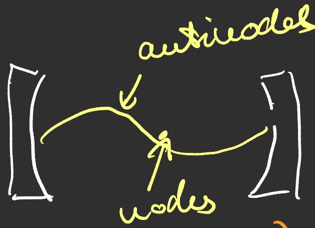
7.2 Jaynes–Cummings model

→ similar to Rabi model with a quantised EM field

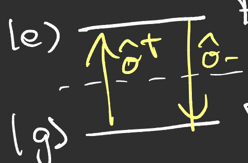


$$\hat{H}_p = \hbar\omega\, \hat{a}^\dagger \hat{a}$$

sin(kz)

$$\hat{H}' = -\hat{\vec{d}}\cdot\hat{\vec{E}}, \qquad \hat{\vec{E}} = \hat{e}\,\sin(kz)\,\underbrace{\sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}}_{}\cdot(\hat{a}+\hat{a}^\dagger)$$

E_0: vacuum field amplitude



$E_e = \hbar\omega_0/2$

$E = 0$

$E_g = -\hbar\omega_0/2$

$$\hat{H}_0 = \frac{\hbar\omega_0}{2}\,\hat{\sigma}_3\ ;\ \hat{\sigma}_3 = |e\rangle\langle e| - |g\rangle\langle g|$$

$$\hat{\vec{d}}\cdot\hat{e} = d\,[\underbrace{|e\rangle\langle g|}_{\hat{\sigma}_+} + \underbrace{|g\rangle\langle e|}_{\hat{\sigma}_-}] \quad \text{with } d = \langle e|\hat{d}|g\rangle \in \mathbb{R}$$

atomic basis representation

Useful relations:

$$\hat{\sigma}_+ + \hat{\sigma}_- = \hat{\sigma}_1 \qquad [\hat{\sigma}_3, \hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm$$

$$\hat{\sigma}_\pm = \frac{\hat{\sigma}_1 \pm i\hat{\sigma}_2}{2} \qquad [\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_3$$

$$\boxed{\hat{H} = \frac{\hbar\omega_0}{2}\hat{\sigma}_3 + \hbar\omega\hat{n} + \hbar\lambda(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^\dagger)}$$

$$\lambda = -\sqrt{\frac{\hbar\omega_0}{\varepsilon_0 V}}\sin(kz)\cdot\frac{d}{\hbar}$$

Quantum Rabi model (no RWA here!)

→ evaluate free (uncoupled) evolution:

$$\dot{\hat{a}} = i[\hat{H}(\lambda=0), \hat{a}] = -\hbar\omega\hat{a} \rightarrow \hat{a}_0(t) = \hat{a}(0)e^{-i\omega t}$$

$$\dot{\hat{\sigma}}_\pm = i[\hat{H}(\lambda=0), \hat{\sigma}_\pm] = \pm i\hbar\omega_0\hat{\sigma}_\pm \rightarrow \hat{\sigma}_\pm(0)e^{\pm i\omega_0 t}$$

Interaction terms

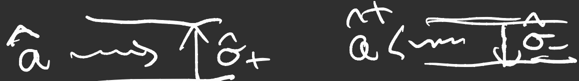
$$\hat{a}\hat{\sigma}_- \sim e^{-i(\omega+\omega_0)t} \xleftrightarrow{\text{FAST}} \hat{a}^\dagger\hat{\sigma}_+ \sim e^{i(\omega+\omega_0)t}$$

$$\hat{a}^\dagger\hat{\sigma}_- \sim e^{-i(-\omega+\omega_0)t} \qquad \hat{a}\hat{\sigma}_+ \sim e^{i(-\omega+\omega_0)t}$$

→ RWA: drop fast terms

$$\boxed{\hat{H}_{JC} = \frac{\hbar\omega_0}{2}\hat{\sigma}_3 + \hbar\omega\hat{n} + \hbar\lambda(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-)}$$

Jaynes-Cummings Hamiltonian



*) exactly solvable quantum system → p-set 5

*) RWA safe when $\Delta \ll \omega, \omega_0$ and $\Omega_0 \ll \omega_0$

↳ otherwise: Bloch-Siegert shift

*) Analytical solution is possible because

$$[\hat{H}_{JC}, \hat{C}] = 0 \quad \text{with} \quad \hat{C} = \hat{a}^\dagger \hat{a} + \hat{\sigma}_+ \hat{\sigma}_-$$

gives a conserved quantity

*) Quantum Rabi model (no RWA) was thought to have no analytical solution

↳ PRL 107, 100401 (2011)

↳ allows investigations of the limitations of the JC model

Details of JC model → P-set 5

Essential: dynamics occurs in a subspaces spanned by

$|i\rangle \equiv |e\rangle |n\rangle$ → bare energy E_i

$|f\rangle \equiv |g\rangle |n+1\rangle$ → " " E_f

$$|\psi(t)\rangle = c_i(t)\, e^{-iE_i t/\hbar}\,|i\rangle + c_f(t)\, e^{-iE_f t/\hbar}\,|f\rangle$$

consider: $c_i(0) = 1, \quad c_f(0) = 0$

PSET

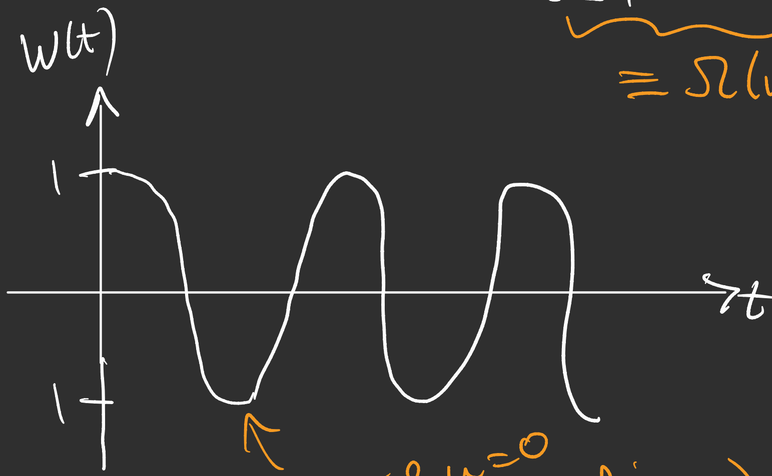
$$\Rightarrow c_i(t) = \cos\left(\lambda\sqrt{n+1}\, t\right), \quad c_f(t) = -i\sin\left(\lambda\sqrt{n+1}\, t\right)$$

define $W(t) = \langle\psi|\hat{\sigma}_3|\psi\rangle$ inversion $(\hat{R}_3)$

$$= \cos^2\left(\lambda\sqrt{n+1}\, t\right) - \sin^2\left(\lambda\sqrt{n+1}\, t\right)$$

$$= 2\cos^2\left(\lambda\sqrt{n+1}\, t\right) - 1$$

$$= \cos\left(\underbrace{2\lambda\sqrt{n+1}\, t}_{\equiv\, \Omega(n)\cdot t}\right)$$



even if $n = 0$

(quantum Rabi oscillation)

ASIDE

$|\psi(t)\rangle = |g,0\rangle$

$i\partial_t|\psi\rangle = H|\psi\rangle \xrightarrow{\mathcal{H}} i\partial_t c_{g,0} = -\frac{\omega_0}{2} c_{g,0}$

Quantum Rabi model

$c_{g,0}(t) = e^{+i\frac{\omega_0}{2}t}$

$$i\partial_t c_{g,0} = -\frac{\omega_0}{2} c_{g,0} + \lambda c_{e,1}(t)$$

need: $i\partial_t c_{e,1} = \dots$

Cummings collapse   CPRA 140, A1051 (1965)

→ generalize $|n\rangle \to \sum_{n=0}^{\infty} c_n |n\rangle$

↳ $W(t) = \sum_{n=0}^{\infty} |c_n|^2 \cos(\underbrace{2\lambda t \sqrt{n+1}}_{=\Omega(n)\cdot t})$

*) $|\psi(t)\rangle \to |\alpha\rangle$ (coherent state): $c_n = \frac{e^{-|\alpha|^2/2}\,\alpha^n}{\sqrt{n!}}$

recall: $\bar{n} = |\alpha|^2$, $\Delta n = \sqrt{\bar{n}} = |\alpha|$

Different Rabi frequencies will dephase$^{\text{(collapse)}}$ at about a time t_c : $t_c \cdot \Delta\Omega \simeq 1$

$\hookrightarrow t_c = \dfrac{1}{2\lambda}$