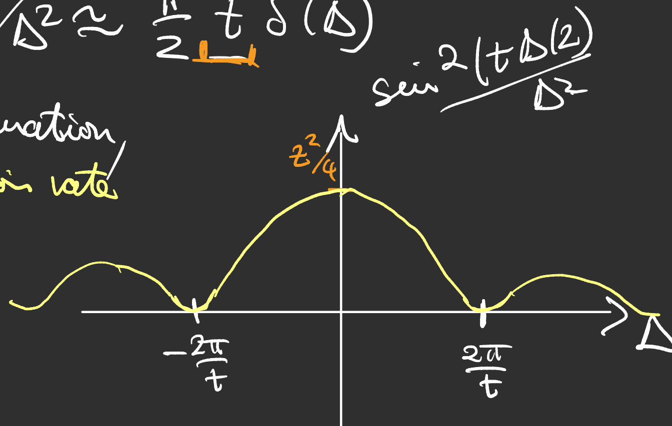
$$P^{(1)}_{i\to m}(t) = \frac{|\vec{d}_{mi}\cdot\vec{E}_0|^2}{\hbar^2}\,\frac{\sin^2(t\Delta/2)}{\Delta^2}$$

For $t \gg \frac{2\pi}{|\Delta|}$:

$$\sin^2(t\Delta/2)/\Delta^2 \simeq \frac{\pi}{2}\,t\,\delta(\Delta)$$

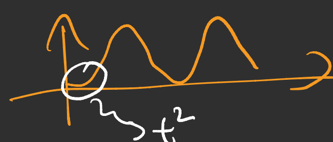
Under this approximation we obtain a transition rate



$$\boxed{W_{i\to m}(t) = \frac{P_{i\to m}(t)}{t} = \frac{\pi}{2}\,\frac{|\vec{d}_{mi}\cdot\vec{E}_0|^2}{\hbar^2}\,\delta(\omega-\omega_{mi})}$$

(Fermi's Golden Rule)

→ typically applies to discrete → continuum transition

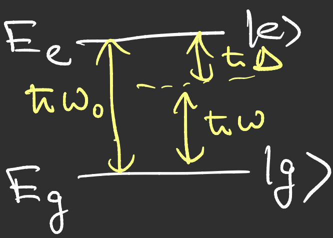


→ Comment: energy is conserved for $t \gg \frac{\hbar}{|\Delta E|}$

→ Comment 2: applies to time-independent perturbations as well: $\delta(E_f - E_i)$

6.3 Rabi model

→ one of the rare examples of an exactly solvable model



$\omega_0 = \frac{E_e - E_g}{\hbar}$ ... transition freq.

$\Delta = \omega_0 - \omega$ detuning, $\omega, \omega_0 > 0$

$$\hat{H}'(t) = -\hat{\vec{d}} \cdot \vec{E}_0 \cos(\omega t) = \hat{V}_0 \cos(\omega t)$$

Express general atomic state as a superposition of $\{|g\rangle, |e\rangle\}$

$$|\psi(t)\rangle = c_g(t)\, e^{-i\omega_g t} |g\rangle + c_e(t)\, e^{-i\omega_e t} |e\rangle$$

$$c_g(0) = 1, \quad c_e(0) = 0$$

Insert into TDSE: $i\partial_t |\psi(t)\rangle = [\hat{H}_0 + \hat{H}'(t)] |\psi(t)\rangle$
and project onto state $\langle g|$ and $\langle e|$

→ We assume atomic states to be of definite parity:

$$\langle g|\hat{d}|g\rangle = \langle e|\hat{d}|e\rangle = 0$$

and define $V = \langle e|\hat{V}_0|g\rangle = -\vec{d}_{eg} \cdot \vec{E}_0$

→ We can assume V as real: $V = V^*$

→ Equations of motion:

$$\dot{c}_g = -\frac{i}{\hbar} \mathcal{V} \cos(\omega t)\, e^{-i\omega_0 t}\, c_e$$

$$\dot{c}_e = -\frac{i}{\hbar} \mathcal{V} \cos(\omega t)\, e^{i\omega_0 t}\, c_g$$

$$e^{i(\omega_0 \pm \omega)t}$$

RWA:

$$\boxed{\begin{aligned} \dot{c}_g &= -\frac{i}{2\hbar} \mathcal{V}\, e^{i(\omega-\omega_0)t}\, c_e \\ \dot{c}_e &= -\frac{i}{2\hbar} \mathcal{V}\, e^{-i(\omega-\omega_0)t}\, c_g \end{aligned}}$$

→ form an equivalent 2^{nd}-order ODE:

$$\ddot{c}_e \cancel{e^{i(\omega-\omega_0)t}} + i(\omega-\omega_0)\, c_e \cancel{e^{i(\omega-\omega_0)t}} = -\frac{i}{2\hbar}\dot{c}_g$$

$$= -\frac{\mathcal{V}^2}{(2\hbar)^2} \cancel{e^{i(\omega-\omega_0)t}}\, c_e$$

$$\boxed{\ddot{c}_e + i(\omega-\omega_0)\,\dot{c}_e + \frac{1}{4}\frac{\mathcal{V}^2}{\hbar^2} c_e = 0}$$

→ Try solutions of type $c_e(t) = A_+ e^{i\lambda_+ t} + A_- e^{i\lambda_- t}$

from initial conditions

$$-\lambda^2 - (\omega-\omega_0)\lambda + \frac{1}{4}\frac{\upsilon^2}{t_0^2} = 0$$

$$\Leftrightarrow \lambda = \frac{\underbrace{(\omega-\omega_0)}_{-\Delta} \pm \sqrt{\underbrace{(\omega-\omega_0)}_{-\Delta}^2 + \upsilon^2/t_0^2}}{-2}$$

$$\lambda_\pm = \frac{1}{2}\left(\Delta \pm \sqrt{\Delta^2 + \upsilon^2/t_0^2}\right)$$

We define : $\boxed{\Omega_R = \sqrt{\Delta^2 + \frac{\upsilon^2}{t_0^2}}}$ generalized Rabi frequency

$\Delta = \sqrt{3}\,\frac{\upsilon}{t_0}$

→ with initial conditions

$$c_e(t) = i\frac{\upsilon}{\Omega_R t_0} e^{it\Delta/2} \sin\left(\frac{\Omega_R t}{2}\right)$$

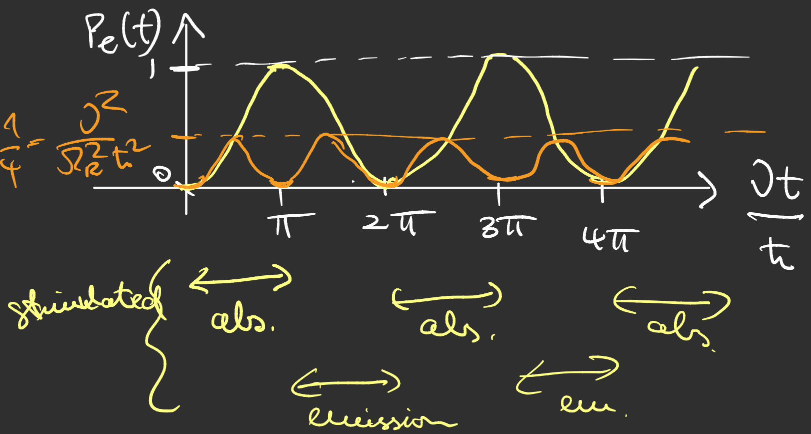
The probability that the atom is in the excited state is:

$$P_e(t) = |c_e(t)|^2 = \frac{\upsilon^2}{\Omega_R^2 t_0^2} \sin^2\left(\frac{\Omega_R t}{2}\right) \xrightarrow[\Delta=0]{\text{resonance}} \sin^2\left(\frac{\upsilon t}{2t_0}\right)$$

transfer is reduced by detuning

Rabi oscillations

$\Delta = 0$



$\Delta = \sqrt{3}\ \Omega/\hbar$

$\Omega_R \rightarrow 2\Omega_R$ compared to $\Delta = 0$

Rabi frequency is not applied frequency, but rather a measure of the coupling strength of the EM field with the "atom" (2-level system)

Typical values of Ω_R

$1\,eV \sim 2.4 \cdot 10^{14}\,Hz$

* In InGaAs QD (quantumdots)

$\hbar\Omega_R \sim 100\,\mu eV$ at $3.8\,K$, cw $\sim$ PRL 87, 246401 (2001)

$\hookrightarrow \frac{\Omega_R}{2\pi} \sim 24\,GHz$ (microwave)

$I \simeq 25\,\frac{kW}{cm^2}$

$|e\rangle = |X^*\rangle$

$\updownarrow 20.2\,meV$

$|g\rangle = |X\rangle$

* transmon qubit $\sim$ PRX 10, 031032 (2020)

transition $\sim 7.8\,GHz$, typical Rabi freq. $\frac{\Omega_R}{2\pi} \sim 1\,MHz$