B.Sc./Grad. Dip.: Probability Models and Time Series
MAS programmes: Stochastic Models and Time Series

Solutions 5

1. In case (i) the AR characteristic equation is

$$1 - \frac{3}{10}z - \frac{1}{25}z^2 = 0.$$

Its roots are -10, $\frac{5}{2}$, which both lie outside the unit circle in the complex plane. Thus the stationarity condition is satisfied.

In case (ii) the AR characteristic equation is

$$1 - \frac{2}{5}z + \frac{1}{16}z^2 = 0.$$

Its roots are $\frac{4}{5}(4 \pm 3i)$, which both lie outside the unit circle in the complex plane. Thus again the stationarity condition is satisfied.

(a) The infinite moving average representation,

$$Y_t = \sum_{i=0}^{\infty} \psi_i \epsilon_{t-i}$$

has its coefficients determined by $\psi_0 = 1$, $\psi_1 = \phi_1$ and the equation

$$\psi_i = \phi_1 \psi_{i-1} + \phi_2 \psi_{i-2}, \qquad i \geq 2. \tag{1}$$

The general solution of Equation (1) is of the form

$$\psi_j = B_1 \alpha_1^j + B_2 \alpha_2^j, \qquad j \geq 0,$$

where α_1 and α_2 are the inverses of the roots of the AR characteristic equation. In the case (i),

$$\psi_j = \frac{1}{5}\left(4\left(\frac{2}{5}\right)^j + \left(-\frac{1}{10}\right)^j\right).$$

In the case (ii),

$$\psi_j = \left(\frac{1}{2} - \frac{2i}{3}\right)\left(\frac{1}{5} + \frac{3i}{20}\right)^j + \left(\frac{1}{2} + \frac{2i}{3}\right)\left(\frac{1}{5} - \frac{3i}{20}\right)^j.$$

(b) The autocorrelations are determined by the Yule-Walker equations

$$\rho_\tau = \phi_1 \rho_{\tau-1} + \phi_2 \rho_{\tau-2}, \qquad \tau \geq 1 \tag{2}$$

together with the conditions $\rho_0 = 1$ and $\rho_1 = \rho_{-1}$.
In particular, $\rho_1 = \phi_1/(1-\phi_2)$. Using Equation (2) iteratively, we obtain the values (correct to 4 d.p.) in the table below.

autocorrelations

lag	case (i)	case (ii)
1	0.3125	0.3765
2	0.1338	0.0881
3	0.0526	0.0117
4	0.0211	-0.0008

(c) The general solution of Equation (2) is the same as that of Equation (1). Using the initial conditions, we obtain, in the case (i),

$$\rho_\tau = \frac{1}{40}\left(33\left(\frac{2}{5}\right)^\tau + 7\left(-\frac{1}{10}\right)^\tau\right).$$

In case (ii),

$$\rho_\tau = \left(\frac{1}{2} - \frac{10i}{17}\right)\left(\frac{1}{5} + \frac{3i}{20}\right)^\tau + \left(\frac{1}{2} + \frac{10i}{17}\right)\left(\frac{1}{5} - \frac{3i}{20}\right)^\tau.$$

2. (a) From the equation satisfied by $\{Y_t\}$,

$$\begin{aligned} Y_t - Y_{t-1} &= \frac{1}{4}Y_{t-1} + \frac{1}{8}Y_{t-2} - \frac{3}{8}Y_{t-3} + \epsilon_t \\ &= \frac{1}{4}(Y_{t-1} - Y_{t-2}) + \frac{3}{8}(Y_{t-2} - Y_{t-3}) + \epsilon_t. \end{aligned}$$

Thus the equation satisfied by $\{W_t\}$ is

$$W_t = \frac{1}{4}W_{t-1} + \frac{3}{8}W_{t-2} + \epsilon_t.$$

(b) The equation for $\{W_t\}$ is that of an AR(2) model. For the process to be stationary we require that all the roots of the characteristic equation be greater than one in modulus. But the characteristic equation is

$$1 - \frac{1}{4}z - \frac{3}{8}z^2 = 0,$$

whose roots are 4/3 and -2. Thus $\{W_t\}$ is a stationary process and hence $\{Y_t\}$ is an ARIMA(2,1,0) process.