1 a)

Friedmann equation:

$$(1) \quad H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

At time $t = t_0$ (current time),
$H = H_0$, $a = 1$, $\rho = \Omega_0 \rho_c$

$$\rightarrow \quad H_0^2 = \frac{8\pi G}{3}\Omega_0 \rho_{c,0} - k \qquad (2)$$

Now, ρ_c is the density that makes $k=0$ so

$$H^2 = \frac{8\pi G}{3}\rho_c \quad \text{by } (1)$$

$$t=t_0 \rightarrow \quad H_0^2 = \frac{8\pi G}{3}\rho_{c,0} \rightarrow \rho_{c,0} = \frac{3H_0^2}{8\pi G}$$

$$(2) \quad H_0^2 = \frac{8\pi G}{3}\Omega_0 \cdot \frac{3H_0^2}{8\pi G} - k = \Omega_0 H_0^2 - k$$

$$\rightarrow \quad \boxed{k = H_0^2(\Omega_0 - 1)}$$

b)

Substitute the result from a) in to the Friedmann equation:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{H_0^2(\Omega_0 - 1)}{a^2}$$

We derived in class that $\rho \propto a^{-3(1+w)}$ for fluids defined by the E.O.S $p = w\rho$.

Hence,
$$\frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\frac{\rho}{\rho_0}\rho_0 = \frac{8\pi G}{3}\frac{a^{-3(1+w)}}{a_0^{-3(1+w)}}\rho_0$$

As $a_0 = a(t_0) \equiv 1$, we get

$$\frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\frac{1}{a^{3+3w}}\rho_0$$

Thus,

$$H^2 = \frac{8\pi G}{3} \frac{1}{a^{3+3w}} \rho_0 - \frac{H_0^2 (\Omega_0 - 1)}{a^2}$$

In c), we derived that:

$$\rho_0 = \Omega_0 \rho_{c,0} = \Omega_0 \frac{3H_0^2}{8\pi G}$$

$$\rightarrow H^2 = \frac{H_0^2 \Omega_0}{a^{3+3w}} - H_0^2 \frac{(\Omega_0 - 1)}{a^2}$$

$$\rightarrow \boxed{H^2 = H_0^2 \left(\frac{\Omega_0}{a^{3+3w}} + \frac{1 - \Omega_0}{a^2} \right)}$$

2 a)

$$1-\Omega = \frac{1-\Omega_0}{1-\Omega_0+\Omega_{0,\Lambda}a^2+\Omega_{0,m}a^{-1}+\Omega_{0,r}a^{-2}}$$

For $a = 10^{-3}$, $\Omega_{0,r}=0$ we get

$\Omega_{0,m}$	$\Omega_{0,\Lambda}$	$\Omega(a)$
0.32	0.0	0.9979
0.32	0.68	1.0000
1.0	0.0	1.0000
5.0	0.0	1.0008

At $a=10^{-3}$ the term $\Omega_{0,m}a^{-1}$ dominates so $1-\Omega \approx 0 \to \Omega \simeq 1$ in all cases. However, for $\Omega_{0,m}$ sufficiently large, $\Omega > 1$ and for $\Omega_{0,m}$ sufficiently small $\Omega < 1$. When $\Omega_{0,m}+\Omega_{0,\Lambda}=1$ $\Omega=0$. So there are strong geometrical consequences of the values of these parameters.

(The plot is attached on a different page)
We see that $\Omega(a) \to 1$ for small a in all 4 cases. If $\Omega = 1$ at $a = 10^{-3}$ it will stay $= 1$ forever. This is the orange and green lines (the cases where $\Omega_0 = 1$). When $\Omega > 1$ initially, $\Omega(a)$ blows up exponentially and when $\Omega < 1$ $\Omega(a)$ decays exponentially. This is an example of the flatness problem.

b)

Friedmann Equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

Matter-dominated: $\Omega = \Omega_m$, $w = 0$, $\rho \propto a^{-3}$

By question 1, $k = H_0^2(\Omega_0 - 1)$

Thus,

closed universe $\Leftrightarrow k > 0 \Leftrightarrow \Omega_0 > 1$
open universe $\Leftrightarrow k < 0 \Leftrightarrow \Omega_0 < 1$

Derived in question 1b:

$$H^2 = H_0^2\left(\frac{\Omega_0}{a^3} + \frac{1-\Omega_0}{a^2}\right)$$

$$a^2H^2 = \dot{a}^2 = H_0^2\left(\frac{\Omega_0}{a} + 1 - \Omega_0\right)$$

This is a seperable ODE that we will integrate from $t=0$ to $t=t$ and $a=0$ to $a=a(t)\equiv a$.

$$\frac{da}{dt} = H_0\sqrt{\frac{\Omega_0}{a} + 1 - \Omega_0}$$

$$\int_0^a \frac{da'}{\sqrt{\frac{\Omega_0}{a'} + 1 - \Omega_0}} = \int_0^t dt'\, H_0 = H_0 t$$

~~Worked~~ ~~Page~~

Open universe: $1-\Omega_0 > 0$

We can rewrite the denominator in the integral on the left-hand side:

$$\sqrt{\frac{\Omega_0}{a} + 1-\Omega_0} = \sqrt{1-\Omega_0}\sqrt{\frac{\Omega_0}{1-\Omega_0}\frac{1}{a} + 1}$$

$$= \frac{\sqrt{1-\Omega_0}}{a}\sqrt{\frac{\Omega_0}{1-\Omega_0}a + a^2} \quad (*)$$

Set $\frac{\Omega_0}{1-\Omega_0} \equiv 2b$.

$$(*) = \frac{\sqrt{1-\Omega_0}}{a}\sqrt{a^2+2ab} = \frac{\sqrt{1-\Omega_0}}{a}\sqrt{(a+b)^2 - b^2}$$

$$= \frac{\sqrt{1-\Omega_0}}{a} b\sqrt{\left(\frac{a+b}{b}\right)^2 - 1}$$

Set $\cosh x \equiv (a+b)/b$.

$$(*) = \sqrt{1-\Omega_0}\,\frac{b}{a}\sqrt{\cosh^2 x - 1} = \sqrt{1-\Omega_0}\,\frac{b}{a}\sinh x$$

Thus,

$$\int_0^a \frac{da'}{\sqrt{\frac{\Omega_0}{a'} + 1-\Omega_0}} = \frac{1}{b\sqrt{1-\Omega_0}}\int_0^a \frac{a'\,da'}{\sinh x} \quad (**).$$

$$\cosh x = \frac{a+b}{b} \rightarrow a = b(\cosh x - 1) \rightarrow \frac{da}{dx} = b\sinh x$$

$$\rightarrow a\,da = b^2(\cosh x - 1)\sinh x\,dx$$

$$\rightarrow (**) = \frac{b}{\sqrt{1-\Omega_0}}\int_0^{x(a)} (\cosh x' - 1)\,dx'$$

The solution to the ODE is given by

$$\frac{b}{\sqrt{1-\Omega_0}} \int_0^{x(a)} (\cosh x' - 1)\, dx' = H_0 t$$

$$\frac{b}{\sqrt{1-\Omega_0}} (\sinh x' - x') \Big|_{x'=0}^{x'=x} = H_0 t$$

$$\frac{b}{\sqrt{1-\Omega_0}} (\sinh x - x) = H_0 t$$

Since $\frac{d}{dx}(\sinh x) = \cosh x$ and $\frac{d}{dx}(\cosh x) = \sinh x$, we can easily Taylor expand $\sinh x$. We use that $\sinh(0) = 0$, $\cosh(0) = 1$.

$$\sinh x = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{(2k+1)}.$$

Thus, $\sinh x - x = \sum_{k=1}^{\infty} \frac{1}{(2k+1)!} x^{(2k+1)!}$.

Hence $(\sinh x - x)$ is strictly increasing and small values of t correspond to small values of x.

For early times (very small t), we can therefore write

$$H_0 t = \frac{b}{\sqrt{1-\Omega_0}} \left(x^3/3! + O(x^5) \right) \approx \frac{b}{3!\sqrt{1-\Omega_0}} x^3$$

We note that $\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \ldots$
$\rightarrow$ for small x, $(\cosh x - 1) \approx x^2/2$

Combining these, we got the following relations at small t:

$$x^3 = \frac{6\sqrt{1-\Omega_0}}{b} H_0 t$$

$$a = b(\cosh x - 1) = \frac{b}{2} x^2$$

$$\Rightarrow a = \frac{b}{2}\left(\frac{6\sqrt{1-\Omega_0}}{b} H_0\right)^{2/3} t^{2/3}$$

$$\rightarrow a \propto t^{2/3}$$

In the closed case, $1-\Omega_0 < 0$.

We would therefore write

$$\sqrt{\frac{\Omega_0}{a} + 1 - \Omega_0} = \sqrt{\Omega_0 - 1}\sqrt{\frac{\Omega_0}{\Omega_0 - 1}\cdot\frac{1}{a} - 1}$$

In this case, $\frac{\Omega_0}{\Omega_0 - 1} = 2b$

and we get

$$\sqrt{\frac{\Omega_0}{a}+1-\Omega_0} = \frac{\sqrt{\Omega_0 - 1}}{a}\, b\sqrt{1-\left(\frac{a-b}{b}\right)^2}$$

Now, we let $\cos x = -(a-b)/b$ and get

$$\frac{\sqrt{\Omega_0-1}}{a}\, b\sqrt{1-(-\cos x)^2} = \frac{\sqrt{\Omega_0-1}}{a}\, b \sin x$$

Thus, $$\int_0^a \frac{da'}{\sqrt{\frac{\Omega_0}{a'}+1-\Omega_0}} = \frac{1}{b\sqrt{\Omega_0-1}}\int_0^a \frac{a'\,da'}{\sin x}$$

With $\cos x = -(a-b)/b$, we get

$$a = b(1-\cos x) \rightarrow da = b\sin x\, dx$$

$$a = 0 \rightarrow x = 0$$

$$\rightarrow \frac{1}{b\sqrt{\Omega_0-1}}\int_0^{x(a)} \frac{b^2(1-\cos x')\sin x'}{\sin x'}\,dx' = \frac{b}{\sqrt{\Omega_0-1}}(x'-\sin x')\Big|_0^{x(a)}$$

$$= \frac{b}{\sqrt{\Omega_0-1}}(x-\sin x)$$

As $x \geq \sin x \ \forall x \geq 0$, we know that $x-\sin x$ is increasing and again small t corresponds to small x.

~~Small t approximation~~

$$\frac{b}{\sqrt{\Omega_0-1}}(x-\sin x) = H_0 t \rightarrow H_0 t \approx \frac{b}{\sqrt{\Omega_0-1}}\frac{x^3}{3!}, \text{ for small } t$$

$$\Rightarrow x^3 \approx \frac{6\sqrt{\Omega_0-1}}{b} H_0 t$$

$\cos x = -(a-b)/b = 1 - a/b$

For small x, $\cos x \simeq 1 - x^2/2$

$\rightarrow 1 - x^2/2 \simeq 1 - a/b$

$\rightarrow x^2 \simeq 2a/b$

Putting everything together:

$$\frac{2a}{b} = x^2 = (x^3)^{2/3} = \left(\frac{6\sqrt{\Omega_0 - 1}}{b} H_0 t\right)^{2/3}$$

Finally, we conclude:

In the open case:

$$a = \frac{b}{2}\left(\frac{6\sqrt{1-\Omega_0}}{b} H_0\right)^{2/3} t^{2/3}, \quad b = \frac{1}{2}\frac{\Omega_0}{1-\Omega_0}$$

In the closed case

$$a = \frac{b}{2}\left(\frac{6\sqrt{\Omega_0 - 1}}{b} H_0\right)^{2/3} t^{2/3}, \quad b = \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1}$$

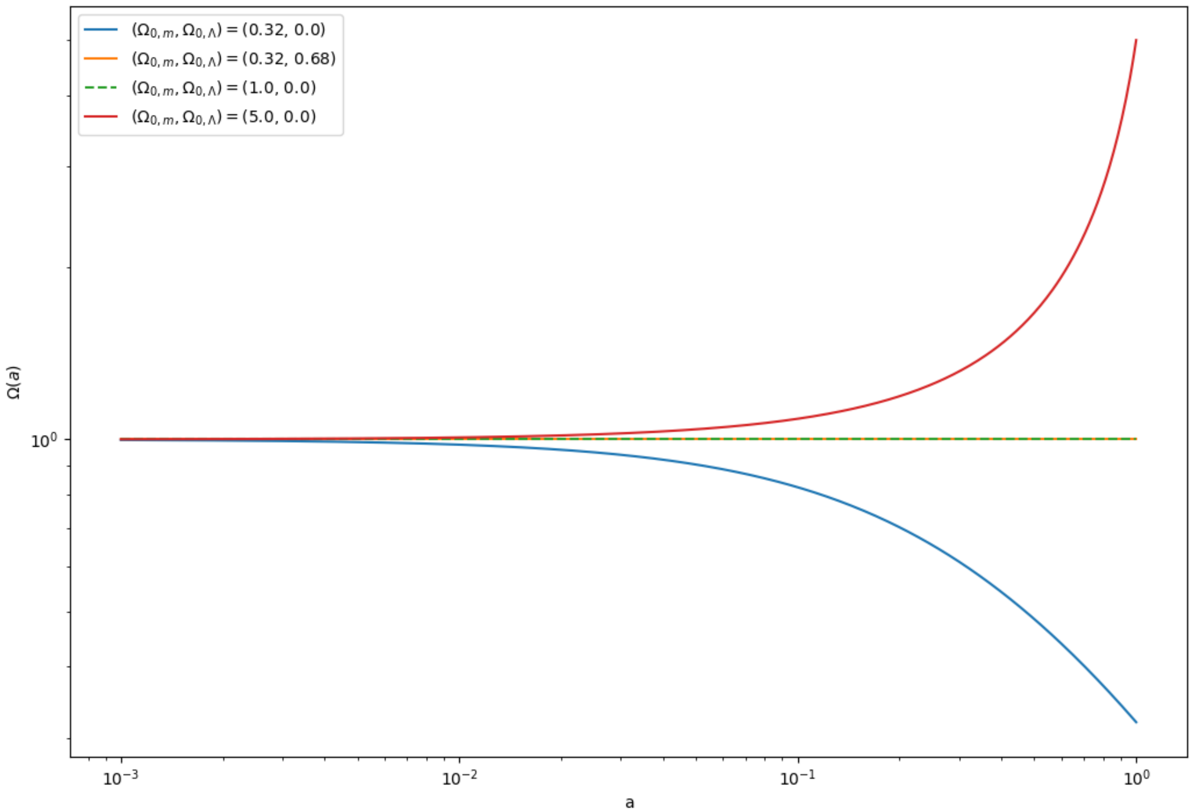
In general (open or closed):

$$a = \frac{b}{2}\left(\frac{6\sqrt{|1-\Omega_0|}}{b} H_0\right)^{2/3} t^{2/3}, \quad b = \frac{1}{2}\frac{\Omega_0}{|1-\Omega_0|}$$

$$\rightarrow \boxed{a \propto t^{\beta} \text{ with } \beta = 2/3}$$

This is the same as the matter-dominated flat case. So all cases should be the same at early times, which is constent with the plot in a).

Question 2a



$(\Omega_{0,m}, \Omega_{0,\Lambda}) = (0.32, 0.0)$
$(\Omega_{0,m}, \Omega_{0,\Lambda}) = (0.32, 0.68)$
$(\Omega_{0,m}, \Omega_{0,\Lambda}) = (1.0, 0.0)$
$(\Omega_{0,m}, \Omega_{0,\Lambda}) = (5.0, 0.0)$
$\Omega(a)$
10^0
10^{-3}
10^{-2}
10^{-1}
10^0
a

3 a)

We derived in question 1:

$$H^2 = H_0^2\left(\frac{\Omega_0}{a^{3(1+w)}} + \frac{1-\Omega_0}{a^2}\right)$$

In terms of the components:

$$H^2 = H_0^2\left(\frac{\Omega_{0,r}}{a^4} + \frac{\Omega_{0,m}}{a^3} + \frac{1-\Omega_0}{a^2} + \Omega_{0,\Lambda}\right)$$

$$\rightarrow \dot{a}^2 = H_0^2\left(\Omega_{0,r}\, a^{-2} + \Omega_{0,m}\, a^{-1} + 1-\Omega_0 + \Omega_{0,\Lambda}\, a^2\right)$$

Differentiate both sides wrt time:

$$2\dot{a}\ddot{a} = H_0^2\left(-2\Omega_{0,r}\, a^{-3}\dot{a} - \Omega_{0,m}\, a^{-2}\dot{a} + 2\Omega_{0,\Lambda}\, a\dot{a}\right)$$

$$\ddot{a} = H_0^2\left(-\Omega_{0,r}\, a^{-3} - \tfrac{1}{2}\Omega_{0,m}\, a^{-2} + \Omega_{0,\Lambda}\, a\right) \quad (*)$$

We need to solve $\ddot{a} = 0$ subject to $(*)$:

$$\ddot{a} = 0 \rightarrow -\Omega_{0,r}\, a^{-3} - \tfrac{1}{2}\Omega_{0,m}\, a^{-2} + \Omega_{0,\Lambda}\, a = 0$$

Thu

Dividing by a:

$$-\Omega_{0,r}\, a^{-4} - \frac{1}{2}\Omega_{0,m}\, a^{-3} + \Omega_{0,\Lambda} = 0$$

By question 1, $k = H_0^2(\Omega_0 - 1)$

$$\Rightarrow \Omega_0 = k/H_0^2 + 1$$

$$\Rightarrow \Omega_{0,r} = \Omega_0 - (\Omega_{0,m} + \Omega_{0,\Lambda})$$

$$\Rightarrow \Omega_{0,r} = k/H_0^2 + 1 - (\Omega_{0,m} + \Omega_{0,\Lambda}) \qquad (**)$$

Now, $a = \frac{1}{1+z} \Rightarrow \frac{1}{a} = z+1$

$$\Rightarrow \boxed{-\Omega_{0,r}(z+1)^4 - \frac{1}{2}\Omega_{0,m}(z+1)^3 + \Omega_{0,\Lambda} = 0}$$

b)

We had

$$-\Omega_{0,r}(z+1)^4 - \frac{1}{2}\Omega_{0,m}(z+1)^3 + \Omega_{0,\Lambda} = 0$$

with $\Omega_{0,r} = k/H_0^2 + 1 - (\Omega_{0,m} + \Omega_{0,\Lambda})$

$\Omega_{0,m} = 0.32$, $\Omega_{0,\Lambda} = 0.68$, $k = 0$

$\rightarrow \Omega_{0,r} = 0$

In our universe:

$$-\frac{1}{2}\Omega_{0,m}(z+1)^3 + \Omega_{0,\Lambda} = 0$$

$$\rightarrow (z+1)^3 = \frac{2\Omega_{0,\Lambda}}{\Omega_{0,m}}$$

$$\rightarrow z = \left(\frac{2\Omega_{0,\Lambda}}{\Omega_{0,m}}\right)^{1/3} - 1 \approx \underline{0.62}$$

c)

~~[illegible]~~ The matter-density scales with a^{-3} whereas the Λ density is constant.

We need to solve

$$\Omega_{0,m}\, a^{-3} = \Omega_{0,\Lambda}$$

$$\rightarrow a = (\Omega_{0,m}/\Omega_{0,\Lambda})^{1/3} \approx 0.78$$

$$\rightarrow z = 1/a - 1 \approx 0.28$$

This says that in b) we found that deceleration ended when there was 2 times as much matter as Λ, much before they were equal