CHM 696: Quantum Optics

Problem Set 3 (State representations, Two-level systems)

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Due: October 3, 2023

I Q-function (Husimi) representation

The Q-function representation is a very useful way to represent and visualize quantum states and to calculate many expectation operators. For a given quantum state, defined via its density matrix $\hat{\rho}$, it is defined as

$$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle. \tag{1}$$

where α is a complex number and $|\alpha\rangle$ denotes the corresponding coherent state.

(a) Show that the Q-function of a pure state $|\Psi\rangle$ is given by

$$Q(\alpha) = \frac{1}{\pi}|\langle\alpha|\Psi\rangle|^2$$

(b) Calculate and plot the Q-function for a coherent state $|\alpha'\rangle$, a Fock state $|n\rangle$ and the vacuum state $|0\rangle$. Use a 2D Contour plot with $\mathrm{Im}(\alpha)$ and $\mathrm{Re}(\alpha)$ as axes.

Note: If you use a computer, you can choose any reasonably common program/language. If you have not used a plotting program before, Mathematica can be downloaded for free from Purdue IT.

(c) Calculate the plot the Q-function of the state

$$|\Psi\rangle = N\frac{1}{\sqrt{2}}\left(|\alpha'\rangle + |-\alpha'\rangle\right)$$

What state is this (we have only mentioned it briefly in the lecture)? And why is N in general not unity?

If you use computer tools to solve a problem, please do not only keep out he results with your solution, but also the complete command/computer code which generated the output. It is part of the solution, and it is necessary to understand what you did or tried to do.

II Bonus problem: Wigner functions

In the lecture, we introduced the following definition of the Wigner function

$$W(q,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\langle q+\frac{x}{2}|\hat{\rho}|q-\frac{x}{2}\rangle e^{ipx/\hbar}dx. \tag{2}$$

(a) Show that this definition is equivalent to

$$W(\alpha) = \frac{1}{\pi^2}\int e^{\lambda^*\alpha-\lambda\alpha^*}\mathrm{tr}[\hat{\rho}e^{\lambda\hat{a}^\dagger-\lambda^*\hat{a}}]d^2\lambda \tag{3}$$

with the identification $\alpha = \frac{1}{\sqrt{2\hbar}}(q+ip)$.

Hint: Use the BCH formula $e^{\hat{X}}e^{\hat{Y}} = e^{\hat{Z}}$ *with* $\hat{Z} = \hat{X}+\hat{Y}+\frac{1}{2}[\hat{X},\hat{Y}]+\ldots$. *Further, use the displacement operator* $e^{-\frac{i}{\hbar}x\hat{p}}|q\rangle = |q-x\rangle$

III Numerical exploration of the two-level system in the RWA

From time to time we will have numerical exercises. This one is meant as a first warm-up. You will need a linear differential equation solver for 'brute-force' solution of the Schrödinger equation for time evolution. Any programming language is fine, as long as you write the code yourself; in particular, it is probably easiest to use Mathematica, or the Qutip (qutip.org) package for Python 3, which is specifically built for solving small quantum systems. Please attach your code along with the plots you produce. We consider the Hamiltonian of a driven two-level system, which using the rotating-wave approximation (RWA) can be written in the form

$$\hat{H}_{\mathrm{RWA}} = -\frac{\hbar\delta}{2}\hat{\sigma}_z - \frac{\hbar\Omega_0}{2}\hat{\sigma}_x \tag{4}$$

with the Pauli-operators $\hat{\sigma}_z = |1\rangle\langle 1| - |2\rangle\langle 2|$ and $\hat{\sigma}_x = |1\rangle\langle 2| + |2\rangle\langle 1|$, Ω_0 is the Rabi frequency and δ is the detuning between the laser frequency and the transition frequency. The driven two-level problem will be solved analytically in the lecture. In this exercise we want to solve it numerically. Construct the equations of motion from the time-independent Hamiltonian of Eq. (4), feed it to the solver of your choice (subject to the atom initially being in the ground state), and plot the probability to find an atom in the ground state as a function of time for $\delta = 0$ and $\delta = \Omega_0$.

Hint: in qutip, the function you want is "mesolve()".