MAS programmes - Statistical Inference

Solutions 4

1. (i) The likelihood ratio

$$\frac{f_1(\mathbf{x})}{f_0(\mathbf{x})} = \frac{\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu_1)^2\right)}{\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu_0)^2\right)} = \exp\left(\frac{\mu_1-\mu_0}{\sigma^2}\sum_{i=1}^n x_i + \frac{n}{2\sigma^2}(\mu_0^2-\mu_1^2)\right).$$

Any likelihood ratio test has critical region specified by

$$\exp\left(\frac{(\mu_1-\mu_0)n\bar{x}}{\sigma^2} + \frac{n}{2\sigma^2}(\mu_0^2-\mu_1^2)\right) > c$$

for some constant c, which is equivalent to

$$\frac{(\mu_1-\mu_0)n\bar{x}}{\sigma^2} + \frac{n}{2\sigma^2}(\mu_0^2-\mu_1^2) > \ln(c),$$

which is equivalent to

$$\bar{x} \geq k$$

for a constant k.

(ii) Since $\bar{X} \sim N(\mu, \sigma^2/n)$,

$$\mathbb{P}(\bar{X} \geq k) = 1 - \Phi\left(\frac{k-\mu}{\sigma/\sqrt{n}}\right) = 1 - \Phi\left(\frac{(k-\mu)\sqrt{n}}{\sigma}\right) = \Phi\left(\frac{(\mu-k)\sqrt{n}}{\sigma}\right).$$

When $\mu = \mu_0$, this gives us α, and when $\mu = \mu_1$, it gives us β.

(iii) We require

$$\Phi\left(\frac{(k-\mu_0)\sqrt{n}}{\sigma}\right) = 0.95 \quad \text{and} \quad \Phi\left(\frac{(\mu_1-k)\sqrt{n}}{\sigma}\right) = 0.95.$$

Thus

$$k - \mu_0 = \frac{\sigma}{\sqrt{n}}\Phi^{-1}(0.95)$$

and

$$\mu_1 - k = \frac{\sigma}{\sqrt{n}}\Phi^{-1}(0.95).$$

Adding the two above equations,

$$\mu_1 - \mu_0 = \frac{2\sigma}{\sqrt{n}}\Phi^{-1}(0.95).$$

It follows that

$$n = \frac{4\sigma^2}{(\mu_1-\mu_0)^2}[\Phi^{-1}(0.95)]^2.$$

From tables or by using a statistical package we find that $\Phi^{-1}(0.95) = 1.6449$. Hence the required result.

2. From Chapter 1 of the notes,

$$\ln L(\mu, \sigma^2; \mathbf{x}) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(s_x^2 + n(\bar{x}-\mu)^2),$$

which is maximized by taking $\mu = \bar{x},\ \sigma^2 = s_x^2/n$. Thus the maximized likelihood is equal to

$$-\frac{n}{2}\left(1 + \ln\left(\frac{2\pi s_x^2}{n}\right)\right).$$

Hence

$$\text{AIC} = n\left(1 + \ln\left(\frac{2\pi s_x^2}{n}\right)\right) + 4,$$

which reduces to the given expression.

3. For σ^2 known,

$$\ln L(\mu; \mathbf{x}) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(s_x^2 + n(\bar{x}-\mu)^2),$$

which is maximized by taking $\mu = \bar{x}$. Hence

$$\begin{aligned} 2\ln\lambda(x) &= 2\ln L(\bar{x}; \mathbf{x}) - 2\ln L(\mu_0; \mathbf{x}) \\ &= \frac{n(\bar{x}-\mu_0)^2}{\sigma^2}. \end{aligned}$$

For σ^2 unknown,

$$\ln L(\mu, \sigma^2; \mathbf{x}) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(s_x^2 + n(\bar{x}-\mu)^2),$$

which is maximized by taking $\mu = \bar{x},\ \sigma^2 = \hat{\sigma}^2$, where

$$\hat{\sigma}^2 = \frac{s_x^2}{n}.$$

Under $H_0 : \mu = \mu_0,\ \ln L(\mu_0, \sigma^2; \mathbf{x})$ is maximized by taking $\sigma^2 = \hat{\sigma}_0^2$, where

$$\hat{\sigma}_0^2 = \frac{s_x^2}{n} + (\bar{x}-\mu_0)^2.$$

Hence

$$\begin{aligned} 2\ln\lambda(\mathbf{x}) &= 2\ln L(\bar{x}, \hat{\sigma}^2; \mathbf{x}) - 2\ln L(\mu_0, \hat{\sigma}_0^2; \mathbf{x}) \\ &= n\ln\left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2}\right) \\ &= n\ln\left(1 + \frac{n(\bar{x}-\mu_0)^2}{s_x^2}\right). \end{aligned}$$

The critical regions $\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^n : 2\ln\lambda(\mathbf{x}) \geq c\}$ of the two tests, are equivalent to

$$\mathcal{C} = \left\{\mathbf{x} \in \mathbb{R}^n : \frac{\sqrt{n}\,|\bar{x}-\mu_0|}{\sigma} \geq c_1\right\}$$

and

$$\mathcal{C} = \left\{\mathbf{x} \in \mathbb{R}^n : \frac{\sqrt{n}\,|\bar{x}-\mu_0|}{s} \geq c_2\right\},$$

respectively, for some constants c_1 and c_2, where s is the sample standard deviation.