Problem Set 7

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The code used for this Problem Set can be found on GitHub. If not specified otherwise, we use Einstein summation notation and let Latin letters (i, j, k, ...) range from 1 to 3.

Question 1

The unperturbed fluid equations are:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \partial^i \rho v_i = 0 \tag{1}$$

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} + v^j \partial_j v^i = -\frac{1}{\rho}\partial^i P - \partial^i \Phi \tag{2}$$

$$\partial^j \partial_j \Phi = 4\pi G\rho \tag{3}$$

For an expanding universe, we may define comoving coordinates x^i such that $r^i = ax^i$. We note that x^i is constant in time in the unperturbed case. Thus,

$$v^i \equiv \frac{\mathrm{d}r^i}{\mathrm{d}t} = \frac{\mathrm{d}a}{\mathrm{d}t}x^i = \dot{a}x^i = \frac{\dot{a}}{a}r^i, \tag{4}$$

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} = \ddot{a}x^i = \frac{\ddot{a}}{a}r^i. \tag{5}$$

Combining these equations with the fluid equations above—noting that spatial derivatives of v^i and P vanish in the homogeneous, isotropic case–give:

$$\frac{\ddot{a}}{a}r^i = -\partial^i \Phi \tag{6}$$

$$\partial^j \partial_j \Phi = 4\pi G\rho, \tag{7}$$

and hence,

$$\partial^j(-\frac{\ddot{a}}{a}r_j) = 4\pi G\rho \tag{8}$$

$$-3\frac{\ddot{a}}{a} = 4\pi G\rho \tag{9}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G\rho \tag{10}$$

We compare this with the second Friedmann equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G(\rho + 3P) \tag{11}$$

The difference between Eq. 10 and Eq. 11 is that the perturbed fluid equation implicitly assumes a matter-dominated universe with the equation of state $P = 0$ instead of the more general perfect fluids that allow $P = w\rho$.

Question 2

In this problem we define for a general differentiable function f the following notation for time-derivative and derivative with respect to y:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \dot{f}$$

$$\frac{\mathrm{d}f}{\mathrm{d}y} = f'$$

(a) We showed in class that the linearized evolution of the density perturbations are given by

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\rho_m\delta = 0. \tag{12}$$

In a flat universe, the Friedmann equation yields

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho = \frac{8\pi}{3}G(\rho_m + \rho_r). \tag{13}$$

Letting $y \equiv \rho_m/\rho_r = a/a_{\rm eq}$ gives the following chain rule for any differentiable function f:

$$\dot{f} = \frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}a}\dot{a} = \frac{\mathrm{d}f}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}a}\dot{a} \tag{14}$$

$$\dot{f} = f'\frac{\dot{a}}{a_{\rm eq}} = f'y\frac{\dot{a}}{a} \tag{15}$$

$$\ddot{f} = \frac{\mathrm{d}\dot{f}}{\mathrm{d}t} = \frac{\mathrm{d}(f'\frac{\dot{a}}{a_{\rm eq}})}{\mathrm{d}t} = \frac{\mathrm{d}f'}{\mathrm{d}t}\frac{\dot{a}}{a_{\rm eq}} + f'\frac{\mathrm{d}(\frac{\dot{a}}{a_{\rm eq}})}{\mathrm{d}t} \tag{16}$$

$$\ddot{f} = f''\left(\frac{\dot{a}}{a_{\rm eq}}\right)^2 + f'\frac{\ddot{a}}{a_{\rm eq}} = f''y^2\left(\frac{\dot{a}}{a}\right)^2 + f'y\frac{\ddot{a}}{a} \tag{17}$$

Eq. 15 and Eq. 17 give a recipe for converting derivatives in t to derivatives in y. Combining these with Eq. 13, we can rewrite Eq. 12 as:

$$\left(\delta''y^2\left(\frac{\dot{a}}{a}\right)^2 + \delta'y\frac{\ddot{a}}{a}\right) + 2\frac{\dot{a}}{a}\left(\delta'y\frac{\dot{a}}{a}\right) - \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2\frac{\rho_m}{\rho_m+\rho_r}\delta = 0 \tag{18}$$

$$y^2\left(\frac{\dot{a}}{a}\right)^2\delta'' + y\left(\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2\right)\delta' - \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2\frac{\rho_m}{\rho_m+\rho_r}\delta = 0 \tag{19}$$

$$\delta'' + \frac{1}{y}\left(\frac{a\ddot{a}}{(\dot{a})^2} + 2\right)\delta' - \frac{3}{2y^2}\frac{\rho_m}{\rho_m+\rho_r}\delta = 0 \tag{20}$$

$$\delta'' + \frac{1}{y}\left(\frac{a\ddot{a}}{(\dot{a})^2} + 2\right)\delta' - \frac{3}{2y^2}\frac{y}{y+1}\delta = 0 \tag{21}$$

$$\delta'' + \frac{1}{y}\left(\frac{a\ddot{a}}{(\dot{a})^2} + 2\right)\delta' - \frac{3}{2y(1+y)}\delta = 0 \tag{22}$$

Using the Friedmann Equations for a flat universe given below (note: here we use the equations of state $P = \rho/3$ for radiation and $P = 0$ for matter), we can eliminate the dependence on the scale factor a and its time derivatives as following:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}(\rho_m + \rho_r) \tag{23}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) = -\frac{4\pi G}{3}(\rho_m + 2\rho_r) \tag{24}$$

$$\frac{a\ddot{a}}{(\dot{a})^2} = \left(1\bigg/\left(\frac{\dot{a}}{a}\right)^2\right)\left(\frac{\ddot{a}}{a}\right) = -\frac{3}{8\pi G}\frac{4\pi G}{3}\frac{\rho_m+2\rho_r}{\rho_m+\rho_r} = -\frac{1}{2}\frac{y+2}{y+1} \tag{25}$$

$$\frac{a\ddot{a}}{(\dot{a})^2} + 2 = -\frac{1}{2}\frac{y+2}{y+1} - \frac{4(y+1)}{2(y+1)} = \frac{2+3y}{2(1+y)} \tag{26}$$

We can therefore rewrite Eq. 22 as

$$\delta'' + \frac{2+3y}{2y(1+y)}\delta' - \frac{3}{2y(1+y)}\delta = 0. \tag{27}$$

(b)

$$y \gg 1 \implies \frac{2+3y}{2y(1+y)} \approx \frac{3y}{2y^2} = \frac{3}{2y},\ \frac{3}{2y(1+y)} \approx \frac{3}{2y^2} \tag{28}$$

Thus, Eq. 27 can be written as

$$\delta'' + \frac{3}{2y}\delta' - \frac{3}{2y^2}\delta = 0 \tag{29}$$

We guess a power law solution for Eq. 29 of the form $\delta = Cy^\alpha$. Then,

$$\delta = Cy^\alpha \tag{30}$$
$$\delta' = \alpha C y^{\alpha-1} \tag{31}$$
$$\delta'' = \alpha(\alpha-1)Cy^{\alpha-2}, \tag{32}$$

and Eq. 29 becomes

$$\alpha(\alpha-1)Cy^{\alpha-2} + \frac{3}{2}\alpha C y^{\alpha-2} - \frac{3}{2}Cy^{\alpha-2} = 0 \tag{33}$$
$$\alpha(\alpha-1) + \frac{3}{2}\alpha - \frac{3}{2} = 0 \tag{34}$$
$$\alpha^2 + \frac{1}{2}\alpha - \frac{3}{2} = 0 \tag{35}$$
$$\left(\alpha + \frac{1}{4}\right)^2 - \frac{25}{16} = 0 \tag{36}$$
$$\alpha + \frac{1}{4} = \pm\frac{5}{4} \tag{37}$$
$$\alpha = 1 \vee \alpha = -\frac{3}{2}. \tag{38}$$

Hence we get a growing solution of the form $\delta \propto y$ and a decaying solution of the form $\delta \propto y^{-3/2}$.

(c) Letting $\delta = C\left(y + \frac{2}{3}\right)$ for some constant of proportionality C and substituting back into Eq. 27, we get

$$\frac{2+3y}{2y(1+y)}C - \frac{3}{2y(1+y)}C\left(y + \frac{2}{3}\right) = \frac{C}{2y(1+y)}(2 + 3y - 3y - 2) = 0 \tag{39}$$

Thus, $\delta \propto (y + 2/3)$ is a solution to Eq. 27. In the radiation dominated era, $y \ll 1$ and $\delta \propto 2/3$, hence perturbations cannot grow much in the radiation dominated era.

Question 3

(a) We had for the open, matter-dominated cosmology,

$$\delta \propto \frac{3\sinh\theta(\sinh\theta - \theta)}{(\cosh\theta - 1)^2} - 2, \tag{40}$$
$$a = \frac{\Omega_0}{2(1-\Omega_0)}(\cosh\theta - 1) \tag{41}$$
$$\theta = \operatorname{arcosh}\left(1 + 2a\frac{1-\Omega_0}{\Omega_0}\right). \tag{42}$$

In the flat, matter-dominated cosmology:

$$\delta \propto a. \tag{43}$$

Normalizing to $\delta(a = 10^{-3}) = 10^{-3}$ simply gives $\delta = a$ for the flat case. The open case is more complicated:

$$10^{-3} = a_{\text{ref}} = \frac{\Omega_0}{2(1-\Omega_0)}(\cosh\theta_{\text{ref}} - 1) \tag{44}$$
$$\cosh\theta_{\text{ref}} = 1 + 2 \cdot 10^{-3}\frac{1-\Omega_0}{\Omega_0} \tag{45}$$
$$\theta_{\text{ref}} = \operatorname{arcosh}\left(1 + 2\cdot 10^{-3}\frac{1-\Omega_0}{\Omega_0}\right) \tag{46}$$
$$\delta = 10^{-3}\left(\frac{3\sinh\theta(\sinh\theta-\theta)}{(\cosh\theta-1)^2} - 2\right)\Bigg/\left(\frac{3\sinh\theta_{\text{ref}}(\sinh\theta_{\text{ref}}-\theta_{\text{ref}})}{(\cosh\theta_{\text{ref}}-1)^2} - 2\right) \tag{47}$$

We show the evolution of δ in Fig. 1. In the flat universe, the growth is linear, whereas it slows down at later times in the open case. In general, the larger $\Omega_{0,m}$, the larger the density fluctuations.

Redshift 5 corresponds to $a = 1/6$. If the models predict the same density fluctuations today then we would expect the model with smallest matter density ($\Omega_{0,m} = 0.01$) to have most fluctuations at $z = 5$; a factor of $\sim 5-6$ greater than the flat case, a factor of ~ 2 greater than the $\Omega_{0,m} = 0.3$-case and about ~ 1.5 times greater than the case of $\Omega_{0,m} = 0.1$. This is because the density fluctuations in the $\Omega_{0,m} = 0.01$-universe grow very slowly for $z < 5$ and faster for the other cases (e.g. linearly in the flat case).

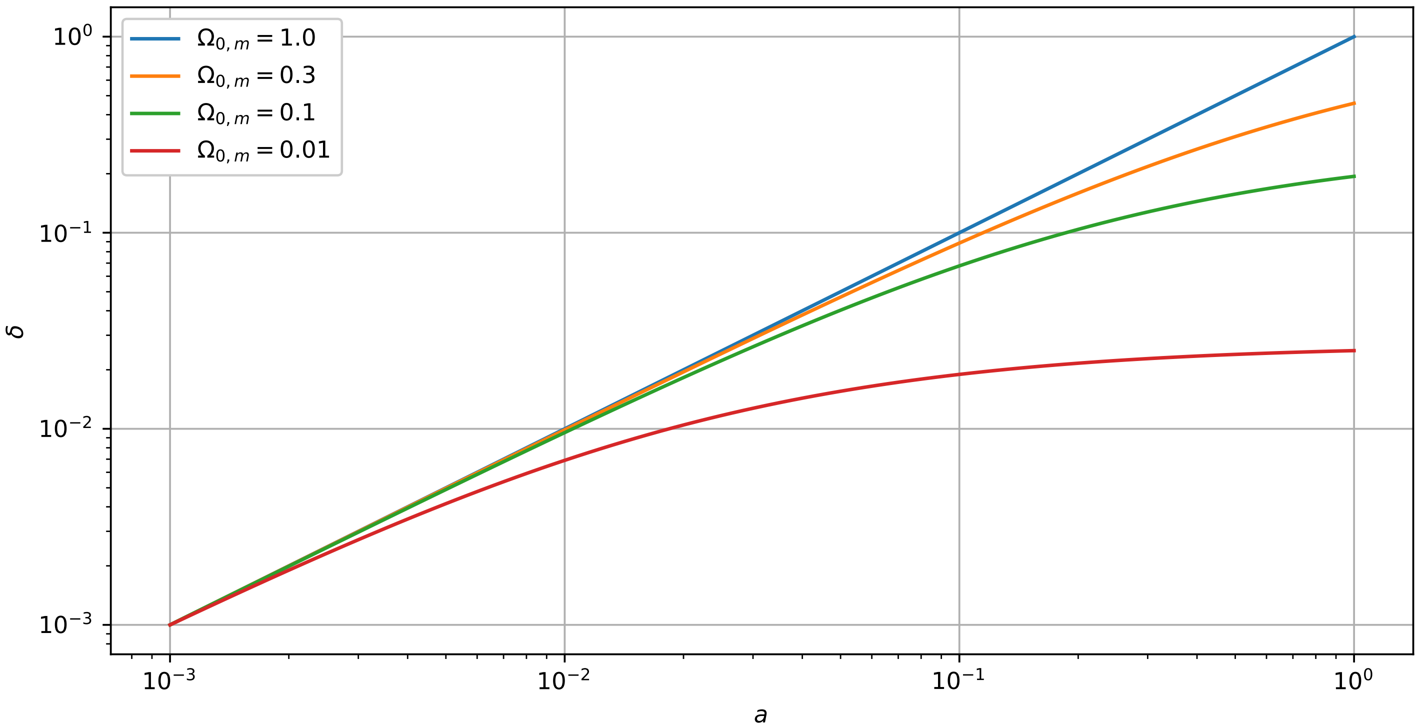

Figure 1: Growth of δ for matter-dominated universes with varying $\Omega_{0,m}$.

(b) We showed in Problem Set 1 that for a flat universe (with negligible radiation density), the Hubble parameter evolves according to

$$H^2 = H_0^2 \left(\frac{\Omega_{0,m}}{a^3} + \Omega_{0,\Lambda} \right). \tag{48}$$

Hence, we may write $\dot{a}$ in terms of a and the density parameters:

$$\left(\frac{\dot{a}}{a} \right)^2 = H_0^2 \left(\frac{\Omega_{0,m}}{a^3} + \Omega_{0,\Lambda} \right) \tag{49}$$

$$\frac{\dot{a}}{a} = H_0 \sqrt{\frac{\Omega_{0,m}}{a^3} + \Omega_{0,\Lambda}} \tag{50}$$

$$\frac{1}{\dot{a}} = \frac{1}{H_0} \frac{1}{a\sqrt{\frac{\Omega_{0,m}}{a^3} + \Omega_{0,\Lambda}}} = \frac{1}{H_0} \frac{1}{\sqrt{\frac{\Omega_{0,m}}{a} + a^2 \Omega_{0,\Lambda}}} \tag{51}$$

Using the same normalization as before, we can write the evolution of δ in the Λ-cosmology as

$$\delta \propto \frac{\dot{a}}{a} \int_0^a \frac{\mathrm{d}a'}{(\dot{a'})^3} \tag{52}$$

$$\delta \propto \sqrt{\frac{\Omega_{0,m}}{a^3} + \Omega_{0,\Lambda}} \int_0^a \mathrm{d}a' \left(\frac{\Omega_{0,m}}{a'} + (a')^2 \Omega_{0,\Lambda} \right)^{-3/2} \equiv f(a) \tag{53}$$

$$\delta = 10^{-3} \frac{f(a)}{f(10^{-3})} \tag{54}$$

The evolution is shown in Fig. 2. The density fluctuations grow with a cosmological constant for universes with the same matter density (evident when comparing purple and orange curves or brown and green curves). If the total density

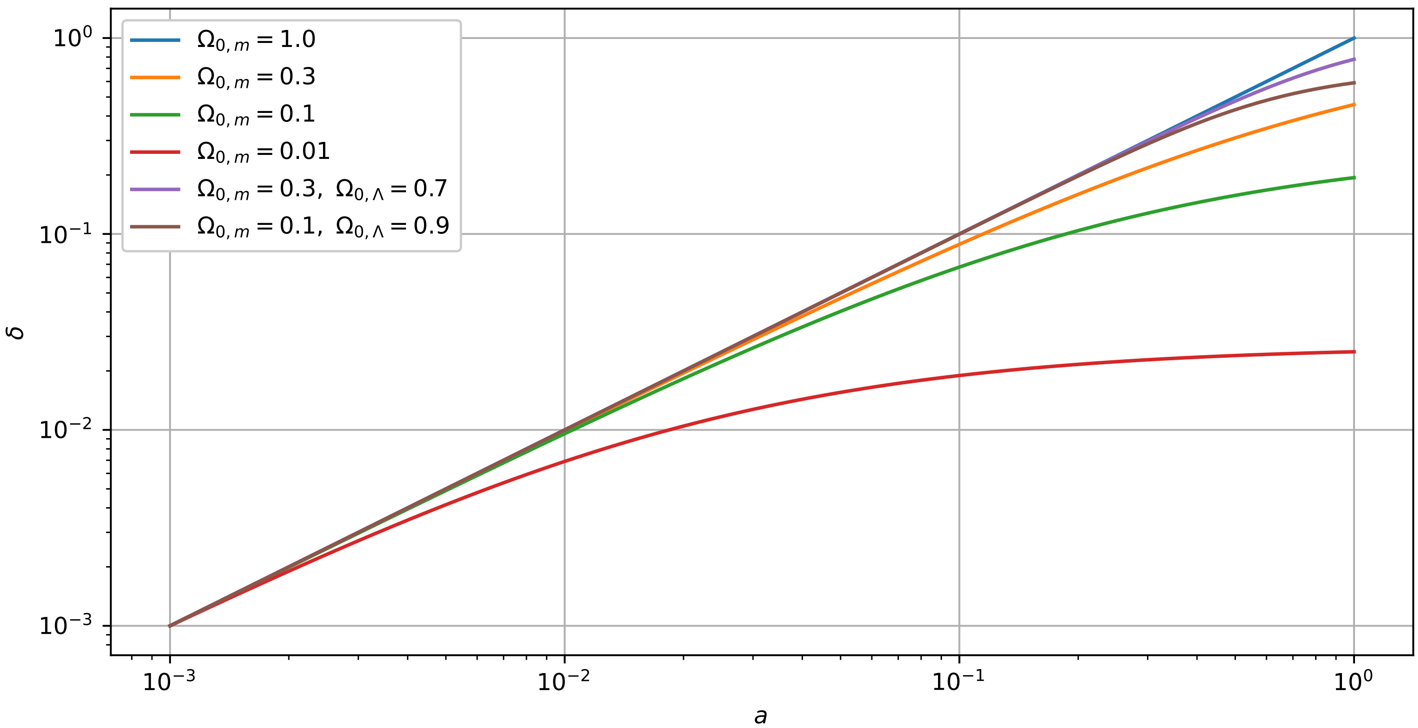

Figure 2: Growth of δ for matter-dominated and flat Λ-cosmologies with varying $\Omega_{0,m}$.

is the same, then increased matter density leads to larger fluctuations (as seen when comparing blue, purple, and brown curves).

Like in the matter-dominated case, it is clear that there would have been most clustering at $z = 5$ in the case with least matter density today ($\Omega_{0,m} = 0.01$). The density fluctuations in the Λ-cosmologies slow down the most at the lowest redshifts so at redshift 5 we would expect the universes with same matter densities to predict comparable clustering if they predict comparable for present time. Hence, models are most sensitive to $\Omega_{0,m}$ in the integrated time $z = 0 - 5$ and it is difficult to distinguish models with different Λ by exclusively comparing clustering at $z = 0$ and $z = 5$.