CHM 696: Quantum Optics

Problem Set 1 (Introduction)

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I Typical photon energies

A photon, that is a quantum of electromagnetic radiation, is described by its energy or frequency. In typical quantum optics experiments, two very different frequency ranges are normally used. Here we will have a look at these frequencies and some typical characteristics.

(a) A typical laser source for quantum optics experiments would be a titanium-sapphire laser with a wavelength of $\lambda = 852$ nm, the wavelength of one of the resonances of the Cesium atom. What is the photon energy in Joule (J) and electron Volt (eV), the frequency of the light in Hertz (Hz) and which temperature would it correspond ($E \approx k_B T$) to?

In many cases one does not need to consider all optical transitions but can restrict oneself to a single transition connecting the ground state and one excited (electronic) state. This approximation is known as the *Two-Level-Atom* and will be used on many occasions throughout the lecture. In atomic Sodium $\left({}^{23}\mathrm{Na}, m = 22.99u, u = 1.66 \cdot 10^{-27}\ \mathrm{kg}\right)$, the most important optical transition has a wavelength of $\lambda = 589$ nm. An atom in the excited state will spontaneously decay back into the ground state by emitting a photon.

(b) Suppose that the relaxation would happen without the photon such that the total energy would be converted into kinetic energy. What would be the final velocity of an atom that was initially at rest?

(c) Even if the atom releases the energy by emitting a photon, the conservation of momentum requires that the atom will nonetheless acquire a velocity, the so-called recoil velocity. How big is this velocity and the corresponding kinetic energy (recoil energy)?

II Mechanical velocity and electromagnetic fields in quantum mechanics

As we will discuss later in the course, we can use the semiclassical Hamiltonian to describe the motion of a charged particle in a classical electromagnetic field:

$$\hat{H} = \frac{1}{2m}(\hat{p} - q\vec{A}(\hat{r}, t))^2 + qU(\hat{r}, t)$$

We want to define a quantum mechanical velocity operator $\hat{v}$ in such a way that its expectation value $\langle \hat{v} \rangle$ corresponds to the time derivative of the expectation value of the position operator $\langle \hat{r} \rangle : \langle \hat{v} \rangle = \frac{d}{dt}\langle \hat{r} \rangle$. The Ehrenfest theorem states that the time derivative of an expectation value is given by the commutator of the observable with the Hamiltonian plus the direct derivative:

$$\frac{d}{dt}\langle \hat{O} \rangle = \frac{1}{i\hbar}\langle [\hat{O}, \hat{H}] \rangle + \langle \frac{\partial}{\partial t}\hat{O} \rangle$$

(a) Which of the following commutators do vanish, and why?

$$[\hat{p}, \vec{A}(\hat{r}, t)], \quad [\hat{r}, \vec{A}(\hat{r}, t)], \quad [\hat{p}, U(\hat{r}, t)], \quad [\hat{r}, U(\hat{r}, t)]$$

(b) Calculate the commutator between the components of the position operator $\hat{r} = (\hat{x}, \hat{y}, \hat{z})$ and the above Hamiltonian and give the corresponding expression for the velocity operator.

(c) By using a component of the velocity, say $\hat{v}_x$, calculate the expectation value of the force $m\frac{d}{dt}\langle \hat{v} \rangle$. What is the classical analogue of this expression?

III Density matrices

Density matrices are a convenient tool to express not only pure states (including superposition states), but also mixed states like e.g. thermal states. An instructive way to think about mixed states is to assume an ensemble of many atoms and then pick out one atom at random.

If you are not familiar with density matrices you can find information on them in every quantum mechanics textbook. In the following, we will assume an ensemble of two-level atoms with the two internal states $|g\rangle$ (ground state) and $|e\rangle$ (excited state).

(a) Assume that you have a an ensemble of atoms that are all in the state $|\Psi\rangle = \sqrt{3}/2|g\rangle - i1/2|e\rangle$. What is the density matrix of an atom in this ensemble in the basis $|g\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|e\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$?

(b) Assume now that you have created an ensemble by mixing 75% ground state atoms and 25% excited state atoms. If you pick one atom at random out of this ensemble, what is its density matrix?

(c) Assume that somebody gave you an ensemble of atoms. How could you experimentally distinguish between the above two situation?

(d) The entropy of a density matrix is defined as $S = -k_B \operatorname{Tr}(\hat{\rho}\log(\hat{\rho}))$. Calculate the entropy of the above two density matrices!

(e) How would the density matrix of a thermal state at $T = 0$ and $T = \infty$ look and what would be its entropy? A thermal density matrix is defined as $\hat{\rho} = \frac{1}{Z}\sum_i e^{-\frac{E_i}{k_B T}}|i\rangle\langle i|$, where Z is the normalization constant which ensures that the sum of the diagonal elements is 1.