Birkbeck
UNIVERSITY OF LONDON

MATHEMATICS AND STATISTICS ASSIGNMENT COVER SHEET

Student to Complete

Name: Christien Gilson	
Degree Programme: MSc. Applied Statistics	
Module: PROBABILITY MODELS AND TIME SERIES [B.Sc./Grad. Dip] STOCHASTIC MODELS AND TIME SERIES [MAS Programmes]	
Lecturer: BROOMS	
Assignment Title/Number: ASSIGNMENT B [B.Sc./Grad. Dip]/ ASSIGNMENT ONE [MAS]	
Due Date: Monday, 23rd March, 2020	

Declaration

I herby testify that, unless otherwise acknowledged, the work submitted here is entirely my own

Signature:

LEAVE BLANK – OFFICIAL USE ONLY
Date Received:

(Unscaled) Mark/Grade	Late – up to 14 days	Late – more than 14 days

Christian Gilson – Spring Markov Chain Assignment 2020

PMTS - Assignment B/SMTS - Assignment 1

Deadline: Monday, 23rd March, 2020

Total marks: [25]. Marks are shown in boxes [ ]. There are 2 questions in this assignment.

1. Consider a Markov chain on the state space $\mathbb{S} = \{1, 2, 3, 4, 5, 6\}$ with corresponding one-step transition matrix given by

$$\mathbf{P} = \begin{array}{c} \\ \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{2}{5} & 0 & 0 & \frac{3}{5} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{array}\right] \\ j = \;\; 1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \;\; 6 \end{array} \begin{array}{l} i=1 \\ i=2 \\ i=3 \\ i=4 \\ i=5 \\ i=6 \\ \\ \end{array}$$

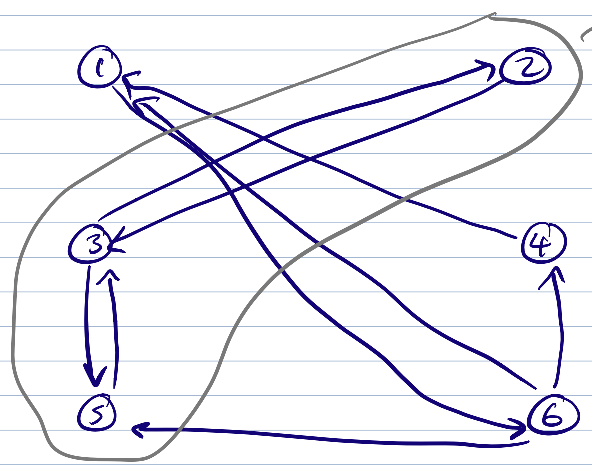
(a) Decompose the state space, $\mathbb{S}$, into the set of transient states and the closed irreducible set(s) of recurrent states. [2]

(b) For each closed irreducible set of recurrent states that has been identified in part (a), determine its periodicity. Justify your answers. [3]

Question 1

1) a) Finite state space $S = \{1, 2, 3, 4, 5, 6\}$

↳ any recurrent states **MUST** be positive recurrent.



$\mathbb{C}_1 = \{2, 3, 5\}$

$\mathbb{T} = \{1, 4, 6\}$.

$S = \mathbb{T} + \mathbb{C}_1 + \mathbb{C}_2 + \dots$

$S = \{1, 4, 6\} \cup \{2, 3, 5\}$

$\boxed{\text{Transient states} = \{1, 4, 6\}. \text{ (Positive) Recurrent states} = \{2, 3, 5\}.}$

b) $\mathbb{C}_1 = \{2, 3, 6\}$.

Periodicity for state i: $d(i) = \gcd(n : P_{ii}(n) > 0)$

Recall, for a fully intercommunicating set of closed states, where $i \leftrightarrow j, \forall i, j \in \mathbb{C}$, then the periodicity is the same for all states, and we classify $\mathbb{C}_1$ as an irreducible set.

So let's calculate the periodicity for state 2, and apply to all states.

Possible ways of starting in state 2 and returning back to state 2:

$2 \to 3 \to 2 \quad (n=2)$

$2 \to 3 \to 5 \to 3 \to 2 \quad (n=4)$

$2 \to 3 \to 5 \to 3 \to 5 \to 3 \to 2 \quad (n=6)$

Similar to the symmetric random walk example, must take an even number of steps to return to original state.

$$\therefore d(2) = \gcd\{2,4,6,\dots\} = 2 \Rightarrow d(i) = 2, \ \forall i \in C_1.$$

Periodicity for fully interconnected, irreducible closed set, C_1, of positive recurrent states $\{2,3,6\}$ is 2, since states 3 and 6 will have the same periodicity as state 2.

Since we have a closed set C_1 with a periodicity that's greater than 1, the Markov Chain will not have a limiting distribution where $p_{ii}(n)$ converges as $n \to \infty$.

---

2. (a) Consider a discrete time Markov chain $\{Y_n\}$ with state space $\mathbb{S} = \{0, 1, 2, \dots\}$ such that

→ go to next state → stay at same state at next stage

$$Y_0 = 0, \quad \mathbb{P}(Y_{n+1} = s+1 | Y_n = s) = p, \quad \mathbb{P}(Y_{n+1} = s | Y_n = s) = 1 - p$$

for $n = 0, 1, 2, \dots$ and $0 < p < 1$.

It can be shown that → binomial, m steps, overall looking for $j-i$ successes, and $m-(j-i)$ failures.

$$p_{ij}(m) = \mathbb{P}(Y_{n+m} = j | Y_n = i) = \binom{m}{j-i} p^{j-i}(1-p)^{m-(j-i)}, \quad i, j \in \mathbb{N}, \quad 0 \le j - i \le m.$$

i) Is the chain $\{Y_n\}$ irreducible? Justify your answer. [2]

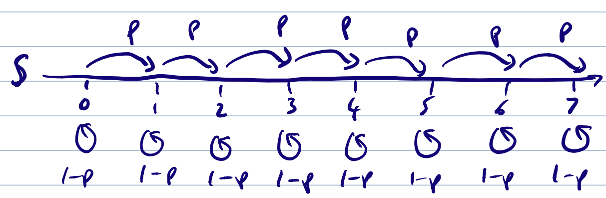
ii) For $i \in \{0, 1, 2, \dots\}$, determine an expression for

$$\sum_{m=0}^{\infty} p_{ii}(m).$$

On the basis of this expression only, deduce whether each of the states within $\mathbb{S}$ is transient or recurrent. [4]

Question 2a)

Q2a) $\mathbb{S}$ is countably infinite. $\mathbb{S} = \{0\} + \mathbb{Z}^+$.



(i) Is $\{Y_n\}$ irreducible?

$\boxed{\text{Criteria: } i \leftrightarrow j, \ \forall i, j \in \mathbb{S}.}$

Two ways of showing that i and j are intercommunicating for all $i, j \in S$:

① $p_{ij}(n) > 0$ for $n \geq 0$ & $p_{ji}(m) > 0$ for $m \geq 0$;

② $f_{ij} > 0$ and $f_{ji} > 0$, $\forall i, j \in S$.

It's clear to see that whilst it's possible to get from any state to an equal or "larger" state in n steps ($n \in \mathbb{N}$), it is impossible to get from a larger state to a smaller state.

Eg. $i = 3$, $j = 2$.

$$f_{ij} = f_{32} = \sum_{n=1}^{\infty} f_{32}(n) = 0$$

Therefore $\{Y_n\}$ IS NOT IRREDUCIBLE.

(ii) For $i \in \{0, 1, 2, \ldots\}$, determine expression for:

$$\boxed{\sum_{m=0}^{\infty} p_{ii}(m).}$$

Will be looking to show whether: $\sum_{m=0}^{\infty} p_{ii}(m) = \infty$ (i is recurrent)

or $\sum_{m=0}^{\infty} p_{ii}(m) < \infty$ (i is transient)

$$p_{ij}(m) = P(Y_{n+m} = j \mid Y_n = i) = \binom{m}{j-i} p^{j-i} (1-p)^{m-(j-i)}$$

$j - i = i - i = 0$

$$p_{ii}(m) = P(Y_{n+m} = i \mid Y_n = i) = \binom{m}{0} p^0 (1-p)^m = (1-p)^m$$

$$\sum_{m=0}^{\infty} p_{ii}(m) = \sum_{m=0}^{\infty} (1-p)^m = \sum_{m=0}^{\infty} q^m \text{, where } 0 < q < 1,$$

$q = 1 - p$,

and this is the geometric series.

$$\sum_{m=0}^{\infty} q^m = \frac{1}{1-q}$$ which converges when $|q| < 1$, which it does.

$$\therefore \boxed{\sum_{m=0}^{\infty} p_{ii}(m) = \sum_{m}^{\infty} (1-p)^m = \sum_{m}^{\infty} q^m = \frac{1}{1-q} = \frac{1}{p}}$$

Since $\sum_{m=0}^{\infty} p_{ii}(m) = \frac{1}{p} < \infty$, for $\forall i \in S$, $0 < p < 1$, we can say that all states within state space S are transient.

(b) Now consider a new Markov chain $\{X_n\}$ with state space $\mathbb{S}' = \{1, 2, 3, 4\}$ such that

$$X_n = \left(Y_n - 4\left\lfloor \frac{Y_n}{4} \right\rfloor\right) + 1$$

with $p = \frac{1}{3}$, where $\lfloor z \rfloor$ represents the integer part of z (i.e. round z **down** to the nearest whole number).

i) Write down the one-step transition matrix, $\mathbf{P}$, for $\{X_n\}$. [3]

ii) Briefly explain, with justification, why the chain is:
(I) irreducible; (II) positive recurrent; (III) aperiodic. [3]

iii) Suppose that at time k, the chain resides in one of the four states according to the mass function $\boldsymbol{\pi}^{(k)} = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$. For the mass function at time $k+2$ given by $\boldsymbol{\pi}^{(k+2)} = (a, b, c, d)$, determine the numerical values of a, b, c and d, expressed as fractions, each in its simplest possible form. [4]

iv) Determine the value of the limiting probability that the chain $\{X_n\}$ resides in state 3, i.e.

$$\lim_{n\to\infty} \pi_3^{(n)}.$$

[4]

Question 2b)

Q2b) State space for $\underset{\sim}{X} = \{X_n\}$ is finite, $S' = \{1,2,3,4\}$

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix}$$

$$\boxed{P_{ij} = P_{ij}(1) = P(X_1 = j \mid X_0 = i)}$$

→ there's a mapping between X_n and Y_n.
→ we have the P_{ij}'s for $\{Y_n\}$
→ probability that $\{Y_n\}$ goes to next state is $1/3$.
→ probability stays where it is is $2/3$.

Examples by hand.

$$X_0 = \left(Y_0 - 4\left\lfloor \frac{Y_0}{4} \right\rfloor\right) + 1 = \left(0 - 4\left\lfloor \frac{0}{4} \right\rfloor\right) + 1 = 1$$

$\boxed{\text{So, } X_0 = 1}$ → $\{X_n\}$ starts at state 1.

If Y_n in state 1:

$$X_n = (1 - 4\underbrace{\left\lfloor \tfrac{1}{4} \right\rfloor}_{0}) + 1 = 2$$

If Y_n in state 2:

$$X_n = (2 - 4\underbrace{\left\lfloor \tfrac{2}{4} \right\rfloor}_{0}) + 1 = 3$$

If Y_n in state 3:

$$X_n = (3 - 4\underbrace{\left\lfloor \tfrac{3}{4} \right\rfloor}_{0}) + 1 = 4$$

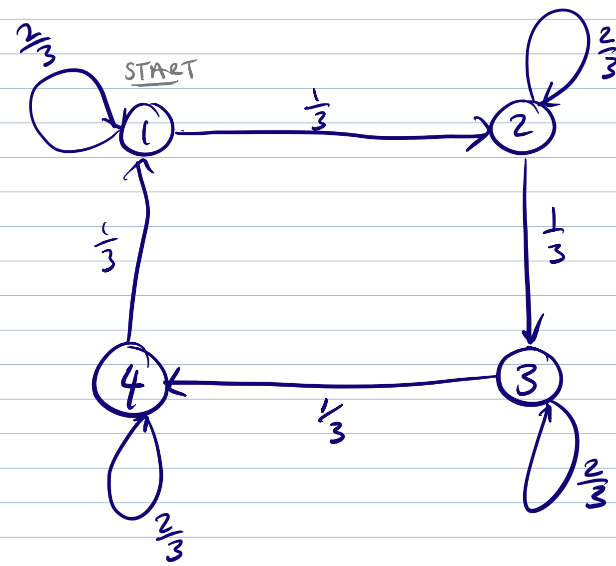
If Y_n in state 4:

$$X_n = (4 - 4\underbrace{\left\lfloor \tfrac{4}{4} \right\rfloor}_{=4}) + 1 = 1$$

And lastly, if Y_n in state 5:

$$X_n = (5 - 4\underbrace{\left\lfloor \tfrac{5}{4} \right\rfloor}_{4}) + 1 = 2.$$

- If $\{Y_n\}$ changes state w/ probability $p = \frac{1}{3}$, so too does $\{X_n\}$
- If $\{Y_n\}$ DOES NOT change state, $1-p = \frac{2}{3}$, then neither does $\{X_n\}$.



Therefore we can write the transition matrix for $\{X_n\}$:

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{bmatrix}$$

Sanity check: expect all rows to sum to 1.

$\hookrightarrow \sum_j P_{ij} = 1$ for $i = 1,2,3,4$ ✓

b) Explain why chain is (I) irreducible; (II) positive recurrent; (III) aperiodic.

(I) $\{X_n\}$ is irreducible because for every state in $S' = \{1, 2, 3, 4\}$, $i \leftrightarrow j$, $\forall i, j \in S'$. This means that every state in the statespace is fully intercommunicating. It's possible to reach each state from any other state, whereby it's clear that $f_{ij} > 0$ $\forall i, j \in S'$ by the fully connected diagram in a).

(II) $\{X_n\}$ has a finite subspace S' ($N = 4$), and therefore every recurrent state MUST be positive recurrent.

We also know that at least one of the states must be recurrent (because we have a finite state space).

But if i = recurrent, then j must ALSO be recurrent if $i \leftrightarrow j$. And because $i \leftrightarrow j$ $\forall i, j$ in S', then every state must be recurrent, and therefore every state in S' must be positive recurrent.

It's also very clear that S' is a closed set, featuring only recurrent states, and no transient states.

$S' = T \cup C = C$.

(III) We know from irreducible Markov Chain property that because $i \leftrightarrow j$, $\forall i, j \in S'$, that $d(i)$ is the same $\forall i \in S'$. i.e. every state has the same period.

So we need to calculate the period of one state and apply it to all of them.

$d(i) = \gcd\{n : p_{ii}(n) > 0\}$

Let's pick state 2:

Possible paths for $2 \to 2$:

$2 \to 3 \to 4 \to 1 \to 2$, $n = 4$

$2 \to 2 \to 3 \to 4 \to 1 \to 2$, $n = 5$

Therefore $d(2) = d(1) = d(3) = d(4) = \gcd\{4, 5, 6, 7, 8, \ldots\} = 1$.

So because state 2 (and thus ALL states in S') have a periodicity of 1, we say the chain is APERIODIC.

Note: because POSITIVE RECURRENT + APERIODIC = ERGODIC, we could also classify $\{X_n\}$ as ergodic, too.

(iii) $\underset{\sim}{\pi}^{(k)} = \left[\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right]$

$\therefore \pi_1^{(k)} = \frac{1}{2}$

$\pi_2^{(k)} = \frac{1}{6}$

$\pi_3^{(k)} = \frac{1}{6}$

$\pi_4^{(k)} = \frac{1}{6}$

$\underset{\sim}{\pi}^{(k+2)} = [a, b, c, d]$ $\rightarrow$ find a, b, c, d.

Recall: $\underset{\sim}{\pi}^{(m+n)} = \underset{\sim}{\pi}^{(m)} P^{(n)} = \underset{\sim}{\pi}^{(m)} P^n$ [Result falls out of applying Chapman-Kolmogorov]

$\rightarrow \underset{\sim}{\pi}^{(k+2)} = \underset{\sim}{\pi}^{(k)} P^2$

Therefore two steps to this: ① Calculate P^2
② $\underset{\sim}{\pi}^k P^2$.

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{bmatrix}$$

①

$$P^2 = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{4}{9} & \frac{4}{9} & \frac{1}{9} & 0 \\ 0 & \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{1}{9} & 0 & \frac{4}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{1}{9} & 0 & \frac{4}{9} \end{bmatrix} \begin{matrix} \sum_j = 1 ✓ \\ \sum_j = 1 ✓ \\ \sum_j = 1 ✓ \\ \sum_j = 1 ✓ \end{matrix}$$

②

$$\underset{\sim}{\pi}^{(k)} P^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} \frac{4}{9} & \frac{4}{9} & \frac{1}{9} & 0 \\ 0 & \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{1}{9} & 0 & \frac{4}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{1}{9} & 0 & \frac{4}{9} \end{bmatrix} = \begin{bmatrix} \frac{17}{54} & \frac{17}{54} & \frac{11}{54} & \frac{1}{6} \end{bmatrix} \quad \sum_i = \pi_i^{(k+2)} = 1 \checkmark$$

$$\boxed{\frac{4}{18} + \frac{1}{54} + \frac{4}{54} = \frac{12}{54} + \frac{1}{54} + \frac{4}{54} = \frac{17}{54}}$$

$$\boxed{\frac{2}{9} + \frac{4}{54} + 0 + \frac{1}{54} = \frac{12}{54} + \frac{4}{54} + \frac{1}{54} = \frac{17}{54}}$$

$$\boxed{\frac{1}{18} + \frac{4}{54} + \frac{4}{54} = \frac{3}{54} + \frac{4}{54} + \frac{4}{54} = \frac{11}{54}}$$

$$\boxed{\frac{1}{54} + \frac{4}{54} + \frac{4}{54} = \frac{9}{54} = \frac{1}{6}}$$

Therefore $\underset{\sim}{\pi}^{(k+2)} = \begin{bmatrix} \frac{17}{54} & \frac{17}{54} & \frac{11}{54} & \frac{1}{6} \end{bmatrix}$,

where $a = \frac{17}{54}$

$b = \frac{17}{54}$

$c = \frac{11}{54}$

$d = \frac{1}{6}$.

(iv) Try the $\underset{\sim}{\pi} = \underset{\sim}{\pi} P$ route.

$\underset{\sim}{\pi} = \underset{\sim}{\pi} P \qquad \underset{\sim}{\pi} = [\pi_1, \pi_2, \pi_3, \pi_4]$

$\underset{\sim}{\pi} I - \underset{\sim}{\pi} P = \underset{\sim}{0}$

$\underset{\sim}{\pi}(I - P) = \underset{\sim}{0}$

$$[\pi_1\ \pi_2\ \pi_3\ \pi_4]\left\{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{bmatrix}\right\} = \underset{\sim}{0}$$

$$\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$\boxed{\text{Know } \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1}$

$$[\pi_1\ \pi_2\ \pi_3\ \pi_4]\,[\ (1)\ \ (2)\ \ (3)\ \ (4)\ ] = [0\ \ 0\ \ 0\ \ 0]$$

(1) $\frac{\pi_1}{3} - \frac{\pi_4}{3} = 0 \rightarrow \pi_1 = \pi_4$

(2) $-\frac{\pi_1}{3} + \frac{\pi_2}{3} = 0 \rightarrow \pi_1 = \pi_2$

(3) $-\frac{\pi_2}{3} + \frac{\pi_3}{3} = 0 \rightarrow \pi_2 = \pi_3$

(4) $-\frac{\pi_3}{3} + \frac{\pi_4}{3} = 0 \rightarrow \pi_3 = \pi_4$

Therefore, $\pi_1 = \pi_2 = \pi_3 = \pi_4$ and $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$.

$$\therefore \quad 4\pi_1 = 1$$

$$\pi_1 = \frac{1}{4}$$

$$\pi_2 = \frac{1}{4}$$

$$\pi_3 = \frac{1}{4}$$

$$\pi_4 = \frac{1}{4}.$$

$$\underset{\sim}{\pi} = [\pi_1 \; \pi_2 \; \pi_3 \; \pi_4] = \left[\tfrac{1}{4} \; \tfrac{1}{4} \; \tfrac{1}{4} \; \tfrac{1}{4}\right]$$

$$\boxed{\text{And thus:} \quad \lim_{n \to \infty} \pi_3^{(n)} = \frac{1}{4}}$$

Which you'd intuitively expect, as you have four equally likely states whereby you either stick with $2/3$ or twist to next state with $1/3$ chance.

End of Assignment. Christian Gilson.