SWI6 Formulation MODFLOW6

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Introduction

Consider Dupuit interface flow in a coastal aquifer. The density of the freshwater is ρ_f and the density of the saltwater is ρ_s. The freshwater head in the fresh water is h_f and the saltwater head in the salt water is h_s. The freshwater and saltwater are separated by an interface. The interface elevation is indicated with ζ.

In the following, equations are presented for three cases:

1. Steady interface flow in a single layer.
2. Transient flow in a single layer where h_s (the saltwater head in the saltwater) is constant and equal to sealevel.
3. Transient flow in a single layer where h_s varies and is a result from a water balance of the saltwater.

It is suggested to follow this same sequence with the implementation using the MODFLOW API.

1 Steady interface flow in a single layer

A steady water balance for the freshwater zone results in the differential equation for steady interface flow in a single layer:

$$\frac{\partial}{\partial x}\left(kb_f\frac{\partial h_f}{\partial x}\right)+\frac{\partial}{\partial x}\left(kb_f\frac{\partial h_f}{\partial y}\right)=-N_f \tag{1}$$

where k is the hydraulic conductivity, b_f is the freshwater thickness, and N_f is the recharge to the freshwater zone. The differential equation is the same as for regular (non-density) flow in MODFLOW, with the exception of the equation for the freshwater thickness. The freshwater thickness is computed as the difference between the top of the freshwater zone $b_{f,t}$ and the bottom of the freshwater zone $b_{f,b}$

$$b_f = b_{f,t} - b_{f,b} \tag{2}$$

where

$$b_{f,t} = \begin{cases} z_t & \text{if} \quad h_f > z_t \\ h_f & \text{if} \quad z_b \le h_f \le z_t \\ z_b & \text{if} \quad h_f < z_b \end{cases} \tag{3}$$

and

$$b_{f,b} = \begin{cases} z_t & \text{if} \quad \zeta > z_t \\ \zeta & \text{if} \quad z_b \le \zeta \le z_t \\ z_b & \text{if} \quad \zeta < z_b \end{cases} \tag{4}$$

where z_t is the elevation of the top of the aquifer and z_b is the elevation of the bottom of the aquifer. The elevation ζ of the interface is computed from the Ghyben-Herzberg equation as

$$\zeta = -\frac{\rho_f}{\rho_s - \rho_f}\left(h_f - \frac{\rho_s}{\rho_f}h_s\right) \tag{5}$$

where h_s is sealevel. This equation for the interface is written as

$$\zeta = \alpha_s h_s - \alpha_f h_f \tag{6}$$

where α_f is

$$\alpha_f = \frac{\rho_f}{\rho_s - \rho_f} \tag{7}$$

and α_s is

$$\alpha_s = \frac{\rho_s}{\rho_s - \rho_f} \tag{8}$$

The MODFLOW API will be used to implement equations (2, 3, 4) for the saturated thickness with equation (6) for ζ. Upstream weighing will be used in combination with Newton's method to obtain a solution.

2 Transient interface flow in a single layer with constant h_s

A transient water balance for the freshwater zone results in the differential equation for transient interface flow in a single layer is

$$\frac{\partial}{\partial x}\left(kb_f\frac{\partial h_f}{\partial x}\right) + \frac{\partial}{\partial x}\left(kb_f\frac{\partial h_f}{\partial y}\right) = S_f\frac{\partial h_f}{\partial t} - S_\zeta\frac{\partial \zeta}{\partial t} - N_f \tag{9}$$

where the freshwater thickness b_f is computed as before, and S_f and S_ζ are storage coefficients. The first term on the right-hand side represents an increase in storage caused by an increase in the freshwater head. The second term on the right-hand side reprsents an increase in storage caused by movement of the interface. The storage coefficient S_f represents elastic storage only when the freshwater head is above the top of the aquifer, and represents both elastic and phreatic storage when there is a groundwater table in the aquifer:

$$S_f = \begin{cases} S_e b_f & \text{if} \quad h_f > z_t \\ \phi + S_e b_f & \text{if} \quad z_b \le h_f \le z_t \\ 0 & \text{if} \quad h_f < z_b \end{cases} \tag{10}$$

where ϕ is the phreatic storage coefficient (the effective porosity or specific yield), and S_e is the specific (elastic) storage coefficient (the more common S_s is not used as the subscript $_s$ is already used for saltwater). The storage coefficient S_ζ is equal to the phreatic storage when the interface is between the top and bottom of the aquifer:

$$S_\zeta = \begin{cases} 0 & \text{if} \quad \zeta > z_t \\ \phi & \text{if} \quad z_b \le \zeta \le z_t \\ 0 & \text{if} \quad h_f < z_b \end{cases} \tag{11}$$

Substitution of the equation for the elevation of the interface (6) for ζ in (9) gives

$$\frac{\partial}{\partial x}\left(kb_f\frac{\partial h_f}{\partial x}\right) + \frac{\partial}{\partial x}\left(kb_f\frac{\partial h_f}{\partial y}\right) = (S_f + \alpha_f S_\zeta)\frac{\partial h_f}{\partial t} - N_f \tag{12}$$

As before, the MODFLOW API will be used to implement equations (2, 3, 4) for the saturated thickness with equation (6) for ζ. In addition, equations (10 and 11) will be used for the storage coefficient. Upstream weighing will be used in combination with Newton's method to obtain a solution.

3 Transient interface flow in a single layer with variable head in the saltwater

Flow in the aquifer is governed by a water balance for the freshwater zone and a water balance for the saltwater zone, resulting in two simultaneous differential equations. The water balance for the freshwater zone was already given by (9)

$$\frac{\partial}{\partial x}\left(kb_f\frac{\partial h_f}{\partial x}\right)+\frac{\partial}{\partial x}\left(kb_f\frac{\partial h_f}{\partial y}\right)=S_f\frac{\partial h_f}{\partial t}-S_\zeta\frac{\partial \zeta}{\partial t}-N_f \tag{13}$$

The water balance for the saltwater zone is

$$\frac{\partial}{\partial x}\left(kb_s\frac{\partial h_s}{\partial x}\right)+\frac{\partial}{\partial x}\left(kb_s\frac{\partial h_s}{\partial y}\right)=S_s\frac{\partial h_s}{\partial t}+S_\zeta\frac{\partial \zeta}{\partial t}-N_s \tag{14}$$

where b_s is the saltwater thickness, S_s is the elastic storage coefficient for the saltwater zone, and N_s is the recharge to the saltwater zone. The saltwater thickness is computed as

$$b_s=\begin{cases} z_t-z_b & \text{if} \quad \zeta>z_t \\ \zeta-z_b & \text{if} \quad z_b\le\zeta\le z_t \\ 0 & \text{if} \quad \zeta<z_b \end{cases} \tag{15}$$

The elastic storage coefficient for the saltwater zone is

$$S_s=S_eb_s \tag{16}$$

The two differential equations (13) and (14) have three dependent variable: the freshwater head in the freshwater zone h_f, the saltwater head in the saltwater zone h_s and the elevation of the interface ζ. These three variables are linked throught the Ghyben-Herzberg equation (6). In the implementation of these two equations, two solution variables have to be chosen, after which the third will follow from (6). The two most logical choices are h_f and h_s, or h_f and ζ. For the case that the differential equations are implemented for h_f and h_s, the following two linked differential equations need to be solved

$$\frac{\partial}{\partial x}\left(kb_f\frac{\partial h_f}{\partial x}\right)+\frac{\partial}{\partial x}\left(kb_f\frac{\partial h_f}{\partial y}\right)=(S_f+\alpha_fS_\zeta)\frac{\partial h_f}{\partial t}-\alpha_sS_\zeta\frac{\partial h_s}{\partial t}-N_f \tag{17}$$

$$\frac{\partial}{\partial x}\left(kb_s\frac{\partial h_s}{\partial x}\right)+\frac{\partial}{\partial x}\left(kb_s\frac{\partial h_s}{\partial y}\right)=(S_s+\alpha_sS_\zeta)\frac{\partial h_s}{\partial t}-\alpha_fS_\zeta\frac{\partial h_f}{\partial t}-N_s \tag{18}$$

Obivously, it will be more complicated to solve these two equations simultaneously using the MODFLOW API. Alternatively, h_f and ζ may be used as the dependent variables, which may converge better, but won't be any easier to implement.