Problem Set 3

Christian Hellum Bye

November 2, 2022

Question 1

(a) The comoving distance to an object can be measured by the time it takes a light-ray to travel from that point to us. We start with the Friedmann–Lemaître–Robertson–Walker metric. Along a radial null-geodesic, this yields

$$\mathrm{d}t^2 = a^2 \frac{\mathrm{d}r^2}{1-kr^2}. \tag{1}$$

The comoving distance χ is then obtained by integrating Eq. 1 from $t = t$ (time of light emission) to $t = t_0$ (present time) and from $r = 0$ to $r = r$. In Problem Set 1, we derived

$$H^2 = H_0^2 \left(\sum_i \frac{\Omega_{0,i}}{a^{3(1+w_i)}} + \frac{1-\Omega_0}{a^2} \right)$$
$$\frac{\mathrm{d}a}{\mathrm{d}t} = H_0 \sqrt{\sum_i \frac{\Omega_{0,i}}{a^{1+3w_i}} + 1 - \Omega_0}$$
$$\mathrm{d}t = \frac{\mathrm{d}a}{H_0\sqrt{\sum_i \frac{\Omega_{0,i}}{a^{1+3w_i}} + 1 - \Omega_0}} = \frac{\mathrm{d}a}{H_0\sqrt{\frac{\Omega_{0,m}}{a} + \Omega_{0,\Lambda}a^2 + 1 - \Omega_0}}$$

Using $a = 1/(1+z)$ and $z(t_0) = 0$, we get:

$$z = \frac{1}{a} - 1 \implies \mathrm{d}z = -\frac{1}{a^2}\mathrm{d}a$$
$$\frac{\mathrm{d}t}{a} = -\frac{\mathrm{d}z}{H_0\sqrt{\frac{\Omega_{0,m}}{a^3} + \Omega_{0,\Lambda} + \frac{1-\Omega_0}{a^2}}}$$

We are now ready to integrate Eq. 1.

$$\int_t^{t_0} \frac{\mathrm{d}t'}{a} = \int_0^r \frac{\mathrm{d}r'}{\sqrt{1-kr'^2}} \tag{2}$$

The left-hand side is:

$$\int_t^{t_0} \frac{\mathrm{d}t'}{a} = \int_0^z \frac{\mathrm{d}z}{H_0\sqrt{\frac{\Omega_{0,m}}{a^3} + \Omega_{0,\Lambda} + \frac{1-\Omega_0}{a^2}}} = \int_0^z \frac{\mathrm{d}z'}{H_0\sqrt{\Omega_{0,m}(1+z')^3 + \Omega_{0,\Lambda} + (1-\Omega_0)(1+z')^2}} \tag{3}$$

For the right-hand side, we let $\mathrm{sinn}x \equiv \sqrt{|k|}r$, where the function $\mathrm{sinn}x$ is as defined in the question. We then evaluate the integral on the right-hand side, noting that $\mathrm{d}(\mathrm{sinn}x) = \sqrt{|k|}\mathrm{d}r$ and $\mathrm{sinn}(0) = 0$.

$k = 0$:

$$\int_0^r \frac{\mathrm{d}r'}{\sqrt{1-kr'^2}} = \int_0^r \mathrm{d}r' = r = \frac{\mathrm{sinn}x}{\sqrt{|k|}} = \frac{x}{\sqrt{|k|}}$$

$k > 0$:

$$\int_0^r \frac{\mathrm{d}r'}{\sqrt{1-kr'^2}} = \int_0^{x(r)} \frac{1}{\sqrt{|k|}} \frac{\mathrm{d}(\mathrm{sinn}x')}{\sqrt{1-\mathrm{sinn}^2x'}} = \int_0^{x(r)} \frac{1}{\sqrt{|k|}} \frac{\mathrm{d}(\sinh x')}{\cosh x'} = \int_0^{x(r)} \frac{\mathrm{d}x'}{\sqrt{|k|}} = \frac{x}{\sqrt{|k|}}$$

$k < 0$:

$$\int_0^r \frac{\mathrm{d}r'}{\sqrt{1-kr'^2}} = \int_0^{x(r)} \frac{1}{\sqrt{|k|}} \frac{\mathrm{d}(\mathrm{sinn}x')}{\sqrt{1+\mathrm{sinn}^2x'}} = \int_0^{x(r)} \frac{1}{\sqrt{|k|}} \frac{\mathrm{d}(\sin x')}{\cos x'} = \int_0^{x(r)} \frac{\mathrm{d}x'}{\sqrt{|k|}} = \frac{x}{\sqrt{|k|}}$$

In any case, the right hand side is equal to $x/\sqrt{|k|}$. Combining this result with Eq. 2 and Eq. 3 gives:

$$\int_0^z \frac{\mathrm{d}z'}{H_0\sqrt{\Omega_{0,m}(1+z')^3+\Omega_{0,\Lambda}+(1-\Omega_0)(1+z')^2}} = \frac{x}{\sqrt{|k|}}$$
$$r = \frac{\mathrm{sinn} x}{\sqrt{|k|}} = \frac{1}{\sqrt{|k|}}\mathrm{sinn}\left(\frac{\sqrt{|k|}}{H_0}\int_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_{0,m}(1+z')^3+\Omega_{0,\Lambda}+(1-\Omega_0)(1+z')^2}}\right)$$

Dimensional analysis reveals that a factor of c is missing in the argument of the sinn-function (this was lost when we set $c = 1$ in Eq. 1). If we add this back in and write out the density parameter in terms of its components, $\Omega_0 = \Omega_{0,m} + \Omega_{0,\Lambda}$, we arrive at the same result as in the question:

$$r = \frac{1}{\sqrt{|k|}}\mathrm{sinn}\left(\frac{c\sqrt{|k|}}{H_0}\int_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_{0,m}(1+z')^3+\Omega_{0,\Lambda}+(1-\Omega_{0,m}-\Omega_{0,\Lambda})(1+z')^2}}\right)$$

(b) We need to Taylor expand the expression from part (a). Define a new function $g(z)$ such that:

$$g(z) = \int_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_{0,m}(1+z')^3+\Omega_{0,\Lambda}+(1-\Omega_{0,m}-\Omega_{0,\Lambda})(1+z')^2}}$$
$$r = \frac{1}{\sqrt{|k|}}\mathrm{sinn}\left(\frac{c\sqrt{|k|}}{H_0}g(z)\right)$$

We need to compute the derivatives of g:

$$g(z) = \int_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_{0,m}(1+z')^3+\Omega_{0,\Lambda}+(1-\Omega_{0,m}-\Omega_{0,\Lambda})(1+z')^2}}$$
$$\frac{\mathrm{d}g}{\mathrm{d}z} = \frac{1}{\sqrt{\Omega_{0,m}(1+z)^3+\Omega_{0,\Lambda}+(1-\Omega_{0,m}-\Omega_{0,\Lambda})(1+z)^2}}$$
$$\frac{\mathrm{d}^2g}{\mathrm{d}z^2} = -\frac{1}{2}\frac{3\Omega_{0,m}(1+z)^2+2(1-\Omega_{0,m}-\Omega_{0,\Lambda})(1+z)}{\left(\Omega_{0,m}(1+z)^3+\Omega_{0,\Lambda}+(1-\Omega_{0,m}-\Omega_{0,\Lambda})(1+z)^2\right)^{3/2}}$$

Evaluated at $z = 0$:

$$g(z=0) = 0$$
$$\frac{\mathrm{d}g}{\mathrm{d}z}(z=0) = 1$$
$$\frac{\mathrm{d}^2g}{\mathrm{d}z^2}(z=0) = -\frac{1}{2}(\Omega_{0,m}-2\Omega_{0,\Lambda}+2) = -\left(\frac{\Omega_{0,m}}{2}-\Omega_{0,\Lambda}+1\right) = -(1+q_0)$$

Hence, for small z, we may expand g as:

$$g(z) = z - \frac{1}{2}(1+q_0)z^2 + \mathcal{O}\left(z^3\right)$$

Now, we can expand r for small z, using that $\sin x \approx x$ and $\sinh x \approx x$ for small x.

$$r = \frac{1}{\sqrt{|k|}}\mathrm{sinn}\left(\frac{c\sqrt{|k|}}{H_0}g(z)\right) = \frac{1}{\sqrt{|k|}}\mathrm{sinn}\left(\frac{c\sqrt{|k|}}{H_0}\left(z-\frac{1}{2}(1+q_0)z^2+\mathcal{O}\left(z^3\right)\right)\right)$$
$$r = \frac{1}{\sqrt{|k|}}\left(\frac{c\sqrt{|k|}}{H_0}\left(z-\frac{1}{2}(1+q_0)z^2+\mathcal{O}\left(z^3\right)\right)\right) = \frac{c}{H_0}\left(z-\frac{1}{2}(1+q_0)z^2\right)+\mathcal{O}\left(z^3\right)$$

Question 2

We need to compute the distance modulus μ for each cosmology.

$$\mu = 5\log\frac{D_L}{10\text{pc}}$$

$$D_L = r(1+z) = \frac{1+z}{\sqrt{|k|}}\text{sinn}\left(\frac{c\sqrt{|k|}}{H_0}\int_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_{0,m}(1+z')^3+\Omega_{0,\Lambda}+(1-\Omega_{0,m}-\Omega_{0,\Lambda})(1+z')^2}}\right)$$

We solve this integral for all three cases.

Flat, matter-dominated universe: In this case, $\Omega_m = 1$, $\Omega_\Lambda = 0$ and $k = 0$. Thus,

$$D_L = (1+z)\frac{c}{H_0}\int_0^z \frac{\mathrm{d}z'}{(1+z')^{3/2}} = (1+z)\frac{c}{H_0}\left(2-\frac{2}{\sqrt{1+z}}\right) = \frac{2c}{H_0}\sqrt{1+z}\left(\sqrt{1+z}-1\right) \tag{4}$$

Flat universe with cosmological constant: In this case, $\Omega_m < 1$, $\Omega_\Lambda = 1-\Omega_m > 0$ and $k = 0$. Thus,

$$D_L = (1+z)\frac{c}{H_0}\int_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_{0,m}(1+z')^3+\Omega_{0,\Lambda}}} \tag{5}$$

This integral does not have an analytical solution so we will numerically integrate this case.

Open, matter-dominated universe: Here, $\Omega_m < 1$, $\Omega_\Lambda = 0$, $k < 0$. Thus,

$$D_L = \frac{1+z}{\sqrt{-k}}\sinh\left(\frac{c\sqrt{-k}}{H_0}\int_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_{0,m}(1+z')^3+(1-\Omega_{0,m})(1+z')^2}}\right)$$

The integral can be solved analytically:

$$\int_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_{0,m}(1+z')^3+(1-\Omega_{0,m})\,(1+z')^2}} = \int_0^z \frac{\mathrm{d}z'}{(1+z')\sqrt{\Omega_{0,m}(1+z')+(1-\Omega_{0,m})}}$$

$$u = \sqrt{\Omega_{0,m}(1+z')+(1-\Omega_{0,m})}$$

$$1+z' = \frac{u^2+\Omega_{0,m}-1}{\Omega_{0,m}}$$

$$\frac{\mathrm{d}u}{\mathrm{d}z'} = \frac{\Omega_{0,m}}{2u} \implies \frac{\mathrm{d}z'}{u(1+z')} = \frac{2\mathrm{d}u}{u^2-(1-\Omega_{0,m})}$$

$$\int_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_{0,m}(1+z')^3+(1-\Omega_{0,m})\,(1+z')^2}} = 2\int_1^{u(z)} \frac{\mathrm{d}u}{u^2-\sqrt{1-\Omega_{0,m}}^2}$$

$$= \frac{1}{\sqrt{1-\Omega_{0,m}}}\int_1^{u(z)}\mathrm{d}u\left(\frac{1}{u-\sqrt{1-\Omega_{0,m}}}-\frac{1}{u+\sqrt{1-\Omega_{0,m}}}\right)$$

$$= \frac{1}{\sqrt{1-\Omega_{0,m}}}\ln\frac{u(z)-\sqrt{1-\Omega_{0,m}}}{u(z)+\sqrt{1-\Omega_{0,m}}} = \frac{1}{\sqrt{1-\Omega_{0,m}}}\ln\frac{\sqrt{\Omega_{0,m}(1+z)+(1-\Omega_{0,m})}-\sqrt{1-\Omega_{0,m}}}{\sqrt{\Omega_{0,m}(1+z)+(1-\Omega_{0,m})}+\sqrt{1-\Omega_{0,m}}}$$

Hence,

$$D_L = \frac{1+z}{\sqrt{-k}}\sinh\left(\frac{c\sqrt{-k}}{H_0}\frac{1}{\sqrt{1-\Omega_{0,m}}}\ln\frac{\sqrt{\Omega_{0,m}(1+z)+(1-\Omega_{0,m})}-\sqrt{1-\Omega_{0,m}}}{\sqrt{\Omega_{0,m}(1+z)+(1-\Omega_{0,m})}+\sqrt{1-\Omega_{0,m}}}\right)$$

Using that $k = (H_0/c)^2\,(\Omega_{0,m}-1)$ and that $\sinh x = (\exp x - \exp(-x))/2$, we get

$$D_L = \frac{1+z}{\sqrt{-k}}\frac{\sqrt{\Omega_{0,m}(1+z)+(1-\Omega_{0,m})}-\sqrt{1-\Omega_{0,m}}}{\sqrt{\Omega_{0,m}(1+z)+(1-\Omega_{0,m})}+\sqrt{1-\Omega_{0,m}}}$$

$$D_L = (1+z)\frac{c}{H_0\sqrt{1-\Omega_{0,m}}}\frac{\sqrt{\frac{\Omega_{0,m}}{1-\Omega_{0,m}}(1+z)+1}-1}{\sqrt{\frac{\Omega_{0,m}}{1-\Omega_{0,m}}(1+z)+1}+1} \tag{6}$$

The code used to generate the plots can be found on GitHub. We see that Fig. 2 is very similar to Fig. 6 in Riess et al (2007). The data in Fig. 2 is the same as the HST data (orange markers) in their plot and the theoretical prediction from the standard cosmology (blue curve in our figure) matches the data similarly in both figures.

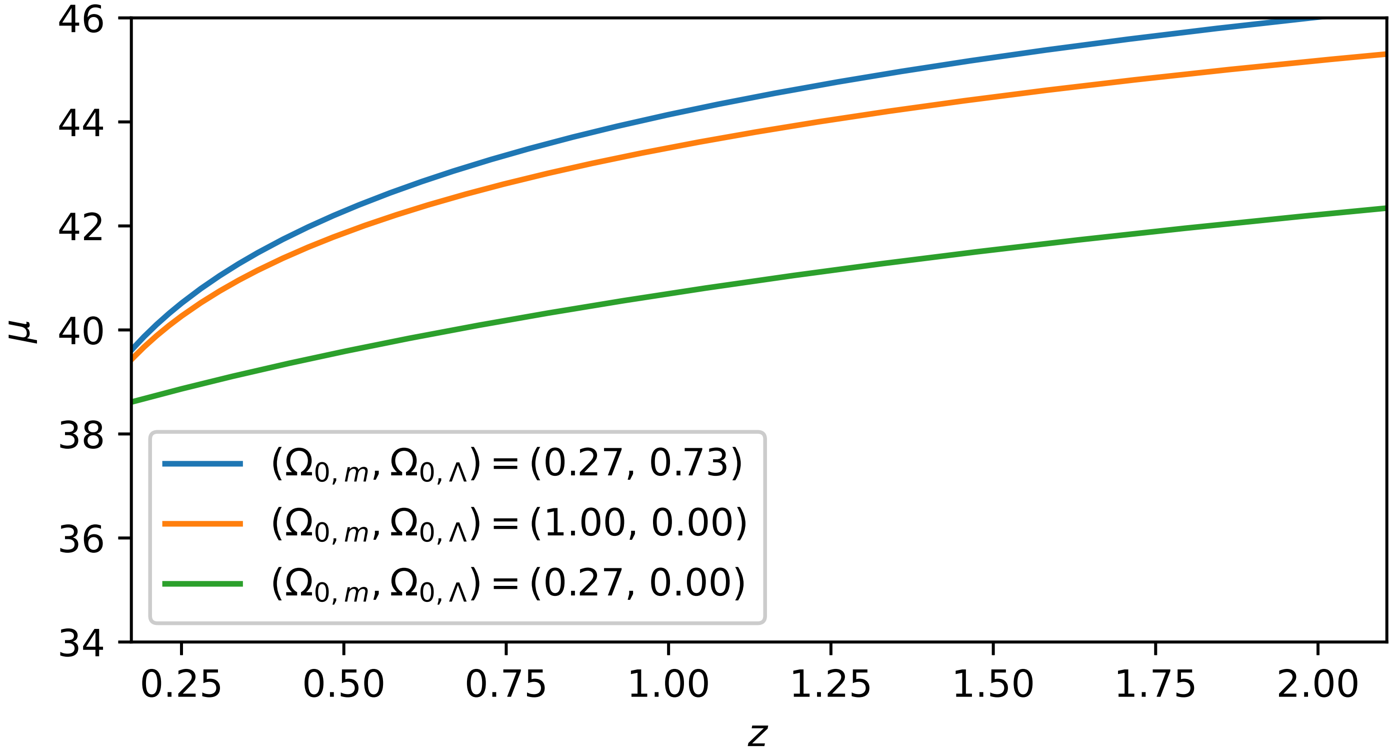

Figure 1: Distance modulus vs redshift for different cosmologies.

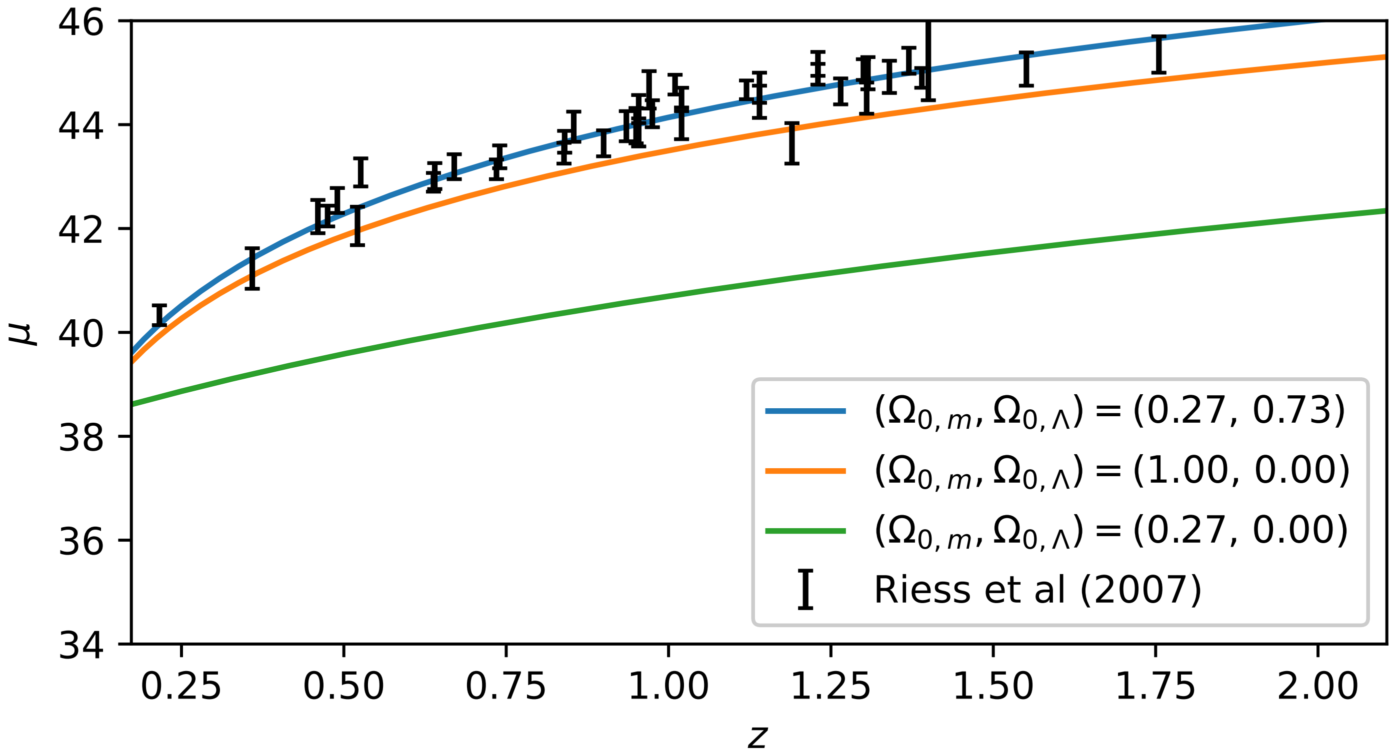

Figure 2: Like Fig. 1 but with data from Riess et al. (2007) overplotted.