Probability and Distribution Theory: Exercises 1

1. Let $\Omega = \{a, b, c, d\}$. Giving justification, determine whether each of the following constitutes a σ-field or not.

   (a) $\mathcal{F}_1 = \{\emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$;

   (b) $\mathcal{F}_2 = \{\emptyset, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$;

   (c) $\mathcal{F}_3 = \{\emptyset, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c, d\}\}$.

2. Let $\mathcal{F}$ be a σ-field of subsets of a set Ω.

   (a) Show that if $A, B \in \mathcal{F}$, then $A \setminus B \in \mathcal{F}$.

   (b) The symmetric difference $A \triangle B$ of two subsets A and B of Ω is defined to be the set of points of Ω which are in either A or B but not both. Show that if $A, B \in \mathcal{F}$, then $A \triangle B \in \mathcal{F}$.

3. Let A and B be independent.

   (a) Show that A^c and B are independent.

   (b) Show that A^c and B^c are independent.

   (c) If $\mathbb{P}(A) = \frac{1}{3}$ and $\mathbb{P}(B^c) = \frac{1}{4}$, find $\mathbb{P}(A \cup B)$.

4. Write down all equalities that must hold for the family of events $\{A, B, C\}$ to be independent.

5. A supplier of a certain testing device claims that his device has high reliability in as much as $\mathbb{P}(A \mid B) = \mathbb{P}(A^c \mid B^c) = 0.95$, where $A = \{$device indicates component is faulty $\}$ and $B = \{$component is faulty$\}$. You hope to use the device to locate the faulty components in a large batch of components of which 5 percent are faulty.

   (a) Find the value of $\mathbb{P}(B \mid A)$.

   (b) Suppose you want $\mathbb{P}(B \mid A) = 0.9$. Let $p = \mathbb{P}(A \mid B) = \mathbb{P}(A^c \mid B^c)$. How large does p have to be?

6. Show that $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A \mid B \cap C)\mathbb{P}(B \mid C)\mathbb{P}(C)$.

7. (a) Prove 4 in Lemma 2 of Chapter 1.

   (b) **Optional question** (for those who have previously studied proof by induction – this technique will not be examinable)
   Give the proof of 5 in Lemma 1 of Chapter 1, by using induction on n.

8. A fair coin is tossed 10 times. Describe the appropriate probability space in detail for the two cases when

   (a) the outcome of every toss is of interest,

   (b) only the total number of tails is of interest.

9. Express the distribution function of $Y = \max\{0, X\}$ in terms of the distribution function F_X of X.