Leveling Up: The Surprising Lessons from 100 Probability Interview Questions (Part 2): Practice Questions Solutions:

Practice Questions & Solutions

Facebook: Facebook has a content team that labels pieces of content on the platform as either spam or not spam. 90% of them are diligent raters and will mark 20% of the content as spam and 80% as non-spam. The remaining 10% are not diligent raters and will mark 0% of the content as spam and 100% as non-spam. Assume the pieces of content are labeled independently of one another, for every rater. Given that a rater has labeled four pieces of content as good, what is the probability this rater is a diligent rater?

Solution:

The intuition for this question is that although 90% of our labelers are diligent we need to account for the probability of non-diligent raters in our final probability.

$P(D) = Dilligent = 0.9$

$P(D') = \text{Non-Diligent} = 0.1$

$P(G|D) =$ Non-Spam | Diligent Rater = 0.8,

$P(G|D') =$ Non-Spam | Non-Diligent Rater = 1.0

In this case the posterior probability will be noted by:

$P(D|G_1, G_2, G_3, G_4)$:

Probability of being a diligent rater if we have 4 non-spam classifications. We know from the prompt that the pieces of content are labeled independently of one another. This means we can take the product of all the classifications for a joint probability.

Applying Bayes Theorem:

$$P(D \mid G_1, G_2, G_3, G_4) = \frac{P(G_1, G_2, G_3, G_4 \mid D) \cdot P(D)}{P(G_1, G_2, G_3, G_4)}$$

$$P(D \mid G_1, G_2, G_3, G_4) = \frac{0.8^4 \cdot 0.9}{0.8^4 \cdot 0.9 + 1^4 \cdot 0.1}$$

Answer:

$$P(D \mid G_1, G_2, G_3, G_4) = \frac{0.36864}{0.3684 + 0.1} = 0.786$$

Intuition:

The probability that the rater is diligent given that they labeled all four pieces of content as good is approximately **0.7866**, or about **78.66%**. This result shows that even though the rater labeled all content as good (which a non-diligent rater always does), the high prior probability of diligence (90%) still makes it more likely that the rater is diligent.

Oracle: Suppose you have two coins, one of which is fair (heads and tails are equally likely) and one of which is biased (heads comes up 75% of the time). You randomly select one of the coins and flip it twice. If both flips come up heads, what is the probability that the selected coin is the fair one?

Solution:

This is a classic Bayes' Theorem question, and requires us to correctly identify the Prior, Likelihood and Marginal probabilities.

Step 1: Identify the probabilities and set-up the Theorem equation

Probabilities:

- $\text{Fair Coin (F)} = \frac{1}{2}$ probability of either Heads (H) or Tail (T)
- $\text{Biased Coin (B)} = \frac{3}{4}$ probability of H
- $\text{Prior Probability} = \frac{1}{2}$ given we select the coins randomly there's an equal chance of picking either F or B.

Bayes Theorem:

$$P(A|B) = \frac{P(B|A) \times P(A)}{P(B)}$$

Or

$$P(F|HH) = \frac{P(HH|F) \times \frac{1}{2}}{P(B)}$$

$$P(B) = P(HH) = P(HH|F) \times \frac{1}{2} + P(HH|B) \times \frac{1}{2}$$

Explanation: We use law of total probability to find all the ways we can observe HH. In our case we can observe both when the coin is Fair and Biased, or in other words look at $F^{\complement} = B$.

Step 2: Arrange Final Formula

$$P(F|HH) = \frac{P(HH|F) \times \frac{1}{2}}{P(HH|F) \times \frac{1}{2} + P(HH|B) \times \frac{1}{2}}$$

Step 3: Plug in numbers and solve

$$P(F|HH) = \frac{\left(\frac{1}{2}\right)^2 \times \frac{1}{2}}{\left(\frac{1}{2}\right)^2 \times \frac{1}{2} + \left(\frac{3}{4}\right)^2 \times \frac{1}{2}} = \frac{\frac{1}{8}}{\frac{1}{8} + \frac{9}{32}} = \frac{\frac{1}{8}}{\frac{13}{32}} = \frac{32}{104} = \frac{4}{13} = 0.307$$

Interpretation: You may wonder why the probability of it being Fair is 0.307, and the reason being is that there's still a high probability of observing HH from a Fair coin in two tosses, and the prior probability was also 50/50.

Lyft: A discount coupon is given to 2 riders. The probability of using a coupon is P. Given that at least one of them uses a coupon, what is the probability that both riders use the coupons?

Solution:

This problem is a bit more tricky than one may assume on first glance. We know that the probability of using a coupon is p, and we are conditioning on at least one rider using a coupon.

One we are trying to find "at least one" this tells us that we can use the complement rule to find all the sequences where coupons used are 1 or greater.

Since we are also observing that the riders are using at least one coupon we need to update our prior belief of selecting 2 coupons, where the prior would be p^2.

Step 1: Set-up the Bayes Theorem and define events

$$P(A|B) = \frac{P(B|A) * P(A)}{P(B)}$$

In this case:

- $P(A|B) = \text{P(Both riders using their coupons|at least one coupon is used)}$
- $P(B) = \text{All the sequences where the riders use one or more coupons.}$
- $P(A) = \text{Prior probability of two riders using a coupon}.$

Step 2: Plug in the definitions to the formula

$$P(B|A) \cdot P(A) = 1 \times p^2$$

Explanation: $P(B|A)$ is a certainty because we know if we observed two coupons we must have used 1 before that. $P(A)$ is the prior probability of two riders using a coupon which is p^2.

$$P(B) = P(\text{At least one coupon used}) = 1 - ((1-p) \cdot (1-p))$$
$$= 1 - (1 - 2p + p^2) = 2p - p^2$$

Step 3: Bring all the terms together

$$P(B|A) = \frac{P(B|A) \cdot P(A)}{P(B)} = \frac{P^2}{2p - p^2} = \frac{p^2}{p(2-p)} = \frac{p}{2-p}$$

Intuition: Although this might not be a straightforward Bayes Theorem application. The trick here is to understand how to calculate $P(B) = P(\text{At least on coupon is used})$.

Jane Street: You flip four coins. At least two are tails. What is the probability that exactly three are tails?

Solution:

Step 1: Set-up the Bayes' Theorem and find the posterior, likelihood, prior, and marginal probabilities.

$$P(A|B) = \frac{P(B|A) \times P(A)}{P(B)}$$

Let's start by organizing the information provided in the question.

$\text{Posterior Probability} = P(A|B) = P(\text{Exactly 3 Tails} \mid \text{At least 2 Tails})$

Given that getting at least 2 tails is a complement of 3 Tails $P(A|B) = 1$.

$\text{Prior Probability} = P(A) = P(\text{Exactly 3 Tails}).$ This represents the probability of 3 Tails without observing at least 2 Tails.

$\text{Likelihood} = P(B|A) = P(\text{At least 2 Tails} \mid \text{Exactly 3 Tails})$

$\text{Marginal Probability} = P(\text{At least 2 Tails})$

Step 2: Bring everything together in the formula and plug in values

$$P(\text{Exactly 3 Tails}) = \frac{1 \times \binom{4}{3} \times \left(\frac{1}{2}\right)^3 \times \frac{1}{2}}{1 - \left(\left(\binom{4}{0} \times \frac{1}{2}^0 \times \frac{1}{2}^{4-0}\right) + \left(\binom{4}{1} \times \frac{1}{2} \times \frac{1}{2}^{4-1}\right)\right)}$$

Simplify

$$P(\text{Exactly 3 Tails}) = \frac{1 \times 4 \times \left(\frac{1}{2}\right)^3 \times \frac{1}{2}}{1 - \left(1 \times 1 \times \frac{1}{2}^4 + 4 \times \frac{1}{2} \times \frac{1}{2}^3\right)}$$

Simplify

$$= \frac{\frac{1}{4}}{1 - \frac{1}{16} + \frac{2}{8}} = \frac{\frac{1}{4}}{\frac{16}{16} - \frac{5}{16}} = \frac{1}{4} \times \frac{16}{11} = \frac{16}{44} = \frac{4}{11}$$

Final answer: There's a $\frac{4}{11}$ chance of getting exactly 3 Tails given we've observed at least 2 Tails.

Intuition

Before observing at least 2 Tails we know that the probability of getting exactly 3 Tails was $\frac{4}{16}$.

However, after observing at least 2 tails we can remove all the sequences, which include 0, or 1 Tails, which come out to be: $\binom{4}{0} = 1 \text{ and } \binom{4}{1} = 4 \text{ which makes up 1+5 combinations.}$ There are still $\binom{4}{3} = 4$ ways to select 3 exactly 3 Tails from 4 flips:

- $[TTTH], [HTTT], [THTT], [TTHT]$

However, after removing the sequences of getting 0 Tails and 1 Tail, we only have 11 outcomes left therefore: $P(\text{Exactly 3 Tails} | \text{ At least 2 Tails}) = \frac{4}{11}$.

D.E. Shaw: You randomly call a family with two kids and ask if there is a kid called Tom. If the answer is yes, what is the probability that the family has two boys?

Solution:

This is a bit of a trick question. You have to be extra careful to properly add the weighting of the name "Tom" in the sample space.

Initially, for a family of 2 children, without knowing anything about their children the possible outcomes are:

- $[BB, GG, BG, GB]$, where B = Boy and G = Girl.

Step 1: Evaluate how sample space changes with new information

If we ask a family with 2 children if they have a boy named "Tom" and the answer is "Yes" then we have the following new sample space:

- [Boy2, Tom], [Tom, Boy2], [Tom, Girl1], [Girl1, Tom]
    - Note: we remove the sequence of [G,G], since we've already observed a boy named "Tom" thus making this sequence impossible.

Step 2: Find Favorable outcomes

- There are two favorable outcomes with 2 boys: [Tom, Boy2] and [Boy2, Tom]

Step 3: Find probability of two boys given Tom

$$P(BB) = \frac{2}{4} = \frac{1}{2}$$

Explanation: There's a 50% chance that there are two boys in the family if one child is a boy and is named Tom.

Zenefits: There are 30 red marbles and 10 black marbles in Urn #1. You have 20 red and 20 black marbles in Urn #2. Randomly you pull a marble from a random urn and find that it is red. What is the probability that it was pulled from Urn #1?

Solution:

You have two urns:

1. **Urn #1**: 30 red marbles, 10 black marbles (40 total)
2. **Urn #2**: 20 red marbles, 20 black marbles (40 total)

You randomly pick **one urn** (50/50 chance) and then draw **one marble**. You observe that the marble is **red**. What's the probability that this red marble came from **Urn #1**?

1. Define Events

- A: "Marble was drawn from Urn #1."
    - $P(A) = \frac{1}{2}$
- A^c: "Marble was drawn from Urn #2."
    - $P(A^c) = \frac{1}{2}$
- B: "The marble drawn is red."

We want $P(A \mid B)$: The probability we picked Urn #1, given that the marble is red.

---

2. Compute $P(B \mid A)$ and $P(B \mid A^c)$

1. $P(B \mid A)$: Probability the marble is red if we chose Urn #1.
    - Urn #1 has 30 red out of 40 total.
        - $P(B \mid A) = \frac{30}{40} = \frac{3}{4} = 0.75.$
2. $P(B \mid A^c)$: Probability the marble is red if we chose Urn #2.
    - Urn #2 has 20 red out of 40 total.
        - $P(B \mid A^c) = \frac{20}{40} = \frac{1}{2} = 0.50.$

---

3. Law of Total Probability for P(B)

To find $P(B)$ (the probability of drawing a red marble overall), we sum over both urn scenarios:

$$P(B) = P(B \mid A)\, P(A) \; + \; P(B \mid A^c)\, P(A^c).$$

Plug in the numbers:

$$P(B) = (0.75) \times \left(\frac{1}{2}\right) + (0.50) \times \left(\frac{1}{2}\right).$$

$$P(B) = 0.375 + 0.25 = 0.625 = \frac{5}{8}.$$

4. Use Bayes' Theorem

Bayes' formula for $P(A \mid B)$ is:

$$P(A \mid B) = \frac{P(B \mid A)\, P(A)}{P(B)}.$$

Substitute the values:

$$P(A \mid B) = \frac{(0.75) \times (0.5)}{0.625} = \frac{0.375}{0.625} = 0.60.$$

So, there's a **60%** chance that the red marble came from **Urn #1**.

LinkedIn: You randomly draw a coin from 100 coins—1 unfair coin (heads on both sides) and 99 fair coins (heads and tails)—and flip it 10 times. If the result is 10 heads, what's the probability that the coin is unfair?

Solution:

Classic Bayes Theorem question. All we need to do is plug in the appropriate values.

P(Fair Coin) = $\frac{99}{100} = 0.99$

P(Biased Coin) = $\frac{1}{100} = 0.01$

Then we know the P(Heads) for the Biased coin is 1, and for the fair coin 0.5.

Step 1: Set-up the bayes theorem and solve:

$$P(\text{Biased}|H^{10}) = \frac{P(H^{10}|\text{Biased}) \times P(\text{Biased})}{P(H^{10})}$$

Step 2: Plug in numbers

$$P(\text{Biased}|H^{10}) = \frac{1 \times 0.01}{1 \times 0.01 + \frac{1}{2}^{10} \times 0.99} = 0.91184$$

Explanation: As expected we would assume that we are flipping the biased coin given observing 10 Heads in a row from a fair coin is extremely unlikely!

D.E. Shaw: A couple has two children. You discover that one of their children is a boy. What is the probability that the second child is also a boy?

Solution:

Approach 1: Sample Space Method (most intuitive)

Step 1: List All Possible Combinations

When a couple has two children, each child can be either a **Boy (B)** or a **Girl (G)**. The possible combinations are:

1. **GG** (Girl, Girl)
2. **GB** (Girl, Boy)
3. **BG** (Boy, Girl)
4. **BB** (Boy, Boy)

Assuming that boys and girls are equally likely, each combination is equally probable.

Step 2: Condition on Observing One Boy

Since we know **one child is a boy**, we can eliminate the **GG** combination.

- **Remaining Combinations:**
  - **GB**
  - **BG**
  - **BB**

These are the only scenarios where at least one child is a boy.

Step 3: Calculate the Probability

Out of the **3** equally likely combinations, only **1** is **BB** (both children are boys).

Therefore, the probability that both children are boys is:

$$P(\text{Both are Boys} \mid \text{At least one is a Boy}) = \frac{1}{3}$$

Approach 2: Bayes' Theorem

Step 1: Define the Events

- Let **B** = Event that at least one child is a boy.
- Let **BB** = Event that both children are boys.

We want to find $P(BB|B)$.

Step 2: Calculate the Prior Probabilities

1. $P(BB)$: That both children are boys.

$$P(BB) = P(\text{First is B}) \times P(\text{Second is B}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

2. $P(B)$: Probability that at least one child is a boy.

$$P(B) = 1 - P(\text{Both are Girls}) = 1 - \left(\frac{1}{2} \times \frac{1}{2}\right) = 1 - \frac{1}{4} = \frac{3}{4}$$

Step 3: Apply Bayes' Theorem

Bayes' Theorem States

$$P(BB|B) = \frac{P(B|BB) \times P(BB)}{P(B)}$$

- $P(B|BB) = 1$: If both children are boys, it's certain that at leas one is a boy.

Plugging in the values:

$$P(BB|B) = \frac{1 \times \frac{1}{4}}{\frac{3}{4}} = \frac{1}{4} \times \frac{4}{3} = \frac{4}{12} = \frac{1}{3}$$

Intuition Behind the Solution's

- **Sample Space Approach:**
    - By listing all possible combinations and eliminating those that don't meet the condition (at least one boy), we see that only **1 out of 3** scenarios results in both children being boys.
- **Bayes' Theorem:**

- We adjust the probability of both children being boys based on the new information (at least one is a boy).
- The prior probability of having two boys is $\frac{1}{4}$, but knowing one child is a boy updates this probability to $\frac{1}{3}$.

Uber: A fair coin is tossed twice, and you are asked to decide whether it is more likely that two heads showed up given that either (a) at least one toss was heads, or (b) the second toss was a head. Does your answer change if you are told that the coin is unfair?

Solution:

This question is a great use case for Bayes theorem and requires a deep understanding on how to apply it correctly.

We are dealing with two scenarios **(A and B):**

Part A

We know what one of the tosses are Heads but not in which toss we obtained heads.

We can formalize the posterior as: $P(H|HH)$, where we are solving for the probability of two Heads in a row given we've observed one head toss.

The prior probability is $P(HH)$ = $\frac{1}{4}$ this as the sample space reduces to $[HH, TT, HT, TH]$ and we only observe HH $\frac{1}{4}$ times.

Lastly, $P(H)$ = $\frac{3}{4}$ this as we have three occurences of Heads in the sample space $[HH, TH, HT, TT]$, as TT is no longer an option after we know one of the tosses = Heads.

Applying the Bayes Theorem

$$P(HH|H) = \frac{P(H|HH) \cdot P(HH)}{P(HH) + P(HT) + P(TH)}$$

Substituting with values

$$P(HH|H) = \frac{1 \cdot \frac{1}{4}}{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}}$$

Solution:

$$\frac{\frac{1}{4}}{\frac{3}{4}} = \frac{4}{12} = \frac{1}{3}$$

Part B

In this case we know that the second toss was a Heads.

We can formalize the posterior as: $P(H|XH)$, where we are solving for the probability of two Heads in a row given we've observed one head in the second toss.

Again, the prior probability is $P(HH)$ = $\frac{1}{4}$ this as the sample space reduces to $[HH, TT, HT, TH]$ and we only observe HH $\frac{1}{4}$ times.

Lastly, $P(H)$ = $\frac{2}{4}$ this as we have now two occurences of Heads in the sample space $[HH, TH]$ as TT and HT is no longer an option after we know the second toss is Heads

Applying the Bayes Theorem

$$P(HH|H) = \frac{P(H|HH) \cdot P(HH)}{P(HH) + P(TH)}$$

Substituting with values

$$P(HH|H) = \frac{1 \cdot \frac{1}{4}}{\frac{1}{4} + \frac{1}{4}}$$

Solution:

$$\frac{\frac{1}{4}}{\frac{2}{4}} = \frac{4}{8} = \frac{1}{2}$$

Thus, the probability of observing HH when we know the second toss is Heads is greater.

When we know the second toss is heads, we're already halfway to HH, so to speak. Only one additional head on the first toss is needed for HH to occur. This makes it more likely that both tosses are heads compared to option (a), where we lack specific information about the position of the head(s).

SIG: Suppose you are given a white cube that is broken into 3×3×3=27 pieces. However, before the cube was broken, all 6 of its faces were painted

green. You randomly pick a small cube and see that 5 faces are white. What is the probability that the bottom face is also white?

Solution:

Before we start this question it is really helpful to visualize the cube and understand its components before it is broken into 3x3x3 = 27 smaller pieces.

Visual Representation

Imagine the 3D cube divided into 27 smaller cubes:

- **Center Cube:** Located at the very core, completely internal, invisible to any face inspection.
- **Face-Center Cubes:** Located at the center of each face, each having one face painted green and the rest white.

When you observe a cube and see that **5 of its faces are white**, visualize:

- **Center Cube:** All faces are white, so any 5 observed faces are white.
- **Face-Center Cubes:** Only 5 faces are white; the green face must be the one not observed.

Applying Bayes Theorem

The next important part is understanding the wording of the question: You randomly pick a small cube and see that **5 faces are white**. What is the probability that the **bottom face is also white**?

$$\mathrm{P(A|B) = P(BottomFaceWhite|5\ white\ faces\ observed)}$$

Then we can find $P(B|A) \cdot P(A)$:

$$P(B|A) \cdot P(A) = 1 \cdot \frac{1}{27}$$

Intuition: In the $P(B|A)$ part we are saying what is the probability of observing 5 white faces given we've observed the bottom face being white, which has to be 1. Then we multiply with the prior probability of satisfying both condition A and B which would be the center piece.

Next Step: Find P(B):

$$\mathrm{P(B)} = 1\frac{1}{27} + \frac{6}{27} \cdot \frac{1}{6}$$

Why do we add: $\frac{6}{27} \cdot \frac{1}{6}$?

- Each Face-Center Cube has 5 white faces and 1 green face.
- To observe 5 white faces, the green face must be the bottom face (unseen).
- There are 6 possible orientations (positions) the cube can be in, one for each face being on the bottom.
- Only **1 out of these 6 orientations** has the green face on the bottom.
- Therefore:

$$\text{Observing 5 White Faces}|\text{Face-Center Cube}) = \frac{1}{6}$$

Solving the problem:

$$P(\text{Bottom Face White}|\text{5 White Faces Observed}) = \frac{\frac{1}{27}}{\frac{1}{27} + \frac{6}{27} \cdot \frac{1}{6}} = \frac{1}{27} \cdot \frac{27}{2} = \frac{1}{2}$$

You’ve reached the end…