Quantum Optics Assignment 1

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1 Typical photon energies

Jupyter notebook used for calculations included at the end of this document.

(a) Photon energies

- Photon energy in Joule: 2.33×10^{-19} J
- Photon energy in eV: 1.46 eV
- Frequency of light: 352 THz
- Temperature: 1.69×10^{4} K

(b) Total energy

- Final velocity: 4.2 km/s

(c) Recoil velocity

- Recoil velocity: 29.5 mm/s
- Recoil energy: 1.66×10^{-29} J

2 Mechanical velocity and electromagnetic fields in quantum mechanics

(a) Commutators

Functions of $\mathbf{r}$ commute with $\mathbf{r}$, and likewise with $\mathbf{p}$.

$$[\mathbf{r}, \mathbf{A}(\mathbf{r}, t)] = 0 \tag{1}$$
$$[\mathbf{r}, U(\mathbf{r}, t)] = 0 \tag{2}$$

(b) Calculating the velocity

$$H = \frac{1}{2m} \sum_{i=x,y,z} \left[p_i^2 + q^2 A_i^2\left(\mathbf{r}, t\right) - q p_i A_i\left(\mathbf{r}, t\right) - q A_i\left(\mathbf{r}, t\right) p_i\right] + qU\left(\mathbf{r}, t\right) \tag{3}$$

$$[x, H] = \frac{1}{2m} \sum_{i=x,y,z} \left\{ \left[x, p_i^2\right] - q\left[x, p_i A_i(\mathbf{r}, t)\right] - q\left[x, A_i(\mathbf{r}, t) p_i\right] \right\} \tag{4}$$

$$= \frac{1}{2m} \left\{ p_x [x, p_x] + [x, p_x] p_x - q [x, p_x] A_x(\mathbf{r}, t) - q A_i(\mathbf{r}, t) [x, p_x] \right\} \tag{5}$$

$$= \frac{1}{2m} \left\{ 2i\hbar p_x - 2qi\hbar A_x(\mathbf{r}, t) \right\} \tag{6}$$

$$= \frac{i\hbar}{m} \left\{ p_x - q A_x(\mathbf{r}, t) \right\} \tag{7}$$

$$\frac{d}{dt} \langle x \rangle = \frac{1}{i\hbar} \langle [x, H] \rangle + \left\langle \frac{\partial}{\partial t} x \right\rangle \tag{8}$$

$$= \frac{1}{m} \langle p_x - q A_x(\mathbf{r}, t) \rangle \tag{9}$$

Similarly,

$$\frac{d}{dt} \langle y \rangle = \frac{1}{m} \langle p_y - q A_y(\mathbf{r}, t) \rangle \tag{10}$$

$$\frac{d}{dt} \langle z \rangle = \frac{1}{m} \langle p_z - q A_z(\mathbf{r}, t) \rangle \tag{11}$$

so that

$$\langle \mathbf{v} \rangle = \frac{1}{m} \langle \mathbf{p} - q \mathbf{A}(\mathbf{r}, t) \rangle \tag{12}$$

(c) The Force

We can find the acceleration by a second application of Ehrenfest's theorem.

$$\frac{d}{dt} \langle \mathbf{v} \rangle = \frac{1}{i\hbar} \langle [\mathbf{v}, H] \rangle + \left\langle \frac{\partial}{\partial t} \mathbf{v} \right\rangle \tag{13}$$

$$= \frac{1}{i\hbar} \langle [\mathbf{v}, qU(\mathbf{r}, t)] \rangle - \frac{q}{m} \left\langle \frac{\partial}{\partial t} \mathbf{A}(\mathbf{r}, t) \right\rangle \tag{14}$$

$$= \frac{1}{i\hbar} \left\langle \left[\frac{\mathbf{p}}{m}, qU(\mathbf{r}, t) \right] \right\rangle - \frac{q}{m} \left\langle \frac{\partial}{\partial t} \mathbf{A}(\mathbf{r}, t) \right\rangle \tag{15}$$

Using $[p, f(x)] = -i\hbar \partial_x f(x)$, this is

$$\frac{d}{dt} \langle \mathbf{v} \rangle = \frac{q}{m} \frac{1}{i\hbar} \langle -i\hbar \nabla U(\mathbf{r}, t) \rangle - \frac{q}{m} \left\langle \frac{\partial}{\partial t} \mathbf{A}(\mathbf{r}, t) \right\rangle \tag{16}$$

$$= -\frac{q}{m} \langle \nabla U(\mathbf{r}, t) \rangle - \frac{q}{m} \left\langle \frac{\partial}{\partial t} \mathbf{A}(\mathbf{r}, t) \right\rangle \tag{17}$$

According to physics stackexchange, $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$. This means that our force $m \frac{d}{dt} \langle \mathbf{v} \rangle$ has a clear classical analogue.

$$m\mathbf{a} = -q \nabla U(\mathbf{r}, t) + q\mathbf{E} \tag{18}$$

3 Density matrices

(a) Pure ensemble

$$|\psi\rangle = \begin{pmatrix} \sqrt{3/4} \\ -i/2 \end{pmatrix} \tag{19}$$

$$\langle \psi | = \begin{pmatrix} \sqrt{3/4} & i/2 \end{pmatrix} \tag{20}$$

$$\rho_{\text{pure}} = |\psi\rangle \langle \psi| = \begin{pmatrix} 3/4 & i\sqrt{3}/4 \\ -i\sqrt{3}/4 & 1/4 \end{pmatrix} \tag{21}$$

(b) Impure ensemble

$$\rho_1 = \frac{3}{4} |g\rangle \langle g| = \frac{3}{4} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{22}$$

$$\rho_2 = \frac{1}{4} |e\rangle \langle e| = \frac{1}{4} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{23}$$

$$\rho_{\text{impure}} = \rho_1 + \rho_2 = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix} \tag{24}$$

(c) Experimental determination

In the pure ensemble, all atoms are in a definite state $|\psi\rangle$. In the impure ensemble, a proportion of atoms are in each state. Therefore, if we could devise an experiment to determine the likelihood of an atom being in $|\psi\rangle$, we would get different answers for the two ensembles. For (a), we would get unity. For (b), we would get an answer based on the projection of the component states onto $|\psi\rangle$. Let $A = |\psi\rangle \langle\psi|$ be the observable that we can measure.

$$\langle A\rangle_{(\text{pure})} = \text{Tr}\left[\rho_{\text{pure}} A\right] = \text{Tr}\left[\rho_{\text{pure}}^2\right] = 1 \tag{25}$$

$$\langle A\rangle_{(\text{impure})} = \text{Tr}\left[\rho_{\text{impure}} A\right] = \text{Tr}\left[\begin{pmatrix} (3/4)^2 & i3^{3/2}/16 \\ -i\sqrt{3}/16 & 1/16 \end{pmatrix}\right] = 10/16 \tag{26}$$

(d) Entropy

If we use a basis in which ρ is diagonal, the entropy is simply

$$S = -k_B \text{Tr}\left[\rho \ln \rho\right] = -k_B \sum_k \rho_k^{(\text{diag})} \ln \rho_k^{(\text{diag})}. \tag{27}$$

For the pure state, the eigenvalues of 1 and 0.

$$S_{\text{pure}} = -k_B \left[1 \cdot \ln 1 + \lim_{x\to 0^+} x \ln x\right]. \tag{28}$$

We can evaluate the limit using L'hopital's rule. Note that we have to write it as a fraction.

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} -\frac{1/x}{1/x^2} = \lim_{x\to 0^+} -x = 0 \tag{29}$$

So the entropy of a pure state is zero!

$$S_{\text{pure}} = 0 \tag{30}$$

For the mixed state,

$$S_{\text{impure}} = -k_B \left[3/4 \cdot \ln 3/4 + 1/4 \cdot \ln 1/4\right] \approx 0.562 k_B \tag{31}$$

(e) Thermal state

In this case, we only consider two state $|g\rangle$ and $|e\rangle$, with energy 0 and 1.

$$\rho = \frac{1}{Z}\left(|g\rangle \langle g| + e^{-1/k_B T} |e\rangle \langle e|\right) \tag{32}$$

At $T = 0$, $e^{-1/k_B T} \to 0$ and the population is purely in the ground state. As $T \to \infty$, $e^{-1/k_B T} \to 1$ and the density matrix is a statistical mixture of ground and excited.

$$\rho_{T\to\infty} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \tag{33}$$

Appendices

Commutator of momentum and a position-dependent function

$$\begin{aligned}
\langle x| \left[\hat{p}, f(\hat{x})\right] |\psi\rangle &= \langle x| \left(\hat{p} f(\hat{x}) - f(\hat{x})\hat{p}\right) |\psi\rangle && (34)\\
&= -i\hbar\partial_x \left(f(x)\psi(x)\right) - i\hbar f(x)\partial_x\psi(x) && (35)\\
&= -i\hbar \left(\partial_x f(x)\right) \psi(x) && (36)
\end{aligned}$$

$$\rightarrow [p, f(x)] = -i\hbar\partial_x f(x) \tag{37}$$