**Probability and Distribution Theory: Exercises 3**

1. Assume that the number $X$ of road accidents occurring on a certain stretch of road in a month has a Poisson distribution with parameter $\lambda$. Let us also assume that the probability is $p$ that each accident will involve a fatality, independently from one another. Let $Y$ denote the number of fatal accidents on the stretch of road in a month. Derive

   (a) the joint distribution of $(X, Y)$,

   (b) the marginal distribution of $Y$, and

   (c) the conditional distribution of $X$, given that $Y = 5$.

2. Let $X$ and $Y$ have joint density function

$$f_{(X,Y)}(x, y) = \lambda^2 e^{-\lambda y} \quad \text{for} \quad 0 \leq x \leq y$$

   for $\lambda > 0$. Find the conditional density function and expectation of $Y$ given $X$.

3. Let the pair $(X, Y)$ of random variables have joint distribution function $F(x, y)$. Show that for any $a < b$ and $c < d$ constants,

$$\mathbb{P}(a < X \leq b,\, c < Y \leq d) = F(b, d) + F(a, c) - F(a, d) - F(b, c).$$

4. The indicator function of an event $A$ is the random variable $\mathbf{1}_A$ defined as

$$\mathbf{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A. \end{cases}$$

   Show that two events $A$ and $B$ are independent if and only if their indicator functions $\mathbf{1}_A$ and $\mathbf{1}_B$ are independent random variables.