Problem Set 2

Christian Hellum Bye

September 23, 2022

Question 1

(a) $1\text{Mpc} = 3.086 \cdot 10^{19}\text{km}$ and $1\text{year} = 3.154 \cdot 10^{7}\text{s}$. Therefore, $H_0 = 70\frac{\text{km}}{\text{Mpc}\cdot\text{s}} = 70 \cdot \frac{1\text{km}}{3.086\cdot 10^{19}\text{km}}\frac{3.154\cdot 10^{7}}{\text{year}} \approx 7.2 \cdot 10^{-11}\text{year}^{-1}$.

(b) Of our three cases, the closed universe is the one with $\Omega_0 = 3.0$. The expansion reaches a maximum when $\dot{a} = 0$ and the Big Crunch occurs when $a = 0$ (for $t \neq 0$ of course). In problem set 1, we derived the formula:

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_0}{a^{3(1+w)}} + \frac{1-\Omega_0}{a^2}.$$

In the matter-dominated case, we can solve for the maximum expansion:

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\frac{\Omega_0}{a^3} + \frac{1-\Omega_0}{a^2}\right)$$
$$(\dot{a})^2 a = H_0^2\left(\Omega_0 + a\left(1-\Omega_0\right)\right)$$
$$\dot{a} = 0 \iff \Omega_0 + a\left(1-\Omega_0\right) = 0 \iff a = \frac{\Omega_0}{\Omega_0 - 1}$$

We derived a parametric solution for the time-evolution of a in the closed universe in problem set 1:

$$\frac{1}{2}\frac{\Omega_0}{\left(\Omega_0-1\right)^{3/2}}(x - \sin x) = H_0 t;\ \cos x = 1 - 2a\frac{\Omega_0 - 1}{\Omega_0}$$
$$a = \frac{\Omega_0}{\Omega_0 - 1} \implies \cos x = -1 \implies x = (2n+1)\pi,\ n \in \mathbb{N}_0 \implies \sin x = 0$$

We showed in problem set 1 that x is increasing with t. We will show below that the Big Crunch occurs at $x = 2\pi$, so greater values of x that maximizes the expansion would not happen before the universe ends. Hence, the maximum expansion happens at $x = \pi$.

$$x = \pi \implies \frac{1}{2}\frac{\Omega_0}{\left(\Omega_0-1\right)^{3/2}}\pi = H_0 t$$
$$t = \frac{1}{H_0}\frac{1}{2}\frac{\Omega_0}{\left(\Omega_0-1\right)^{3/2}}\pi$$

$$\Omega_0 = 3.0,\ H_0 = 7 \cdot 10^{-11}\text{year}^{-1}$$
$$t = \frac{1}{H_0}\frac{1}{2}\frac{3}{2^{3/2}}\pi \approx \frac{1.7}{H_0} = \frac{1.7}{7.2\cdot 10^{-11}\text{year}^{-1}} \approx 2.4 \cdot 10^{10}\text{year}$$

The rate of maximum expansion occurs about 24 billion years after the Big Bang.
To find the time of the Big Crunch, we set $a = 0$. Starting with the parameterization in x, we get:

$$\cos x = 1 - 2a\frac{\Omega_0 - 1}{\Omega_0}$$
$$a = 0 \implies \cos x = 1 \implies x = n2\pi,\ n \in \mathbb{N}_0$$

The Big Bang is obviously the first time ($x = 0$) when $a = 0$, so the Big Crunch is the second time, i.e., the Big Crunch occurs at $x = 2\pi$. We note that the parametric equation for the time t only depends on constants and $x - \sin x$. We found t for the maximum expansion when $x = \pi$ and $x - \sin x = \pi$. Here, $x = 2\pi$ so $x - \sin x = 2\pi$. It follows that the age of the universe at the time of the Big Crunch is simply twice the age at the time of maximum expansion, that is, 48 billion years.

(c) In the flat case, $\Omega_0 = 1$. This gives:

$$\left(\frac{H}{H_0}\right)^2 = \frac{1}{a^{3(1+w)}} = \frac{1}{a^3} \implies (\dot{a})^2 = \frac{H_0^2}{a}$$

$$\int_0^a \sqrt{a'}\mathrm{da}' = \mathrm{H_0 t} \implies \frac{2}{3}\mathrm{a}^{3/2} = \mathrm{H_0 t} \implies \mathrm{a(t)} = \left(\frac{3\mathrm{H_0}}{2}\right)^{2/3} \mathrm{t}^{2/3}$$

$$\dot{a} = \frac{H_0}{\sqrt{a}} = \left(\frac{2}{3}\right)^{1/3} H_0^{2/3} t^{-1/3} \implies \lim_{t\to\infty} \dot{a} = 0$$

(d) The parametric solution we derived was:

$$\frac{1}{2}\frac{\Omega_0}{(1-\Omega_0)^{3/2}}(\sinh x - x) = H_0 t;\ \cosh x = 1 + 2a\frac{1-\Omega_0}{\Omega_0}$$

$$a = \frac{\Omega_0}{1-\Omega_0}\frac{\cosh x - 1}{2} \implies \dot{a} = \frac{\Omega_0}{1-\Omega_0}\frac{\sinh x}{2}\dot{x}$$

First, note that x is approaching ∞ as t approaches ∞ since $t \to \infty \implies \sinh x - x \to \infty \implies x \to \infty$. Thus, to figure out the limit of $\dot{a}$, we need to know the behavior of $\dot{x}$ as x approaches ∞. Differentiating the parametric equation with respect to t gives:

$$\frac{1}{2}\frac{\Omega_0}{(1-\Omega_0)^{3/2}}(\cosh x - 1)\dot{x} = H_0$$

$$\dot{x} = 2H_0\frac{(1-\Omega_0)^{3/2}}{\Omega_0}\frac{1}{\cosh x - 1}$$

$$\dot{a} = \frac{\Omega_0}{1-\Omega_0}\frac{\sinh x}{2}\dot{x} = \frac{\Omega_0}{1-\Omega_0}\frac{\sinh x}{2}2H_0\frac{(1-\Omega_0)^{3/2}}{\Omega_0}\frac{1}{\cosh x - 1}$$

$$\dot{a} = \sqrt{1-\Omega_0}H_0\frac{\sinh x}{\cosh x - 1}$$

$$\lim_{t\to\infty} \dot{a} = \sqrt{1-\Omega_0}H_0 \lim_{x\to\infty}\frac{\sinh x}{\cosh x - 1} = \sqrt{1-\Omega_0}H_0$$

We see that the matter-dominated flat universe eventually stops expanding (the expansion rate approaches 0 asymptotically), whereas the matter-dominated open universe expands forever. In our specfic case ($\Omega_0 = 0.3$), we get $\lim_{t\to\infty}\dot{a} = \sqrt{0.7}H_0 \approx 6.0 \cdot 10^{-11}$year.

(e) We need to set $a = 1$ and compute t.

- Flat case:

$$a(t) = \left(\frac{3H_0}{2}\right)^{2/3} t^{2/3}$$

$$a(t_0) = 1 \implies 1 = \left(\frac{3H_0}{2}\right)^{2/3} t_0^{2/3}$$

$$t_0 = \frac{2}{3H_0} \approx 9.3 \cdot 10^9\text{year}$$

- Closed case:

$$\frac{1}{2}\frac{\Omega_0}{(\Omega_0-1)^{3/2}}(x - \sin x) = H_0 t;\ \cos x = 1 - 2a\frac{\Omega_0 - 1}{\Omega_0}$$

$$\cos x_0 = 1 - 2\frac{\Omega_0 - 1}{\Omega_0} = 1 - 2\cdot\frac{2}{3} = -\frac{1}{3} \implies x_0 \approx 1.91 \vee x_0 \approx 4.37$$

Based on empirical evidence, we know that the universe is expanding. In the closed case, $a = 1$ today and sometime in the future when the universe is contracting (since we found that the maximum value of a is 3/2). We have shown that x increases with t, so today's value of x must be $x_0 = 1.91$.

$$\frac{1}{2}\frac{\Omega_0}{(\Omega_0-1)^{3/2}}(x_0 - \sin x_0) = H_0 t_0$$

$$t_0 = \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0-1)^{3/2}}(x_0 - \sin x_0) = \frac{1}{H_0}\frac{3}{2\cdot 2^{3/2}}(x_0 - \sin x_0) \approx 7.1 \cdot 10^9\text{year}$$

- Open case:

$$\frac{1}{2}\frac{\Omega_0}{(1-\Omega_0)^{3/2}}(\sinh x - x) = H_0 t;\ \cosh x = 1 + 2a\frac{1-\Omega_0}{\Omega_0}$$

$$\cosh x_0 = 1 + 2\frac{1-\Omega_0}{\Omega_0} \approx 5.67 \implies x_0 \approx 2.42$$

$$t_0 = \frac{1}{2H_0}\frac{\Omega_0}{(1-\Omega_0)^{3/2}}(\sinh x_0 - x_0) \approx \frac{0.26}{H_0}(\sinh x_0 - x_0) \approx 11 \cdot 10^9 \text{year}$$

(f) The code used to generate the plot can be found on GitHub.

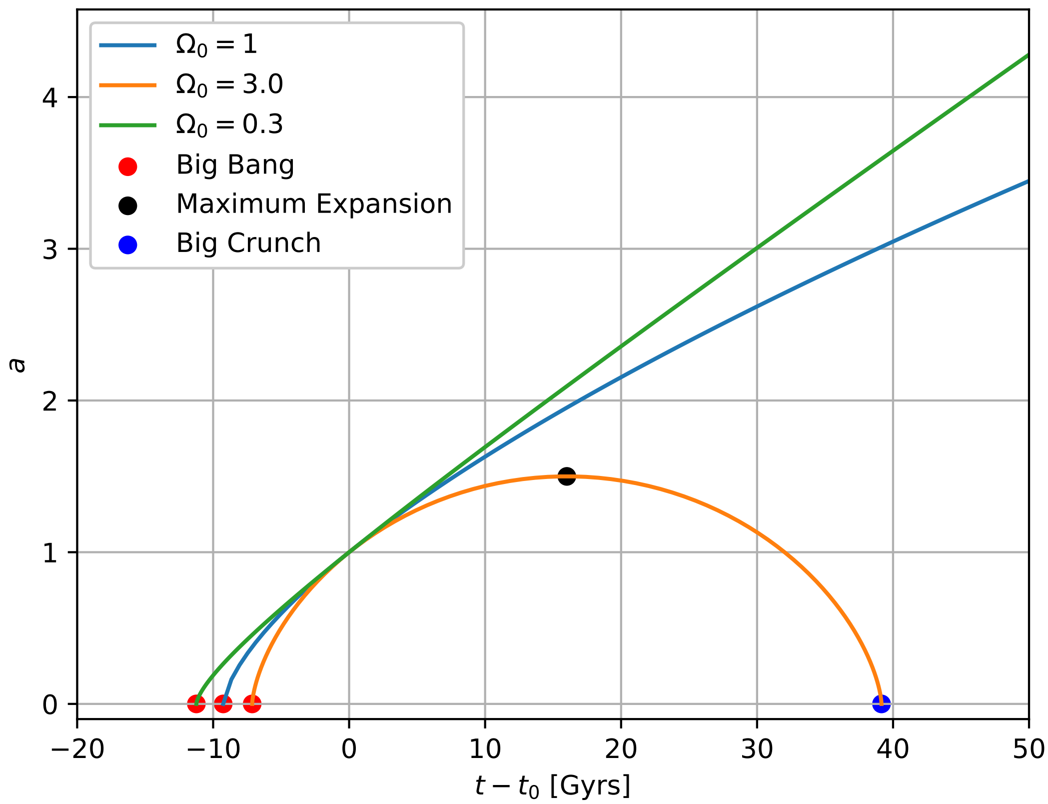

Figure 1: The evolution of the scale parameter given different cosmologies.

2 a)

We had
$$ds^2 = dt^2 - a^2\left(\frac{1}{1-kr^2}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right)$$

We need to find the transformation $r \to \chi$ such that
$$ds^2 = dt^2 - a^2\left(d\chi^2 + S_k^2(\chi)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right).$$

By comparing the two expressions, we see that
$$\frac{1}{1-kr^2}\,dr^2 = d\chi^2, \quad r = S_k(\chi)$$
if $k=0$, obviously $\chi = r$ and $S_k(\chi) = \chi$. If $k \neq 0$:

Integrating:
$$\int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = \chi(r) - \chi(0) \equiv \chi(r)$$
by setting $\chi(0) \equiv 0$ (which we are free to do).

To solve the integral, let $\sin\alpha \equiv \sqrt{k}\, r$. Then $dr = 1/\sqrt{k}\,\cos\alpha\, d\alpha$ and $\sqrt{1-kr^2} = \sqrt{1-\sin^2\alpha} = \cos\alpha$.

$$\chi(r) = \int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = \int_0^{\alpha(r)} \frac{d\alpha'}{\sqrt{k}} = \frac{1}{\sqrt{k}}\alpha(r)$$

$$S_k(\chi) = r = 1/\sqrt{k}\,\sin\alpha = 1/\sqrt{k}\,\sin\left(\sqrt{k}\,\chi\right).$$

Note: if $k<0$: $\sqrt{k} = \sqrt{-|k|} = i\sqrt{|k|}$

$$\to \frac{1}{\sqrt{k}}\sin\left(\sqrt{k}\,\chi\right) = \frac{1}{i\sqrt{|k|}}\sin\left(i\sqrt{|k|}\,\chi\right) = \frac{1}{i\sqrt{|k|}}\,\frac{e^{i^2\sqrt{|k|}\chi} - e^{-i^2\sqrt{|k|}\chi}}{2i}$$

$$= -\frac{1}{\sqrt{|k|}}\,\frac{e^{-\sqrt{|k|}\chi} - e^{\sqrt{|k|}\chi}}{2} = \frac{\sinh\left(\sqrt{|k|}\,\chi\right)}{\sqrt{|k|}}$$

Thus

$$S_k(\chi) = \begin{cases} \chi, & k=0 \\ 1/\sqrt{k}\ \sin(\sqrt{k}\,\chi), & k>0 \\ 1/\sqrt{|k|}\ \sinh(\sqrt{|k|}\,\chi), & k<0 \end{cases}$$

b)

Parametrize a circle in the plane by ~~constant~~ $\chi = \text{constant} \equiv \chi_0$, $\theta = 0$, $0 \le \phi < 2\pi$.

The radius is $a\chi_0$. The circumference is $\int d\ell$, where $d\ell$ is the spatial part of the metric.

Here: $d\ell^2 = a^2 S_k^2(\chi_0)\, d\phi^2$

$$\to \int d\ell = \int_0^{2\pi} a\, S_k(\chi_0)\, d\phi = 2\pi a\, S_k(\chi_0)$$

The ratio is $R = \dfrac{2\pi a\, S_k(\chi)}{a\,\chi_0} = 2\pi\, \dfrac{S_k(\chi_0)}{\chi_0}$.

$$R = \begin{cases} 2\pi, & k=0 \\ 2\pi/\sqrt{k}\ \sin(\sqrt{k}\,\chi_0)/\chi_0, & k>0 \\ 2\pi/\sqrt{|k|}\ \dfrac{\sinh(\sqrt{|k|}\,\chi_0)}{\chi_0}, & k<0 \end{cases}$$

c)

$$x \ll 1/\sqrt{|k|} \rightarrow \sqrt{|k|}\,x \ll 1$$

$k>0$: $$R = \frac{2\pi}{\sqrt{k}} \frac{\sin(\sqrt{k}\,x)}{x} \approx \frac{2\pi}{\sqrt{k}}$$

$k<0$: $$R = \frac{2\pi}{\sqrt{|k|}} \frac{\sinh \sqrt{|k|}\,x}{x} \approx \frac{2\pi}{\sqrt{k}}$$

Here we used that $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \approx x$ for $x \ll 1$ and $\sinh x = x + 1/3!\,x^3 + \frac{1}{5!}x^5 \approx x$ for $x \ll 1$.

d)

$$k<0 \rightarrow R = \frac{2\pi}{\sqrt{|k|}} \frac{\sinh 1/2}{1/2 \cdot 1/\sqrt{|k|}} = 4\pi \sinh\left(\tfrac{1}{2}\right)$$

The ratio between this ratio and that of flat space is

$$\frac{R_{k<0}}{R_{k=0}} = \frac{4\pi \sinh(1/2)}{2\pi} = 2\sinh\left(\tfrac{1}{2}\right) \approx 2\cdot\tfrac{1}{2} = 1$$

(actually ≈ 1.04)

$$k>0 \rightarrow R = \frac{2\pi}{\sqrt{k}} \frac{\sin(1/2)}{1/2 \cdot 1/\sqrt{k}} = 4\pi \sin 1/2$$

$$\rightarrow \frac{R_{k>0}}{R_{k=0}} = 2\sin(1/2) \approx 2 \cdot 1/2 = 1$$ (actually ≈ 0.96)

e)

For $X = 15\,|k|^{-1/2}$:

$k<0$: Ratio is $\sinh(15)/15 \approx 10^5$

$k>0$: Ratio is $\sin(15)/15 \approx 4 \cdot 10^{-2}$

The circumference of a circle increases very fast with radius for $k<0$ and decreases fast (but slower) for $k<0$. When the e)

For $\chi = 15 \; |k|^{-1/2}$:

$k < 0$: Ratio is $\sinh(15)/15 \approx 10^5$

$k > 0$: Ratio is $\sin(15)/15 \approx 4 \cdot 10^{-2}$

The circumference of a circle increases very fast with radius for $k < 0$ and decreases fast (but slower) for $k < 0$. When the radius of the circle is small compared to the curvature radius, it behaves like flat space.