Quantum Optics Assignment 1

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I. TYPICAL PHOTON ENERGIES

(a) Photon energies

For a photon with wavelength λ and frequency $f = c/\lambda$, the energy E and momentum p are as follows.

$$E = \hbar\omega = hf \tag{1}$$

$$p = \frac{h}{\lambda} \tag{2}$$

For a photon with wavelength 852 nm, these give

- Photon energy in Joule: 2.33×10^{-19} J
- Photon energy in eV: 1.46 eV
- Frequency of light: 352 THz

The temperature difference of the atom after emission of the photon is obtained through $E = k_B T$. $T = 1.69 \times 10^4$ K

(b) Total energy

We can get the velocity from $E = hf = \frac{1}{2}mv^2$.

- Final velocity: 4.2 km/s

(c) Recoil velocity

The recoil velocity is obtained with $p = \frac{h}{f} = mv$.

- Recoil velocity: 29.5 mm/s
- Recoil energy: 1.66×10^{-29} J

II. MECHANICAL VELOCITY AND ELECTROMAGNETIC FIELDS IN QUANTUM MECHANICS

It is useful to know the following definitions about the classical vector potential, which I found on Wikipedia. Here, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields.

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{3}$$

$$\mathbf{E} = -\nabla U(\mathbf{r}, t) - \frac{\partial \mathbf{A}}{\partial t} \tag{4}$$

(a) Commutators

Functions of $\mathbf{r}$ commute with $\mathbf{r}$, and likewise with $\mathbf{p}$.

$$[\mathbf{r}, \mathbf{A}(\mathbf{r}, t)] = 0 \tag{5}$$

$$[\mathbf{r}, U(\mathbf{r}, t)] = 0 \tag{6}$$

(b) Calculating velocity

$$H = \frac{1}{2m} \sum_{i=x,y,z} \left[p_i^2 + q^2 A_i^2 (\mathbf{r}, t) - q p_i A_i (\mathbf{r}, t) - q A_i (\mathbf{r}, t) p_i\right] + qU (\mathbf{r}, t) \tag{7}$$

$$\begin{aligned}
[x, H] &= \frac{1}{2m} \sum_{i=x,y,z} \left\{ \left[x, p_i^2\right] - q \left[x, p_i A_i (\mathbf{r}, t)\right] - q \left[x, A_i (\mathbf{r}, t) p_i\right] \right\} && (8)\\
&= \frac{1}{2m} \left\{p_x [x, p_x] + [x, p_x] p_x - q [x, p_x] A_x (\mathbf{r}, t) - q A_i (\mathbf{r}, t) [x, p_x]\right\} && (9)\\
&= \frac{1}{2m} \left\{2i\hbar p_x - 2qi\hbar A_x (\mathbf{r}, t)\right\} && (10)\\
&= \frac{i\hbar}{m} \left\{p_x - q A_x (\mathbf{r}, t)\right\} && (11)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \langle x \rangle &= \frac{1}{i\hbar} \langle [x, H] \rangle + \left\langle \frac{\partial}{\partial t} x \right\rangle && (12)\\
&= \frac{1}{m} \langle p_x - q A_x (\mathbf{r}, t) \rangle && (13)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{d}{dt} \langle y \rangle &= \frac{1}{m} \langle p_y - q A_y (\mathbf{r}, t) \rangle && (14)\\
\frac{d}{dt} \langle z \rangle &= \frac{1}{m} \langle p_z - q A_z (\mathbf{r}, t) \rangle && (15)
\end{aligned}$$

so that

$$\langle \mathbf{v} \rangle = \frac{1}{m} \langle \mathbf{p} - q \mathbf{A} (\mathbf{r}, t) \rangle \tag{16}$$

(c) The force

Based on the result from (b), we can define the velocity operator as follows.

$$\mathbf{v} = \frac{1}{m} \left[\mathbf{p} - q \mathbf{A} (\mathbf{r}, t)\right] \tag{17}$$

We can find the acceleration by a second application of Ehrenfest's theorem.

$$\frac{d}{dt} \langle \mathbf{v} \rangle = \frac{1}{i\hbar} \langle [\mathbf{v}, H] \rangle + \left\langle \frac{\partial}{\partial t} \mathbf{v} \right\rangle \tag{18}$$

There are two things we must be careful of.

- In general, the velocity operator has explicit time dependence through $\mathbf{A} (\mathbf{r}, t)$, so that we cannot write $\left\langle \frac{\partial}{\partial t} \mathbf{v} \right\rangle = \mathbf{0}$.
- While it is true that any operator O commutes with itself $[O, O] = 0$, it is not necessarily the case that a vector of operators $\mathbf{O}$ commutes with its square $[\mathbf{O}, \mathbf{O} \cdot \mathbf{O}] \neq \mathbf{0}$. When you expand into $\left[O_x \hat{\mathbf{x}} + O_y \hat{\mathbf{y}} + \cdots, O_x^2 + O_y^2 + \cdots\right]$, you get terms like $\hat{\mathbf{x}} \left[O_x, O_y^2\right]$ which may not commute. Therefore, we cannot assume that $\left\langle \left[\mathbf{v}, \frac{1}{2} m \mathbf{v} \cdot \mathbf{v}\right] \right\rangle = 0$.

We will start by rewriting the Hamiltonian in terms of the velocity operator.

$$H = \frac{1}{2} m \mathbf{v} \cdot \mathbf{v} + qU (\mathbf{r}, t) \tag{19}$$

Computing $[v_i, v_j]$

It is useful to compute the commutators of the different components of $\mathbf{v}$.

$$[v_i, v_j] = \frac{1}{m^2}\left[p_i - qA_i(\mathbf{r},t), p_j - qA_j(\mathbf{r},t)\right] \tag{20}$$
$$= \frac{1}{m^2}\left\{[p_i, p_j] + q^2\left[A_i(\mathbf{r},t), A_j(\mathbf{r},t)\right] - q\left[p_i, A_j(\mathbf{r},t)\right] - q\left[A_i(\mathbf{r},t), p_j\right]\right\} \tag{21}$$
$$= -\frac{q}{m^2}\left\{\left[p_i, A_j(\mathbf{r},t)\right] + \left[A_i(\mathbf{r},t), p_j\right]\right\} \tag{22}$$

We can use the identity $[p_i, f(x)] = -i\hbar\partial_x f(x)$ [Appx.A] to obtain $[p_i, A_j(\mathbf{r},t)] = -i\hbar\partial_i A_j(\mathbf{r},t)$ and $[A_i(\mathbf{r},t), p_j] = -[p_j, A_i(\mathbf{r},t)] = i\hbar\partial_j A_i(\mathbf{r},t)$ where i, j are x, y, z.

$$[v_i, v_j] = -\frac{q}{m^2}\left\{-i\hbar\partial_i A_j(\mathbf{r},t) + i\hbar\partial_j A_i(\mathbf{r},t)\right\} \tag{23}$$
$$= \frac{i\hbar q}{m^2}\left\{\partial_i A_j(\mathbf{r},t) - \partial_j A_i(\mathbf{r},t)\right\} \tag{24}$$

We can simplify this by recognizing it is in the form of a cross product.

$$(\nabla\times\mathbf{A})_x = \partial_y A_z - \partial_z A_y \tag{25}$$
$$(\nabla\times\mathbf{A})_y = \partial_z A_x - \partial_x A_z \tag{26}$$
$$(\nabla\times\mathbf{A})_z = \partial_x A_y - \partial_y A_x \tag{27}$$

Note: This can be written very compactly with the Levi-Civita tensor ϵ_{ijk}. The rule is that $\epsilon_{ijk} = +1$ if ijk are "in order," that is, $ijk = xyz$, or yzx, or zxy. If ijk are "out of order," that is, $ijk = xzy$ or yxz or zyx, then $\epsilon_{ijk} = -1$. If any indices are repeated, such as $ijk = xxy$, then $\epsilon_{ijk} = 0$. In this notation, any cross product can be written as

$$(\mathbf{A}\times\mathbf{B})_i = \sum_{jk}\epsilon_{ijk}A_j B_k. \tag{28}$$

Computing $[\mathbf{v}, \mathbf{v}\cdot\mathbf{v}]$

Let's begin with one component $[v_x, \mathbf{v}\cdot\mathbf{v}]$.

$$[v_x, \mathbf{v}\cdot\mathbf{v}] = \left[v_x, v_x^2 + v_y^2 + v_z^2\right] \tag{29}$$
$$= \left\{v_y[v_x, v_y] + [v_x, v_y]v_y + v_z[v_x, v_z] + [v_x, v_z]v_z\right\} \tag{30}$$
$$= \frac{i\hbar q}{m^2}\left\{v_y\left[\nabla\times\mathbf{A}(\mathbf{r},t)\right]_z + \left[\nabla\times\mathbf{A}(\mathbf{r},t)\right]_z v_y - v_z\left[\nabla\times\mathbf{A}(\mathbf{r},t)\right]_y - \left[\nabla\times\mathbf{A}(\mathbf{r},t)\right]_y v_z\right\} \tag{31}$$
$$= \frac{i\hbar q}{m^2}\left\{(\mathbf{v}\times(\nabla\times\mathbf{A}(\mathbf{r},t)))_x - ((\nabla\times\mathbf{A}(\mathbf{r},t))\times\mathbf{v})_x\right\} \tag{32}$$

$$[v_i, \mathbf{v}\cdot\mathbf{v}] = \frac{i\hbar q}{m^2}\left\{(\mathbf{v}\times(\nabla\times\mathbf{A}(\mathbf{r},t)))_x - ((\nabla\times\mathbf{A}(\mathbf{r},t))\times\mathbf{v})_x\right\}$$

The total commutator $[\mathbf{v}, \mathbf{v}\cdot\mathbf{v}]$ is simply

$$[\mathbf{v}, \mathbf{v}\cdot\mathbf{v}] = \frac{i\hbar q}{m^2}\left\{(\mathbf{v}\times(\nabla\times\mathbf{A}(\mathbf{r},t))) - ((\nabla\times\mathbf{A}(\mathbf{r},t))\times\mathbf{v})\right\} \tag{33}$$

Computing $[\mathbf{v}, U(\mathbf{r}, t]$

Recall that functions of $\mathbf{r}$ commute with other functions of $\mathbf{r}$ because all of r_x, r_y, r_z commute with each other. Recall also that $[p_x, f(x)] = -i\hbar\partial_x f(x)$ and that the gradient is defined as $\nabla f(\mathbf{r}) = \sum_i \hat{\mathbf{e}}_i\partial_i f(\mathbf{r})$. Here, $\hat{\mathbf{e}}_i$ is the unit vector in the ith direction.

$$[\mathbf{v}, U(\mathbf{r},t)] = \frac{1}{m}\left[\mathbf{p} - q\mathbf{A}\left(\mathbf{r},t\right), U\left(\mathbf{r},t\right)\right] \tag{34}$$

$$= \frac{1}{m}\left[\mathbf{p}, U\left(\mathbf{r},t\right)\right] - \frac{q}{m}\left[\mathbf{A}\left(\mathbf{r},t\right), U\left(\mathbf{r},t\right)\right] \tag{35}$$

$$= \frac{1}{m}\sum_i \hat{\mathbf{e}}_i\left[p_i, U\left(\mathbf{r},t\right)\right] \tag{36}$$

$$= -\frac{i\hbar}{m}\sum_i \hat{\mathbf{e}}_i \partial_i U\left(\mathbf{r},t\right) \tag{37}$$

$$= -\frac{i\hbar}{m}\nabla U\left(\mathbf{r},t\right) \tag{38}$$

Computing $[\mathbf{v}, H]$

We can finally assemble everything.

$$[\mathbf{v}, H] = \left[\mathbf{v}, \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} + qU\left(\mathbf{r},t\right)\right] \tag{39}$$

$$= q\left[\mathbf{v}, U(\mathbf{r},t\right] + \sum_i \hat{\mathbf{e}}_i\left[v_i, \frac{1}{2}m\mathbf{v}\cdot\mathbf{v}\right] \tag{40}$$

$$= -\frac{i\hbar q}{m}\nabla U\left(\mathbf{r},t\right) + \frac{i\hbar q}{2m}\left\{\left(\mathbf{v}\times\left(\nabla\times\mathbf{A}\left(\mathbf{r},t\right)\right)\right) - \left(\left(\nabla\times\mathbf{A}\left(\mathbf{r},t\right)\right)\times\mathbf{v}\right)\right\} \tag{41}$$

The explicit time dependence of $\mathbf{v}$ is all in $\mathbf{A}\left(\mathbf{r},t\right)$.

$$\frac{\partial\mathbf{v}}{\partial t} = \frac{\partial}{\partial t}\left[\frac{\mathbf{p} - q\mathbf{A}\left(\mathbf{r},t\right)}{m}\right] \tag{42}$$

$$= -\frac{q}{m}\frac{\partial\mathbf{A}\left(\mathbf{r},t\right)}{\partial t} \tag{43}$$

$$\frac{d}{dt}\left\langle\mathbf{v}\right\rangle = \frac{1}{i\hbar}\left\langle[\mathbf{v}, H]\right\rangle + \left\langle\frac{\partial\mathbf{v}}{\partial t}\right\rangle \tag{44}$$

$$= -\frac{q}{m}\left\langle\nabla U\left(\mathbf{r},t\right)\right\rangle + \frac{q}{2m}\left\langle\mathbf{v}\times\left[\nabla\times\mathbf{A}\left(\mathbf{r},t\right)\right] - \left[\nabla\times\mathbf{A}\left(\mathbf{r},t\right)\right]\times\mathbf{v}\right\rangle - \frac{q}{m}\left\langle\frac{\partial\mathbf{A}\left(\mathbf{r},t\right)}{\partial t}\right\rangle \tag{45}$$

And that's the answer! In order to make sense of this, we need to employ the relationship between the *classical* electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ with the vector potential.

$$\mathbf{B} = \nabla\times\mathbf{A}\left(\mathbf{r},t\right) \tag{46}$$

$$\mathbf{E} = -\nabla U\left(\mathbf{r},t\right) - \frac{\partial\mathbf{A}\left(\mathbf{r},t\right)}{\partial t} \tag{47}$$

If we substitute these relations, we obtain

$$\left\langle\mathbf{F}\right\rangle = m\frac{d}{dt}\left\langle\mathbf{v}\right\rangle \tag{48}$$

$$= q\left[\left\langle\mathbf{E}\right\rangle + \frac{1}{2}\left\langle\mathbf{v}\times\mathbf{B} - \mathbf{B}\times\mathbf{v}\right\rangle\right] \tag{49}$$

which is exactly the form of the classical Lorentz force $\mathbf{F} = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)$. The only difference is that we have not simplified using $\mathbf{B}\times\mathbf{v} = -\mathbf{v}\times\mathbf{B}$ because $\mathbf{v}$ and $\mathbf{B}$ do not commute.

A. Alternate way of doing-using Levi-Civita tensor

In this section, we will be doing the same computation in a more 'compact way' which will give out the flavor of the Levi-Civita tensor. The basic idea and some identities of the Levi-Civita tensor can be found in Appx. C. The commutator of any component of the velocity $\boldsymbol{v}$ represented by v_i with $\sum_j v_j^2$ where i and j takes value x, y or z can be written as

$$[v_i, \sum_j v_j^2] = \sum_j [v_i, v_j^2] \tag{50}$$

You can show why we can pull out the sum from the commutator pretty easily. Moving forward, we will compute the commutator $[v_i, v_j^2]$, which can be computed as

$$\sum_j [v_i, v_j^2] = \sum_j ([v_i, v_j]v_j + v_j[v_i, v_j]) \tag{51}$$

This breaks down our commutator to commute further and we only now have to compute $[v_i, v_j]$, which we already did that is Eq. (23)-(24)

$$\sum_j [v_i, v_j] = \frac{i\hbar q}{m^2} \sum_j \{\partial_i A_j (\mathbf{r}, t) - \partial_j A_i (\mathbf{r}, t)\} = \frac{i\hbar q}{m^2} \sum_{j,k} \epsilon_{ijk} B_k \tag{52}$$

Where ϵ_{ijk} is the Levi-Civita tensor and $B_k = \epsilon_{kmn} \partial_m A_n$. See Appx. C 1, to actually see why the second equality actually makes sense. Now as we have this, we will look at the $[v_i, v_j^2]$ which will computed as

$$[v_i, v_j^2] = [v_i, v_j]v_j + v_j[v_i, v_j] = \frac{i\hbar q}{m^2} \sum_{j,k} (\epsilon_{ijk} v_j B_k - \epsilon_{ijk} B_j v_k) \tag{53}$$

Now Eq. (53) is written in index form which means we are representing each component and we have defined its general structure. In vector form, we can write

$$\boldsymbol{a}_{\text{Lorentz}} = \frac{i\hbar q}{m^2} (\boldsymbol{v} \times \boldsymbol{B} - \boldsymbol{B} \times \boldsymbol{v}) \tag{54}$$

Note that since $\boldsymbol{v}$ and $\boldsymbol{B}$ do not commute, we have not used the identity for cross products $\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}$. Here, a_{Lorentz} is defined as the acceleration due to Lorentz force, which here is caused only by the magnetic field.

III. DENSITY MATRICES

(a) Pure ensemble

$$|\psi\rangle = \begin{pmatrix} \sqrt{3/4} \\ -i/2 \end{pmatrix} \tag{55}$$

$$\langle\psi| = \begin{pmatrix} \sqrt{3/4} & i/2 \end{pmatrix} \tag{56}$$

$$\rho_{\text{pure}} = |\psi\rangle \langle\psi| = \begin{pmatrix} 3/4 & i\sqrt{3}/4 \\ -i\sqrt{3}/4 & 1/4 \end{pmatrix} \tag{57}$$

(b) Impure ensemble

$$\rho_1 = \frac{3}{4} |g\rangle \langle g| = \frac{3}{4} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{58}$$

$$\rho_2 = \frac{1}{4} |e\rangle \langle e| = \frac{1}{4} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{59}$$

$$\rho_{\text{impure}} = \rho_1 + \rho_2 = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix} \tag{60}$$

Experimental determination

In the pure ensemble, all atoms are in a definite state $|\psi\rangle$. In the impure ensemble, a proportion of atoms are in each state. Therefore, if we could devise an experiment to determine the likelihood of an atom being in $|\psi\rangle$, we would get different answers for the two ensembles. For (a), we would get unity. For (b), we would get an answer based on the projection of the component states onto $|\psi\rangle$. Let $A = |\psi\rangle\langle\psi|$ be the observable that we can measure.

$$\langle A\rangle_{\text{(pure)}} = \text{Tr}\left[\rho_{\text{pure}} A\right] = \text{Tr}\left[\rho_{\text{pure}}^2\right] = 1 \tag{61}$$

$$\langle A\rangle_{\text{(impure)}} = \text{Tr}\left[\rho_{\text{impure}} A\right] = \text{Tr}\left[\begin{pmatrix} (3/4)^2 & i3^{3/2}/16 \\ -i\sqrt{3}/16 & 1/16 \end{pmatrix}\right] = 10/16 \tag{62}$$

(d) Entropy

If we use a basis in which ρ is diagonal, the entropy is simply

$$S = -k_B \text{Tr}\left[\rho \ln \rho\right] = -k_B \sum_k \rho_k^{\text{(diag)}} \ln \rho_k^{\text{(diag)}}. \tag{63}$$

For the pure state, the eigenvalues of 1 and 0.

$$S_{\text{pure}} = -k_B \left[1 \cdot \ln 1 + \lim_{x\to 0^+} x \ln x\right]. \tag{64}$$

We can evaluate the limit using L'hopital's rule. Note that we have to write it as a fraction.

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} -\frac{1/x}{1/x^2} = \lim_{x\to 0^+} -x = 0 \tag{65}$$

So the entropy of a pure state is zero!

$$S_{\text{pure}} = 0 \tag{66}$$

For the mixed state,

$$S_{\text{impure}} = -k_B \left[3/4 \cdot \ln 3/4 + 1/4 \cdot \ln 1/4\right] \approx 0.562 k_B \tag{67}$$

(e) Thermal state

In this case, we only consider two state $|g\rangle$ and $|e\rangle$, with energy 0 and 1.

$$\rho = \frac{1}{Z}\left(|g\rangle\langle g| + e^{-1/k_B T}|e\rangle\langle e|\right) \tag{68}$$

At $T = 0$, $e^{-1/k_B T} \to 0$ and the population is purely in the ground state. As $T \to \infty$, $e^{-1/k_B T} \to 1$ and the density matrix is a statistical mixture of ground and excited.

$$\rho_{T\to\infty} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \tag{69}$$

Appendix A: Commutator of momentum and a position-dependent function

$$\langle x|\left[\hat{p},f(\hat{x})\right]|\psi\rangle = \langle x|\left(\hat{p}f(\hat{x})-f(\hat{x})\hat{p}\right)|\psi\rangle \tag{A1}$$
$$= -i\hbar\partial_x\left(f(x)\psi(x)\right) - i\hbar f(x)\partial_x\psi(x) \tag{A2}$$
$$= -i\hbar\left(\partial_x f(x)\right)\psi(x) \tag{A3}$$

$$\rightarrow [p,f(x)] = -i\hbar\partial_x f(x) \tag{A4}$$

Appendix B: Taylor-series way of obtaining the derivative

In this section we will do the commutators more abstractly, that is without using any representation, like we used the position-the x- representation in the main text, with only the information that $[\hat{p},\hat{x}] = -i\hbar$. We will compute the following commutator $[\hat{p},U(\hat{x},t)]$, The simplest way is to Taylor expand $U(\hat{x},t)$

$$U(\hat{x},t) = \sum_m^\infty U_m\hat{x}^m \tag{B1}$$

The commutator of $\hat{p}$ with this potential gives

$$[\hat{p},U(\hat{x},t)] = \sum_m U_m[\hat{p},\hat{x}^m] \tag{B2}$$

The commutator $[\hat{p},\hat{x}^m]$ can be computed as

$$[\hat{p},\hat{x}^m] = [\hat{p},\hat{x}]\hat{x}^{m-1} + \hat{x}[\hat{p},\hat{x}]\hat{x}^{m-2} + \ldots\hat{x}^{m-1}[\hat{p},\hat{x}] \tag{B3}$$

Substituting $[\hat{p},\hat{x}] = -i\hbar$ gives the commutator as

$$[\hat{p},\hat{x}^m] = -i\hbar m\hat{x}^{m-1} \tag{B4}$$

Using this commutator we get the commutator between $\hat{p}$ and $U(\hat{x},t)$ as

$$[\hat{p},U(\hat{x},t)] = -i\hbar\left[\sum_m mU_m\hat{x}^{m-1}\right] \tag{B5}$$

The term in the parenthesis is actually the derivative of $U(x,t)$ with respect to x (see Eq. (B1)) if we go to the x-representation of the operators. Therefore we can write

$$[\hat{p},U(x,t)] = -i\hbar\frac{\partial U(x,t)}{\partial x} \tag{B6}$$

Appendix C: The deal with Levi-Civita

The Levi-Civita are simply a compact way of expressing terms more generally. The Levi-Civita is more of a Tensor (A 3D Array for our case). It is defined as

$$\epsilon_{ijk} = \begin{cases} +1 & i,j,k \text{ in cyclic order } i.e \text{ } 123,\ 231 \text{ or } 312 \\ -1 & i,j,k \text{ not in cyclic order } i.e \text{ } 213,\ 321 \text{ or } 132 \\ 0 & \text{any two of the index are same like } 112,\ 221 \text{ and etc.} \end{cases} \tag{C1}$$

Using this tensor (or symbol) we can write the cross product as

$$C_i = [\boldsymbol{A}\times\boldsymbol{B}]_i = \sum_{j,k}\epsilon_{ijk}A_jB_k \tag{C2}$$

You can plug-in x, y, z (a substitute for 1,2,3) to verify that this indeed gives the right expression for each component. One important identity which will be useful for such tensors is the following

$$\sum_k \epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jk} \tag{C3}$$

Were δ_{ij} is the Kronecker delta. The proof can be done by inspection and if you want to be more rigourous it can be done using the fact that the terms in right hand side actually forms the basis of all the third rank tensors.

1. Use in problem 2

We used this in our problem when we got the expression

$$c_i = \sum_j [v_i, v_j] = \sum_j (\partial_i A_j - \partial_j A_i) = \sum_j \left(\sum_{l,m} \delta_{li}\delta_{mj}(\partial_l A_m) - \sum_{l,m} \delta_{lj}\delta_{mi}(\partial_l A_m) \right) = \sum_{j,l,m} (\delta_{li}\delta_{mj} - \delta_{lj}\delta_{mi})\partial_l A_m \tag{C4}$$

Now invoking the property that is in Eq. (C3) we get

$$c_i = [v_i, v_j] = \sum_{j,k,l,m} \epsilon_{ijk}\epsilon_{klm}\partial_l A_m \tag{C5}$$

Now we know $B_k = [\nabla \times \boldsymbol{A}]_k = \sum_k \epsilon_{klm}\partial_l A_m$. So, we get

$$c_i = \sum_{j,k} \epsilon_{ijk} B_k \tag{C6}$$