8. Squeezed States

→ recall: $[\hat{A}, \hat{B}] = i\hat{C}$

$$\Rightarrow \langle (\underbrace{\Delta \hat{A}}_{\hat{A} - \langle \hat{A} \rangle})^2 \rangle \cdot \langle (\Delta \hat{B})^2 \rangle \geq \frac{1}{4} |\langle \hat{C} \rangle|^2$$

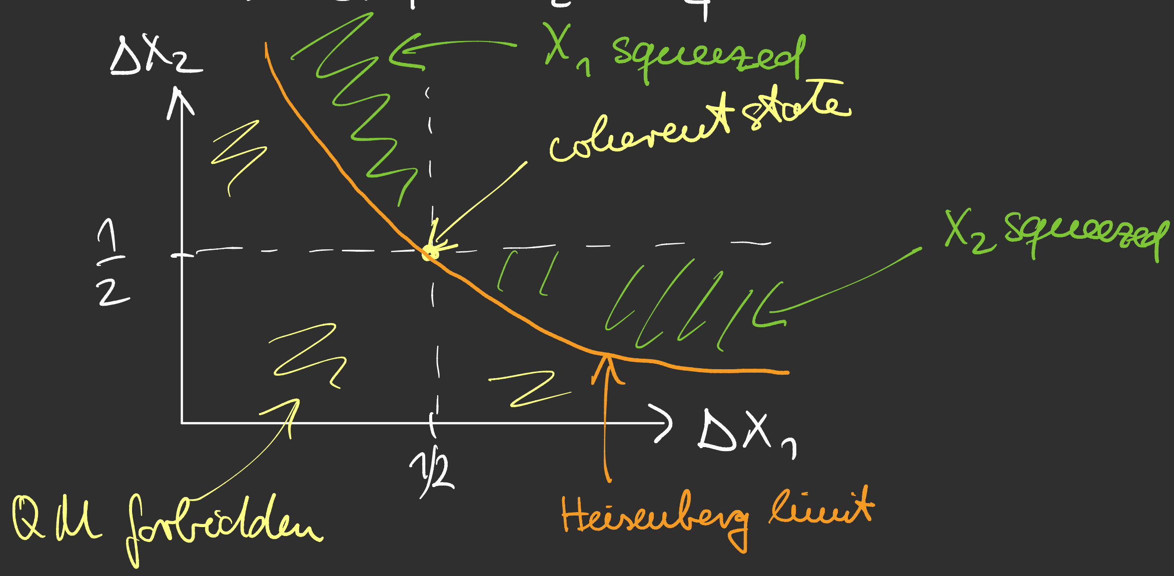
Def.: state $|\psi\rangle$ is "squeezed" when

$$(\Delta \hat{A})_\psi \leq \sqrt{\frac{1}{2} \langle \hat{C} \rangle_\psi}$$

→ ψ has less "noise" than $|\alpha\rangle$

Consider quadrature operators $\hat{X}_{1,2}$: $[\hat{X}_1, \hat{X}_2] = \frac{i}{2}$

$$\Rightarrow \Delta X_1 \cdot \Delta X_2 \geq \frac{1}{4}$$



Squeezing operator: $\hat{S}(\xi) = e^{\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{+2})}$

$\xi = r e^{i\phi}$ ← $0 \leq \phi \leq 2\pi$

r ← squeezing parameter $r \in [0, \infty)$

→ Note: the presence of $\hat{a}^2$, $\hat{a}^{+2}$ indicates two-photon process (NLO) ← non-linear optics

Properties:

*) $\hat{S}$ is unitary: $\hat{S}^+(\xi) = e^{\frac{1}{2}(\xi \hat{a}^{+2} - \xi^* \hat{a}^2)} = S(-\xi)$

$= -\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{+2})$

$\Rightarrow \hat{S}^+(\xi)\hat{S}(\xi) = \hat{S}(\xi)\hat{S}^+(\xi) = \mathbb{1}$

*) action of $\hat{S}$ on vacuum state: $|\xi\rangle = \hat{S}(\xi)|0\rangle$

→ to find $(\Delta X_1)_\xi$, $(\Delta X_2)_\xi$ need

$\langle \hat{a} \rangle_\xi$, $\langle \hat{a}^+ \rangle_\xi$, $\langle \hat{a}^2 \rangle_\xi$, $\langle \hat{a}^+ \rangle_\xi$

general approach for an operator $\hat{B}$:

BCH: $e^{i\lambda\hat{A}} \hat{B} e^{-i\lambda\hat{A}} = \hat{B} + i\lambda[\hat{A}, \hat{B}] + \frac{(i\lambda)^2}{2}[\hat{A}, [\hat{A}, \hat{B}]] + \dots$

Here, we have, e.g., $\langle \hat{a} \rangle_\chi = \langle \underbrace{\hat{S}^\dagger(\chi)}_{\hat{S}(-\chi)} \hat{a} \hat{S}(\chi) \rangle$

$$\lambda = \frac{i}{2} r$$

$$\hat{A} = e^{-i\phi} \hat{a}^2 - e^{i\phi} \hat{a}^{\dagger 2}$$

$$\hat{B} = \hat{a}$$

$$\Rightarrow [\hat{A}, \hat{B}] = 2 e^{i\phi} \hat{a}^\dagger$$

$$[\hat{A}, [\hat{A}, \hat{B}]] = 4 \hat{a}$$

$$[\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] = -8 e^{i\phi} \hat{a}^\dagger, \ldots$$

$$\Rightarrow \hat{S}^\dagger(\chi) \hat{a} \hat{S}(\chi) = \hat{a} \Big[\overbrace{1 + \frac{r^2}{2!} + \frac{r^4}{4!} + \ldots}^{\cosh r} \Big]$$

$$- e^{i\phi} \hat{a}^\dagger \Big[\underbrace{r + \frac{r^3}{3!} + \frac{r^5}{5!} + \ldots}_{\sinh r} \Big]$$

$\Rightarrow$ can derive $\langle \hat{a}^\dagger \rangle_\chi$ similarly

$$\hat{S}^\dagger(\chi) \hat{a}^\dagger \hat{S}(\chi) = \hat{a}^\dagger \cosh r - e^{-i\phi} \hat{a} \sinh r$$

$$\Rightarrow \hat{S}^{+} \hat{a}^{2} \hat{S} = \underbrace{\hat{S}^{+} \hat{a} \hat{S}}\,\underbrace{\hat{S}^{+}}_{=\mathbb{1}} \hat{a} \hat{S}$$

$$= \left(\hat{a}\cosh r - e^{-i\phi}\hat{a}^{+}\sinh r\right)^{2}$$

(and similarly for $\hat{a}^{+2}$)

$\langle S^{+}(x)\, \hat{a}\, \hat{S}(x) \rangle$

$$\Rightarrow \langle \hat{X}_1 \rangle = \frac{1}{2}\left(\langle \hat{a} \rangle_x + \langle \hat{a}^{+} \rangle_x\right)$$

$$= \langle 0 | \hat{a}\cosh r - e^{i\phi}\hat{a}^{+}\sinh r | 0 \rangle$$

$$= \cosh r \langle 0 | \hat{a} | 0 \rangle - e^{i\phi}\sinh r \langle 0 | \hat{a}^{+} | 0 \rangle = 0$$

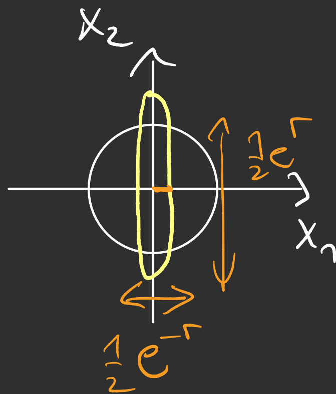
$$\rightarrow \langle \hat{X}_1 \rangle_x = \langle \hat{X}_2 \rangle_x = 0$$

but: $\langle \hat{a}^{2} \rangle_x \neq 0$ (evaluate!)

$$\boxed{\begin{aligned}(\Delta \hat{X}_1)_x &= \frac{1}{2}\left[\cosh^{2} r + \sinh^{2} r - 2\sinh r \cosh r \cos\phi\right]^{1/2}\\ (\Delta \hat{X}_2)_x &= \frac{1}{2}\left[\cosh^{2} r + \sinh^{2} r + 2\sinh r \cosh r \cdot \cos\phi\right]^{1/2}\end{aligned}}$$

i) $\phi = 0$ (X_1 squeezed)

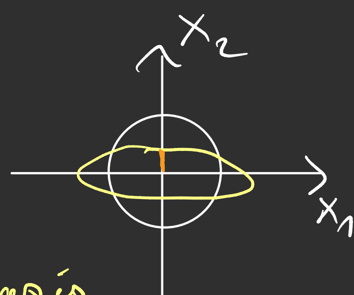
$$(\Delta X_1) = \frac{1}{2} e^{-r}, \quad (\Delta X_2) = \frac{1}{2} e^{r}$$



$$(\Delta X_1)(\Delta X_2) = \frac{1}{4} \quad \text{(min. uncertainty)}$$

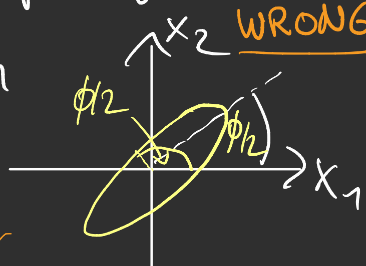
ii) $\phi = \pi$ (X_2 squeezed)

$$(\Delta X_1) : \frac{1}{2} e^{r}, \quad (\Delta X_2) = \frac{1}{2} e^{-r}$$



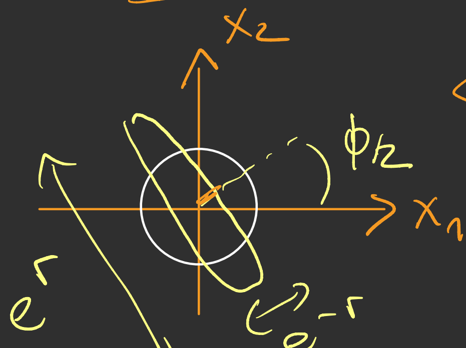
For arbitrary ϕ, the ~~ellipse~~ minor axis forms an angle $\measuredangle\, \phi/2$ with X_1

WRONG!



→ minor axis?

CORRECT



8.1 Squeezed coherent state

$$|\alpha, \chi\rangle = \underbrace{\hat{D}(\alpha)}_{e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}} \hat{S}(\chi) |0\rangle , \qquad \alpha = |\alpha| e^{i\theta}$$
$$\chi = r e^{i\phi}$$

Note: $[\hat{D}(\alpha), \hat{S}(\chi)] \neq 0 \Rightarrow$ first squeeze vacuum, then displace it

$$\langle \hat{a} \rangle_{\alpha,\chi} = \langle 0 | \hat{S}^\dagger \underbrace{\hat{D}^\dagger \hat{a} \hat{D}} \hat{S} |0\rangle$$

BCH: $\hat{D}^\dagger \hat{a} \hat{D} = \hat{a} - \alpha \underbrace{[\hat{a}^\dagger, \hat{a}]}_{-1} + \frac{\alpha^2}{2!} \underbrace{[\hat{a}^\dagger [\hat{a}^\dagger, \hat{a}]]}_{0}$

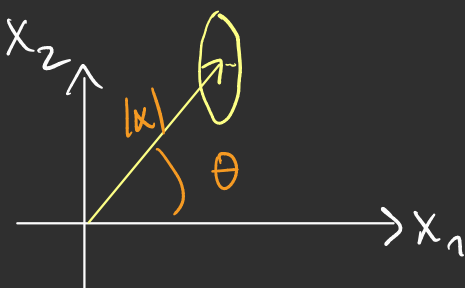
$$= \hat{a} + \alpha \mathbb{1}$$

$$\boxed{\begin{aligned} \hat{D}^\dagger \hat{a} \hat{D} &= \hat{a} + \alpha \mathbb{1} \\ \hat{D}^\dagger \hat{a}^\dagger \hat{D} &= \hat{a}^\dagger + \alpha^* \mathbb{1} \\ \hat{D}^\dagger \hat{a}^2 \hat{D} &= \hat{a}^2 + 2\alpha \hat{a} + \alpha^2 \mathbb{1} \end{aligned}}$$

$$\Rightarrow \langle \hat{a} \rangle_{\alpha,r} = \langle 0 | \hat{S}^{+} \underbrace{\underbrace{\hat{D}^{+} \hat{a} \hat{D}}_{\hat{a} + \alpha \mathbb{1}} \hat{S}}_{\left[\hat{a} \cosh r - e^{i\phi} \hat{a}^{+} \sinh r\right] + \alpha \mathbb{1}} | 0 \rangle = \alpha$$

Similarly: $\langle \hat{a}^2 \rangle_{\alpha,r} = \alpha^2 - e^{i\phi} \sinh r \cosh r$

here $\phi = 0$ (X_1 squeezed)



(time evolution?)

ASIDE

$$\hat{U} = e^{i\hat{H}} \rightarrow \hat{U}^{+} = e^{-i\hat{H}}$$

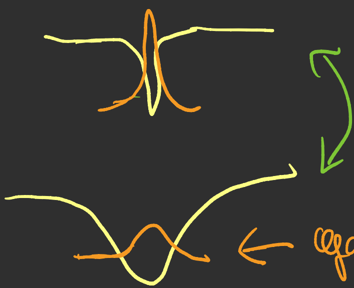
$$\hookrightarrow \hat{U}^{+}\hat{U} = \mathbb{1}$$

Friday 3-4 pm

(online)

link

$$\hat{A} = e^{\alpha\hat{a} + \alpha^{*}\hat{a}^{+}}$$



$\hat{a}$

Gaussian

$$\hat{A} = \hat{A}^{+}$$

$$\hat{A}\hat{A}^{+} \neq \mathbb{1}$$

$|\alpha\rangle$