Probability and Distribution Theory: Solutions 3

1. (a) From the information given, we know that the probability distribution for the number of accidents is:

$$\mathbb{P}(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad \text{for } x = 0, 1, 2, \ldots$$

and that the number of fatal accidents, y, is conditional on the number of accidents, x, hence the probability distribution for the number of fatal accidents is a binomial distribution:

$$\mathbb{P}(Y = y|X = x) = \binom{x}{y} p^y (1-p)^{x-y} \qquad \text{for } y = 0, 1, 2, \ldots x$$

Therefore, the joint distribution of (X, Y) is

$$\begin{aligned}
\mathbb{P}(X = x, Y = y) &= \mathbb{P}(Y = y|X = x)\mathbb{P}(X = x) \\
&= \binom{x}{y} p^y (1-p)^{x-y} \frac{e^{-\lambda}\lambda^x}{x!} \\
&= \frac{p^y (1-p)^{x-y} e^{-\lambda}\lambda^x}{y!(x-y)!}
\end{aligned}$$

(b) The marginal distribution of Y is

$$\begin{aligned}
\mathbb{P}(Y = y) &= \sum_{x=0}^{\infty} \mathbb{P}(X = x, Y = y) \\
&= \sum_{x=y}^{\infty} \frac{p^y (1-p)^{x-y} e^{-\lambda}\lambda^x}{y!(x-y)!} \\
&= \frac{p^y e^{-\lambda}}{y!} \sum_{x=y}^{\infty} \frac{\lambda^x (1-p)^{x-y}}{(x-y)!} \\
&= \frac{(\lambda p)^y e^{-\lambda}}{y!} \sum_{x=y}^{\infty} \frac{[\lambda(1-p)]^{x-y}}{(x-y)!} \\
&= \frac{(\lambda p)^y e^{-\lambda}}{y!} \sum_{r=0}^{\infty} \frac{[\lambda(1-p)]^r}{r!} \qquad (r = x - y) \\
&= \frac{(\lambda p)^y e^{-\lambda}}{y!} e^{\lambda(1-p)} = \frac{(\lambda p)^y e^{-\lambda p}}{y!}
\end{aligned}$$

Thus Y has a Poisson distribution with parameter λp, i.e., $Y \sim \text{Poisson}(\lambda p)$.

(c) The conditional distribution of X, given that $Y = 5$ is

$$\mathbb{P}(X = x|Y = 5) \quad = \quad \frac{\mathbb{P}(X = x, Y = 5)}{\mathbb{P}(Y = 5)} = \frac{[\lambda(1-p)]^{x-5}e^{-\lambda(1-p)}}{(x-5)!}$$

2. The marginal density function of X is

$$f_X(x) = \int_x^\infty \lambda^2 e^{-\lambda y} dy = \lambda e^{-\lambda x}, \quad 0 \le x \le \infty,$$

so that

$$f_{Y|X}(y|x) = \frac{f_{(X,Y)}(x,y)}{f_X(x)} = \lambda e^{\lambda(x-y)}, \quad 0 \le x \le y < \infty$$

The conditional expectation of Y given $X = x$ is

$$\mathbb{E}[Y|X = x] = \int_x^\infty y f_{Y|X}(y|x) dy = x + \frac{1}{\lambda}$$

so that $\mathbb{E}[Y|X] = X + \frac{1}{\lambda}$.

3.

$$\begin{aligned}
\mathbb{P}(a < X \le b,\, c < Y \le d) &= \mathbb{P}(X \le b,\, c < Y \le d) - \mathbb{P}(X \le a,\, c < Y \le d) \\
&= \mathbb{P}(X \le b,\, Y \le d) - \mathbb{P}(X \le b,\, Y \le c) \\
&\quad - \mathbb{P}(X \le a,\, Y \le d) + \mathbb{P}(X \le a,\, Y \le c) \\
&= F(b,d) - F(b,c) - F(a,d) + F(a,c).
\end{aligned}$$

4. First we suppose that events A and B are independent. We want to show that for each $a, b \in \{0, 1\}$ we have $\mathbb{P}(\mathbf{1}_A = a,\, \mathbf{1}_B = b) = \mathbb{P}(\mathbf{1}_A = a)\mathbb{P}(\mathbf{1}_B = b)$. For the case $a = b = 1$, this reduces to showing

$$\mathbb{P}(\{\omega : \omega \in A \cap B\}) = \mathbb{P}(\{\omega : \omega \in A\})\mathbb{P}(\{\omega : \omega \in B\}),$$

which is the same as $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, which is true by assumption of independent of A and B. For the other cases of a and b the argument is similar once we show that A and B being independent implies also that A^c and B^c are independent (also A^c and B are independent and A and B^c are independent). This can be shown by simple algebra:

$$\begin{aligned}
\mathbb{P}(A^c \cap B^c) &= \mathbb{P}(A^c) - \mathbb{P}(A^c \cap B) \\
&= \mathbb{P}(A^c) - (\mathbb{P}(B) - \mathbb{P}(A \cap B)) \\
&= \mathbb{P}(A^c) - \mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(B) \\
&= \mathbb{P}(A^c)\mathbb{P}(B^c).
\end{aligned}$$

Next we suppose that random variables $\mathbf{1}_A$ and $\mathbf{1}_B$ are independent. Then in particular we have

$$\mathbb{P}(\mathbf{1}_A = 1,\, \mathbf{1}_B = 1) = \mathbb{P}(\mathbf{1}_A = 1)\mathbb{P}(\mathbf{1}_B = 1),$$

which is equivalent to $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, i.e. events A and B are independent.