B.Sc./Grad. Dip.: Probability Models and Time Series
MAS programmes: Stochastic Models and Time Series

Solutions 3

1. (a) i. All the states intercommunicate and so they constitute a single closed irreducible set.

   ii. Since the state space is finite, then there exists at least one recurrent state and that state is positive recurrent. However, since the chain itself is irreducible, then all the states are positive recurrent.

   iii. It just so happens that the chain can remain in the same state from one stage to the next. Since the chain is irreducible, it follows that all states have period equal to one.

   (b) Going from end of December 31$^{\text{st}}$ 2010 to the end of June 30$^{\text{th}}$ 2011 traverses two quarters. Therefore, we need to calculate $p_{BC}(2)$. By the Chapman-Kolmogorov equations, this is given by

$$p_{BC}(2) = [\mathbf{P}^{(2)}]_{BC} = [\mathbf{P}^2]_{BC} = \sum_k p_{Bk}p_{kC}$$

$$= \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} = (\frac{1}{4} \times 0) + (\frac{1}{4} \times \frac{1}{2}) + (\frac{1}{2} \times \frac{1}{4}) + (0 \times \frac{1}{2}) = \frac{1}{4}.$$

   (c) The stationary distribution would have to satisfy $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$ or $\boldsymbol{\pi}(\mathbf{I} - \mathbf{P}) = \mathbf{0}$. This corresponds to

$$\begin{bmatrix} \pi_A & \pi_B & \pi_C & \pi_D \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & 0 & 0 \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} & 0 \\ -\frac{1}{4} & 0 & \frac{3}{4} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}.$$

   From 1st column:
$$\frac{2}{3}\pi_A - \frac{1}{4}\pi_B - \frac{1}{4}\pi_C = 0.$$

   From 2nd column:
$$-\frac{2}{3}\pi_A + \frac{3}{4}\pi_B = 0.$$

   From 3rd column:
$$-\frac{1}{2}\pi_B + \frac{3}{4}\pi_C - \frac{1}{2}\pi_D = 0.$$

   From 4th column:
$$-\frac{1}{2}\pi_C + \frac{1}{2}\pi_D = 0.$$

   The result follows.

(d) Need to find the limiting distribution corresponding to state A, i.e. $\lim_{n\to\infty} p_{iA}(n)$ for all $i \in A, B, C, D$. Since all the states are positive recurrent, then there exists a unique stationary distribution π (which can be found by solving the equations in part (c)). Since all the states are both positive recurrent and aperiodic, then the limiting probability for state A is given by π_A.

Thus solving for the equations we find:

$$\pi_B = \frac{8}{9}\pi_A \quad \pi_B = \frac{16}{9}\pi_A; \quad \pi_D = \frac{16}{9}\pi_A.$$

Hence

$$\boldsymbol{\pi} = k \times \left(1, \frac{8}{9}, \frac{16}{9}, \frac{16}{9}\right).$$

But since $\pi_A + \pi_B + \pi_C + \pi_D = 1$, then $k = \frac{9}{49}$, and so

$$\boldsymbol{\pi} = \left(\frac{9}{49}, \frac{8}{49}, \frac{16}{49}, \frac{16}{49}\right).$$

The limiting probability of customer having credit rating A is $\frac{9}{49}$.