B.Sc./Grad. Dip.: Probability Models and Time Series
MAS programmes: Stochastic Models and Time Series

Solutions 2

1. (a)

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

(b)

$$\mathbf{P}^2 = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{7}{12} & \frac{1}{12} & \frac{1}{4} & \frac{1}{12} \\ \frac{1}{8} & \frac{1}{6} & \frac{7}{24} & \frac{1}{8} & \frac{7}{24} \\ \frac{1}{12} & \frac{1}{3} & \frac{1}{12} & \frac{5}{12} & \frac{1}{12} \\ \frac{1}{8} & \frac{1}{6} & \frac{7}{24} & \frac{1}{8} & \frac{7}{24} \end{pmatrix}$$

Hence, by reading off the entries from the second line-fourth column, and the second line-second column of $\mathbf{P}^2$, respectively, we find that

$$\mathbb{P}(X_2 = D | X_0 = B) = \frac{1}{4}, \qquad \mathbb{P}(X_2 = B | X_0 = B) = \frac{7}{12}.$$

(c) Solving the equation

$$(\pi_A, \pi_B, \pi_C, \pi_D, \pi_E) \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix} = (\pi_A, \pi_B, \pi_C, \pi_D, \pi_E)$$

i.e.

$$(\pi_A, \pi_B, \pi_C, \pi_D, \pi_E) \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & -\frac{1}{3} & 1 & -\frac{1}{3} \\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 \end{pmatrix} = (0, 0, 0, 0, 0).$$

It can be deduced that

$$\pi_B = 4\pi_A, \quad \pi_C = 2\pi_A, \quad \pi_D = 3\pi_A, \quad \pi_E = 2\pi_A.$$

But since $\pi_A + \pi_B + \pi_C + \pi_D + \pi_E = 1$, it follows that

$$(\pi_A, \pi_B, \pi_C, \pi_D, \pi_E) = \left(\frac{1}{12}, \frac{1}{3}, \frac{1}{6}, \frac{1}{4}, \frac{1}{6} \right).$$

2. Escape from states/nodes A and E is no longer possible, so we adjust the transition matrix $\mathbf{P}$ accordingly, to yield:

$$\mathbf{P}_{\text{new}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$