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## MATHEMATICS AND STATISTICS ASSIGNMENT COVER SHEET

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*Student to Complete*

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Degree Programme:

MSc. Applied Statistics

Module:

PROBABILITY MODELS AND TIME SERIES [B.Sc./Grad. Dip]  
STOCHASTIC MODELS AND TIME SERIES [MAS Programmes]

Lecturer:

BROOMS

Assignment Title/Number:

ASSIGNMENT B [B.Sc./Grad. Dip]/ ASSIGNMENT ONE [MAS]

Due Date:

Monday, 23<sup>rd</sup> March, 2020

*Declaration*

I hereby testify that, unless otherwise acknowledged, the work submitted here is entirely my own

Signature:



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Date Received:

(Unscaled) Mark/Grade	Late – up to 14 days	Late – more than 14 days

# Christian Lütsen - Spring Markov Chain Assignment 2020

## PMTS - Assignment B/SMTS - Assignment 1

Deadline: Monday, 23<sup>rd</sup> March, 2020

Total marks: [25]. Marks are shown in boxes [ ]. There are 2 questions in this assignment.

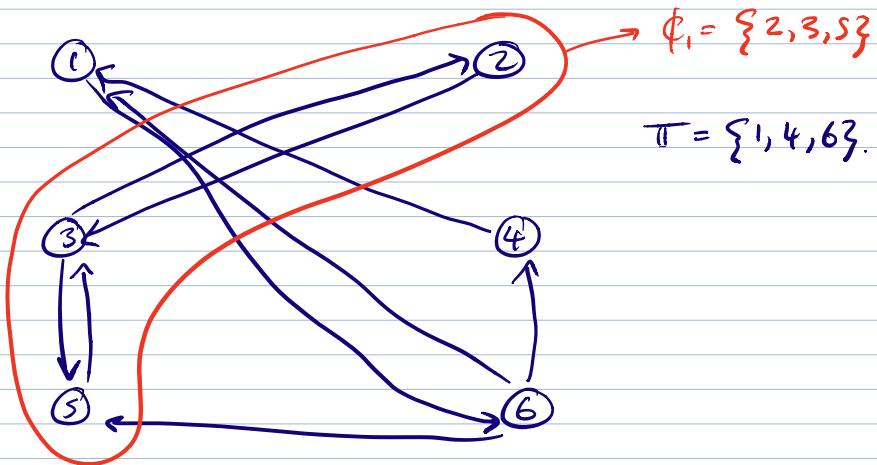
1. Consider a Markov chain on the state space  $S = \{1, 2, 3, 4, 5, 6\}$  with corresponding one-step transition matrix given by

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix} \begin{array}{l} i=1 \\ i=2 \\ i=3 \\ i=4 \\ i=5 \\ i=6 \end{array}$$

- (a) Decompose the state space,  $S$ , into the set of transient states and the closed irreducible set(s) of recurrent states. [2]  
 (b) For each closed irreducible set of recurrent states that has been identified in part (a), determine its periodicity. Justify your answers. [3]

### Question 1

i) a) Finite state space  $S = \{1, 2, 3, 4, 5, 6\}$   
 ↳ any recurrent states MUST be positive recurrent.



$$S = \Pi + \Phi_1 + \Phi_2 + \dots$$

$$S = \{1, 4, 6\} \cup \{2, 3, 5\}$$

Transient states =  $\{1, 4, 6\}$ . (Positive) Recurrent states =  $\{2, 3, 5\}$ .

b)  $\Phi_1 = \{2, 3, 5\}$ .

Periodicity for state  $i$ :  $d(i) = \gcd(n : P_{ii}(n) > 0)$

Recall, for a fully intercommunicating set of closed states, where  $i \leftrightarrow j, \forall i, j \in \Phi_1$ , then the periodicity is the same for all states, and we classify  $\Phi_1$  as an irreducible set.

So let's calculate the periodicity for state 2, and apply to all states.

Possible ways of starting in state 2 and returning back to state 2:

$$2 \rightarrow 3 \rightarrow 2 \quad (n=2)$$

$$2 \rightarrow 3 \rightarrow 5 \rightarrow 3 \rightarrow 2 \quad (n=4)$$

$$2 \rightarrow 3 \rightarrow 5 \rightarrow 3 \rightarrow 5 \rightarrow 3 \rightarrow 2 \quad (n=6)$$

Similar to the symmetric random walk example, must take an even number of steps to return to original state.

$$\therefore d(2) = \gcd\{2, 4, 6, \dots\} = 2 \Rightarrow d(i) = 2, \forall i \in \mathbb{N}_i.$$

Periodicity for fully interconnected, irreducible closed set,  $\mathbb{N}_i$ , of positive recurrent states  $\{2, 3, 6\}$  is 2, since states 3 and 6 will have the same periodicity as state 2.

Since we have a closed set  $\mathbb{N}_i$  with a periodicity that's greater than 1, the Markov chain will not have a limiting distribution where  $p_{ii}(n)$  converges as  $n \rightarrow \infty$ .

2. (a) Consider a discrete time Markov chain  $\{Y_n\}$  with state space  $\mathbb{S} = \{0, 1, 2, \dots\}$  such that

$$Y_0 = 0, \quad \mathbb{P}(Y_{n+1} = s+1 | Y_n = s) = p, \quad \mathbb{P}(Y_{n+1} = s | Y_n = s) = 1-p$$

for  $n = 0, 1, 2, \dots$  and  $0 < p < 1$ .

It can be shown that

$$p_{ij}(m) = \mathbb{P}(Y_{n+m} = j | Y_n = i) = \binom{m}{j-i} p^{j-i} (1-p)^{m-(j-i)}, \quad i, j \in \mathbb{N}, \quad 0 \leq j-i \leq m.$$

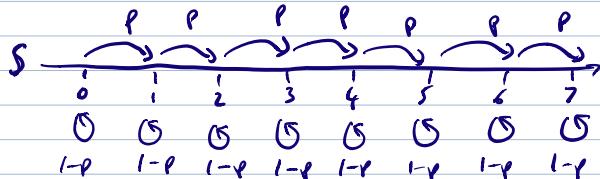
- i) Is the chain  $\{Y_n\}$  irreducible? Justify your answer. [2]  
ii) For  $i \in \{0, 1, 2, \dots\}$ , determine an expression for

$$\sum_{m=0}^{\infty} p_{ii}(m).$$

On the basis of this expression only, deduce whether each of the states within  $\mathbb{S}$  is transient or recurrent. [4]

Question 2a)

Q2a)  $\mathbb{S}$  is countably infinite.  $\mathbb{S} = \{0\} + \mathbb{Z}^+$ .



(i) Is  $\{Y_n\}$  irreducible?

Criterion:  $i \leftrightarrow j, \forall i, j \in \mathbb{S}$ .

Two ways of showing that  $i$  and  $j$  are intercommunicating for all  $i, j \in S$ :

$$\textcircled{1} \quad p_{ij}(n) > 0 \text{ for } n \geq 0 \quad \textcircled{2} \quad p_{ji}(m) > 0 \text{ for } m \geq 0;$$

$$\textcircled{2} \quad f_{ij} > 0 \text{ and } f_{ji} > 0, \forall i, j \in S.$$

It's clear to see that whilst it's possible to get from any state to an equal or "larger" state in  $n$  steps ( $n \in \mathbb{N}$ ), it is impossible to get from a larger state to a smaller state.

$$\text{Eg. } i = 3, j = 2.$$

$$f_{ij} = f_{23} = \sum_{n=1}^{\infty} f_{23}(n) = 0$$

Therefore  $\{Y_n\}$  is NOT irreducible.

(ii) for  $i \in \{0, 1, 2, \dots\}$ , determine expression for:

$$\sum_{m=0}^{\infty} p_{ii}(m).$$

Will be looking to show whether:  $\sum_{m=0}^{\infty} p_{ii}(m) = \infty$  ( $i$  is recurrent)

$\textcircled{1} \leq \sum_{m=0}^{\infty} p_{ii}(m) < \infty$  ( $i$  is transient)

$$p_{ij}(m) = P(Y_{n+m} = j \mid Y_n = i) = \binom{m}{j-i} p^{j-i} (1-p)^{m-(j-i)} \Big|_{\substack{j-i \\ = i-i \\ = 0}} = 0$$

$$p_{ii}(m) = P(Y_{n+m} = i \mid Y_n = i) = \binom{m}{0} p^0 (1-p)^m = (1-p)^m$$

$$\sum_{m=0}^{\infty} p_{ii}(m) = \sum_{m=0}^{\infty} (1-p)^m = \sum_{m=0}^{\infty} q^m, \text{ where } 0 < q < 1,$$

$$q = 1-p,$$

and this is the geometric series.

$$\sum_{m=0}^{\infty} q^m = \frac{1}{1-q} \quad \text{which converges when } |q| < 1, \text{ which it does.}$$

$$\therefore \sum_{m=0}^{\infty} p_{ii}(m) = \sum_{m=0}^{\infty} (1-p)^m = \sum_{m=0}^{\infty} q^m = \frac{1}{1-q} = \frac{1}{p}$$

Since  $\sum_{m=0}^{\infty} p_{ii}(m) = \frac{1}{p} < \infty$ , for  $\forall i \in S, 0 < p < 1$ , we can

say that all states within state space  $S$  are transient.

- (b) Now consider a new Markov chain  $\{X_n\}$  with state space  $S' = \{1, 2, 3, 4\}$  such that

$$X_n = \left( Y_n - 4 \left\lfloor \frac{Y_n}{4} \right\rfloor \right) + 1$$

with  $p = \frac{1}{3}$ , where  $\lfloor z \rfloor$  represents the integer part of  $z$  (i.e. round  $z$  down to the nearest whole number).

- Write down the one-step transition matrix,  $\mathbf{P}$ , for  $\{X_n\}$ . [3]
- Briefly explain, with justification, why the chain is:  
(I) irreducible; (II) positive recurrent; (III) aperiodic. [3]
- Suppose that at time  $k$ , the chain resides in one of the four states according to the mass function  $\pi^{(k)} = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$ . For the mass function at time  $k+2$  given by  $\pi^{(k+2)} = (a, b, c, d)$ , determine the numerical values of  $a, b, c$  and  $d$ , expressed as fractions, each in its simplest possible form. [4]
- Determine the value of the limiting probability that the chain  $\{X_n\}$  resides in state 3, i.e.

$$\lim_{n \rightarrow \infty} \pi_3^{(n)}.$$

[4]

### Question 2b)

Q2b) State space for  $X = \{X_n\}$  is finite,  $S' = \{1, 2, 3, 4\}$

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix}$$

$$p_{ij} = P(X_i=j | X_0=i)$$

→ there's a mapping between  $X_n$  and  $Y_n$ .  
→ we have the  $p_{ij}$ 's for  $\{Y_n\}$

→ probability that  $\{Y_n\}$  goes to next state is  $\frac{1}{3}$ .  
→ probability stays where it is is  $\frac{2}{3}$ .

Example by hand.

$$X_0 = \left( Y_0 - 4 \left\lfloor \frac{Y_0}{4} \right\rfloor \right) + 1 = (0 - 4 \left\lfloor \frac{0}{4} \right\rfloor) + 1 = 1$$

So,  $X_0 = 1 \rightarrow \{X_n\}$  starts at state 1.

If  $Y_n$  in state 1:  $\textcircled{1} \frac{2}{3}$

$$\frac{1}{3} \quad X_n = (1 - 4 \left\lfloor \frac{1}{4} \right\rfloor) + 1 = 2$$

$\underbrace{\phantom{0}}_0$

If  $Y_n$  in state 2:  $\textcircled{2} \frac{2}{3}$

$$\frac{1}{3} \quad X_n = (2 - 4 \left\lfloor \frac{2}{4} \right\rfloor) + 1 = 3$$

$\underbrace{\phantom{0}}_0$

If  $Y_n$  in state 3:  $\textcircled{3} \frac{2}{3}$

$$\frac{1}{3} \quad X_n = (3 - 4 \left\lfloor \frac{3}{4} \right\rfloor) + 1 = 4$$

$\underbrace{\phantom{0}}_0$

If  $Y_n$  in state 4:  $\textcircled{4} \frac{2}{3}$

$$\frac{1}{3} \quad X_n = (4 - 4 \left\lfloor \frac{4}{4} \right\rfloor) + 1 = 1$$

$= 4$

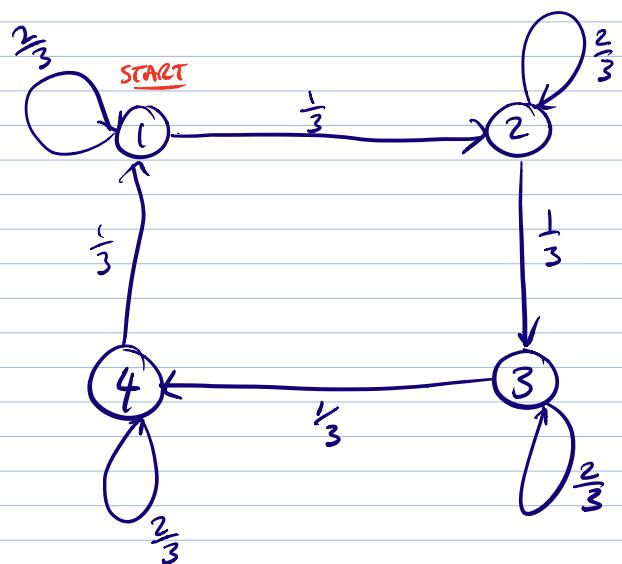
And lastly, if  $Y_n$  in state 5:  $\textcircled{5} \frac{2}{3}$

$$X_n = (5 - 4 \left\lfloor \frac{5}{4} \right\rfloor) + 1 = 2.$$

$\underbrace{\phantom{0}}_4$

- If  $\{Y_n\}$  changes state w/ probability  $P = \frac{1}{3}$ , so too does  $\{X_n\}$

- If  $\{Y_n\}$  does NOT change state,  $1-P = \frac{2}{3}$ , then neither does  $\{X_n\}$ .



Therefore we can write the transition matrix for  $\{X_n\}$ :

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{bmatrix}$$

Sanity check: expect all rows to sum to 1.

$$\hookrightarrow \sum_j p_{ij} = 1 \quad \text{Q.E.D.}$$

b) Explain why chain is (I) irreducible; (II) positive recurrent; (III) aperiodic.

(I)  $\{X_n\}$  is irreducible because for every state in  $S' = \{1, 2, 3, 4\}$ ,

$i \leftrightarrow j, \forall i, j \in S'$ . This means that every state in the statespace

is fully intercommunicating. It's possible to reach each state from any other state, whereby it's clear that  $f_{ij} > 0 \quad \forall i, j \in S'$  by the fully connected diagram in a).

(II)  $\{X_n\}$  has a finite subspace  $S' (N=4)$ , and therefore every recurrent state must be positive recurrent.

We also know that at least one of the states must be recurrent (because we have a finite state space).

But if  $i = \text{recurrent}$ , then  $j$  must ALSO be recurrent if  $i \leftrightarrow j$ . And because  $i \leftrightarrow j, \forall i, j \in S'$ , then every state must be recurrent, and therefore every state in  $S'$  must be positive recurrent.

→ It's also very clear that  $S'$  is a closed set, featuring only recurrent states, and no transient states.

$$\hookrightarrow S' = \pi \cup \emptyset = \emptyset.$$

(III) We know from irreducible Markov Chain property that because  $i \leftrightarrow j, \forall i, j \in S'$ , then  $d(i)$  is the same  $\forall i \in S'$ . i.e. every state has the same period.

So we need to calculate the period of one state and apply it to all of them.

$$d(i) = \gcd \{n : p_{ii}(n) > 0\}$$

Let's pick state 2:

Possible paths for  $2 \rightarrow 2$ :

$$2 \rightarrow 3 \rightarrow 4 \rightarrow 1 \rightarrow 2, \quad n = 4$$

$$2 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1 \rightarrow 2, \quad n = 5$$

$$\text{Therefore } d(2) = d(1) = d(3) = d(4) = \gcd \{4, 5, 6, 7, 8, \dots\} = 1.$$

So because state 2 (and thus ALL states in  $S'$ ) have a periodicity of 1, we say the chain is APERIODIC.

↪ Note: because POSITIVE RECURRENT + APERIODIC = ERGODIC, we could also classify  $\{X_n\}$  as ergodic, too.

$$(iii) \quad \underline{\pi}^{(k)} = \left[ \frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6} \right]$$

$$\therefore \underline{\pi}_1^{(k)} = \frac{1}{2}$$

$$\underline{\pi}_2^{(k)} = \frac{1}{6}$$

$$\underline{\pi}_3^{(k)} = \frac{1}{6}$$

$$\underline{\pi}_4^{(k)} = \frac{1}{6}$$

$$\underline{\pi}^{(k+2)} = [a, b, c, d] \rightarrow \text{find } a, b, c, d.$$

Recall:  $\underline{\pi}^{(mn)} = \underline{\pi}^{(m)} p^{(n)} = \underline{\pi}^{(m)} p^n$  [result falls out of applying Chapman-Kolmogorov]  
 $\rightarrow \underline{\pi}^{(k+2)} = \underline{\pi}^{(k)} p^2$

Therefore two steps to find:  
① Calculate  $p^2$   
②  $\underline{\pi}^{(k)} p^2$ .

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{bmatrix}$$

$$\textcircled{1} \quad p^2 = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{4}{9} & \frac{4}{9} & \frac{1}{9} & 0 \\ 0 & \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{1}{9} & 0 & \frac{4}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{1}{9} & 0 & \frac{4}{9} \end{bmatrix} \begin{matrix} \sum_j = 1 \\ \sum_j = 1 \\ \sum_j = 1 \\ \sum_j = 1 \end{matrix}$$

②

$$\begin{bmatrix} \frac{4}{9} & \frac{4}{9} & \frac{1}{9} & 0 \\ 0 & \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{1}{9} & 0 & \frac{4}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{1}{9} & 0 & \frac{4}{9} \end{bmatrix}$$

$$\tilde{\pi}^{(k+2)} = \left[ \frac{1}{2} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \right] \left[ \frac{17}{54} \quad \frac{17}{54} \quad \frac{11}{54} \quad \frac{1}{6} \right]$$

$$\sum_i = \tilde{\pi}_i^{(k+2)} = 1 \checkmark$$

$$\frac{1}{54} + \frac{4}{54} + \frac{4}{54} = \frac{9}{54} = \frac{1}{6}$$

$$\frac{4}{18} + \frac{1}{54} + \frac{4}{54} = \frac{12}{54} + \frac{1}{54} + \frac{4}{54} = \frac{17}{54}$$

$$\frac{1}{18} + \frac{4}{54} + \frac{4}{54} = \frac{3}{54} + \frac{4}{54} + \frac{4}{54} = \frac{11}{54}$$

$$\frac{2}{9} + \frac{4}{54} + 0 + \frac{1}{54} = \frac{12}{54} + \frac{4}{54} + \frac{1}{54} = \frac{17}{54}$$

$$\text{Therefore } \tilde{\pi}^{(k+2)} = \left[ \frac{17}{54} \quad \frac{17}{54} \quad \frac{11}{54} \quad \frac{1}{6} \right],$$

$$\text{where } a = \frac{17}{54}$$

$$b = \frac{17}{54}$$

$$c = \frac{11}{54}$$

$$d = \frac{1}{6}.$$

(iv) Try the  $\underline{\pi} = \underline{\pi}\rho$  rule.

$$\underline{\pi} = \underline{\pi}\rho$$

$$\underline{\pi} = [\pi_1, \pi_2, \pi_3, \pi_4]$$

$$\underline{\pi}\underline{I} - \underline{\pi}\rho = \underline{0}$$

$$\underline{\pi}(\underline{I} - \rho) = \underline{0}$$

$$[\pi_1, \pi_2, \pi_3, \pi_4] \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{bmatrix} \right\} = \underline{0}$$

$$\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Know  $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$

$$[\pi_1, \pi_2, \pi_3, \pi_4] [① \quad ② \quad ③ \quad ④] = [0 \quad 0 \quad 0 \quad 0]$$

$$① \quad \frac{\pi_1}{3} - \frac{\pi_4}{3} = 0 \rightarrow \pi_1 = \pi_4$$

$$② \quad -\frac{\pi_1}{3} + \frac{\pi_2}{3} = 0 \rightarrow \pi_1 = \pi_2$$

$$③ \quad -\frac{\pi_2}{3} + \frac{\pi_3}{3} = 0 \rightarrow \pi_2 = \pi_3$$

$$④ \quad -\frac{\pi_3}{3} + \frac{\pi_4}{3} = 0 \rightarrow \pi_3 = \pi_4$$

Therefore,  $\pi_1 = \pi_2 = \pi_3 = \pi_4$  and  $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$ .

$$\therefore 4\pi_1 = 1$$

$$\pi_1 = \frac{1}{4}$$

$$\pi_2 = \frac{1}{4}$$

$$\pi_3 = \frac{1}{4}$$

$$\pi_4 = \frac{1}{4}.$$

$$\underline{\pi} = [\pi_1 \ \pi_2 \ \pi_3 \ \pi_4] = \left[ \frac{1}{4} \ \frac{1}{4} \ \frac{1}{4} \ \frac{1}{4} \right]$$

And thus:

$$\lim_{n \rightarrow \infty} \pi_3^{(n)} = \frac{1}{4}$$

which you'd intuitively expect, as you have four equally likely states whereby you either stick with  $\frac{2}{3}$  or twist to next state with  $\frac{1}{3}$  chance.

END OF ASSIGNMENT. CHRISTIAN GILLSEN.

