) called a "string equation". (The trivial $k = 1$ case had no such equation, just the exact relation $u(x) = -x$.) Let's derive it for the quartic case $k = 2$.

We must keep all terms in the equation (66), and Taylor expand to give:

$$R(X)\left(2+4g\left[3R(X)+\epsilon^2\frac{\partial^2 R(X)}{\partial X^2}+\cdots\right]\right)=\gamma X\ , \tag{82}$$

and now inserting the scaling relations (77) yields that orders δ^0 and δ^2 cancel to zero, and a non-trivial equation appears at order δ^4, which is:

$$-\frac{\hbar^2}{3}\frac{\partial^2 u(x)}{\partial x^2}+u(x)^2=-x\ . \tag{83}$$

The "string equation" of equation (83) is in fact the Painlevé I non-linear ODE. In principle, such equations yield both perturbative and non-perturbative information about the model, which was one of the great discoveries of the classic double-scaling limit papers (Brezin-Kazakov,Gross-Migdal,Douglas-Shenker). Let's look at the perturbative expansion:

$$u(x)=\sqrt{-x}-\frac{1}{24}\frac{\hbar^2}{x^2}-\frac{49}{1152}\frac{\hbar^4}{x^4}+\cdots \tag{84}$$

which yields, after integrating twice according to (79):

$$Z(\mu)=\frac{4}{15}\frac{\mu^{\frac{5}{2}}}{\hbar^2}+\frac{1}{24}\log|\mu|-\frac{7}{1440}\frac{\hbar^2}{\mu^{\frac{5}{2}}}+\cdots \tag{85}$$

corresponding to the sphere, torus, and double torus orders in perturbation theory.

The multicritical models that can be obtained from the higher order potentials in general have for their leading behaviour at genus zero $u_0(x) = (-x)^{1/k}$ (for the Gaussian case, k=1, it is simply $u(x)$=$-x$ to all orders.). Indeed, on integrating this leading behaviour for $u_0(x)$ twice we get the $|\mu|^{2+\frac{1}{k}}$ form discussed beneath equation (9), with γ_s=$-\frac{1}{k}$, as anticipated already.

The ODEs that arise by following the steps from before to for higher k can be written as:

$$\mathcal{R}=0\ ,\quad \text{where}\quad \mathcal{R}\equiv\tilde{R}_k[u]+x\ . \tag{86}$$

and the $\tilde{R}_k[u]$ are kth order polynomials in $u(x)$ and its x derivatives normalized such that their $u(x)^k$ term has coefficient unity. They have a pure derivative piece given by $2k-2$ derivatives acting on $u(x)$, and then various mixed non-linear pieces. The first three of these "Gel'fand-Dikii"polynomials are:

$$\begin{aligned}&\tilde{R}_1[u]=u\ ,\qquad \tilde{R}_2[u]=u^2-\frac{1}{3}u''\ ,\quad \text{and}\\ &\tilde{R}_3[u]=u^3-\frac{1}{2}(u')^2-uu''+\frac{1}{10}u''''\ ,\end{aligned} \tag{87}$$

where here a prime indicates an x-derivative times a factor of $\hbar$. The higher ones can be generated using a recursion relation **(which I might insert later)**.

As an exercise, I strongly recommend working out the $k = 3$ example just to see how it works. You'll get a non-trivial equation emergining at order δ^6 (it is δ^{2k} in general).

2.8 A larger family

I'll put some notes on the fact that the linearity of the structure of the equations (not the equations themselves) allow for potentials to be "added" with coefficients $\sim t_k$. The resulting more general string equation is:

$$\mathcal{R} = 0\,, \quad \text{where} \quad \mathcal{R} \equiv \sum_k t_k \tilde{R}_k[u] + x\ . \tag{88}$$

These will be very useful later.

2.9 A useful quantum mechanics emerges

Returning to the main story, it is useful to look more closely at the organization of the physics the orthogonal polynomials provide. The normalized polynomials, with an additional factor from the measure absorbed into them, $e^{-\frac{N}{\gamma}V(\lambda)/2}P_n(\lambda)/\sqrt{h_n}$, will become a function denoted $\psi(x,E)$ in the double-scaling limit, about which more will be said shortly. The basis denoted previously $|n\rangle$ becomes $|X\rangle$ in the limit. An important question to ask is what becomes, in the limit, of the operation of multiplying the orthogonal polynomials by λ. This is interesting and important to work out, and follows from the recursion relation (54). First, dividing throughout by $\sqrt{h_n}$ and then using the relation $h_n = h_{n+1}/R_{n+1}$ multiple times gives:

$$\lambda|n\rangle = \sqrt{R_{n+1}}|n+1\rangle + \sqrt{R_n}|n-1\rangle\ , \tag{89}$$

and preparing for the large N limit this is:

$$\lambda|X\rangle = \sqrt{R(X+\epsilon)}|X+\epsilon\rangle + \sqrt{R(X)}|X-\epsilon\rangle\ , \tag{90}$$

and so multiplying by λ can be written as an operator $\hat{\lambda}$ on the X dependence in the following way:

$$\hat{\lambda} = \sqrt{R(X+\epsilon)}\exp\left\{\epsilon\frac{\partial}{\partial X}\right\} + \sqrt{R(X)}\exp\left\{-\epsilon\frac{\partial}{\partial X}\right\} \tag{91}$$

Taylor expanding and substituting the scaling relations (77) and (80) gives $\hat{\lambda}=2-\mathcal{H}\delta^2+\cdots$, where:

$$\mathcal{H} = -\hbar^2\frac{\partial^2}{\partial x^2} + u(x)\,, \tag{92}$$

is the operator whose eigenvalue is the scaled energy E. The scaled wavefunctions in the limit are denoted $\psi(x,E)$ and are thus the eigenfunctions of $\mathcal{H}$:

$$\mathcal{H}\psi(E,x) = E\psi(E,x)\,, \tag{93}$$

With a known $u(x)$, (*that must extend over all of* x–we'll discuss this later) this then fully defines a quantum mechanics problem. This quantum mechanical system naturally provides the tools with which to excavate the *fully non-perturbative* physics. We will do a simple exactly solvable soon to see how this all works.

2.10 Looking Ahead

We've achieved one major goal, which is to define to all orders in genus expansion (and possibly beyond?) Liouville gravity coupled to (a class of) matter. This is a big deal. We are also very close to begin understanding JT gravity using these same tools! We also have to talk about non-perturbative physics now that we have the tools to extract it. We have a lot to do!