Statistical Inference

Solutions 1

1.

$$
\begin{aligned}
L(\lambda,\mu;\mathbf{x}) &= \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \left(\prod_{i=1}^{n} x_i\right)^{-\frac{3}{2}} \exp\left(-\frac{\lambda}{2\mu^2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{x_i}\right) \\
&= \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \exp\left(\frac{n\lambda}{\mu}\right) \exp\left(-\frac{\lambda}{2\mu^2}\sum_{i=1}^{n} x_i - \frac{\lambda}{2}\sum_{i=1}^{n}\frac{1}{x_i}\right) \left(\prod_{i=1}^{n} x_i\right)^{-\frac{3}{2}}.
\end{aligned}
$$

Using the Factorization Theorem, the statistic

$$
\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n}\frac{1}{X_i}\right)
$$

is sufficient for (λ,μ).

2. (i) The probability generating function $G_{X_i}(s)$ of X_i is given by

$$
G_{X_i}(z) = e^{\theta(z-1)}.
$$

Hence the p.g.f. $G_T(z)$ of T is given by

$$
G_T(s) = G_{X_i}(z)^n = \left(e^{\theta(z-1)}\right)^n = e^{n\theta(z-1)},
$$

the p.g.f. of the Poisson distribution with mean $n\theta$.

(ii)

$$
f(\mathbf{x},t;\theta) = \frac{e^{-n\theta}\theta^t}{\prod x_i!},
$$

where $t=\sum_{i=1}^{n} x_i$. From (i), the p.m.f. of T is given by

$$
q(t;\theta) = \frac{e^{-n\theta}(n\theta)^t}{t!}.
$$

Hence, writing $T(\mathbf{x}) \equiv \sum_{i=1}^{n} x_i = t$,

$$
\mathbb{P}(\mathbf{X}=\mathbf{x}|T=t) = \frac{f(\mathbf{x},t;\theta)}{q(t;\theta)} = \frac{t!}{x_1!x_2!\ldots x_n!}\left(\frac{1}{n}\right)^t .
$$

This is a multinomial distribution that does not depend upon θ. Hence, by Theorem 1 of Chapter 1, $T \equiv \sum_{i=1}^{n} X_i$ is sufficient for θ.

(iii) We may write

$$
f(\mathbf{x};\theta) = \left(e^{-n\theta}\theta^{\sum x_i}\right)\left(\frac{1}{\prod x_i!}\right).
$$

Taking $g(\sum x_i;\theta) = e^{-n\theta}\theta^{\sum x_i}$, $h(\mathbf{x}) = (\prod x_i!)^{-1}$, it follows immediately from the Factorization Theorem that $T \equiv \sum_{i=1}^{n} X_i$ is sufficient for θ.

3.

$$\begin{aligned} f(x;\mu,\sigma^2) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2} + \frac{\mu x}{\sigma^2} - \frac{\mu^2}{2\sigma^2}\right), \end{aligned}$$

which is in the form of Equation (8) of Section 1 with $k = 2$,

$$h(x) = 1, \qquad A(\mu,\sigma^2) = \frac{\mu^2}{2\sigma^2} + \frac{1}{2}\ln(2\pi\sigma^2),$$

$$\eta_1(\mu,\sigma^2) = \frac{1}{\sigma^2}, \qquad t_1(x) = -\frac{1}{2}x^2,$$

$$\eta_2(\mu,\sigma^2) = \frac{\mu}{\sigma^2}, \qquad t_2(x) = x.$$

Hence the natural parameters as stated in the question.