MAS programmes - Statistical Inference

Examples 5

1. Let $\mathbf{X} \equiv (X_1, X_2, \ldots, X_n)$ denote a random sample of size n from a distribution with probability function

$$f(\mathbf{x}|\theta) = (1-\theta)\theta^x \qquad\qquad x = 0, 1, 2, \ldots,$$

where the parameter θ such that $0 < \theta < 1$ is unknown.

(i) Prove that the statistic

$$T(\mathbf{X}) \equiv \sum_{i=1}^{n} X_i$$

is sufficient for θ.

(ii) In a Bayesian approach to inference, the prior distribution adopted for θ is a beta distribution with parameters $a > 0$ and $b > 0$, i.e., with p.d.f.

$$\pi(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1} \qquad\qquad 0 < \theta < 1.$$

Find the posterior distribution of θ given the above sample.

(iii) Deduce that the Bayes estimator $\hat{\theta}$ of θ, the mean of the posterior distribution, is given by

$$\hat{\theta}(\mathbf{X}) = \frac{a+T}{a+b+n+T}.$$

(iv) For any given value of T, find the value $\check{\theta}$ that maximizes the posterior density as a function of θ, taking account of and commenting upon the special case $a + T \leq 1$.

(v) State and comment upon the special case in which $\check{\theta}$ reduces to the maximum likelihood estimator of θ.

2. Consider a random sample $X_1, X_2, \ldots, X_n$ of size n from a $N(\mu, \sigma^2)$ distribution, where σ^2 is known. Adopting a Bayesian approach, let the prior distribution for μ be $N(\nu, \tau^2)$. Prove that the posterior density for μ is normal with mean

$$\frac{\tau^2}{\tau^2 + \sigma^2/n}\,\bar{x} + \frac{\sigma^2/n}{\tau^2+\sigma^2/n}\,\nu$$

and variance

$$\left(\frac{n}{\sigma^2} + \frac{1}{\tau^2}\right)^{-1}.$$

3. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from an exponential distribution with parameter θ, i.e., a distribution with p.d.f.

$$f(x|\theta) = \theta e^{-\theta x} \qquad\qquad x \geq 0.$$

where the parameter $\theta \in (0, \infty)$ is to be estimated.

(i) Write down the likelihood function and show that the statistic $T \equiv \sum_{i=1}^{n} X_i$ is a sufficient statistic for θ.

(ii) In a Bayesian approach to the problem of inference, the prior distribution for θ is taken to be a gamma distribution with parameters $\nu > 0$ and $\lambda > 0$, i.e., the p.d.f. of the prior distribution is given by

$$\pi(\theta) = \frac{\lambda^{\nu} \theta^{\nu - 1} e^{-\lambda\theta}}{\Gamma(\nu)} \qquad\qquad \theta > 0.$$

Show that the posterior distribution for θ is then also a gamma distribution, specifying the parameters.

(iii) Explain what in Bayesian inference is meant by a family of conjugate prior distributions and how this relates to the present example.

(iv) State what is meant by a *Bayes estimator* and write down what it is in the present case.

(v) A further observation X_{n+1} is to be taken from the same exponential distribution as above. Obtain the p.d.f. of the *predictive distribution* for X_{n+1} and deduce that

$$\mathbb{P}(X_{n+1} > x) = \left(\frac{\lambda + t}{\lambda + t + x} \right)^{\nu + n} \qquad\qquad x \geq 0.$$