Problem Set 3

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Question 1

(a) The age of the universe is inversely proportional to the matter density, i.e. it decreases when the matter density increases.

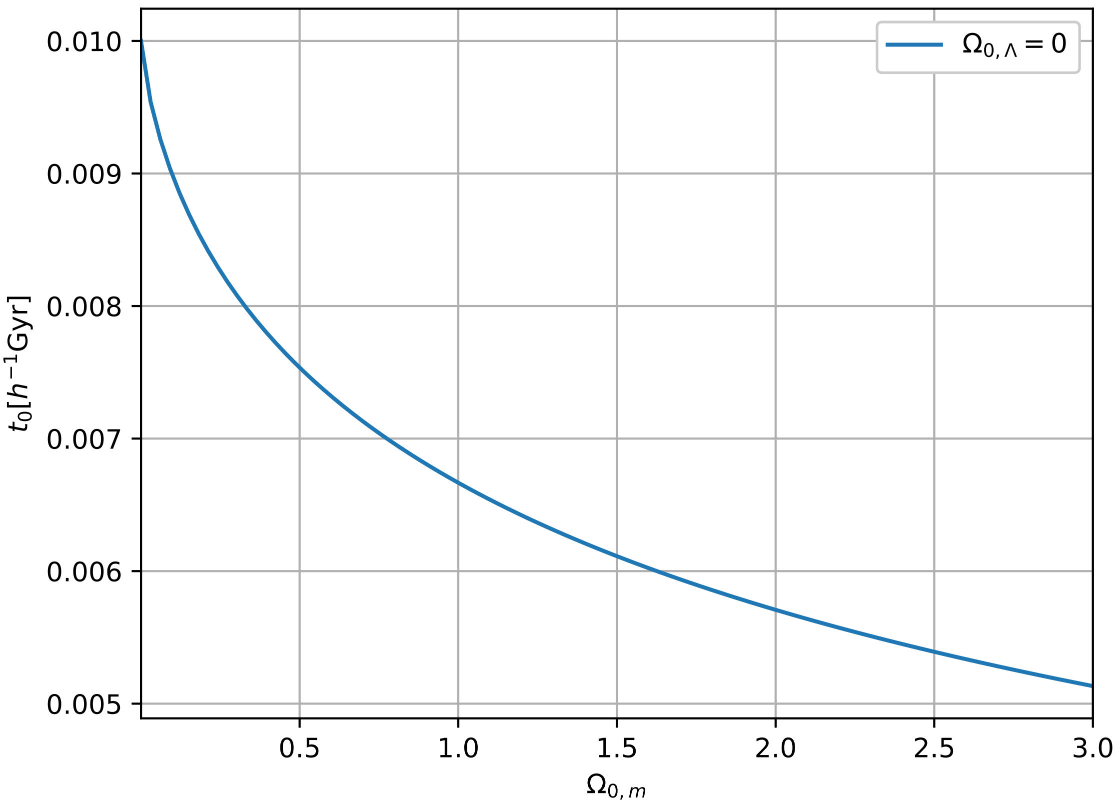


Figure 1: 1a

(b) We have computed t_0 for the matter-dominated universe before. In this case, the universe is flat, but has a cosmological constant. Hence $\Omega_0 = \Omega_{0,m} + \Omega_{0,\Lambda} = 1$. In problem set 1, we derived the formula:

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_0}{a^{3(1+w)}} + \frac{1-\Omega_0}{a^2} = \frac{\Omega_{0,m}}{a^3} + \Omega_{0,\Lambda} = \frac{\Omega_{0,m}}{a^3} + 1 - \Omega_{0,m} \tag{1}$$

Writing $H = \dot{a}/a$, we can invert and solve Eq. 1 for t_0 defined by $a(t_0) \equiv 1$. If $\Omega_{0,m} = 1$, then the universe is matter-dominated and we recover the solution from Problem set 2:

$$\Omega_{0,m} = 1 \implies t_0 = \frac{2}{3H_0} \tag{2}$$

If $\Omega_{0,m} = 0$, then

$$\left(\frac{H}{H_0}\right)^2 = 1 \implies \dot{a} = H_0 a \implies a(t) = e^{H_0(t-t_0)}, \tag{3}$$

which implies that $t_0 = \infty$ in order to satisfy the initial value condition $a(0) = 0$.

If $0 < \Omega_{0,m} < 1$:

$$(\dot{a})^2 = H_0^2 \left(\frac{\Omega_{0,m}}{a} + a^2 \left(1 - \Omega_{0,m}\right) \right) \tag{4}$$

$$\frac{\mathrm{d}a}{\mathrm{d}t} = H_0 \sqrt{\frac{\Omega_{0,m}}{a} + a^2 \left(1 - \Omega_{0,m}\right)} \tag{5}$$

$$\frac{1}{H_0} \int_0^1 \frac{\mathrm{d}a}{\sqrt{\frac{\Omega_{0,m}}{a} + a^2 \left(1 - \Omega_{0,m}\right)}} = \int_0^{t_0} \mathrm{d}t = t_0 \tag{6}$$

$$\frac{1}{H_0 \sqrt{\Omega_{0,m}}} \int_0^1 \frac{\sqrt{a}\mathrm{d}a}{\sqrt{1 + a^3 \frac{1-\Omega_{0,m}}{\Omega_{0,m}}}} = t_0 \tag{7}$$

In Eq. 7, we define $u(a)$ such that:

$$u = \sqrt{1 + a^3 \frac{1 - \Omega_{0,m}}{\Omega_{0,m}}} \tag{8}$$

$$u(0) = 1,\ u(1) = \frac{1}{\sqrt{\Omega_{0,m}}} \tag{9}$$

$$\frac{\mathrm{d}u}{\mathrm{d}a} = \frac{1}{2\sqrt{1 + a^3 \frac{1-\Omega_{0,m}}{\Omega_{0,m}}}} 3a^2 \frac{1 - \Omega_{0,m}}{\Omega_{0,m}} \tag{10}$$

$$\frac{\sqrt{a}\mathrm{d}a}{\sqrt{1 + a^3 \frac{1-\Omega_{0,m}}{\Omega_{0,m}}}} = \frac{2}{3} \frac{\Omega_{0,m}}{1 - \Omega_{0,m}} \frac{\mathrm{d}u}{a^{3/2}} \tag{11}$$

If we invert Eq. 8, we can write the right-hand side of Eq. 11 and thus integrate Eq. 7 in u.

$$u = \sqrt{1 + a^3 \frac{1 - \Omega_{0,m}}{\Omega_{0,m}}} \implies u^2 = 1 + a^3 \frac{1 - \Omega_{0,m}}{\Omega_{0,m}} \implies \frac{1}{a^3} = \frac{1}{u^2 - 1} \frac{1 - \Omega_{0,m}}{\Omega_{0,m}} \tag{12}$$

$$\frac{\sqrt{a}\mathrm{d}a}{\sqrt{1 + a^3 \frac{1-\Omega_{0,m}}{\Omega_{0,m}}}} = \frac{2}{3} \sqrt{\frac{\Omega_{0,m}}{1 - \Omega_{0,m}}} \frac{\mathrm{d}u}{\sqrt{u^2 - 1}} \tag{13}$$

$$\frac{2}{3H_0 \sqrt{1 - \Omega_{0,m}}} \int_1^{\sqrt{1/\Omega_{0,m}}} \frac{\mathrm{d}u}{\sqrt{u^2 - 1}} = t_0 \tag{14}$$

Now, we can solve Eq. 14 by yet another substitution:

$$\cosh x = u \implies \frac{\mathrm{d}u}{\sqrt{u^2 - 1}} = \frac{\sinh x \mathrm{d}x}{\sqrt{\cosh^2 x - 1}} = \mathrm{d}x \tag{15}$$

$$u = 1 \implies \cosh x = 1 \implies x = 0 \tag{16}$$

$$u = \frac{1}{\sqrt{\Omega_{0,m}}} \implies \cosh x = \frac{1}{\sqrt{\Omega_{0,m}}} \implies x = \operatorname{arcosh} \frac{1}{\sqrt{\Omega_{0,m}}} = \ln \left(\frac{1 + \sqrt{1 - \Omega_{0,m}}}{\sqrt{\Omega_{0,m}}} \right) \tag{17}$$

Thus, Eq. 14 becomes:

$$t_0 = \frac{2}{3H_0 \sqrt{1 - \Omega_{0,m}}} \left(\operatorname{arcosh} \frac{1}{\sqrt{\Omega_{0,m}}} - 0 \right) = \frac{2}{3H_0 \sqrt{1 - \Omega_{0,m}}} \ln \left(\frac{1 + \sqrt{1 - \Omega_{0,m}}}{\sqrt{\Omega_{0,m}}} \right) \tag{18}$$

A flat universe with a cosmological constant is older than an open universe with the same matter density.

(c) We can invert Eq. 18 to get an expression for H_0:

$$H_0 = \frac{2}{3t_0 \sqrt{1 - \Omega_{0,m}}} \ln \left(\frac{1 + \sqrt{1 - \Omega_{0,m}}}{\sqrt{\Omega_{0,m}}} \right) \tag{19}$$

Given this (see Figure 3), we conclude that $\Omega_{0,m} \lesssim 0.6$. If $\Omega_{0,m} = 1$ and $t_0 > 11.5$Gyr, H_0 would have to be less than ~ 57 km/s/Mpc, which is significantly smaller than the observed value of ~ 70 km/s/Mpc.

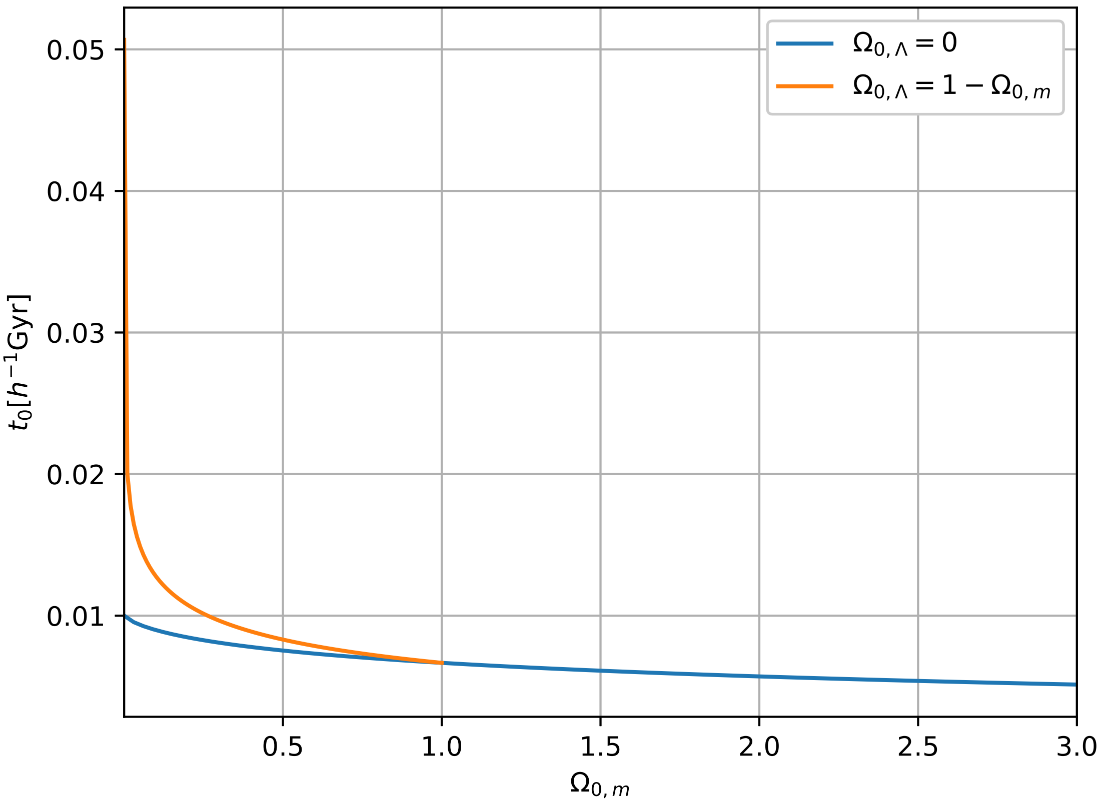


Figure 2: 1b

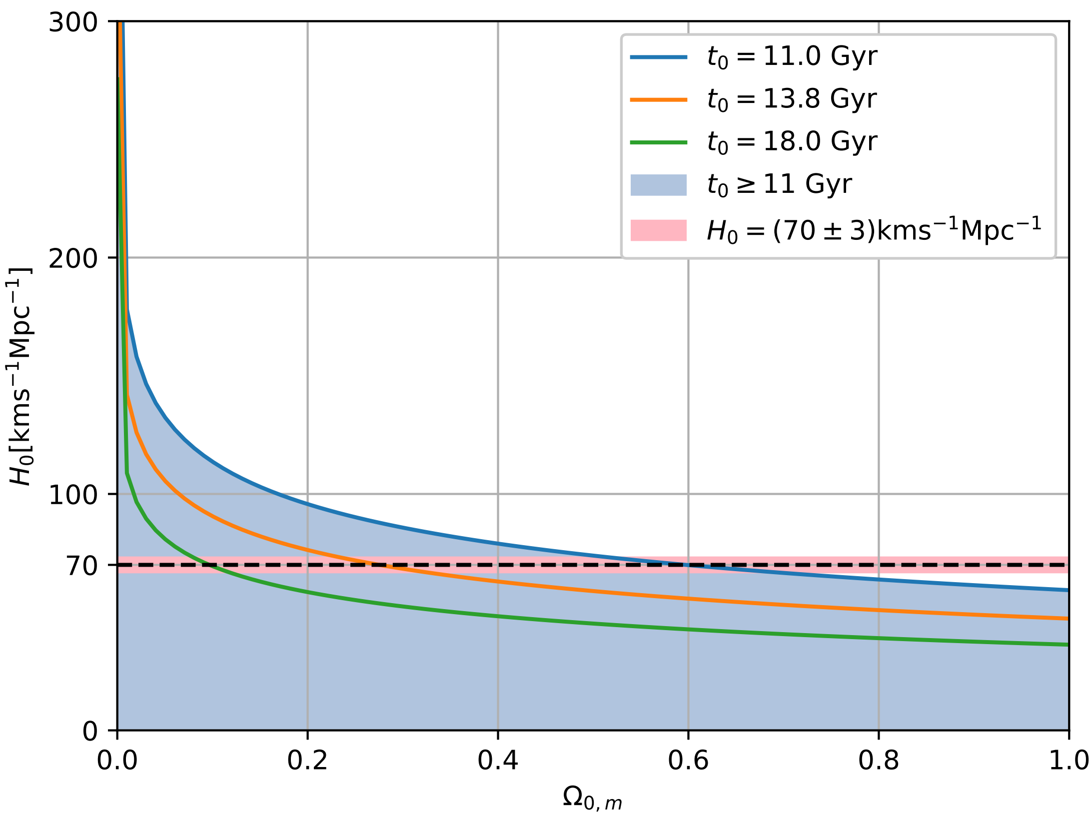


Figure 3: 1c

Question 2

Derivation of the time-redshift relations

We derived in Problem Set 3 a relation between time and redshift:

$$\frac{\mathrm{d}t}{\mathrm{d}z} = -\frac{1}{H_0(1+z)\sqrt{\Omega_{0,m}(1+z)^3 + (1-\Omega_0)(1+z)^2 + \Omega_{0,\Lambda}}} \tag{20}$$

Here, we consider the special cases of a flat universe ($\Omega_0 = 1$) and a matter-dominated universe ($\Omega_{0,\Lambda} = 0$). Denote the redshift of emission by z_*.

$\underline{\Omega_{m,0} = 1, \Omega_{0,\Lambda} = 0:}$

$$\frac{\mathrm{d}t}{\mathrm{d}z} = -\frac{1}{H_0(1+z)\sqrt{(1+z)^3}} = -\frac{1}{H_0(1+z)^{5/2}} \tag{21}$$

$$t = -\frac{1}{H_0}\int_\infty^{z_*} \mathrm{d}z(1+z)^{-5/2} \tag{22}$$

$$t = \frac{2}{3H_0}(1+z)^{-3/2}\Big|_\infty^{z_*} = \frac{2}{3H_0}(1+z_*)^{-3/2} \tag{23}$$

$\underline{\Omega_{m,0} < 1, \Omega_{0,\Lambda} = 1 - \Omega_{m,0}:}$

$$\frac{\mathrm{d}t}{\mathrm{d}z} = -\frac{1}{H_0(1+z)\sqrt{\Omega_{0,m}(1+z)^3 + \Omega_{0,\Lambda}}} \tag{24}$$

$$t = -\frac{1}{H_0}\int_\infty^{z_*} \frac{\mathrm{d}z}{(1+z)\sqrt{\Omega_{0,m}(1+z)^3 + \Omega_{0,\Lambda}}} \tag{25}$$

$$u = \sqrt{\Omega_{0,m}(1+z)^3 + \Omega_{0,\Lambda}} \tag{26}$$

$$\Omega_{0,m}(1+z)^3 = u^2 - \Omega_{0,\Lambda} \tag{27}$$

$$\frac{\mathrm{d}u}{\mathrm{d}z} = \frac{3\Omega_{0,m}(1+z)^2}{2\sqrt{\Omega_{0,m}(1+z)^3 + \Omega_{0,\Lambda}}} \tag{28}$$

$$\frac{\mathrm{d}z}{(1+z)\sqrt{\Omega_{0,m}(1+z)^3 + \Omega_{0,\Lambda}}} = \frac{2\mathrm{d}u}{3\Omega_{0,m}(1+z)^3} = \frac{2\mathrm{d}u}{3\left(u^2 - \Omega_{0,\Lambda}\right)} \tag{29}$$

$$t = -\frac{2}{3H_0}\int_\infty^{u(z_*)} \frac{\mathrm{d}u}{u^2 - \Omega_{0,\Lambda}} = \frac{2}{3H_0}\int_{u(z_*)}^\infty \frac{\mathrm{d}u}{\left(u - \sqrt{\Omega_{0,\Lambda}}\right)\left(u + \sqrt{\Omega_{0,\Lambda}}\right)} \tag{30}$$

$$t = \frac{1}{3H_0\sqrt{\Omega_{0,\Lambda}}}\int_{u(z_*)}^\infty \mathrm{d}u\left(\frac{1}{u - \sqrt{\Omega_{0,\Lambda}}} - \frac{1}{u + \sqrt{\Omega_{0,\Lambda}}}\right) \tag{31}$$

$$t = \frac{1}{3H_0\sqrt{\Omega_{0,\Lambda}}}\ln\left(\frac{u(z_*) + \sqrt{\Omega_{0,\Lambda}}}{u(z_*) - \sqrt{\Omega_{0,\Lambda}}}\right) \tag{32}$$

$$t = \frac{1}{3H_0\sqrt{\Omega_{0,\Lambda}}}\ln\left(\frac{\sqrt{\Omega_{0,m}(1+z_*)^3 + \Omega_{0,\Lambda}} + \sqrt{\Omega_{0,\Lambda}}}{\sqrt{\Omega_{0,m}(1+z_*)^3 + \Omega_{0,\Lambda}} - \sqrt{\Omega_{0,\Lambda}}}\right) \tag{33}$$

$\underline{\Omega_{m,0} < 1, \Omega_{0,\Lambda} = 0:}$

This is the matter-dominated open universe. In Problem set 1, we derived a parametric solution to the Friedmann equation:

$$t = \frac{1}{2H_0}\frac{\Omega_{0,m}}{(1-\Omega_{0,m})^{3/2}}(\sinh x - x) \tag{34}$$

$$\cosh x = 1 + 2a\frac{1-\Omega_{0,m}}{\Omega_{0,m}} = 1 + \frac{2}{1+z_*}\frac{1-\Omega_{0,m}}{\Omega_{0,m}} \tag{35}$$

Universe Age at Time of CMB Emission ($z_* = 1100$)

We can now compute the age of the universe when the CMB was released.

$\underline{\Omega_{m,0} = 0.32, \Omega_{0,\Lambda} = 0.68:}$ Using Eq. 33, we get $t = 448,000$ years.

$\underline{\Omega_{m,0} = 0.32, \Omega_{0,\Lambda} = 0:}$ $t = 447,800$ years.

$\underline{\Omega_{m,0} = 1., \Omega_{0,\Lambda} = 0:}$ $t = 253,000$ years.

Universe Age at Time of JWST Galaxy Emission ($z_* = 13$)

$\Omega_{m,0} = 0.32, \Omega_{0,\Lambda} = 0.68$: $t = 0.312$ Gigayears.
$\Omega_{m,0} = 0.32, \Omega_{0,\Lambda} = 0$: $t = 0.299$ Gigayears.
$\Omega_{m,0} = 1., \Omega_{0,\Lambda} = 0$: $t = 0.177$ Gigayears.

Question 3

(a) The angular diameter is a function of the physical size of the object and the distance to the object. The size L is the size of the object at the time of emission t_*. Hence, the distance has to be given by the physical distance at the time of emission. Denote the coming distance by r (which is a function of redshift derived in Problem Set 3), then the physical distance is ar and the angular diameter θ is given by:

$$\theta = \frac{L}{a(t_*)r(z_*)} = \frac{L(1+z_*)}{|k|^{-1/2}\mathrm{sinn}\left(\frac{c|k|^{1/2}}{H_0}\int_0^{z_*}\frac{\mathrm{d}z}{\sqrt{\Omega_{0,m}(1+z)^3+(1-\Omega_{0,m})(1+z)^2}}\right)}, \tag{36}$$

where sinn$x$ is as defined in Problem Set 3 and $k = (H_0/c)^2(\Omega_{0,m} - 1)$ as derived in Problem Set 1. We want θ in units of h arcsec and L in units of kpc. Thus c must have units such that the length dimension is *kpc* and the time dimension is inverse h units. Hence, we need

$$c = c_{\mathrm{m/s}}\frac{\mathrm{m}}{\mathrm{s}} = c_{\mathrm{m/s}}\frac{\mathrm{m}\cdot\mathrm{Mpc}}{\mathrm{kpc}\cdot\mathrm{km}}\mathrm{kpc}\frac{\mathrm{km}}{\mathrm{Mpc}\cdot\mathrm{s}} = c_{\mathrm{m/s}}\mathrm{kpc}\frac{\mathrm{km}}{\mathrm{Mpc}\cdot\mathrm{s}} \tag{37}$$

That is, c has the same value in SI units and in units of kpc$\frac{\mathrm{km}}{\mathrm{Mpc}\cdot\mathrm{s}}$, ie. $3.00\cdot 10^8$.

$$\theta = \frac{100(Lh/c)(1+z_*)\sqrt{|\Omega_{0,m}-1|}}{\mathrm{sinn}\left(\sqrt{|\Omega_{0,m}-1|}\int_0^{z_*}\frac{\mathrm{d}z}{\sqrt{\Omega_{0,m}(1+z)^3+(1-\Omega_{0,m})(1+z)^2}}\right)} \tag{38}$$

$$\theta[h\mathrm{arcsec}] = 1.2\cdot 10^{-3}\frac{(1+z_*)\sqrt{|\Omega_{0,m}-1|}}{\mathrm{sinn}\left(\sqrt{|\Omega_{0,m}-1|}\int_0^{z_*}\frac{\mathrm{d}z}{\sqrt{\Omega_{0,m}(1+z)^3+(1-\Omega_{0,m})(1+z)^2}}\right)}L[\mathrm{kpc}] \tag{39}$$

(b) We need to solve the integral inside the sinn-function for flat, open, and closed universes. If the universe is flat, we get:

$$\int_0^{z_*}\frac{\mathrm{d}z}{\sqrt{\Omega_{0,m}(1+z)^3+(1-\Omega_{0,m})(1+z)^2}} = \int_0^{z_*}\frac{\mathrm{d}z}{\sqrt{(1+z)^3}} = 2 - \frac{2}{\sqrt{1+z_*}} \tag{40}$$

$$\theta = 1.2\cdot 10^{-3}\frac{1+z_*}{2-2/\sqrt{1+z_*}}L \tag{41}$$

If it is open or closed, the integral is more complicated:

$$\int_0^{z_*}\frac{\mathrm{d}z}{\sqrt{\Omega_{0,m}(1+z)^3+(1-\Omega_{0,m})(1+z)^2}} = \int_0^{z_*}\frac{\mathrm{d}z}{(1+z)\sqrt{\Omega_{0,m}(1+z)+(1-\Omega_{0,m})}} \tag{42}$$

$$u = \sqrt{\Omega_{0,m}(1+z)+(1-\Omega_{0,m})} \tag{43}$$

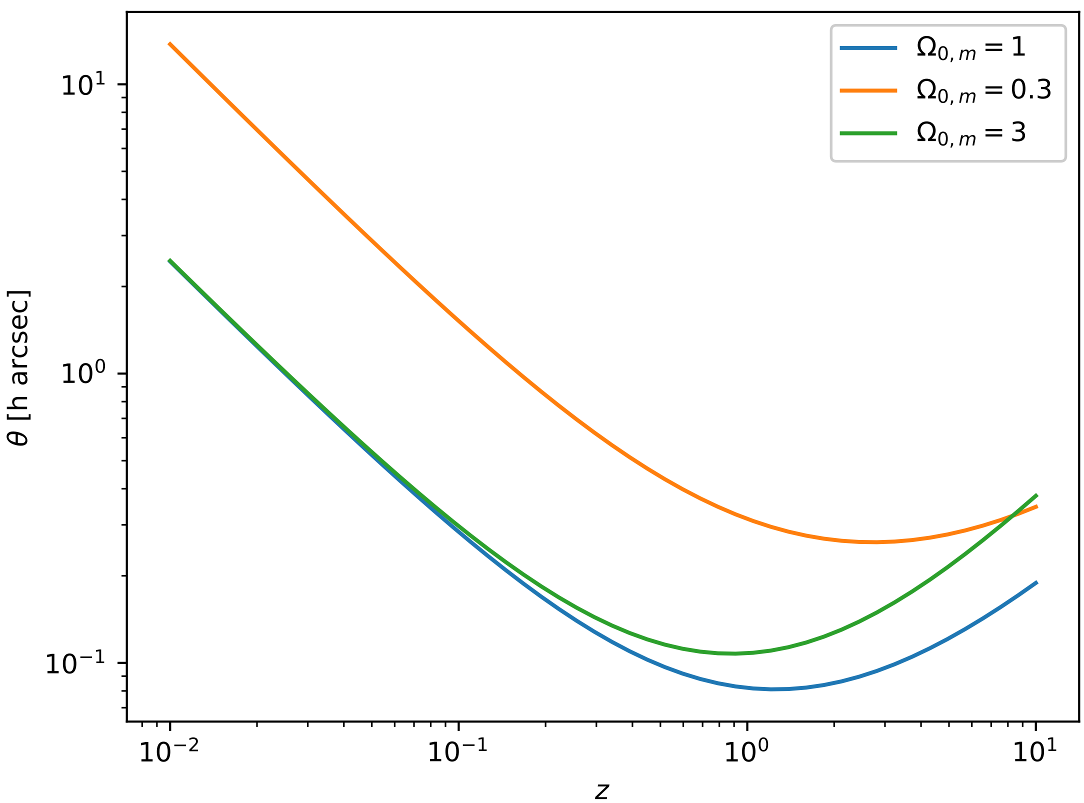
$$1+z = \frac{u^2+\Omega_{0,m}-1}{\Omega_{0,m}} \tag{44}$$

$$\frac{\mathrm{d}u}{\mathrm{d}z} = \frac{\Omega_{0,m}}{2u} \implies \frac{\mathrm{d}z}{u(1+z)} = \frac{2}{|\Omega_{0,m}-1|}\frac{\mathrm{d}u}{\left(u/\sqrt{|\Omega_{0,m}-1|}\right)^2+1} \tag{45}$$

$$\int_0^{z_*}\frac{\mathrm{d}z}{\sqrt{\Omega_{0,m}(1+z)^3+(1-\Omega_{0,m})(1+z)^2}} = \frac{2}{\Omega_{0,m}-1}\int_1^{u(z_*)}\frac{\mathrm{d}u}{\left(u/\sqrt{|\Omega_{0,m}-1|}\right)^2+1} \tag{46}$$

$$= \frac{2}{\sqrt{|\Omega_{0,m}-1|}}\left(\arctan\left(u(z_*)/\sqrt{|\Omega_{0,m}-1|}\right) - \arctan\left(1/\sqrt{|\Omega_{0,m}-1|}\right)\right) \tag{47}$$

We see that the angular size of an object appears to decrease with distance until a certain point (depending on the cosmology), where things that are further appear larger. This is because light from high-z objects was emitted when the universe was much smaller - hence countering the effect of them being further away (the expansion is not linear).



$\Omega_{0,m} = 1$
$\Omega_{0,m} = 0.3$
$\Omega_{0,m} = 3$
10^1
10^0
10^{-1}
θ [h arcsec]
10^{-2}
10^{-1}
10^0
10^1
z