Examples:

* Pure coherent states: $\hat{\rho} = |\gamma\rangle\langle\gamma|$, $\gamma \in \mathbb{C}$

Nomenclature $|\alpha\rangle \to |\beta\rangle$

$$\langle -\beta|\hat{\rho}|\beta\rangle = \langle -\beta|\gamma\rangle\langle\gamma|\beta\rangle$$
$$= e^{-|\beta|^2} e^{-|\gamma|^2} e^{-\beta^*\gamma - \gamma^*\beta}$$

$$\Rightarrow P_\gamma(\alpha) = e^{|\alpha|^2} e^{-|\gamma|^2} \frac{1}{\pi^2} \int \underbrace{e^{-\beta^*\gamma - \gamma^*\beta + \beta^*\alpha - \beta\alpha^*}}_{e^{\beta^*(\alpha-\gamma) - \beta(\gamma^* - \alpha^*)}} d^2\beta$$

$$= e^{2i\,\mathrm{Im}(\beta^*(\alpha-\gamma))}$$
$$= e^{2i[\beta_R(\alpha_I - \gamma_I) - \beta_I(\alpha_R - \gamma_R)]}$$

$$\boxed{\int_{-\infty}^{\infty} e^{ikx}\,dx = 2\pi\delta(k)}$$

$$= e^{|\alpha|^2} e^{-|\gamma|^2} \Bigg(\underbrace{\frac{1}{\pi}\int_{-\infty}^{\infty} d\beta_R\, e^{2i\beta_R(\alpha_I - \gamma_I)}}_{=\delta(\alpha_I - \gamma_I)}\Bigg)\Bigg(\underbrace{\frac{1}{\pi}\int_{-\infty}^{\infty} d\beta_I\, e^{-2i\beta_I(\alpha_R - \gamma_R)}}_{\delta(\alpha_R - \gamma_R)}\Bigg)$$

$$= \delta^{(2)}(\alpha - \gamma)$$ [as expected]

* Pure number state $\hat{\rho} = |n\rangle\langle n|$

$$\langle -\beta|\hat{\rho}|\beta\rangle = \langle -\beta|n\rangle\underbrace{\langle n|\beta\rangle}_{\frac{e^{-|\beta|^2/2}\beta^n}{\sqrt{n!}}} = e^{-|\beta|^2}\frac{(-\beta^*\beta)^n}{n!}$$

$$\Rightarrow P_n(\alpha) = \frac{e^{|\alpha|^2}}{n!}\frac{1}{\pi^2}\int(-\beta^*\beta)^n e^{\beta^*\alpha-\beta\alpha^*}d^2\beta$$

Formal trick

$$= \frac{e^{|\alpha|^2}}{n!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\underbrace{\frac{1}{\pi^2}\int e^{\beta^*\alpha-\beta\alpha^*}d^2\beta}_{\frac{1}{\pi^2}\int e^{2i\,\mathrm{Im}(\beta^*\alpha)}d^2\beta \overset{\text{as above}}{=} \delta^{(2)}(\alpha)}$$

$$\Rightarrow \boxed{P_n(\alpha) = \frac{e^{|\alpha|^2}}{n!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\delta^{(2)}(\alpha)}$$

Comments: 1) can only be interpreted under integral.
2) more singular than δ-function.

5.2 Optical Equivalence Theorem

Define: normal ordering superoperator

$$:\hat{O}(\hat{a},\hat{a}^{+}): \equiv \hat{O}^{N}(\hat{a},\hat{a}^{+})$$

collect all creation operators to the left and all annihilation operators to the right

→ often useful!

e.g. $:\hat{n}^2: = :\hat{a}^{+}\hat{a}\hat{a}^{+}\hat{a}: = \hat{a}^{+}\hat{a}^{+}\hat{a}\hat{a}$

Optical Equivalence Theorem

→ consider a generic normal-ordered function of $\hat{a}$ and $\hat{a}^{+}$

$$\hat{G}^{N}(\hat{a},\hat{a}^{+}) = \sum_{n}\sum_{m} c_{nm}\, \hat{a}^{+n}\, \hat{a}^{m}$$

(any expression can be brought into this form)

Its expectation value is

$$\langle \hat{G}^{N}(\hat{a},\hat{a}^{+})\rangle = \mathrm{Tr}\left[\hat{G}^{N}(\hat{a},\hat{a}^{+})\hat{\rho}\right]$$

$\hat{\rho} = \int P(\alpha)|\alpha\rangle\langle\alpha| d^2\alpha$

$$= \mathrm{Tr}\left[\int P(\alpha)\sum_{nm} c_{nm}\,\hat{a}^{+n}\hat{a}^{m}|\alpha\rangle\langle\alpha|\, d^2\alpha\right]$$

$$= \int P(\alpha)\sum_{nm} c_{nm} \sum_{l}\langle l|\hat{a}^{+n}\hat{a}^{m}|\alpha\rangle\langle\alpha|l\rangle\, d^2\alpha$$

remove $\sum_{l} |l\rangle\langle l|$

$$= \int P(\alpha) \sum_{nm} c_{nm} \langle \alpha | \underbrace{\hat{a}^{\dagger n}}_{\langle\alpha|(\alpha^*)^n} \underbrace{\hat{a}^{m} | \alpha \rangle}_{\alpha^m|\alpha\rangle} d^2\alpha$$

$$\Rightarrow \langle \hat{G}^{(N)}(\hat{a}, \hat{a}^\dagger) \rangle = \int P(\alpha) \underbrace{\sum_{nm} c_{nm} \alpha^{*n} \alpha^{m}} d^2\alpha$$

$$= \int P(\alpha) \, G^{(N)}(\alpha, \alpha^*) \, d^2\alpha$$

→ The expectation value of a normal ordered operator is the P-function weighted average of the function obtained by

$$\hat{a} \to \alpha \quad ; \quad \hat{a}^\dagger \to \alpha^*$$

→ easy way of computing expectation values

e.g.: $\langle : \hat{n}^2 : \rangle = \langle (\hat{a}^\dagger)^2 \hat{a}^2 \rangle = \int P(\alpha) |\alpha|^4 d^2\alpha$

example:

$$\langle : \hat{n}^2 : \rangle_\gamma = \int \underbrace{P_\gamma(\alpha)}_{\delta^{(2)}(\gamma - \alpha)} |\alpha|^4 d^2\alpha = |\gamma|^4$$

coherent state

How to make an operator normal ordered?

$$\hat a^+ \underbrace{\hat a \hat a^+}_{\hat a^+ \hat a + 1} \hat a = \hat a^+ \hat a^+ \hat a \hat a + \hat a^+ \hat a$$

$$\rightarrow \langle : \hat a^+ \hat a \hat a^+ \hat a : \rangle_\gamma = |\gamma|^4 + |\gamma|^2$$

$$[\hat a, \hat a^+] = \hat a \hat a^+ - \hat a^+ \hat a = 1$$

$$\hat a \hat a^+ = \hat a^+ \hat a + \underbrace{[\hat a, \hat a^+]}_{=1}$$

5.2 Wigner function

"Phase space" representation of a quantum state

here 1D: $$W(q,p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \langle q + \tfrac{x}{2} | \hat\rho | q - \tfrac{x}{2} \rangle e^{ipx/\hbar} dx$$

Example: pure state $\hat\rho = |\psi\rangle\langle\psi|$

$$W(q,p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \underbrace{\langle q + \tfrac{x}{2} | \psi \rangle}_{\psi(q+\frac{x}{2})} \underbrace{\langle \psi | q - \tfrac{x}{2} \rangle}_{\psi^*(q-x/2)} e^{ipx/\hbar} dx$$

Test 1: Integrate out momentum degree of freedom

$$\int_{-\infty}^{\infty} W(q,p) dp = \int_{-\infty}^{\infty} dx\, \psi^*(q - \tfrac{x}{2}) \psi(q + \tfrac{x}{2}) \underbrace{\frac{1}{2\pi\hbar} \int_{-\infty} e^{ipx/\hbar} dp}_{=\delta(x)}$$

$$= |\psi(q)|^2$$

→ probability density over position q

marginal: