CHM 696: Quantum Optics

Problem Set 5 (Jaynes-Cummings model)

Valentin Walther
vwalther@purdue.edu

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I Fourier-Transformation, Natural Linewidth and Uncertainty

Unlike inhomogeneously broadened systems, dynamical decoupling cannot combat homogeneous dephasing. Here, we consider atomic decay as an example of a homogeneous broadening mechanism. A very useful mathematical tool to analyze functions is the Fourier transformation, which is a linear integral transformation that relates a function $f(t)$ of one real variable (e.g. time) to a a function $F(\omega)$ of another variable (e.g. frequency):

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt$$

Now consider a two-level atom with transition frequency ω_a that is initially in the excited state. In a classical picture it will emit an electromagnetic wave that is exponentially damped in time:

$$E(t) = E_0 \Theta(t) e^{-\frac{\Gamma}{2} t} e^{i\omega_a t}$$

Here Γ denotes the decay constant of the transition and $\Theta(t)$ the Heaviside function with $\Theta(t) = 1$ for $x \geq 0$ and $\Theta(t) = 0$ for $x < 0$.

(a) What is the relation between the decay constant and the lifetime of the upper level?

(b) Calculate the Fourier transform of the emitted light.

(c) Calculate and sketch or plot the resulting intensity spectrum.

II Jaynes-Cummings-Hamiltonian

The Hamiltonian of the Jaynes-Cummings-Model is given by

$$\hat{H} = \hbar\omega_{21}\hat{\sigma}^\dagger\hat{\sigma} + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar g\left(\hat{a}\hat{\sigma}^\dagger + \hat{a}^\dagger\hat{\sigma}\right), \tag{1}$$

where the uncoupled states of the atomic two-level-system are given by $|e\rangle$ and $|g\rangle$, while $|n\rangle$ denotes the Fock-states of the (single-mode, single-frequency) light field. *Note:* ($|e, n\rangle = |e\rangle \otimes |n\rangle$).

(a) What is the effect of the atomic operators $\hat{\sigma}$ and $\hat{\sigma}^\dagger$ on the atomic states $|g\rangle$ and $|e\rangle$?

(b) What are the eigenstates and eigenenergies of the system in the uncoupled case $g = 0$?

(c) Show that the interaction term in $\hat{H}$ only couples the following pairs of states: $|e, n\rangle \leftrightarrow |g, n+1\rangle$ and $|e, n-1\rangle \leftrightarrow |g, n\rangle$.

(d) Calculate the matrix elements of $\hat{H}$ in the basis $\{|e, n\rangle, |g, n+1\rangle\}$.

(e) Calculate the eigenenergies $E_n^\pm$in the coupled case ($g > 0$). Use the following replacement: $\Omega_n^\Delta = \sqrt{\Delta^2 + 4g^2(n+1)}, \Delta = \omega - \omega_{21}$. You should get:

$$E_n^\pm = \frac{\hbar}{2}\omega_{21} + \hbar\omega\left(n + \frac{1}{2}\right) \pm \frac{\hbar}{2}\Omega_n^\Delta$$

(f) Show that the state $|g, 0\rangle$ is also an eigenstate with eigenenergy 0. Why couldn't you obtain this eigenenergy in the above calculation?

(g) All eigenstates can be written as:

$$|+, n\rangle = \cos\left(\frac{\theta}{2}\right)|e, n\rangle + \sin\left(\frac{\theta}{2}\right)|g, n+1\rangle$$
$$|-, n\rangle = \cos\left(\frac{\theta}{2}\right)|g, n+1\rangle - \sin\left(\frac{\theta}{2}\right)|e, n\rangle$$

Calculate θ or $\tan(\theta)$.

III Dressed States and Energy Shifts

We look at the properties of the Jaynes-Cummings Hamiltonian

$$\hat{H} = \hbar\omega_{21}\hat{\sigma}^\dagger\hat{\sigma} + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar g\left(\hat{a}\hat{\sigma}^\dagger + \hat{a}^\dagger\hat{\sigma}\right). \tag{2}$$

(a) Our starting points are once again the eigenenergies in the case of vanishing coupling ($g = 0$). Using the notation from the last problem set ($|x, n\rangle; x \in \{e, g\}, n \in \mathbb{N}$), draw an energy diagram of the lowest four eigenenergies for the two cases $\Delta < 0$ and $\Delta > 0$.

(b) The difference between the eigenenergies with coupling and without coupling is called the AC Stark shift.

$$\Delta E_n^\pm = E_n^\pm(g) - E_n^\pm(g = 0). \tag{3}$$

Show that, in the case of large detuning $\left(\frac{g^2(n+1)}{\Delta^2} \ll 1\right)$, the AC Stark shift takes the form:

$$\Delta E_n^\pm = \pm\hbar\frac{g^2(n+1)}{\Delta} \tag{4}$$

(c) Add the Stark shift schematically to the diagram from (a).

(d) Show that in the limit of large detuning and large photon number n the Stark shift is proportional to the light intensity. For this, you can replace the coupling constant g with the formula from the lecture. *Hint: intensity is energy per volume multiplied by propagation speed.*

(e) Show that expression Eq. (3) predicts a finite Stark shift even for photon number $n = 0$.

IV Journal Club Questions

The paper for the next Journal Club has the simple title "One-Atom Maser", by D. Meschede, H. Walther, and G. Müller, Phys. Rev. Lett. 54, 551 (1985). There are some very technical details of constructing the cavity (second page, second column, second paragraph, mostly) which are not so important.

(a) What is a "maser"?

(b) What is a Rydberg atom, and why are they being used?

(c) Why is the Q-value of the cavity crucial in this experiment?

(d) Why are Rabi oscillations not observed in this experiment?

(e) How do the authors show that their maser works?

As usual, please also look at the usual questions: what is the purpose of the experiment; how does the measurement method work (for an experimental paper); what is (are) the main result(s) and what is the argumentation that leads to it; etc...