Probability and Distribution Theory: Exercises 4

1. *Log-normal distribution.* Let $Y = e^X$ where X is $N(0,1)$. Find the density function of Y.

2. Let X_1 and X_2 be independent random variables each having the density function

$$f_{X_i}(x_i) \quad = \quad 4x_i e^{-2x_i} \qquad 0 < x_i < \infty \qquad\qquad i = 1, 2.$$

Define $Y_1 = \frac{X_1}{X_1+X_2}$ and $Y_2 = X_1 + X_2$. Calculate

(a) the joint density of (X_1, X_2);

(b) the joint density of (Y_1, Y_2);

(c) the marginal densities of Y_1 and Y_2 and check for independence.

3. A discrete random variable, X, has a PGF given by

$$G_X(s) = E\left[s^X\right] = \sum_{x=0}^{\infty} s^x \mathbb{P}(X = x).$$

Show that:

$$\frac{\mathrm{d}^k G_X(s)}{\mathrm{d}s^k}|_{s=0} = k! p_k; \qquad \frac{\mathrm{d}^k G_X(s)}{\mathrm{d}s^k}|_{s=1} = E\left[\frac{X!}{(X-k)!}\right]$$

for $k = 1, 2, 3, \cdots$.

4. A discrete random variable, X, has a PGF

$$G_X(s) = \{\theta s + (1-\theta)\}^n$$

where $\theta + (1-\theta) = 1$ and $0 < \theta < 1$. Use the results in Q3 to: (a) compute the probabilities $\mathbb{P}(X = x) = p_x$, for $x = 0, 1, 2, 3, \ldots$; (b) find $\mathbb{E}[X]$ and $\mathrm{var}(X)$; (c) identify the distribution from the results obtained.

5. Suppose a random variable, X, has p.d.f., $f_X(x)$ for $-\infty < x < \infty$ and $g(X)$ is a function of X :

   (a) Write down an expression for $M_{g(X)}(t)$, the MGF of $g(X)$, where t is a dummy variable.

   (b) Show that if $g(X) = c \cdot h(X)$, where c is a constant and $h(X)$ is another function of X, then $M_{g(X)}(t) = M_{h(X)}(ct)$.

   (c) Show that if $g(X) = h(X) + c$, then $M_{g(X)}(t) = e^{ct} M_{h(X)}(t)$.

   (d) If $X \sim N(\mu, \sigma^2)$ and $Z = \frac{X-\mu}{\sigma}$, find $M_Z(t)$, and hence find $\mathbb{E}[Z]$ and $\text{var}(Z)$.

   (e) If $S = \sum_{i=1}^{n} Z_i$ where the Z_i are i.i.d. $N(0,1)$, find $M_S(t)$.