B.Sc./Grad. Dip.: Probability Models and Time Series
MAS programmes: Stochastic Models and Time Series

Solutions 4

1. In the Cauchy-Schwarz inequality, put $n = T - \tau$ and

$$a_t = y_{t+\tau} - \bar{y},\ b_t = y_t - \bar{y}, \qquad 1 \le t \le T - \tau.$$

It then follows that

$$\begin{aligned} T^2 c_\tau^2 &= \left(\sum_{t=1}^{T-\tau} a_t b_t\right)^2 \\ &\le \sum_{t=1}^{T-\tau} a_t^2 \sum_{t=1}^{T-\tau} b_t^2 \\ &= \sum_{t=\tau+1}^{T} (y_t - \bar{y})^2 \sum_{t=1}^{T-\tau} (y_t - \bar{y})^2 \\ &\le \sum_{t=1}^{T} (y_t - \bar{y})^2 \sum_{t=1}^{T} (y_t - \bar{y})^2 \\ &= T^2 c_0^2. \end{aligned}$$

2. (a) The autocorrelation function declines geometrically with ratio -0.4. The model that gives rise to this is the AR(1) model with parameter $\phi = -0.4$.

   (b) The cut-off in the autocorrelation function at lag 1 indicates a MA(1) model. The parameter θ must satisfy

$$\frac{\theta}{1 + \theta^2} = 0.4.$$

Hence

$$0.4\theta^2 - \theta + 0.4 = 0,$$

a quadratic equation with roots $\theta = 0.5$ and 2. To give an invertible model, we may choose the parameter value $\theta = 0.5$.

3. Using the Cauchy-Schwarz inequality,

$$\begin{aligned} \gamma_\tau &= \sigma^2 \sum_{j=\tau}^{\infty} \psi_j \psi_{j-\tau} \\ &\le \sigma^2 (\sum_{j=\tau}^{\infty} \psi_j^2)^{1/2} (\sum_{j=0}^{\infty} \psi_j^2)^{1/2} \\ &\to 0 \end{aligned}$$

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as $\tau \to \infty$, since

$$\sum_{j=0}^{\infty} \psi_j^2 < \infty$$

and

$$\sum_{j=\tau}^{\infty} \psi_j^2 \to 0$$

so, when $\sum_{\to\infty}^{\infty}$ something, this appears to $\to 0$

as $\tau \to \infty$.

4. Writing $\rho_1(\theta)$ to indicate explicitly the dependence of ρ_1 upon θ, recall that $\rho_1(\theta) = \theta/(1+\theta^2)$. Recall also that $\rho_1(\theta) = \rho_1(1/\theta)$, so that to investigate the range of possible values of $\rho_1(\theta)$ it is sufficient to restrict attention to values of θ in the closed interval [-1,1]. From the expression

$$\boxed{\frac{d\rho_1}{d\theta} = \frac{1-\theta^2}{(1+\theta^2)^2},}$$

it follows that $\rho_1(\theta)$ is strictly monotonic increasing on [-1,1]. But $\rho_1(-1) = -1/2$ and $\rho_1(1) = 1/2$. Hence the range of possible values of ρ_1 is the closed interval [-1/2,1/2]. If the MA(1) process is to be invertible then the range of values of θ is restricted to the open interval (-1,1) and hence the range of possible values of ρ_1 to the open interval (-1/2,1/2).