Assignment 4

November 20, 2023

Problem 1

Problem 1c

In the $R_1, R_{2,}R_3$ representation,

$$\rho = \frac{1}{2}\begin{pmatrix} 1+R_3 & R_1 - iR_2 \\ R_1 + R_2 & 1 - R_3 \end{pmatrix} \tag{1}$$

So we have the initial condition

$$\rho(0) = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \tag{2}$$

Problem 2

Problem 3

Unitary transformation

Hamiltonians

The Hamiltonians are

$$H = H_0 + H_L \tag{3}$$

$$H_0 = \hbar\omega_0\sigma^\dagger\sigma \tag{4}$$

$$H_L = -\frac{\hbar\Omega_R}{2}\left(e^{i\Delta t}\sigma + e^{-i\Delta t}\sigma^\dagger\right) \tag{5}$$

Unitary operator

The unitary operator is

$$U = e^{i\delta t\sigma^\dagger\sigma} = \sum_n \frac{(i\delta t)^n}{n!}\sigma^\dagger\sigma \tag{6}$$

Transformation

The transformations are

$$\tilde{H} = UHU^\dagger + i\hbar\frac{\partial U}{\partial t}U^\dagger \tag{7}$$

$$i\hbar\frac{\partial U}{\partial t}U^\dagger = i\hbar\left(i\delta\sigma^\dagger\sigma\right) = -\hbar\delta\sigma^\dagger\sigma \tag{8}$$

The transformed unperturbed Hamiltonian is

$$UH_0U^\dagger = \hbar\left(\omega_0 - \delta\right)\sigma^\dagger\sigma \tag{9}$$

$$\frac{UH_LU^\dagger}{-\hbar\Omega_R/2} = e^{i\Delta t}U\sigma U^\dagger + \text{H.c.} \tag{10}$$

We have

$$U\sigma U^\dagger = U\sigma\left(\sum_n \frac{(-i\delta t)^n}{n!}\sigma^\dagger\sigma\right) \tag{11}$$

We can use

$$\left(\sigma\sigma^\dagger\right)\sigma = \left(1+\sigma^\dagger\sigma\right)\sigma \tag{12}$$

to obtain

$$\begin{aligned} U\sigma U^\dagger &= U\left(\sum_n \frac{(-i\delta t)^n}{n!}\left(1+\sigma^\dagger\sigma\right)\right)\sigma && (13)\\ &= U\left(U^\dagger e^{-i\delta t}\right)\sigma && (14)\\ &= e^{-i\delta t}\sigma && (15)\end{aligned}$$

The transformed driving Hamiltonian is

$$\frac{UH_LU^\dagger}{-\hbar\Omega_R/2} = e^{-i(\Delta-\delta)t}\sigma + \text{H.c.} \tag{16}$$

so that the total transformed Hamiltonian is

$$\tilde{H} = -\hbar\delta\sigma^\dagger\sigma - \frac{\hbar\Omega_R}{2}\left(e^{-i(\Delta-\delta)t}\sigma + \text{H.c.}\right) \tag{17}$$

If we choose $\delta = \Delta$,

$$\tilde{H} = -\hbar\Delta\sigma^\dagger\sigma - \frac{\hbar\Omega_R}{2}\left(\sigma+\sigma^\dagger\right) \tag{18}$$