Quantum Optics Assignment 1

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1 Typical photon energies

(a)

2 Mechanical velocity and electromagnetic fields in quantum mechanics

3 Density matrices

(a) Pure ensemble

$$|\psi\rangle = \begin{pmatrix} \sqrt{3/4} \\ -i/2 \end{pmatrix} \tag{1}$$

$$\langle\psi| = \begin{pmatrix} \sqrt{3/4} & i/2 \end{pmatrix} \tag{2}$$

$$\rho_{\text{pure}} = |\psi\rangle\langle\psi| = \begin{pmatrix} 3/4 & i\sqrt{3}/4 \\ -i\sqrt{3}/4 & 1/4 \end{pmatrix} \tag{3}$$

(b) Impure ensemble

$$\rho_1 = \frac{3}{4}|g\rangle\langle g| = \frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{4}$$

$$\rho_2 = \frac{1}{4}|e\rangle\langle e| = \frac{1}{4}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{5}$$

$$\rho_{\text{impure}} = \rho_1 + \rho_2 = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix} \tag{6}$$

(c) Experimental determination

In the pure ensemble, all atoms are in a definite state $|\psi\rangle$. In the impure ensemble, a proportion of atoms are in each state. Therefore, if we could devise an experiment to determine the likelihood of an atom being in $|\psi\rangle$, we would get different answers for the two ensembles. For (a), we would get unity. For (b), we would get an answer based on the projection of the component states onto $|\psi\rangle$. Let $A = |\psi\rangle\langle\psi|$ be the observable that we can measure.

$$\langle A\rangle_{(\text{pure})} = \text{Tr}\left[\rho_{\text{pure}}A\right] = \text{Tr}\left[\rho_{\text{pure}}^2\right] = 1 \tag{7}$$

$$\langle A\rangle_{(\text{impure})} = \text{Tr}\left[\rho_{\text{impure}}A\right] = \text{Tr}\left[\begin{pmatrix} (3/4)^2 & i3^{3/2}/16 \\ -i\sqrt{3}/16 & 1/16 \end{pmatrix}\right] = 10/16 \tag{8}$$

(d) Entropy

If we use a basis in which ρ is diagonal, the entropy is simply

$$S = -k_B \mathrm{Tr}\left[\rho \ln \rho\right] = -k_B \sum_k \rho_k^{(\mathrm{diag})} \ln \rho_k^{(\mathrm{diag})}. \tag{9}$$

For the pure state, the eigenvalues of 1 and 0.

$$S_{\mathrm{pure}} = -k_B \left[1 \cdot \ln 1 + \lim_{x \to 0^+} x \ln x\right]. \tag{10}$$

We can evaluate the limit using L'hopital's rule. Note that we have to write it as a fraction.

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} -\frac{1/x}{1/x^2} = \lim_{x \to 0^+} -x = 0 \tag{11}$$

So the entropy of a pure state is zero!

$$S_{\mathrm{pure}} = 0 \tag{12}$$

For the mixed state,

$$S_{\mathrm{impure}} = -k_B \left[3/4 \cdot \ln 3/4 + 1/4 \cdot \ln 1/4\right] \approx 0.562 k_B \tag{13}$$