# Problem Set 5

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The code used for this Problem Set can be found on GitHub.

## Question 1

(a) We showed in class that the number density of bosons is given by

$$n_B = \frac{g}{\pi^2}\zeta(3)\left(\frac{kT}{\hbar c}\right)^3. \tag{1}$$

Here, $g = 2$ and $T = T_{\text{CMB}} = 2.725\text{K}$ so Eq. 1 becomes

$$n_\gamma = \frac{2}{\pi^2}\zeta(3)\left(\frac{kT_{\text{CMB}}}{\hbar c}\right)^3 \approx 410.5\,\text{cm}^{-3}. \tag{2}$$

(b) The critical density of the universe is given by the Friedmann equations as the density required to make the universe flat.

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G} = \frac{3750h^2}{\pi G}\left(\frac{\text{km}}{\text{Mpc}\cdot\text{s}}\right)^2 \approx 1.878\cdot 10^{-32}h^2\frac{\text{kg}}{\text{cm}^3} \tag{3}$$

We showed in class that the energy density of bosons is given by

$$u_B = \frac{\pi^2}{30}g\frac{(kT)^4}{(\hbar c)^3}, \tag{4}$$

which for the CMB is equal to

$$u_\gamma = \frac{\pi^2}{15}\frac{(kT_{\text{CMB}})^4}{(\hbar c)^3} \approx 4.172\cdot 10^{-20}\frac{\text{J}}{\text{cm}^3}. \tag{5}$$

We can now calculate the present radiation density parameter as the ratio between the CMB photon energy density and the critical density:

$$\Omega_{0,\gamma} = \frac{u_\gamma}{\rho_{c,0}c^2} \approx 2.471\cdot 10^{-5}h^{-2}. \tag{6}$$

The universe today is matter dominated since $\Omega_{0,m} = \mathcal{O}(1)$ and $\Omega_{0,\gamma} = \mathcal{O}(10^{-5})$, where we used that $h = \mathcal{O}(1)$.

## Question 2

(a) The ratio between the neutrino temperature and the photon temperature is given by the ratio of the number of degenerate states. This was derived in class, using the assumption that entropy was conserved by electron-positron annihilation (here $g$ denotes degrees of freedom, $s$ is entropy density):

$$s \propto gT^3 \implies g_\gamma T_\gamma^3 = g_\nu T_\nu^3 \implies T_\nu = \left(\frac{g_\gamma}{g_\nu}\right)^{1/3} T_\gamma \tag{7}$$

As above, $g_\gamma = 2$. The neutrino number of degrees of freedom is the sum of the degrees of freedom in the reaction that produced neutrinos (electron-positron annihilation), i.e. $g_\nu = 2 + 2\cdot 7/4 = 11/2$. Finally, the neutrinos cool at the same rate as photons so we can use the present CMB temperature to determine the neutrino temperature. This yields:

$$T_\nu = \left(\frac{2}{11/2}\right)^{1/3} T_{\text{CMB}} = \left(\frac{4}{11}\right)^{1/3} 2.725\,\text{K} \approx 1.945\,\text{K} \tag{8}$$

(b) We have shown that the number density of relativistic fermions is related to the number density of relativistic bosons by a factor of 3/4. Moreover, it scales with $T^3$, yielding:

$$n(\nu_e) + n(\bar{\nu}_e) = \frac{3}{4}\left(\frac{T_\nu}{T_\gamma}\right)^3 n_\gamma = \frac{3}{11}n_\gamma \approx 112.0\,\mathrm{cm}^{-3} \tag{9}$$

(c) In this case, all three species would be relativistic at the time of decoupling. Hence, they would all have the same number density set by the neutrino temperature. Assuming that there is an equal number of neutrinos and anti-neutrinos, this number is $n \approx 112.0/2\,\mathrm{cm}^{-3} = 56\,\mathrm{cm}^{-3}$ per species.

(d) Neutrinos would behave like relativistic particles today if their rest mass energy were much less than the thermal energy, i.e.:

$$m_\nu c^2 \ll kT_\nu = k(1.945\,\mathrm{K}) \approx 1.676 \cdot 10^{-4}\,\mathrm{eV} \tag{10}$$

Since all neutrinos in this assumption has the same number density and temperature (by part c), we may write:

$$\Omega_{0,r} = \frac{u_\gamma + 3u_\nu}{\rho_{c,0}c^2}, \tag{11}$$

where $u_\nu$ is understood to be the energy density of each neutrino flavor (both the particle and anti-particle). The energy density of relativistic fermions is 7/8 times the energy density of relativistic bosons of the same temperature. Moreover, the energy scales with $T^4$.

$$u_\nu = \frac{7}{8}\left(\frac{T_\nu}{T_\gamma}\right)^4 u_\gamma = \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} u_\gamma = \frac{7}{2^{1/3}11^{4/3}}u_\gamma \tag{12}$$

$$\Omega_{0,r} = \frac{u_\gamma + 3u_\nu}{\rho_{c,0}c^2} = \frac{u_\gamma}{\rho_{c,0}c^2}\left(1 + \frac{7}{2^{1/3}11^{4/3}}\right) \tag{13}$$

$$\Omega_{0,r} = \Omega_{0,\gamma}\left(1 + \frac{7}{2^{1/3}11^{4/3}}\right) \approx \left(2.471 \cdot 10^{-5}h^{-2}\right)(1.227) \tag{14}$$

$$\Omega_{0,r} \approx 3.032 \cdot 10^{-5}h^{-2} \tag{15}$$

# Question 3

Let $m$ be the mass of individual baryons (approximately equal to the proton mass).

$$\Omega_{0,B}h^2 = \frac{\rho_B}{\rho_{c,0}}h^2 = \frac{n_B m}{\rho_{c,0}}h^2 = \frac{\eta n_\gamma m}{\rho_{c,0}}h^2 \equiv \beta\gamma \tag{16}$$

We calculate $\beta$ in Eq. 16 using values from Eq. 2 and Eq. 3:

$$\beta = \frac{n_\gamma m}{\rho_{c,0}h^{-2}} \approx \frac{\left(410.5\,\mathrm{cm}^{-3}\right)\left(1.673 \cdot 10^{-27}\,\mathrm{kg}\right)}{1.878 \cdot 10^{-32}\,\mathrm{kg/cm}^3} \approx 3.656 \cdot 10^7. \tag{17}$$

Thus, we can constrain the present baryon density parameter:

$$\eta_{\min}\beta \leq \Omega_{0,B}h^2 \leq \eta_{\max}\beta \tag{18}$$

$$0.02175 \leq \Omega_{0,B}h^2 \leq 0.02314 \tag{19}$$

## Question 4

(a) The matter density scales with $(1+z)^3$ whereas the radiation density scales with $(1+z)^4$. Thus, the equality redshift is given by:

$$\Omega_{0,m}(1+z_{\mathrm{eq}})^3 = \Omega_{0,r}(1+z_{\mathrm{eq}})^4 \tag{20}$$

$$z_{\mathrm{eq}} = \frac{\Omega_{0,m}}{\Omega_{0,r}} - 1 = \frac{0.32}{3.032 \cdot 10^{-5} h^{-2}} - 1 \approx 5200 \tag{21}$$

(b) We showed in Problem Set 1 that the matter-$\Lambda$ equality redshift was $z \approx 0.28$.