
MATHEMATICS AND STATISTICS ASSIGNMENT COVER SHEET

Student to Complete

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MSc. Applied Statistics

Module:

PROBABILITY MODELS AND TIME SERIES [B.Sc./Grad. Dip]
STOCHASTIC MODELS AND TIME SERIES [MAS Programmes]

Lecturer:

BROOMS

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ASSIGNMENT C [B.Sc./Grad. Dip]/ ASSIGNMENT TWO [MAS]

Due Date:

Thursday, 30th April, 2020

Declaration

I hereby testify that, unless otherwise acknowledged, the work submitted here is entirely my own

Signature:



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Date Received:

(Unscaled) Mark/Grade	Late – up to 14 days	Late – more than 14 days
25 out of 25		

Christian Gilsen - 30 April 2020 Time Series Assignment

Deadline: Thursday, 30th April, 2020

Total marks: [25]. Marks are shown in boxes []. There are 2 questions in this assignment.

1. Consider a process $\{Y_t\}$ that satisfies the model equation

AR(3) model

$$Y_t = \frac{21}{5}Y_{t-1} - \frac{19}{5}Y_{t-2} + \frac{3}{5}Y_{t-3} + \epsilon_t, \quad t \in \mathbb{Z}.$$

- (a) Find the equation satisfied by the process, $\{W_t\}$, of first differences, where $W_t = Y_t - Y_{t-1}$. [2]
- (b) Is $\{W_t\}$ a stationary process? [3]

Question 1

1a) Subtract Y_{t-1} from both sides

$$W_t = Y_t - Y_{t-1} = \frac{21}{5}Y_{t-1} - \frac{5}{5}Y_{t-1} - \frac{19}{5}Y_{t-2} + \frac{3}{5}Y_{t-3} + \epsilon_t$$

$$= \frac{16}{5}Y_{t-1} - \frac{16}{5}Y_{t-2} - \frac{3}{5}Y_{t-2} + \frac{3}{5}Y_{t-3} + \epsilon_t$$

$$= \frac{16}{5}(Y_{t-1} - Y_{t-2}) - \frac{3}{5}(Y_{t-2} - Y_{t-3}) + \epsilon_t$$

2
Clearly, if $W_t = Y_t - Y_{t-1}$, then $W_{t-1} = Y_{t-1} - Y_{t-2}$, and $W_{t-2} = Y_{t-2} - Y_{t-3}$

$$\therefore W_t = \frac{16}{5}W_{t-1} - \frac{3}{5}W_{t-2} + \epsilon_t$$

$\hookrightarrow W_t = Y_t - Y_{t-1} = Y_t - LY_t$
 $= (1-L)Y_t$
 $= \Delta Y_t.$

b) Necessary and sufficient condition for stationary process:

- All roots of AR characteristic eq. $\phi(z)$ are greater than unit length in the complex plane.

AR characteristic polynomial equation:

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p = 0$$

$$\phi_1 = \frac{16}{5}, \quad \phi_2 = -\frac{3}{5}$$

[Subbing ϕ_1 and ϕ_2 from $\{W_t\}$ equations into AR characteristic poly.]

$$1 - \frac{16}{5}z - \left(-\frac{3}{5}\right)z^2 = 0$$

$$\textcircled{1} \quad \frac{3}{5}z^2 - \frac{16}{5}z + 1 = 0$$

[Multiply \textcircled{1} through by 5]

$$\textcircled{2} \quad 3z^2 - 16z + 5 = 0$$

$$(3z - 1)(z - 5) = 0$$

Check: $3z^2 - 15z - z + 5 = 0$

$$\therefore 3z_1 = 1 \quad , \quad z_2 = 5$$

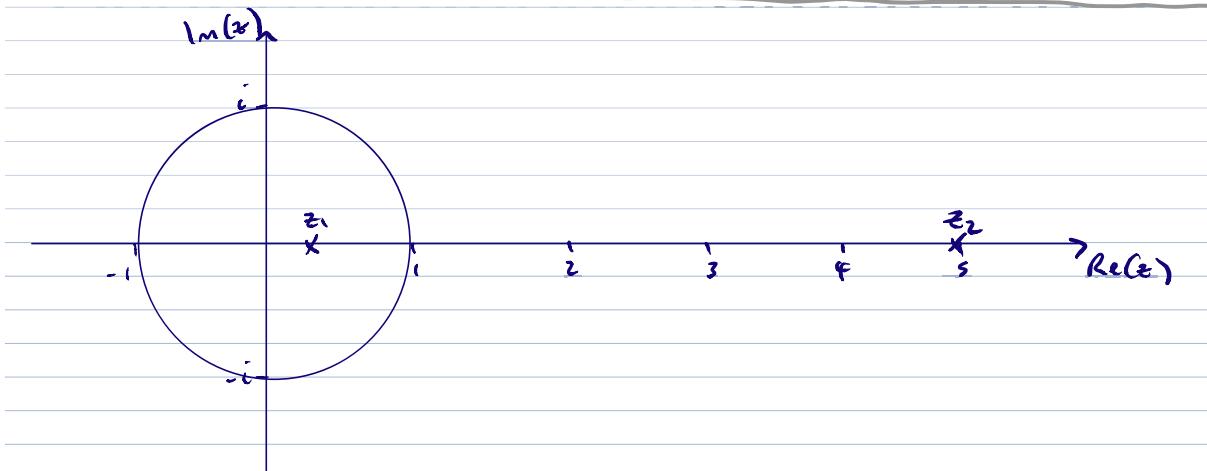
$$z_1 = \frac{1}{3}$$

5/5

3

So, we have 2 real roots: $z_1 = \frac{1}{3}$ and $z_2 = 5$.

As not ALL roots are greater than 1, $\{w_n\}$ is NOT a stationary process.



2. Let $\{Z_t\}$ be an ARIMA(1,0,3) process which satisfies the relation $= \text{ARIMA}(1,0,3)$

$$(1 - \phi L)Z_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \theta_3 \epsilon_{t-3}, \quad t \in \mathbb{Z},$$

where L is the lag operator such that $LZ_t = Z_{t-1}$, $\{\epsilon_t\}$ is a white noise process such that the ϵ_t are NID($0, \sigma^2$), ϕ is the autoregressive parameter, and θ_1, θ_2 , and θ_3 , are moving average parameters. Let z_1, z_2, \dots, z_T denote an observed realization of the process up to time T . Consider the forecasting problem with T the origin of the forecasts, h the lead time and $\hat{z}_T(h)$ the corresponding minimum mean square error forecast, for lead times $h \geq 1$.

- (a) Dede that an iteration scheme for the forecasts can be characterized by the following relations:

$$\hat{z}_T(1) = \phi z_T + \theta_1 \epsilon_T + \theta_2 \epsilon_{T-1} + \theta_3 \epsilon_{T-2}$$

$$\hat{z}_T(2) = \phi \hat{z}_T(1) + \theta_2 \epsilon_T + \theta_3 \epsilon_{T-1}$$

$$\hat{z}_T(3) = \phi \hat{z}_T(2) + \theta_3 \epsilon_T$$

$$\hat{z}_T(h) = \phi \hat{z}_T(h-1), \quad h \geq 4.$$

[8]

Question 2a

a) Recall minimum mean square forecast equation:

$$\hat{z}_T(h) = E[Z_{T+h} | \mathcal{H}_T]$$

Plan: ① First we want to express the ARIMA(1,0,3) process in terms of Z_{T+h}

② Apply minimum mean square forecast equation

③ produce expressions at $h=1, 2, 3, 4$

(Could probably attack this using the recursive method which also produces the forecast error terms - but think that'll be overkill for this Q)

$$① (1 - \phi L)Z_t = Z_t - \phi L Z_t = Z_t - \phi Z_{t-1} = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \theta_1 \epsilon_{t-2} + \theta_3 \theta_2 \epsilon_{t-3}$$

$$Z_t = \phi Z_{t-1} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \theta_3 \epsilon_{t-3}$$

[in terms of Z_{T+h} : $t \rightarrow T+h$]

$$Z_{T+h} = \phi Z_{T+h-1} + \epsilon_{T+h} + \theta_1 \epsilon_{T+h-1} + \theta_2 \epsilon_{T+h-2} + \theta_3 \epsilon_{T+h-3}$$

[now take conditional expectation w.r.t. the observed history, \mathcal{H}_T]

$$\begin{aligned} \textcircled{2} \quad \hat{z}_T(h) &= E[z_{T+h} | \mathcal{H}_T] \\ &= E[\phi z_{T+h-1} + \varepsilon_{T+h} + \theta_1 \varepsilon_{T+h-1} + \theta_2 \varepsilon_{T+h-2} + \theta_3 \varepsilon_{T+h-3} | \mathcal{H}_T] \end{aligned}$$

\textcircled{3} Let's go through this by hand for $h=1, 2, 3, 4$, but first noting what the conditional expectations will look like for the z terms and the ε white noise terms:

$$E[z_{T+j} | \mathcal{H}_T] = \begin{cases} z_{T+j}, & j \leq 0 \quad (\text{realised observation}) \\ \hat{z}_T(j), & j > 0 \quad (\text{forecast value}) \end{cases}$$

$$E[\varepsilon_{T+j} | \mathcal{H}_T] = \begin{cases} \varepsilon_{T+j}, & j \leq 0 \quad (\text{realised observation, NOT a random variable}) \\ 0, & j > 0 \quad (\varepsilon_{T+j} \text{ independent of } \mathcal{H}_T, \text{ therefore: } E[\varepsilon_{T+j} | \mathcal{H}_T] = E[\varepsilon_{T+j}], \text{ and } \varepsilon_{T+j} \text{ is a mean zero white noise term, so } E[\varepsilon_{T+j}] = 0). \end{cases}$$

When $h=1$:

$$\begin{aligned} \hat{z}_T(1) &= E[\phi z_T + \varepsilon_{T+1} + \theta_1 \varepsilon_T + \theta_2 \varepsilon_{T-1} + \theta_3 \varepsilon_{T-2} | \mathcal{H}_T] \\ &= \phi E[z_T | \mathcal{H}_T] + \cancel{E[\varepsilon_{T+1} | \mathcal{H}_T]} + \theta_1 E[\varepsilon_T | \mathcal{H}_T] \\ &\quad + \theta_2 E[\varepsilon_{T-1} | \mathcal{H}_T] + \cancel{\theta_3 E[\varepsilon_{T-2} | \mathcal{H}_T]} \end{aligned}$$

$$\hat{z}_T(1) = \phi z_T + \theta_1 \varepsilon_T + \theta_2 \varepsilon_{T-1} + \theta_3 \varepsilon_{T-2}$$

when $h=2$:

$$\begin{aligned}\hat{z}_T(2) &= E[\phi z_{T+1} + \varepsilon_{T+2} + \theta_1 \varepsilon_{T+1} + \theta_2 \varepsilon_T + \theta_3 \varepsilon_{T-1} | \mathcal{H}_T] \\ &= \phi E[z_{T+1} | \mathcal{H}_T] + E[\varepsilon_{T+2} | \mathcal{H}_T] + \cancel{\theta_1 E[\varepsilon_{T+1} | \mathcal{H}_T]} \\ &\quad + \theta_2 E[\varepsilon_T | \mathcal{H}_T] + \cancel{\theta_3 E[\varepsilon_{T-1} | \mathcal{H}_T]}\end{aligned}$$

$$\hat{z}_T(2) = \phi \hat{z}_T(1) + \theta_2 \varepsilon_T + \theta_3 \varepsilon_{T-1}$$

when $h=3$:

$$\begin{aligned}\hat{z}_T(3) &= E[\phi z_{T+2} + \varepsilon_{T+3} + \theta_1 \varepsilon_{T+2} + \theta_2 \varepsilon_{T+1} + \theta_3 \varepsilon_T | \mathcal{H}_T] \\ &= \phi E[z_{T+2} | \mathcal{H}_T] + E[\varepsilon_{T+3} | \mathcal{H}_T] + \cancel{\theta_1 E[\varepsilon_{T+2} | \mathcal{H}_T]} \\ &\quad + \cancel{\theta_2 E[\varepsilon_{T+1} | \mathcal{H}_T]} + \theta_3 E[\varepsilon_T | \mathcal{H}_T]\end{aligned}$$

$$\hat{z}_T(3) = \phi \hat{z}_T(2) + \theta_3 \varepsilon_T$$

when $h=4$:

$$\begin{aligned}\hat{z}_T(4) &= E[\phi z_{T+3} + \varepsilon_{T+4} + \theta_1 \varepsilon_{T+3} + \theta_2 \varepsilon_{T+2} + \theta_3 \varepsilon_{T+1} | \mathcal{H}_T] \\ &= \phi E[z_{T+3} | \mathcal{H}_T] + E[\varepsilon_{T+4} | \mathcal{H}_T] + \cancel{\theta_1 E[\varepsilon_{T+3} | \mathcal{H}_T]} \\ &\quad + \cancel{\theta_2 E[\varepsilon_{T+2} | \mathcal{H}_T]} + \cancel{\theta_3 E[\varepsilon_{T+1} | \mathcal{H}_T]}\end{aligned}$$

$$\hat{z}_T(4) = \phi \hat{z}_T(3)$$

superfluous in view of what you do
on the next page

Generalising when $h \geq 4$:

Can see how for $h \geq 4$, that all of the moving average parameters disappear, and therefore the forecast for the next (h) time step will be the current ($h-1$) forecast multiplied by the autoregressive parameter, ϕ .

$$\therefore \hat{z}_T(h) = \phi \hat{z}_T(h-1), h \geq 4$$

should write this out like you did with $h=1, 2, 3$.
But, on balance,
it's good enough.

Summarising above result:

$$\hat{z}_T(h) = \begin{cases} \phi z_T + \theta_1 \varepsilon_T + \theta_2 \varepsilon_{T-1} + \theta_3 \varepsilon_{T-2}, & h=1 \\ \phi \hat{z}_T(1) + \theta_2 \varepsilon_T + \theta_3 \varepsilon_{T-1}, & h=2 \\ \phi \hat{z}_T(2) + \theta_3 \varepsilon_T, & h=3 \\ \phi \hat{z}_T(h-1), & h \geq 4 \end{cases}$$

- (b) By working with an appropriately explicit expression for Z_{T+2} , derive an explicit formula for each of the following:

$e_T(2)$

- i) the forecast error at lead time $h = 2$, formulated in terms of one or more members of the white noise process; [6]

$v(2)$

- ii) the forecast error variance at lead time $h = 2$, namely $V(2)$ (involving nothing more than σ^2 , θ_1 , and ϕ). [2]

b)

Question 2b

(i) Recall forecast error: $e_T(h) = Z_{T+h} - \hat{Z}_T(h)$

o want $e_T(2)$, therefore want Z_{T+2} , can use $\hat{Z}_T(2)$ from (a).

o will want to be able to do $Z_{T+2} - \hat{Z}_T(2)$, so we'll want to eliminate Y_t terms, and thus have comparable values that we can subtract from another. Can't do $Y_{T+1} - \hat{Y}_T(1)$, but can do $Y_t - y_T$ (since they're both realised observations).

o So will put $\hat{Z}_T(2)$ in terms of z_T , and Z_{T+2} in terms of Z_T .

$\hat{Z}_T(2)$

$$\hat{Z}_T(2) = \phi \hat{Z}_T(1) + \theta_1 \varepsilon_T + \theta_2 \varepsilon_{T-1}$$

[sub $\hat{Z}_T(1) = \phi z_T + \theta_1 \varepsilon_T + \theta_2 \varepsilon_{T-1} + \theta_3 \varepsilon_{T-2}$ into above]

$$\hat{Z}_T(2) = \phi (\phi z_T + \theta_1 \varepsilon_T + \theta_2 \varepsilon_{T-1} + \theta_3 \varepsilon_{T-2}) + \theta_2 \varepsilon_T + \theta_3 \varepsilon_{T-1}$$

$$\begin{aligned} \hat{Z}_T(2) &= \phi^2 z_T + \phi \theta_1 \varepsilon_T + \phi \theta_2 \varepsilon_{T-1} + \phi \theta_3 \varepsilon_{T-2} \\ &\quad + \theta_2 \varepsilon_T + \theta_3 \varepsilon_{T-1} \end{aligned}$$

\hat{z}_{T+2}

$$z_t = \phi z_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3}$$

[$t \rightarrow T+2$]

$$z_{T+2} = \phi z_{T+1} + \varepsilon_{T+2} + \theta_1 \varepsilon_{T+1} + \theta_2 \varepsilon_T + \theta_3 \varepsilon_{T-1}$$

$$z_{T+1} = \underbrace{\phi z_T + \varepsilon_{T+1} + \theta_1 \varepsilon_T + \theta_2 \varepsilon_{T-1} + \theta_3 \varepsilon_{T-2}}_{\check{z}_T} \quad \check{z}_T \checkmark$$

[Sub \check{z}_{T+1} into expression for z_{T+2}]

$$\begin{aligned} z_{T+2} &= \phi(\phi z_T + \varepsilon_{T+1} + \theta_1 \varepsilon_T + \theta_2 \varepsilon_{T-1} + \theta_3 \varepsilon_{T-2}) \\ &\quad + \varepsilon_{T+2} + \theta_1 \varepsilon_{T+1} + \theta_2 \varepsilon_T + \theta_3 \varepsilon_{T-1} \end{aligned}$$

$$\begin{aligned} z_{T+2} &= \phi^2 z_T + \phi \varepsilon_{T+1} + \phi \theta_1 \varepsilon_T + \phi \theta_2 \varepsilon_{T-1} + \phi \theta_3 \varepsilon_{T-2} \\ &\quad + \varepsilon_{T+2} + \theta_1 \varepsilon_{T+1} + \theta_2 \varepsilon_T + \theta_3 \varepsilon_{T-1} \end{aligned}$$

↳ Note: The z_T here is a realized observation from H_T .

Could factorize both expressions for $\hat{z}_T(z)$ and z_{T+2} , but purposefully leaving both expressions as is to make subtracting terms from each other easier.

$$e_T(2) = z_{T+2} - \hat{z}_T(2)$$

$$\begin{aligned} e_T(2) &= \cancel{\phi z_T} + \phi e_{T+1} + \phi \cancel{\theta_1 e_T} + \phi \cancel{\theta_2 e_{T-1}} + \phi \cancel{\theta_3 e_{T-2}} \\ &\quad + e_{T+2} + \theta_1 e_{T+1} + \theta_2 \cancel{e_T} + \theta_3 \cancel{e_{T-1}} \end{aligned}$$

$$\begin{aligned} &- (\cancel{\phi z_T} + \phi \cancel{\theta_1 e_T} + \phi \cancel{\theta_2 e_{T-1}} + \phi \cancel{\theta_3 e_{T-2}} \\ &\quad + \theta_2 \cancel{e_T} + \theta_3 \cancel{e_{T-1}}) \end{aligned}$$

6

$$e_T(2) = \phi e_{T+1} + \theta_1 e_{T+1} + e_{T+2}$$

$$e_T(2) = (\phi + \theta_1) e_{T+1} + e_{T+2}$$

(ii) The forecast error variance at $h=2$, $V(2)$.

$$V(2) = \text{Var}(e_T(2)) = \text{Var}((\phi + \theta_1) e_{T+1} + e_{T+2})$$

The white noise terms are INDEPENDENT: $e_t \sim N(0, \sigma^2)$, and therefore the variance of the sum = sum of the variances.

$$V(2) = \text{Var}((\phi + \theta_1) e_{T+1} + e_{T+2})$$

$$= \text{Var}((\phi + \theta_1) e_{T+1}) + \text{Var}(e_{T+2})$$

$$= (\phi + \theta_1)^2 \text{Var}(e_{T+1}) + \text{Var}(e_{T+2})$$

$$V(2) = (\phi + \theta_1)^2 \sigma^2 + \sigma^2$$

2

(won't factorise this any more as the Q requires the answer to only include ϕ, θ_1, σ^2 .)

(c) Consider a stationary process $\{Y_t\}$ that satisfies the following model equation

ARMA(1,3)

$$Y_t = \phi Y_{t-1} + \epsilon_t + \omega_1 \epsilon_{t-1} + \omega_2 \epsilon_{t-2} + \omega_3 \epsilon_{t-3}, \quad t \in \mathbb{Z},$$

where ϕ, ω_1, ω_2 , and ω_3 , are parameters.

Suppose that $\hat{y}_T(2) = 12.904$, the estimated white noise variance $\hat{\sigma}^2 = 0.297$, and that the estimated values of ϕ, ω_1, ω_2 , and ω_3 , are 0.56, -1.383, 0.533, and -0.0625, respectively.

Stating any assumptions you make, and with reference to appropriate percentage points from statistical tables, find a 99% prediction interval for Y_{T+2} . [4]

Question 2c

(c)

- o So, this is the same ARMA(1,3) model as in (a) and (b).
- o we can use our expression for $\hat{\sigma}^2(2)$ to calculate the estimated forecast error variance.
- o When producing the forecast equation in part (a) and the forecast error variance in (b), we assume that we knew the real parameter values for $\phi, \theta_1, \theta_2, \theta_3$. Actually we just have estimates, $\hat{\phi}, \hat{\omega}_1(\hat{\theta}_1), \hat{\omega}_2(\hat{\theta}_2)$ and $\hat{\omega}_3(\hat{\theta}_3)$. These estimates should be useful for making forecasts if the history, H_T , is deep enough. (The theory often assumes that H_T goes back to the infinite past).
- o We're also assuming that the process $\{Y_t\}$ is normally distributed given that the non-systematic part of the model - the white noise term - are assumed to be normally distributed.
↳ this assumption was declared in the Q, where $\epsilon_t \sim N(0, \sigma^2)$.
- o We therefore use the normal distributions to deduce how many standard errors ($\sqrt{V(h)}$) we will use to form the probability limits.

- we can then calculate the $100(1-\alpha)\%$ prediction interval via:

$$\hat{Y}_T(z) \pm z_{\alpha/2} \sqrt{V(z)}$$

α here is 0.01, and therefore $\alpha/2 = 0.005$

The reason we divide α by 2 is because the normal distribution is symmetric and we looking at the regions of the normal distribution of equal or greater extremeness - which is in both tails.

$$z_{0.005} = 2.5758 \quad (\text{from Lindley & Scott})$$

Could also get the value from R via: `qnorm(0.005, lower.tail=FALSE)`

$$\begin{aligned} V(z) &= (\phi + \theta_1)^2 \sigma^2 + \sigma^2 \\ &= (0.56 - 1.383)^2 \times 0.297 + 0.297 \\ &= 0.4982 \end{aligned}$$

$$\sqrt{V(z)} = 0.7058$$

4

$$12.904 \pm 2.5758 \times 0.4982$$

$$\rightarrow 12.904 \pm 1.818$$

20/20

And thus the 99% prediction interval for Y_{T+2} :

$$[11.086, 14.722]$$

END OF ASSIGNMENT.

