10 Nested Models

10.1 Introduction

Example 10.1: Quality Control of raw materials (*Montgomery (1997)*)

A company buys raw material in batches from 3 different suppliers. It is required to determine whether the variation in the purity of the raw material is attributable to the use of different suppliers, and/or the variability between batches.

Carry out an appropriate analysis and advise the company director on the course of action to take in order to reduce the variation in purity.

Investigation:
Four batches of material are randomly selected from each supplier, and 3 determinations of purity are carried out on each batch. The data are presented below:

	Supplier 1				Supplier 2				Supplier 3			
Batches	1	2	3	4	1	2	3	4	1	2	3	4
	1	-2	-2	1	1	0	-1	0	2	-2	1	3
	-1	-3	0	4	-2	4	0	3	4	0	-1	2
	0	-4	1	0	-3	2	-2	2	0	2	2	1
Batch Totals $y_{ij.}$	0	-9	-1	5	-4	6	-3	5	6	0	2	6
Supplier Totals $y_{i..}$	-5				4				14			

The readings, the $\{y_{ijk}\}$, are given by the actual purity-93.[1]

This is an example of a 2-stage nested design.

10.2 Two Stage Nested Designs

10.2.1 Design and Linear Statistical Model

The two-stage *Nested Design* (or hierarchical design) is characterized by the following:
• The responses $\{y_{ijk}\}$ can be classified according to the levels of factor A and factor B.
• The levels of factor B, say, at a particular level of factor A are not identical to the corresponding levels of factor B at another level of factor A, although they are similar in nature.

The linear statistical model for the *two stage nested design* can be written as:

$$y_{ijk} = \mu + \tau_i + \beta_{(i)j} + \epsilon_{(ij)k} \qquad i=1,\ldots,a; j=1,\ldots,b; k=1,\ldots,n \tag{10.1}$$

where y_{ijk} represents the reading or observation taken on the k-th replicate or observation taken from the j-th level of factor B nested within the i-th level of factor A;

[1] It can be shown that the sums of squares and resulting F-statistics in an ANOVA model are the same when a constant is subtracted from each value of the original data set.

μ is the overall mean[2],
the τ_i are the effects for factor A,
the $\beta_{(i)j}$ are the effects for factor B (nested within each of the levels of factor A), and
the $\epsilon_{(ij)k} \sim \text{NID}(0, \sigma^2)$ are the error terms.

10.3 Analysis of Variance

The total variation in the data is given by

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} (y_{ijk} - \overline{y}_{...})^2$$

It can be shown that this quantity can be decomposed to yield the identity

$$SS_T = SS_A + SS_{B(A)} + SS_R$$

where

$$SS_A = bn \sum_{i=1}^{a} (\overline{y}_{i..} - \overline{y}_{...})^2$$

$$SS_{B(A)} = n \sum_{i=1}^{a} \sum_{j=1}^{b} (\overline{y}_{ij.} - \overline{y}_{i..})^2$$

$$SS_R = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} (y_{ijk} - \overline{y}_{ij.})^2.$$

SS_A is the *sum-of-squares* due to factor A,
$SS_{B(A)}$ is the *sum-of-squares* due to factor B nested within A, and
SS_R is the *residual sum-of-squares*.

For easy computation, these quantities can be expressed in terms of the (sub-) totals of the actual observations $\{y_{ijk}\}$, so that

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} y_{ijk}^2 - C_f$$

$$SS_A = \frac{1}{bn} \sum_{i=1}^{a} y_{i..}^2 - C_f$$

$$SS_{B(A)} = \left\{ \frac{1}{n} \sum_{i=1}^{a} \sum_{j=1}^{b} y_{ij.}^2 - C_f \right\} - SS_A$$

$$SS_R = SS_T - SS_A - SS_{B(A)}$$

where $C_f = \dfrac{y_{...}^2}{abn}$.

We can summarize this information in the form of an *Analysis of Variance* table:

[2] when using the sum-to-zero constraints.

ANOVA TABLE

Source	SS	DF	MS
A	SS_A	$a-1$	$MS_A = SS_A/(a-1)$
B(A)	$SS_{B(A)}$	$a(b-1)$	$MS_{B(A)} = SS_{B(A)}/a(b-1)$
Residual	SS_R	$ab(n-1)$	$MS_R = SS_R/ab(n-1)$
Total	SS_T	$abn-1$	

The way in which we test for the significance of a particular effect will depend on whether (i) both A and B are fixed, (ii) A is fixed, but B is random, (iii) both A and B are random.[3]

We deal with each of these three cases separately by considering the form of the expectations of the *mean sum-of-squares*.

Case (i): both A and B fixed

We assume that $\sum_{i=1}^{a} \tau_i = 0$ and $\sum_{j=1}^{b} \beta_{(i)j} = 0$ for $i=1,\ldots,a$.
It can be shown that

$$\mathrm{E}[MS_A] = \sigma^2 + \frac{bn\sum_{i=1}^{a}\tau_i^2}{a-1} \tag{10.2}$$

$$\mathrm{E}[MS_{B(A)}] = \sigma^2 + \frac{n\sum_{i=1}^{a}\sum_{j=1}^{b}\beta_{(i)j}^2}{a(b-1)} \tag{10.3}$$

$$\mathrm{E}[MS_R] = \sigma^2. \tag{10.4}$$

The meaningful questions to be asked here are whether there are differences in the observations due to the various levels of factor A, and whether there are differences due to the levels of factor B (nested within those of factor A).

More formally, we seek tests for

H_{0A}: $\tau_1 = \tau_2 = \ldots\ldots \tau_a = 0$
vs.
H_{1A}: $\tau_i \neq 0$ for some i, $i=1,\ldots,a$,

and

H_{0B}: $\beta_{(i)j} = 0$, $i=1,\ldots,a$, $j=1,\ldots,b$
vs.
H_{1B}: $\beta_{(i)j} \neq 0$ for some i,j, $i=1,\ldots,a$, $j=1,\ldots,b$.

Now consider the statistic $F_A = MS_A/MS_R$, which has been constructed from the *mean squares* from lines 1 and 3 of the ANOVA table.

[3]The situation where A is random and B is fixed is not to be considered here.

Both MS_A under H_{0A}, and MS_R, are independent unbiased estimates of σ^2; however, under H_{1A}, $\mathrm{E}[MS_A] > \mathrm{E}[MS_R]$, as can be seen from (10.2) and (10.4).

Furthermore, under H_{0A},

$$F_A \sim F_{a-1,ab(n-1)}.$$

This suggests that when H_{0A} is true, then F_A is 'close' to 1, whereas when H_{0A} is false, then F_A is somewhat large. So, we reject H_{0A} (in favour of the alternative) at the $100\alpha\%$ level of significance if

$$F_A > F_{a-1,ab(n-1),\alpha}$$

and accept H_{0A} otherwise.

Next consider $F_B = MS_{B(A)}/MS_R$ constructed from lines 2 and 3 of the ANOVA.

$MS_{B(A)}$ under H_{0B}, and MS_R, provide independent unbiased estimates of σ^2; however under H_{1B}, $\mathrm{E}[MS_{B(A)}] > \mathrm{E}[MS_R]$.

Also, under H_{0B},

$$F_B \sim F_{a(b-1),ab(n-1)}.$$

Thus, we reject H_{0B} at the $100\alpha\%$ level of significance if

$$F_B > F_{a(b-1),ab(n-1),\alpha}$$

and accept H_{0B} otherwise.

Case (ii): A fixed and B random

Here we suppose that $\sum_{i=1}^{a} \tau_i = 0$ and $\beta_{(i)j} \sim \mathrm{NID}(0, \sigma_B^2)$, $i=1,\ldots,a$, $j=1,\ldots,b$ (which are independent of the $\epsilon_{(ij)k}$).

The meaningful hypotheses to test are:

H_{0A}: $\tau_1 = \tau_2 = \ldots = \tau_a = 0$
vs.
H_{1A}: $\tau_i \neq 0$ for some i, $i = 1, \ldots, a$,

and

H_{0B}: $\sigma_B^2 = 0$
vs.
H_{1B}: $\sigma_B^2 > 0$.

It can be shown that

$$\mathrm{E}[MS_A] = \sigma^2 + n\sigma_B^2 + \frac{bn}{a-1}\sum_{i=1}^{a} \tau_i^2 \tag{10.5}$$

$$\mathrm{E}[MS_{B(A)}] = \sigma^2 + n\sigma_B^2 \tag{10.6}$$

$$\mathrm{E}[MS_R] = \sigma^2. \tag{10.7}$$

To test H_{0A}, consider the statistic $F_A = MS_A/MS_{B(A)}$, corresponding to lines 1 and 2 of the ANOVA table.

MS_A under H_{0A}, and $MS_{B(A)}$, are independent unbiased estimates of $\sigma^2 + n\sigma_B^2$; otherwise

$$E[MS_A] > E[MS_{B(A)}].$$

Also, under H_{0A},

$$F_A \sim F_{a-1,a(b-1)}.$$

So we should reject H_{0A} at the $100\alpha\%$ level of significance if

$$F_A > F_{a-1,a(b-1),\alpha}$$

and accept otherwise.

To test H_{0B}, we note that $MS_{B(A)}$ under H_{0B}, and MS_R, are independent unbiased estimates of σ^2, and, under H_{0B},

$$F_B \sim F_{a(b-1),ab(n-1)}$$

where $F_B = MS_{B(A)}/MS_R$, constructed from lines 2 and 3 of the ANOVA table.

Otherwise, under H_{1B}, $E[MS_{B(A)}] > E[MS_R]$.

So, we should reject H_{0B} at the $100\alpha\%$ level if

$$F_B > F_{a(b-1),ab(n-1),\alpha}$$

and accept otherwise.

Case (iii): both A and B random

Here it is assumed that $\tau_i \sim \mathrm{NID}(0, \sigma_A^2)$, $i = 1, \ldots, a$, and $\beta_{(i)j} \sim \mathrm{NID}(0, \sigma_B^2)$, $i = 1, \ldots, a$, $j = 1, \ldots, b$; also the $\{\tau_i\}$, $\{\beta_{(i)j}\}$, and $\{\epsilon_{(ij)k}\}$ are independent of each other.

Meaningful hypotheses to test for are:

H_{0A}: $\sigma_A^2 = 0$ vs. H_{1A}: $\sigma_A^2 > 0$

H_{0B}: $\sigma_B^2 = 0$ vs. H_{1B}: $\sigma_B^2 > 0$.

It can be shown that

$$\mathrm{E}[MS_A] = \sigma^2 + n\sigma_B^2 + bn\sigma_A^2 \tag{10.8}$$
$$\mathrm{E}[MS_{B(A)}] = \sigma^2 + n\sigma_B^2 \tag{10.9}$$
$$\mathrm{E}[MS_R] = \sigma^2 \tag{10.10}$$

In order to test the first set of hypotheses, consider $F_A = MS_A/MS_{B(A)}$, which is constructed from (10.8) and (10.9), and thus lines 1 and 2 of the ANOVA table.

Both MS_A under H_{0A}, and $MS_{B(A)}$, provide independent unbiased estimates of $\sigma^2 + n\sigma_B^2$; otherwise $\mathrm{E}[MS_A] > \mathrm{E}[MS_{B(A)}]$. Also, under H_{0A},

$$F_A \sim F_{a-1,a(b-1)}.$$

So we reject H_{0A} at the $100\alpha\%$ level of significance if

$$F_A > F_{a-1,a(b-1),\alpha}.$$

To test the second set of hypotheses, consider $F_B = MS_{B(A)}/MS_R$, constructed from (10.9) and (10.10), corresponding to lines 2 and 3 of the ANOVA table. $MS_{B(A)}$ under H_{0B}, and MS_R, provide unbiased estimates of σ^2; otherwise $\mathrm{E}[MS_{B(A)}] > E[MS_R]$.

Also, under H_{0B},

$$F_B \sim F_{a(b-1),ab(n-1)}$$

Thus, we reject H_{0B} at the $100\alpha\%$ level of significance if

$$F_B > F_{a(b-1),ab(n-1),\alpha}.$$

10.4 Point Estimates

Estimates for the fixed effects, and for the variances of the random effects are presented. Cases (i)-(iii) are considered in turn.

Case (i): both A and B fixed

Using the usual least squares procedures, it can be shown that the estimated values of the effects are given by

$$\hat{\tau}_i = \overline{y}_{i..} - \overline{y}_{...} \qquad i=1,\ldots,a$$
$$\hat{\beta}_{(i)j} = \overline{y}_{ij.} - \overline{y}_{i..} \qquad i=1,\ldots,a,\ j=1,\ldots,b$$

Case (ii): A fixed and B random

A least squares procedure can be used to show that

$$\hat{\tau}_i = \overline{y}_{i..} - \overline{y}_{...} \qquad i=1,\ldots,a.$$

Using (10.6) and (10.7), it can be shown that

$$\hat{\sigma}_B^2 = \frac{1}{n}\left\{MS_{B(A)} - MS_R\right\}$$

is an unbiased estimator of σ_B^2.

Case (iii): both A and B random

Using (10.8) and (10.9), it can be shown that

$$\hat{\sigma}_A^2 = \frac{1}{bn}\left\{MS_A - MS_{B(A)}\right\}$$

is unbiased for σ_A^2.

Similarly, using (10.9) and (10.10), it follows that

$$\hat{\sigma}_B^2 = \frac{1}{n}\left\{MS_{B(A)} - MS_R\right\}$$

is unbiased for σ_B^2.

10.5 Some Examples

Example 10.1 ctd...

Here, factor A can be thought of as being the 'supplier' factor, and factor B as the 'batch' factor (nested within levels of factor A). We treat A as fixed, and B as random.

We fit the following model to the data:

$$y_{ijk} = \mu + \tau_i + \beta_{(i)j} + \epsilon_{(ij)k}$$

$i=1,\ldots,3,\ j=1,\ldots,4,\ k=1,\ldots,3,$

where $\sum_{i=1}^{a}\tau_i=0$, and $\beta_{(i)j} \sim \text{NID}(0,\sigma_B^2)$ and $\epsilon_{(ij)k} \sim \text{NID}(0,\sigma^2)$ independently.

The standard formulae can be used to show that

$$\begin{aligned} SS_T &= 148.31 \\ SS_A &= 15.06 \\ SS_{B(A)} &= 69.92 \\ SS_R &= 63.33 \end{aligned}$$

Thus, the ANOVA table takes the following form:

ANOVA Table

Source	Sums of Squares	Degrees of Freedom	Mean Square
Suppliers	15.06	2	7.53
Batches, within Suppliers	69.92	9	7.77
Residual	63.33	24	2.64
Total	148.31	35	

Now,

$$F_{Suppliers} = \frac{7.53}{7.77} = 0.97$$

However, $F_{2,9,0.05} = 4.256$

Also,

$$F_{Batches} = \frac{7.77}{2.64} = 2.94$$

But $F_{9,24,0.05} = 2.300$ and $F_{9,24,0.01} = 3.256$

Conclusion: No evidence that the variation in purity is due to the use of different suppliers. However, there is evidence (at the 5%, but not the 1%, level) that batches coming from within each supplier are reasonably variable. So the MD is advised to encourage the suppliers to tighten up their quality control.

Furthermore,

$$\hat{\sigma}^2_{batches} = \frac{MS_{B(A)} - MS_R}{n} = \frac{7.77 - 2.64}{3} = 1.71$$
$$\hat{\sigma}^2 = MS_R = 2.64$$

Also

$$\overline{y}_{1..} = -5/12 = -0.4166667, \quad \overline{y}_{2..} = 4/12 = 0.3333333, \quad \overline{y}_{3..} = 14/12 = 1.166667$$

and $\overline{y}_{...} = 13/36 = 0.3611111$. Thus

$$\hat{\tau}_1 = -0.4166667 - 0.3611111 = -0.7777778, \quad \hat{\tau}_2 = -0.0277778, \quad \hat{\tau}_3 = 0.8055559$$

```
# F-quantiles
> qf(0.95, 2, 9)
[1] 4.256495
> qf(0.95, 9, 24)
[1] 2.300244
> qf(0.99, 9, 24)
[1] 3.255985
```

Nested Models in R

These results can be easily obtained using R. Here, the data frame used (`purity`) consists of 3 columns - labelled `code`, `supplier`, and `batch`; the latter two are declared as type **factor**.

```
> code <- c(1, -1, 0, -2, -3, -4, -2, 0, 1, 1, 4, 0,
            1, -2, -3, 0, 4, 2, -1, 0, -2, 0, 3, 2,
            2, 4, 0, -2, 0, 2, 1, -1, 2, 3, 2, 1)
> s <- rep(1:3, rep(12, 3))
> b <- rep(rep(1:4, rep(3, 4)), 3)
> supplier <- factor(s)
> batch <- factor(b)
> purity <- data.frame(code, supplier, batch)
purity.aov <- aov(code ~ supplier/batch, data = purity)
summary(purity.aov)
               Df Sum Sq Mean Sq F value Pr(>F)
supplier        2  15.06   7.528   2.853 0.0774 .
supplier:batch  9  69.92   7.769   2.944 0.0167 *
Residuals      24  63.33   2.639
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

Note the following:

- The notation used in R for a factor B nested within A is `A/B`.

- The tests here are for fixed effects. Thus the test for batches within suppliers is correct (significant differences between batches) but that for suppliers is incorrect. Instead, comparing with the `supplier:batch Mean Sq` we obtain F = 0.97 on (2, 9) d.f., which is not significant.

- The correct F-statistic for suppliers is much smaller than that reported in the R output; the latter suggested that the supplier effect was almost significant. So, if we incorrectly regarded batches as fixed, we might draw the erroneous conclusion that the variation in purity is partly due to differences between the suppliers. Actually, the correct analysis shows that it is due to the inconsistency of batches within suppliers.

```
> model.tables(purity.aov)

Tables of effects

 supplier
supplier
      1       2       3
-0.7778 -0.0278  0.8056

 supplier:batch
        batch
supplier 1       2       3       4
       1  0.4167 -2.5833  0.0833  2.0833
       2 -1.6667  1.6667 -1.3333  1.3333
       3  0.8333 -1.1667 -0.5000  0.8333

> model.tables(purity.aov, type = "means")

Tables of means
Grand mean

0.3611111

 supplier
supplier
      1       2       3
-0.4167  0.3333  1.1667

 supplier:batch
        batch
supplier 1       2       3       4
       1  0.0000 -3.0000 -0.3333  1.6667
       2 -1.3333  2.0000 -1.0000  1.6667
       3  2.0000  0.0000  0.6667  2.0000
```

Example 10.2. Effect of daylight on growth of plants (*Adapted from Steel & Torrie (1980)*).

An experiment is performed to measure the effect that exposure to daylight has on the growth of plants.

There are 6 different conditions corresponding to 3 different exposures to sunlight (8, 12 and 16 hours) at two different glasshouse conditions (low and high night temperatures).

• Pots with soil (which have been sampled from a large population) are randomly assigned to each condition, with 3 in each one.
• 4 plants are taken at random at a time (from a large population), and placed in each of the $6 \times 3 = 18$ pots.

The growth in c.m. of the stems were measured after one week. The data are presented below.

Pot	Plant	Cond. 1	Cond. 2	Cond. 3	Cond. 4	Cond. 5	Cond. 6
1	1	3.5	5.0	5.0	8.5	6.0	7.0
	2	4.0	5.5	4.5	6.0	5.5	9.0
	3	3.0	4.0	5.0	9.0	3.5	8.5
	4	4.5	3.5	4.5	8.5	7.0	8.5
2	1	2.5	3.5	5.5	6.5	6.0	6.0
	2	4.5	3.5	6.0	7.0	8.5	7.0
	3	5.5	3.0	5.0	8.0	4.5	7.0
	4	5.0	4.0	5.0	6.5	7.5	7.0
3	1	3.0	4.5	5.5	7.0	6.5	11.0
	2	3.0	4.0	4.5	7.0	6.5	7.0
	3	2.5	4.0	6.5	7.0	8.5	9.0
	4	3.0	5.0	5.5	7.0	7.5	8.0

Let y_{ijk} be the week's growth for plant k in pot j in growth condition i,
$i = 1, \dots, 6$, $j = 1, \dots, 3$ and $k = 1, \dots, 4$.

The appropriate model to fit to the data is;

$$y_{ijk} = \mu + \tau_i + \beta_{(i)j} + \epsilon_{(ij)k} \tag{10.11}$$

with $\sum_{i=1}^{6} \tau_i = 0$, and $\beta_{(i)j} \sim \text{NID}(0, \sigma_B^2)$ and $\epsilon_{(ij)k} \sim \text{NID}(0, \sigma^2)$ independently.

The ANOVA table takes the following form:

ANOVA Table

Source	Sums of Squares	Degrees of Freedom	Mean Square
Between Conditions	179.64	5	35.93
Between pots, within conditions	25.83	12	2.15
Residual	50.44	54	0.93
Total	255.91	71	

Now,

$$F_{Conds} = \frac{35.93}{2.15} = 16.7$$

But $F_{5,12,0.01} = 5.064$. Thus there is strong evidence that there are differences in the effects of the 6 conditions.

Also,

$$F_{Pots} = \frac{2.15}{0.93} = 2.3$$

But

$$F_{12,54,0.05} = 1.93631, \quad F_{12,54,0.01} = 2.53282$$

So there is evidence of variability between pots at the 5% (and almost at the 1%) level of significance.

This is confirmed in the corresponding analysis in R:

```
> growth <- c(35, 40, 30, 45, 25, 45, 55, 50, 30, 30, 25, 30,
              50, 55, 40, 35, 35, 35, 30, 40, 45, 40, 40, 50,
              50, 45, 50, 45, 55, 60, 50, 50, 55, 45, 65, 55,
              85, 60, 90, 85, 65, 70, 80, 65, 70, 70, 70, 70,
              60, 55, 35, 70, 60, 85, 45, 75, 65, 65, 85, 75,
              70, 90, 85, 85, 60, 70, 70, 70, 110, 70, 90, 80)/10
> c <- rep(1:6, rep(12, 6))
> p <- rep(rep(1:3, rep(4, 3)), 6)
> condition <- factor(c)
> pot <- factor(p)
> daylight <- data.frame(growth, condition, pot)

> daylight.raov <- aov(growth ~ condition/pot, data = daylight)
> summary(daylight.raov)
              Df Sum Sq Mean Sq F value Pr(>F)
condition      5 179.64   35.93  38.466 <2e-16 ***
condition:pot 12  25.83    2.15   2.305 0.0186 *
Residuals     54  50.44    0.93
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

> p.value.condition <- pf(35.92847/2.15278, 5, 12, lower.tail = F)
> p.value.condition
[1] 4.881089e-05
```

10.6 Generalization to the m-stage nested design

The formulae and results of the previous sections can be extended (in an obvious way) to deal with m completely nested factors. To motivate how this may done intuitively, we present here the linear statistical model and the ANOVA table for the 3-stage nested design.

The responses, $\{y_{ijkl}\}$, satisfy

$$y_{ijkl} = \mu + \tau_i + \beta_{(i)j} + \gamma_{(ij)k} + \epsilon_{(ijk)l}$$

$i=1,\ldots,a,\ \ j=1,\ldots,b,\ \ k=1,\ldots,c,\ \ l=1,\ldots,n.$

ANOVA TABLE

Source	SS	DF	MS
A	$SS_A = bcn\sum_{i=1}^a(\overline{y}_{i...} - \overline{y}_{....})^2$	$a-1$	$MS_A = \frac{SS_A}{a-1}$
B (within A)	$SS_{B(A)} = cn\sum_{i=1}^a\sum_{j=1}^b(\overline{y}_{ij..} - \overline{y}_{i...})^2$	$a(b-1)$	$MS_{B(A)} = \frac{SS_{B(A)}}{a(b-1)}$
C (within B)	$SS_{C(B)} = n\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c(\overline{y}_{ijk.} - \overline{y}_{ij..})^2$	$ab(c-1)$	$MS_{C(B)} = \frac{SS_{C(B)}}{ab(c-1)}$
Residual	$SS_R = \sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\sum_{l=1}^n(y_{ijkl} - \overline{y}_{ijk.})^2$	$abc(n-1)$	$MS_R = \frac{SS_R}{abc(n-1)}$
Total	$SS_T = \sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\sum_{l=1}^n(y_{ijkl} - \overline{y}_{....})^2$	$abcn-1$	

Depending on which factors are fixed, and which are random, one can construct appropriate test statistics for the situation at hand: this requires careful consideration of the form of the *expected mean squares*.

So, for example, suppose that A and B are both fixed, but C is random. Then

$$\begin{aligned}
E[MS_A] &= \sigma^2 + n\sigma_C^2 + \frac{bcn\sum_{i=1}^a \tau_i^2}{a-1} \\
E[MS_{B(A)}] &= \sigma^2 + n\sigma_C^2 + \frac{cn\sum_{i=1}^a\sum_{j=1}^b \beta_{(i)j}^2}{a(b-1)} \\
E[MS_{C(B)}] &= \sigma^2 + n\sigma_C^2 \\
E[MS_R] &= \sigma^2.
\end{aligned}$$

Thus to test
$H_{0A} : \tau_1 = \tau_2 = \ldots = \tau_a = 0$
vs.
$H_{1A} : \tau_i \neq 0$ for some i

we use $F_A = MS_A/MS_{C(B)}$, which is derived from lines 1 and 3 of the table.

To test
$H_{0B} : \beta_{(i)j} = 0$ for all i, j
vs.
$H_{1B} : \beta_{(i)j} \neq 0$ for some i, j

we use $F_B = MS_{B(A)}/MS_{C(B)}$, which is derived from lines 2 and 3 of the table.

On the other hand, to test
$H_{0C} : \sigma_C^2 = 0$
vs.
$H_{1C} : \sigma_C^2 > 0$,

we use $F_C = MS_{C(B)}/MS_R$, derived from lines 3 and 4 of the table.