7. Interaction of an "atom" with a quantized EM field

→ semiclassical treatment does not keep track of photons

→ cannot account for spont. emission

Hamiltonian:

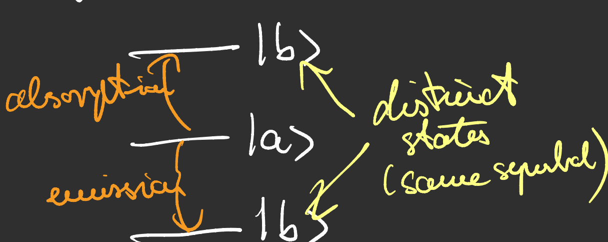
$$\hat{H}' = -\hat{\vec{d}} \cdot \hat{\vec{E}}$$ ← now an operator

$$\rightarrow \hat{\vec{E}} = i \underbrace{\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\, \hat{e}}_{\vec{\varepsilon}_0} (\hat{a} - \hat{a}^\dagger) \longrightarrow \hat{H}' = -\hat{\vec{d}}\, \vec{\varepsilon}_0 (\hat{a} - \hat{a}^\dagger)$$

Full Hamiltonian: $$\hat{H} = \hat{H}_{atom} + \underbrace{\hat{H}_{field}}_{\hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2})} + \hat{H}'$$ drop ZPE

7.1 Stimulated and spontaneous emission

initial state:



initial: $|a\rangle \otimes |n\rangle$ (photon state)

$|b\rangle \otimes |n-1\rangle$ (abs.)

$|b\rangle \otimes |n+1\rangle$ (em.)

Corresponding energies:

$$E_i = E_a + n\hbar\omega$$
$$E_{fA} = E_b + (n-1)\hbar\omega$$
$$E_{fE} = E_b + (n+1)\hbar\omega$$

→ in first-order perturbation (one-photon process)

$$|\psi(t)\rangle = c_i(t)\, e^{-iE_i t/\hbar} |i\rangle + c_{f_A}(t)\, e^{-iE_{f_A} t/\hbar} |f_A\rangle$$
$$+ c_{f_E}(t)\, e^{-iE_{f_E} t/\hbar} |f_E\rangle$$

subject to $|\psi(t=0)\rangle = |i\rangle \Leftrightarrow c_i(t=0) = 1$, others are 0

→ Insert into Schrödinger equation + project into final state amplitudes f_j, $j \in \{A, E\}$

$$\circledast \quad \dot{c}^{(1)}_{f_j} = -\frac{i}{\hbar} \langle f_j | \hat{H}' | i\rangle\, e^{i(E_{f_j} - E_i)t/\hbar}\, \underbrace{c_i^{(0)}(t)}_{=1}$$

$\langle b | \vec{d} | a\rangle$

$$\langle f_A | \hat{H}' | i\rangle = -\vec{d}_{ba}\, \vec{\mathcal{E}}_0 \langle n-1 | \hat{a} - \hat{a}^\dagger | n\rangle = -\vec{d}_{ba} \vec{\mathcal{E}}_0 \sqrt{n}$$

$$\langle f_E | \hat{H}' | i\rangle = -\vec{d}_{ba}\, \vec{\mathcal{E}}_0 \langle n+1 | \hat{a} - \hat{a}^\dagger | n\rangle = +\vec{d}_{ba} \vec{\mathcal{E}}_0 \sqrt{n+1}$$

Integrate $\circledast$, using $c^{(1)}_{f_j}(t=0) = 0$

$$\rightarrow c^{(1)}_{f_A}(t) = \frac{i}{\hbar} \vec{d}_{ba}\, \vec{\mathcal{E}}_0 \sqrt{n} \int_0^t dt' e^{i[E_b + (n-1)\hbar\omega - E_a - n\hbar\omega] t'/\hbar}$$

$$\omega_{ba} = \frac{E_b - E_a}{\hbar}$$

$$= \frac{\vec{d}_{ba}}{\hbar}\, \vec{\mathcal{E}}_0 \sqrt{n}\; \frac{e^{i(\omega_{ba} - \omega)t} - 1}{\omega_{ba} - \omega}$$

same integral $\longrightarrow$ $$c^{(1)}_{f_E}(t) = \frac{\vec{d}_{ba}}{\hbar}\underbrace{(-\vec{\varepsilon}_0)}_{=\vec{\varepsilon}_0^{\,*}}\sqrt{n+1}\;\frac{e^{i(\omega_{ba}-\omega)t}}{\omega_{ba}-\omega}$$

The final states are orthogonal, i.e. $\langle f_A | f_E\rangle = 0$

$$\Rightarrow P_b(t) = |c^{(1)}_{f_A}(t)|^2 + |c^{(1)}_{f_E}(t)|^2$$

stimulated

$$= \frac{|\vec{d}_{ba}\cdot\vec{\varepsilon}_0|^2}{\hbar^2}\left[\underbrace{n\cdot\frac{|e^{i(\omega_{ba}-\omega)t}-1|^2}{(\omega_{ba}-\omega)^2}}_{\text{absorption}} + \underbrace{(n+1)\frac{|e^{i(\omega_{ba}-\omega)t}-1|^2}{(\omega_{ba}-\omega)^2}}_{\text{emission}}\right]$$

spontaneous

The rates of emission + absorption are proportional to the absolute square of the matrix element (Fermi's Golden Rule)

$$\boxed{\frac{W_E}{W_A} = \frac{|\langle f_E|\hat{H}'|i\rangle|^2}{|\langle f_A|\hat{H}'|i\rangle|^2} = \frac{n+1}{n}}$$

→ Imbalance due to spontaneous emission

→ For $n=0$, vacuum fluctuations trigger emission while no "spontaneous absorption" is possible.

Recall: $\bar{n} = \dfrac{1}{e^{\frac{\hbar\omega}{k_B T}}-1}$ (thermal statistics of photons)

at room T

i.e. $\dfrac{1}{n} = e^{\frac{\hbar\omega}{k_B T}}-1$

$\lambda \simeq 400\,\mathrm{nm}$: $\dfrac{1}{n} \Rightarrow$ large [Spont. emission is dominant]

$$\Rightarrow \quad \frac{W_E}{W_A} = \frac{n+1}{n} \Rightarrow 1+\frac{1}{n} = e^{\frac{\hbar\omega}{k_B T}}$$

μ-wave: $\dfrac{1}{n} \to 0$

→ [spontaneous emission negligible]

exam: $\boxed{\text{Nov. 20} \quad 6^{15}\,\text{pm} - 8^{15}\,\text{pm}}$