Probability and Distribution Theory:
Solutions to Supplementary Exercises

Probability Theory

1. (a) $A \cap (B \cup C \cup D)^c = A \cap (B^c \cap C^c \cap D^c)$.
   (b) $(A \cup B \cup C \cup D)^c = A^c \cap B^c \cap C^c \cap D^c$.
   (c) $A \cap B \cap C \cap D$.
   (d) $A^c \cup B^c \cup C^c \cup D^c = (A \cap B \cap C \cap D)^c$.
   (e) $(A \cap B \cap C) \cup (A \cap C \cap D) \cup (B \cap C \cap D) \cup (A \cap B \cap D)$.
   (f) $(A^c \cap B \cap C \cap D) \cup (A \cap B^c \cap C \cap D) \cup (A \cap B \cap C^c \cap D) \cup (A \cap B \cap C \cap D^c)$.
   (g) $(A \cap B^c \cap C^c \cap D^c) \cup (A^c \cap B \cap C^c \cap D^c) \cup (A^c \cap B^c \cap C \cap D^c) \cup (A^c \cap B^c \cap C^c \cap D)$.
   (h) $(A \cap B^c \cap D^c) \cup (A^c \cap B \cap D^c)$.

2. Let A be the event that the randomly selected student uses Shotmail, and let B be the event that the student uses Payserve. Then

$$\mathbb{P}(A) = 0.5, \quad \mathbb{P}(B) = 0.3, \quad \mathbb{P}(A \cap B) = 0.25.$$

   (a) Need to find $\mathbb{P}(A \cup B)$. In fact,

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 0.5 + 0.3 - 0.25 = 0.55.$$

   Hence, statement is TRUE.

   (b) $\mathbb{P}((A \cap B)^c) = 1 - \mathbb{P}(A \cap B) = 1 - 0.25 = 0.75$. TRUE.
   (c) $\mathbb{P}((A \cup B)^c) = 1 - \mathbb{P}(A \cup B) = 1 - 0.55 = 0.45$. FALSE.
   (d) $\mathbb{P}((A \cup B) \setminus (A \cap B)) = \mathbb{P}(A \cup B) - \mathbb{P}(A \cap B) = 0.55 - 0.25 = 0.3$, where the 1st equality follows from the fact that $A \cap B \subseteq A \cup B$. FALSE.

3. (a) Set of outcomes whose sum is divisible by 4:

$$A = \{(1,3), (2,2), (2,6), (3,1), (3,5), (4,4), (5,3), (6,2), (6,6)\}.$$

   $\mathbb{P}(A) = \frac{\#A}{36} = \frac{9}{36} = \frac{1}{4}$.

(b) Set of outcomes in which a 3 turns up exactly once:

$$B = \{(3,1),(3,2),(3,4),(3,5),(3,6),(1,3),(2,3),(4,3),(5,3),(6,3)\}.$$

$\mathbb{P}(B) = \frac{10}{36} = \frac{5}{18}$.

(c) Set of outcomes whose sum is 5:

$$C = \{(1,4),(2,3),(3,2),(4,1)\}.$$

$\mathbb{P}(C) = \frac{4}{36} = \frac{1}{9}$.

(d) Set of outcomes for which both numbers are even:

$$D = \{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}.$$

$\mathbb{P}(D) = \frac{9}{36} = \frac{1}{4}$.

4. Let A be the event that the sum is 4.
Let B be the event that the sum is ≥ 4.
$\mathbb{P}(A) = \mathbb{P}(\{(1,3),(2,2),(3,1)\}) = \frac{3}{25}$.
$\mathbb{P}(B) = \mathbb{P}(\Omega \setminus \{(1,1),(1,2),(2,1)\}) = \mathbb{P}(\Omega) - \mathbb{P}(\{(1,1),(1,2),(2,1)\}) = 1 - \frac{3}{25} = \frac{22}{25} > 0$.
$A \cap B = \{(1,3),(2,2),(3,1)\}$.
$\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} = \frac{3/25}{22/25} = \frac{3}{22}$.

5. A and B independent $\Rightarrow \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

(a) $\mathbb{P}(A^c \cap B) = \mathbb{P}(B \setminus (A \cap B)) = \mathbb{P}(B) - \mathbb{P}(A \cap B) = \mathbb{P}(B) - \mathbb{P}(A)\mathbb{P}(B)$, where the last equality follows by the independence of A and B; hence $\mathbb{P}(A^c \cap B) = (1 - \mathbb{P}(A))\mathbb{P}(B) = \mathbb{P}(A^c)\mathbb{P}(B)$.

(b) $\mathbb{P}(A^c \cap B^c) = \mathbb{P}((A \cup B)^c)$, by De-Morgan's Laws, which is equal to $1 - \mathbb{P}(A \cup B) = 1 - \{\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)\} = 1 + \mathbb{P}(A \cap B) - \mathbb{P}(A) - \mathbb{P}(B)$, which, by independence of A and B, is equal to $1 + \mathbb{P}(A)\mathbb{P}(B) - \mathbb{P}(A) - \mathbb{P}(B)$ $= (1 - \mathbb{P}(A))(1 - \mathbb{P}(B)) = \mathbb{P}(A^c)\mathbb{P}(B^c)$.

<u>Univariate Distribution Theory</u>

6. (a) $1 = \int_{-1}^{1} c(1-x^2)dx = c\int_{-1}^{1}(1-x^2)dx = c[x - \frac{x^3}{3}]_{-1}^{1}$
$= c\{(1 - \frac{1}{3}) - (-1 + \frac{1}{3})\} = c\{2 - \frac{2}{3}\} = c \times \frac{4}{3} \Rightarrow c = \frac{3}{4}$.

(b) $1 = c\int_5^\infty \frac{1}{x^2} = -c[\frac{1}{x}]_5^\infty = c[\frac{1}{x}]_\infty^5 = \frac{c}{5} - 0 = \frac{c}{5}$
$\Rightarrow c = 5.$

7. (a)

$$E[X] = \int_0^\infty x f_X(x)dx = \int_0^\infty x\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}dx = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty x^\alpha e^{-\beta x}dx.$$
$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \times \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}}\int_0^\infty \frac{\beta^{\alpha+1}}{\Gamma(\alpha+1)}x^{(\alpha+1)-1}e^{-\beta x}dx$$
$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \times \frac{\alpha\Gamma(\alpha)}{\beta^{\alpha+1}} \times 1$$

since the integrand is the p.d.f. corresponding to a r.v. from the $\Gamma(\alpha+1,\beta)$ distribution. Hence

$$E[X] = \frac{\alpha}{\beta}.$$

$$E[X^2] = \int_0^\infty x^2 f_X(x)dx = \int_0^\infty x^2\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}dx = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{(\alpha+2)-1}e^{-\beta x}dx$$
$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \times \frac{\Gamma(\alpha+2)}{\beta^{\alpha+2}}\int_0^\infty \frac{\beta^{\alpha+2}}{\Gamma(\alpha+2)}x^{(\alpha+2)-1}e^{-\beta x}dx$$
$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \times \frac{(\alpha+1)\alpha\Gamma(\alpha)}{\beta^{\alpha+2}} \times 1$$

since the integrand is the p.d.f. of a r.v. from the $\Gamma(\alpha+2,\beta)$ distribution. Hence,

$$E[X^2] = \frac{\alpha(\alpha+1)}{\beta^2}.$$

It follows that

$$\text{var}(X) = \frac{\alpha(\alpha+1)}{\beta^2} - \left(\frac{\alpha}{\beta}\right)^2 = \frac{\alpha}{\beta^2}.$$

(b)

$$E[X] = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 x.x^{\alpha-1}(1-x)^{\beta-1}dx$$
$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \times \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}\int_0^1 \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta)}x^{(\alpha+1)-1}(1-x)^{\beta-1}dx$$
$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \times \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \times 1$$

since the integrand is the p.d.f. of the $Beta(\alpha+1,\beta)$ distribution. Hence,

$$E[X] = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \times \frac{\alpha\Gamma(\alpha)\Gamma(\beta)}{(\alpha+\beta)\Gamma(\alpha+\beta)} = \frac{\alpha}{\alpha+\beta}.$$

In a similar way, we can show that

$$E[X^2] = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}.$$

Hence

$$\mathrm{var}(X) = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2$$

which, after some manipulation, can be shown to be equal to

$$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

8. We need to check the 2 conditions which characterize p.d.f.'s.
Since $\frac{1}{\sqrt{2\pi\sigma^2}} > 0$ and $e^{-\frac{(x-\mu)^2}{2\sigma^2}} > 0$ for $-\infty < x < \infty$, then clearly $f_X(x) > 0$ for $-\infty < x < \infty$, which establishes the first condition.
Now

$$\int_{-\infty}^{\infty} f_X(x)dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

By using the substitution $y = \frac{x-\mu}{\sigma}$ and that consequently $\ldots dx = \ldots \sigma dy$, the RHS of the above equation can be shown to be equal to

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy = \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} = 1,$$

as required to show that the second condition is satisfied.

<u>Joint and Conditional Distributions</u>

9. (a) $R_{(X,Y)} = \{(x,y) : 0 < x < 1,\ 0 < y < 1\}$

(b)

$$1 = \int_0^1\int_0^1 f_{(X,Y)}(x,y)dxdy = \int_0^1 c\left[\frac{x^3}{3}\right]_0^1 dy = \int_0^1 \frac{c}{3}dy = \frac{c}{3}[y]_0^1 = \frac{c}{3} \Rightarrow c = 3.$$

(c) $f_X(x) = \int_0^1 3x^2 dy = [3x^2 y]_{y=0}^{y=1} = 3x^2 \quad 0<x<1$, and
$f_Y(y) = \int_0^1 3x^2 dx = 3\left[\frac{x^3}{3}\right]_{x=0}^{x=1} = 1 \quad 0<y<1.$

10. (a) $f_{(X,Y)}(x,y) = c$, therefore

$$1 = \int_{-1}^1\left(\int_{-1}^1 f_{(X,Y)}(x,y)dx\right)dy = c\int_{-1}^1 [x]_{-1}^1 dy = 2c\int_{-1}^1 dy = 2c[y]_{-1}^1 = 4c \Rightarrow c = \frac{1}{4}.$$

$f_{(X,Y)} = 1/4$ for $(x,y) \in R_{(X,Y)}$.

(b) i.

$$\mathbb{P}(2X > Y) = \iint_{2X>Y} f_{(X,Y)}(x,y)dxdy = \iint_{R_1} f_{(X,Y)}(x,y)dxdy + \iint_{R_2} f_{(X,Y)}(x,y)dxdy$$
$$= \int_{x=-1/2}^{1/2}\left(\int_{y=-1}^{2x} \frac{1}{4}dy\right)dx + \int_{x=1/2}^{1}\left(\int_{y=-1}^{1} \frac{1}{4}dy\right)dx$$
$$= \frac{1}{4}\left\{\int_{-1/2}^{1/2} [y]_{-1}^{2x} dx + \int_{1/2}^{1} [y]_{-1}^{1} dx\right\} = \frac{1}{4}\left\{\int_{-1/2}^{1/2} (2x+1)dx + \int_{1/2}^{1} 2dx\right\}$$
$$= \frac{1}{4}\left\{[x^2+x]_{-1/2}^{1/2} + [2x]_{1/2}^{1}\right\} = \frac{1}{4}\left\{\left(\frac{1}{4}+\frac{1}{2}\right) - \left(\frac{1}{4}-\frac{1}{2}\right) + 2 - 1\right\} = \frac{1}{2}.$$

ii.

$$\iint_{X^2+Y^2<1} f_{(X,Y)}(x,y)dxdy = \frac{1}{4}\int_{X^2+Y^2<1} dxdy = \frac{1}{4}\pi \times 1^2 = \frac{\pi}{4}.$$

iii. $|x+y| < 1 \Leftrightarrow x+y < 1$ and $x+y > -1$ i.e. $y < 1-x$ and $y > -1-x$.

$$\mathbb{P}(|X+Y|<1) = \iint_{|X+Y|<1} f_{(X,Y)}(x,y)dxdy$$
$$= \frac{1}{4}\left\{\int_{x=-1}^{0}\left(\int_{y=-1-x}^{1} dy\right)dx + \int_{x=0}^{1}\left(\int_{y=-1}^{1-x} dy\right)dx\right\}$$
$$= \frac{1}{4}\left\{\int_{x=-1}^{0}(2+x)dx + \int_{x=0}^{1}(2-x)dx\right\} = \frac{1}{4}\left\{\left[2x+\frac{x^2}{2}\right]_{-1}^{0} + \left[2x-\frac{x^2}{2}\right]_{0}^{1}\right\}$$
$$= \frac{1}{4}\left\{-\left(-2+\frac{1}{2}\right) + \left(2-\frac{1}{2}\right)\right\} = \frac{3}{4}.$$

11. (a)

$$p_X(1) = \sum_{y=1}^{3} p_{(X,Y)}(1,y) = 1/6 + 1/6 + 0 = 1/3.$$
$$p_X(2) = \sum_{y=1}^{3} p_{(X,Y)}(2,y) = 1/12 + 0 + 2/9 = 11/36.$$
$$p_X(3) = \sum_{y=1}^{3} p_{(X,Y)}(3,y) = 1/12 + 1/6 + 1/9 = 13/36.$$

Similarly

$$p_Y(1) = 1/6 + 1/12 + 1/12 = 1/3.$$
$$p_Y(2) = 1/6 + 0 + 1/6 = 1/3.$$
$$p_Y(3) = 0 + 2/9 + 1/9 = 1/3.$$

(b) First note that $R_{XY} = \{1,2,3,4,6,9\}$[1]. The distribution of XY can be found by adding the total mass corresponding to particular values of XY from the table on the examples sheet. Hence

1	2	3	4	6	9
1/6	1/12	1/12			
	1/6		0	1/6	
		0		2/9	1/9
1/6	1/4	1/12	0	7/18	1/9

The p.m.f. of XY is given in the final row.

$E[XY] = \frac{1}{36}\{(1\times 6)+(2\times 9)+(3\times 3)+(4\times 0)+(6\times 14)+(9\times 4)\} = \frac{1}{36}\{6+18+9+0+84+36\} = \frac{153}{36} = 4.25.$

(c) $\mathbb{P}(Y=2|X>1) = \frac{\mathbb{P}(Y=2,X>1)}{\mathbb{P}(X>1)} = \frac{\mathbb{P}(Y=2,X>1)}{1-\mathbb{P}(X=1)} = \frac{0+1/6}{1-1/3} = \frac{1/6}{2/3} = \frac{1}{4}.$

12. (a) $f_{(X,Y)}(x,y) = f_{Y|X}(y|x)f_X(x) = \frac{2y}{x^2}\times 7x^6 = 14x^4y$ with $R_{(X,Y)} = \{(x,y) : 0<x<1, 0<y<x\}$.

(b) $f_Y(y) = \int_{x=y}^1 f_{(X,Y)}(x,y)dx = 14y\int_y^1 x^4dx = 14y\left[\frac{x^5}{5}\right]_y^1 = \frac{14}{5}y(1-y^5) \qquad y\in R_Y = (0,1).$

13. To save time, for $n,m\in\mathbb{Z}^+$, can compute

$$\begin{aligned} E[X^nY^m] &= \int_0^1\int_0^1 x^ny^m(x+y)dxdy = \int_0^1\left\{\int_0^1(x^{n+1}y^m + x^ny^{m+1})dx\right\}dy \\ &= \int_0^1\left[\frac{x^{n+2}}{n+2}y^m + \frac{x^{n+1}}{n+1}y^{m+1}\right]_0^1 dy = \int_0^1\left(\frac{y^m}{n+2}+\frac{y^{m+1}}{n+1}\right)dy \\ &= \left[\frac{y^{m+1}}{(m+1)(n+2)} + \frac{y^{m+2}}{(m+2)(n+1)}\right]_0^1 = \frac{1}{(m+1)(n+2)} + \frac{1}{(m+2)(n+1)}. \end{aligned}$$

Plugging in $n=1, m=0$ yields:

$$E[X] = \frac{1}{3} + \frac{1}{2\times 2} = \frac{1}{3}+\frac{1}{4} = \frac{7}{12}.$$

$m=1, n=0$:

$$E[Y] = \frac{7}{12}.$$

$m=1, n=1$

$$E[XY] = \frac{1}{3}.$$

[1]Note: i.e. the range space of $Z = XY$; this is **not** the same as $R_{(X,Y)}$.

$m = 2, n = 0$ and $m = 0, n = 2$:

$$E[Y^2] = E[X^2] = \frac{5}{12}.$$

(a)

$$\mathrm{cov}(X,Y) = E[XY] - E[X]E[Y] = \frac{1}{3} - \frac{7}{12} \times \frac{7}{12} = -\frac{1}{144}.$$

(b)

$$\mathrm{var}(X) = \mathrm{var}(Y) = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}.$$

Therefore

$$\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)\mathrm{var}(Y)}} = -\frac{1}{11}.$$

(c)

$$\mathrm{var}(2X - Y + 4) = \mathrm{var}(2X - Y) = \mathrm{var}(2X) + \mathrm{var}(Y) + 2(1)(-1)\mathrm{cov}(2X, Y)$$

$$= 2^2\mathrm{var}(X) + \mathrm{var}(Y) - 2 \times 2\mathrm{cov}(X,Y) = 4\mathrm{var}(X) + \mathrm{var}(Y) - 4\mathrm{cov}(X,Y) = \frac{59}{144}.$$

14. (a)

$$f_X(x) = \int_x^2 \tfrac{1}{2} dy = [\tfrac{y}{2}]_x^2 = \tfrac{2-x}{2} \quad x \in [0,2].$$

$$f_Y(y) = \int_0^y \tfrac{1}{2} dx = [\tfrac{x}{2}]_0^y = \tfrac{y}{2} \quad y \in [0,2].$$

(b) One proof for the fact that X and Y are not independent is that the region $R_{(X,Y)}$ is not 'rectangular'. Another proof is that the joint p.d.f. of X and Y is not equal to the product of their marginal p.d.f.'s, i.e.

$$f_X(x)f_Y(y) = \tfrac{2-x}{2} \times \tfrac{y}{2} \neq \tfrac{1}{2} = f_{(X,Y)}(x,y)$$

for $(x,y) \in R_{(X,Y)}$.

Transformations of Random Variables

15. (a)

$$F_X(x) = \mathbb{P}(-\frac{1}{\lambda}\ln U \leq x) = \mathbb{P}(\ln U \geq -\lambda x) = \mathbb{P}(U \geq e^{-\lambda x}) = 1 - F_U(e^{-\lambda x})$$

Hence,

$$f_X(x) = F_X^{'}(x) = -f_U(e^{-\lambda x}) \times -\lambda e^{-\lambda x} = -1 \times -\lambda e^{-\lambda x} = \lambda e^{-\lambda x}$$

for $x \in R_X = \{x : x \geq 0\}$.

(b) Set $Y = 1 - e^{-\lambda X} \sim U(0,1)$.

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(1 - e^{-\lambda X} \leq y) = \mathbb{P}(-e^{-\lambda X} \leq y - 1) = \mathbb{P}(e^{-\lambda X} \geq 1 - y)$$
$$= \mathbb{P}(-\lambda X \geq \ln(1-y)) = \mathbb{P}(X \leq -\frac{1}{\lambda}\ln(1-y)).$$

$$f_Y(y) = F_Y^{'}(y) = f_X(-\frac{1}{\lambda}\ln(1-y)) \times -\frac{1}{\lambda(1-y)} \times -1 = \frac{1}{\lambda(1-y)}\lambda e^{\ln(1-y)} = \frac{1-y}{1-y} = 1$$

for $y \in R_X = (0,1)$, i.e.
$Y \sim U(0,1)$.

16. $\mathbb{P}(W > w) = \mathbb{P}(\min(X,Y) > w) = \mathbb{P}(X > w, Y > w)$, which, by independence, is equal to
$\mathbb{P}(X > w)\mathbb{P}(Y > w) = e^{-\lambda w} \times e^{-\mu w} = e^{-(\lambda+\mu)w}$
Hence $F_W(w) = \mathbb{P}(W \leq w) = 1 - \mathbb{P}(W > w) = 1 - e^{-(\lambda+\mu)w}$, and so
$f_W(w) = \frac{d}{dw}F_W(w) = \frac{d}{dw}(1 - e^{-(\lambda+\mu)w}) = (\lambda+\mu)e^{-(\lambda+\mu)w}$ with $R_W = \{w : w \geq 0\}$, since both x, y are greater than or equal to zero.

17. (a) The transformation maps (U,V) to (X,Y). The inverse transformation can be expressed

$$U = X \qquad\qquad V = Y - X$$

with the values taken by each of the 4 random variables represented by their lower case equivalents. The Jacobian (not "Jacobean", which relates to the period of James I of England, 1603-1625) of the transformation from (U,V) to (X,Y) is given by

$$J(x,y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1.$$

In order to remember which element goes where in the Jacobian, I suggest that you start with the diagonal elements first: that will then give a clue as to what the off-diagonal partial derivatives should be.

(b)

$$f_{(X,Y)}(x,y) = f_{(U,V)}(x, y-x) \times 1 = \lambda^2 e^{-\lambda(x+(y-x))} \times 1$$
$$= \lambda^2 e^{-\lambda y}$$

for $0 \leq x \leq y < \infty$.

Generating Functions for univariate distributions

18. (a) Coefficient of θ^1 is equal to $\mathbb{P}(X=1)=0.65$.

(b) Coefficient of θ^{10} is clearly equal to 0, and so $\mathbb{P}(X=10)=0$.

(c) Coefficient of θ^0 is the constant term 0.35, i.e. $\mathbb{P}(X=0)=0.35$.

(d)

$$E[X]=G_X^{'}(1)=0.65.$$

19.

$$M_X(t)=\int_0^\infty e^{tx}f_X(x)dx=\int_0^\infty e^{tx}\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}dx=\frac{\beta^\alpha}{\Gamma(\alpha)}\times\frac{\Gamma(\alpha)}{(\beta-t)^\alpha}\int_0^\infty\frac{(\beta-t)^\alpha}{\Gamma(\alpha)}e^{-(\beta-t)x}x^{\alpha-1}dx$$

$$=\frac{\beta^\alpha}{\Gamma(\alpha)}\times\frac{\Gamma(\alpha)}{(\beta-t)^\alpha}\times 1=\left(\frac{\beta}{\beta-t}\right)^\alpha$$

for $t<\beta$.

$$M_X^{'}(t)=\beta^\alpha\times-\frac{\alpha}{(\beta-t)^{\alpha+1}}\times-1=\frac{\alpha}{\beta}\left(\frac{\beta}{\beta-t}\right)^{\alpha+1}.$$

$$M_X^{''}(t)=\frac{\alpha}{\beta}\beta^{\alpha+1}\times-(\alpha+1)\frac{1}{(\beta-t)^{\alpha+2}}\times-1=\alpha\beta^\alpha\frac{\alpha+1}{(\beta-t)^{\alpha+2}}=\frac{\alpha(\alpha+1)}{\beta^2}\left(\frac{\beta}{\beta-t}\right)^{\alpha+2}.$$

Hence

$$E[X]=M_X^{'}(0)=\frac{\alpha}{\beta},\qquad M_X^{''}(0)=\frac{\alpha(\alpha+1)}{\beta^2}$$

$$\text{var}(X)=E[X^2]-E[X]^2=\frac{\alpha(\alpha+1)}{\beta^2}-\left(\frac{\alpha}{\beta}\right)^2=\frac{\alpha}{\beta^2}.$$

20.

$$M_W^{'}(t)=\frac{-20}{(1-7t)^{21}}\times-7=\frac{140}{(1-7t)^{21}}.$$

$$M_W^{''}(t)=\frac{140\times-21}{(1-7t)^{22}}\times-7=\frac{20,580}{(1-7t)^{22}}.$$

Hence

$$E[W]=M_W^{'}(0)=140,\qquad E[W^2]=M_W^{''}(0)=20580.$$

$$\text{var}(W)=E[W^2]-E[W]^2=20580-(140)^2=980.$$

21. (a)

$$M_X(t)=\int_0^\infty e^{tx}108x^2e^{-6x}dx=\int_0^\infty 108x^2e^{-(6-t)x}dx.$$

Assuming that $t < 6$, there are at least 2 ways to proceed with the computation of this integral.

Integration by Parts:

$$M_X(t) = \left[108\tfrac{x^2}{-(6-t)}e^{-(6-t)x}\right]_0^\infty + \textstyle\int_0^\infty 108 \times 2x\tfrac{e^{-(6-t)x}}{6-t}dx$$
$$= (0-0) + 216\textstyle\int_0^\infty \tfrac{xe^{-(6-t)x}}{(6-t)}dx = \left[216\tfrac{x}{-(6-t)^2}e^{-(6-t)x}\right]_0^\infty + \tfrac{216}{(6-t)^2}\textstyle\int_0^\infty e^{-(6-t)x}dx$$
$$= -\tfrac{216}{(6-t)^3}\left[e^{-(6-t)x}\right]_0^\infty = \tfrac{216}{(6-t)^3}.$$

Slick method:

$$M_X(t) = 108\textstyle\int_0^\infty x^{3-1}e^{-(6-t)x}dx = 108\tfrac{\Gamma(3)}{(6-t)^3}\textstyle\int_0^\infty \tfrac{(6-t)^3}{\Gamma(3)}x^{3-1}e^{-(6-t)x}dx.$$

However, the integrand is the p.d.f. of a r.v. from the $Gamma(3, 6-t)$ distribution. Also, from the handout on the Gamma function, we know that $\Gamma(n) = (n-1)!$ for $n \in \mathbb{Z}^+$. Hence

$$M_X(t) = 108\tfrac{2!}{(6-t)^3} \times 1 = \tfrac{216}{(6-t)^3}.$$

(b)

$$M'_X(t) = \tfrac{216\times-3\times-1}{(6-t)^4} = \tfrac{648}{(6-t)^4},$$
$$M''_X(t) = \tfrac{2592}{(6-t)^5},$$
$$M^{(3)}_X(t) = \tfrac{12{,}960}{(6-t)^6}.$$

Hence

$$E[X^3] = M^{(3)}_X(0) = \tfrac{5}{18} = 0.278 \text{ (to 3 s.f.)}$$

22. (a)

$$\Psi_X(\theta) = \sum_{k=0}^{1} e^{i\theta k}p_X(k) = e^0 \times (1-p) + e^{i\theta}p$$

(b)

$$\Psi_Y(\theta) = \Psi_{\sum_{j=1}^n X_j}(\theta) = \prod_{j=1}^n \Psi_{X_j}(\theta) = \prod_{j=1}^n((1-p)+pe^{i\theta}) = ((1-p)+pe^{i\theta})^n.$$

<u>Central Limit Theorem</u>

23.

$$\mathbb{P}(245 \leq \sum_{i=1}^{40} X_i \leq 255) = \mathbb{P}(\frac{245}{40} \leq \overline{X} \leq \frac{255}{40})$$

$$= \mathbb{P}(6.125 \leq \overline{X} \leq 6.375) = \mathbb{P}(6.125 - 6.53 \leq \overline{X} - \mu \leq 6.375 - 6.53)$$

since $\mu = 6.53$.

Also, since $\sigma^2 = 10$, then $\mathrm{var}(\overline{X}) = \sigma^2/n = 10/40 = 0.25$, and the above probability is equal to

$$\mathbb{P}\left(\frac{6.125 - 6.53}{0.5} \leq \frac{\overline{X} - \mu}{\sqrt{\sigma^2/n}} \leq \frac{6.375 - 6.53}{0.5}\right)$$

which is **approximately** equal to $\mathbb{P}(-0.81 \leq Z \leq -0.31)$, where $Z \sim N(0,1)$. In turn, this can be written as

$$\Phi(-0.31) - \Phi(-0.81) = 1 - \Phi(0.31) - \{1 - \Phi(0.81)\} = \Phi(0.81) - \Phi(0.31)$$

$$= 0.7910 - 0.6217 = 0.1693.$$

24.

$$E[X] = \int_0^1 x.3(1-x)^2 dx = 3\int_0^1 \left\{x - 2x^2 + x^3\right\} dx = \frac{1}{4}.$$

$$E[X^2] = 3\int_0^1 x^2.3(1-x)^2 dx = 3\int_0^1 \left\{x^2 - 2x^3 + x^4\right\} dx = \frac{1}{10}.$$

Thus $\mu = 0.25$ and $\sigma^2 = \mathrm{var}(X) = E[X^2] - E[X]^2 = 0.1 - 0.25^2 = 0.0375$.

$$\mathbb{P}\left(\sum_{i=1}^{625} X_i < 170\right)$$

$$= \mathbb{P}\left(\frac{\sum_{i=1}^{625} X_i - n\mu}{\sigma\sqrt{n}} < \frac{170 - 625 \times 0.25}{\sqrt{0.0375} \times 25}\right)$$

$$= \mathbb{P}\left(\frac{\sum_{i=1}^{625} X_i - n\mu}{\sigma\sqrt{n}} < \frac{13.75}{4.841229}\right) \approx \Phi(2.84).$$