MAS programmes - Statistical Inference

Examples 4

All the following examples concern a random sample $\mathbf{X} \equiv (X_1, X_2, \ldots, X_n)$ of size n from a $N(\mu, \sigma^2)$ distribution.

1. Assuming that σ^2 is known, consider testing $H_0 : \mu = \mu_0$ against $H_1 : \mu = \mu_1$, where $\mu_1 > \mu_0$.

(i) Show that any likelihood ratio test reduces to a test with critical region

$$\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^n : \bar{x} \geq k\}$$

for some constant k.

(ii) Show that the size α and power β of the above test are given by

$$\alpha = \Phi\left(\frac{(\mu_0 - k)\sqrt{n}}{\sigma}\right) \;,\; \beta = \Phi\left(\frac{(\mu_1 - k)\sqrt{n}}{\sigma}\right),$$

where Φ is the standard normal distribution function.

(iii) Deduce that, to be able to construct a test with $\alpha = 0.05$ and $\beta = 0.95$, the required sample size is approximately

$$\frac{10.82\sigma^2}{(\mu_1 - \mu_0)^2}.$$

2. Given an observed set of data, the AIC (Akaike information criterion), which is sometimes used to compare the goodness of fit of different models, is defined by

$$\text{AIC} = -2 \text{ maximized log-likelihood} + 2 \text{ number of parameters}.$$

Show that in the present case, with both μ and σ^2 unknown,

$$\text{AIC} = n \ln(s_x^2/n) + n[1 + \ln(2\pi)] + 4,$$

where s_x^2 denotes the sample corrected sum of squares.

3. Consider testing $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$. Show that if $\lambda(\mathbf{x})$ denotes the generalized likelihood ratio then $2 \ln \lambda(\mathbf{x})$ is given by

$$\frac{n(\bar{x} - \mu_0)^2}{\sigma^2},$$

in the case where σ^2 is known, and by

$$n \ln\left(1 + \frac{n(\bar{x} - \mu_0)^2}{s_x^2}\right),$$

in the case where σ^2 is unknown.

Deduce that in both cases the generalized likelihood ratio test reduces to the usual standard test for testing H_0 against H_1.