Probability and Distribution Theory: Solutions 2

1. Hint: check if (1) $f_X(x) \geqslant 0$ and (2) $\int_{-\infty}^{\infty} f_X(x)dx = 1$.

   (a) It is obvious that $f_X(x) \geqslant 0$ when $c \geq 0$. Firstly, we notice when $d \leq 1$, $\int_1^{\infty} cx^{-d}dx$ doesn't converge, that is, the integral doesn't exist. When $d > 1$, we have

   $$\int_{-\infty}^{\infty} f_X(x)dx = \int_1^{\infty} cx^{-d}dx = \frac{c}{1-d}x^{1-d}\Big|_1^{\infty} = \frac{-c}{1-d}.$$

   The integration must be 1, so we need the restriction $d > 1$ and $c = d - 1$.

   (b) It is easy to see that $f_X(x) \geq 0$ when $c \geq 0$.

   $$\begin{aligned}\int_{-\infty}^{\infty} f_X(x)dx = \int_{-\infty}^{\infty} ce^x(1+e^x)^{-2}dx &= \int_{-\infty}^{\infty} c(1+e^x)^{-2}d(1+e^x) \\ &= \frac{-c}{1+e^x}\Big|_{-\infty}^{\infty} = c.\end{aligned}$$

   The integration must be 1, so we need $c = 1$.

2.

$$\begin{aligned}\int_0^{\infty}[1-F(x)]dx &= \int_0^{\infty}\mathbb{P}(X > x)dx \\ &= \int_0^{\infty}\left(\int_x^{\infty} f(y)dy\right)dx \\ &= \int_0^{\infty}\left(\int_0^{y} dx\right)f(y)dy \quad (\textit{change the order of integration}) \\ &= \int_0^{\infty} yf(y)dy = \mathbb{E}[X].\end{aligned}$$

3. Let X_1 be the number of passengers turning up for their *Symphony Airways* seats and X_2 the number of passengers turning up their *Harmony Airways* seats. Then $X_1 \sim \text{Bin}(10, 0.9)$ and $X_2 \sim \text{Bin}(20, 0.9)$.

   That *Symphony Airways* is overbooked means $X_1 = 10$ and that *Harmony Airways* is overbooked means $X_2 = 20$. Since

   $$\mathbb{P}(X_1 = 10) = 0.9^{10} \quad > \quad \mathbb{P}(X_2 = 20) = 0.9^{20},$$

   Symphony Airways is more often overbooked.