Quantum Optics Assignment 1

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In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt
```

In [ ]:
```
### constants ###
c = 299792458
h_si = 6.626e-34  # J s
h_ev = 4.136e-15  # eV s
kB = 1.381e-23  # J / K
```

Problem 1a

In [ ]:
```
wl = 852e-9  # wavelength of cesium in nm
f = c / wl
E_joules = h_si * f  # J
E_ev = h_ev * f  # eV
T = E_joules / kB

print("Problem 1a")
print(f"Photon energy: {E_joules:.03} J or {E_ev:.03} eV")
print(f"Photon frequency: {f*1e-12:.04n} THz")
print(f"Temperature: {T:.03} K")
```

```
Problem 1a
Photon energy: 2.33e-19 J or 1.46 eV
Photon frequency: 351.9 THz
Temperature: 1.69e+04 K
```

Problem 1b

In [ ]:
```
wl = 589e-9  # wavelength of sodium in nm
u = 1.66e-27  # kg
m = 22.99 * u

f = c / wl
E = f * h_si
v = np.sqrt(2 * E / m)
print(f"Velocity: {v:.03} m/s")
```

```
Velocity: 4.2e+03 m/s
```

Problem 1c

In [ ]:
```
p = h_si / wl
v = p / m
E = 1/2 * m * v**2
print(f"Recoil velocity: {v:.03} m/s")
```

```
print(f"Recoil energy: {E:.03} J or {E/h_si*1e12} THz")
print(f"Recoil temperature: {E/kB*1e6:.03n} uK")
```

```
Recoil velocity: 0.0295 m/s
Recoil energy: 1.66e-29 J or 2.5023254157294932e+16 THz
Recoil temperature: 1.2 uK
```

Problem 2a

Functions of $\mathbf{r}$ commute with $\mathbf{r}$, and likewise with $\mathbf{p}$.

$$[\mathbf{r}, \mathbf{A}(\mathbf{r}, t)] = 0$$
$$[\mathbf{r}, U(\mathbf{r}, t)] = 0$$

Problem 2b

$$H = \frac{1}{2m} \sum_{i=x,y,z} \left[p_i^2 + q^2 A_i^2 (\mathbf{r}, t) - q p_i A_i (\mathbf{r}, t) - q A_i (\mathbf{r}, t) p_i\right] + qU (\mathbf{r}, t)$$

$$\begin{aligned}
[x, H] &= \frac{1}{2m} \sum_{i=x,y,z} \left\{ \left[x, p_i^2\right] - q \left[x, p_i A_i (\mathbf{r}, t)\right] - q \left[x, A_i (\mathbf{r}, t) p_i\right] \right\} \\
&= \frac{1}{2m} \{p_x [x, p_x] + [x, p_x] p_x - q [x, p_x] A_x (\mathbf{r}, t) - q A_i (\mathbf{r}, t) [x, p_x]\} \\
&= \frac{1}{2m} \{2i\hbar p_x - 2qi\hbar A_x (\mathbf{r}, t)\} \\
&= \frac{i\hbar}{m} \{p_x - q A_x (\mathbf{r}, t)\}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \langle x \rangle &= \frac{1}{i\hbar} \langle [x, H] \rangle + \left\langle \frac{\partial}{\partial t} x \right\rangle \\
&= \frac{1}{m} \langle p_x - q A_x (\mathbf{r}, t) \rangle
\end{aligned}$$

Similarly for y and z,

$$\frac{d}{dt} \langle y \rangle = \frac{1}{m} \langle p_y - q A_y (\mathbf{r}, t) \rangle$$
$$\frac{d}{dt} \langle z \rangle = \frac{1}{m} \langle p_z - q A_z (\mathbf{r}, t) \rangle$$

The velocity is therefore

$$\langle \mathbf{v} \rangle = \frac{1}{m} \langle \mathbf{p} - q \mathbf{A} (\mathbf{r}, t) \rangle$$

Problem 2c

$$
\begin{aligned}
\frac{d}{dt}\langle \mathbf{v}\rangle &= \frac{1}{i\hbar}\langle[\mathbf{v}, H]\rangle + \left\langle \frac{\partial}{\partial t}\mathbf{v}\right\rangle \\
&= \frac{1}{i\hbar}\langle[\mathbf{v}, qU(\mathbf{r},t)]\rangle - \frac{q}{m}\left\langle \frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t)\right\rangle \\
&= \frac{1}{i\hbar}\left\langle\left[\frac{\mathbf{p}}{m}, qU(\mathbf{r},t)\right]\right\rangle - \frac{q}{m}\left\langle \frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t)\right\rangle
\end{aligned}
$$

Using $[p, f(x)] = -i\hbar\partial_x f(x)$, this is

$$
\begin{aligned}
\frac{d}{dt}\langle \mathbf{v}\rangle &= \frac{q}{m}\frac{1}{i\hbar}\langle -i\hbar\nabla U(\mathbf{r},t)\rangle - \frac{q}{m}\left\langle \frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t)\right\rangle \\
&= -\frac{q}{m}\langle \nabla U(\mathbf{r},t)\rangle - \frac{q}{m}\left\langle \frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t)\right\rangle
\end{aligned}
$$

According to physics stackexchange, $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$. This means that our force $m\frac{d}{dt}\langle \mathbf{v}\rangle$ has a clear classical analogue.

$$
m\mathbf{a} = -q\nabla U(\mathbf{r},t) + q\mathbf{E}
$$

Problem 3a

$$
\begin{aligned}
|\psi\rangle &= \begin{pmatrix} \sqrt{3/4} \\ -i/2 \end{pmatrix} \\
\langle\psi| &= \begin{pmatrix} \sqrt{3/4} & i/2 \end{pmatrix} \\
\rho_{\text{pure}} = |\psi\rangle\langle\psi| &= \begin{pmatrix} 3/4 & i\sqrt{3}/4 \\ -i\sqrt{3}/4 & 1/4 \end{pmatrix}
\end{aligned}
$$

Problem 3b

$$
\begin{aligned}
\rho_1 &= \frac{3}{4}|g\rangle\langle g| = \frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
\rho_2 &= \frac{1}{4}|e\rangle\langle e| = \frac{1}{4}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

$$
\rho_{\text{impure}} = \rho_1 + \rho_2 = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}
$$

Problem 3c

In the pure ensemble, all atoms are in a definite state $|\psi\rangle$. In the impure ensemble, a proportion of atoms are in each state. Therefore, if we could devise an experiment to determine the likelihood of an atom being in $|\psi\rangle$, we would get different answers for the two ensembles. For (a), we would get unity. For (b), we would get an answer based on the projection of the component states onto $|\psi\rangle$. Let $A = |\psi\rangle\langle\psi|$ be the observable that we can measure.

$$\langle A\rangle_{(\text{pure})} = \text{Tr}\left[\rho_{\text{pure}} A\right] = \text{Tr}\left[\rho_{\text{pure}}^2\right] = 1$$

$$\langle A\rangle_{(\text{impure})} = \text{Tr}\left[\rho_{\text{impure}} A\right] = \text{Tr}\left[\begin{pmatrix} (3/4)^2 & i3^{3/2}/16 \\ -i\sqrt{3}/16 & 1/16 \end{pmatrix}\right] = 10/16$$

Problem 3d

If we use a basis in which ρ is diagonal, the entropy is simply

$$S = -k_B \text{Tr}\left[\rho \ln \rho\right] = -k_B \sum_k \rho_k^{(\text{diag})} \ln \rho_k^{(\text{diag})}.$$

For the pure state, the eigenvalues are 1 and 0.

$$S_{\text{pure}} = -k_B \left[1 \cdot \ln 1 + \lim_{x\to 0^+} x \ln x\right].$$

We can evaluate the limit using L'hopital's rule. Note that we have to write it as a fraction.

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} -\frac{1/x}{1/x^2} = \lim_{x\to 0^+} -x = 0$$

So the entropy of a pure state is zero!

$$S_{\text{pure}} = 0$$

For the mixed state,

$$S_{\text{impure}} = -k_B \left[3/4 \cdot \ln 3/4 + 1/4 \cdot \ln 1/4\right] \approx 0.562 k_B$$

In [ ]:
```
S = -(3/4 * np.log(3/4) + 1/4 * np.log(1/4))
print(f"Entropy: {S:.03} kB")
```

```
Entropy: 0.562 kB
```

Problem 3e

In this case, we only consider two states $|g\rangle$ and $|e\rangle$, with energy 0 and 1.

$$\rho = \frac{1}{Z}\Big(|g\rangle\langle g| + e^{-1/k_BT}\,|e\rangle\langle e|\Big)$$

At $T = 0$, $e^{-1/k_BT} \to 0$ and the population is purely in the ground state. As $T \to \infty$, $e^{-1/k_BT} \to 1$ and the density matrix is a statistical mixture of ground and excited.

$$\rho_{T\to\infty} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Appendices

Commutator of momentum and a position-dependent function

$$\begin{aligned} \langle x|\,[\hat{p}, f(\hat{x})]\,|\psi\rangle &= \langle x|\,(\hat{p}f(\hat{x}) - f(\hat{x})\hat{p})\,|\psi\rangle \\ &= -i\hbar\partial_x\left(f(x)\psi(x)\right) - i\hbar f(x)\partial_x\psi(x) \\ &= -i\hbar\left(\partial_x f(x)\right)\psi(x) \end{aligned}$$