MAS programmes - Statistical Inference

Solutions 5

1. (i)

$$f(\mathbf{x}|\theta) = (1-\theta)^n \theta^t,$$

where $t = \sum_{i=1}^n x_i$. Hence, by the factorization criterion, $T(\mathbf{X}) \equiv \sum X_i$ is sufficient for θ.

(ii)

$$\pi(\theta|\mathbf{x}) \propto \pi(\theta) f(\mathbf{x}|\theta) \propto \theta^{a+T-1}(1-\theta)^{b+n-1}.$$

Hence the posterior distribution is a beta distribution with parameters $a+T$ and $b+n$.

(iii) The mean of the posterior distribution is

$$\frac{a+T}{(a+T)+(b+n)} = \frac{a+T}{a+b+n+T}.$$

(iv)

$$\ln \pi(\theta|\mathbf{x}) = \text{const.} + (a+T-1)\ln\theta + (b+n-1)\ln(1-\theta).$$

Hence

$$\frac{\partial \ln \pi(\theta|\mathbf{x})}{\partial\theta} = \frac{a+T-1-(a+b+n+T-2)\theta}{\theta(1-\theta)}.$$

If $a+T \leq 1$, for which it must be the case that $T=0$ and $a \leq 1$, $\check{\theta}(\mathbf{X}) = 0$. Strictly speaking, there is no maximizing value of θ in the range $0 < \theta < 1$. If $a+T > 1$ then

$$\check{\theta}(\mathbf{X}) = \frac{a+T-1}{a+b+n+T-2}.$$

(v) When the prior distribution is uniform ($a=1, b=1$) then $\pi(\theta|\mathbf{x}) \propto L(\theta|\mathbf{x})$ and we obtain the MLE $T/(n+T)$. This **may** be interpreted as the case of prior ignorance about θ.

2.

$$\begin{aligned}
\pi(\mu|\mathbf{x}) &\propto \pi(\mu)L(\mu|\mathbf{x}) \\
&\propto \exp\left(-\frac{(\mu-\nu)^2}{2\tau^2}\right)\exp\left(-\frac{1}{2\sigma^2}(s_x^2 + n(\bar{x}-\mu)^2)\right) \\
&\propto \exp\left(-\frac{(\mu-\nu)^2}{2\tau^2} - \frac{n(\bar{x}-\mu)^2)}{2\sigma^2}\right) \\
&= \exp\left(-\frac{\sigma^2(\mu-\nu)^2 + n\tau^2(\bar{x}-\mu)^2)}{2\sigma^2\tau^2}\right) \\
&\propto \exp\left(-\frac{(\sigma^2+n\tau^2)\mu^2 - 2(n\tau^2\bar{x}+\sigma^2\nu)\mu}{2\sigma^2\tau^2}\right),
\end{aligned}$$

which is proportional to the density of the $N(\nu^*, \tau^{*2})$ distribution, where ν^* and τ^{*2} are the mean and variance of the posterior distribution as specified in the question.

3. (i)

$$L(\theta|\mathbf{x}) = \theta^n \exp\left(-\theta\sum_{i=1}^{n} x_i\right) = \theta^n e^{-\theta t} \ ,$$

where $t = \sum_{i=1}^{n} x_i$. Hence, by the factorization criterion, T as specified in the question is a sufficient statistic.

(ii)

$$\begin{aligned}\pi(\theta|\mathbf{x}) &\propto \pi(\theta)L(\theta|\mathbf{x}) \\ &\propto \theta^{\nu-1}e^{-\lambda\theta}\ \theta^n e^{-\theta t} \\ &\propto \theta^{\nu+n-1}e^{-(\lambda+t)\theta} \ .\end{aligned}$$

This is proportional to the p.d.f. of a gamma distribution with parameters $\nu+n$ and $\lambda+t$. Hence the posterior must actually be the gamma distribution with these parameters.

(iii) A class Π of prior distributions $\pi(\theta)$ is a *conjugate family* for a family of p.d.f.s f if the posterior distribution $\pi(\theta|\mathbf{x})$ is in the class Π for all $\pi(\theta) \in \Pi$ and all possible sets of observed values $x_1, x_2, \ldots, x_n$.

In this example, we have shown that the family of gamma distributions with parameters ν and λ is a family of conjugate priors for the family of exponential distributions.

(iv) The Bayes estimator $\hat{\theta}$ is the mean of the posterior distribution, in this case

$$\hat{\theta}(\mathbf{X}) = \frac{\nu+n}{\lambda+T} \ .$$

(v) The predictive distribution has p.d.f

$$\begin{aligned}&\int_0^\infty f(x|\theta)\pi(\theta|x_1, x_2, \ldots, x_n)d\theta \\ =\ &\int_0^\infty \theta e^{-\theta x}\ \frac{(\lambda+t)^{\nu+n}\theta^{\nu+n-1}e^{-(\lambda+t)\theta}}{\Gamma(\nu+n)}d\theta \\ =\ &\frac{(\lambda+t)^{\nu+n}}{(\lambda+t+x)^{\nu+n+1}}\,\frac{\Gamma(\nu+n+1)}{\Gamma(\nu+n)}\int_0^\infty \frac{(\lambda+t+x)^{\nu+n+1}\theta^{\nu+n}e^{-(\lambda+t+x)\theta}}{\Gamma(\nu+n+1)}d\theta \\ =\ &\frac{(\nu+n)(\lambda+t)^{\nu+n}}{(\lambda+t+x)^{\nu+n+1}} \qquad x \geq 0\end{aligned}$$

since the integrand is the p.d.f. of the Gamma distribution with parameters $\nu+n+1$ and $\lambda+t+x$.

Hence

$$\begin{aligned}\mathbb{P}(X_{n+1} > x) &= \int_x^\infty \frac{(\nu+n)(\lambda+t)^{\nu+n}}{(\lambda+t+u)^{\nu+n+1}}du \\ &= \left[\frac{-(\lambda+t)^{\nu+n}}{(\lambda+t+u)^{\nu+n}}\right]_x^\infty \\ &= \left(\frac{\lambda+t}{\lambda+t+x}\right)^{\nu+n} \qquad x \geq 0.\end{aligned}$$