MAS programmes - Statistical Inference

Exercises 2

1. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a distribution with p.d.f.

$$f(x; \lambda) = \lambda e^{-\lambda x} \qquad x \geq 0,$$

where $\lambda > 0$. Let $T = \sum_{i=1}^{n} X_i$ and consider the class of estimators of λ of the form

$$\hat{\lambda} = \frac{a}{T}$$

for some constant a.

Note: in answering this question, you may make use of the fact that the distribution of T is the gamma distribution with p.d.f.

$$\frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \qquad t \geq 0.$$

(i) Using the factorization criterion, show that T is a sufficient statistic for making inferences about λ.

(ii) Show that the value $a = n$ of the constant gives the estimator obtained both by the method of moments and the method of maximum likelihood.

(iii) Show that for any integer k with $k = 1, \ldots, n-1$

$$E\left[\frac{1}{T^k}\right] = \frac{(n-k-1)!\lambda^k}{(n-1)!}.$$

Deduce that the value $a = n - 1$ gives an unbiased estimator of λ and show that the variance of this estimator is given by

$$\frac{\lambda^2}{n-2}.$$

(iv) Show that the mean square error of the estimator a/T for arbitrary a is given by

$$\left[\frac{a^2}{(n-1)(n-2)} - \frac{2a}{n-1} + 1\right]\lambda^2.$$

Hence find the value of a that minimizes the mean square error.

2. Consider a random sample $X_1, X_2, \ldots, X_n$ of size n from the Uniform$(0, \theta)$ distribution, where $\theta > 0$.

(i) Show that the p.d.f. $q(t; \theta)$ of the sufficient statistic

$$T(\mathbf{X}) \equiv \max_{1 \le i \le n} X_i$$

is given by

$$q(t; \theta) = \frac{nt^{n-1}}{\theta^n} \qquad 0 \le t \le \theta.$$

Deduce expressions for the mean and variance of T.

(ii) Show that the method of moments gives the estimator $\theta^* \equiv 2\bar{X}$ of θ. Comment on this estimator with regard to the sufficiency principle and show that it can yield estimates of θ that are incompatible with the sample data, i.e. that there are cases in which it is not possible for θ^* to attain the value θ.

(iii) Write down the maximum likelihood estimator of θ and show that it is not unbiased.

(iv) Find the constant a such that aT is an unbiased estimator of θ.

(v) Find the mean squared errors of the estimators of parts (ii), (iii) and (iv), respectively, and comment on their relative values.