Standard deviation

$$\langle\alpha|\hat{E}_x^2(\vec{r},t)|\alpha\rangle = -\frac{\hbar\omega}{2\varepsilon_0 V}\langle\alpha|\hat{a}\hat{a}e^{2i(\vec{k}\vec{r}-\omega t)} - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger e^{-2i(\vec{k}\vec{r}-\omega t)}|\alpha\rangle$$

$$= \frac{\hbar\omega}{2\varepsilon_0 V}\left(1+2|\alpha|^2 - 2\,\mathrm{Re}\left(\alpha^2 e^{2i(\vec{k}\vec{r}-\omega t)}\right)\right)$$

$$= \frac{\hbar\omega}{2\varepsilon_0 V}\left(1+2|\alpha|^2\left[1-\cos\left(2(\vec{k}\vec{r}-\omega t)+\theta\right)\right]\right)$$

$1-\cos(2\alpha) = 2\sin^2\alpha$

$$= \frac{\hbar\omega}{2\varepsilon_0 V}\left(1+4|\alpha|^2\sin^2\left(\omega t-\vec{k}\vec{r}+\theta\right)\right)$$

$$\langle\Delta E\rangle_\alpha = \sqrt{\langle\hat{E}_x^2\rangle - \langle\hat{E}_x\rangle^2} = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}$$

(independent of α and of $t, \vec{r}$)

→ comment: same uncertainty as for $|0\rangle$

→ attain minimum uncertainty product in quadratures

$$\langle(\Delta X_1)^2\rangle = \langle(\Delta X_2)^2\rangle_\alpha = \frac{1}{4}$$

Number basis representation of $|\alpha\rangle$

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \rightarrow \underbrace{\hat{a}|\alpha\rangle}_{\alpha \sum_{n=0}^{\infty} c_n |n\rangle} = \sum_{n=1}^{\infty} c_n \underbrace{\hat{a}|n\rangle}_{\sqrt{n}\,|n-1\rangle}$$

$\hat{a}$ $\hat{a}$

$$= \sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1}\, |n\rangle$$

$\Rightarrow$ get a recursive relation:

$$\alpha c_n = \sqrt{n+1}\, c_{n+1} \quad \text{for } n = 0, 1, \ldots$$

$\rightarrow$ can generate coefficients from

$$c_n = \frac{\alpha^n}{\sqrt{n!}} c_0 \qquad n = 1,$$

$\rightarrow$ Determine c_0 from normalization $\langle\alpha|\alpha\rangle = 1$

$$|c_0|^2 \sum_{n,m=0}^{\infty} \frac{(\alpha^*)^n \alpha^m}{\sqrt{n!}\sqrt{m!}} \underbrace{\langle n|m\rangle}_{=\delta_{nm}} = 1$$

$$= |c_0|^2 \sum_{n=0}^{\infty} \frac{(|\alpha|^2)^n}{n!} \Longrightarrow \boxed{c_0 = e^{-|\alpha|^2/2}}$$

$$\boxed{|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle}$$

Number statistics of $|\alpha\rangle$

$$\bar{n}_\alpha = \langle\alpha|\hat{n}|\alpha\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2$$

$$\langle\alpha|\hat{n}^2|\alpha\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\alpha\rangle = \dots = |\alpha|^2 + |\alpha|^4$$

$$(\Delta\hat{n})_\alpha = \sqrt{\langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2} = |\alpha| = \sqrt{\bar{n}_\alpha}$$

$$(\Delta n)^2_\alpha = \bar{n}_\alpha \quad (\text{variance} = \text{mean})$$

↳ Poisson distribution

Thus $\frac{\Delta n}{\bar{n}} = \frac{1}{\sqrt{\bar{n}}} \xrightarrow{\bar{n}\to\infty} 0$

→ relative fluctuations decrease (classical limit)

The number probability distribution P_n

$$P_n(\alpha) = |\langle n|\alpha\rangle|^2 = |c_n|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} \xrightarrow{\bar{n}=|\alpha|^2}$$

$$\boxed{P_n(\bar{n}) = \frac{e^{-\bar{n}}\,\bar{n}^n}{n!}}$$

Poisson distribution

A typical above threshold ($\bar{n} \gg 1$) converges to the Poisson distribution.

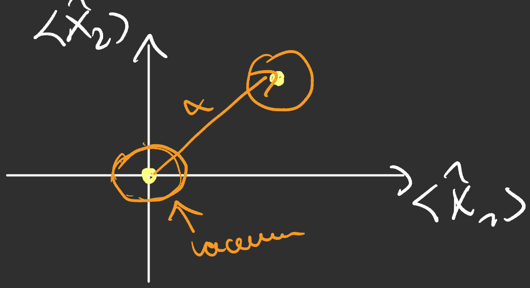
Phase space vs. complex α space

recall: $\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger)$, $\hat{X}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger)$

$$\langle \hat{X}_1 \rangle_\alpha = \frac{1}{2} \langle \alpha | \hat{a} + \hat{a}^\dagger | \alpha \rangle = \mathrm{Re}\{\alpha\}$$

$$\langle \hat{X}_2 \rangle_\alpha = \frac{1}{2i} \langle \alpha | \hat{a} - \hat{a}^\dagger | \alpha \rangle = \mathrm{Im}\{\alpha\}$$

→ for coherent states, $\langle \hat{X}_1 \rangle_\alpha$, $\langle \hat{X}_2 \rangle_\alpha$ are sufficient labels of the state



coherent states have the same $\Delta X_{1,2}$ as vacuum. They can be understood as "displaced" vacuum.

$$|0\rangle \xrightarrow{\hat{D}} |\alpha\rangle$$

Displacement operator: $\hat{D}(\alpha) \equiv e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}$

$$\overset{\text{B-C-H}}{=} e^{-|\alpha|^2/2} \, e^{\alpha \hat{a}^\dagger} \, e^{-\alpha^* \hat{a}}$$

BCH: $e^X e^Y = e^Z$, $Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X,[X,Y]] + \dots$

$$\mathcal{D}^{+}(\alpha) = e^{\alpha^* \hat{a} - \alpha \hat{a}^{+}} = \mathcal{D}(-\alpha) = \mathcal{D}^{-1}(\alpha)$$

$\Rightarrow$ $\mathcal{D}(\alpha)$ is a unitary operator

$$\boxed{|\alpha\rangle = \hat{\mathcal{D}}(\alpha)\,|0\rangle}$$

because $\mathcal{D}(\alpha) = e^{-|\alpha|^2/2}\, e^{\alpha \hat{a}^{+}} \underbrace{e^{-\alpha^* \hat{a}}|0\rangle}_{|0\rangle}$

$$= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} (\hat{a}^{+})^n |0\rangle$$

$$= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \underbrace{\frac{(\hat{a}^{+})^n}{\sqrt{n!}}|0\rangle}$$

$= |\alpha\rangle$

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p})$$

$$\hat{a}^{+} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} - i\hat{p})$$

Let's express $\hat{\mathcal{D}}(\alpha)$ in terms of $\hat{q}$ and $\hat{p}$

$$\hat{\mathcal{D}}(\alpha) = e^{\alpha\hat{a}^{+} - \alpha^*\hat{a}} = e^{\frac{\alpha}{\sqrt{2\hbar\omega}}(\omega\hat{q} - i\hat{p}) - \frac{\alpha^*}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p})}$$

$$= e^{(\alpha - \alpha^*)\sqrt{\frac{\omega}{2\hbar}}\,\hat{q} - i(\alpha + \alpha^*)\sqrt{\frac{1}{2\hbar\omega}}\,\hat{p}}$$

B-CH

$$\downarrow$$

$$= e^{-(\alpha^{*2} - \alpha^2)/4}\, e^{(\alpha - \alpha^*)\sqrt{\frac{\hbar}{2\hbar}}\,\hat{q}}\, e^{-i\frac{(\alpha+\alpha^*)}{\sqrt{2\hbar\omega}}\,\hat{p}}$$

$\Rightarrow$ $\hat{\mathcal{D}}$ is a placement operator in phase space

Time evolution of coherent states

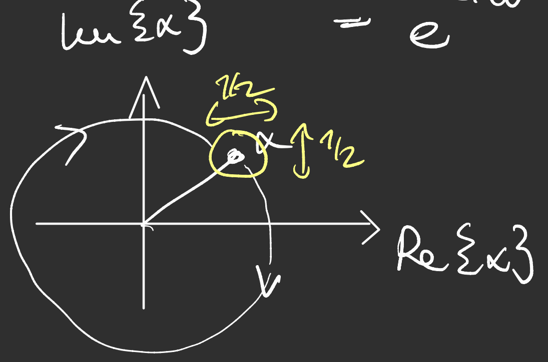
$$|\alpha(t)\rangle = e^{-i\hat{H}t/\hbar}|\alpha\rangle, \quad H = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)$$

$$= e^{-i\omega(\hat{n}+1/2)t}|\alpha\rangle$$

$$= e^{-i\omega_{/2} t}\, e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-i\omega \overbrace{\hat{n}}^{t\,n}} |n\rangle$$

$$= e^{-i\omega_{/2} t}\, e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}} |n\rangle$$

$$= e^{-i\omega t/2} |\alpha e^{-i\omega t}\rangle$$



Comments:

* Coherent states are not orthogonal

$$\langle\alpha|\alpha'\rangle \neq 0 \text{ for } \alpha \neq \alpha'$$

* form an over-complete basis

$$\frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle\langle\alpha| = \mathbb{1}$$