# 5. Representations of Quantum States

*) finite systems → density matrix tomography

*) infinite systems (e.g. phot. states)

↳ visual alternatives

## 5.1 P-representation (coherent-state representation)

→ use completeness of coherent states

$$\boxed{\frac{1}{\pi}\int d^2\alpha \, |\alpha\rangle\langle\alpha| = \hat{\mathbb{1}}}$$

### Completeness of coherent states

$$\int \frac{d^2\alpha}{\pi} \underset{\mathbb{1}}{|\alpha\rangle}\underset{\mathbb{1}}{\langle\alpha|} = \frac{1}{\pi}\int e^{-|\alpha|^2} \sum_n \sum_m \frac{\alpha^n \alpha^{*m}}{\sqrt{n!\,m!}} |n\rangle\langle m| d^2\alpha$$

use polar coordinates $\alpha = r e^{i\theta}$, $d^2\alpha = r\,dr\,d\theta$

$$= \frac{1}{\pi}\sum_{nm} \frac{|n\rangle\langle m|}{\sqrt{n!\,m!}} \int_0^\infty dr\, e^{-r^2} r^{n+m+1} \underbrace{\int_0^{2\pi} d\theta\, e^{i(n-m)\theta}}_{=2\pi\delta_{n,m}}$$

$$= \frac{1}{\pi}\sum_n \frac{|n\rangle\langle n|}{n!} \cdot 2\pi \underbrace{\int_0^\infty dr\, e^{-r^2} r^{2n+1}}$$

substitute: $y = r^2$ : $\frac{1}{2}\int_0^\infty e^{-y} y^n = \frac{1}{2}\Gamma(n+1) = \frac{1}{2} n!$

$$= \sum_n |n\rangle\langle n| = \hat{\mathbb{1}}$$

## Over-completeness

$|\beta\rangle$ : coherent state

$$\Rightarrow |\beta\rangle = \frac{1}{\pi}\int \frac{d^2\alpha}{\pi}\, |\alpha\rangle \underbrace{\langle\alpha|\beta\rangle}_{*}$$

$$^{*}\langle\alpha|\beta\rangle = e^{-\frac{|\alpha|^2}{2}-\frac{|\beta|^2}{2}} \sum_{nm} \frac{(\alpha^*)^n \beta^m}{\sqrt{n!\,m!}} \underbrace{\langle m|n\rangle}_{=\delta_{nm}}$$

$$= e^{-\frac{|\alpha|^2}{2}-\frac{|\beta|^2}{2}} \underbrace{\sum_n \frac{(\alpha^*\beta)^n}{n!}}_{e^{\alpha^*\beta}}$$

$$= e^{-\frac{|\alpha|^2}{2}-\frac{|\beta|^2}{2}+\alpha^*\beta}$$

$$= e^{-\frac{1}{2}|\beta-\alpha|^2}\, e^{\frac{1}{2}\underbrace{(\alpha^*\beta-\beta^*\alpha)}_{\text{phase}}}$$

$$|\langle\alpha|\beta\rangle|^2 = e^{-|\beta-\alpha|^2} \neq 0$$

$$|\beta\rangle = \int \frac{d^2\alpha}{\pi}\, |\alpha\rangle\langle\alpha|\beta\rangle \Rightarrow \text{not linearly independent}$$

$\Rightarrow$ basis is overcomplete

$$\left[ \langle\beta|\hat{E}|\beta\rangle \approx \langle\beta|\hat{a}+\hat{a}^{\dagger}|\beta\rangle = \beta+\beta^* \right]$$

$$\boxed{\hat{\rho} = \int P(\alpha)\, |\alpha\rangle\langle\alpha|\, d^2\alpha}$$ (P-function / Glauber–Sudarshan fct.)

↑ real because $\hat{\rho}$ is hermitian

Check: $\mathrm{Tr}(\hat{\rho}) = 1$ if $\int P(\alpha)\, d^2\alpha = 1$

"Proof:" $= \mathrm{Tr} \int P(\alpha) |\alpha\rangle\langle\alpha| d^2\alpha$

$$= \int \sum_n P(\alpha) \langle n|\alpha\rangle\langle\alpha|n\rangle\, d^2\alpha$$

$\sum_n |n\rangle\langle n|$

$$= \int P(\alpha)\, d^2\alpha$$

But: $P(\alpha) < 0$ is possible! (measure of non-classicality)

↳ "quasi-probability distr."

How to compute $P(\alpha)$? → Mehta, PRL 18, 752 (1967)

coherent state

$$\langle -u|\hat{\rho}|u\rangle = \int P(\alpha) \langle -u|\alpha\rangle \langle\alpha|u\rangle\, d^2\alpha$$

see above

$$= \int P(\alpha)\, e^{-\frac{1}{2}|u|^2-\frac{1}{2}|\alpha|^2-u^*\alpha}\, e^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}|u|^2+\alpha^* u}\, d^2\alpha$$

$$= e^{-|u|^2}\int d^2\alpha\, P(\alpha)\, e^{-|\alpha|^2}\, e^{\alpha^* u-\alpha u^*}$$

$\alpha = x+iy$

$u = x'+iy'$

$$\Rightarrow \alpha^* u - \alpha u^* = (x-iy)(x'+iy') - (x+iy)(x'-iy') = 2i\underbrace{(xy'-x'y)}_{\text{real}}$$

$\rightarrow$ this is a 2D Fourier transform

$$= e^{-|u|^2}\int dx \int dy\, P(\alpha)\, e^{-|\alpha|^2}\, e^{i(2y')x}\, e^{i(-2x')y}$$

Apply inverse Fourier transforms

$$\Rightarrow P(\alpha)\, e^{-|\alpha|^2} = \frac{1}{\pi^2}\int dx' \int dy'\, e^{|u|^2}\langle -u|\hat{\varrho}|u\rangle e^{-i(2y')x}\, e^{-i(-2x')y}$$

$$= \frac{1}{\pi^2}\int d^2u\, e^{|u|^2}\langle -u|\hat{\varrho}|u\rangle e^{u^*\alpha-u\alpha^*}$$

$$\boxed{P(\alpha) = \frac{e^{|\alpha|^2}}{\pi^2}\int d^2u\, e^{|u|^2}\langle -u|\hat{\varrho}|u\rangle e^{u^*\alpha-u\alpha^*}}$$