B.Sc./Grad. Dip.: Probability Models and Time Series
MAS programmes: Stochastic Models and Time Series

Examples 3

1. A high street bank has been experimenting with a model in which the credit rating of each customer is re-classified every three months (i.e. quarterly at December 31, March 31, June 30, September 30 each year) according to the letter codes A, B, C and D in the order of highest to lowest. Thus a rating of A is used for the least risky customer, and D for the most risky. The rating comes into effect at midday on each of the above dates.

   Let X_n be the rating of a customer at the end of the n-th quarter. The process is thought to behave in accordance with a time-homogeneous Markov chain with one-step transition matrix given by

$$\mathbf{P} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

   corresponding to the states A, B, C and D (in that order).

   (a) i. Classify the states into those which are transient and/or those which are closed and irreducible.
       ii. Which states are positive recurrent?
       iii. What is the period of each state?

   (b) Suppose that at the end of December 31$^{\text{st}}$ 2010, a customer has a credit rating of B. What is the probability that at the end of June 30$^{\text{th}}$ 2011, the customer will have a credit rating of C?

   (c) If a stationary distribution, $\boldsymbol{\pi} = (\pi_A, \pi_B, \pi_C, \pi_D)$, for the Markov chain were to exist, show that it would have to satisfy the following set of equations:

$$\begin{aligned} \frac{2}{3}\pi_A &= \frac{1}{4}(\pi_B + \pi_C) \\ \frac{2}{3}\pi_A &= \frac{3}{4}\pi_B \\ \frac{3}{4}\pi_C &= \frac{1}{2}(\pi_B + \pi_D) \\ \frac{1}{2}\pi_C &= \frac{1}{2}\pi_D. \end{aligned}$$

   (d) What is the probability that a customer who has been with the bank continuously for a sufficiently long period of time will have a credit rating of A?