Probability and Distribution Theory: Solutions 1

1. (a) Clearly $\emptyset \in \mathcal{F}_1$;
   $\{a,b\} \cup \{c,d\} = \{a,b,c,d\}$; $\{a,b\} \cup \{a,b,c,d\} = \{a,b,c,d\}$;
   $\{c,d\} \cup \{a,b,c,d\} = \{a,b,c,d\}$;
   $\{c,d\}^c = \{a,b\}$; $\{a,b\}^c = \{c,d\}$; $\emptyset^c = \{a,b,c,d\}$; $\{a,b,c,d\}^c = \emptyset$.

   (b) Clearly $\emptyset \in \mathcal{F}_2$;
   Straightforward, but tedious to show that all unions of members of $\mathcal{F}_2$ are members of $\mathcal{F}_2$;
   Straightforward, but tedious to show that all complements of members of $\mathcal{F}_2$ are members of $\mathcal{F}_2$.

   (c) $\{a,b\} \cup \{a,c\} \equiv \{a,b,c\} \notin \mathcal{F}_3$

2. (a) $A \setminus B = A \cap B^c$, but $B^c \in \mathcal{F}$ and $\mathcal{F}$ is closed under finite intersections.

   (b) $A \triangle B = (A \cup B) \setminus (A \cap B)$. But $A \cup B \in \mathcal{F}$ and $A \cap B \in \mathcal{F}$ so follows using part (a).

3. (a) Show $\mathbb{P}(A^c \cap B) = \mathbb{P}(A^c)\mathbb{P}(B)$.

$$\begin{aligned}\mathbb{P}(A^c \cap B) &= \mathbb{P}((\Omega \setminus A) \cap B) = \mathbb{P}(B \setminus (A \cap B)) = \mathbb{P}(B) - \mathbb{P}(A \cap B) \\ &= \mathbb{P}(B) - \mathbb{P}(A)\mathbb{P}(B) = (1 - \mathbb{P}(A))\mathbb{P}(B) = \mathbb{P}(A^c)\mathbb{P}(B).\end{aligned}$$

   (b) Using result of (a):

$$\begin{aligned}\mathbb{P}(A^c \cap B^c) &= \mathbb{P}(A^c \cap (\Omega \setminus B)) = \mathbb{P}(A^c \setminus (A^c \cap B)) \\ &= \mathbb{P}(A^c) - \mathbb{P}(A^c \cap B) = \mathbb{P}(A^c) - \mathbb{P}(A^c)\mathbb{P}(B) \\ &= \mathbb{P}(A^c)(1 - \mathbb{P}(B)) = \mathbb{P}(A^c)\mathbb{P}(B^c).\end{aligned}$$

   (c) Using $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ and noting $\mathbb{P}(B) = 1 - \mathbb{P}(B^c)$ and $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, we have

$$\begin{aligned}\mathbb{P}(A \cup B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A)\mathbb{P}(B) \\ &= \frac{1}{3} + \frac{3}{4} - \frac{1}{3} \cdot \frac{3}{4} = \frac{5}{6}.\end{aligned}$$

4. $\mathbb{P}(A\cap B\cap C)=\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$, $\mathbb{P}(A\cap B)=\mathbb{P}(A)\mathbb{P}(B)$, $\mathbb{P}(A\cap C)=\mathbb{P}(A)\mathbb{P}(C)$, $\mathbb{P}(B\cap C)=\mathbb{P}(B)\mathbb{P}(C)$.

5. (a) Use Bayes' Theorem.

   It is easy to see that

$$\mathbb{P}(A|B^c)=1-\mathbb{P}(A^c|B^c)=0.05 \quad \text{and} \quad \mathbb{P}(\mathrm{B}^c)=0.95,$$

   then we have

$$\begin{aligned}\mathbb{P}(B|A) &= \frac{\mathbb{P}(B\cap A)}{\mathbb{P}(A)} \\ &= \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A|B)\mathbb{P}(B)+\mathbb{P}(A|B^c)\mathbb{P}(B^c)} \\ &= \frac{0.95\times 0.05}{0.95\times 0.05+0.05\times 0.95}=\frac{1}{2}\end{aligned}$$

   (b) Substituting the parameter p into the Bayes'expression in (a), we have

$$\frac{p\times 0.05}{p\times 0.05+(1-p)\times 0.95}=0.9.$$

   Solving the equation, we get $p\simeq 0.994$. So p has to be larger than 0.994.

6. $\mathbb{P}(A\cap B\cap C)=\mathbb{P}(A\mid B\cap C)\mathbb{P}(B\cap C)=\mathbb{P}(A\mid B\cap C)\mathbb{P}(B\mid C)\mathbb{P}(C)$.

7. (a) proof of 4 in Lemma 2.

   Noting that $\mathbb{P}(\cdot|B)$ is a probability function and

$$A_1\cup A_2=(A_1-A_1\cap A_2)\cup A_2 \quad \text{and} \quad (\mathrm{A}_1-\mathrm{A}_1\cap\mathrm{A}_2)\cap\mathrm{A}_2=\varnothing,$$

   it follows that

$$\begin{aligned}\mathbb{P}(A_1\cup A_2|B) &= \mathbb{P}((A_1-A_1\cap A_2)\cup A_2|B)=\mathbb{P}(A_1-A_1\cap A_2|B)+\mathbb{P}(A_2|B) \\ &= \mathbb{P}(A_1|B)+\mathbb{P}(A_2|B)-\mathbb{P}(A_1\cap A_2|B).\end{aligned}$$

   (b) proof of 5 in Lemma 1.

   Firstly, we will prove the expression holds for $n=2$. Noting that

$$A\cup B=(A\setminus A\cap B)\cup B \quad \text{and} \quad (\mathrm{A}\setminus(\mathrm{A}\cap\mathrm{B}))\cap\mathrm{B}=\varnothing,$$

it follows that

$$\begin{aligned}\mathbb{P}(A \cup B) &= \mathbb{P}((A \setminus A \cap B) \cup B) = \mathbb{P}(A \setminus (A \cap B)) + \mathbb{P}(B) \\ &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).\end{aligned}$$

Assuming that the expression holds for $n-1$, that is,

$$\begin{aligned}\mathbb{P}(\bigcup_{i=1}^{n-1} A_i) &= \sum_{i=1}^{n-1} \mathbb{P}(A_i) - \sum_{i<j\le n-1} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k\le n-1} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + \\ &\quad (-1)^n \mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_{n-1}),\end{aligned}$$

we now prove that the expression holds for n. Noting that

$$\begin{aligned}\mathbb{P}(\bigcup_{i=1}^{n} A_i) &= \mathbb{P}((A_1 \cup \cdots \cup A_{n-1}) \cup A_n) \\ &= \mathbb{P}((A_1 \cup \cdots \cup A_{n-1}) + \mathbb{P}(A_n) - \mathbb{P}((A_1 \cup \cdots \cup A_{n-1}) \cap A_n)\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}((A_1 \cup \cdots \cup A_{n-1}) \cap A_n) &= \mathbb{P}((A_1 \cap A_n) \cup \cdots \cup (A_{n-1} \cap A_n)) \\ &= \sum_{i=1}^{n} \mathbb{P}(A_i \cap A_n) - \sum_{i<j\le n-1} \mathbb{P}(A_i \cap A_j \cap A_n) - \cdots + \\ &\quad (-1)^n \mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_{n-1} \cap A_n),\end{aligned}$$

the expression holds for n by applying again the assumption for $n-1$ to $\mathbb{P}((A_1 \cup \cdots \cup A_{n-1})$ in the expansion of $\mathbb{P}(\bigcup_{i=1}^n A_i)$.

8. (a) $\Omega = \{H, T\}^{10}$, i.e. the set of vectors of length 10 with each element from $\{H, T\}$, $\mathcal{F} = 2^\Omega$ and $\mathbb{P}(\omega) = 2^{-10}$ for each $\omega \in \Omega$.

   (b) $\Omega = \{0, 1, \ldots, 10\}$, $\mathcal{F} = 2^\Omega$ and $\mathbb{P}(\omega) = \binom{10}{\omega} 2^{-10}$ for each $\omega \in \Omega$.

9. If $y < 0$ then clearly $F_Y(y) = 0$. So suppose $y \ge 0$. We have

$$\begin{aligned}F_Y(y) &= \mathbb{P}(Y \le y) = \mathbb{P}(\max\{0, X\} \le y) = \mathbb{P}(\{\omega :\, \max\{0, X(\omega)\} \le y\}) \\ &= \mathbb{P}(\{\omega :\, 0 < X(\omega) \le y\} \cup \{\omega :\, X(\omega) \le 0\}) \\ &= \mathbb{P}(0 < X \le y) + \mathbb{P}(X \le 0) \\ &= F_X(y) - F_X(0) + F_X(0) = F_X(y).\end{aligned}$$