CHM 696: Quantum Optics

Problem Set 1 (Coherent States)

Valentin Walther
vwalther@purdue.edu

Due: September 19, 2023

I Field operators

In the quantized case, the strength of the electric field at a certain position is an observable and is given by the field operator $\hat{E}(\mathbf{r})$, which has been given in the lecture. If we restrict ourself to a single mode l of the electromagnetic field, the field operator of the electric field can be written as:

$$\hat{E}(\chi) = E_0 \left(\hat{a}e^{-i\chi} + \hat{a}^\dagger e^{i\chi}\right)$$

(a) Calculate the expectation value $\langle \hat{E} \rangle$ and the variance $(\Delta E)^2 = \left\langle \hat{E}^2 \right\rangle - \langle \hat{E} \rangle^2$ for the vacuum state $|0\rangle$, the n-photon Fock state $|n\rangle$ and the coherent state $|\alpha\rangle$.

(b) How does E_0 scale with the frequency ω and the considered volume V?

(c) The Hamiltonian of the field is given by

$$\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)$$

Calculate for the same states $|0\rangle, |n\rangle$ and $|\alpha\rangle$ the expectation value $\langle \hat{H} \rangle$ of the energy, the energy uncertainty ΔH and the relative uncertainty $\Delta H / \langle \hat{H} \rangle$ and interpret the different results.

II Quadrature Components and fields

(a) Show that every classical oscillation of the form $A\cos(\omega t + \phi)$ can be written in the form $b\cos(\omega t) + c\sin(\omega t)$ and calculate b and c.

The "position" and "momentum" operators (which are proportional to the quadratures) are defined as:

$$\hat{x} = \sqrt{\frac{\hbar}{2\omega}} \left(\hat{a} + \hat{a}^\dagger\right) \tag{1}$$

$$\hat{p} = -i\sqrt{\frac{\hbar\omega}{2}} \left(\hat{a} - \hat{a}^\dagger\right) \tag{2}$$

(b) Calculate the expectation values $\langle \hat{x} \rangle, \langle \hat{p} \rangle, \left\langle \hat{x}^2 \right\rangle, \left\langle \hat{p}^2 \right\rangle$ for a coherent state $|\alpha\rangle$.

(c) Show that such a coherent state describes a minimum uncertainty state: $\Delta x \Delta p = \hbar/2$. How does this property relate to "classical" states?

(d) Does every possible state of the quantized electromagnetic field have a classical analogue and conversely, can every possible classical field by described in the quantized photon picture? (Think about field strengths, volumes, photon distributions, phases, etc.)

III Fock states and coherent states in the EM field Hamiltonian

We consider only a single mode of the radiation field. In this mode, Fock states are states with a well defined photon number and are given by:

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle$$

(a) Show that the set of Fock states for one mode forms an orthonormal set, i.e. $\langle n \mid n' \rangle = \delta_{n,n'}$.

(b) How do these orthonormality conditions transfer to the many-mode case?

(c) We again consider coherent states of the form

$$|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle,$$

which are eigenstates of the annihilation operator $\hat{a}$ as discussed earlier. Can an analogue eigenstate of the creation operator $\hat{a}^\dagger$ exist?

(d) What is the expectation value of the photon number operator $\langle\hat{N}\rangle$ for the above coherent state?

(e) Show that the coherent state $|\alpha\rangle$ can also be expressed as:

$$|\alpha\rangle = \exp\left(\alpha\hat{a}^\dagger - \frac{1}{2}|\alpha|^2\right)|0\rangle$$

IV Bonus problem: Harmonic oscillator and the Glauber states

As you know, the harmonic oscillator ground state is a Gaussian wave function. The higher energy eigenstates $|n\rangle$ have the shape of the higher Hermite functions:

$$\Psi_n(x) = \left(\frac{a}{\sqrt{\pi}2^n n!}\right)^{\frac{1}{2}} H_n(ax)e^{-\frac{1}{2}a^2x^2}$$

(a) Show that the real space wave functions $\Psi(x)$ corresponding to the Glauber states of the harmonic oscillator are always Gaussian-shaped, even for $\alpha \neq 0$ (which means that, if you would plot the amplitude of the complex-valued wave function without the phase, or the probability distribution, you would get Gaussians). *You can use the following identity:*

$$e^{2zt-t^2} = \sum_{n=0}^{\infty} H_n(z)\frac{t^n}{n!}$$

(b) Show that such Gaussian wave functions are also a Gaussian in momentum space, and prove that in this case the uncertainty relation $\Delta x\Delta p \geq \hbar/2$ actually is an equality $\Delta x\Delta p = \hbar/2$.