MSc: Statistical Analysis (Spring Term)

Generalized Linear Models: Exercises 1 - Solutions

1. (a)

$$\begin{aligned} f(y) &= \lambda e^{-\lambda y} \\ &= \exp(\log\lambda)\exp(-\lambda y) \\ &= \exp\{\log(\lambda) - \lambda y\} \\ &= \exp\{y(-\lambda) - (-\log\lambda)\} \end{aligned}$$

Setting

$$\theta = -\lambda, \quad b(\theta) = -\log\lambda = -\log(-\theta), \quad a(\phi) = \phi = 1 \quad \text{and} \quad c(y,\phi) = 0$$

we see

$$f(y) = \exp\left\{\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right\}$$

establishing the required result.

(b)

$$E[Y] = b'(\theta) = \frac{-1}{-\theta} \times -1 = -\frac{1}{\theta}.$$

Defining $\mu = E[Y]$, then

$$\mu = -\frac{1}{\theta} \quad \Rightarrow \quad \theta = -\frac{1}{\mu}$$

Hence the canonical link is $g(\mu) = -1/\mu$.

2. (a)

$$\begin{aligned} f(y) &= \frac{1}{\Gamma(\nu)}\left(\frac{\nu}{\mu}\right)^{\nu} y^{\nu-1}\exp\left(-\frac{\nu y}{\mu}\right) = \exp\left\{\log\left[\frac{1}{\Gamma(\nu)}\left(\frac{\nu}{\mu}\right)^{\nu} y^{\nu-1}\exp\left(-\frac{\nu y}{\mu}\right)\right]\right\} \\ &= \exp\left\{\nu\log\left(\frac{\nu}{\mu}\right) + (\nu-1)\log(y) - \frac{\nu y}{\mu} + \log\left(\frac{1}{\Gamma(\nu)}\right)\right\} \\ &= \exp\left\{-\nu\log(\mu) - \frac{\nu y}{\mu} + \nu\log(\nu) + (\nu-1)\log(y) - \log(\Gamma(\nu))\right\} \\ &= \exp\left\{\frac{(-\mu^{-1})y + log(\mu^{-1})}{v^{-1}} + \nu\log(\nu) + (\nu-1)\log(y) - \log(\Gamma(\nu))\right\} \end{aligned}$$

Setting

$$\theta = -\mu^{-1} = -\frac{1}{\mu}, \quad b(\theta) = -\log(\mu^{-1}) = -\log(-\theta), \quad a(\phi) = \phi = \nu^{-1}, \text{ and}$$

$$c(y,\phi) = c(y,\nu^{-1}) = \nu\log(\nu) + (\nu-1)\log(y) - \log(\Gamma(\nu))$$

establishes the required result.

(b) We have,

$$E[Y] = b'(\theta) = -\frac{1}{\theta} = \mu$$

and,

$$\mathrm{Var}[Y] = a(\phi)b''(\theta) = \frac{1}{\nu}b''(\theta) = \frac{1}{\nu}\mu^2$$

as we would expect, substituting $\lambda = \nu/\mu$ in the expressions for $\mathrm{E}[Y]$ and $\mathrm{Var}(Y)$ given in the question.

The coefficient of variation $\mathrm{CV}(Y)$ is given by

$$\mathrm{CV}(Y) \;=\; \frac{\mathrm{SD}(Y)}{\mathrm{E}[Y]} \;=\; \frac{\frac{1}{\sqrt{\nu}}\mu}{\mu} \;=\; \frac{1}{\sqrt{\nu}},$$

which is constant as required.

(c) As with the exponential distribution, we have

$$\mu = \mathrm{E}[Y] = b'(\theta) = -\frac{1}{\theta}$$

so that,

$$\theta = -\frac{1}{\mu}$$

Hence, the canonical link is also $g(\mu) = -1/\mu$. (The *reciprocal* function).

A Note on the use of the Gamma and Exponential distributions in GLMs

• As with the normal distribution, the Gamma distribution has a constant scale parameter, $\phi = 1/\nu$.

• The coefficient of variation,

$$CV = \frac{\text{standard deviation}}{\text{mean}}.$$

As shown on the previous page, for the Gamma distribution, we have

$$CV = \frac{\sqrt{\frac{1}{\nu}\mu^2}}{\mu} = \frac{1}{\sqrt{\nu}}$$

which is constant.

Hence, the Gamma distribution is appropriate for modelling continuous data which is strictly positive, and has a constant coefficient of variation.

• Also recall the following interpretations of the Exponential/Gamma distributions with regard to (random) events in a Poisson process. That is, the time between events in a Poisson process with mean rate λ follow an exponential distribution with $\alpha = \lambda$. If $\nu = n$, then the time to the nth event follows the Gamma distribution $\mathrm{G}(\lambda, n)$.

• The exponential distribution can be modelled as a Gamma distribution with a fixed dispersion parameter $\phi = 1$. This is the method adopted in R and other software packages for GLMs involving the exponential distribution.

In R, we fit, for example

```
data.glm(formula,family=Gamma(link=reciprocal))
```

but use

```
summary(data.glm,dispersion=1)
```

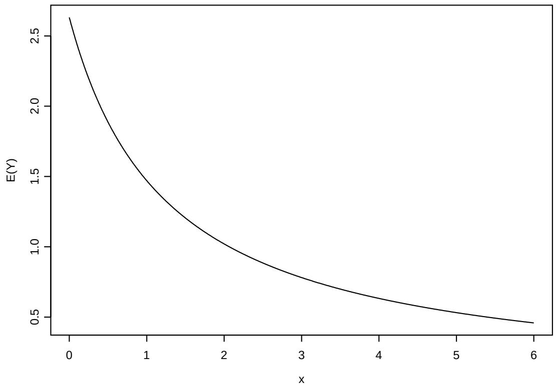
to ensure the presented terms use the correct dispersion parameter.

• In R, the `reciprocal` link fits $\frac{1}{\mathrm{E}[Y]} = \sum_{i=1}^{n} \beta_i x_i$.

3. (a) For (i) use, for example, the R program

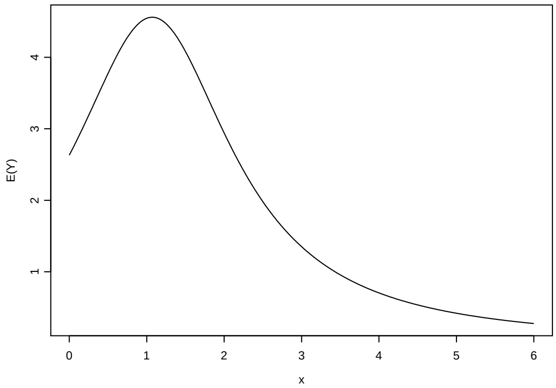
```
x <- seq(0,6,0.005)
beta1 <- 0.3
beta0 <- 0.38
y <- 1/(beta0 + beta1*x)
plot(x,y,type="l",xlab="x",ylab="E(Y)")
```

to obtain

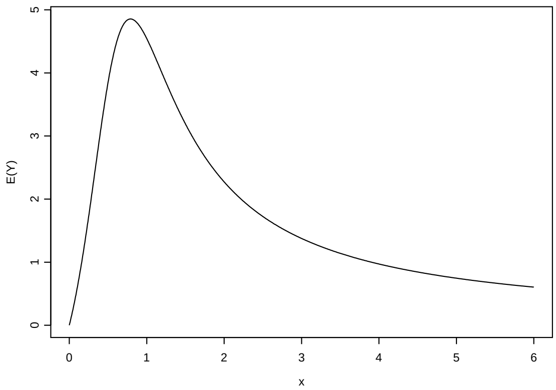


Similarly (with appropriately chosen parameter values) we obtain

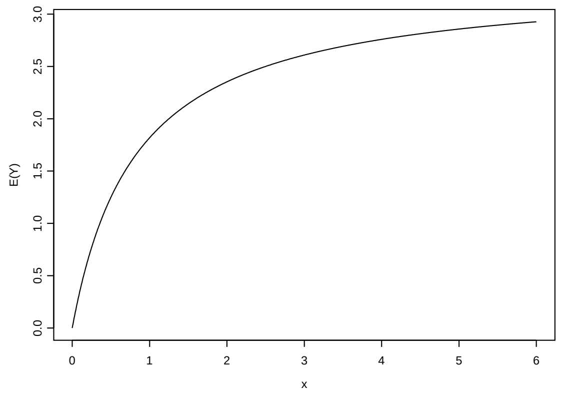
(ii)



(iii)



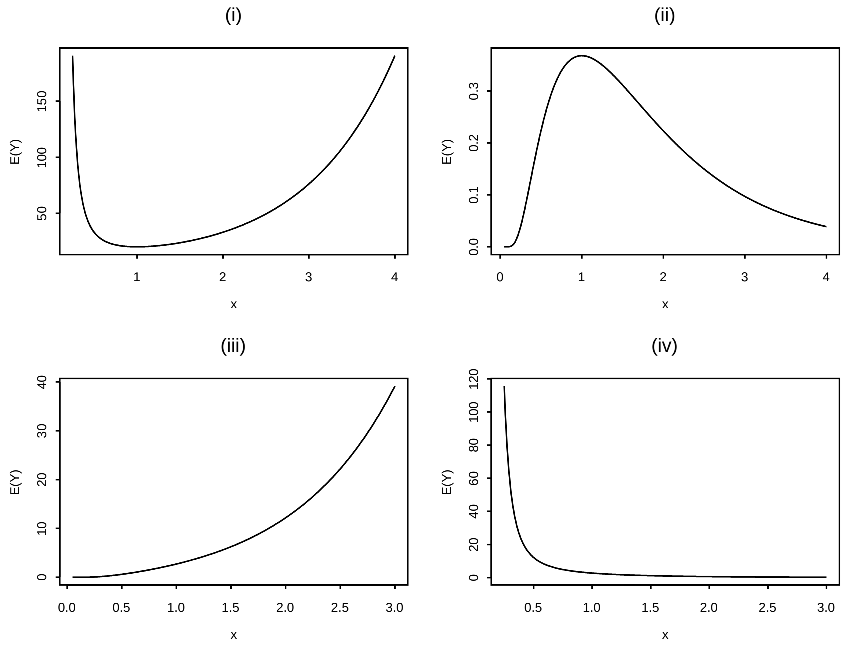
(iv)



4. Plots for (i) to (iv) can be obtained using the code

```
par(mfrow = c(2, 2))

beta0 <- 1

x <- seq(0.25,4,0.005)
beta1 <- 1
beta2 <- 1
y <- exp(beta0 + beta1*x + beta2*(1/x))
plot(x,y,type="l",xlab="x",ylab="E(Y)")
title(main="(i)")

x <- seq(0.05,4,0.005)
beta1 <- -1
beta2 <- -1
y <- exp(beta0 + beta1*x + beta2*(1/x))
plot(x,y,type="l",xlab="x",ylab="E(Y)")
title(main="(ii)")

x <- seq(0.05,3,0.005)
beta1 <- 1
beta2 <- -1
y <- exp(beta0 + beta1*x + beta2*(1/x))
plot(x,y,type="l",xlab="x",ylab="E(Y)")
title(main="(iii)")

x <- seq(0.25,3,0.005)
beta1 <- -1
beta2 <- 1
y <- exp(beta0 + beta1*x + beta2*(1/x))
plot(x,y,type="l",xlab="x",ylab="E(Y)")
title(main="(iv)")
```

to obtain


(i)
(ii)
(iii)
(iv)
E(Y)
x