E5076 - Topics in Time Series Analysis
Assignment #7

Due on April 27^{th}, 2020 at 01:45pm

Christian Hilscher

First way

I first rewrite $f_x(\lambda)$ and then use the insight from slide 127 to rewrite the power function. Furthermore I use that in a $MA(q)$ process the variance is given by $\gamma(0) = \sigma^2(1 + \sum_{i=1}^{q} c_i^2)$ where c_i represent the coefficients (similar to slide 30).

$$\begin{aligned}
f_x(\lambda) &= \frac{1}{2\pi}\sigma^2 T_c(\lambda) \\
&= \frac{1}{2\pi}\sigma^2 \left| \sum_{j=0}^{q} b_j e^{-ij\lambda} \right|^2 \\
&= \frac{1}{2\pi}\sigma^2 \left(\sum_{j=0}^{q} b_j^2 + 2\sum_{h=1}^{q}\sum_{j=0}^{q} b_j b_{j+h} cos(\lambda h) \right) \\
&= \frac{1}{2\pi}\sigma^2 \left(1 + \sum_{j=1}^{q} b_j^2 \right) + \frac{1}{2\pi}\sigma^2 \left(2\sum_{h=1}^{q}\sum_{j=0}^{q} b_j b_{j+h} cos(\lambda h) \right) \\
&= \frac{1}{2\pi}\gamma(0) + \frac{1}{2\pi}\sigma^2 \left(2\sum_{h=1}^{q} \underbrace{\sum_{j=0}^{q} b_j b_{j+h}}_{A} cos(\lambda h) \right) \qquad (1)
\end{aligned}$$

Since almost everything looks like it should be, except the part called A. That's why I look at what $\gamma(m)$ is for our process and whether I can simplify it further and maybe replace A by something different.

$$\begin{aligned}
\gamma(h) &= \int_{-\pi}^{\pi} \frac{1}{2\pi}\sigma^2 f_x(\lambda) e^{ih\lambda} d\lambda \\
&= \frac{1}{2\pi}\sigma^2 \int_{-\pi}^{\pi} \left(\sum_{j=0}^{q} b_j^2 + \sum_{k=1}^{q}\sum_{j=0}^{q} b_j b_{j+k} cos(\lambda k) \right) e^{ih\lambda} d\lambda \\
&= \frac{1}{2\pi}\sigma^2 \sum_{j=0}^{q} b_j^2 \underbrace{\int_{-\pi}^{\pi} e^{ih\lambda} d\lambda}_{=0} + \frac{1}{2\pi}\sigma^2 \int_{-\pi}^{\pi} \sum_{k=1}^{q}\sum_{j=0}^{q} b_j b_{j+k} (e^{ik\lambda} + e^{-ik\lambda}) e^{ih\lambda} d\lambda \\
&= \frac{1}{2\pi}\sigma^2 \int_{-\pi}^{\pi} \left(\sum_{j=0}^{q} b_j b_{j+h} (e^{2ih\lambda} + 1) d\lambda \right) + \frac{1}{2\pi}\sigma^2 \int_{-\pi}^{\pi} \sum_{k\neq h}^{q}\sum_{j=0}^{q} b_j b_{j+k} (e^{ik\lambda} + e^{-ik\lambda}) e^{ih\lambda} d\lambda \\
&= \frac{1}{2\pi}\sigma^2 \int_{-\pi}^{\pi} \left(\sum_{j=0}^{q} b_j b_{j+h} (e^{2ih\lambda} + 1) d\lambda \right) + \frac{1}{2\pi}\sigma^2 \sum_{k\neq h}^{q}\sum_{j=0}^{q} b_j b_{j+k} \underbrace{\int_{-\pi}^{\pi} cos(\lambda(h+k)) d\lambda}_{=0} \\
&= \frac{1}{2\pi}\sigma^2 \sum_{j=0}^{q} b_j b_{j+h} \left(\underbrace{\int_{-\pi}^{\pi} e^{2ih\lambda} d\lambda}_{=0} + \underbrace{\int_{-\pi}^{\pi} 1 d\lambda}_{=2\pi} \right) \\
&= \sigma^2 \sum_{j=0}^{q} b_j b_{j+h} \;\; = \;\; \sigma^2 A
\end{aligned}$$

Inserting this into (1) then yields the final result

$$f_x(\lambda) = \frac{1}{2\pi}\gamma(0) + \frac{1}{2\pi}\left(2\sum_{h=1}^{q} \gamma(h) cos(\lambda h) \right)$$

Different approach

Here I used the information from slide 128 and 129 and approximated $f_x(\lambda)$ by a Fourier series and then rearranged things a little bit.

$$\begin{aligned}
f_x(\lambda) &= \frac{1}{2\pi} \sum_{h=-q}^{q} \gamma(h) e^{-i\lambda h} \\
&= \frac{1}{2\pi} \left(\sum_{h=-q}^{-1} e^{-i\lambda h}\gamma(h) + \gamma(0) e^{-i\lambda 0} + \sum_{h=1}^{q} e^{-i\lambda h}\gamma(h) \right) \\
&= \frac{1}{2\pi}\gamma(0) + \frac{1}{2\pi} \left(\sum_{h=-q}^{-1} e^{-i\lambda h}\gamma(h) + \sum_{h=1}^{q} e^{-i\lambda h}\gamma(h) \right) \\
&= \frac{1}{2\pi}\gamma(0) + \frac{1}{2\pi} \left(\sum_{h=1}^{q} (e^{-i\lambda h} + e^{i\lambda h})\gamma(h) \right) \\
&= \frac{1}{2\pi}\gamma(0) + \frac{1}{2\pi} \left(\sum_{h=1}^{q} 2cos(\lambda h)\gamma(h) \right)
\end{aligned}$$