Inference Assignment

Deadline: Friday, 17$^{\text{th}}$ January, 2020

Total marks: [45]. Marks are shown in boxes [ ]. There are three questions on this assignment.

1. Let $X_1, \dots, X_n$ be a random sample from the $N(\theta, 1)$ distribution.

   (a) Show that this family belongs to the exponential family of distributions and determine the canonical sufficient statistic for θ. [4]

   (b) Find the maximum likelihood estimator (MLE) of θ. [4]

   (c) Consider the function $u : \mathbb{R} \to \{0, 1\}$ defined as

   $$u(x) = \begin{cases} 1 & \text{if } x \leq c \\ 0 & \text{if } x > c. \end{cases}$$

   Show that $u(X_1)$ is unbiased for $\Phi(c - \theta)$ where $\Phi(x) = \mathbb{P}(Z \leq x)$ for $Z \sim N(0, 1)$. [2]

   (d) Show that the joint distribution of X_1 and $\sum_{i=1}^n X_i$ is bivariate normal with parameters $\mu_1 = \theta$, $\mu_2 = n\theta$, $\sigma_1^2 = 1$, $\sigma_2^2 = n$. Specify the correlation ρ. [6]
   Hint: Consider defining a random variable $Y = X_2 + \cdots + X_n$ and making the transformation $(X_1, Y) \to (X_1, X_1 + Y)$.

   (e) Hence, show that the conditional pdf of X_1 given $\sum_{i=1}^n X_i = z$ is $N(z/n, 1 - 1/n)$. [5]

   (f) Deduce that

   $$\mathbb{E}\left[u(X_1) \,\middle|\, \sum_{i=1}^n X_i = z\right] = \Phi\left(\frac{\sqrt{n}(c - z/n)}{\sqrt{n-1}}\right),$$

   where u and Φ are the functions defined in part (c). [4]
   Hint: Consider using integration by substitution.

   (g) Hence, or otherwise, (and with reference to any appropriate theorem(s)) determine the unique minimum variance unbiased estimator of $\Phi(c-\theta)$ and compare it with the MLE of $\Phi(c - \theta)$ for large n. [5]

2. Let $X_1, \dots, X_n$ be a random sample from $N(0, \sigma^2)$.

   (a) Find the most powerful test for the simple hypothesis $H_0 : \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 = \sigma_1^2$ where $\sigma_1^2 > \sigma_0^2$. [4]

   (b) Let $\alpha = P(\sum_{i=1}^n X_i^2 > c | \sigma_0^2)$ where c is an unknown parameter to be determined. Given $\sigma_0^2 = 4$ and $n = 15$, find the value of c such that the significance level $\alpha = 0.05$. [4]

(c) Given $n = 15$ and c is the value found in part (b), find the approximate value of $\beta = P(\sum_{i=1}^{n} x_i^2 < c|\sigma_1^2 = 16)$. [3]

3. The probability density function of gamma distribution with parameters $\alpha > 0$ and $\theta > 0$ is given by

$$f(x) = \frac{x^{\alpha-1}e^{-x/\theta}}{\Gamma(\alpha)\theta^{\alpha}}, x > 0.$$

Let $X_1, \ldots, X_n$ be a random sample from a gamma distribution with parameter $\alpha = 1$ and unknown θ. Let $\tau = \log(\theta)$. Suppose that the prior density function of τ is an improper uniform prior, i.e.

$$\pi(\tau) \propto 1, \tau \in (-\infty, \infty).$$

(a) Find the prior density function of θ up to a constant of proportionality. [2]

(b) Find the posterior density function of θ up to a constant of proportionality. [2]

Important Note:

• Please read the current version of the *Mathematics & Statistics Coursework Policy*. Copies can be obtained from the course website, or in hardcopy from the programme administrator.