Probability and Distribution Theory: Solutions 4

1. From $Y = e^X$, we have that

$$x = \log y \quad for \quad y > 0.$$

The Jacobian determinant is

$$J(y) = \frac{dx}{dy} = \frac{1}{y}.$$

So the density function of Y is

$$f_Y(y) = f_X(x(y))|J(y)| = \frac{1}{y\sqrt{2\pi}} e^{-\frac{1}{2}(\log y)^2}, \quad y > 0.$$

2. (a) Since X_1 and X_2 are independent, the joint density of (X_1, X_2) is

$$\begin{aligned} f_{(X_1,X_2)}(x_1, x_2) &= f_{X_1}(x_1) f_{X_2}(x_2) \\ &= 4x_1 e^{-2x_1} 4x_2 e^{-2x_2} \\ &= 16 x_1 x_2 e^{-2(x_1+x_2)} \quad \text{for} \quad 0 < x_1, x_2 < \infty. \end{aligned}$$

(b) Since $Y_1 = \frac{X_1}{X_1+X_2}$ and $Y_2 = X_1 + X_2$ and $0 < X_1 < \infty$ and $0 < X_2 < \infty$, we get

$$0 < Y_1 < 1; \quad 0 < Y_2 < \infty$$

and

$$X_1 = Y_1 Y_2; \quad X_2 = Y_2(1 - Y_1).$$

The Jacobian determinant is given by

$$J(y_1, y_2) = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ -y_2 & 1 - y_1 \end{vmatrix} = y_2.$$

Therefore the joint density function of Y_1 and Y_2 is

$$\begin{aligned} f_{(Y_1,Y_2)}(y_1, y_2) &= f_{(X_1,X_2)}(x_1(y_1, y_2), x_2(y_1, y_2))|J(y_1, y_2)| \\ &= 16 y_1 y_2 (y_2(1 - y_1)) e^{-2(y_1 y_2 + y_2 - y_1 y_2)} \times y_2 \\ &= 16 y_1 (1 - y_1) y_2^3 e^{-2y_2} \quad for \quad 0 < y_1 < 1, 0 < y_2 < \infty. \end{aligned}$$

(c) The marginal density of Y_1 is

$$\begin{aligned} f_{Y_1}(y_1) &= \int_0^\infty 16y_1(1-y_1)y_2^3e^{-2y_2}dy_2 \\ &= 16y_1(1-y_1)\int_0^\infty y_2^3e^{-2y_2}dy_2 \\ &= 6y_1(1-y_1) \quad \text{for} \quad 0 < y_1 < 1. \end{aligned}$$

The computation above involves integrating by parts three times or you can use a property of gamma function($\Gamma(n) = (n-1)!$). The marginal density of Y_2 is

$$\begin{aligned} f_{Y_2}(y_2) &= \int_0^1 16y_1(1-y_1)y_2^3e^{-2y_2}dy_1 \\ &= y_2^3e^{-2y_2}\int_0^1 16y_1(1-y_1)dy_1 \\ &= y_2^3e^{-2y_2}\left[16\frac{y_1^2}{2} - 16\frac{y_1^3}{3}\right]_0^1 = \frac{8y_2^3e^{-2y_2}}{3}, \quad 0 < y_2 < \infty. \end{aligned}$$

It can be verified that

$$f_{(Y_1,Y_2)}(y_1,y_2) = f_{Y_1}(y_1)f_{Y_2}(y_2).$$

So Y_1 and Y_2 are independent.

3. (a) Noting that

$$G_X(s) = \mathbb{E}\left(s^X\right) = \sum_{x=0}^\infty s^x\mathbb{P}(X=x) = \sum_{x=0}^\infty s^x p_x,$$

we have

$$\frac{\mathrm{d}G_X(s)}{\mathrm{d}s} = \sum_{x=1}^\infty xs^{x-1}p_x \implies \frac{\mathrm{d}G_X(s)}{\mathrm{d}s}|_{s=0} = p_1 = 1!p_1.$$

$$\frac{\mathrm{d}^2G_X(s)}{\mathrm{d}s^2} = \sum_{x=2}^\infty x(x-1)s^{x-2} \implies \frac{\mathrm{d}^2G_X(s)}{\mathrm{d}s^2}|_{s=0} = 2p_2 = 2!p_2.$$

Similarly, we can get

$$\frac{\mathrm{d}^kG_X(s)}{\mathrm{d}s^k} = \sum_{x=k}^\infty x(x-1)\cdots(x-k+1)s^{x-k} \implies \frac{\mathrm{d}^kG_X(s)}{\mathrm{d}s^k}|_{s=0} = k!p_k.$$

(b)

$$\frac{\mathrm{d}G_X(s)}{\mathrm{d}s} = \sum_{x=1}^{\infty} x s^{x-1} p_x \implies \frac{\mathrm{d}G_X(s)}{\mathrm{d}s}|_{s=1} = \sum_{x=1}^{\infty} x p_x$$
$$= E(X) = E\left\{\frac{X!}{(X-1)!}\right\}.$$
$$\frac{\mathrm{d}^2G_X(s)}{\mathrm{d}s^2} = \sum_{x=2}^{\infty} x(x-1) s^{x-2} p_x \implies \frac{\mathrm{d}^2G_X(s)}{\mathrm{d}s^2}|_{s=1} = \sum_{x=2}^{\infty} x(x-1) p_x$$
$$= E\{X(X-1)\} = E\left\{\frac{X!}{(X-2)!}\right\}.$$

Similarly, we can get

$$\begin{aligned}
\frac{\mathrm{d}^kG_X(s)}{\mathrm{d}s^k} &= \sum_{x=k}^{\infty} x(x-1)(x-2)\cdots(x-k+1)s^{x-k}p_x \\
\frac{\mathrm{d}^kG_X(s)}{\mathrm{d}s^k}|_{s=1} &= \sum_{x=k}^{\infty} x(x-1)(x-2)\cdots(x-k+1)p_x \\
&= E\{X(X-1)(X-2)\cdots(X-k+1)\} \\
&= E\left\{\frac{X!}{(X-k)!}\right\}.
\end{aligned}$$

Remark: You can see from the result in (1) that if we know the PGF of X, the probability mass function of X can be found by

$$p_k = \frac{\mathrm{d}^kG_X(s)}{\mathrm{d}s^k}|_{s=0}/k!.$$

4. (a) It can be verified that

$$\frac{\mathrm{d}^kG_X(s)}{\mathrm{d}s^k} = n(n-1)(n-2),\ldots,(n-k+1)\{\theta s + (1-\theta)\}^{n-k}\theta^k,$$

so the mass function of X is

$$\begin{aligned}
p_k = \frac{\mathrm{d}^kG_X(s)}{\mathrm{d}s^k}|_{s=0}/k! &= \frac{n(n-1)(n-2)\cdots(n-k+1)(1-\theta)^{n-k}}{k!}\theta^k \\
&= \begin{pmatrix} n \\ k \end{pmatrix} (1-\theta)^{n-k}\theta^k
\end{aligned}$$

(b) The expectation and variance of X can be found by

$$\begin{aligned}\mathbb{E}(X) &= G'_X(1) = n\theta \\ Var(X) &= G''_X(1) + G'_X(1) - [G'_X(1)]^2 = n\theta(1-\theta)\end{aligned}$$

(c) X has a binomial distribution.

5. (a)

$$M_{g(X)}(t) = E\{e^{tg(X)}\} = \int_{-\infty}^{\infty} e^{tg(x)} f_X(x)\mathrm{d}x$$

(b) The MGF of $h(X)$ is $M_{h(X)}(t) = \int_{-\infty}^{\infty} e^{th(x)} f_X(x)\mathrm{d}x$ and

$$M_{g(X)}(t) = E\left\{e^{tg(X)}\right\} = E\left\{e^{tch(X)}\right\} = \int_{-\infty}^{\infty} e^{tch(x)} f_X(x)\mathrm{d}x.$$

It is easy to see that

$$M_{g(X)}(t) = M_{h(X)}(ct).$$

(c)

$$M_{g(X)}(t) = E\left\{e^{tg(X)}\right\} = E\left[e^{t\{c+h(X)\}}\right] = \int_{-\infty}^{\infty} e^{t\{c+h(x)\}} f_X(x)\mathrm{d}x = e^{ct} \int_{-\infty}^{\infty} e^{th(x)} f_X(x)\mathrm{d}x$$

It is easy to see that

$$M_{g(X)}(t) = e^{ct} M_{h(X)}(t).$$

(d) The MGF of X is found by

$$\begin{aligned}M_X(t) &= E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)dx \\ &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{(2\pi\sigma^2)}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} \mathrm{d}x = e^{\mu t + \frac{1}{2}\sigma^2 t^2}.\end{aligned}$$

Applying the results in (1) and (2), we get

$$M_Z(t) = M_{\frac{X-\mu}{\sigma}}(t) = e^{-\frac{\mu}{\sigma}t} M_X\left(\frac{t}{\sigma}\right) = e^{-\frac{\mu}{\sigma}t} e^{\frac{\mu}{\sigma}t + \frac{1}{2}\sigma^2\frac{t^2}{\sigma^2}} = e^{\frac{1}{2}t^2}.$$

We use MGF here to find $\mathbb{E}(Z)$ and $Var(Z)$.

$$\begin{aligned}M'_Z(t) = te^{\frac{1}{2}t^2} &\implies \mathbb{E}(Z) = M'_Z(0) = 0 \\ M''_Z(t) = t^2 e^{\frac{1}{2}t^2} + e^{\frac{1}{2}t^2} &\implies Var(Z) = M''_Z(0) - [M'_Z(0)]^2 = 1.\end{aligned}$$

(e) Noting that S is the sum of Z_i's and Z_i's are i.i.d., we can get

$$\begin{aligned} M_S(t) &= E(e^{tS}) = E(e^{t\sum Z_i}) \\ &= \prod_{i=1}^{n} E(e^{tZ_i}) \qquad \text{by independence} \\ &= \left\{E(e^{tZ})\right\}^n \qquad \text{because all the } Z_i\text{'s are identically distributed;} \\ &= \{M_Z(t)\}^n = \left(e^{\frac{1}{2}t^2}\right)^n = e^{\frac{1}{2}nt^2}. \end{aligned}$$