MAS programmes - Statistical Inference

Solutions 2

1. (i)

$$f(x_1, x_2, \ldots, x_n; \lambda) = \lambda^n \exp\left(-\lambda \sum_{i=1}^{n} x_i\right).$$

According to the factorization criterion, a statistic T is sufficient for λ if and only if there exist functions $g(t;\lambda)$ and $h(\mathbf{x})$ such that

$$f(\mathbf{x};\lambda) = g(T(\mathbf{x});\lambda)h(\mathbf{x}),$$

where $\mathbf{x}$ here denotes $(x_1, x_2, \ldots, x_n)$. The factorization criterion is satisfied in the present case by taking $T = \sum_{i=1}^{n} X_i$, $g(t;\lambda) = \lambda^n e^{-\lambda t}$ and $h(\mathbf{x}) = 1$.

(ii) The mean of the exponential distribution is $1/\lambda$. According to the method of moments, the mean of the distribution is equated with the sample mean to give the estimator. Thus $1/\hat{\lambda} = t/n$ and hence $\hat{\lambda} = n/t$.

The log-likelihood function is given by

$$\ln f(\mathbf{x};\lambda) = n\ln\lambda - \lambda t.$$

Hence

$$\frac{d\ln f}{d\lambda} = \frac{n}{\lambda} - t.$$

Thus the log-likelihood function is a unimodal function with its maximum at $\lambda = n/t$. According to the method of maximum likelihood, this gives us the maximum likelihood estimator.

(iii) By the law of the unconscious statistician

$$\begin{aligned} E\left[\frac{1}{T^k}\right] &= \int_0^\infty \frac{\lambda^n t^{n-k-1} e^{-\lambda t}}{(n-1)!} dt \\ &= \frac{(n-k-1)!\lambda^k}{(n-1)!} \int_0^\infty \frac{\lambda^{n-k} t^{n-k-1} e^{-\lambda t}}{(n-k-1)!} dt = \frac{(n-k-1)!\lambda^k}{(n-1)!}. \end{aligned}$$

Hence $E[1/T] = \lambda/(n-1)$ and $E[(n-1)/T] = \lambda$.

$$\text{var}\left(\frac{1}{T}\right) = E\left[\frac{1}{T^2}\right] - \left(\frac{\lambda}{n-1}\right)^2 = \frac{\lambda^2}{(n-1)(n-2)} - \frac{\lambda^2}{(n-1)^2} = \frac{\lambda^2}{(n-1)^2(n-2)}.$$

Hence

$$\text{var}\left(\frac{n-1}{T}\right) = \frac{\lambda^2}{n-2}.$$

(iv)

$$\begin{aligned} \text{MSE}(a/T) &= \text{var}(a/T) + [\text{bias}(a/T)]^2 \\ &= \frac{a^2\lambda^2}{(n-1)^2(n-2)} + \left(\frac{a}{n-1} - 1\right)^2 \lambda^2 \\ &= \left[\frac{a^2}{(n-1)(n-2)} - \frac{2a}{n-1} + 1\right]\lambda^2. \end{aligned}$$

Differentiating with respect to a, we find that the minimizing value of a is $n-2$.

2. (i)

$$\mathbb{P}(T \leq t) = \prod_{i=1}^{n} \mathbb{P}(X_i \leq t) = \left(\frac{t}{\theta}\right)^n \qquad 0 \leq t \leq \theta.$$

Hence, differentiating,

$$q(t;\theta) = \frac{nt^{n-1}}{\theta^n} \qquad (0 \leq t \leq \theta).$$

$$E[T;\theta] = \int_0^\theta \frac{nt^n}{\theta^n} dt = \frac{n\theta}{n+1}.$$

$$E[T^2;\theta)] = \int_0^\theta \frac{nt^{n+1}}{\theta^n} dt = \frac{n\theta^2}{n+2}.$$

Hence

$$\text{var}(T;\theta) = \frac{n\theta^2}{(n+1)^2(n+2)}.$$

(ii) First note that $E[X_i] = \theta/2$ and $\text{var}(X_i) = \theta^2/12$. It follows that $E[\bar{X}] = \theta/2$ and $\text{var}(\bar{X}) = \theta^2/(12n)$.

The method of moments estimator θ^* is given by equating sample and population means, i.e., by solving the equation $\bar{X} = \frac{1}{2}\theta^*$. Thus $\theta^*(\mathbf{X}) = 2\bar{X}$. Because $\bar{X}$ is not a sufficient statistic for θ, the estimator $\theta^*(\mathbf{X})$ does not utilize all the information in the sample about θ.

To find an example where θ^* gives an estimate that is incompatible with the sample data, we search for a set of data such that $\theta^* \equiv 2\bar{x} < \max x_i \leq \theta$.

Suppose that $n = 3$ and we have the sample values 0.1, 0.2, 1.2 , so that $\bar{x} = 0.5$. It follows that $\theta^* = 1.0$, but the data imply that $\theta > 1.2$.

(iii) Since

$$L(\theta;\mathbf{x}) = \begin{cases} \theta^{-n} & \theta \geq t \\ \\ 0 & \text{(otherwise)}, \end{cases}$$

it follows that the maximum likelihood estimator of θ is $\hat{\theta}(\mathbf{X}) \equiv T$ since the smallest value of θ actually maximizes $L(\theta;\mathbf{x})$ in this case. But from (i), $E[T|\theta] \neq \theta$.

(iv)

$$E[aT;\theta] = \frac{an\theta}{n+1},$$

which is equal to θ if and only if $a = (n+1)/n$. Thus

$$\breve{\theta} \equiv \frac{(n+1)T}{n}$$

is an unbiased estimator of θ.

(v)

$$\mathrm{MSE}(\theta^*;\theta) = \mathrm{var}(2\bar{X};\theta) = 4\mathrm{var}(\bar{X};\theta) = \frac{\theta^2}{3n}.$$

$$\begin{aligned}
\mathrm{MSE}(\hat{\theta};\theta) &= \mathrm{var}(\hat{\theta};\theta) + b(\theta)^2 \\
&= \mathrm{var}(T;\theta) + \left(\frac{\theta}{n+1}\right)^2 \\
&= \frac{n\theta^2}{(n+1)^2(n+2)} + \frac{\theta^2}{(n+1)^2} \\
&= \frac{2\theta^2}{(n+1)(n+2)}. \\
\mathrm{MSE}(\check{\theta};\theta) &= \left(\frac{n+1}{n}\right)^2 \mathrm{var}(T;\theta) \\
&= \frac{\theta^2}{n(n+2)}.
\end{aligned}$$

For $n > 2$ and for all $\theta > 0$,

$$\mathrm{MSE}(\check{\theta};\theta) < \mathrm{MSE}(\hat{\theta};\theta) < \mathrm{MSE}(\theta^*;\theta).$$