B.Sc./Grad. Dip.: Probability Models and Time Series
MAS programmes: Stochastic Models and Time Series

5 General autoregressive processes

5.1 An example of a second order autoregressive process

Consider a particular second order autoregressive process (an AR(2) process), a stationary process $\{Y_t\}$ that satisfies the relation

$$Y_t = \frac{1}{3}Y_{t-1} + \frac{2}{9}Y_{t-2} + \epsilon_t \qquad t \in \mathbb{Z}, \tag{1}$$

where $\{\epsilon_t\}$ is a white noise process with mean 0 and variance σ^2, and the infinite moving average representation of $\{Y_t\}$,

$$Y_t = \sum_{i=0}^{\infty} \psi_i \epsilon_{t-i}, \tag{2}$$

where the sequence of coefficients $\{\psi_i : i \geq 0\}$ is to be determined.

Substituting the expression (2) into Equation (1),

$$\begin{aligned}\sum_{i=0}^{\infty} \psi_i \epsilon_{t-i} &= \frac{1}{3}\sum_{i=0}^{\infty} \psi_i \epsilon_{t-1-i} + \frac{2}{9}\sum_{i=0}^{\infty} \psi_i \epsilon_{t-2-i} + \epsilon_t \\ &= \frac{1}{3}\sum_{i=1}^{\infty} \psi_{i-1} \epsilon_{t-i} + \frac{2}{9}\sum_{i=2}^{\infty} \psi_{i-2} \epsilon_{t-i} + \epsilon_t.\end{aligned}$$

Equating coefficients of $\epsilon_{t-i},\ i \geq 0$,

$$\psi_0 = 1 \qquad [i = 0], \tag{3}$$

$$\psi_1 = \frac{1}{3}\psi_0 = \frac{1}{3} \qquad [i = 1], \tag{4}$$

$\psi_2 = \frac{1}{3}\psi_1 + \frac{2}{9}\psi_0 = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$

$$\psi_i = \frac{1}{3}\psi_{i-1} + \frac{2}{9}\psi_{i-2} \qquad [i \geq 2]. \tag{5}$$

Starting with the initial values given in Equations (3) and (4), we may use Equation (5) to calculate iteratively the values of the ψ_i for integers $i \geq 2$. Thus $\psi_2 = 1/3, \psi_3 = 5/27, \ldots$. But it will be of more fundamental interest to find the general solution of the difference equation (5), which is of the form

$$\boxed{\psi_i = A_1\alpha_1^i + A_2\alpha_2^i \qquad i \geq 0,} \tag{6}$$

where first of all the constants α_1 and α_2 are to be determined. Substituting the trial solution $\psi_i = \alpha^i$ into Equation (5), we obtain

$$\alpha^i = \frac{1}{3}\,\alpha^{i-1} + \frac{2}{9}\,\alpha^{i-2} \qquad i \geq 2.$$

Dividing through by α^{i-2}, we see that we have a solution of Equation (5) if and only if α satisfies the *auxiliary equation*,

$$\alpha^2 = \frac{1}{3}\alpha + \frac{2}{9}. \tag{7}$$

The constants α_1 and α_2 in the general solution (6) are the roots of Equation (7), which factorizes as

$$\left(\alpha - \frac{2}{3}\right)\left(\alpha + \frac{1}{3}\right) = 0,$$

so that the roots are $\alpha_1 = 2/3$ and $\alpha_2 = -1/3$. Hence the general solution of the difference equation (5) is

$$\psi_i = A_1\left(\frac{2}{3}\right)^i + A_2\left(-\frac{1}{3}\right)^i \qquad i \geq 0. \tag{8}$$

The appropriate values of the constants A_1 and A_2 are determined using the initial conditions of Equations (3) and (4). Thus

$$A_1 + A_2 = 1$$

and

$$\frac{2}{3}A_1 - \frac{1}{3}A_2 = \frac{1}{3},$$

that is,

$$2A_1 - A_2 = 1.$$

Hence $A_1 = 2/3,\ A_2 = 1/3$ and

$$\psi_i = \left(\frac{2}{3}\right)^{i+1} - \left(-\frac{1}{3}\right)^{i+1} \qquad i \geq 0. \tag{9}$$

Thus the infinite moving average representation of $\{Y_t\}$ is

$$Y_t = \sum_{i=0}^{\infty}\left[\left(\frac{2}{3}\right)^{i+1} - \left(-\frac{1}{3}\right)^{i+1}\right]\epsilon_{t-i}. \tag{10}$$

Note that the infinite moving average representation of Equation (10) is well defined, because the condition $\sum_{i=0}^{\infty}|\psi_i| < \infty$, given in the discussion of linear processes in Section 4.3, is satisfied when the ψ_i are as specified in Equation (9) (see Appendix). Thus the particular AR(2) model being considered here really does define a stationary process.

From the results of Section 4.3,

$$\text{var}(Y_t) = \gamma_0 = \sigma^2\sum_{i=0}^{\infty}\psi_i^2.$$

Substituting in the values of the ψ_i from Equation (9), and noting that $\sum_{i=0}^{\infty}\psi_i^2 = \frac{567}{440}$ (see Appendix), we obtain

$$\gamma_0 = \sigma^2\sum_{i=0}^{\infty}\left[\left(\frac{2}{3}\right)^{i+1} - \left(-\frac{1}{3}\right)^{i+1}\right]^2 = \frac{567}{440}\sigma^2.$$

The autocovariance and autocorrelation function of the process $\{Y_t\}$ may be calculated using its infinite moving average expression and the approach described in Section 4.3, but a somewhat simpler method is based upon the use of the model equation (1).

Recall the result of Section 4.3 that

$$E[Y_s \epsilon_t] = 0 \qquad s < t.$$

Multiplying Equation (1) by $Y_{t-\tau}$ $\tau \geq 1$, and taking expectations,

$$\gamma_\tau = \frac{1}{3}\gamma_{\tau-1} + \frac{2}{9}\gamma_{\tau-2} \qquad \tau \geq 1.$$

Dividing through by γ_0,

$$\rho_\tau = \frac{1}{3}\rho_{\tau-1} + \frac{2}{9}\rho_{\tau-2} \qquad \tau \geq 1. \tag{11}$$

The Equations (11) are the *Yule-Walker equations*, which, together with the initial conditions,

$$\rho_0 = 1 \tag{12}$$

and

$$\rho_1 = \rho_{-1}, \tag{13}$$

may be used to compute the autocorrelation function $\{\rho_\tau\}$. We may use the Yule-Walker equations iteratively to calculate successive values of ρ_τ for $\tau \geq 1$. Putting $\tau = 1$ in Equation (11) and using Equation (13) we obtain

$$\rho_1 = \frac{1}{3} + \frac{2}{9}\rho_1.$$

Hence

$$\rho_1 = \frac{3}{7}.$$

Putting $\tau = 2$ in Equation (11) ,

$$\rho_2 = \frac{1}{3}\rho_1 + \frac{2}{9}\rho_0 = \frac{23}{63},$$

and so on, for successive values of τ.

Alternatively, adopting a more mathematically analytical approach, note that Equations (11) are another set of difference equations, similar in form to the Equations (5) and hence have a general solution similar in form to Equation (8),

$$\rho_\tau = B_1\left(\frac{2}{3}\right)^\tau + B_2\left(-\frac{1}{3}\right)^\tau \qquad \tau \geq -1.$$

Only the initial conditions are now different, as specified by Equations (12) and (13). They yield a pair of equations for the constants B_1 and B_2:

$$B_1 + B_2 = 1$$

and

$$\frac{2}{3}\,B_1 - \frac{1}{3}\,B_2 = \frac{3}{2}\,B_1 - 3B_2,$$

that is,

$$B_2 = \frac{5}{16}\,B_1.$$

Hence $B_1 = 16/21,\; B_2 = 5/21$ and the autocorrelation function is given by

$$\rho_\tau = \frac{16}{21}\left(\frac{2}{3}\right)^\tau + \frac{5}{21}\left(-\frac{1}{3}\right)^\tau \qquad \tau \geq 0.$$

Note the essentially geometric decrease of the sequence $\{\rho_\tau\}$.

5.2 Definition of the general autoregressive process

Definition 5.2.1 (AR(p) process)
An *autoregressive process of order* p, an AR(p) process, is a zero mean stationary process $\{Y_t\}$ which satisfies the relation

$$Y_t = \sum_{k=1}^{p} \phi_k Y_{t-k} + \epsilon_t \qquad t \in \mathbb{Z}, \tag{14}$$

where $\{\epsilon_t\}$ is a white noise process with mean 0 and variance σ^2 and where $\phi_p \neq 0$. □

However, as we shall see, not all sets of values of the autoregressive parameters $\phi_1, \phi_2, \ldots, \phi_p$ can be used to define a stationary process.

We may define a corresponding *AR characteristic polynomial* $\phi(z)$, which may be thought of as a generating function for the autoregressive coefficients, by

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \ldots - \phi_p z^p. \tag{15}$$

Equation (14) for the AR(p) model may then be written compactly in terms of the lag operator L as

$$\phi(L)Y_t = \epsilon_t. \tag{16}$$

5.3 Infinite moving average representation for the AR(p) model

We look for an infinite moving average representation of $\{Y_t\}$,

$$Y_t = \sum_{i=0}^{\infty} \psi_i \epsilon_{t-i} \tag{17}$$

where the sequence of coefficients $\{\psi_i : i \geq 0\}$ is to be determined. We generalize the method used for the example of the AR(2) process. Substituting the expression (17) into Equation

(14),

$$\begin{aligned}
\sum_{i=0}^{\infty} \psi_i \epsilon_{t-i} &= \sum_{k=1}^{p} \phi_k \sum_{i=0}^{\infty} \psi_i \epsilon_{t-k-i} + \epsilon_t \\
&= \sum_{k=1}^{p} \phi_k \sum_{i=k}^{\infty} \psi_{i-k} \epsilon_{t-i} + \epsilon_t \\
&= \sum_{i=1}^{\infty} \left(\sum_{k=1}^{\min\{i,p\}} \phi_k \psi_{i-k} \right) \epsilon_{t-i} + \epsilon_t .
\end{aligned}$$

Equating coefficients of $\{\epsilon_{t-i} : i \geq 0\}$,

$$\psi_0 = 1, \tag{18}$$

$$\psi_i = \sum_{k=1}^{i} \phi_k \psi_{i-k} \qquad 1 \leq i \leq p-1, \tag{19}$$

$$\psi_i = \sum_{k=1}^{p} \phi_k \psi_{i-k} \qquad i \geq p. \tag{20}$$

Equation (20) is a linear difference equation of order p for the ψ_i, with p initial conditions provided by the Equations (18) and (19). The general solution of Equation (20) is of the form

$$\psi_i = \sum_{k=1}^{p} A_k \alpha_k^i \qquad i \geq 0, \tag{21}$$

where the $\{\alpha_k : k = 1, \ldots, p\}$ are the roots of the *auxiliary equation*,

$$\alpha^p = \sum_{k=1}^{p} \phi_k \alpha^{p-k}. \tag{22}$$

The constants $\{A_k : 1 \leq k \leq p\}$ could be determined from Equations (18) and (19), although that is not our primary concern at present.

- The auxiliary equation, Equation (22), is a polynomial equation of order p, which, if we allow complex roots, will have exactly p roots, some of which may be repeated. The proof of the result that Equation (21) gives the general solution of Equation (20), and the extension to the case when the auxiliary equation has repeated roots, may be found, for example, in Biggs, *Discrete Mathematics*.

Equation (17) corresponds to a well defined, stationary, process if and only if $|\alpha_k| < 1, k = 1, \ldots, p$. Another way of saying this is that all the roots of the auxiliary equation, Equation (22), must lie strictly within the unit circle in the complex plane. Yet another way of expressing this condition comes from introducing the *AR characteristic equation*, $\phi(z) = 0$, where the polynomial ϕ is the characteristic polynomial as defined in Equation (15). Thus the characteristic equation is

$$1 - \phi_1 z - \phi_2 z^2 - \ldots - \phi_p z^p = 0. \tag{23}$$

Writing $z = \alpha^{-1}$, it is easy to check that the roots of Equation (23) are precisely the inverses of the roots of Equation (22). Hence a condition that is equivalent to the previous one is that all the roots of the characteristic equation should lie strictly outside the unit circle in the complex plane.

We have shown that, if and only if this condition is satisfied, does there exist a linear process $\{Y_t\}$, with the infinite moving average expression of Equation (17), that satisfies the Equation (14) of the AR(p) model. To summarize:

Theorem 5.3.1 (Stationarity condition)
A necessary and sufficient condition for there to exist a unique stationary solution to the AR(p) model specified by autoregressive parameters $\phi_1, \phi_2, \ldots, \phi_p$, and which is expressible as a linear process, is that all the roots of the characteristic equation, Equation (23), lie strictly outside the unit circle in the complex plane (i.e., all the roots are greater than one in modulus/absolute value). □

Example In the case of the example of Section 5.1 of an AR(2) process, the characteristic equation is

$$1 - \frac{1}{3}z - \frac{2}{9}z^2 = 0,$$

that is,

$$\left(1 - \frac{2}{3}z\right)\left(1 + \frac{1}{3}z\right) = 0,$$

which has roots $z_1 = 3/2$ and $z_2 = -3$. The roots are outside the unit circle in the complex plane, so the stationarity condition is satisfied. Note also that the roots of the characteristic equation really are the inverses of the roots that we found of the auxiliary equation, $\alpha_1 = 2/3$ and $\alpha_2 = -1/3$.

Example For the AR(1) process introduced in Section 3.6, the characteristic equation is

$$1 - \phi z = 0,$$

whose root is $z_1 = 1/\phi$. For this root to lie outside the unit circle in the complex plane, we require $|\phi| < 1$, as found previously.

Returning to the general case, if the stationarity condition is satisfied then we may write the infinite moving average representation of $\{Y_t\}$ as

$$Y_t = \psi(L)\epsilon_t, \tag{24}$$

where ψ is the generating function of the coefficients of the infinite moving average representation,

$$\psi(z) = \sum_{i=0}^{\infty} \psi_i z^i.$$

We may also invert Equation (16) to write

$$Y_t = \phi(L)^{-1}\epsilon_t. \tag{25}$$

It follows from comparing Equations (24) and (25) that we may write $\psi(z)$ in terms of the characteristic polynomial $\phi(z)$ as

$$\psi(z) = \phi(z)^{-1},$$

so that

$$\phi(z)\psi(z) = 1.$$

There is a duality between the idea of expressing an invertible finite MA process as an infinite order autoregression and the idea of expressing a stationary finite AR process as an infinite moving average. Note that the conditions for invertibility of a MA process, given in Section 4.6, and for the stationarity of an AR process are similar.

For the purposes of modelling observed time series, we use models with a finite number of parameters, whose values can then be estimated.

5.4 The autocorrelation function for the AR(p) model

Let $\{Y_t\}$ be an AR(p) process with zero mean, so that the stationarity condition is satisfied and $\{Y_t\}$ is a stationary process that satisfies Equation (14). Multiplying Equation (14) by $Y_{t-\tau}$, $\tau \geq 1$, taking expectations and using again the result of Section 4.3 that $E[Y_s\epsilon_t] = 0,\ s < t$, we find that

$$\gamma_\tau = \sum_{k=1}^{p} \phi_k \gamma_{\tau-k}, \qquad \tau \geq 1.$$

Dividing through by γ_0,

$$\rho_\tau = \sum_{k=1}^{p} \phi_k \rho_{\tau-k}, \qquad \tau \geq 1. \tag{26}$$

The Equations (26) are the *Yule-Walker equations*, which may be used to compute the autocorrelation function, $\{\rho_\tau\}$. In solving the Yule-Walker equations, we need to use the conditions

$$\rho_0 = 1 \tag{27}$$

and

$$\rho_\tau = \rho_{-\tau}, \qquad 1 \leq \tau \leq p-1. \tag{28}$$

Substituting the conditions (27) and (28) into the first $p-1$ of the Equations (26), we obtain the equations

$$\begin{aligned}
\rho_1 &= \phi_1 + \phi_2\rho_1 + \phi_3\rho_2 + \ldots + \phi_p\rho_{p-1}, \\
\rho_2 &= \phi_1\rho_1 + \phi_2 + \phi_3\rho_1 + \ldots + \phi_p\rho_{p-2}, \\
\ldots & \\
\rho_{p-1} &= \phi_1\rho_{p-2} + \phi_2\rho_{p-3} + \ldots + \phi_{p-1} + \phi_p\rho_1.
\end{aligned}$$

This set of $p-1$ simultaneous linear equations can be solved numerically to obtain $\rho_1, \rho_2, \ldots, \rho_{p-1}$. Thereafter, the Equations (26) can be solved iteratively for $\tau \geq p$ to obtain $\rho_p, \rho_{p+1}, \ldots$.

Alternatively, note that Equations (26) are another set of difference equations similar in form to Equations (20) and hence have a general solution similar to the one in Equation (21),

$$\rho_\tau = \sum_{k=1}^{p} B_k \alpha_k^\tau, \qquad \tau \geq 1 - p, \tag{29}$$

where the α_k are again the roots of Equation (22) and the values of the coefficients $B_1, B_2, \ldots, B_p$ may be determined by using the conditions (27) and (28).

Note that the stationarity condition in the form $|\alpha_k| < 1, \ 1 \leq k \leq p$, implies that, for an AR(p) process, $\rho_\tau \to 0$ geometrically as $\tau \to \infty$.

Appendix

In this section we demonstrate that if

$$\psi_i = \left(\frac{2}{3}\right)^{i+1} - \left(-\frac{1}{3}\right)^{i+1} \qquad i \geq 0$$

then
(a)

$$\sum_{i=0}^{\infty} |\psi_i| < \infty$$

and
(b)

$$\sum_{i=0}^{\infty} \psi_i^2 = \frac{567}{440}.$$

For (a)

$$\begin{aligned}
\sum_{i=0}^{\infty} \left| \left(\frac{2}{3}\right)^{i+1} - \left(-\frac{1}{3}\right)^{i+1} \right| &\leq \sum_{i=0}^{\infty} \left\{ \left| \left(\frac{2}{3}\right)^{i+1} \right| + \left| - \left(-\frac{1}{3}\right)^{i+1} \right| \right\} \\
\leq \sum_{i=0}^{\infty} \left\{ \left(\frac{2}{3}\right)^{i+1} + \left(\frac{1}{3}\right)^{i+1} \right\} &= \left(\frac{2}{3}\right) \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^{i} + \left(\frac{1}{3}\right) \sum_{i=0}^{\infty} \left(\frac{1}{3}\right)^{i} \\
\leq \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^{i} + \sum_{i=0}^{\infty} \left(\frac{1}{3}\right)^{i} &= \frac{1}{1/3} + \frac{1}{2/3} = 3 + \frac{3}{2} = \frac{9}{2}
\end{aligned}$$

where the first inequality follows from the "triangle inequality". Thus $\sum_{i=0}^{\infty} |\psi_i| < \infty$.

For (b), notice that

$$\begin{aligned}
\psi_i^2 &= \left(\frac{2}{3}\right)^{2(i+1)} + \left(-\frac{1}{3}\right)^{2(i+1)} - 2\left(-\frac{2}{9}\right)^{i+1} \\
&= \frac{4}{9} \times \left(\frac{4}{9}\right)^{i} + \frac{1}{9} \times \left(\frac{1}{9}\right)^{i} + \frac{4}{9} \times \left(-\frac{2}{9}\right)^{i}.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{i=0}^{\infty} \psi_i^2 &= \frac{4}{9} \sum_{i=0}^{\infty} \left(\frac{4}{9}\right)^{i} + \frac{1}{9} \sum_{i=0}^{\infty} \left(\frac{1}{9}\right)^{i} + \frac{4}{9} \sum_{i=0}^{\infty} \left(-\frac{2}{9}\right)^{i} \\
&= \left(\frac{4}{9} \times \frac{1}{1 - \frac{4}{9}}\right) + \left(\frac{1}{9} \times \frac{1}{1 - \frac{1}{9}}\right) + \left(\frac{4}{9} \times \frac{1}{1 + \frac{2}{9}}\right) \\
&= \frac{4}{9} \times \frac{1}{5/9} + \frac{1}{9} \times \frac{1}{8/9} + \frac{4}{9} \times \frac{1}{11/9} \\
&= \frac{4}{5} + \frac{1}{8} + \frac{4}{11} = \frac{352 + 55 + 160}{440} = \frac{567}{440}.
\end{aligned}$$