Test 2: Integrate over position degree of freedom

$$\int_{-\infty}^{\infty} W(q,p)\,dq = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dq\,dx\; \psi^*\left(q-\frac{x}{2}\right)\psi\left(q+\frac{x}{2}\right)e^{ipx/\hbar}$$

Change of variables:

$$u_1 = q - \frac{x}{2} \qquad \Big| \qquad x = u_2 - u_1$$

$$u_2 = q + \frac{x}{2} \qquad \Big| \qquad q = \frac{u_1+u_2}{2}$$

Jacobian: $$J = \begin{pmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial x}{\partial u_2} \\ \frac{\partial q}{\partial u_1} & \frac{\partial q}{\partial u_2} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1/2 & 1/2 \end{pmatrix}$$

$$\Rightarrow \int_{-\infty}^{\infty} W(q,p)\,dq = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} du_1 \int_{-\infty}^{\infty} du_2\, \underbrace{|\det J|}_{=1}\, \psi^*(u_1)\psi(u_2)\, e^{ip(u_1-u_2)/\hbar}$$

$$= \underbrace{\left(\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} du_1\, \psi^*(u_1)e^{ipu_1/\hbar}\right)}_{=\tilde{\psi}^*(p)} \underbrace{\left(\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty} du_2\, \psi(u_2)e^{-ipu_2/\hbar}\right)}_{\tilde{\psi}(p)}$$

$$= |\psi(p)|^2 \quad \rightarrow$$ probability distribution over momentum p

Properties of the Wigner function (both for pure and mixed states)

* $W(q,p) \in \mathbb{R}$

Proof: $W^*(q,p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \langle q+\frac{x}{2}|\hat{\rho}|q-\frac{x}{2}\rangle^* e^{-ipx/\hbar} dx$

$$\overset{y=-x}{=} \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \underbrace{\langle q-\frac{y}{2}|\hat{\rho}|q+\frac{y}{2}\rangle^*}_{\langle q+\frac{y}{2}|\underbrace{\hat{\rho}^\dagger}_{=\hat{\rho}}|q-\frac{y}{2}\rangle} e^{ipy/\hbar} dy$$

$$= W(q,p)$$

* Wigner functions are normalised

$$\int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp \, W(q,p) = \mathrm{Tr}(\hat{\rho}) = 1$$

Proof: $\int dq \int dp \underbrace{\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \langle q+\frac{x}{2}|\hat{\rho}|q-\frac{x}{2}\rangle e^{ipx/\hbar} dx}_{\sum_k p_k |\psi_k(q)|^2 = 1}$

* WF ≥ 0 for classical states

↳ if WF < 0 → shows "quantumness"

* For expectation values of non-commuting operator products, use symmetrized representation

$$\hat{q}\hat{p} \to \frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q})$$

5.3 Q – representation (Husimi rep.) → anti-normal-ordered operators

Def.: $Q(\alpha) = \frac{1}{\pi} \langle \alpha | \hat{\rho} | \alpha \rangle$

* Normalized $\int Q(\alpha)\, d^2\alpha = 1$
* non-neg. + bounded : $0 \leq Q(\alpha) \leq \frac{1}{\pi}$

$$\frac{1}{\pi}\langle \alpha | \hat{\rho} | \alpha \rangle = \sum_k p_k \frac{1}{\pi} \underbrace{|\langle \psi_k | \alpha \rangle|^2}_{\leq 1} \leq \frac{1}{\pi}$$

* still, not a true probability distr.

P vs. Q representation

1) assume operator $\hat{B}(\hat{a}, \hat{a}^\dagger)$ has P-representation

$$\hat{B} = \int B_p(\alpha, \alpha^*) |\alpha\rangle\langle\alpha| \, d^2\alpha$$

$$\Rightarrow \langle \hat{B} \rangle = \mathrm{Tr}(\hat{B}\hat{\rho}) = \sum_u \langle u | \int B_p(\alpha,\alpha^*) |\alpha\rangle\langle\alpha| \hat{\rho} |u\rangle \, d^2\alpha$$

$$\sum_u |u\rangle\langle u| \rightarrow \; = \int d^2\alpha \; B_p(\alpha,\alpha^*) \underbrace{\langle\alpha|\hat{\rho}|\alpha\rangle}_{=\pi Q(\alpha)}$$

2) assume we have P-representation of $\hat{\rho}$

$$\langle \hat{B} \rangle = \mathrm{Tr}(\hat{B}\hat{\rho}) = \mathrm{Tr}\left(\hat{B} \int P(\alpha) |\alpha\rangle\langle\alpha| \, d^2\alpha\right)$$

$$= \int \underbrace{\langle \alpha | \hat{B} | \alpha \rangle}_{B_Q(\alpha,\alpha^*)} P(\alpha) \, d^2\alpha$$

$B_Q(\alpha, \alpha^*)$: Q-representation of B

Conclusion: P-rep. of $\hat{g}$ $\longleftrightarrow$ Q-repr. of B

Q-rep. of $\hat{g}$ $\longleftrightarrow$ P-repr. of B

$\Rightarrow$ more info: Scully, Chapt. 3, 4, pp. 81-85

Yeung/Knight 3.8, pp. 65-71

6. Interaction of EM fields and matter

→ matter is used to create all interesting states of light

→ start with semiclassical picture (matter quant. + light class.)

Atom field interactions

$$\hat{H}_0 |n\rangle = \underbrace{E_n}_{\equiv \hbar \omega_n} |n\rangle$$

unperturbed electric energy levels

Electric dipole coupling

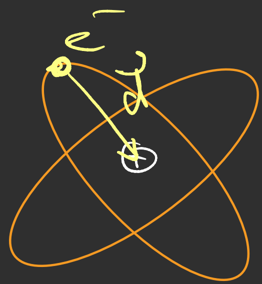
$$\hat{H}' = -\hat{d} \cdot \vec{E}(t)$$

electric field of radiation in the long-wavelength limit $\lambda \gg \langle r \rangle$:

$e^{i\vec{k}\vec{r}}$ part of field does not appreciably over the size of the atom

$$\hat{\vec{d}} = -e\,\hat{\vec{r}}$$

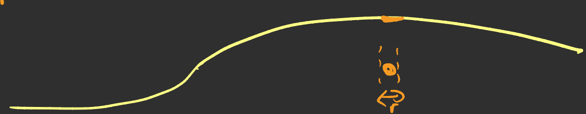
dipole moment of the atom



SIDE:

→ More is different

Phil. Anderson



→ full Hamiltonian: $\boxed{\hat{H} = \hat{H}_0 + \hat{H}'}$

Formal ("exact") solution

→ time-dependent Schrödinger equation (TDSE)

$$i\hbar \partial_t |\psi(t)\rangle = \hat{H} |\psi(t)\rangle \; ; \text{ subject to } |\psi(t_0)\rangle$$

We expand $|\psi(t)\rangle$ in terms of eigenstates of $\hat{H}_0 : \{|n\rangle\}$ with time-dependent coefficients $c_n(t)$ ← to be determined

$$|\psi(t)\rangle = \sum_n c_n(t) e^{-i\omega_n t} |n\rangle$$

extracting these factors is convenient as it eliminates the free evolution from the c_n's.