All Solutions

Recommended Problems Solutions:

1. Google: Two teams play a series of games (best of 7 — whoever wins 4 games first) in which each team has a 50% chance of winning any given round (no draws allowed). What is the probability that the series goes to 7 games?

Method 1

This is a relatively straightforward problem. We know that no draws are allowed, and we are looking to solve for the probability of playing 7 games. In order to reach 7 games we need the two teams to be equal after 6 games. Each team must then have 3 wins and 3 losses. So, the question becomes how many times can we choose 3 wins from 6 games, and the number of total outcomes in 6 games.

$\binom{6}{3} =$ 6 games choose 3, which will give us the number of ways a team can win 3 games.

$2^6 =$ number of possible out comes for 6 played games. Think about in this way: If played two games we could have this sample space [T1W,T1W], [T2W,T2W], [T1W,T2W], [T2W,T1W] = 4 possible outcomes.

Solution 1:

$$\frac{\binom{6}{3}}{2^6} = \frac{20}{64} = 0.3125$$

Method 2

We also know that it is a discrete probability distribution with a fixed probability of 0.5 win/loss per game since they are independent of each other.

Using the binomial theorem we can express this as:

$$\binom{n}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{n-k} = \binom{6}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{6-3} = 20 \cdot \left(\frac{1}{2}\right)^6 = 0.3125$$

Both methods come to the same conclusion: There's a 31.25% chance we get to a 7th game!

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2. JP Morgan: Say you roll a die three times. What is the probability of getting two sixes in a row?

The question asks us what is the probability of getting two sixes in a row in three die rolls. The key to solving this question is to think about the sample space of favorable outcomes.

We can obtain two sixes over three rolls in these ways:

- $[X, 6, 6]$, $[6, 6, X]$, $[6, 6, 6]$ where X is any other number than 6.

These are mutually exclusive events since the sequences can't occur simultaneously. This allows us to use the law of total probability to add up all the individual probabilities for observing two sixes, **calculated by summing these mutually exclusive events:**

Accounting for $[X, 6, 6]$ and $[6, 6, X]$:

$$P([X,6,6])+P([6,6,X])=2\cdot\left(\frac{1}{6}\right)^2\cdot\left(\frac{5}{6}\right)=\frac{10}{216}$$

Accounting for $[6,6,6]$:

$$P([6,6,6])=\left(\frac{1}{6}\right)^3=\frac{1}{216}$$

Final step: Add the three probabilities together:

$$P(\text{Two 6's in a row})=\frac{10}{216}+\frac{1}{216}=\frac{11}{216}=0.0509$$

Intuition Note:

To find the probability of getting two consecutive sixes in three die rolls, consider the **distinct ways** this can happen. We list the favorable sequences— $[X,6,6]$, $[6,6,X]$, and $[6,6,6]$—where each sequence represents a unique path to achieving two sixes in a row. These sequences are **mutually exclusive** (they can't occur simultaneously), so we use the **law of total probability** to add up the individual probabilities of each sequence. This gives the overall probability of observing two sixes in a row across all possible scenarios.

Note: The sequence $[6,X,6]$ is excluded because the two sixes are not consecutive.

3. Zenefits: Assume you have a deck of 100 cards with values ranging from 1 to 100, and that you draw two cards at random without replacement. What is the probability that the number of one card is precisely double that of the other?

In this problem, we have a deck of 100 cards numbered from 1 to 100, and we are sampling two cards without replacement. Our goal is to find the probability that one of the two chosen cards has a value that is exactly double the value of the other.

Step 1: Define the Total Number of Possible Outcomes

Since we are choosing two distinct cards from a deck of 100, the total number of ways to do this can be represented by the combination formula: $\binom{100}{2}$, which accounts for all possible pairs:

$$\text{Total Pairs}=\frac{100!}{2!(100-2)!}=\frac{100\cdot 99}{2}=4,950$$

This value represents the total number of unique ways to select two cards from the deck, forming our sample space.

Step 2: Count the Favorable Outcomes

To satisfy the condition that one card's value is exactly double the other, we need to identify pairs $(\mathrm{x},2\mathrm{x})$ such that both values are within the range of 1 to 100. This constraint limits the possible values for x to integers from 1 up to 50, since $2\mathrm{x}$ must be at most 100.

Each value x from 1 to 50 corresponds to a unique valid pair $(\mathrm{x},2\mathrm{x})$, so there are exactly 50 pairs that meet the condition. These pairs include $(1,2),(2,4),(3,6),\ldots,(50,100)$.

Step 3: Calculate the Probability

The probability of drawing a pair where one card is exactly double the other is the ratio of favorable outcomes to the total outcomes:

$$\text{Probability} = \frac{\text{Favorable Outcomes}}{\text{Total Outcomes}} \frac{\binom{50}{1}}{\binom{100}{2}} = \frac{50}{4950} = \frac{1}{99}$$

Intuition

There are 50 specific ways to select pairs that satisfy the condition (x,2x), where one card's value is precisely double the other's, out of the **4950** possible unique pairs in the deck. For example, drawing cards numbered $[10, 20]$ would meet the criteria, while a pair like [$[7, 6]$ would not. Thus, each of these 50 favorable pairs represents one of the possible outcomes out of the 4950 total pairs, giving us a refined probability of $\frac{1}{99}$.

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4. D.E. Shaw: A couple has two children. You discover that one of their children is a boy. What is the probability that the second child is also a boy?

Approach 1: Sample Space Method (most intuitive)

Step 1: List All Possible Combinations

When a couple has two children, each child can be either a **Boy (B)** or a **Girl (G)**. The possible combinations are:

1. **GG** (Girl, Girl)
2. **GB** (Girl, Boy)
3. **BG** (Boy, Girl)
4. **BB** (Boy, Boy)

Assuming that boys and girls are equally likely, each combination is equally probable.

Step 2: Condition on Observing One Boy

Since we know **one child is a boy**, we can eliminate the **GG** combination.

- **Remaining Combinations:**
  - **GB**
  - **BG**
  - **BB**

These are the only scenarios where at least one child is a boy.

Step 3: Calculate the Probability

Out of the **3** equally likely combinations, only **1** is **BB** (both children are boys).

Therefore, the probability that both children are boys is:

$$P(\text{Both are Boys} \mid \text{At least one is a Boy}) = \frac{1}{3}$$

Approach 2: Bayes' Theorem

Step 1: Define the Events

- Let **B** = Event that at least one child is a boy.
- Let **BB** = Event that both children are boys.

We want to find $P(BB|B)$.

Step 2: Calculate the Prior Probabilities

1. $P(BB)$: That both children are boys.

$$P(BB) = P(\text{First is B}) \times P(\text{Second is B}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

2. $P(B)$: Probability that at least one child is a boy.

$$P(B) = 1 - P(\text{Both are Girls}) = 1 - \left(\frac{1}{2} \times \frac{1}{2}\right) = 1 - \frac{1}{4} = \frac{3}{4}$$

Step 3: Apply Bayes' Theorem

Bayes' Theorem States

$$P(BB|B) = \frac{P(B|BB) \times P(BB)}{P(B)}$$

- $P(B|BB) = 1$: If both children are boys, it's certain that at leas one is a boy.

Plugging in the values:

$$P(BB|B) = \frac{1 \times \frac{1}{4}}{\frac{3}{4}} = \frac{1}{4} \times \frac{4}{3} = \frac{4}{12} = \frac{1}{3}$$

Intuition Behind the Solution's

- **Sample Space Approach:**
  - By listing all possible combinations and eliminating those that don't meet the condition (at least one boy), we see that only **1 out of 3** scenarios results in both children being boys.
- **Bayes' Theorem:**
  - We adjust the probability of both children being boys based on the new information (at least one is a boy).
  - The prior probability of having two boys is $\frac{1}{4}$, but knowing one child is a boy updates this probability to $\frac{1}{3}$.

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5. SIG: Suppose you have ten fair dice. If you randomly throw these dice simultaneously, what is the probability that the sum of all the top faces is divisible by 6?

At first glance this problem can take you in many directions. However, the key here is to recognize we can use modulo 6 for each die roll to find the probability of "no remainder." For instance, if we roll one die the possible mod 6 outcomes are:

$$[0, 1, 2, 3, 4, 5] = \text{Remainder after mod 6}.$$

Explanation: $\text{If we roll a 6, then 6 mod 6} = 0\text{, meaning there is no remainder, for } 5 \pmod 6 = \frac{5}{6} = \text{Remainder of 5, and so on.}$

What is interesting is that each remainder has the same probability of happening $\frac{1}{6}$.

Let's start by looking at a smaller example

Imagine we are rolling two die:

$$\text{Total Outcomes for two Die} = 6 \times 6 = 36.$$

Since each remainder across these 36 possible outcome is equally likely we can express the probabilities in this way:

- S = Remainder
  - Each possible total remainder S occurs with the following frequencies:

S	Number of Combinations	Probability P(S)
0	6	6/36 = 1/6
1	6	6/36 = 1/6
2	6	6/36 = 1/6
3	6	6/36 = 1/6
4	6	6/36 = 1/6
5	6	6/36 = 1/6

- For instance the remainder 0 can be achieved with these combinations: $[5,1],[1,5],[4,2],[2,4],[6,6],[3,3] = 6 \text{ combos}.$ The same is true for all the other remainders, but with different combinations.

Extend the idea to the 10 dies

Now we are dealing with 10 simultaneous die rolls, but the same logic as with 2 dies still applies.

Step 1: Find total number of outcomes

$$\text{Total Outcomes for 10 Die} = 6^{10}.$$

Step 2: Find the number of possible combinations for each remainder

We know that each remainder will have the same amount of combinations. Then there has to be 6^9 combinations per remainder, because $6^9 \cdot 6^1 = 6^{10}$.

Step 3: Set-up final equation

$$\text{P(Remainder} = 0) = \frac{6^9}{6^{10}} = \frac{1}{6}$$

Explanation: We have for each remainder an equal amount of combinations, so the answer simplifies to just $\frac{1}{6}$.

Method 2 (more intuitive)

If we just consider the first nine dices, and their sums, and after mod 6, we can get:

[0,1,2,3,4,5]. The tenth die will contain one number which can make the remainder = 0 for all the sums from the 9 first die. For instance, the first 9 die sum to 45, then we need a 3 in the tenth die roll to get to 48 mod 6 = 0. For each time we roll these die simultaneously there's a $\frac{1}{6}$ chance of obtaining the number leading to a remainder of 0!

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6. D.E. Shaw: Say you have 150 friends, and 3 of them have phone numbers that have the last four digits with some permutation of the digits 0, 1, 4, and 9. What's the probability of this occurring?

You have 150 friends. We are interested in finding the probability that **exactly 3** of them have phone numbers where the **last four digits** are a **permutation of the digits 0, 1, 4, and 9**.

This is a tricky problem and requires several layers of steps to find the expected output. The first caveat in the problem is that we are looking for “exactly” the probability of 3 friends having a phone number, which ends with a permutation of 0, 1, 4, and 9. We also know that we will be selecting our 3 friends from 150 friends, so we have $\binom{150}{3}$ ways to select 3 friends.

Step 1: Find the probability of getting a permutation off 0,1, 4 and 9.

Each digit ranges from 0-9, which gives us 10 possible digits at any time. Given that each digit is independent of the other, meaning that digit 1 does not depend on digit 2 and so on we can express the total combination of 4 digits with 10 numbers as:

$$10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10,000$$

We have 10,000 combinations of these 4 digits.

Step 2: Find the probability of these digits fall include 0,1, 4 and 9 where ordering matters.

Explanation:

Since we are interested in all possible **orderings** of these four digits, we calculate the permutations of 4 distinct items, which is 4!.

$$\frac{4!}{10^4} = \frac{24}{10000} = \frac{3}{1250} = 0.0024$$

This tells us the number of ways we can order the 4 digits, which are our favorable outcomes, and dividing that by the total number of combinations of digits.

Step 3: Find the exact probability of 3 friends with phone numbers including the permutation of 0,1, 4 and 9.

Since we are looking for an exact probability we can leverage the binomial theorem:

$$\binom{n}{k} \cdot (P)^3 \cdot (1-P)^{n-k}$$

Plug the numbers in:

$$\binom{150}{3} \cdot (0.0024)^3 \cdot (1-0.0024)^{150-3} = 0.0053$$

Note: As expected this is a improbable event to take place given the sheer magnitude of friends to choose from and the number of digit combinations!

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7. Doordash: A group of 10 friends randomly choose seats at a movie theater with 20 seats in a row. What is the probability that all 10 friends end up sitting next to each other?

This is a fairly straightforward question, if one get’s the assumptions correct. The first assumption is that the friends are indistinguishable. Meaning that if we arrange them in a block of 10 seats, so they all sit next to each other, rearranging their order would not count as an additional arrangement. If they are distinguishable (e.g., specific seat assignments matter), the solution would change significantly because permutations of the friends in the same block would also need to be accounted for.

However, it is necessary to check this assumption with the interviewer.

Step 1: Find the total number of ways 10 friends can sit together in a row of 20 seats

Imagine we fill all the seats up from the left, which counts as one arrangement.

$$[\text{F,F,F,F,F,F,F,F,F,F}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$$

Second Arrangement

$$[0,F,F,F,F,F,F,F,F,F, F, 0, 0, 0, 0, 0, 0, 0, 0, 0]$$

Third Arrangement

$$[0,0,F,F,F,F,F,F,F,F, F, F, 0, 0, 0, 0, 0, 0, 0, 0]$$

Count all the way up to the friends being fully shifted over to the right:

$$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0,F,F,F,F,F,F,F,F,F,F]$$

The number of distinct arrangements of the block of 10 friends in 20 seats, where the friends are treated as a single block, is calculated as:

$$21 - 10 = 11$$

This accounts for the 11 possible positions of the block of friends in the row, starting from the leftmost position (seats 1-10) to the rightmost position (seats 11-20).

Step 2: Find the total number of ways the friends can choose 10 out of 20 seats

$$\text{Seating Arrangements} = \binom{20}{10} \text{ ways the friends can sit in the theater.}$$

This is the number of ways to select any 10 seats for the friends, without the constraint of being contiguous.

Step 3: Find the final probability

$$\text{Probability of all friends sitting together} = \frac{11}{\binom{20}{10}} = 0.00005$$

is extremely low, reflecting the rarity of this specific arrangement.

Additional Considerations

- If the friends are distinguishable, the total number of arrangements where the friends are contiguous would be:

$$11 \times 10!$$

where 10! accounts for the permutations of the friends within the block.

- For distinguishable friends, the probability would be:

$$P(\text{All friends together, distinguishable}) = \frac{11 \times 10!}{20!}$$

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8. Asana: You are playing a game with a friend where you roll two fair six-sided dice. If the sum of the two dice is 7 or 11, you win; otherwise, your friend wins. What is the probability that you win on your first roll?

A straightforward problem which involves simple counting to get it correct.

Step 1: Find the number of ways for two dice to sum either 7 or 11

$$\text{Sum of 7 } = (6,1),(1,6),(5,2),(2,5),(4,3),(3,4) = 6 \text{ combinations.}$$

$$\text{Sum of 11} = (6,5),(5,6) = 2 \text{ combinations.}$$

Step 2: Find the total number of two dice sum combinations

$$\text{Number of possible outcomes} = 6 \times 6 = 36$$

Step 3: Final equation

$$\text{P(of winning game on first die roll)} = \frac{8}{36} = \frac{2}{9}$$

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9. Goldman Sachs: You are given a bag containing 10 balls, of which 3 are red and 7 are blue. You randomly draw 3 balls without replacement. What is the probability that at least 2 of the balls are red?

This is a great question and tests your ability to incorporate draws without replacement into the final probability.

Step 1: Figure out the initial probabilities of selecting a red or blue ball

$$P(r) = \text{Probability of selecting a red ball} = \frac{r}{r+b} = \frac{3}{10}$$

$$P(b) = \text{Probability of selecting a blue ball} = \frac{b}{r+b} = \frac{7}{10}$$

Step 2: Find the favorable sequences

We are looking to satisfy the criteria where $r \geq 2$ in 3 draws without replacement. This can happen in two scenarios:

- **Scenario 1:** $[r, r, r]$ – Drawing 3 red balls in a row.
- **Scenario 2:** Drawing 2 red balls and 1 blue ball. There are $\binom{3}{2}$ = 3 ways to arrange these draws:
- $[b, r, r]$
- $[r, b, r]$
- $[r, r, b]$

Scenario 1:

$$\text{P(RRR)} = \frac{3}{10} \times \frac{2}{9} \times \frac{1}{8} = \frac{6}{720} = \frac{1}{120}$$

Scenario 2:

$$\text{P(2R and 1B)} = 3 \times \left(\frac{3}{10} \times \frac{2}{9} \times \frac{7}{8}\right) = 3 \times \frac{42}{720} = 3 \times \frac{7}{120} = \frac{21}{120}$$

Step 3: Add the probabilities together

$$\text{P(RRR)+P(2R and 1B)} = \frac{1}{120} + \frac{21}{120} = \frac{22}{120} = \frac{11}{60} \approx 0.1833$$

Intuition Behind the Solution:

To determine the probability of drawing at least 2 red balls out of 3 without replacement:

1. **Identify All Favorable Outcomes:** Calculate the probabilities for each scenario where the condition is met.
2. **Use Combinatorics:** Determine the number of ways each favorable outcome can occur.
3. **Apply Probability Rules:** Multiply the probabilities of sequential draws and sum them up, ensuring that overlapping scenarios are accounted for correctly.

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10. Google: There are four people in an elevator that is about to make four stops on four different floors of the building. What is the probability that each person gets off on a different floor?

This is a classic interview question, which requires us to understand independent probability, permutations, or conditional probability. There are two ways to solve the question, and both approaches are equally as valid.

Approach 1: Counting favorable and total outcomes

We assume that each person is equally as likely to get off on any floor, and they are independent off each other. This as the question does not state otherwise.

Step 1: Finding the number of total outcomes

Given independence of each person they all have 4 ways to get off the elevator. Either floor 1, floor 2, floor 3, or floor 4. If helpful you can also think of this as having 4 independent die rolls, but imagine we have a 4 sided die.

We know then that the total number of ways to get off the floors for 4 people are:

$$\text{Number of ways to get off elevator} = 4 \times 4 \times 4 \times 4 = 4^4 = 256$$

Step 2: Find the number of favorable outcomes

To find the number of ways these 4 people can get out at different floors we need to find the number of ways we arrange 4 people with no repeats. Here, the people are distinguishable, so we can use permutations.

$$\text{Number of ways to order 4 people} = 4! = 24$$

Explanation of 4!:

- The factorial 4! means we multiply the total number of positions by decreasing values:

  $4 \times 3 \times 2 \times 1 = 24$

- Each position can be filled by one of the remaining individuals:
  1. The first position has 4 choices.
  2. The second position has 3 choices (one person is already placed).
  3. The third position has 2 choices.
  4. The fourth position has 1 choice.

This results in 24 unique arrangements.

Step 3: Find the probability

$$\text{Probability of getting of at different floors} = \frac{4!}{4^4} = \frac{24}{256} = \frac{3}{32}$$

Approach 2: Counting direct probabilities (Intuitive)

This is perhaps the more intuitive method and is straightforward.

- We know the first person can get off at any floor $\frac{4}{4}$
- The second person has $\frac{3}{4}$ floors left to choose
- The third person has $\frac{2}{4}$ floors left to choose
- The fourth person has $\frac{1}{4}$ floors left to choose

Step 1: Find the probability of this sequence happening

$$\text{Probability of selecting unique floors} = 1 \times \frac{3}{4} \times \frac{2}{4} \times \frac{1}{4} = \frac{6}{64} = \frac{3}{32}$$

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11. Optiver: Two people flip a coin 4 times each. What is the probability that the two people have the same number of heads at the end?

This is a tricky question, because it requires you to connect the dots between independent trials (Bernoulli trials) and mutually exclusive events to get the final probability.

How does one think about this problem?

- We can assume that the coin being flipped is fair, and each coin flip is independent.

At first glance, one might think that since each flip is independent and there are only two outcomes (same number of heads or not), the probability is $\frac{1}{2}$. However, this overlooks the variability in the number of heads each person can obtain over multiple flips.

Approach 1: Mechanical Direct Counting

Step 1: Identify the instances when flipping two fair coins 4 times where there are an equal amount of heads.

Let's denote the first person flipping the coin as X and the second person as Y. Where both X and Y refer to the number of heads each person gets.

Since X and Y can range from 0 to 4 heads, there are 5 possible matching sequences:

- Scenario 1: $P(X = 0)$ and $P(Y = 0)$, probability of 0 heads.
- Scenario 2: $P(X = 1)$ and $P(Y = 1)$, probability of 1 head each.
- Scenario 3: $P(X = 2)$ and $P(Y = 2)$, probability of 2 heads each.
- Scenario 4: $P(X = 3)$ and $P(Y = 3)$, probability of 3 heads each.
- Scenario 5: $P(X = 4)$ and $P(Y = 4)$, probability of 4 heads each.

Explanation: These are the possible scenarios where we have the same amount heads in 4 coin flips across the two persons. However, it does not account yet for the number of sequences each scenario can form.

Step 2: Find the probability of each scenario and determine the probability of both happening across the 4 flips.

One thing you may have noticed is that scenario for both player, and then the probability of these independent scenarios happening sequentially can be found through the product of the first persons scenario and second. In essence, we'd sum up the product of each persons scenario happening across all the scenarios.

Since each scenario finds k successes in n trials with a constant probability we'll use the binomial theorem:

- **Clarification:** Squaring the probability comes from multiplying the independent probabilities of $X = k$ and $Y = k$.

- **For $k = 0$:**

$$\left[\binom{4}{0} \times \left(\frac{1}{2}\right)^0 \times \left(\frac{1}{2}\right)^4\right]^2 = \left[1 \cdot 1 \cdot \frac{1}{16}\right]^2 = \left(\frac{1}{16}\right)^2 = \frac{1}{256}$$

- **For $k = 1$:**

$$\left[\binom{4}{1} \times \left(\frac{1}{2}\right)^1 \times \left(\frac{1}{2}\right)^3\right]^2 = \left[4 \cdot \frac{1}{2} \cdot \frac{1}{8}\right]^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

- **For $k = 2$:**

$$\left[\binom{4}{2} \times \left(\frac{1}{2}\right)^2 \times \left(\frac{1}{2}\right)^2\right]^2 = \left[6 \cdot \frac{1}{4} \cdot \frac{1}{4}\right]^2 = \left(\frac{3}{8}\right)^2 = \frac{9}{64}$$

- **For $k = 3$:**

$$\left[\binom{4}{3} \times \left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^1\right]^2 = \left[4 \cdot \frac{1}{8} \cdot \frac{1}{2}\right]^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

- **For $k = 4$:**

$$\left[\binom{4}{4} \times \left(\frac{1}{2}\right)^4 \times \left(\frac{1}{2}\right)^0\right]^2 = \left[1 \cdot \frac{1}{16} \cdot 1\right]^2 = \left(\frac{1}{16}\right)^2 = \frac{1}{256}$$

Step 3: Add the probabilities together

$$P(X = Y) = \frac{1}{256} + \frac{1}{16} + \frac{9}{64} + \frac{1}{16} + \frac{1}{256}$$

Finding common denominators:

$$P(X = Y) = \frac{1}{256} + \frac{16}{256} + \frac{36}{256} + \frac{16}{256} + \frac{1}{256} = \frac{70}{256} = \frac{35}{128} \approx 0.273$$

Intuition:

Since each person's coin flips are independent, the probability that both obtain k heads is the product of their individual probabilities for that k. We then sum these joint probabilities across all possible k because these scenarios are mutually exclusive.

Approach 2: Using symmetries in Binomial Coefficient

One may also recognize that we can count the number of combinations directly, and due to the symmetry between the two persons flipping the coin 4 times we only need to do square the combinations.

Step 1: Compute each term

- $\left(\binom{4}{0}\right)^2 = 1^2 = 1$
- $\left(\binom{4}{1}\right)^2 = 4^2 = 16$
- $\left(\binom{4}{2}\right)^2 = 6^2 = 36$

- $\left(\binom{4}{3}\right)^2 = 4^2 = 16$
- $\left(\binom{4}{4}\right)^2 = 1^2 = 1$

Sum: $1 + 16 + 36 + 16 + 1 = 70$

Step 2: Calculate the probability

$$P(X = Y) = \frac{\text{Favorable outcomes}}{\text{Total outcomes}} = \frac{70}{(2^4)^2} = \frac{70}{16^2} = \frac{70}{256} = \frac{35}{128}$$

12. Jane Street: What is the probability of getting a sum of 10 if you roll a die three times?

Step 1: Find the total number of outcomes for 3 die rolls

$$\text{Total number of outcomes} = 6 \times 6 \times 6 = 6^3 = 216$$

Step 2: Find the combinations across three dies that sum to 10

Combinations:

- [1,3,6]
- [4,4,2]
- [5,1,4]
- [4,3,3]
- [3,5,2]
- [2,6,2]

Step 2: Find the number of ways to permute each combination without overcounting

Given that we have combinations with two of the same number simply permuting $3! = 6$ for each combination would lead to overcounting. We are only interested in unique combinations.

Multinomial Coefficient

$$\text{Number of combinations} = \frac{3!}{1!1!1} + \frac{3!}{2!1!} + \frac{3!}{1!1!1!} + \frac{3!}{2!1!} + \frac{3!}{1!1!1!} + \frac{3!}{2!1!}$$

Add the combinations

$$\text{Number of combinations} = 6 + 3 + 6 + 3 + 6 + 3 = 27$$

Step 3: Compute the final probability

$$\frac{\text{Favorable outcomes}}{\text{Total outcomes}} = \frac{27}{216} = \frac{9}{72} = \frac{3}{24} = \frac{1}{8}$$

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13. Jane Street: You have a drawer with an infinite number of socks of two colors, which exist with equal probability. What is the expected number of attempts of taking out individual socks before a matching pair is found?

The fact that there are an infinite amount of socks is not highly relevant. What we care about is that there are exactly 2 colors, and both are equally likely to be selected. Let's assume we have 2 colors: Black and White.

Step 1: Find the required number of draws to get 2 of the same color socks.

Let's assume that the number of draws = X

Scenario 1: $X = 2$

We can draw two socks of the same color in a row, or two different colored socks. In 2 draws we have the following sample space:

$$\text{Sample space for 2 sock draws} = 2^2 = [BB, WW, BW, WB]$$

Note: Only 2 sock draws will satisfy the criteria: $[BB, WW]$, $X = 2$ happens $\frac{1}{2}$.

Scenario 2: $X = 3$

We proceed to a third draw when we don't get two of the same colored socks on the first two draws:

$$\text{Sample space for 3 sock draws} = 2^3 = 8$$

The outcomes: $[BBB, WWW, BBW, WWB, WBW, BWB, WBB, BWW]$

Note: only $[WBW, BWB, WBB, BWW]$ match our criteria of a matching third pair. Therefore again $P(X = 3)\frac{4}{8} = \frac{1}{2}$

Step 2: Calculate expected number of draws combining both scenarios

$$\text{Expected number of draws for 2 matching pairs} = 2 \times \frac{1}{2} + 3 \times \frac{1}{2} = 2.5$$

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14. Chicago Trading Company: Toss a coin until you get 2 heads. What's the expected number of tosses to achieve this goal?

Step 1: Introduce Formula for calculating expected number of rolls:

$$\text{Expected Number of rolls (r)} = \frac{r}{p}$$

Note: r refers to the number of heads we want, and p refers to the probability of Heads.

Step 2: Plug in numbers

$r = 2$

$p = 0.5$

$$\text{Expected Number of Rolls to get 2 heads} = \frac{2}{0.5} = 4$$

15. Meta: What is the probability of getting a pair by drawing 2 cards in a 52-card deck?

A standard 52-card deck consists of **4 suits** (hearts, diamonds, spades, clubs), each containing **13 ranks** (2 through Ace). For example, within the suit of hearts, you have the 2 of hearts, 3 of hearts, ..., up to the Ace of hearts.

Step 1: Find the Total Number of Ways to Draw a Pair

- **Number of Ranks:** 13
- **Number of Ways to Choose 2 Suits Out of 4 for a Given Rank:** $\binom{4}{2} = 6$.

  For each rank (e.g., two 2s), there are 6 possible pairs based on different suit combinations.

- **Total Number of Pairs Across All Ranks:**

We are looking specifically selecting two cards with the same suit, leaving the following favorable combinations:

$$\text{Total Pairs} = \binom{4}{2} \times 13 = 78$$

Step 2: Find the Number of Total Outcomes

- **Total Number of Ways to Choose 2 Cards from 52:**

$$\binom{52}{2} = \frac{52 \times 51}{2} = 1326$$

Step 3: Compute the Final Probability

- **Probability of Drawing a Pair:**

$$P(\text{Pair}) = \frac{\text{Number of Favorable Outcomes}}{\text{Total Number of Outcomes}} = \frac{78}{1326} \approx 0.0588$$

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16. Capital One: What is more likely: getting at least one six in 6 rolls, at least two sixes in 12 rolls, or at least 100 sixes in 600 rolls?

To determine which of the following events is **more likely**:

1. **Getting at least one six in 6 rolls**
2. **Getting at least two sixes in 12 rolls**
3. **Getting at least 100 sixes in 600 rolls**

we'll calculate the probability of each event and compare them.

We'll assume that each die roll is independent and that the die is fair, meaning the probability of rolling a six on any given roll is $p = \frac{1}{6}$

1. Probability of Getting at Least One Six in 6 Rolls

$$\text{P(At least one 6 in six rolls)} = 1 - \left(\binom{6}{0} \times \frac{1}{6}^0 \times \frac{5}{6}^6\right) = 1 - 0.3348 = 0.6651$$

2. Probability of Getting at Least Two Sixes in 12 Rolls

Find P(Zero six)

$$\binom{12}{0} \times \frac{1}{6}^0 \times \frac{5}{6}^{12} \approx 0.1121$$

Find P(One six)

$$\binom{12}{1} \times \frac{1}{6}^1 \times \frac{5}{6}^{11} \approx 0.269$$

Add both together to find and subtract by 1 for: $P(\text{At least two sixes})$**:**

$$1 - (0.1121 + 0.269) \approx 0.6189$$

3. Probability of Getting at Least 100 Sixes in 600 Rolls

For large numbers of trials, the **Normal Approximation** to the Binomial Distribution is suitable.

Step 1: Define the parameters for the binomial distribution.

$\mu = \frac{1}{6} \times 600 = 100$

$\sigma = 600 \times \frac{1}{6} \times \frac{5}{6} = 83.33, \sqrt{83.33} = 9.128$

Step 2: Apply the **Normal Approximation** with continuity correction.

Compute Z-score:

$$z = \frac{(100 - 0.5) - 100}{9.128} = \frac{-0.5}{9.128} = -0.0548$$

Step 3: Use the standard normal distribution table to find the probability.

CDF of Z: $\phi(z) = 0.5219$

Comparing all three, scenario 1 is most likely!

Based on the calculations:

- **Most Likely:** Getting **at least one six in 6 rolls** $\approx 66.51\%$
- **Next:** Getting **at least two sixes in 12 rolls** $\approx 61.89\%$
- **Least Likely:** Getting **at least 100 sixes in 600 rolls** $\approx 52.19\%$

Therefore, **getting at least one six in 6 rolls** is the most likely event among the three.

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17. AXA: If a jar has X red balls and Y blue balls, what is the minimum number of draws that is necessary to ensure that you have one ball of each color?

To determine the **minimum number of draws** required to ensure obtaining at least one red ball and one blue ball, we analyze the **worst-case scenario**. This approach ensures that, no matter how the balls are drawn, the desired outcome is achieved.

Step-by-Step Reasoning

1. Understanding the Worst-Case Scenario

The worst-case scenario occurs when you draw as many balls as possible of **one color** before drawing a ball of the other color. To ensure you have at least one ball of each color, you must account for this scenario.

- **Scenario A:** Drawing all X red balls first, followed by a blue ball.
- **Scenario B:** Drawing all Y blue balls first, followed by a red ball.

Both scenarios represent the maximum number of draws required before obtaining at least one ball of the other color.

2. Calculating the Minimum Number of Draws

To cover both scenarios, the minimum number of draws *n* needed to ensure at least one ball of each color is:

$$n = max(X, Y) + 1$$

Explanation:

- **max(X,Y):** Determines the larger of the two quantities, ensuring that you account for the possibility of drawing all balls of the more abundant color first.

- **+1:** Guarantees that after drawing all balls of the majority color, the next draw will yield a ball of the minority color.

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18. HealthTap: There are 25 horses. You can race any 5 of them at once, and all you get is the order they finished. How many races would you need to find the 3 fastest horses?

Step-by-Step Reasoning

Step 1: Divide and Race in Groups

Action:

- **Divide** the 25 horses into **5 groups** of **5 horses each**. Let's label them as **Group A, B, C, D,** and **E**.
- **Race** each group separately, resulting in **5 races** (Race 1 to Race 5).
- **Record** the finishing order within each group.

Example:

Race	Group	1st Place	2nd Place	3rd Place	4th Place	5th Place
1	A	**A1**	A2	A3	A4	A5
2	B	**B1**	B2	B3	B4	B5
3	C	**C1**	C2	C3	C4	C5
4	D	**D1**	D2	D3	D4	D5
5	E	**E1**	E2	E3	E4	E5

- **Bolded** horses (**A1, B1, C1, D1, E1**) are the winners of their respective groups.

Step 2: Race the Group Winners

Action:

- **Race 6:** Race the **5 group winners** against each other: **A1, B1, C1, D1, E1**.
- **Record** the finishing order of this race.

Example Outcome of Race 6:

Position	Horse
1st	**A1**
2nd	B1
3rd	C1
4th	D1
5th	E1

- **A1** is the fastest among the group winners, making it the **fastest horse overall**.

Step 3: Determine Potential Candidates for 2nd and 3rd Fastest

Action:

Based on the outcomes of the first 6 races, we can deduce which horses could still be in the top 3. Here's how:

1. **Eliminate Certain Horses:**
    - Any horse that finished **behind** the 3rd place in Race 6 cannot be in the top 3.
    - Specifically, **D1** and **E1**, along with all horses in Groups D and E, can be eliminated.

2. **Identify Remaining Candidates:**
    - The potential candidates for the **2nd and 3rd fastest horses** are:
        - **B1 and B2:** Since B1 finished 2nd in Race 6, B2 could be faster than others.
        - **C1:** Finished 3rd in Race 6.
        - **A2 and A3:** As Group A's second and third place finishers, they could be faster than other groups' second-place finishers.
3. **Summary of Candidates:**
    - **A2, A3, B1, B2, C1**

Step 4: Final Race Among Potential Candidates

Action:

- **Race 7:** Race the remaining **5 candidates**: **A2, A3, B1, B2, C1**.
- **Record** the finishing order.

Example Outcome of Race 7:

Position	Horse
1st	**B1**
2nd	**A2**
3rd	C1
4th	A3
5th	B2

- The **top 2 finishers** in this race are the **2nd and 3rd fastest horses overall**.

Step 5: Consolidate Results

Final Standings:

1. **A1** - Fastest horse (from Race 6)
2. **B1** - 2nd fastest horse (from Race 7)
3. **A2** - 3rd fastest horse (from Race 7)

Total Number of Races: 7

- **5 initial races** to determine group winners.
- **1 race** among group winners to identify the fastest horse.
- **1 final race** among potential candidates to determine the 2nd and 3rd fastest horses.
