B.Sc./Grad. Dip.: Probability Models and Time Series
MAS programmes: Stochastic Models and Time Series

Solutions 7

1. Trying to prove that

$$Y_{T+h} = ch + Y_T + \theta\epsilon_T + \epsilon_{T+h} + (1+\theta)\sum_{i=1}^{h-1}\epsilon_{T+h-i} \qquad h \geq 1. \quad (1)$$

Setting $t = T+1$ in the model equation,

$$Y_{T+1} = c + Y_T + \theta\epsilon_T + \epsilon_{T+1},$$

so that the required result holds for $h = 1$. Now assume that the result holds for some given $h \geq 1$. Setting $t = T+h+1$ in the model equation and using the result for the given h,

$$\begin{aligned} Y_{T+h+1} &= c + Y_{T+h} + \epsilon_{T+h+1} + \theta\epsilon_{T+h} \\ &= c + [ch + Y_T + \theta\epsilon_T + \epsilon_{T+h} + (1+\theta)\sum_{i=1}^{h-1}\epsilon_{T+h-i}] + \epsilon_{T+h+1} + \theta\epsilon_{T+h} \\ &= c(h+1) + Y_T + \theta\epsilon_T + \epsilon_{T+h+1} + (1+\theta)\sum_{i=1}^{h}\epsilon_{T+h+1-i}. \end{aligned}$$

Thus the result holds also for $h+1$ and the induction proof is complete.

It follows that

$$\hat{y}_T(h) = ch + Y_T + \theta\epsilon_T \quad (2)$$

and

$$e_T(h) = \epsilon_{T+h} + (1+\theta)\sum_{i=1}^{h-1}\epsilon_{T+h-i}. \quad (3)$$

2. (a) Just to note first of all that the ARIMA(3,1,0) model appears to provide an adequate fit to the data. Some of the output for the ARIMA(3,1,0) model was given in Solutions 6.

From the lectures notes, we know that the general form of the forecast function is

$$\hat{y}_T(h) = \sum_{k=1}^{p} A_k \alpha_k^h + \sum_{k=0}^{d-1} B_k h^k + \frac{c}{d!\,(1 - \sum_{k=1}^{p} \phi_k)}\ h^d,$$

where p is the number of autoregressive terms, d the order of differencing and c the constant term in the model, if present.

For the ARIMA(3,1,0) model, this becomes

$$\hat{y}_T(h) = B_0 + A_1\alpha_1^h + A_2\alpha_2^h + A_3\alpha_3^h$$

which involves a sum of geometric terms, which leads to a relatively complex oscillatory behaviour. The values B_0, A_1, A_2, A_3 would need to be calibrated against the data to hand.

(b) From the above expression, we can see that, in view of the fact that $|\alpha_k| < 1$, $k = 1, 2$ and 3, $\hat{y}_T(h) \longrightarrow B_0$ as $h \longrightarrow \infty$. Thus we deduce that the eventual forecast function is a constant; the value of the constant can be obtained from the sequence of forecasts for large lead times.

```
> sheep310.fore <- predict(sheep310, 70)
> sheep310.fore
$pred
Time Series:
Start = 1940
End = 2009
Frequency = 1
 [1] 1777.996 1718.869 1695.985 1704.067 1730.084 1746.371 1745.518 1733.953
 [9] 1724.299 1722.829 1727.678 1732.954 1734.644 1732.815 1730.098 1728.809
[17] 1729.371 1730.695 1731.531 1731.445 1730.837 1730.344 1730.286 1730.545
[25] 1730.817 1730.896 1730.796 1730.655 1730.592 1730.624 1730.693 1730.735
[33] 1730.729 1730.697 1730.672 1730.670 1730.684 1730.698 1730.701 1730.696
[41] 1730.689 1730.685 1730.687 1730.691 1730.693 1730.693 1730.691 1730.690
[49] 1730.690 1730.690 1730.691 1730.691 1730.691 1730.691 1730.690 1730.690
[57] 1730.691 1730.691 1730.691 1730.691 1730.691 1730.691 1730.691 1730.691
[65] 1730.691 1730.691 1730.691 1730.691 1730.691 1730.691
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$se
Time Series:
Start = 1940
End = 2009
Frequency = 1
 [1]  69.16201 120.17563 154.17498 168.88593 175.79008 181.80851 190.79519
 [8] 202.99565 215.56924 226.07118 234.44444 241.87221 249.43197 257.49084
[15] 265.73259 273.65273 281.03090 287.98783 294.76553 301.51769 308.24405
[22] 314.85685 321.28551 327.52745 333.62837 339.63517 345.56615 351.41290
[29] 357.16045 362.80410 368.35205 373.81738 379.20937 384.53077 389.78062
[36] 394.95859 400.06699 405.11000 410.09172 415.01489 419.88100 424.69117
[43] 429.44690 434.15014 438.80298 443.40724 447.96440 452.47570 456.94237
[50] 461.36568 465.74695 470.08742 474.38823 478.65043 482.87500 487.06290
[57] 491.21508 495.33246 499.41590 503.46624 507.48425 511.47070 515.42631
[64] 519.35179 523.24782 527.11506 530.95413 534.76564 538.55018 542.30831
```

The forecasts for the years 1940 to 1944 are 1777.996, 1718.869, 1695.985, 1704.067 and 1730.084, respectively.

We find that the forecast converges to 1730.691.