9 Blocking in Designs

9.1 Introduction

So far the *units* in the designs we have considered have had no structure. That is, they have been assumed to be from a single homogeneous population. The randomization was over the design as a whole, without taking account of any groupings of the units. Often however, the population of units is not homogeneous. For example,

• an agricultural experiment might involve several different fields, or parts of a field, all with different underlying levels of fertility;

• an industrial experiment may need to be conducted on several different days with different batches of material;

• structure may also be imposed artificially, by trying to form sets of similar units (and perhaps also subsets), with the aim of decreasing the variability of the experiment.

These structures (or *blocks*) should then be reflected in the way that the randomization is carried out and the treatments are applied. In a sense the 'blocking' factors may be thought of as *nuisance* factors which are in some sense *controllable*.

The simplest example of this is the *randomized complete block design*.

9.1.1 The randomized complete block design

Model and Design

	Block 1	**Block 2**			**Block b**
Treatment 1	y_{11}	y_{12}			y_{1b}
Treatment 2	y_{21}	y_{22}			y_{2b}
.	.	.	...	...	.
.	.	.	...	...	.
.	.	.	...	...	.
Treatment a	y_{a1}	y_{a2}			y_{ab}

The *randomized complete block design* is characterized by:

• a treatments and b blocks
• one observation per treatment in each block [1]
• the *order* in which treatments are run in each block is determined randomly[2].

In order to test for differences between the treatments under this design, we define the following model:

$$y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij} \qquad i=1,\ldots,a;\ j=1,\ldots,b \tag{9.1}$$

[1]However the design can be extended to incorporate replications.
[2]We say that the blocks represent a restriction on the randomization.

where
μ is the *overall mean*,
τ_i is the i-th *treatment effect*,
β_j is the j-th *block effect*, and
$\epsilon_{ij} \sim \text{NID}(0, \sigma^2)$ is a random error term.

Treatments and blocks are here considered to be *fixed* factors. Note also that there is assumed to be no interaction between the blocks and treatments. Consistent with the notion that μ represents the overall mean, we impose the (intuitively reasonable '*sum to zero*') constraints:

$$\textstyle\sum_{i=1}^{a} \tau_i = 0 \tag{9.2}$$
$$\textstyle\sum_{j=1}^{b} \beta_j = 0. \tag{9.3}$$

Note that we effectively have the same model as the '*no interaction model*' of Section 8.8, without replication (although we interpret these models differently). The parameter estimates are given by

$$\hat{\mu} = \frac{y_{..}}{ab} = \overline{y}_{..} \tag{9.4}$$
$$\hat{\tau}_i = \frac{y_{i.}}{b} - \hat{\mu} = \overline{y}_{i.} - \overline{y}_{..} \qquad i=1,\ldots,a \tag{9.5}$$
$$\hat{\beta}_j = \frac{y_{.j}}{a} - \hat{\mu} = \overline{y}_{.j} - \overline{y}_{..} \qquad j=1,\ldots,b. \tag{9.6}$$

(c/f equations (8.18)-(8.20)), and the fitted values are

$$\begin{aligned} \hat{y}_{ij} &= \hat{\mu} + \hat{\tau}_i + \hat{\beta}_j \\ &= \overline{y}_{..} + (\overline{y}_{i.} - \overline{y}_{..}) + (\overline{y}_{.j} - \overline{y}_{..}) \\ &= \overline{y}_{i.} + \overline{y}_{.j} - \overline{y}_{..} \qquad\qquad i=1,\ldots,a,\ j=1,\ldots,b \end{aligned}$$

Analysis of Variance

To determine whether or not there are 'differences' between the treatments, we carry out a formal hypothesis test based upon *Analysis of Variance.*

Consider the following pair of hypotheses

$$\begin{aligned} &H_0 : \mu_1 = \mu_2 = \ldots = \mu_a \\ &H_1 : \text{at least one } \mu_i \neq \mu_j,\ i \neq j \end{aligned}$$

We can re-write these hypotheses as:

$$\begin{aligned} &H_0 : \tau_1 = \tau_2 = \ldots = \tau_a = 0 \\ &H_1 : \tau_i \neq 0, \text{for at least one } i \end{aligned}$$

and undertake the test of no differences between the treatments by reference to the following ANOVA Table:

ANOVA TABLE

Source	DF	SS	MS	F
Treatments	$a-1$	$b\sum_{i=1}^{a}(\bar{y}_{i.}-\bar{y}_{..})^2$	$SS_{Trts}/(a-1)$	MS_{Trts}/MS_R
Blocks	$b-1$	$a\sum_{j=1}^{b}(\bar{y}_{.j}-\bar{y}_{..})^2$	$SS_{Blocks}/(b-1)$	MS_{Blocks}/MS_R
Residual	$(a-1)(b-1)$	By subtraction	$SS_R/(a-1)(b-1)$	
Total	$ab-1$	$\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{..})^2$		

Under H_0, the statistic $F_0 = \dfrac{MS_{Trts}}{MS_R}$ has an $F_{a-1,(a-1)(b-1)}$ distribution, i.e. we reject H_0 if $F_0 > F_{a-1,(a-1)(b-1);\alpha}$.

Remarks

• The expected mean squares are given by

$$\begin{aligned}
\mathrm{E}[MS_{Trts}] &= \sigma^2 + \frac{b\sum_{i=1}^{a}\tau_i^2}{a-1}\\
\mathrm{E}[MS_{Blocks}] &= \sigma^2 + \frac{a\sum_{j=1}^{b}\beta_j^2}{b-1}\\
\mathrm{E}[MS_R] &= \sigma^2.
\end{aligned}$$

(again, c/f two-way ANOVA). Under H_0, both MS_{Trts} and MS_R are *unbiased* estimators of σ^2, but under H_1, $\mathrm{E}[MS_{Trts}] > E[MS_R]$. We expect, therefore, that realizations of F_0 take values that are 'close' to 1 under H_0, but values that are somewhat 'larger' than 1 under the alternative H_1. In fact, we can develop a rigorous test based on this statistic by noting the following (distributional) results:

(i) SS_R/σ^2 has a chi-square distribution with $(a-1)(b-1)$ degrees of freedom *always*;
(ii) SS_{Trts}/σ^2 has a chi-square distribution with $a-1$ degrees of freedom *under* H_0;
(iii) SS_R/σ^2 and SS_{Trts}/σ^2 are independent.

Thus, **under** H_0,

$$F_0 = \frac{MS_{Trts}}{MS_R} = \frac{SS_{Trts}/(a-1)}{SS_R/(a-1)(b-1)} \tag{9.7}$$

$$= \frac{\frac{SS_{Trts}}{\sigma^2}/(a-1)}{\frac{SS_R}{\sigma^2}/(a-1)(b-1)} \sim \frac{\chi^2_{a-1}/(a-1)}{\chi^2_{(a-1)(b-1)}/(a-1)(b-1)} \tag{9.8}$$

i.e. under H_0, F_0 has a $F_{a-1,(a-1)(b-1)}$ distribution.

9.2 Example: Comparison of Types of Drill Bit

A manufacturer of factory power equipment is trying to compare the hardness levels of several types of drill bit for their new power drill (which is supposed to be able to handle sheet metal).

The choice is between four available types of drill bit.

Approach 1: Use the completely randomized (one-way) design.

Four drill bits are available from each of the 4 types. We assign each of the $4 \times 4 = 16$ drill bits to its own standard strip of metal in random order. The hardness reading of a bit is ascertained by connecting it to a special machine that drills it through the metal strip, and takes a reading related to the 'Rockwell C hardness scale'.

After the results are collected, we can fit the one-way analysis of variance model to the data, and draw conclusions.

PROBLEM: The strips of metal happen to be sourced from 4 different steel companies. The metals from the different companies may differ somewhat in their levels of hardness, and so may make a contribution to the variability in the hardness data for the drill bits; this in turn may mask any true differences between the types of drill bit.

Approach 2: Use the randomized complete block design.

Instead, we test each type of tip (1 to 4) on each of the metals from the 4 different companies (A,B,C, and D). The tip types are tested in random order for metals sourced from within a particular company. The data arising from this design are given in the table below.

		Company			
		A	**B**	**C**	**D**
Type of Tip	**1**	9.3	9.4	9.6	10.0
	2	9.4	9.3	9.8	9.9
	3	9.2	9.4	9.5	9.7
	4	9.7	9.6	10.0	10.2

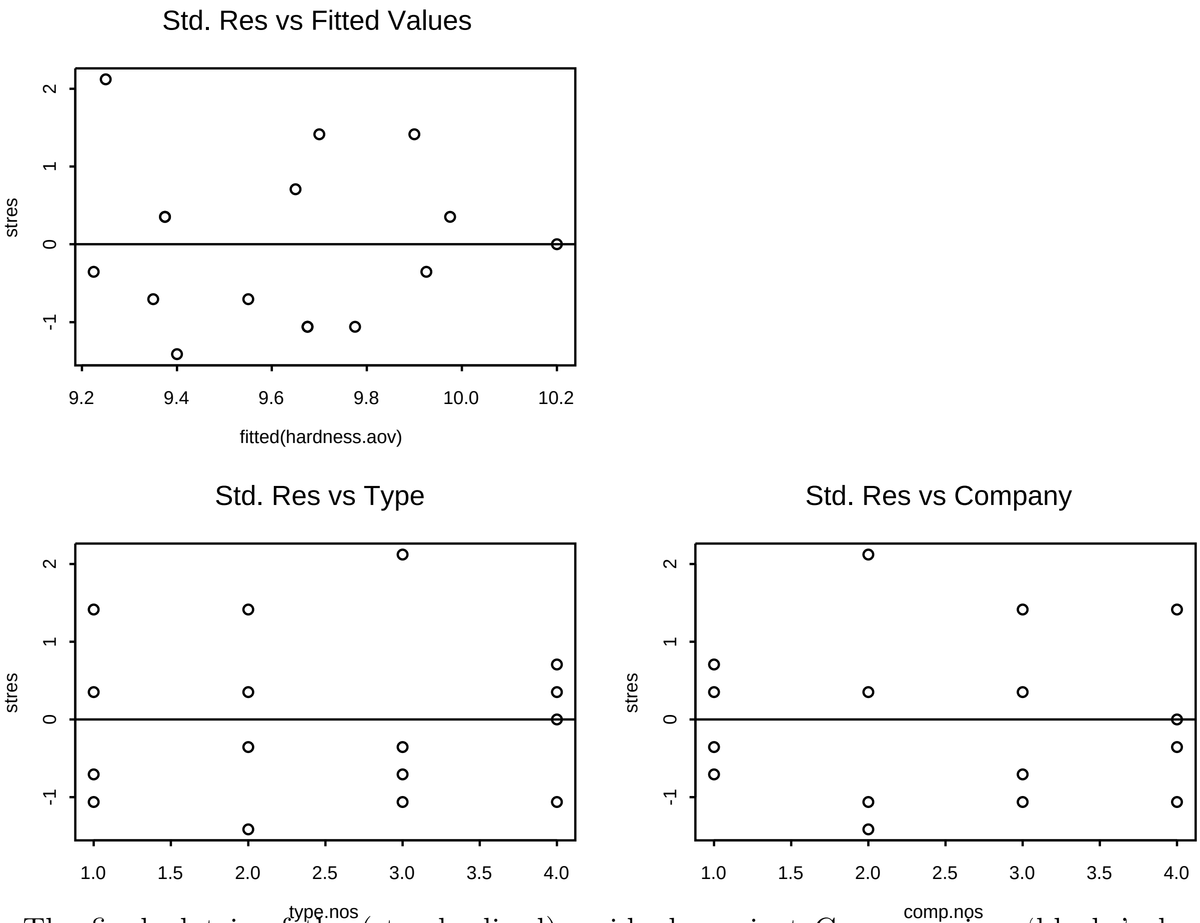
The data can be entered in R, and the analysis carried out as follows.

```
> reading <- c(9.3, 9.4, 9.6, 10., 9.4, 9.3, 9.8, 9.9, 9.2, 9.4, 9.5, 9.7, 9.7, 9.6, 10., 10.2)
> type.nos <- rep(1:4, rep(4, 4))
> comp <- rep(c("A", "B", "C", "D"), 4)
> type <- factor(type.nos)
> company <- factor(comp)
> hardness <- data.frame(reading, type, company)
> hardness.aov <- aov(reading ~ type + company, data = hardness, qr = T)
> summary(hardness.aov)
          Df Sum Sq Mean Sq F value   Pr(>F)
type       3  0.385 0.12833   14.44 0.000871 ***
company    3  0.825 0.27500   30.94 4.52e-05 ***
Residuals  9  0.080 0.00889
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

There is very strong evidence ($p << 0.01$) of 'real' differences in the hardnesses of the 4 types of tip.

Plotting the residuals (in the manner suggested earlier) does not appear to indicate anything untoward about the fit of the model.

```
library(MASS)
stres <- stdres(hardness.aov)
# Define comp.nos because we need a 'numeric' vector for plotting
comp.nos <- rep(1:4, 4)
par(mfrow = c(2, 2))
plot(fitted(hardness.aov), stres, main = "Std. Res vs Fitted Values")
abline(h=0)
plot(type.nos, stres, main = "Std. Res vs Type")
abline(h=0)
plot(comp.nos, stres, main = "Std. Res vs Company")
abline(h=0)
```



The final plot is of the (standardized) residuals against Company, i.e. 'blocks': here A=1, B=2, C=3, D=4. (i.e. we check the assumption of the model that the variances are equal across blocks).

Appendix

Computational Issues in the Randomized Complete Block Design

Define $C_f = y_{..}^2/N$, which we call the *correction factor*. Then

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}^2 - C_f$$

$$SS_{Trts} = \frac{1}{b}\sum_{i=1}^{a} y_{i.}^2 - C_f$$

$$SS_{Blocks} = \frac{1}{a}\sum_{j=1}^{b} y_{.j}^2 - C_f$$

and, as usual, SS_R can be found via subtraction:

$$SS_R = SS_T - SS_{Trts} - SS_{Blocks}.$$