MAS programmes - Statistical Inference

Exercises 3

1. A random sample $X_1, X_2, \ldots, X_n$ of size n is taken from a gamma distribution with p.d.f.

$$f(x;\theta) = \frac{x^{\nu-1}e^{-x/\theta}}{\theta^{\nu}\Gamma(\nu)} \qquad x > 0,$$

where $\nu > 0$ is known but the parameter $\theta > 0$ is unknown.

(i) Find the maximum likelihood estimator $\hat{\theta}$ of θ in terms of the sample mean $\bar{X}$.

(ii) Show that $\hat{\theta}$ is an unbiased estimator of θ and find its variance as a function of θ.

(iii) Find the Fisher information $I(\theta)$ corresponding to the above estimation problem.

(iv) Stating carefully any general result that you use, show that $\hat{\theta}$ is the minimum variance unbiased estimator of θ.

2. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a distribution with p.d.f.

$$f(x;\theta) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x} \qquad x \geq 0,$$

where θ is an unknown parameter with $\theta > 0$.

(i) Write down the likelihood function and, stating carefully any criterion that you use, deduce that the sample mean $\bar{X}$ is a sufficient statistic for making inferences about θ.

(ii) Find an expression for the method of moments estimator $\tilde{\theta}$ of θ.

(iii) Write down the likelihood equation and deduce that the maximum likelihood estimator of θ is identical with the method of moments estimator.

(iv) Obtain an expression for the Fisher information and, quoting any appropriate asymptotic properties of maximum likelihood estimators, deduce the approximate distribution of $\hat{\theta}$ for large n.

3. Let $Y_1, \ldots, Y_n$ be a random sample from the Bernoulli(p) distribution and suppose $n \geq 3$.

(i) Show the Bernoulli distribution is a member of the exponential family and find the canonical statistic based on the above random sample.

(ii) Show that the variance of the MLE of p attains the Cramér–Rao lower bound.

(iii) Show that the product $Y_1Y_2Y_3$ is an unbiased estimator of p^3.

(iv) Find the conditional expectation

$$\mathbb{E}\left[Y_1Y_2Y_3 \;\middle|\; \sum_{i=1}^{n} Y_i = y\right],$$

for each $y \in \{0, 1, 2, \ldots\}$.

(v) Hence, or otherwise, find the unique minimum variance unbiased estimator of p^3.

4. Let $Y_1, \ldots, Y_n$ be a random sample from the Binomial(k, p) distribution with k known and p unknown. Consider estimation of the probability of exactly one success, i.e. we wish to estimate

$$\tau(p) = \mathbb{P}(Y_1 = 1) = kp(1-p)^{k-1}.$$

(i) Show the Binomial distribution is a member of the exponential family and find the canonical statistic T based on the above random sample.

(ii) Find an unbiased estimator $\hat{\tau}(\mathbf{Y})$ of $\tau(p)$ which is just a function of Y_1.

(iii) Find the conditional expectation $\mathbb{E}\left[\hat{\tau}(\mathbf{Y}) \mid T = t\right]$, for each t.

(iv) Hence, or otherwise, show that the unique minimum variance unbiased estimator of $\tau(p)$ is

$$k\frac{\binom{k(n-1)}{\sum_{i=1}^{n} Y_i - 1}}{\binom{kn}{\sum_{i=1}^{n} Y_i}}.$$