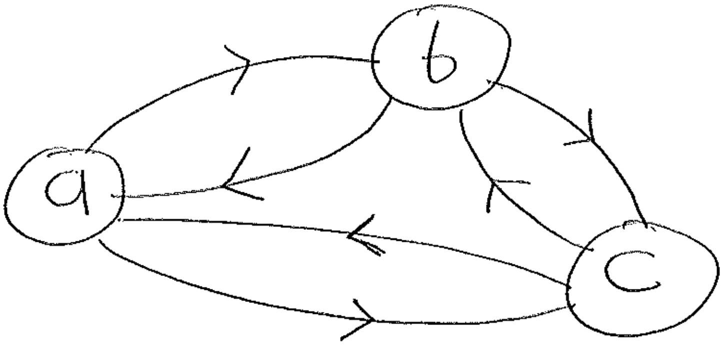
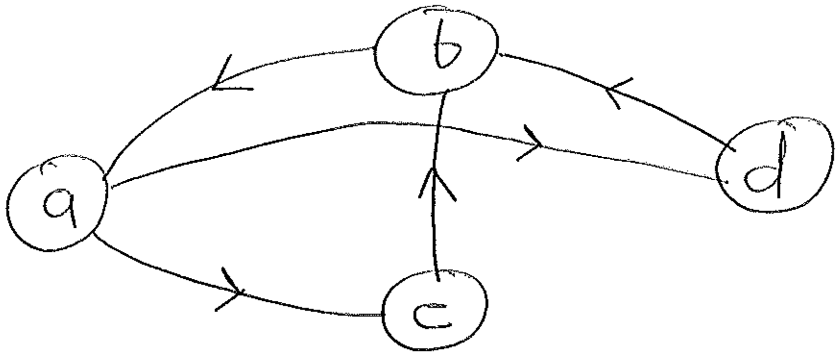
B.Sc./Grad. Dip.: Probability Models and Time Series
MAS programmes: Stochastic Models and Time Series

Solutions 1

1. (a) Irreducible aperiodic Markov chain. Since the chain is finite (and irreducible), then all states are positive recurrent.



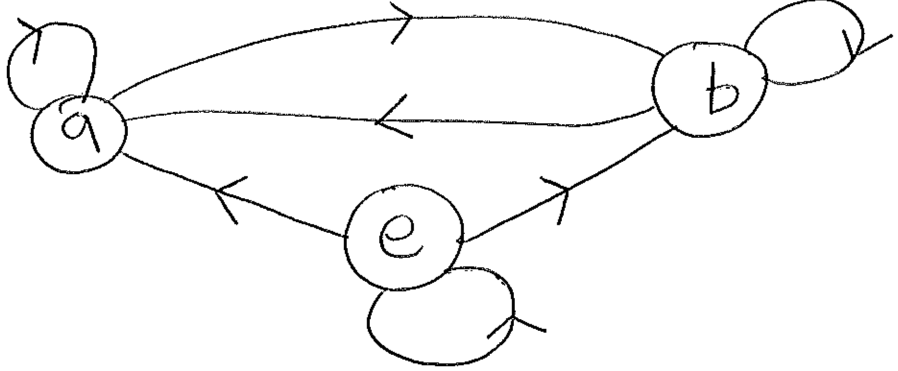
(b) Irreducible Markov chain with all states having period 3. Since the chain is finite (and irreducible), then all states are positive recurrent.

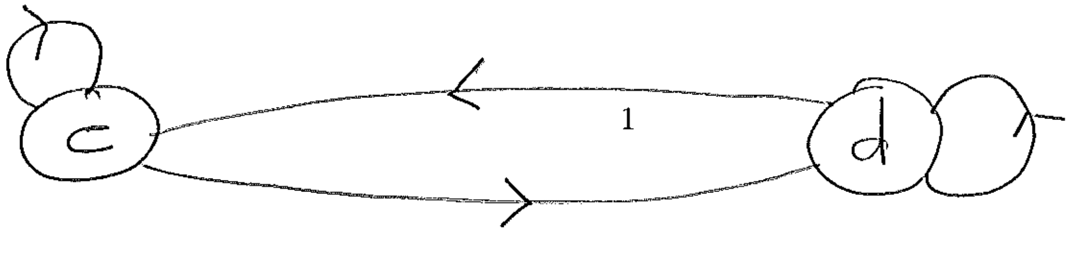


(c) There is one transient class, $\mathbb{T} = \{e\}$. There are two closed irreducible classes:

$$\mathbb{C}_1 = \{a, b\} \qquad \mathbb{C}_2 = \{c, d\}$$

both of which are aperiodic, and positive recurrent.





2. Consider the vector (x, y) where x represents the ball that is drawn from the first bucket and y the ball from the second bucket

   Suppose that $X_n = i$.

   Let B represent a blue ball, and G a green one.

   Then the following outcomes are possible at stage $n+1$:

$$(B, B) \text{ with probability } \frac{i}{B} \times \frac{B-i}{B}$$

$$(G, B) \text{ with probability } \frac{B-i}{B} \times \frac{B-i}{B}$$

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$$(G, G) \text{ with probability } \frac{B-i}{B} \times \frac{i}{B}$$

Thus

$$p_{ii} = \mathbb{P}(X_{n+1} = i | X_n = i) = \mathbb{P}(\{(B, B), (G, G)\}) = \mathbb{P}(\{(B, B)\}) + \mathbb{P}(\{(G, G)\}) = 2\frac{i(B-i)}{B^2}.$$

$$p_{i,i-1} = \mathbb{P}(X_{n+1} = i-1 | X_n = i) = \mathbb{P}(\{(B, G)\}) = \frac{i^2}{B^2}.$$

$$p_{i,i+1} = \mathbb{P}(X_{n+1} = i+1 | X_n = i) = \mathbb{P}(\{(G, B)\}) = \frac{(B-i)^2}{B^2}.$$

Thus, in summary

$$p_{ij} = \begin{cases} \frac{2i(B-i)}{B^2} & \text{if } j = i \\ \frac{i^2}{B^2} & \text{if } j = i-1 \\ \frac{(B-i)^2}{B^2} & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}.$$