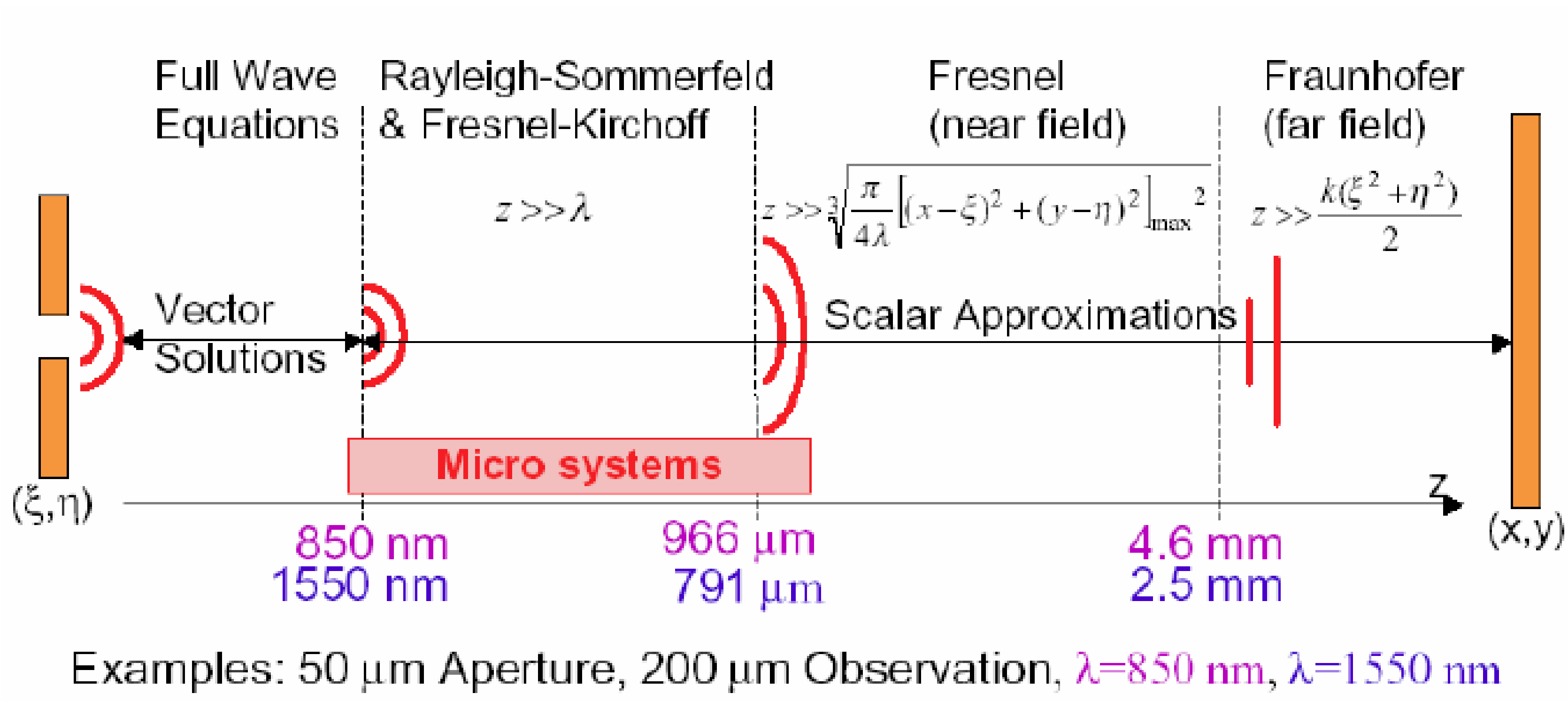
13. Fresnel diffraction

Augustin Jean Fresnel (1788-1827)
Made contributions to transverse nature of light and diffraction theory

Josef von Fraunhofer (1787-1826)
Developed diffraction gratings and increased understanding of diffraction

Remind! Diffraction regimes



Fraunhofer Approximation - Assume planar wavefronts

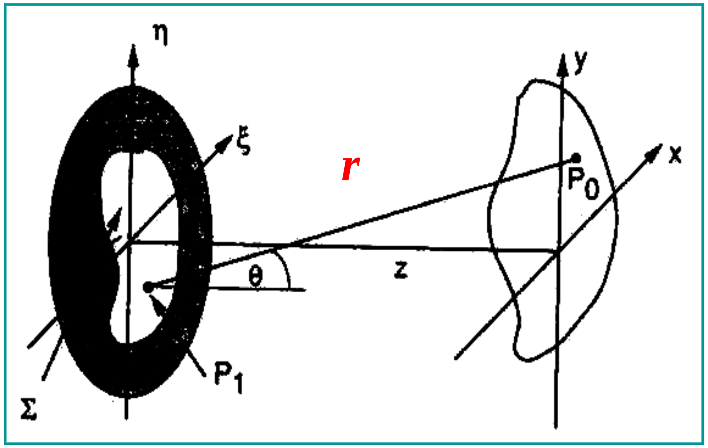
Fresnel Approximation - Assume parabolic wavefronts

Rayleigh-Sommerfeld Formulation - Spherical wavefronts

Fresnel-Kirchhoff diffraction formula

$$E(P_0)=\frac{E_0}{i\lambda}\iint_{\Sigma}\frac{\exp(ikr)}{r}F(\theta)dA$$

Obliquity factor : $F(\theta)=\cos\theta=\dfrac{z}{r}$



Aperture (ξ,η) Screen (x,y)

$$E(x,y)=\frac{zE_0}{i\lambda}\iint_{\Sigma}\frac{\exp(ikr)}{r^2}\,d\xi\,d\eta$$

$$r=\sqrt{z^2+(x-\xi)^2+(y-\eta)^2}$$

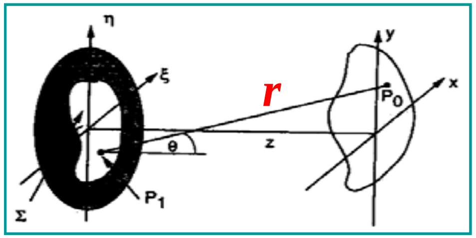
$$\approx z\left[1+\frac{1}{2}\left(\frac{x-\xi}{z}\right)^2+\frac{1}{2}\left(\frac{y-\eta}{z}\right)^2\right]=z+\frac{(x-\xi)^2}{2z}+\frac{(y-\eta)^2}{2z}$$

$$=z+\left(\frac{x^2}{2z}+\frac{y^2}{2z}\right)+\left(\frac{\xi^2}{2z}+\frac{\eta^2}{2z}\right)-\left(\frac{x\xi}{z}+\frac{y\eta}{z}\right)$$

$$E(x,y)=\frac{E_0}{i\lambda z}\exp(ikz)\exp\left[i\frac{k}{2z}\left(x^2+y^2\right)\right]$$

$$\times\iint_{\Sigma}\exp\left[i\frac{k}{2z}\left(\xi^2+\eta^2\right)\right]\exp\left[-i\frac{k}{z}(x\xi+y\eta)\right]\,d\xi\,d\eta$$

$$E(x,y) = \frac{E_0}{i\lambda z}\exp(ikz)\exp\left[i\frac{k}{2z}\left(x^2+y^2\right)\right]$$
$$\times \iint_{\Sigma} \exp\left[i\frac{k}{2z}\left(\xi^2+\eta^2\right)\right]\exp\left[-i\frac{k}{z}\left(x\xi+y\eta\right)\right]\,d\xi\,d\eta$$
$$= C\iint_{\Sigma} \exp\left[i\frac{k}{2z}\left(\xi^2+\eta^2\right)\right]\exp\left[-i\frac{k}{z}\left(x\xi+y\eta\right)\right]\,d\xi\,d\eta$$



Aperture (ξ,η) Screen (x,y)

Fresnel diffraction

$$E(x,y) = C\iint U(\xi,\eta)\exp\left[i\frac{k}{2z}\left(\xi^2+\eta^2\right)\right]\exp\left[-i\frac{k}{z}\left(x\xi+y\eta\right)\right]\,d\xi\,d\eta$$

$$\boxed{E(x,y) \propto \mathcal{F}\left\{U(\xi,\eta)\,e^{j\frac{k}{2z}\left(\xi^2+\eta^2\right)}\right\}}$$

Fraunhofer diffraction

$$E(x,y) = C\iint U(\xi,\eta)\exp\left[-i\frac{k}{z}\left(x\xi+y\eta\right)\right]\,d\xi\,d\eta$$
$$= C\iint U(\xi,\eta)\exp\left[-ik\left(\xi\sin\theta_{\xi}+\eta\sin\theta_{\eta}\right)\right]\,d\xi\,d\eta$$

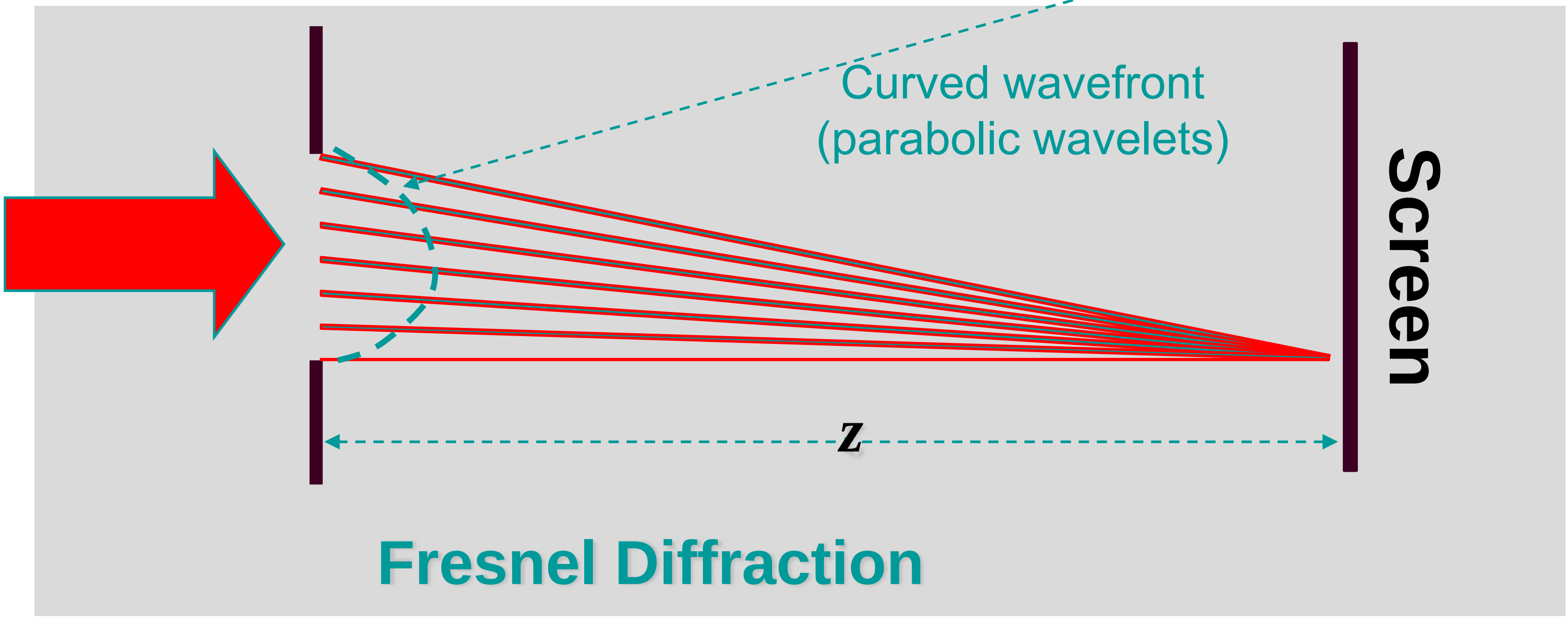
$$\boxed{E(x,y) \propto \mathcal{F}\left\{U(\xi,\eta)\right\}}$$

Fresnel (near-field) diffraction

This is most general form of diffraction

- **No restrictions on optical layout**
  - **near-field diffraction**
  - **curved wavefront**
- **Analysis somewhat difficult**

$$U(x,y) \approx \mathcal{F}\left\{U(\xi,\eta)e^{j\frac{k}{2z}\left(\xi^2+\eta^2\right)}\right\}$$



Accuracy of the Fresnel Approximation

$$z^3 \gg \frac{\pi}{4\lambda}\left[(x-\xi)^2+(y-\eta)^2\right]^2_{\max}$$

- Accuracy can be expected for much shorter distances

for $U(\xi,\eta)$ *smooth & slow varing function;* $2\left|x-\xi\right| = D \le 4\sqrt{\lambda z}$

$$\boxed{z \ge \frac{D^2}{16\lambda}}$$ Fresnel approximation

In summary, Fresnel diffraction is …

Assume: $z >> x_1, y_1; x_0, y_0$

$$\iint_{\Sigma} \rightarrow \iint_{-\infty}^{\infty} \qquad \text{(U = 0 outside the aperture)}$$

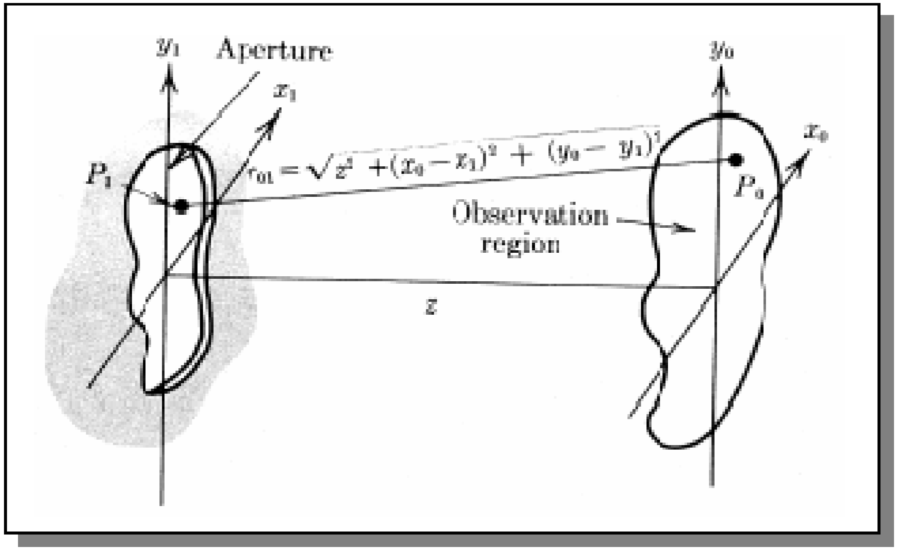
Fresnel's approximation:

In the exponent:

$$r = \sqrt{z^2 + (x_0 - x_1)^2 + (y_0 - y_1)^2}$$

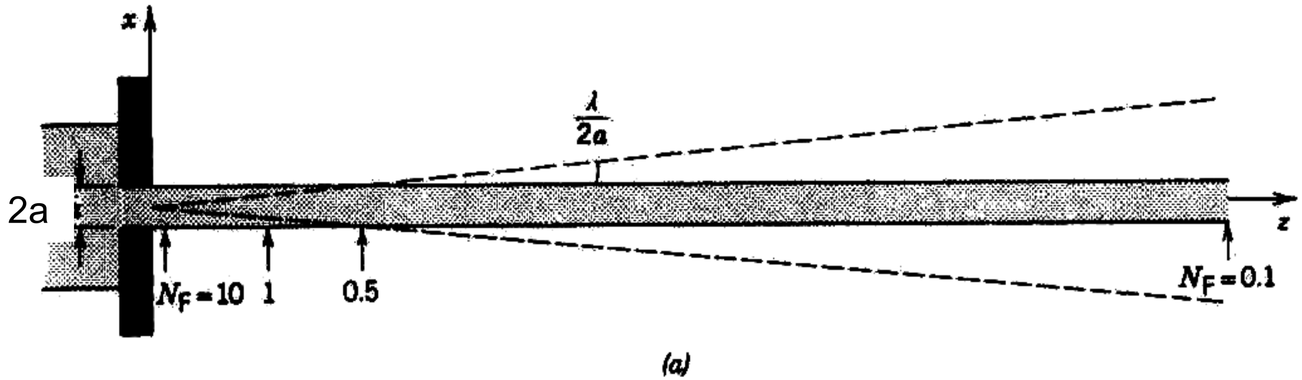
$$\approx z\left[1 + \frac{1}{2}\left(\frac{x_0 - x_1}{z}\right)^2 + \frac{1}{2}\left(\frac{y_0 - y_1}{z}\right)^2\right] \qquad \left(\sqrt{1+x} \approx 1 + \frac{1}{2}x\right)$$

In the denominator: $r \rightarrow z$

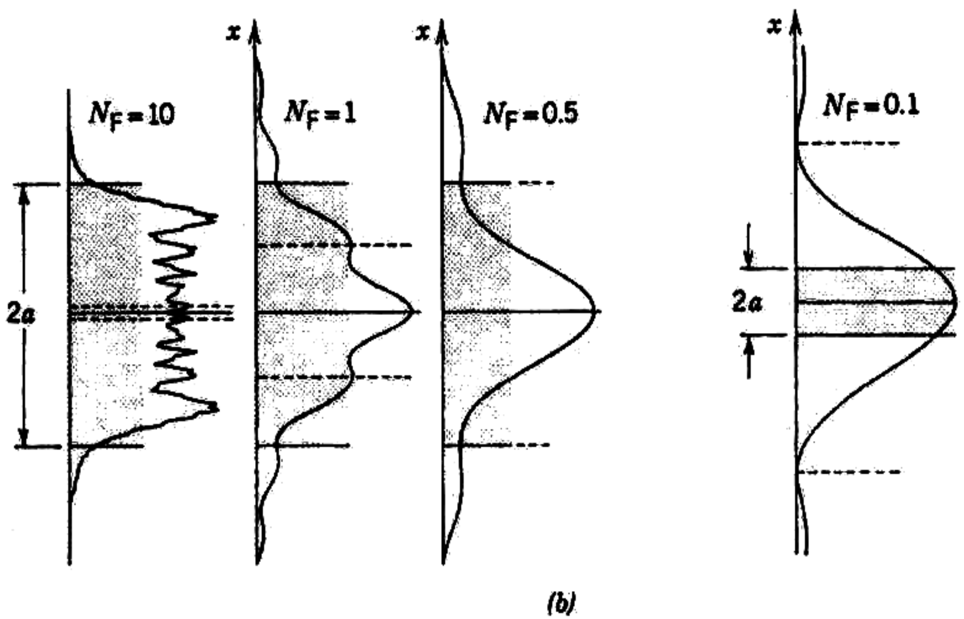


$$U(x_0, y_0) = -\frac{ik}{2\pi}\frac{e^{ikz}}{z}\iint_{-\infty}^{\infty} U(x_1, y_1)\, e^{\frac{ik}{2z}\left[(x_0 - x_1)^2 + (y_0 - y_1)^2\right]}\, dx_1 dy_1$$

13-7. Fresnel Diffraction by Square Aperture



Fresnel Diffraction from a slit of width $D = 2a$. (a) Shaded area is the geometrical shadow of the aperture. The dashed line is the width of the Fraunhofer diffracted beam.



(b) Diffraction pattern at four axial positions marked by the arrows in (a) and corresponding to the Fresnel numbers N_F=10, 1, 0.5, and 0.1. The shaded area represents the geometrical shadow of the slit. The dashed lines at $|x| = (\lambda / D)d$ represent the width of the Fraunhofer pattern in the far field. Where the dashed lines coincide with the edges of the geometrical shadow, the Fresnel number N_F=0.5.

$$N_F = a^2 / \lambda z \quad : \text{ Fresnel number}$$

$$U(x,y)=\frac{e^{jkz}}{j\lambda z}\int\int_{-\infty}^{\infty}U(\xi,\eta)\exp\left\{j\frac{k}{2z}\left[(x-\xi)^2+(y-\eta)^2\right]\right\}d\xi d\eta$$

The Fresnel Formula

$$U_P = \frac{-ikU_0}{2\pi z z'}\, e^{ik|PS|} \int_A \exp\left(\frac{ik}{2z_a}\left[(x_0-x_m)^2+(y_0-y_m)^2\right]\right) dS$$

- let z' $\rightarrow \infty$ and use a point on the axis x,y = 0

$$U_P = B\int_A \exp\left(\frac{ik}{2z}\left[x_0^2+y_0^2\right]\right) dS$$

$$= B\int_{x_1}^{x_2}\exp\left(\frac{ikx_0^2}{2z}\right) dx_0 \int_{y_1}^{y_2}\exp\left(\frac{iky_0^2}{2z}\right) dy_0$$

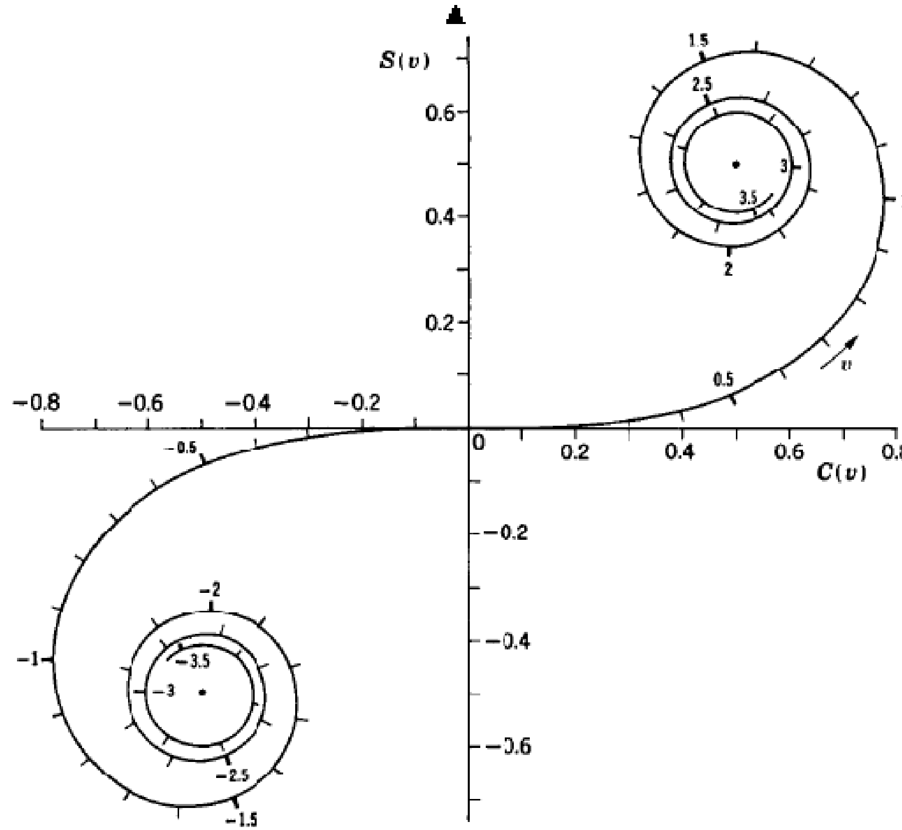
$$= B\int_{x_1}^{x_2}\exp\left(\frac{i\pi u^2}{2}\right) dx_0 \int_{y_1}^{y_2}\exp\left(\frac{i\pi v^2}{2}\right) dy_0$$

- where $u^2 = kx_0^2/(\pi z)$, $v^2 = ky_0^2/(\pi z)$

The Fresnel Integral

$$\int_{s_1}^{s_2} \exp(i\pi w^2/2)\,dw = \int_{s_1}^{s_2} \cos(\pi w^2/2)\,dw + i\int_{s_1}^{s_2} \sin(\pi w^2/2)\,dw$$
$$= C(s) + iS(s)$$

- Cornu spiral
- As s → ± ∞, C(s),S(s)→ ± ½
- Value of integral can be evaluated numerically



Fresnel Integral Definitions

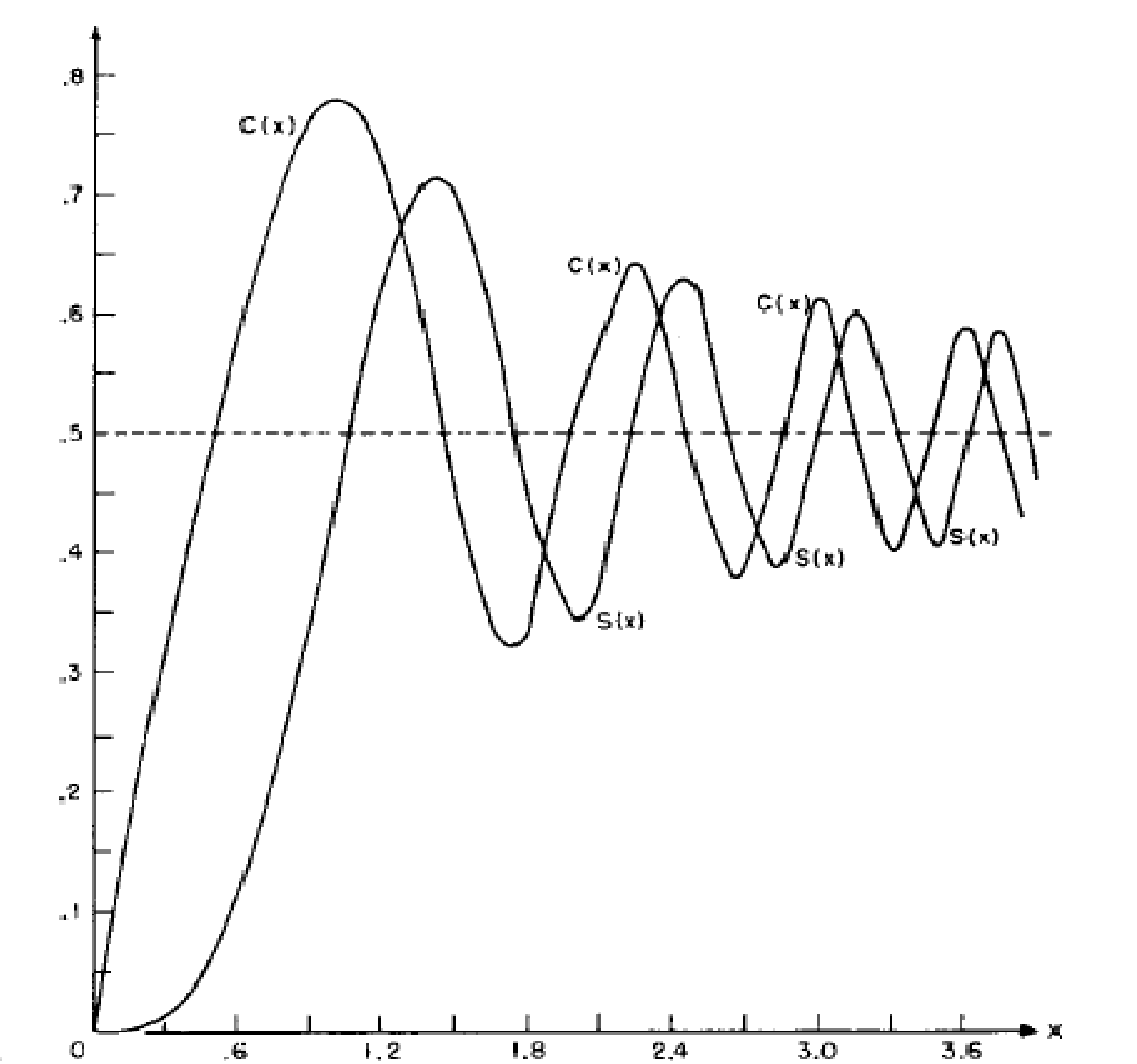
$$I(\xi) \equiv \int_0^{\xi} e^{+\frac{j\pi}{2}t^2}\, dt \quad \text{[Complex Fresnel Integral } I\text{]}$$

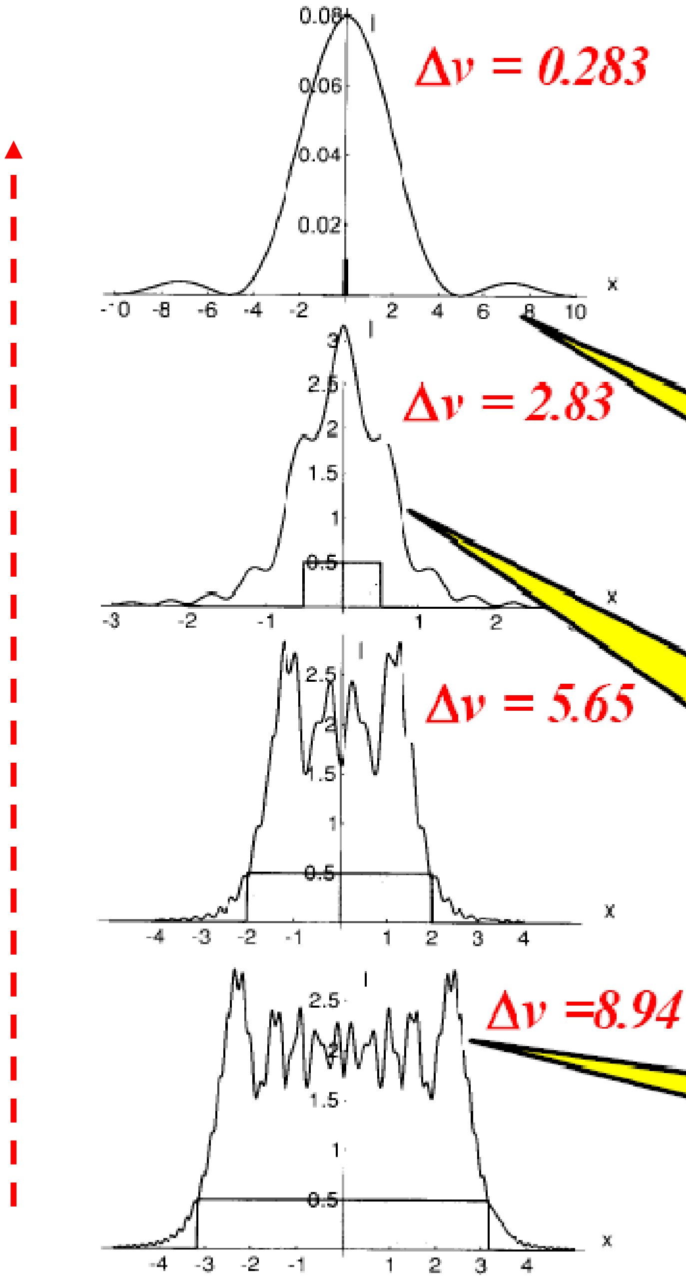
$$I(\xi) = C(\xi) + jS(\xi) \quad \text{[Fresnel Integrals } C \text{ and } S\text{]}$$

$$C(\xi) = \int_0^{\xi} \cos\left(\frac{\pi}{2}t^2\right) dt$$

$$S(\xi) = \int_0^{\xi} \sin\left(\frac{\pi}{2}t^2\right) dt$$

Plot of Fresnel Integrals





Fresnel Slit Diffraction Evolution

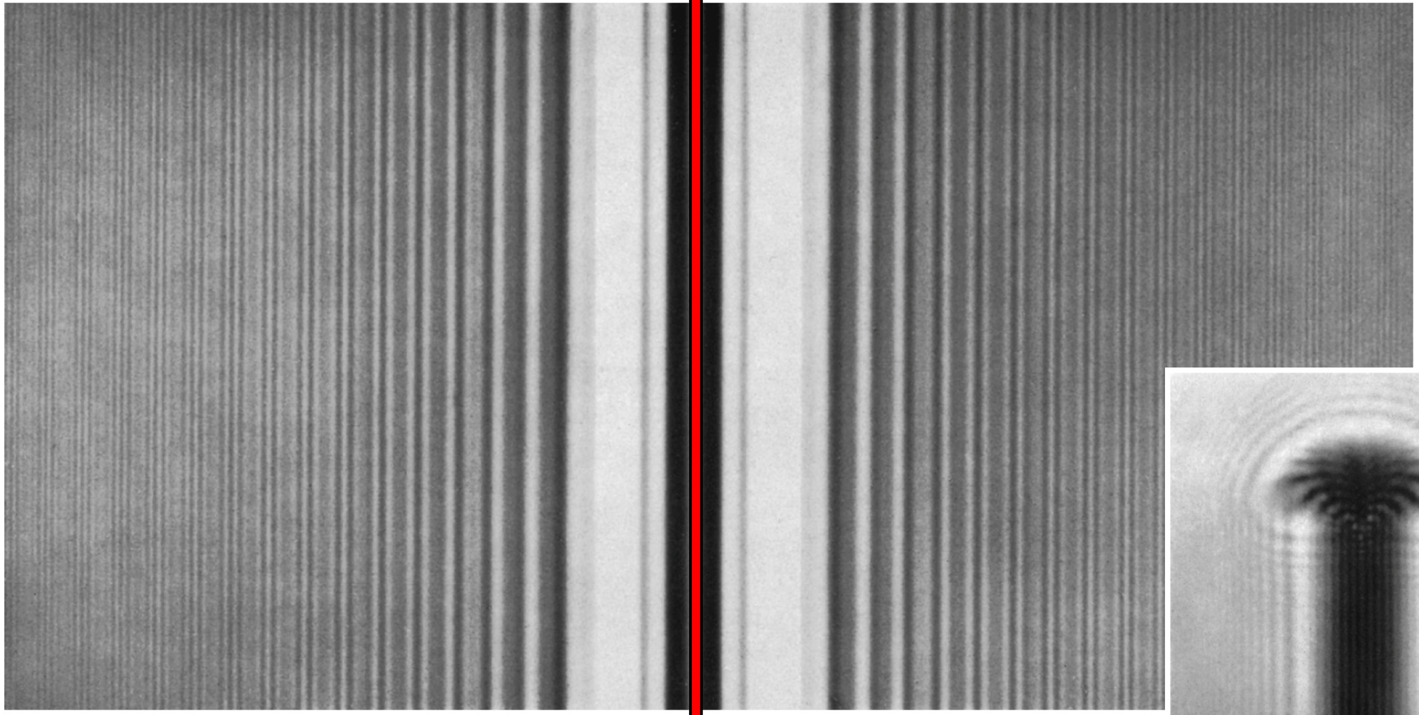
Fourier transform-like result near far-zone

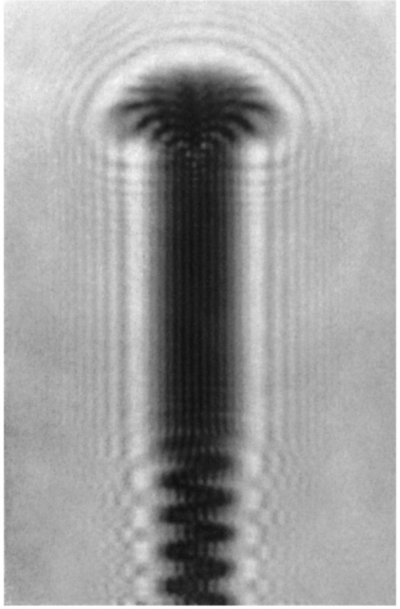
Sloughing off of diffraction "shoulders" with distance

Geometric "image" of slit near aperture

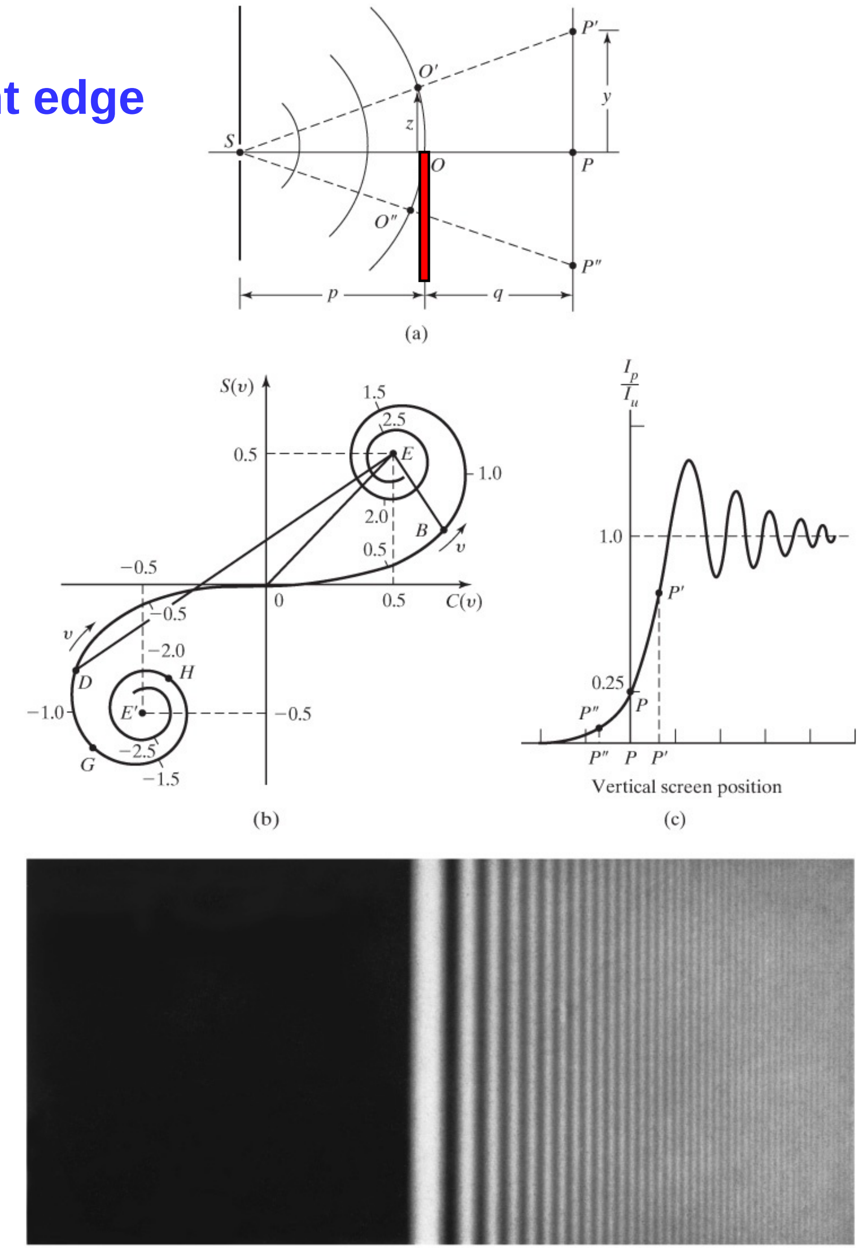
$$\Delta v = N_F = a^2 / \lambda z \;\; : \text{ Fresnel number}$$

Fresnel diffraction from a wire





Fresnel diffraction from a straight edge

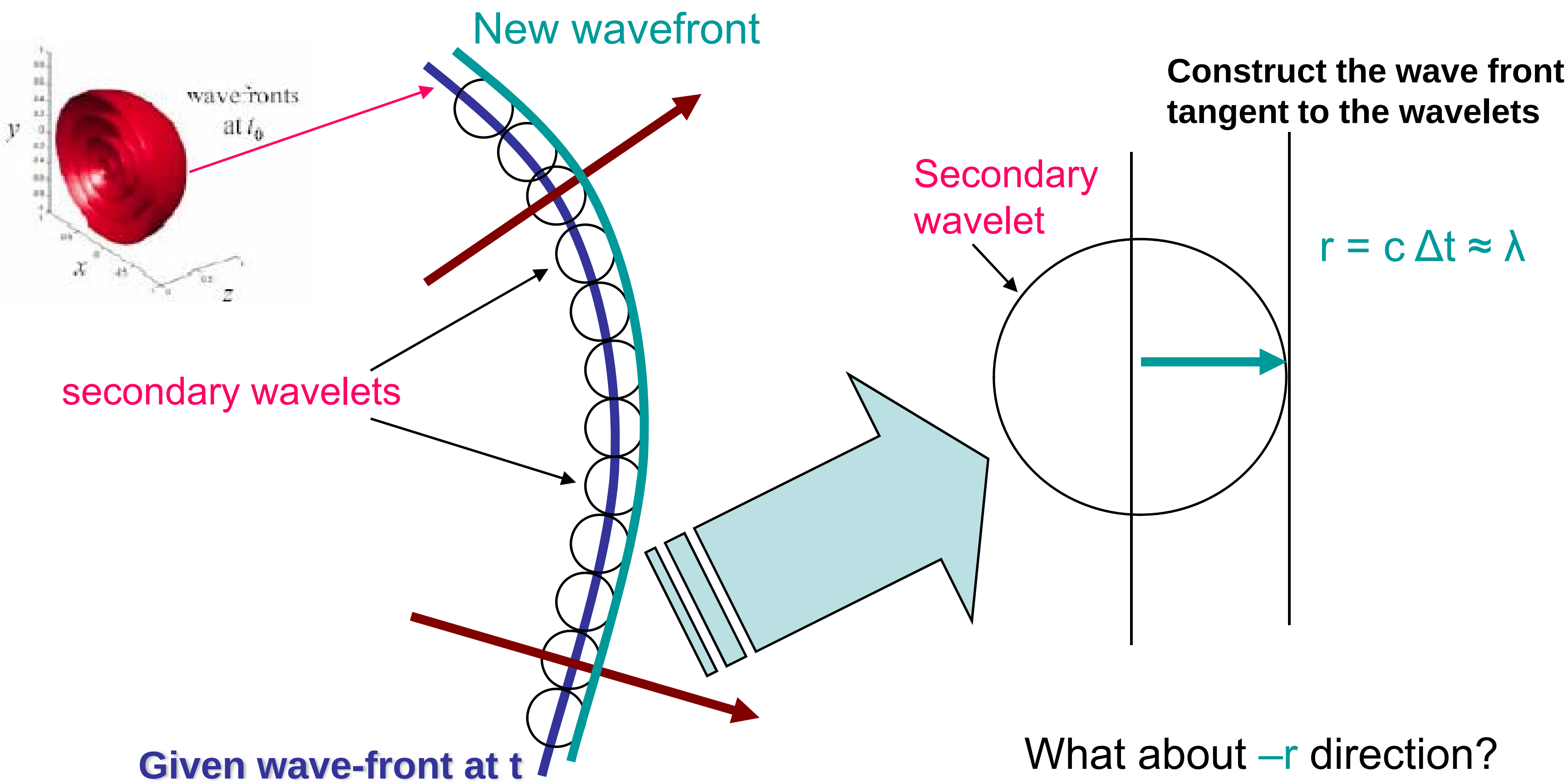


(d)

From Huygens' principle
to Fresnel-Kirchhoff diffraction

Huygens’ principle

Every <u>point</u> on a wave front is a source of secondary wavelets.
i.e. particles in a medium excited by electric field (E) re-radiate in all directions
i.e. in vacuum, E, B fields associated with wave act as sources of additional fields



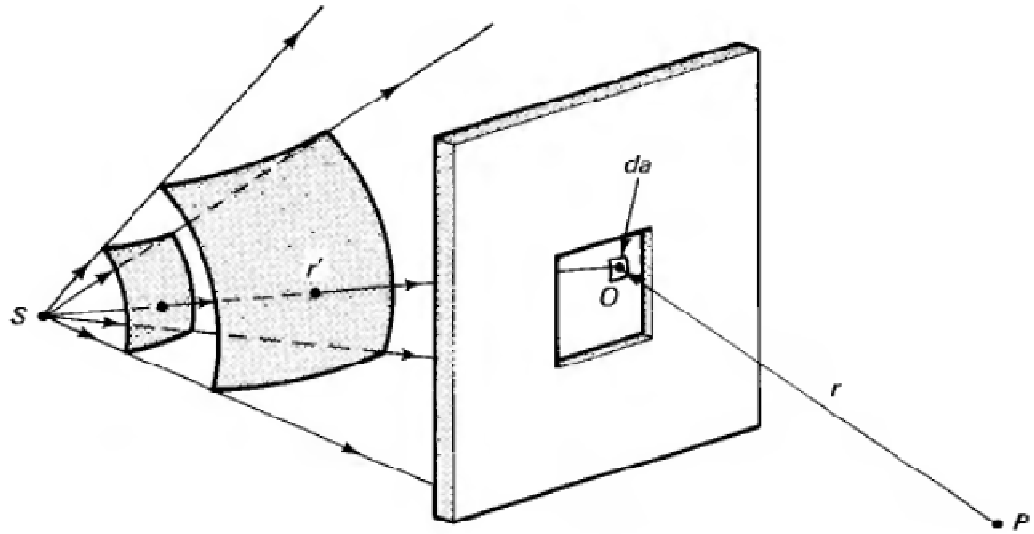
What about $-r$ direction?

(π-phase delay when the secondary wavelets, Hecht, 3.5.2, 3nd Ed)

Allow wavelets to evolve for time Δt

Huygens' wave front construction

the electric field at P due to a superposition of all the Huygens' wavelets from the wavefront at the aperture,



$$dE_P = \left(\frac{dE_0}{r}\right)e^{ikr}$$

$$dE_0 \propto E_L\, da$$

The amplitude E_L at point O is the amplitude of the spherical wave originating at the source,

$$E_L = \left(\frac{E_s}{r'}\right)e^{ikr'}$$

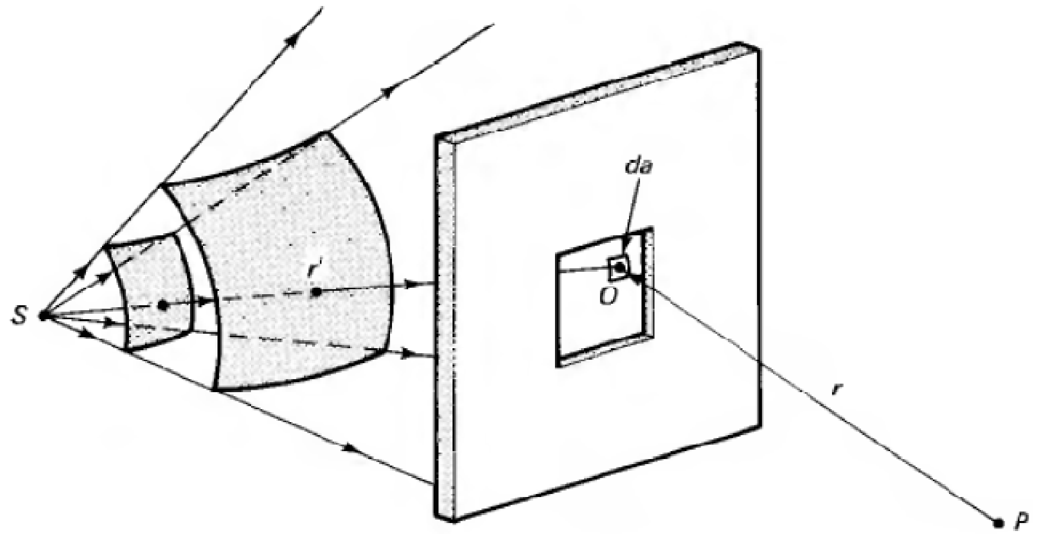
$$\Rightarrow \quad dE_P = \left(\frac{E_s}{rr'}\right)e^{ik(r+r')}\, da$$

The field at P due to the secondary wavelets from the entire aperture is the surface integral

$$\Rightarrow \quad E_P = E_s \iint_{A_p} \left(\frac{1}{rr'}\right)e^{ik(r+r')}\, da$$

Incompleteness of Huygens' principle

$$E_P = E_s \iint_{A_P} \left(\frac{1}{rr'}\right) e^{ik(r+r')}\, da$$



incomplete in two ways

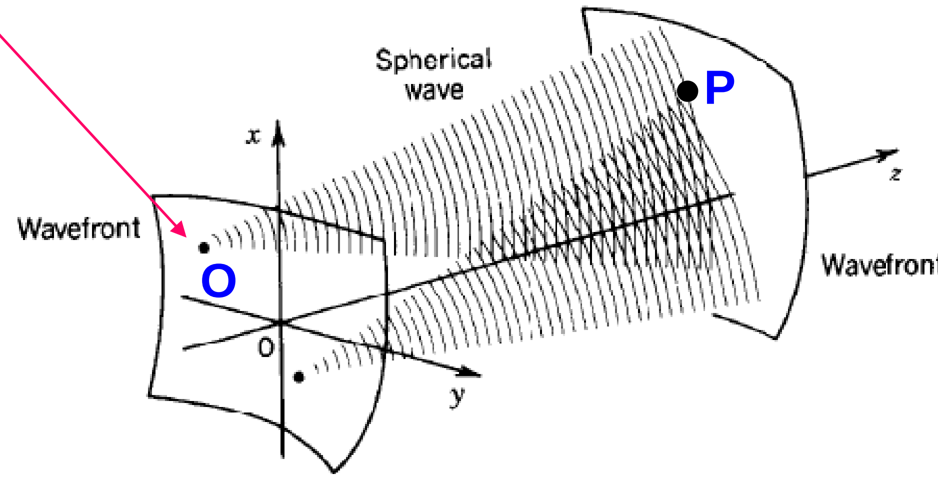
- First, it does not take into account the $F(\theta)$,
  obliquity factor, which attenuates the diffracted waves according to their direction

- Second, it does not take into account a curious requirement,
  a 90° phase shift of the diffracted waves relative to the primary incident wave.

Fresnel's modification ➔ Huygens-Fresnel principle

Huygens-Fresnel principle

The Huygens–Fresnel principle. Each point on a wavefront generates a spherical wave.



$$E_P = E_s \iint_{A_P} \left(\frac{1}{rr'}\right) e^{ik(r+r')}\, da$$

$$\Rightarrow \quad E_p = E_s \iint_{A_p} \frac{1}{rr'} e^{ik(r+r')} F(\theta)\, da = E_s \left[\frac{1}{r'} e^{ikr'}\right] \iint_{A_p} \frac{1}{r} e^{ikr} F(\theta)\, da$$

Spherical wave from the point source S

Obliquity factor:
unity where $\theta=0$
zero where $\theta = \pi/2$

Huygens' Secondary wavelets on the wavefront surface O

Kirchhoff modification

Fresnel's shortcomings :

He did not mention the existence of backward secondary wavelets, however, there also would be a reverse wave traveling back toward the source. He introduce a quantity of the obliquity factor, but he did little more than conjecture about this kind.

$$E_p = E_s \frac{1}{r'} e^{ikr'} \iint_{A_p} \frac{1}{r} e^{ikr} F(\theta) da \, , \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2} \right)$$

Gustav Kirchhoff : Fresnel-Kirchhoff diffraction theory

A more rigorous theory based directly on the solution of the differential wave equation. He, although a contemporary of Maxwell, employed the older elastic-solid theory of light. He found $F(\theta) = (1 + \cos\theta)/2$. $F(0) = 1$ in the forward direction, $F(\pi) = 0$ with the back wave.

Fresnel-Kirchhoff diffraction formula

$$\Rightarrow \quad \boxed{E_P = \frac{-ikE_S}{2\pi} \iint F(\theta) \frac{e^{ik(r+r')}}{rr'} da}$$

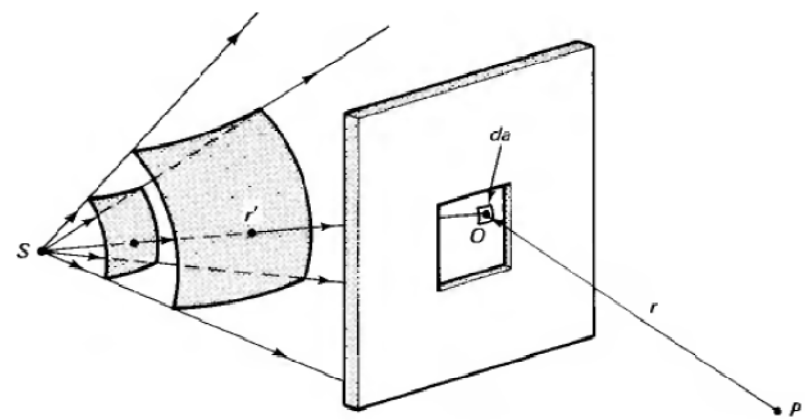
where the factor $-i = e^{-i\pi/2}$ represents the required phase shift, and $F(\theta) = \dfrac{1 + \cos\theta}{2}$

Fresnel-Kirchhoff diffraction integral

$$E_p = \frac{-ikE_s}{2\pi}\iint_{A_p}\left\{\frac{1+\cos\theta}{2}\right\}\frac{1}{rr'}e^{ik(r+r')}da\,,\ \left(-\pi<\theta<\pi\right)$$

Arnold Johannes Wilhelm Sommerfeld : Rayleigh-Sommerfeld diffraction theory
A very rigorous solution of partial differential wave equation.
The first solution utilizing the electromagnetic theory of light.

$$E_p = \frac{1}{i\lambda}\iint_{A_p} E_O \frac{e^{ikr}}{r}\cos\theta\, da$$



This final formula looks similar to the Fresnel-Kirchhoff formula, therefore, now we call this the revised Fresnel-Kirchhoff formula, or, just call the Fresnel-Kirchhoff diffraction integral.

HUYGENS-FRESNEL CONSTRUCTION : Fresnel Zones

The total contribution to the disturbance at P is expressed as an area integral over the primary wavefront,

$$\psi(P) = A\,\frac{\exp\left[-i(\omega t - kr_0)\right]}{r_0} \iint_S \frac{\exp(iks)}{s} K(\chi)\, dS \qquad (8)$$

Spherical wave from source P_o

Obliquity factor:
unity where $\chi=0$ at C
zero where $\chi=\pi/2$ at high enough zone index

Huygens' Secondary wavelets on the wavefront surface S

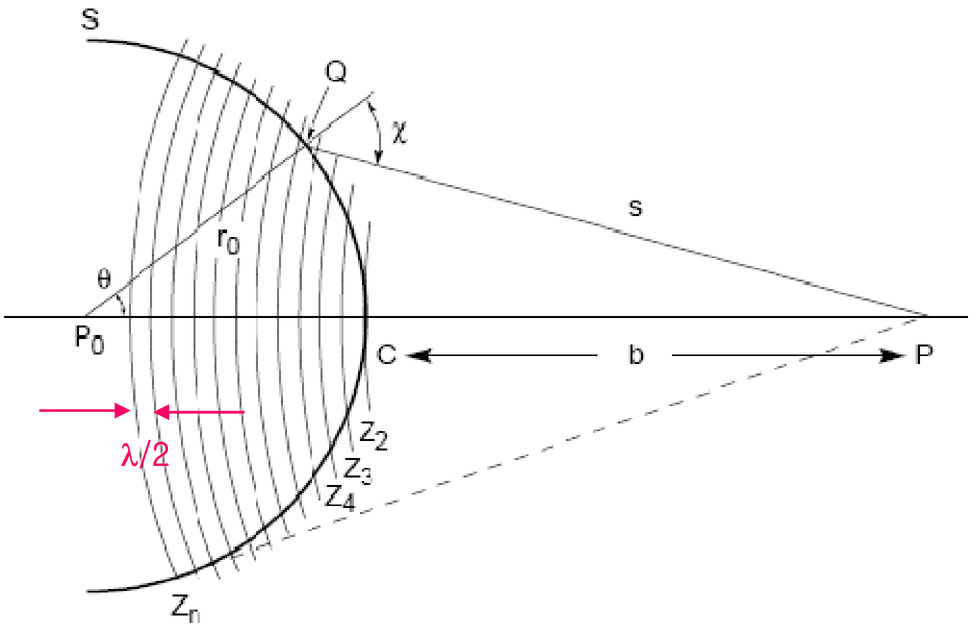
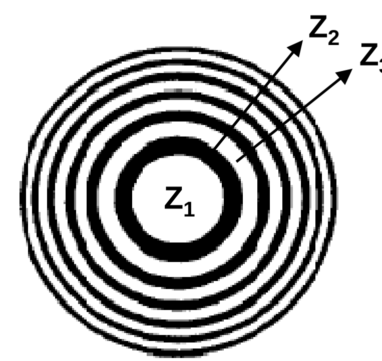
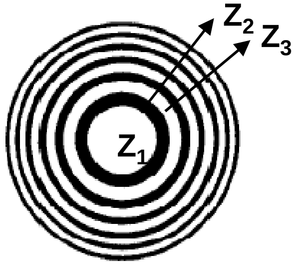


FIGURE 2 Fresnel zone construction. P_0: point source. S: wavefront. r_0: radius of the wavefront b: distance CP. s: distance QP. (*After Born and Wolf*[1].)



HUYGENS-FRESNEL CONSTRUCTION : Fresnel Zones

$$\psi(P) = A\,\frac{\exp[-i(\omega t - kr_0)]}{r_0}\iint_S \frac{\exp(iks)}{s}K(\chi)\,dS$$



The average distance of successive zones from P differs by λ/2 -> half-period zones.
Thus, the contributions of the zones to the disturbance at P alternate in sign,

$$\psi(P) = \psi_1 - \psi_2 + \psi_3 - \psi_4 + \psi_5 - \psi_6 + \ldots$$

where ψ_j stands for the contribution of the jth zone, $j = 1, 2, 3, \ldots$ The contribution of each annular zone is directly proportional to the zone area and is inversely proportional to the average distance of the zone to the point of observation P. The ratio of the zone area to its average distance from P is independent of the zone index j. Thus, in summing the contributions of the zones we are left with only the variation of the obliquity factor, $K(\chi)$. To a good approximation, the obliquity factors for any two adjacent zones are nearly equal and for a large enough zone index j the obliquity factor becomes negligible. The total disturbance at the point of observation P may be approximated by

$$\psi(P) = 1/2(\psi_1 \pm \psi_n)$$

(1/2 means averaging of the possible values, more details are in 10-3, Optics, Hecht, 2nd Ed)

For an unobstructed wave, the last term $\psi_n = 0$.

$$\psi(P) = 1/2\psi_1$$
$$= \frac{A}{r_0 + b}\lambda \exp\{-i[\omega t - k(r_0 + b) - \pi/2]\}$$

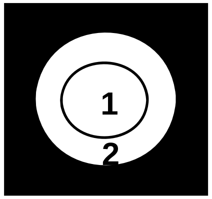
Whereas, a freely propagating spherical wave from the source P_o to P is

$$\psi(P) = \frac{A}{r_0 + b}\exp\{-i[\omega t - k(r_0 + b)]\}$$

Therefore, one can assume that the complex amplitude of $\exp(iks)/s$

$$[1/\lambda \exp(-i\pi/2)]\ \exp(iks)/s$$
$$= \frac{1}{i\lambda}\left(\frac{\exp(iks)}{s}\right)$$

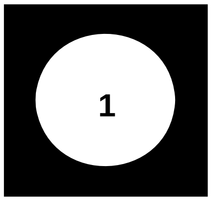
HUYGENS-FRESNEL CONSTRUCTION : Diffraction of light from circular apertures and disks



(a) The first two zones are uncovered,

$$\psi(P) = \psi_1 - \psi_2 = 0!$$ (consider the point P at the on-axis P)

since these two contributions are nearly equal,



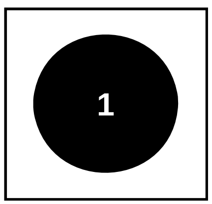
(b) The first zone is uncovered if point P is placed father away,

$$\psi(P) = \psi_1$$

which is twice what it was for the unobstructed wave!

: Babinet principle

$$\psi_S(P) + \psi_{CS}(P) = \psi_{UN}(P)$$



(c) Only the first zone is covered by an opaque disk,

$$\psi(P) = -\psi_2 + \psi_3 - \psi_4 + \psi_5 - \psi_6 + \cdots = -\tfrac{1}{2}\psi_2 \approx \tfrac{1}{2}\psi_1$$

which is the same as the amplitude of the unobstructed wave.

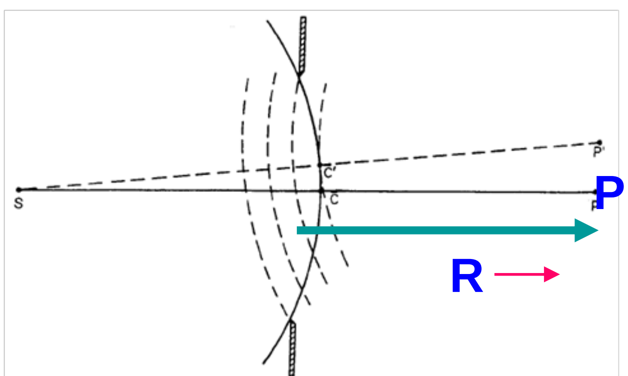


FIGURE 4 The redrawn zone structure for use with an off-axis point P'. (*After Andrews.*[8])

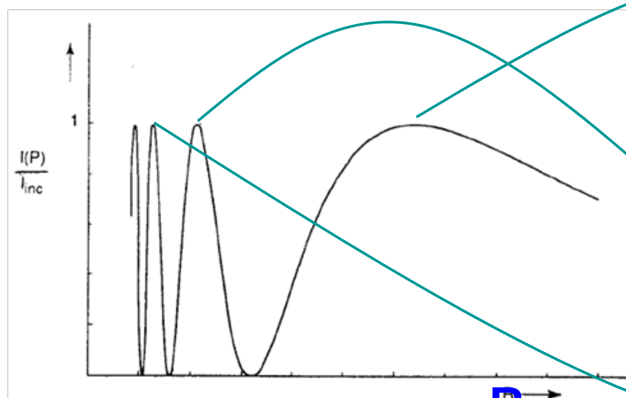
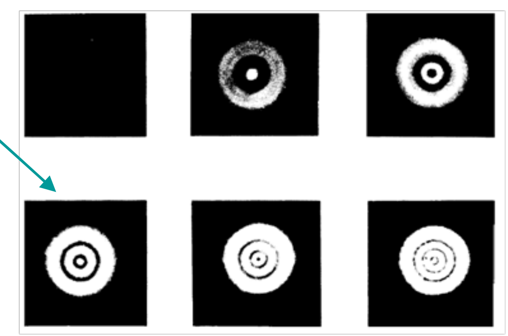


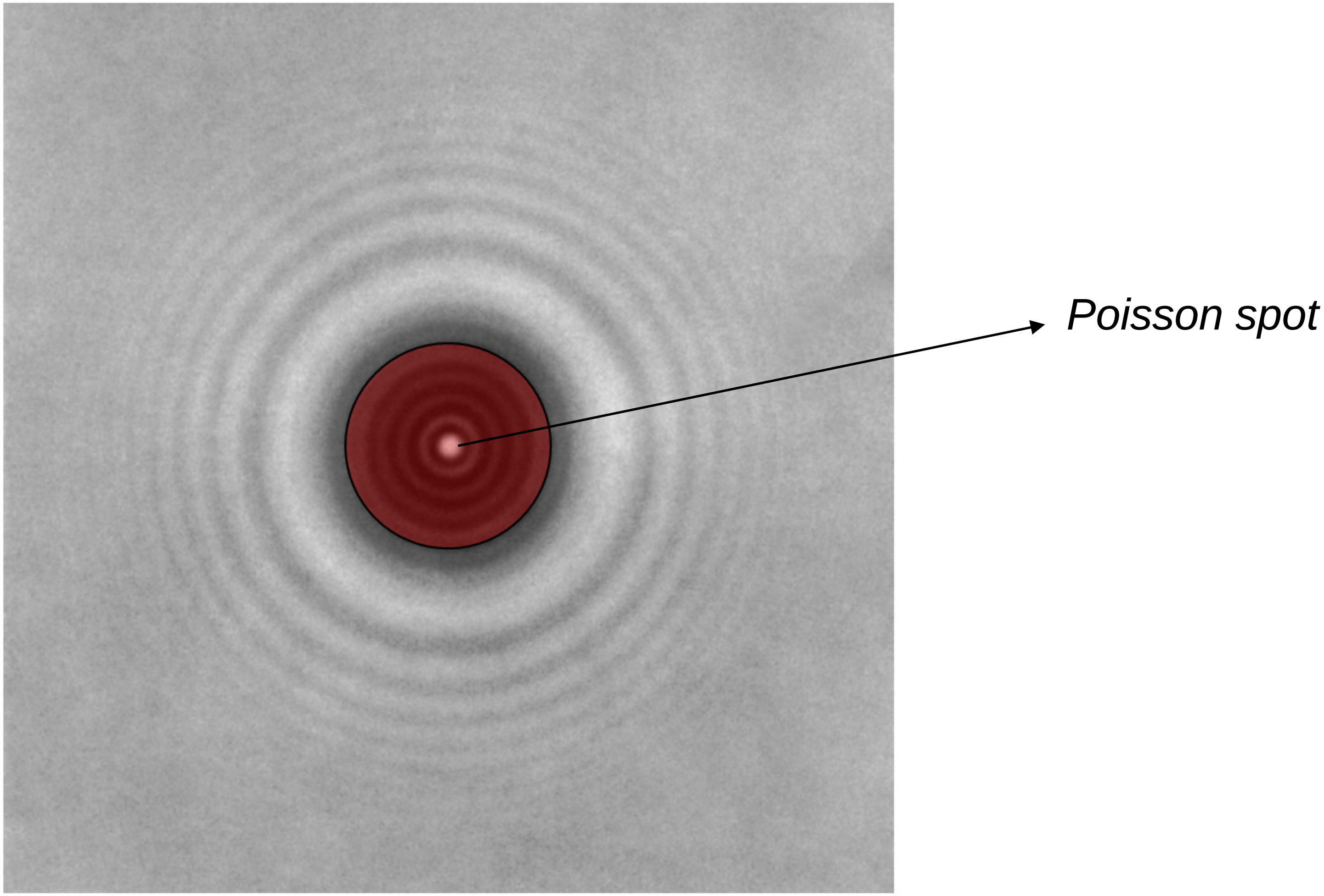
FIGURE 6 ... ng the axis behind ...

Variation of on-axis irradiance



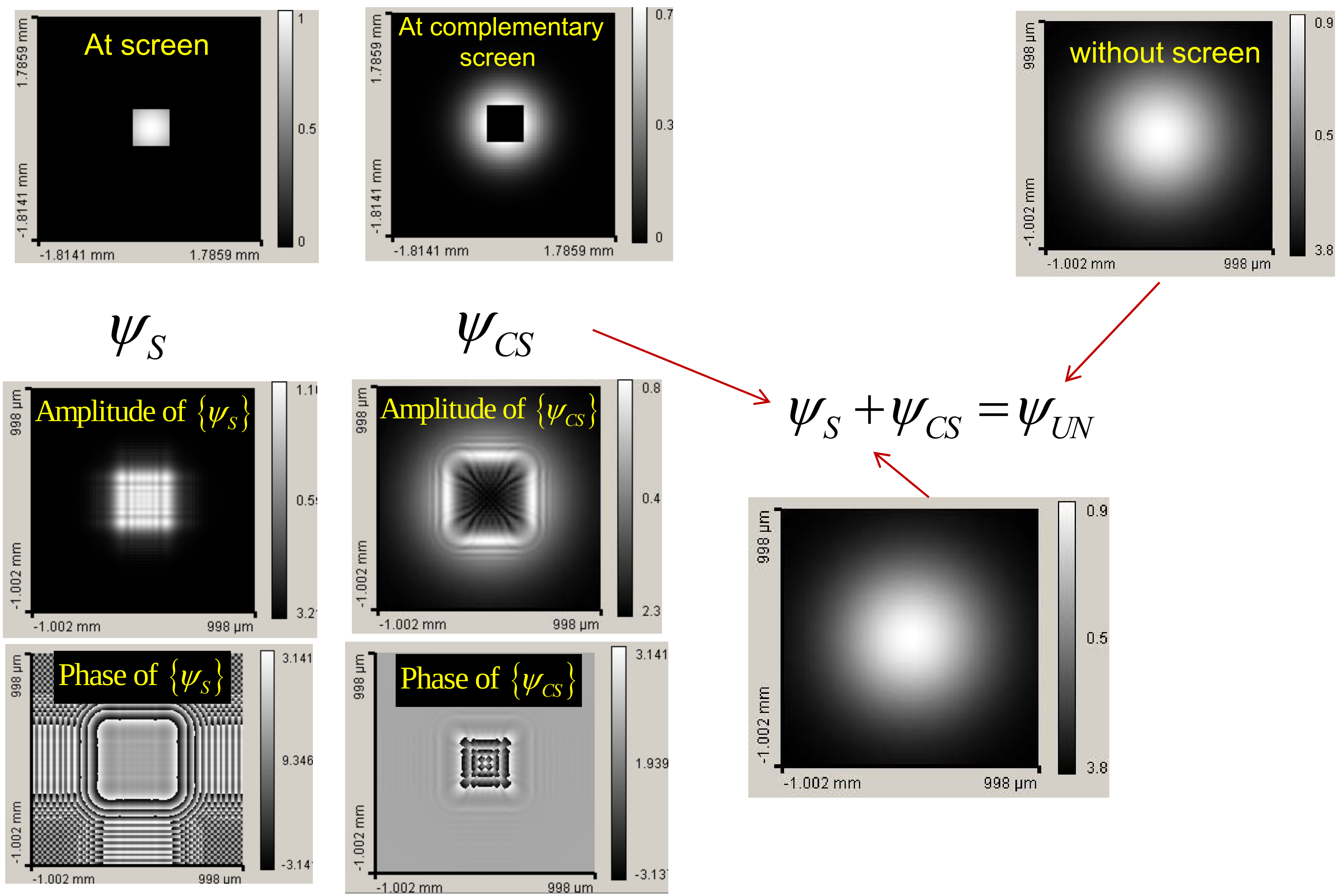
Diffraction patterns from circular apertures

Fresnel diffraction from a circular aperture

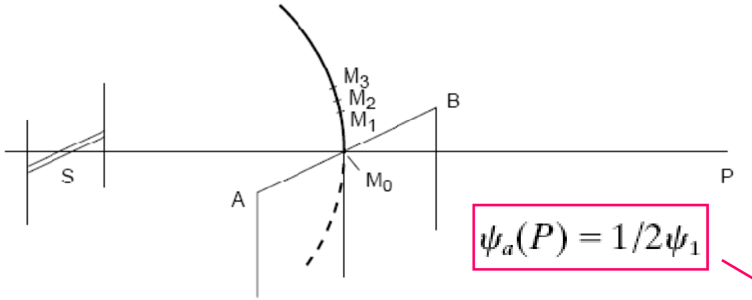


Babinet principle

$$\psi_S(P) + \psi_{CS}(P) = \psi_{UN}(P)$$



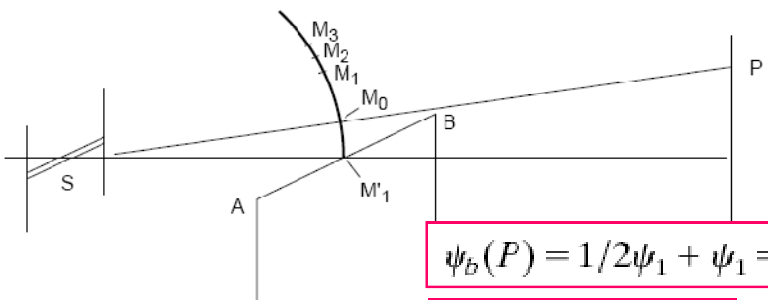
HUYGENS-FRESNEL CONSTRUCTION : Straight edge



$$\psi_a(P) = 1/2\psi_1$$

(a)

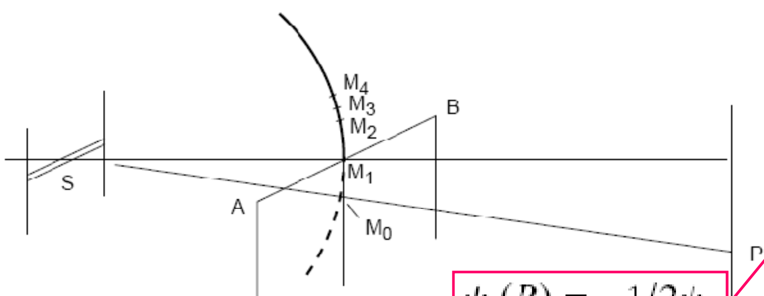
At the edge



$$\psi_b(P) = 1/2\psi_1 + \psi_1 = 3/2\psi_1$$

(b)

$$\psi(P) = 3/2\psi_1 - \psi_2$$

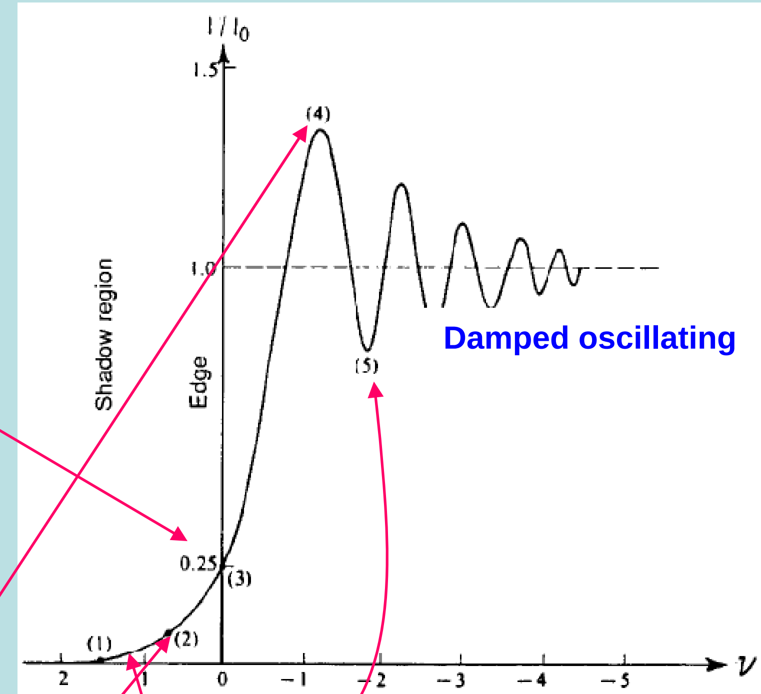


$$\psi_c(P) = -1/2\psi_2$$

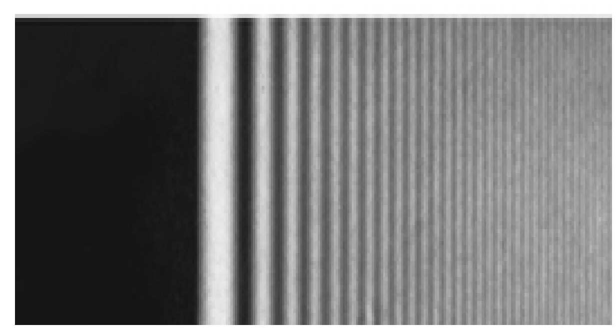
(c)

$$\psi(P) = 1/2\psi_3$$

Monotonically decreasing



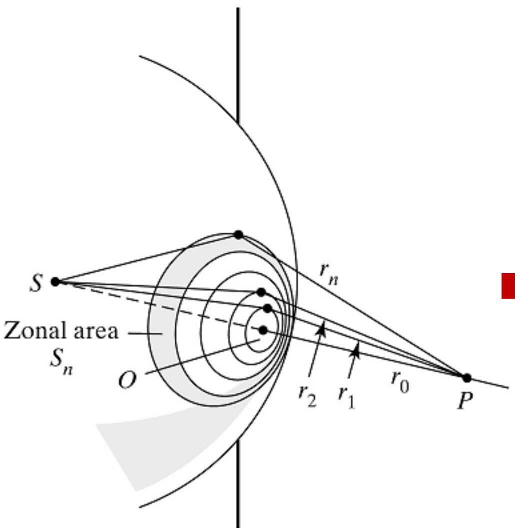
Damped oscillating



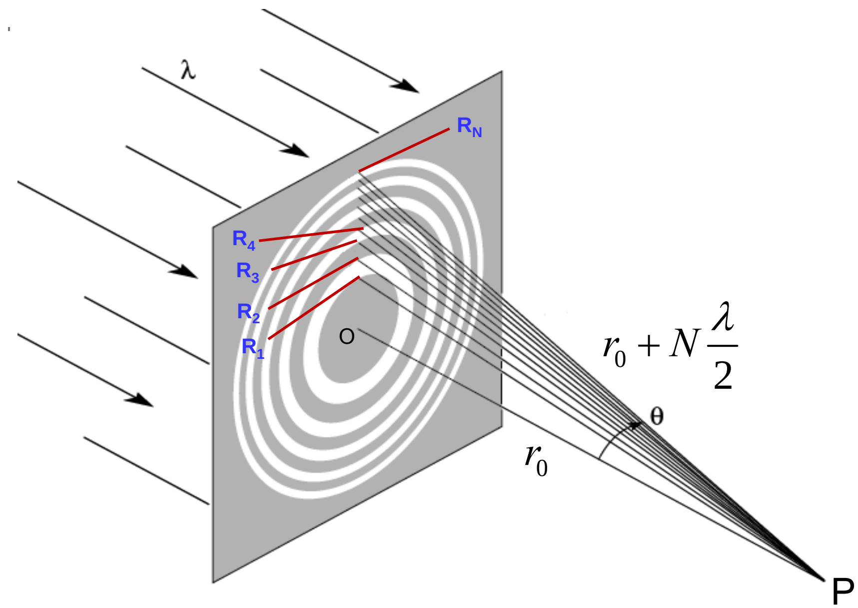
13-6. The Fresnel zone plate

The average distance of successive zones from P differs by λ/2 -> half-period zones.
Thus, the contributions of the zones to the disturbance at P alternate in sign,

$$\psi(P) = \psi_1 - \psi_2 + \psi_3 - \psi_4 + \psi_5 - \psi_6 + \ldots$$

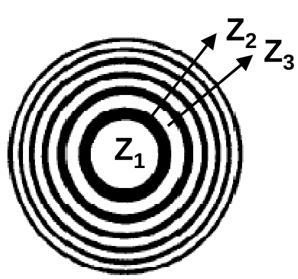


Assume
plane wavefronts



$$R_n^2 = \left(r_0 + n\frac{\lambda}{2}\right)^2 - r_0^2 = r_0^2\left[n\frac{\lambda}{r_0} + \frac{n^2}{4}\left(\frac{\lambda}{r_0}\right)^2\right] \quad \Rightarrow \quad R_n \approx \sqrt{n r_0 \lambda} \quad (r_0 >> \lambda)$$

If the even zones (n=even) are blocked

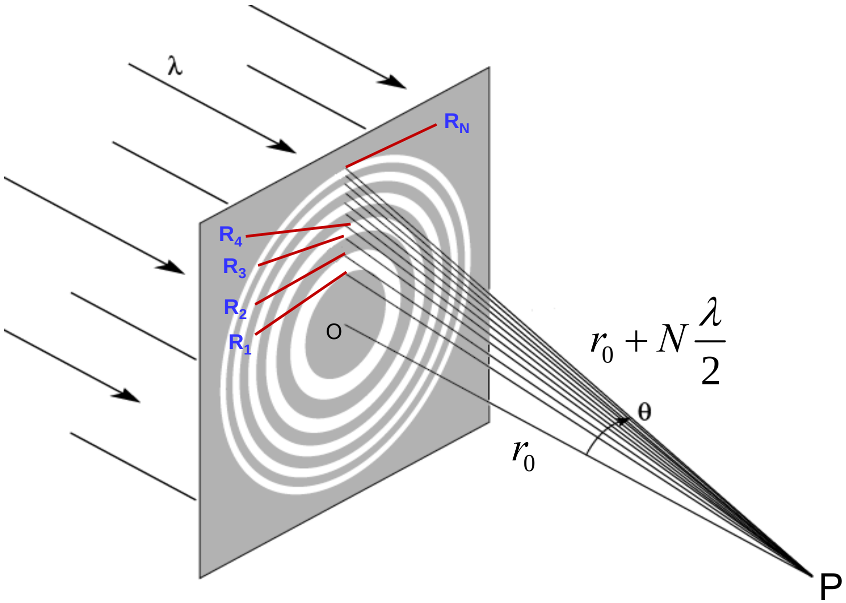


$$\psi(P) = \psi_1 + \psi_3 + \psi_5 + \cdots$$

Bright spot at P

It acts as a lens!

Fresnel zone-plate lens

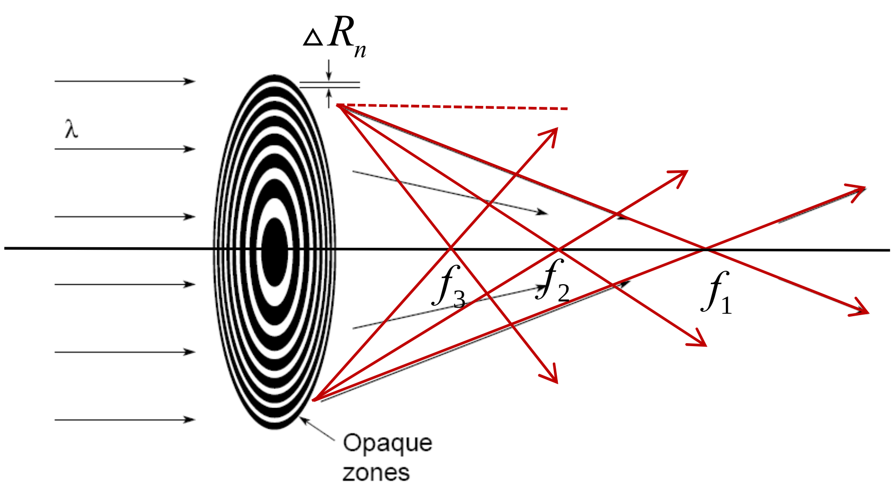


$$R_n \approx \sqrt{n r_0 \lambda} \quad (r_0 >> \lambda)$$

$$r_0 = \frac{R_n^2}{n\lambda}$$

$$f_1 = r_0(n=1) = \frac{R_1^2}{\lambda}$$

Fresnel zone-plate lens has multiple foci.



$$\triangle R_n = \sqrt{r_0\lambda}\,\frac{\triangle n}{2\sqrt{n}} = R_1 \frac{1}{2\sqrt{n}}$$

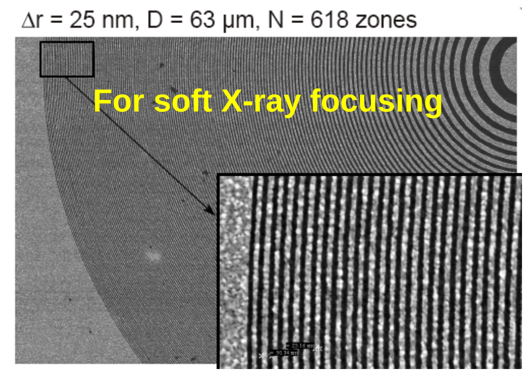
$$(\triangle R_n)\sin\theta_m = m\lambda \Rightarrow \sin\theta_m \sim \tan\theta_m = \frac{R_n}{f_m} = \frac{m\lambda}{\triangle R_n}$$

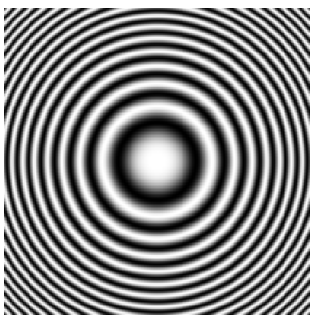
$$f_m = (R_n)(\triangle R_n)\frac{1}{m\lambda} = \left(\sqrt{n}R_1\right)\left(\frac{R_1}{2\sqrt{n}}\right)\frac{1}{m\lambda}$$

$$\boxed{f_m = \frac{R_1^2}{m\lambda}}$$

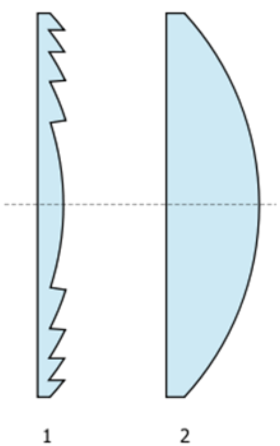
Fresnel zone-plate lens

Binary zone plate:
The areas of each ring, both light and dark, are equal.
It has multiple focal points.





Sinusoidal zone plate:
This type has a single focal point.



Fresnel lens:
This type has a single focal point.
Focusing efficiency approaches 100%.