B.Sc./Grad. Dip.: Probability Models and Time Series
MAS programmes: Stochastic Models and Time Series
SPRING 2020

PMTS - Assignment B/SMTS - Assignment 1

Deadline: Monday, 23rd March, 2020

Total marks: [25]. Marks are shown in boxes [ ]. There are 2 questions in this assignment.

1. Consider a Markov chain on the state space $\mathbb{S} = \{1, 2, 3, 4, 5, 6\}$ with corresponding one-step transition matrix given by

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{2}{5} & 0 & 0 & \frac{3}{5} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}.$$

   (a) Decompose the state space, $\mathbb{S}$, into the set of transient states and the closed irreducible set(s) of recurrent states. [2]

   (b) For each closed irreducible set of recurrent states that has been identified in part (a), determine its periodicity. Justify your answers. [3]

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2. (a) Consider a discrete time Markov chain $\{Y_n\}$ with state space $\mathbb{S} = \{0, 1, 2, \ldots\}$ such that

$$Y_0 = 0, \quad \mathbb{P}(Y_{n+1} = s+1 | Y_n = s) = p, \quad \mathbb{P}(Y_{n+1} = s | Y_n = s) = 1 - p$$

for $n = 0, 1, 2, \ldots$ and $0 < p < 1$.
It can be shown that

$$p_{ij}(m) = \mathbb{P}(Y_{n+m} = j | Y_n = i) = \binom{m}{j-i} p^{j-i} (1-p)^{m-(j-i)}, \quad i, j \in \mathbb{N}, \quad 0 \le j - i \le m.$$

i) Is the chain $\{Y_n\}$ irreducible? Justify your answer. [2]

ii) For $i \in \{0, 1, 2, \ldots\}$, determine an expression for

$$\sum_{m=0}^{\infty} p_{ii}(m).$$

On the basis of this expression only, deduce whether each of the states within $\mathbb{S}$ is transient or recurrent. [4]

(b) Now consider a new Markov chain $\{X_n\}$ with state space $\mathbb{S}' = \{1, 2, 3, 4\}$ such that

$$X_n = \left(Y_n - 4 \left\lfloor \frac{Y_n}{4} \right\rfloor \right) + 1$$

with $p = \frac{1}{3}$, where $\lfloor z \rfloor$ represents the integer part of z (i.e. round z **down** to the nearest whole number).

i) Write down the one-step transition matrix, $\mathbf{P}$, for $\{X_n\}$. [3]

ii) Briefly explain, with justification, why the chain is:
(I) irreducible; (II) positive recurrent; (III) aperiodic. [3]

iii) Suppose that at time k, the chain resides in one of the four states according to the mass function $\boldsymbol{\pi}^{(k)} = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$. For the mass function at time $k+2$ given by $\boldsymbol{\pi}^{(k+2)} = (a, b, c, d)$, determine the numerical values of a, b, c and d, expressed as fractions, each in its simplest possible form. [4]

iv) Determine the value of the limiting probability that the chain $\{X_n\}$ resides in state 3, i.e.

$$\lim_{n \to \infty} \pi_3^{(n)}.$$

[4]

Important Note:
• Please read the current version of the *Mathematics & Statistics Coursework Policy*. Copies can be obtained from the course website, or in hardcopy from the programme administrator.