Probability and Distribution Theory:
Supplementary Exercises

Probability Theory

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let A, B, C and D be events on Ω, i.e. A, B, C, D belong to the collection $\mathcal{F}$. Express the following statements in set theoretic terms.

   (a) A occurs but neither B nor C nor D occurs.

   (b) None of the events occur.

   (c) All occur.

   (d) At most, three occur.

   (e) At least three of them occur.

   (f) Exactly three occur.

   (g) Exactly one of them occurs.

   (h) Exactly one of A and B occurs, but D does not occur.

2. A survey of the students at a college who use email found that 50% have a Shotmail account, 30% have a Payserve account, and 25% have both Shotmail and Payserve accounts.
   Which of the following statements are correct?

   (a) 0.55 that s/he has a Shotmail or a Payserve account.

   (b) 0.75 that s/he does not have both a Shotmail and a Payserve account.

   (c) 0.75 that s/he has neither a Shotmail nor a Payserve account.

   (d) 0.25 that s/he has either a Shotmail or a Payserve account, but not both.

3. A pair of *fair* dice are thrown once. What is the probability that

   (a) the sum of the scores is divisible by 4?

   (b) a three turns up exactly once?

   (c) the sum of the scores is 5?

   (d) both numbers are even?

4. A pair of *fair* five-sided spinners are spun, each one being numbered 1,2,3,4, or 5. What is the probability that the sum is 4, given that a sum of 4 or more is obtained?

5. Suppose that A and B are independent events. Show that:

   (a) A^c and B are independent events;

   (b) A^c and B^c are independent events.

<u>Univariate Distribution Theory</u>

6. Determine the constant c for the following distributions.

   (a)
$$f_X(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

   (b)
$$f_X(x) = \begin{cases} \frac{c}{x^2} & x > 5 \\ 0 & \text{o.w.} \end{cases}$$

7. Calculate the mean and variance of the following distributions.

   (a)
$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \qquad x > 0$$

   where $\alpha, \beta > 0$,

   i.e. $X \sim Gamma(\alpha, \beta)$.

   (b)
$$f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \qquad 0 < x < 1$$

   where $\alpha, \beta > 0$

   i.e. $X \sim Beta(\alpha, \beta)$.

   You may use the fact that $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$ *and* $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ *for* $\alpha > 0$.

8. THE NORMAL DISTRIBUTION

Assuming that

$$\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\,dy = \sqrt{2\pi},$$

show that

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty$$

is a p.d.f. for any $\mu \in \mathbb{R}$, $\sigma > 0$.

[Hint: Consider using the substitution $y = \frac{x-\mu}{\sigma}$ for part of your analysis].

Joint and Conditional Distributions

9. Suppose X and Y are continuous random variables with p.d.f.

$$f_{(X,Y)}(x, y) = \begin{cases} cx^2 & 0<x<1,\ 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

   (a) Specify $R_{(X,Y)}$.

   (b) Find the constant c.

   (c) Find the marginal p.d.f.'s of X and Y, i.e. $f_X(\cdot)$ and $f_Y(\cdot)$ respectively.

10. Suppose (X, Y) is a continuous random vector which is 'uniformly' distributed over $R_{(X,Y)} = \{(x, y) : -1 \leq x \leq 1,\ -1 \leq y \leq 1\}$ (i.e. the joint p.d.f. is constant over the entire region).

   (a) Specify the joint p.d.f. $f_{(X,Y)}$.

   (b) Find:

      i. $\mathbb{P}(2X > Y)$

      ii. $\mathbb{P}(X^2 + Y^2 < 1)$

      iii. $\mathbb{P}(|X + Y| < 1)$.

11. The discrete random vector (X, Y) has a p.m.f. given by the table below:

Y \ X	1	2	3
1	1/6	1/12	1/12
2	1/6	0	1/6
3	0	2/9	1/9

(So, for e.g., $p_{(X,Y)}(2,1) = \frac{1}{12}$).

(a) Find the marginal p.m.f.'s of X and Y,
i.e. $p_X(k)$ for each $k \in R_X$ and $p_Y(k)$ for each $k \in R_Y$.

(b) Find the p.m.f. of XY and hence $E[XY]$.

(c) Calculate $\mathbb{P}(Y = 2|X > 1)$.

12. The conditional p.d.f. of Y, given $X = x$, is

$$f_{Y|X}(y|x) = \begin{cases} \frac{2y}{x^2} & 0<x<1,\ 0<y<x \\ 0 & \text{o.w.} \end{cases}$$

and the marginal p.d.f. of X is

$$f_X(x) = \begin{cases} 7x^6 & 0<x<1 \\ 0 & \text{o.w.} \end{cases}.$$

(a) Find the joint p.d.f. of (X, Y).

(b) Find the marginal p.d.f. of Y.

13. Let the random variables X and Y have joint p.d.f.

$$f_{(X,Y)}(x,y) = \begin{cases} x+y & \text{if } 0<x<1,\ 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

Find

(a) $\text{cov}(X, Y)$

(b) $\rho(X, Y)$ and

(c) $\text{var}(2X - Y + 4)$.

14. Suppose that the joint density function of X and Y is given by

$$f_{(X,Y)}(x,y) = \begin{cases} \frac{1}{2} & 0 < x < y < 2 \\ 0 & \text{o.w.} \end{cases}$$

(a) Find the marginal probability density functions of X and Y.

(b) Are X and Y independent? Justify your answer.

Transformations of Random Variables

15. (a) Show that if $U \sim Uniform(0,1)$, then

$$-\frac{1}{\lambda}\ln U \sim Exp(\lambda)$$

where $\lambda > 0$.

(b) Now suppose that $X \sim Exp(\lambda)$, where $\lambda > 0$. Show that

$$1 - e^{-\lambda X} \sim U(0,1).$$

16. Suppose that $X \sim Exp(\lambda)$ and $Y \sim Exp(\mu)$ are independent r.v.'s. Show that $W = \min(X,Y) \sim Exp(\lambda + \mu)$.

17. Suppose that U and V are independently distributed non-negative random variables with joint p.d.f. given by $f_{(U,V)}(\cdot,\cdot)$. Define the random variables X and Y via the transformations

$$X = U, \qquad\qquad Y = U + V.$$

Let the joint p.d.f. of X and Y be denoted by $f_{(X,Y)}(\cdot,\cdot)$.

(a) State the numerical value of $J(x,y)$ in the expression

$$f_{(X,Y)}(x,y) = f_{(U,V)}(x, y-x)|J(x,y)|.$$

(b) Suppose that

$$f_{(U,V)}(u,v) = \lambda^2 e^{-\lambda(u+v)}, \qquad u \geq 0,\ v \geq 0.$$

Find an expression in terms of x and/or y for $f_{(X,Y)}(x,y)$. What are the set of values (x,y) for which $f_{(X,Y)}(x,y)$ is positive?

Generating Functions for univariate distributions

18. Suppose that X is a discrete r.v. with p.g.f.

$$G_X(\theta) = 0.35 + 0.65\theta.$$

Find the following quantities:

(a) $\mathbb{P}(X = 1)$,

(b) $\mathbb{P}(X = 10)$,

(c) $\mathbb{P}(X = 0)$,

(d) $E[X]$.

19. Suppose that $X \sim Gamma(\alpha, \beta)$ i.e.

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \qquad x > 0$$

$\alpha, \beta > 0$.
Compute the m.g.f. of X, and hence calculate $E[X]$ and $\text{var}(X)$.

20. Suppose that the r.v. W has m.g.f.

$$M_W(t) = (1 - 7t)^{-20}.$$

Find the mean and variance of W.

21. Let X be the time it takes, starting from midnight, for a total of 3 buses to leave Trafalgar Square. It is assumed that X has the following probability density function

$$f_X(x) = \begin{cases} 108x^2 e^{-6x} & x > 0 \\ 0 & \text{o.w.} \end{cases}$$

(a) Find the moment generating function, $M_X(t)$, of X (for t in a suitably defined range).

(b) Compute $M_X^{(3)}(t)$ and hence find $E[X^3]$.

22. (a) Suppose that the random variable $X \sim Bernoulli(p)$ where $0 < p < 1$. Show that the characteristic function of X is given by

$$\Psi_X(\theta) = (1 - p) + pe^{i\theta}.$$

(b) Suppose that $X_j \sim Bernoulli(p)$, $i = 1, \ldots, n$, are independent. Let $Y = \sum_{j=1}^n X_j$. Show that the characteristic function of Y is given by

$$\Psi_Y(\theta) = \left((1 - p) + pe^{i\theta}\right)^n.$$

<u>Central Limit Theorem</u>

23. Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n = 40$, with population mean $\mu = 6.53$, and population variance $\sigma^2 = 10$.

    By using the *Central Limit Theorem*, find the approximate probability that $\sum_{i=1}^{40} X_i$ lies between 245 and 255.

24. The random variables $X_1, \ldots, X_{625}$ are independent and identically distributed each with p.d.f. $f_X(x) = 3(1-x)^2$ for $0 \leq x \leq 1$. Approximate

$$\mathbb{P}(X_1 + X_2 + \ldots + X_{625} < 170)$$

in terms of the distribution function, $\Phi(\cdot)$, of the standard normal distribution.