Quantum light

Photon properties Energy:$E = \hbar\omega = hf \to \frac{1}{2}mv^2, k_B T$, momentum: $p = h/\lambda$

Single-photon Rabi freq $g = \mu\sqrt{\frac{\omega}{2\hbar\epsilon_0 V}}$.

Quantized field amplitude $E_0 = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}$.

Spontaneous emission The ratio of spontaneous emission (A) and stimulated emission (B) is $\frac{A}{B} \approx e^{\frac{\hbar\omega}{k_B T}}$.

Variances For a field operator $E(\chi) = E_0\left(ae^{-i\chi} + \text{H.c.}\right)$,

- $\langle E\rangle_n = 0$
- $\langle E^2\rangle_n = (\Delta E)_n^2 = 2E_0^2(n+1)$
- $\langle E\rangle_\alpha = E_0^2\left(\alpha^2 e^{-2i\chi} + \alpha^{*2}e^{2i\chi} + 2|\alpha|^2\right)$
- $\langle E^2\rangle_\alpha = E_0^2\left(\alpha^2 e^{-2i\chi} + 1 + 2|\alpha|^2 + \alpha^{*2}e^{2i\chi}\right)$
- $(\Delta E)_\alpha^2 = E_0^2$

Von-Neumann Entroyp $S = -k_B\sum_k \rho_k^{(\text{diag})}\ln\rho_k^{(\text{diag})}$ where $\rho_k^{(\text{diag})}$ is diagonalized.

Pure entropy $S = -k_B \text{Tr}\left[\rho\ln\rho\right] = 0$.

Mixed entropy $S = -k_B \text{Tr}\left[\lambda_k \ln\lambda_k\right] > 0$.

Variance $(|n\rangle)$ $\left\langle(\Delta n)^2\right\rangle = 0$.

Variance $(|\alpha\rangle)$ $\left\langle(\Delta n)^2\right\rangle = |\alpha|^2 = \overline{n}$.

Autocorrelation $(|n\rangle)$ $g^{(2)}(\tau) = 1 - \frac{1}{n}$.

Autocorrelation $(|\alpha\rangle)$ $g^2(\tau) = 1$.

Autocorrelation (single mode)

$$g^{(2)}(\tau) = \frac{\left\langle(\Delta n)^2\right\rangle + \langle n\rangle^2 - \langle n\rangle}{\langle n\rangle^2}$$

Coherent state $a|\alpha\rangle = \alpha|\alpha\rangle$,

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

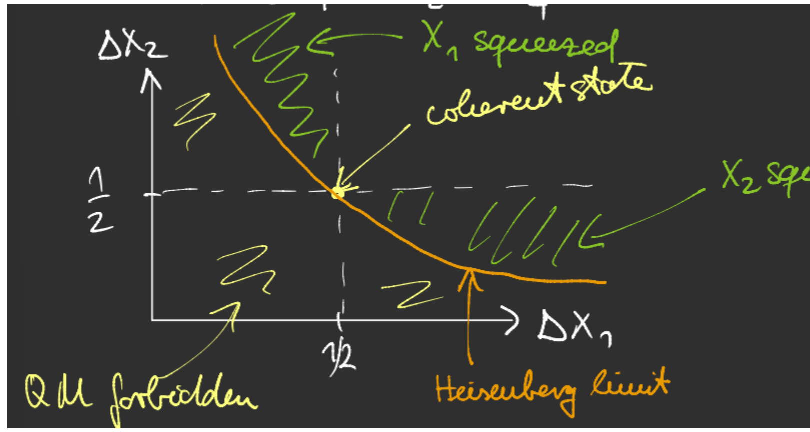
$$\alpha = e^{-\frac{|\alpha|^2}{2}}e^{\alpha a^\dagger}e^{-\alpha^* a}|0\rangle.$$

Completeness $\mathbb{I} = \frac{1}{\pi}\int d^2\alpha\,|\alpha\rangle\langle\alpha|$.

Overcompleteness $\langle\alpha|\beta\rangle \neq \delta^{(2)}(\alpha - \beta)$.

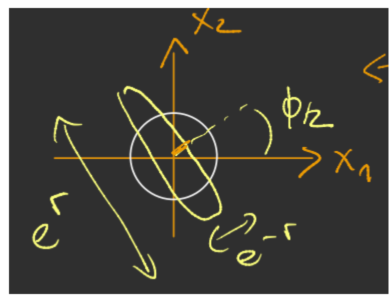
P-representation $P_\gamma(\alpha) = \delta^{(2)}(\alpha - \gamma)$.

Squeezed state State $|\psi\rangle$ is squeezed when $(\Delta A)_\psi = (A - \langle A\rangle)_\psi \le \sqrt{\frac{1}{2}\langle C\rangle_\psi}$ where $[A, B] = iC$ and $\left\langle(\Delta A)^2\right\rangle\left\langle(\Delta B)^2\right\rangle \ge \frac{1}{4}|\langle C\rangle|^2$.



Quadrature $x_1 = \frac{1}{2}\left(a + a^\dagger\right)$, $x_2 = \frac{1}{2i}\left(a - a^\dagger\right)$.

Squeezing operator $S(x) = \exp\left[\frac{1}{2}\left(x^* a^2 - x a^{\dagger 2}\right)\right]$ where $x = re^{i\varphi}$. r is "squeezing parameter".



1 Representations

P-representation $\rho = \int P(\alpha)|\alpha\rangle\langle\alpha|\,d^2\alpha$, where

$$P(\alpha) = \frac{e^{|\alpha|^2}}{\pi^2}\int e^{|u|^2}\langle -u|\rho|u\rangle e^{u^*\alpha - u\alpha^*}d^2u$$

Note: $P(\alpha) < 0$ implies nonclassicality.

Wigner

$$W(q,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\left\langle q + \frac{x}{2}\right|\rho\left|q - \frac{x}{2}\right\rangle e^{ipx/\hbar}dx$$

Q-function

$$Q_\rho(\alpha) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle \tag{1}$$

Optical-Equivalence theorem For a normal ordered function $G(a, a^\dagger)$,

$$\left\langle G\left(a, a^\dagger\right)\right\rangle = \langle G(\alpha, \alpha^*)\rangle.$$

Q-function (Fock)

$$Q_n(\alpha) = \frac{1}{\pi}\left|\langle\,\alpha|\,n\rangle\right|^2 = \frac{1}{\pi}\exp\left(-\frac{|\alpha|^2}{2}\right)\frac{|\alpha|^{2n}}{\sqrt{n!}} \tag{2}$$

Q-function (coherent)

$$Q_\beta\left(\alpha\right) = \frac{1}{\pi}\exp\left(-\frac{|\alpha|^2+|\beta|^2}{2}\right)\sum_{n,m}\frac{\alpha^n\beta^m}{\sqrt{n!m!}}\delta_{nm} \tag{3}$$
$$= \frac{1}{\pi}\exp\left(-\frac{|\alpha|^2+|\beta|^2}{2}\right)\left(\alpha\beta\right)^n \tag{4}$$

Optical Bloch equations

Regular representation

$$\begin{aligned}
\dot\rho_{ee} &= -\Gamma\rho_{ee} + i\frac{\Omega}{2}\left(\rho_{eg}-\rho_{ge}\right)\\
\dot\rho_{gg} &= 1-\rho_{ee}\\
\dot\rho_{eg} &= -\left(\gamma_\perp - i\Delta\right)\rho_{eg} + i\frac{\Omega}{2}\left(\rho_{ee}-\rho_{gg}\right)\\
\dot\rho_{ge} &= -\left(\gamma_\perp + i\Delta\right)\rho_{ge} - i\frac{\Omega}{2}\left(\rho_{ee}-\rho_{gg}\right).
\end{aligned}$$

Bloch-vector representation

$$\begin{aligned}
\partial_t\left\langle\sigma_x\right\rangle &= \Delta\left\langle\sigma_y\right\rangle - \gamma_\perp\left\langle\sigma_x\right\rangle\\
\partial_t\left\langle\sigma_y\right\rangle &= -\Delta\left\langle\sigma_x\right\rangle - \Omega\left\langle\sigma_z\right\rangle - \gamma_\perp\left\langle\sigma_y\right\rangle\\
\partial_t\left\langle\sigma_z\right\rangle &= \Omega\left\langle\sigma_y\right\rangle - \Gamma\left(\left\langle\sigma_z\right\rangle+1\right)
\end{aligned}$$

where

$$\begin{aligned}
\left\langle\sigma_x\right\rangle &= \rho_{eg}+\rho_{ge}\\
\left\langle\sigma_y\right\rangle &= i\left(\rho_{eg}-\rho_{ge}\right)\\
\left\langle\sigma_z\right\rangle &= \rho_{ee}-\rho_{gg}
\end{aligned}$$

Example: no drive If we take $\Delta = 0$ and $\Omega = 0$,

$$\partial_t\left\langle\sigma_x\right\rangle = -\gamma_\perp\left\langle\sigma_x\right\rangle \tag{5}$$
$$\partial_t\left\langle\sigma_y\right\rangle = -\gamma_\perp\left\langle\sigma_y\right\rangle \tag{6}$$
$$\partial_t\left\langle\sigma_z\right\rangle = -\Gamma\left(\left\langle\sigma_z\right\rangle+1\right) \tag{7}$$

with initial condition $\left\langle\sigma_z\right\rangle = 1$, we have

$$\left\langle\sigma_z\right\rangle = 2e^{-\Gamma t}-1. \tag{8}$$

Atom-field interactions

Rotating-wave-approximation RWA is safe when $\Delta \ll \omega, \omega_0$ and $\Omega_0 \ll \omega_0$.

Rabi excited state

$$\rho_{ee}\left(t\right) = \frac{\nu^2}{\Omega_R^2\hbar^2}\sin^2\left(\frac{\Omega_R t}{2}\right) \xrightarrow{\Delta=0} \sin^2\left(\frac{\nu t}{2\hbar}\right) \tag{9}$$

π-pulse $|g\rangle \to |e\rangle$, $\Delta = 0$, $T_\pi = \frac{1}{2}\left(\frac{2\pi}{\Omega_R}\right)$.

$\frac{\pi}{2}$-pulse $|g\rangle \to \frac{1}{\sqrt{2}}\left(|g\rangle + |e\rangle\right)$,$T_{\pi/2} = \frac{1}{4}\left(\frac{2\pi}{\Omega_R}\right)$.

Linewidth For a field that decays as

$$E(t) = E_0\Theta(t)e^{-\frac{\Gamma}{2}t}e^{i\omega_0 t},$$

We can take a Fourier transform,

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\omega t}f(t)dt$$

to obtain

$$E(\omega) = -\frac{E_0}{\sqrt{2\pi}}\frac{1}{i\left(\omega_0-\omega\right)-\frac{\Gamma}{2}}$$

so that

$$I\left(\omega\right) = \left|E(\omega)\right|^2 = \frac{E_0^2}{2\pi}\frac{1}{\left(\omega_0-\omega\right)^2-\left(\frac{\Gamma}{2}\right)^2}.$$

Other

	$N=1$	$N\geq 1$
no RWA	Quantum Rabi	Dicke
RWA	Jaynes-Cummings	Tavis-Cummings

Baker-Campbell-Hausdorff $e^x e^y = e^z$ where

$$z = x + y + \frac{1}{2}\left[x,y\right] + \frac{1}{12}\left(\left[x,\left[x,y\right]\right] + \left[y,\left[y,x\right]\right]\right) + \cdots$$

One-atom maser (a) Rydberg states are strongly coupled to the radiation field. (b) Rydberg states have transitions to neighboring levels in the region of millimeter waves, allowing the use of cavities with large low-order modes for long interaction times. (c) Rydberg atoms have long spontaneous lifetimes. They measured a linewidth broadening due to the cavity interaction.

Vacuum fluctuations (a) The question is whether vacuum fluctuations an be measured in fre space. (b) The idea is to measure fluctuations via changes of the refractive index of a material. (c) The authors vary mode width and temporal length to measure fluctuations. (d) The problem with shot noise is that it is an additional fluctuation source.

Jaynes-Cummings For a Hamiltonian $H = \begin{pmatrix}\Delta/2 & J\\ J & -\Delta/2\end{pmatrix}$, the eigenstates are $|e\rangle = \begin{pmatrix}1\\1\end{pmatrix}$, $|g\rangle = \begin{pmatrix}0\\0\end{pmatrix}$,$|+\rangle = \cos\theta\begin{pmatrix}1\\0\end{pmatrix} + \sin\theta\begin{pmatrix}0\\1\end{pmatrix}$,$|-\rangle = \cos\theta\begin{pmatrix}0\\1\end{pmatrix} - \sin\theta\begin{pmatrix}1\\0\end{pmatrix}$ where $E_{e,g} = 0$, $E_\pm = \pm\frac{\Delta}{2}$. Also, $\tan\theta = \frac{2J}{\Delta+\Delta'}$ where $\Delta' = \sqrt{\Delta^2+4J^2}$.

2x2 matrices The eigenvalues of a 2x2 are $m \pm \sqrt{m^2-p}$ where m is the mean of the diagonal elements and p is the determinant.