Problem Set 1, Ex. 3

a) $|\psi\rangle = \frac{\sqrt{3}}{2}|g\rangle - i\frac{1}{2}|e\rangle \hat{=} \begin{pmatrix} \sqrt{3}/2 \\ -i/2 \end{pmatrix}$

$$\hat{\rho}_1 = |\psi\rangle\langle\psi| = \begin{pmatrix} \sqrt{3}/2 \\ -i/2 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & i/2 \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & \sqrt{3}\,i/4 \\ -i\frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}$$

b) $\hat{\rho}_2 = \sum_k p_k |\psi_k\rangle\langle\psi_k| = \frac{3}{4}|g\rangle\langle g| + \frac{1}{4}|e\rangle\langle e|$

$$\hat{=} \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}$$

c) measure off-diagonal elements, e.g. $\langle \sigma_y \rangle$

d) $S = -k_B \mathrm{Tr}(\hat{\rho} \log \hat{\rho})$

$\hat{\rho}$ is hermitian

↳ $\lambda_1 = 0, 1$ eigenvalues $\Rightarrow S_1 = 0$

$\rightarrow S_2 = -k_B \left(\frac{3}{4}\log\frac{3}{4} + \frac{1}{4}\log\frac{1}{4}\right)$

e) $\hat{\rho} = \frac{1}{Z} \sum_i e^{-\frac{E_i}{k_B T}} |i\rangle\langle i|$

$T=0 : \hat{\rho} \to |i_0\rangle\langle i_0|$ with E_{i_0} minimal

$\hookrightarrow S = 0$

$T=\infty : \hat{\rho} = \frac{1}{N} \sum_i |i\rangle\langle i| \to S = -k_B \sum_{i=1}^{N} \frac{1}{N} \log \frac{1}{N}$

$= k_B \log N$