6 ARMA and ARIMA processes

6.1 Mixed autoregressive moving average processes

We consider a generalization of the moving average and autoregressive models, that explicitly includes a non-zero process mean.

Definition 6.1.1 (ARMA(p, q))
A *mixed autoregressive moving average process of order* (p, q), an ARMA(p, q) process, is a stationary process $\{Y_t\}$ which satisfies the relation

$$Y_t = \mu + \sum_{k=1}^{p} \phi_k (Y_{t-k} - \mu) + \epsilon_t + \sum_{i=1}^{q} \theta_i \epsilon_{t-i} \qquad t \in \mathbb{Z}, \tag{1}$$

where μ is the process mean, $\{\epsilon_t\}$ is a white noise process with mean 0 and variance σ^2, $\phi_p \neq 0$ and $\theta_q \neq 0$. $\square$

Alternatively, the model (1) may be written as

$$Y_t = c + \sum_{k=1}^{p} \phi_k Y_{t-k} + \epsilon_t + \sum_{i=1}^{q} \theta_i \epsilon_{t-i} \qquad t \in \mathbb{Z},$$

where the constant c is given by

$$c = \left(1 - \sum_{k=1}^{p} \phi_k\right) \mu.$$

Another way of writing the model (1) is as

$$\phi(L)(Y_t - \mu) = \theta(L)\epsilon_t, \tag{2}$$

where $\phi(z)$ is the AR characteristic polynomial,

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \ldots - \phi_p z^p,$$

and $\theta(z)$ is the MA characteristic polynomial,

$$\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \ldots + \theta_q z^q.$$

From now on we shall assume that the AR and MA characteristic polynomials have no common factors, since, otherwise, the model would be over-parameterized and the common factors could be cancelled out in Equation (2) to obtain an equivalent model of lower order with no common factors. Note that ARMA$(p, 0) \equiv$ AR(p) and ARMA$(0, q) \equiv$ MA(q).

The ARMA(p, q) model defines a stationary, linear process if and only if all the roots of the AR characteristic equation $\phi(z) = 0$ lie strictly outside the unit circle in the complex plane, which is precisely the condition for the corresponding AR(p) model to define a stationary process. The resulting process is invertible if and only if all the roots of the MA characteristic equation $\theta(z) = 0$ lie strictly outside the unit circle in the complex plane, which is precisely the condition for the corresponding MA(q) process to be invertible. We shall require both the stationarity and invertibility conditions to be satisfied.

Having assumed for an ARMA model that the AR and MA characteristic polynomials have no common factors and that the process is stationary and invertible, it follows that the model and its parameter values (apart from the process mean μ) are uniquely identifiable from its autocovariance function. It also follows from Equation (2) that the infinite moving average expression for $\{Y_t\}$ is given by

$$Y_t = \mu + \sum_{i=0}^{\infty} \psi_i \epsilon_{t-i},$$

i.e.,

$$Y_t = \mu + \psi(L)\epsilon_t,$$

where the generating function ψ is given by

$$\psi(z) = \phi(z)^{-1}\theta(z). \tag{3}$$

The ARMA processes all belong to the family of linear processes as defined in Section 4.3 (slightly generalized by the addition of the term μ for the process mean). What is important about the ARMA processes for practical purposes is that they are characterized by a finite number, $p+q+1$, of parameters — p autoregressive parameters, q moving average parameters and one parameter μ for the process mean — which can be estimated from the observed time series data to which the model is being fitted.

6.2 Autocovariances and autocorrelations for ARMA processes

To investigate the autocovariances of an ARMA(p, q) process, we may without loss of generality take the process mean to be zero. Setting $\mu = 0$ in Equation (1), multiplying through by $Y_{t-\tau}$ and taking expectations,

$$\gamma_\tau = \sum_{k=1}^{p} \phi_k E[Y_{t-k}Y_{t-\tau}] + E[\epsilon_t Y_{t-\tau}] + \sum_{i=1}^{q} \theta_i E[\epsilon_{t-i}Y_{t-\tau}]. \tag{4}$$

If $\tau \geq q+1$ then, by Equation (10) of Section 4.3,

$$E[\epsilon_{t-i}Y_{t-\tau}] = 0, \qquad 0 \leq i \leq q.$$

Hence if $\tau \geq q+1$ then Equation (4) may be written as

$$\gamma_\tau = \sum_{k=1}^{p} \phi_k \gamma_{\tau-k}.$$

Dividing through by γ_0,

$$\rho_\tau = \sum_{k=1}^{p} \phi_k \rho_{\tau-k}, \qquad \tau \geq q+1. \tag{5}$$

Equation (5) is similar to Equation (26) of Section 5.4 for the autocorrelation function of an AR(p) process — it is the same difference equation but with a more restricted range of validity. Hence the general form of solution for the autocorrelation function of an ARMA(p, q) process, as a sum of geometric terms, is similar to that for the corresponding AR(p) process, but the determination of the arbitrary constants in the general solution is more complicated.

6.3 The ARMA$(1, 1)$ process

We shall obtain more explicit results for the special case of the ARMA$(1, 1)$ model with process mean 0 , i.e.,

$$Y_t = \phi Y_{t-1} + \epsilon_t + \theta \epsilon_{t-1} \qquad t \in \mathbb{Z}, \tag{6}$$

which may be written alternatively as

$$(1 - \phi L) Y_t = (1 + \theta L) \epsilon_t. \tag{7}$$

The condition that the AR and MA characteristic polynomials have no common factors here reduces to the condition $\phi + \theta \neq 0$. (If $\phi + \theta = 0$ then the model (6) reduces to $Y_t = \epsilon_t$, so that the process $\{Y_t\}$ is just white noise.)

The stationarity condition is that $|\phi| < 1$ and the invertibility condition is that $|\theta| < 1$. We check the former by obtaining the infinite moving average representation of the ARMA$(1, 1)$ process. From Equation (7),

$$\begin{aligned} Y_t &= (1 - \phi L)^{-1} (1 + \theta L) \epsilon_t \\ &= (1 + \theta L) \sum_{i=0}^{\infty} \phi^i L^i \epsilon_t \\ &= \epsilon_t + \sum_{i=1}^{\infty} (\theta + \phi) \phi^{i-1} L^i \epsilon_t. \end{aligned}$$

Thus

$$Y_t = \epsilon_t + (\theta + \phi) \sum_{i=1}^{\infty} \phi^{i-1} \epsilon_{t-i}, \tag{8}$$

where the sum converges with probability 1, due to the absolute summability condition specified in Section 4.3, so that the process is stationary, if and only if $|\phi| < 1$.

Assuming that the process is stationary, we obtain its autocorrelation function. One way of doing this is to use the infinite moving average representation given in Equation (8) together with Equation (9) of Section 4.3. Instead, we shall adopt an alternative approach, working directly from the model equation (6).

Multiplying Equation (6) by $Y_{t-\tau}$, $\tau \geq 2$, taking expectations, and recalling that $Y_{t-\tau}$ is uncorrelated with ϵ_t and ϵ_{t-1}, we obtain

$$\gamma_\tau = \phi\gamma_{\tau-1}, \qquad \tau \geq 2.$$

Dividing by γ_0,

$$\rho_\tau = \phi\rho_{\tau-1}, \qquad \tau \geq 2. \tag{9}$$

It remains to evaluate ρ_1, which we shall do by first finding γ_0 and γ_1. Recall that ϵ_{t-1} and Y_{t-1} are uncorrelated with ϵ_t. Squaring both sides of Equation (6) and taking expectations,

$$\gamma_0 = \phi^2\gamma_0 + (1+\theta^2)\sigma^2 + 2\theta\phi E[Y_{t-1}\epsilon_{t-1}]. \tag{10}$$

Multiplying Equation (6) by ϵ_t and taking expectations,

$$E[Y_t\epsilon_t] = \sigma^2.$$

Hence also, by the stationarity,

$$E[Y_{t-1}\epsilon_{t-1}] = \sigma^2,$$

and substituting into Equation (10) we obtain

$$\gamma_0 = \frac{(1+2\theta\phi+\theta^2)\sigma^2}{1-\phi^2}.$$

Multiplying Equation (6) by Y_{t-1} and taking expectations,

$$\gamma_1 = \phi\gamma_0 + \theta E[Y_{t-1}\epsilon_{t-1}].$$

Hence

$$\begin{aligned}\gamma_1 &= \left[\phi\frac{(1+2\theta\phi+\theta^2)}{1-\phi^2} + \theta\right]\sigma^2 \\ &= \frac{(1+\theta\phi)(\theta+\phi)\sigma^2}{1-\phi^2}\end{aligned}$$

and

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{(1+\theta\phi)(\theta+\phi)}{1+2\theta\phi+\theta^2}.$$

Hence, recalling Equation (9),

$$\rho_\tau = \frac{(1+\theta\phi)(\theta+\phi)}{1+2\theta\phi+\theta^2}\,\phi^{\tau-1}, \qquad \tau \geq 1.$$

Note that the autocorrelation function decreases geometrically as in the special case $\theta = 0$ of the AR(1) process, the only difference being that for the ARMA(1,1) process the ratio ρ_1/ρ_0 is different from the ratio $\rho_\tau/\rho_{\tau-1}$ for $\tau \geq 2$.

6.4 The random walk

We begin our discussion of non-stationary processes by considering a process $\{Y_t\}$ defined by the relation

$$Y_t = Y_{t-1} + \epsilon_t \qquad t \in \mathbb{Z}, \tag{11}$$

where $\{\epsilon_t\}$ is a white noise process with mean zero and variance σ^2. Such a process $\{Y_t\}$ is known as a *random walk*. If Y_t denotes the position of a particle on the real line at time t, Equation (11) states that the position of the particle at time t is its position at time $t-1$ plus a random displacement. The particle may be said to perform a random walk.

Equation (11) is identical with the equation satisfied by an AR(1) process if the autoregressive parameter ϕ is set equal to 1. But the AR(1) process is a well-defined stationary process if and only if $|\phi| < 1$. Thus the random walk is *not* a stationary process. We can also demonstrate this more directly.

Suppose that a process $\{Y_t\}$ that satisfies Equation (11) is stationary. Taking variances of both sides of Equation (11) and using the fact that Y_{t-1} and ϵ_t are uncorrelated, we find that

$$\text{var}(Y_t) = \text{var}(Y_{t-1}) + \sigma^2,$$

so that $\text{var}(Y_t) \neq \text{var}(Y_{t-1})$, which contradicts the assumed stationarity. Thus Equation (11) cannot be used to define a stationary process.

However, if we define the variables $\{W_t\}$ by

$$W_t = Y_t - Y_{t-1}, \qquad t \in \mathbb{Z} \tag{12}$$

then Equation (11) reduces to

$$W_t = \epsilon_t \qquad t \in \mathbb{Z},$$

so that $\{W_t\}$ is a white noise process and hence stationary.

6.5 The difference operator

Using the lag operator L, we may write W_t, defined in Equation (12), as

$$W_t = (1 - L)Y_t$$

or as

$$W_t = \triangle Y_t,$$

where the *difference operator* $\triangle$ is defined by

$$\triangle = 1 - L.$$

Thus Equation (11) may be rewritten as

$$(1 - L)Y_t = \epsilon_t$$

or as

$$\triangle Y_t = \epsilon_t.$$

The random walk is the simplest example of a non-stationary process which can be converted into a stationary process by the operation of taking differences.

In general, it is often possible to transform non-stationary processes, including processes with various forms of trend, into stationary processes by taking first, second or possibly higher order differences. The dth difference is represented by the operator

$$\triangle^d = (1 - L)^d.$$

For example, the second difference of Y_t is

$$(Y_t - Y_{t-1}) - (Y_{t-1} - Y_{t-2}) = Y_t - 2Y_{t-1} + Y_{t-2}.$$

Equivalently,

$$\triangle^2 Y_t = (1 - L)^2 Y_t = (1 - 2L + L^2)Y_t = Y_t - 2Y_{t-1} + Y_{t-2}.$$

6.6 The ARIMA Models

To deal with a process $\{Y_t\}$ with trend, consider a process $\{W_t\}$ which is related to $\{Y_t\}$ by

$$W_t = \triangle^d Y_t \equiv (1 - L)^d Y_t \tag{13}$$

for some integer $d \geq 0$. Suppose that $\{W_t\}$ is stationary, an ARMA(p, q) process with mean μ which satisfies

$$\phi(L)(W_t - \mu) = \theta(L)\epsilon_t, \tag{14}$$

where $\phi(z)$ is the AR characteristic polynomial,

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \ldots - \phi_p z^p,$$

and $\theta(z)$ is the MA characteristic polynomial,

$$\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \ldots + \theta_q z^q.$$

Substituting from Equation (13) into Equation (14), we find that $\{Y_t\}$ satisfies the equation

$$\phi(L)(1 - L)^d Y_t = c + \theta(L)\epsilon_t, \tag{15}$$

where

$$c = \left(1 - \sum_{k=1}^{p} \phi_k\right)\mu.$$

A process $\{Y_t\}$ that satisfies Equation (15) is said to be an *autoregressive integrated moving average process of order* (p, d, q), an ARIMA(p, d, q) process.

The term "integrated" is used because the process $\{W_t\}$ of d-th differences is an ARMA process. Summation/integration is the inverse of the difference operator, and so the process $\{Y_t\}$ is obtained by "integration" of the ARMA process.

If $d \geq 1$ then the ARIMA(p, d, q) process is not stationary, but

$$\begin{aligned} \text{ARIMA}(p, 0, q) &\equiv \text{ARMA}(p, q), \\ \text{ARIMA}(p, 0, 0) &\equiv \text{AR}(p), \\ \text{ARIMA}(0, 0, q) &\equiv \text{MA}(q). \end{aligned}$$

In fitting an ARMA model to stationary data, we normally include a process mean μ in the model; but, in fitting an ARIMA model with $d \geq 1$ to non-stationary data, it often turns out to be more satisfactory to take $\mu = 0$.

6.7 Examples

An ARIMA(0,1,1) model with zero mean, for which $\{\triangle Y_t\}$ is MA(1), is of the form

$$(1 - L)Y_t = (1 + \theta L)\epsilon_t,$$

i.e.,

$$Y_t = Y_{t-1} + \epsilon_t + \theta\epsilon_{t-1}.$$

The above equation is of the same form as that which defines an ARMA(1,1) process, but the value of the parameter, $\phi = 1$, is such that the present process is not stationary.

An ARIMA(1,1,0) model with zero mean, for which $\{\triangle Y_t\}$ is AR(1), is of the form

$$(1 - \phi L)(1 - L)Y_t = \epsilon_t,$$

i.e.,

$$Y_t = (1 + \phi)Y_{t-1} - \phi Y_{t-2} + \epsilon_t,$$

where $|\phi| < 1$.

The above equation is of the same form as that which defines an AR(2) process, but the values of the parameters, $\phi_1 = 1 + \phi$ and $\phi_2 = -\phi$, are such that the present process is not stationary.

6.8 Unit roots

If $\{Y_t\}$ is an ARIMA(p, d, q) process that satisfies Equation (15) then we may rewrite this model equation as

$$\phi^\star(L)Y_t = c + \theta(L)\epsilon_t, \tag{16}$$

where

$$\phi^\star(z) = \phi(z)(1 - z)^d. \tag{17}$$

If we were considering Equation (16) in terms of fitting an ARMA model to the process $\{Y_t\}$ then, from the expression of Equation (17), the corresponding AR characteristic equation would be

$$\phi(z)(1 - z)^d = 0,$$

which has unit roots. Thus we would discover that we were not dealing with a stationary process but that differencing was needed to transform the process $\{Y_t\}$ to stationarity.

To address the question of whether differencing is necessary to transform an observed time series to stationarity, so called "unit root tests" have been constructed for testing whether the AR characteristic equation has unit roots. However, we shall adopt an alternative approach in a later chapter.

Now suppose that the process $\{Y_t\}$ is a stationary process, an ARMA(p, q) process that satisfies the equation

$$\phi(L)Y_t = c + \theta(L)\epsilon_t. \tag{18}$$

Consider the process $\{W_t\}$ of differences of order d with

$$W_t = (1 - L)^d Y_t.$$

Multiplying through by $(1 - L)^d$ in Equation (18), we find that $\{W_t\}$ satisfies the equation

$$\phi(L)W_t = \theta^\star(L)\epsilon_t,$$

where

$$\theta^\star(z) = \theta(z)(1 - z)^d.$$

It follows that $\{W_t\}$ is an ARMA$(p, q + d)$ process with MA characteristic polynomial $\theta^\star(z)$. Thus the MA characteristic equation now has unit roots and $\{W_t\}$ is non-invertible.

If we are dealing with an observed time series which we have differenced a number of times and we find that the MA characteristic equation appears to have unit roots then this may indicate that we have over-differenced the series.

6.9 Multiplicative seasonal models

Consider a time series exhibiting seasonality with period s. For monthly data with an annual cycle, for example, we would take $s = 12$. Define $\triangle_s$, the *seasonal difference operator of period* s, by

$$\triangle_s = 1 - L^s.$$

Thus

$$\triangle_s Y_t = (1 - L^s)Y_t = Y_t - Y_{t-s}.$$

In the extreme case where a time series has a regular seasonal pattern that is repeated exactly year after year, so that $y_t = y_{t-s}$ for all t, the seasonal difference operator will eliminate the seasonality completely.

If, in addition, the series has a long-term trend, $\triangle_s$ may at least partly eliminate it; but $\triangle_s$ is usually taken in combination with the difference operator $\triangle$ to eliminate trend.

Other less regular seasonal effects may be modelled by seasonal autoregressive and moving average terms at lag s.

For integers $d \geq 0$ and $D \geq 0$ define

$$W_t = \triangle^d \triangle_s^D Y_t \tag{19}$$

and suppose that $\{W_t\}$ is an ARMA process satisfying an equation of the form

$$\phi(L)\Phi(L^s)(W_t - \mu) = \theta(L)\Theta(L^s)\epsilon_t, \tag{20}$$

where the characteristic polynomials ϕ and θ are defined as before, the *seasonal AR characteristic polynomial* $\Phi(z^s)$, is defined by

$$\Phi(z^s) = 1 - \Phi_1 z^s - \Phi_2 z^{2s} - \ldots - \Phi_P z^{Ps},$$

and the *seasonal MA characteristic polynomial* $\Theta(z^s)$ by

$$\Theta(z^s) = 1 + \Theta_1 z^s + \Theta_2 z^{2s} + \ldots + \Theta_Q z^{Qs}.$$

Substituting from Equation (19) into Equation (20), we find that $\{Y_t\}$ satisfies the equation

$$\phi(L)\Phi(L^s)(1-L)^d(1-L^s)^D Y_t = c + \theta(L)\Theta(L^s)\epsilon_t, \tag{21}$$

where the constant c is a function of μ , ϕ and Φ.

Equation (21) specifies what is known as a *multiplicative seasonal ARIMA model*, an ARIMA$(p, d, q) \times (P, D, Q)_s$ model. All the models discussed previously are special cases of this model.

The family of models (21) is so large that it may be difficult to identify what might be the most appropriate model to use for a particular set of seasonal data. In practice, D is rarely set to values greater than 1 and is usually combined with $d = 0$ or $d = 1$. Very often a relatively simple model such as the ARIMA$(0, 1, 1) \times (0, 1, 1)_s$, for example, may be fitted.