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## MATHEMATICS AND STATISTICS ASSIGNMENT COVER SHEET

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*Student to Complete*

Name:

*Chrishen Gilson*

Degree Programme:

*MSc. Applied Statistics*

Module:

**PROBABILITY MODELS AND TIME SERIES [B.Sc./Grad. Dip]  
STOCHASTIC MODELS AND TIME SERIES [MAS Programmes]**

Lecturer:

**BROOMS**

Assignment Title/Number:

**ASSIGNMENT B [B.Sc./Grad. Dip]/ ASSIGNMENT ONE [MAS]**

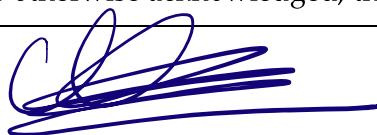
Due Date:

**Monday, 23<sup>rd</sup> March, 2020**

*Declaration*

I hereby testify that, unless otherwise acknowledged, the work submitted here is entirely my own

Signature:



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Date Received:

(Unscaled) Mark/Grade	Late – up to 14 days	Late – more than 14 days
<b>25 out of 25</b>		

# Christian Gilson - Spring Markov Chain Assignment 2020

## PMTS - Assignment B/SMTS - Assignment 1

Deadline: Monday, 23<sup>rd</sup> March, 2020

Total marks: [25]. Marks are shown in boxes [ ]. There are 2 questions in this assignment.

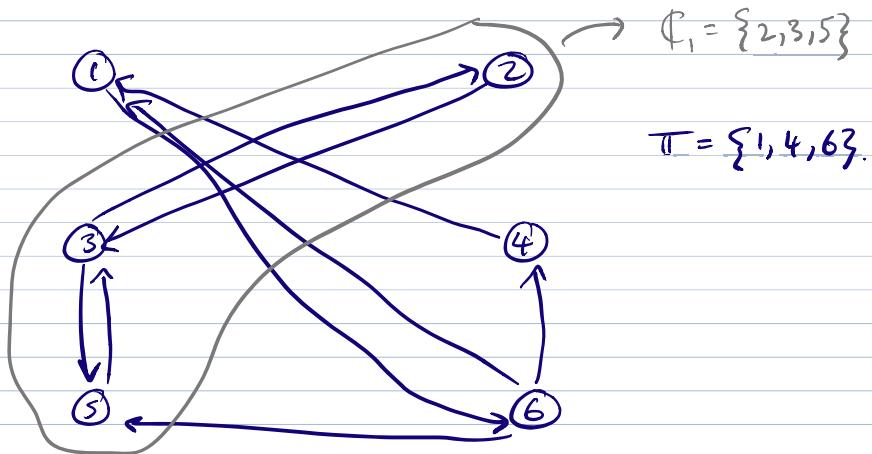
1. Consider a Markov chain on the state space  $S = \{1, 2, 3, 4, 5, 6\}$  with corresponding one-step transition matrix given by

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 \end{bmatrix} \quad \begin{array}{l} i=1 \\ i=2 \\ i=3 \\ i=4 \\ i=5 \\ i=6 \end{array}$$

- (a) Decompose the state space,  $S$ , into the set of transient states and the closed irreducible set(s) of recurrent states. [2]  
 (b) For each closed irreducible set of recurrent states that has been identified in part (a), determine its periodicity. Justify your answers. [3]

### Question 1

1) a) Finite state space  $S = \{1, 2, 3, 4, 5, 6\}$ .  
 Any recurrent states MUST be positive recurrent.



2

$$\begin{aligned} S &= T + \mathbb{P}_1 + \mathbb{P}_2 + \dots \\ S &= \{1, 4, 6\} \cup \{2, 3, 5\} \end{aligned}$$

Transient states =  $\{1, 4, 6\}$ . (Positive) Recurrent states =  $\{2, 3, 5\}$ .

b)  $\mathbb{P}_1 = \{2, 3, 6\}$  ? slip of the pen: in view of later context, no penalty.

Periodicity for state  $i$ :  $d(i) = \gcd(n : P_{ii}(n) > 0)$

Recall, for a fully intercommunicating set of closed states, where  $i \leftrightarrow j, \forall i, j \in \mathbb{P}_1$ , then the periodicity is the same for all states, and we classify  $\mathbb{P}_1$  as an irreducible set.

superfluous:  
 no penalty though.

So let's calculate the periodicity for state 2, and apply to all states.

Possible ways of starting in state 2 and returning back to state 2:

$$2 \rightarrow 3 \rightarrow 2 \quad (n=2)$$

$$2 \rightarrow 3 \rightarrow 5 \rightarrow 3 \rightarrow 2 \quad (n=4)$$

5/5

$$2 \rightarrow 3 \rightarrow 5 \rightarrow 3 \rightarrow 5 \rightarrow 3 \rightarrow 2 \quad (n=6)$$

Similar to the symmetric random walk example, must take an even number of steps to return to original state.

$$\therefore d(2) = \gcd\{2, 4, 6, \dots\} = 2 \Rightarrow d(i) = 2, \forall i \in \mathbb{Q}_i.$$

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Periodicity for fully interconnected, irreducible closed set,  $\mathbb{Q}_i$ , of positive recurrent states  $\{2, 3, 6\}$  is 2, since states 3 and 6 will have the same periodicity as state 2.

Since we have a closed set  $\mathbb{Q}_i$  with a periodicity that's greater than 1, the Markov chain will not have a limiting distribution where  $p_{ii}(n)$  converges as  $n \rightarrow \infty$ .

quit while  
you're ahead!

2. (a) Consider a discrete time Markov chain  $\{Y_n\}$  with state space  $S = \{0, 1, 2, \dots\}$  such that

$$Y_0 = 0, \quad \mathbb{P}(Y_{n+1} = s+1|Y_n = s) = p, \quad \mathbb{P}(Y_{n+1} = s|Y_n = s) = 1-p$$

for  $n = 0, 1, 2, \dots$  and  $0 < p < 1$ .

It can be shown that

$$p_{ij}(m) = \mathbb{P}(Y_{n+m} = j|Y_n = i) = \binom{m}{j-i} p^{j-i} (1-p)^{m-(j-i)}, \quad i, j \in \mathbb{N}, \quad 0 \leq j-i \leq m.$$

and  $m - (j-i)$  failures.

- i) Is the chain  $\{Y_n\}$  irreducible? Justify your answer.  
ii) For  $i \in \{0, 1, 2, \dots\}$ , determine an expression for

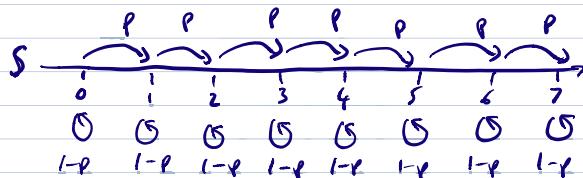
$$\sum_{m=0}^{\infty} p_{ii}(m).$$

On the basis of this expression only, deduce whether each of the states within  $S$  is transient or recurrent.

[4]

Question 2a)

Q2a)  $S$  is countably infinite.  $S = \{0\} + \mathbb{Z}t$ .



(i) Is  $\{Y_n\}$  irreducible?

Criteria:  $i \leftrightarrow j, \forall i, j \in S$

Two ways of showing that  $i$  and  $j$  are intercommunicating for all  $i, j \in S$ :

$$\textcircled{1} \quad p_{ij}(n) > 0 \text{ for } n \geq 0 \quad \textcircled{2} \quad p_{ji}(m) > 0 \text{ for } m \geq 0;$$

$$\textcircled{2} \quad f_{ij} > 0 \text{ and } f_{ji} > 0, \forall i, j \in S.$$

It's clear to see that whilst it's possible to get from any state to an equal or "larger" state in  $n$  steps ( $n \in \mathbb{N}$ ), it is impossible to get from a larger state to a smaller state. ✓ with +ve probability.

2

$$\text{Ex. } i = 3, j = 2.$$

$$f_{ij} = f_{32} = \sum_{n=1}^{\infty} f_{32}(n) = 0$$

Therefore  $\{Y_n\}$  is NOT irreducible. ✓

(ii) For  $i \in \{0, 1, 2, \dots\}$ , determine expression for:

$$\sum_{m=0}^{\infty} p_{ii}(m).$$

Will be looking to show whether:  $\sum_{m=0}^{\infty} p_{ii}(m) = \infty$  ( $i$  is recurrent)

$\Leftrightarrow \sum_{m=0}^{\infty} p_{ii}(m) < \infty$  ( $i$  is transient)

$$p_{ij}(n) = P(Y_{n+m} = j \mid Y_n = i) = \binom{m}{j-i} p^{j-i} (1-p)^{m-(j-i)}$$

$$p_{ii}(n) = P(Y_{n+m} = i \mid Y_n = i) = \binom{m}{0} p^0 (1-p)^m = (1-p)^m$$

$$\sum_{m=0}^{\infty} p_{ii}(m) = \sum_{m=0}^{\infty} (1-p)^m = \sum_{m=0}^{\infty} q^m, \text{ where } 0 < q < 1,$$

$$q = 1-p,$$

and this is the geometric series.

$$\sum_{m=0}^{\infty} q^m = \frac{1}{1-q} \quad \text{which converges when } |q| < 1, \text{ which it does.}$$

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$$\therefore \sum_{m=0}^{\infty} p_{ii}(m) = \sum_{m=0}^{\infty} (1-p)^m = \sum_{m=0}^{\infty} q^m = \frac{1}{1-q} = \frac{1}{p}$$

Since  $\sum_{m=0}^{\infty} p_{ii}(m) = \frac{1}{p} < \infty$ , for  $\forall i \in S, 0 < p < 1$ , we can

say that all states within state space  $S$  are transient. ✓

(b) Now consider a new Markov chain  $\{X_n\}$  with state space  $S' = \{1, 2, 3, 4\}$  such that

$$X_n = \left( Y_n - 4 \left\lfloor \frac{Y_n}{4} \right\rfloor \right) + 1$$

with  $p = \frac{1}{3}$ , where  $\lfloor z \rfloor$  represents the integer part of  $z$  (i.e. round  $z$  down to the nearest whole number).

- Write down the one-step transition matrix,  $P$ , for  $\{X_n\}$ . [3]
- Briefly explain, with justification, why the chain is:  
(I) irreducible; (II) positive recurrent; (III) aperiodic. [3]
- Suppose that at time  $k$ , the chain resides in one of the four states according to the mass function  $\pi^{(k)} = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$ . For the mass function at time  $k+2$  given by  $\pi^{(k+2)} = (a, b, c, d)$ , determine the numerical values of  $a, b, c$  and  $d$ , expressed as fractions, each in its simplest possible form. [4]
- Determine the value of the limiting probability that the chain  $\{X_n\}$  resides in state 3, i.e.

$$\lim_{n \rightarrow \infty} \pi_3^{(n)}.$$

[4]

Question 2b)

Q2b) State space for  $X = \{X_n\}$  is finite,  $S' = \{1, 2, 3, 4\}$

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix}$$

$$p_{ij} = P_{ij}(1) = P(X_i=j | X_0=i)$$

→ There's a mapping between  $X_n$  and  $Y_n$ .  
→ we have the  $p_{ij}$ 's for  $\{Y_n\}$

→ probability that  $\{Y_n\}$  goes to next state is  $\frac{1}{3}$ .  
→ probability stays where it is is  $\frac{2}{3}$ .

Example by hand.

$$X_0 = \left( Y_0 - 4 \left\lfloor \frac{Y_0}{4} \right\rfloor \right) + 1 = \left( 0 - 4 \left\lfloor \frac{0}{4} \right\rfloor \right) + 1 = 1$$

So,  $X_0 = 1 \rightarrow \{X_n\}$  starts at state 1.

seen

If  $Y_n$  in state 1:  $\text{D}_{\frac{2}{3}}$

$$X_n = \left(1 - 4\left\lfloor \frac{1}{4} \right\rfloor\right) + 1 = 2$$

If  $Y_n$  in state 2:  $\text{D}_{\frac{2}{3}}$

$$X_n = \left(2 - 4\left\lfloor \frac{2}{4} \right\rfloor\right) + 1 = 3$$

If  $Y_n$  in state 3:  $\text{D}_{\frac{2}{3}}$

$$X_n = \left(3 - 4\left\lfloor \frac{3}{4} \right\rfloor\right) + 1 = 4$$

If  $Y_n$  in state 4:  $\text{D}_{\frac{2}{3}}$

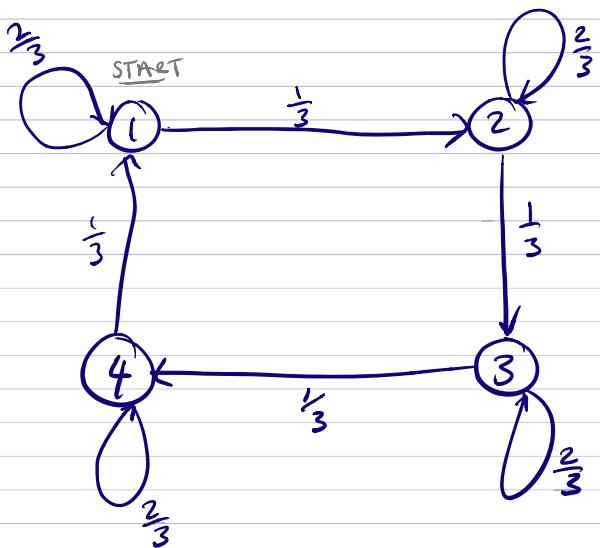
$$X_n = \left(4 - 4\left\lfloor \frac{4}{4} \right\rfloor\right) + 1 = 1$$

And lastly, if  $Y_n$  in state 5:

$$X_n = \left(5 - 4\left\lfloor \frac{5}{4} \right\rfloor\right) + 1 = 2.$$

- If  $\{Y_n\}$  changes state w/ probability  $P = \frac{1}{3}$ , so too does  $\{X_n\}$

- If  $\{Y_n\}$  does NOT change state,  $1-P = \frac{2}{3}$ , then neither does  $\{X_n\}$ .



Therefore we can write the transition matrix for  $\{X_n\}$ :

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{bmatrix}$$

3

Sanity check: expect all rows to sum to 1.

$$\hookrightarrow \sum_j p_{ij} = 1 \quad \text{for } i=1,2,3,4 \quad \checkmark$$

determined by inspection...presumably.

b) Explain why chain is (I) irreducible; (II) positive recurrent; (III) aperiodic.

(I)  $\{X_n\}$  is irreducible because for every state in  $S' = \{1, 2, 3, 4\}$ ,

$i \leftrightarrow j, \forall i, j \in S'$ . This means that every state in the statespace is fully intercommunicating. It's possible to reach each state from any other state, whereby it's clear that  $f_{ij} > 0$  if  $i, j \in S'$  by the fully connected diagram in a).

(II)  $\{X_n\}$  has a finite subspace  $S'$  ( $N=4$ ), and therefore every recurrent state must be positive recurrent.

We also know that at least one of the states must be recurrent (because we have a finite statespace).

But if  $i$  is recurrent, then  $j$  must ALSO be recurrent if  $i \leftrightarrow j$ . And because  $i \leftrightarrow j, \forall i, j \in S'$ , then every state must be recurrent, and therefore every state in  $S'$  must be positive recurrent.

It's also very clear that  $S'$  is a closed set, featuring only recurrent states, and no transient states.

$$\hookrightarrow S' = \pi \cup \emptyset = \emptyset.$$

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(III) We know from irreducible Markov Chain property that because  $i \leftrightarrow j, \forall i, j \in S'$ , that  $d(i)$  is the same  $\forall i \in S'$ . i.e. every state has the same period.

So we need to calculate the period of one state and apply it to all of them.

$$d(i) = \text{gcd} \{ n : p_{ii}(n) > 0 \}$$

Let's pick state 2:

Possible paths for  $2 \rightarrow 2$ :

$$2 \rightarrow 3 \rightarrow 4 \rightarrow 1 \rightarrow 2, n = 4$$

$$2 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1 \rightarrow 2, n = 5$$

$$\text{Therefore } d(2) = d(1) = d(3) = d(4) = \text{gcd}\{4, 5, 6, 7, 8, \dots\} = 1.$$

So because state 2 (and thus ALL states in  $S'$ ) have a periodicity of 1, we say the chain is APERIODIC.

$\hookrightarrow$  Note: because POSITIVE RECURRENT + APERIODIC = ERGODIC, we could also classify  $\{X_n\}$  as ergodic, too.

$$(iii) \quad \underline{\pi}^{(k)} = \left[ \frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6} \right]$$

$$\therefore \pi_1^{(k)} = \frac{1}{2}$$

$$\pi_2^{(k)} = \frac{1}{6}$$

$$\pi_3^{(k)} = \frac{1}{6}$$

$$\pi_4^{(k)} = \frac{1}{6}$$

$$\underline{\pi}^{(k+2)} = [a, b, c, d] \rightarrow \text{find } a, b, c, d.$$

Recall:  $\underline{\pi}^{(n+m)} = \underline{\pi}^{(n)} p^m = \underline{\pi}^{(n)} p^n$  [Result falls out of applying Chebyshev-Kolmogorov]  
 $\rightarrow \underline{\pi}^{(k+2)} = \underline{\pi}^{(k)} p^2$

Therefore two steps to this: ① Calculate  $p^2$   
 ②  $\underline{\pi}^k p^2$ .

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{bmatrix}$$

$$① \quad p^2 = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{4}{9} & \frac{4}{9} & \frac{1}{9} & 0 \\ 0 & \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{1}{9} & 0 & \frac{4}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{1}{9} & 0 & \frac{4}{9} \end{bmatrix} \quad \begin{array}{l} \sum_j = 1 \\ \sum_j = 1 \\ \sum_j = 1 \\ \sum_j = 1 \end{array}$$

②

$$\begin{pmatrix} \frac{4}{9} & \frac{4}{9} & \frac{1}{9} & 0 \\ 0 & \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{1}{9} & 0 & \frac{4}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{1}{9} & 0 & \frac{4}{9} \end{pmatrix}$$

$$\tilde{\pi}^{(k+2)} P^2 = \left[ \frac{1}{2} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \right] \left[ \frac{17}{54} \quad \frac{17}{54} \quad \frac{11}{54} \quad \frac{1}{6} \right]$$

$$\sum_i \tilde{\pi}_i^{(k+2)} = 1 \checkmark$$

$$\frac{1}{54} + \frac{4}{54} + \frac{4}{54} = \frac{9}{54} = \frac{1}{6}$$

$$\frac{4}{18} + \frac{1}{54} + \frac{4}{54} = \frac{12}{54} + \frac{1}{54} + \frac{4}{54} = \frac{17}{54}$$

$$\frac{1}{18} + \frac{4}{54} + \frac{4}{54} = \frac{3}{54} + \frac{4}{54} + \frac{4}{54} = \frac{11}{54}$$

$$\frac{2}{9} + \frac{4}{54} + 0 + \frac{1}{54} = \frac{12}{54} + \frac{4}{54} + \frac{1}{54} = \frac{17}{54}$$

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$$\text{Therefore } \tilde{\pi}^{(k+2)} = \left[ \frac{17}{54} \quad \frac{17}{54} \quad \frac{11}{54} \quad \frac{1}{6} \right],$$

$$\text{where } a = \frac{17}{54}$$

$$b = \frac{17}{54}$$

$$c = \frac{11}{54}$$

$$d = \frac{1}{6}.$$

(iv) Try the  $\underline{\pi} = \underline{\pi} P$  route.

$$\underline{\pi} = \underline{\pi} P \quad \checkmark \quad \underline{\pi} = [\pi_1, \pi_2, \pi_3, \pi_4]$$

$$\underline{\pi} I - \underline{\pi} P = \underline{0}$$

$$[\pi_1, \pi_2, \pi_3, \pi_4] \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{bmatrix} \right\} = \underline{0}$$

$$\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$\boxed{\text{Know } \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1}$$

$$[\pi_1, \pi_2, \pi_3, \pi_4] \begin{bmatrix} ① & ② & ③ & ④ \end{bmatrix} = [0 \ 0 \ 0 \ 0]$$

$$① \frac{\pi_1}{3} - \frac{\pi_4}{3} = 0 \rightarrow \pi_1 = \pi_4$$

$$② -\frac{\pi_1}{3} + \frac{\pi_2}{3} = 0 \rightarrow \pi_1 = \pi_2$$

$$③ -\frac{\pi_2}{3} + \frac{\pi_3}{3} = 0 \rightarrow \pi_2 = \pi_3$$

$$④ -\frac{\pi_3}{3} + \frac{\pi_4}{3} = 0 \rightarrow \pi_3 = \pi_4$$

Therefore,  $\pi_1 = \pi_2 = \pi_3 = \pi_4$  and  $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$ .

$$\therefore 4\pi_i = 1$$

$$\pi_1 = \frac{1}{4}$$

$$\pi_2 = \frac{1}{4}$$

$$\pi_3 = \frac{1}{4}$$

$$\pi_4 = \frac{1}{4}$$

$$\underline{\pi} = [\pi_1 \ \pi_2 \ \pi_3 \ \pi_4] = \left[ \frac{1}{4} \ \frac{1}{4} \ \frac{1}{4} \ \frac{1}{4} \right]$$

And thus:

$$\lim_{n \rightarrow \infty} \pi_3^{(n)} = \frac{1}{4}$$

Which you'd intuitively expect, as you have four equally likely states whereby you either stick with  $\frac{2}{3}$  or twist to next state with  $\frac{1}{3}$  chance.

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END OF ASSIGNMENT. CHESTON ALLAN.



I guess so: I never thought of it like that.

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