

Computability of Bass-Serre structures

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Introduction

Grzegorzcyk [Grz53] introduced a hierarchy of classes of recursive functions delineated by the number of times unbounded recursion is used in the construction of their members. Following Rabin [Rab60], Cannonito [Can66] defined a notion of computability for groups in the context of this hierarchy. Cannonito and Gatterdam [CG73] [Gat73] showed that, when taking a free product with amalgamation or an HNN extension of group(s) represented by functions computable at a particular level of the hierarchy, one creates a group with corresponding functions computable at most one level higher.

We describe the Grzegorzcyk hierarchy and present a clear notation for defining functions within it, along with constructions of many useful elementary functions.

Section 1 defines the notation used in this paper. Section 2 defines the Grzegorzcyk hierarchy of computable functions and gives definitions and positions on the hierarchy for many useful functions. Section 3 repeats Serre's formulation of graphs, and Section 4 gives conditions for a tree to be computable at a certain level of the Grzegorzcyk hierarchy.

In section 5 we restate [Can66]'s definition of an \mathcal{E}^n -computable group and \mathcal{E}^3 -computable index of the free group on countably many generators. We also discuss a flaw in [CG73]'s concept of a “standard” group.

Section 6 consists of a statement and proof of the main result of this paper – that the fundamental group of a graph of \mathcal{E}^n groups is \mathcal{E}^{n+1} -computable, generalising the results of [CG73]. Additionally, we show that the fundamental group of a graph of finitely-generated \mathcal{E}^n -computable groups is also \mathcal{E}^n -computable, and give an example of a graph of \mathcal{E}^3 -computable groups whose fundamental group is at least \mathcal{E}^4 -computable.

Finally, Section 7 defines computability for Stallings' pregroups [Sta71] in the context of the Grzegorzcyk hierarchy, generalising the results above.

1 Building blocks

1.1 Notation

When defining expressions or new notation, $E \equiv E' := D$ means I will write E or E' to mean D .

1.2 Constructing functions

Let $\mathbb{Z}_{\geq 0}$ denote the non-negative integers. A function $f(x_1, \dots, x_n)$ is a map,

$$f : \mathbb{Z}_{\geq 0}^n \rightarrow \mathbb{Z}_{\geq 0},$$

taking $n \geq 0$ parameters and resulting in a single value.

Functions in general can take any number of parameters, so it is clearly impractical to write out definitions of each operation for each possible arity. When the arity of a function f doesn't matter, I will write it as $f(\mathbf{x})$, where \mathbf{x} represents an arbitrarily big vector. When a function has to take at least a specific number of parameters y_1, \dots, y_n , and potentially more, I will write something of the form $f(\mathbf{x}, y_1, \dots, y_n)$

1.2.1 The primitive functions

We will begin by defining the *primitive functions*, which are functions of one of the following three types:

- *The zero function.* The zero function is a nullary function which returns the number 0. For brevity I will write 0 to mean the zero function.
- *The successor function.* The successor function $s(x) := x + 1$ is a unary function which adds 1 to its input.
- *The projection functions.* For every $n > 0$ and every $1 \leq i \leq n$ there is a projection function $P_n^i(x_1, \dots, x_n) := x_i$. P_n^i is an n -ary function which returns its i^{th} parameter.

There are two operations we shall use to create new functions from these primitives.

1.2.2 Composition

A function h is constructed by *composition* of functions f and g if $h = f \circ g$ where

$$f \circ g(\mathbf{x}) \equiv f(g(\mathbf{x})) := C(f, g)(\mathbf{x}).$$

In general, given an n -ary function f and m -ary functions g_1, \dots, g_n , the composition operator C constructs a new m -ary function

$$f(g_1(\mathbf{x}), \dots, g_n(\mathbf{x})) := C(f, g_1, \dots, g_n)(\mathbf{x}).$$

We will omit the generalised parameters for most function definitions, like so:

$$\begin{aligned} f \circ g &:= C(f, g), \\ f(g_1, \dots, g_n) &:= C(f, g_1, \dots, g_n). \end{aligned}$$

The mixing of functions of different arities can be dealt with systematically. For every n -ary function f , and every $m > n$, we can define an m -ary function f_m ,

$$f_m(x_1, \dots, x_n, \dots, x_m) := C(f, P_m^1, \dots, P_m^n,$$

such that, for all x_1, \dots, x_m , $f_m(x_1, \dots, x_n, \dots, x_m) \equiv f(x_1, \dots, x_n)$.

Variable substitution can also be dealt with systematically:

$$f(x_1, \dots, y, \dots, x_n) := f(x_1, \dots, y(x_1, \dots, x_n), \dots, x_n) = C(f, P_n^1, \dots, y, \dots, P_n^n).$$

1.2.3 Recursion

A function h is constructed by *recursion* from functions f and g if

$$\begin{aligned} h(\mathbf{x}, 0) &:= f(\mathbf{x}), \\ h(\mathbf{x}, n+1) &:= g(n, h(\mathbf{x}, n), \mathbf{x}), \end{aligned}$$

in which case we write $h = R(f, g)$.

Note that the parameter of recursion n can be made to be in any position by composition with the projection functions. If we have a function $h(x_1, \dots, x_m, n)$, we can define a new function,

$$f'(x_1, \dots, x_k, n, x_{k+1}, \dots, x_m) := C(f, P_{m+1}^1, \dots, P_{m+1}^k, P_{m+1}^{k+2}, \dots, P_{m+1}^m, P_{m+1}^{k+1}),$$

so that

$$f'(x_1, \dots, x_k, n, x_{k+1}, \dots, x_m) := f(x_1, \dots, x_m, n).$$

1.3 Further notation

1.3.1 Bound variables

A *bound variable* is one which is used to define a function, not passed as a parameter into it. For instance, in the porjection function P_n^i , n and i are bound variables. Bound variables do not contribute to the arity of a function.

1.3.2 Iteration

A function which evaluates n iterations of a function f , where n is a bound variable, can be constructed by composing f with itself $n - 1$ times.

$$f^{(n)}(\mathbf{x}) := \underbrace{C(f, C(f, C(f, \dots C(f, f)))}_{n-1 \text{ compositions}}(\mathbf{x}).$$

1.3.3 Infix operators

When using infix notation, binary operators are written between their operands.

For a binary function f_* , representing the operation $*$, we will use $x*y$ to denote $f_*(x, y)$, whenever the former notation is more conventional.

$$x * y := f_*(x, y).$$

1.3.4 Sets

A set $S \subseteq \mathbb{N}^d$ is defined by its characteristic function χ_S .

$$\chi_S(\mathbf{x}) := \begin{cases} 1 & \mathbf{x} \in S, \\ 0 & \mathbf{x} \notin S. \end{cases}$$

1.3.5 Relations

If $R \subseteq \mathbb{N}^d$ is a relation, R is defined by χ_R . In addition, for binary relations I will write

$$xRy := \chi_R(x, y).$$

1.3.6 Logic

The concept of falsity will be represented by the number 0, and all other values will represent truth. With this in mind, predicates $P(\mathbf{x})$ can be written out as ordinary functions once the logical operations have been defined.

2 The Grzegorczyk hierarchy

A class \mathcal{C} is *closed under finite composition* if, whenever f, g_1, \dots, g_m all belong to \mathcal{C} , then $C(f, g_1, \dots, g_m)$ also belongs to \mathcal{C} . Therefore a function defined by finitely many compositions of elements of \mathcal{C} is also an element of \mathcal{C} .

A function f is said to be *bounded* by another function g if, for all \mathbf{x} , $f(\mathbf{x}) < g(\mathbf{x})$.

A class \mathcal{C} is *closed under bounded recursion* if, whenever f, g, h belong to \mathcal{C} and $R(f, g)(\mathbf{x})$ is bounded by $h(\mathbf{x})$, then $R(f, g)$ also belongs to \mathcal{C} .

The *Grzegorczyk hierarchy* is an infinite nested hierarchy of classes \mathcal{E}^n of functions which are each closed under finite composition and bounded recursion.

\mathcal{E}^0 consists of just the primitive functions, and whatever else can be obtained by finitely many applications of the composition operator and bounded recursion.

\mathcal{E}^{n+1} , for $n \geq 0$, is the smallest class which

- contains all of \mathcal{E}^n ;
- contains every function obtained by a single unbounded recursion on \mathcal{E}^n functions;
- is closed under finite composition and bounded recursion.

2.1 The stockpile of functions

Below I will give explicit definitions for the operations of elementary arithmetic, plus some other useful things, using the notation defined above. I have tried to use definitions which, though not the most obvious, place functions as low down on the hierarchy as possible.

2.2 \mathcal{E}^0 functions

Add a constant n to x by iterating the successor function n times on x :

$$+_n(x) \equiv x + n := s^{(n)}(x). \quad (1)$$

Note that n is a bound variable, that is, there is a different function $+_n$ for each n .

A nullary function returning any single constant can be constructed by composing $+_n$ with z

$$c_n() := 0 + n. \quad (2)$$

From now on I will just write the number n to mean the function c_n , when appropriate.

Decrement by 1:

$$\begin{aligned} dec(0) &:= 0, \\ dec(n+1) &:= n. \end{aligned} \tag{3}$$

Proper subtraction:

$$\begin{aligned} x \dot{-} 0 &:= x, \\ x \dot{-} (y+1) &:= dec(x \dot{-} y). \end{aligned} \tag{4}$$

$x \dot{-} y$ is bounded by $P_2^1 \equiv x$, so belongs to \mathcal{E}^0 .

Signature:

$$\begin{aligned} sg(0) &:= 0, \\ sg(x+1) &:= 1. \end{aligned} \tag{5}$$

And reverse signature:

$$\begin{aligned} \overline{sg}(0) &:= 1, \\ \overline{sg}(x+1) &:= 0. \end{aligned} \tag{6}$$

Logical AND:

$$\begin{aligned} x \wedge 0 &:= 0, \\ x \wedge (n+1) &:= x. \end{aligned} \tag{7}$$

\wedge is bounded by $P_2^1 \equiv x$, so is in \mathcal{E}^0 .

Logical OR:

$$\begin{aligned} x \vee 0 &:= x, \\ x \vee (n+1) &:= 1. \end{aligned} \tag{8}$$

\vee is bounded by $s \circ P_2^1 \equiv x+1$, so is in \mathcal{E}^0 .

Logical NOT:

$$\neg x := \overline{sg}(x). \tag{9}$$

Ordering:

$$\begin{aligned} x > y &:= x \dot{-} y, \\ x < y &:= y \dot{-} x, \\ x \geq y &:= \neg(y > x), \\ x \leq y &:= \neg(x > y). \end{aligned} \tag{10}$$

Equality:

$$x = y := (x \leq y) \wedge (y \leq x) \quad (11)$$

The smallest of two numbers:

$$\min(x, y) := x \dot{-} (x \dot{-} y). \quad (12)$$

Remainder when dividing x by y :

$$\begin{aligned} 0 \bmod y &:= 0, \\ (x + 1) \bmod y &:= rm'(y \dot{-} ((x \bmod y) + 1), x \bmod y). \end{aligned} \quad (13)$$

$$\begin{aligned} rm'(0, y) &:= 0, \\ rm'(x + 1, y) &:= y + 1. \end{aligned}$$

rm' is bounded by $y + 1$ so belongs to \mathcal{E}^0 . $x \bmod y$ is defined by recursion on \mathcal{E}^0 functions and is bounded by $y - 1$, so belongs to \mathcal{E}^0 .

2.3 \mathcal{E}^1 functions

Adding two numbers together is achieved by a recursion on the successor function.

$$\begin{aligned} x + 0 &:= x, \\ x + (y + 1) &:= (x + y) + 1. \end{aligned} \quad (14)$$

Because every function in \mathcal{E}^0 is bounded by a function of the form $x + n$, where n is a bound variable, addition must belong to \mathcal{E}^1 and not \mathcal{E}^0 .

We can define another version of $+_n$ where n is unbound:

$$+_n(y) \equiv f_+(n, y) := n + y. \quad (15)$$

This version is in \mathcal{E}^1 . This is really an abuse of notation, but it will be useful in a little while.

The biggest of two numbers:

$$\max(x, y) := x + (y \dot{-} x). \quad (16)$$

Note that $\max(x, y)$ is bounded by one of P_2^1 or P_2^2 , but I don't think there's a way of defining it without using addition, so it has to be in \mathcal{E}^1 .

Absolute difference:

$$|x - y| := (x \dot{-} y) + (y \dot{-} x). \quad (17)$$

Any constant multiple $n \cdot x$ is produced by n iterations of addition:

$$n \cdot x \equiv 0 + n + \dots + n := +_x^{(n)}(0). \quad (18)$$

So we can also get any positive- and constant-coefficient linear polynomial $a_0 + a_1x_1 + a_2x_2 + \dots a_nx_n$ by finite compositions.

Remember that the functions we are considering have domain \mathbb{N}^d , so computations involving subtraction can go wrong. The key point is that proper subtraction is not associative. For example, $(4 + 3) - 5 \neq 4 + (3 - 5)$. For polynomials with negative coefficients, computing all the positive terms and then subtracting the negative terms using proper subtraction will give the closest match to the actual value.

Quotient:

$$\begin{aligned} \left\lfloor \frac{0}{y} \right\rfloor &:= 0, \\ \left\lfloor \frac{x+1}{y} \right\rfloor &:= \left\lfloor \frac{x}{y} \right\rfloor + \overline{sg}((x+1) \bmod y). \end{aligned} \tag{19}$$

Does x divide y :

$$x \mid y := \overline{sg}(x \bmod y). \tag{20}$$

Bounded minimisation: given a predicate P and a bound y this returns the least i strictly less than y that satisfies P . If there is no such i , y is returned.

$$\begin{aligned} \min_{i < 0} P(\mathbf{x}, i) &:= 0, \\ \min_{i < y+1} P(\mathbf{x}, i) &:= \left(\min_{i < y} P(\mathbf{x}, i) \right) + sg \left(\left(\min_{i < y} P(\mathbf{x}, y) = y \right) \wedge (P(\mathbf{x}, y) = 0) \right). \end{aligned} \tag{21}$$

Bounded maximisation:

$$\begin{aligned} \max_{i < y} P(\mathbf{x}, i) &:= y - \min_{i < y} P'(\mathbf{x}, y, i), \\ P'(\mathbf{x}, y, i) &:= P(\mathbf{x}, y - i). \end{aligned} \tag{22}$$

If P is an \mathcal{E}^1 function then so are bounded minimisations and maximisations over P , because they are defined by a recursion over \mathcal{E}^1 functions, bounded by y .

2.4 \mathcal{E}^2 functions

Multiplication is an unbounded recursion on the addition operation:

$$\begin{aligned} x \cdot 0 &:= 0, \\ x \cdot (y + 1) &:= (x \cdot y) + x. \end{aligned} \tag{23}$$

We can get x^n , when n is constant, by n compositions of multiplication by x , and hence any (positive-coefficient) polynomial by finite compositions of addition and multiplication. So polynomials belong to \mathcal{E}^2 .

Series sum:

$$\begin{aligned} \sum_{i < 0} f(\mathbf{x}, i) &:= 0, \\ \sum_{i < y+1} f(\mathbf{x}, i) &:= f(\mathbf{x}, y+1) + \sum_{i < y} f(\mathbf{x}, i). \end{aligned} \tag{24}$$

Note that the level of the sum on the hierarchy really depends on the level of $f(\mathbf{x}, i)$. However, we can say that if f is in \mathcal{E}^2 then $\sum_{i < y} f(\mathbf{x}, i)$ is also in \mathcal{E}^2 because it entails y additions. If y is a bound variable and f is in \mathcal{E}^1 , then we could even place the sum in \mathcal{E}^1 by defining the sum as y compositions of addition.

Bounded existential quantifier:

$$\exists_{i < y} P(\mathbf{x}, i) := \sum_{i < y} P(\mathbf{x}, i). \tag{25}$$

Definition by cases: if $P_i(\mathbf{x})$, $1 \leq i \leq k$ is a finite collection of mutually exclusive, exhaustive predicates, then we can define

$$f(\mathbf{x}) := \begin{cases} g_1(\mathbf{x}) & \text{if } P_1(\mathbf{x}), \\ \vdots & \vdots \\ g_k(\mathbf{x}) & \text{if } P_k(\mathbf{x}). \end{cases}$$

In our language, this is:

$$f(\mathbf{x}) := \sum_{i < k+1} g_i(\mathbf{x}) \cdot sg(P_i(\mathbf{x})), \tag{26}$$

where k is a bound variable. If all the P_i and g_i are in \mathcal{E}^2 , then so is f , because we are just doing k multiplications and additions.

2.5 \mathcal{E}^3 functions

\mathcal{E}^3 is where just about all the really useful operations are.

First of all, we can perform exponentiation by unbounded recursion on the multiplication operation:

$$\begin{aligned} x^0 &:= 1, \\ x^{y+1} &:= x \cdot x^y. \end{aligned} \tag{27}$$

Factorial:

$$\begin{aligned} 0! &:= 1, \\ (x+1)! &:= (x+1) \cdot x!. \end{aligned} \tag{28}$$

Series product:

$$\begin{aligned} \prod_{i < 0} f(\mathbf{x}, i) &:= 1, \\ \prod_{i < y+1} f(\mathbf{x}, i) &:= f(\mathbf{x}, y+1) \cdot \prod_{i < y} f(\mathbf{x}, i). \end{aligned} \tag{29}$$

If f is in \mathcal{E}^3 then so is the product $\prod_{i < y} f(\mathbf{x}, i)$, as it consists of y multiplications. If y is a bound variable, then we can define the product using y compositions of multiplication instead of the recursion, so if f is in \mathcal{E}^2 then the series product is too.

Bounded universal quantifier:

$$\forall_{i < y} P(\mathbf{x}, i) := \prod_{i < y} P(\mathbf{x}, i). \tag{30}$$

Primality:

$$x \text{ prime} := (x > 1) \wedge \forall_{z < x} ((z < 2) \vee \neg(z \mid x)). \tag{31}$$

The first prime after x :

$$\text{nextprime}(x) := \min_{i \leq x!+1} (i \text{ prime}) \wedge (i > x). \tag{32}$$

The bound on this function comes from Euclid or some such bearded Greek. Note that this is an exceptionally inefficient way of finding primes.

The n^{th} prime:

$$\begin{aligned} p_0 &\equiv p(0) := 2, \\ p_{n+1} &\equiv p(n+1) := \text{nextprime}(p_n). \end{aligned} \tag{33}$$

Exponent of x in the decomposition of y :

$$[y]_x := \max_{i < y} (x^i \mid y). \tag{34}$$

2.6 Lists

We will be working with n -tuples, or lists, extensively. All of the list operations can be defined as \mathcal{E}^3 functions on numbers by way of the Gödel encoding.

Gödel encoding of an n -tuple:

$$\mathbf{x} \equiv [x_0, \dots, x_n] := \prod_{0 \leq i \leq n} p_i^{x_i+1}. \quad (35)$$

The $+1$ in the exponent is so we can reliably determine the length of a list. The bold \mathbf{x} is shorthand so I don't have to write out the brackets formation. It shouldn't be confused with the use of \mathbf{x} previously to mean an arbitrary number of parameters in a function application. An encoded list \mathbf{x} is just a single number.

Value of n^{th} position in an encoded list:

$$(\mathbf{x})_i := [\mathbf{x}]_{p_i} - 1. \quad (36)$$

Length of an encoded list:

$$|\mathbf{x}| := \min_{i < \mathbf{x}} ([\mathbf{x}]_{p_i} = 0). \quad (37)$$

The index of the first occurrence of a given value in an encoded list:

$$\text{find}(\mathbf{x}, n) := \min_{i < |\mathbf{x}|} (\mathbf{x})_i = y. \quad (38)$$

Whether an encoded list contains a given value:

$$y \in \mathbf{x} := \text{find}(\mathbf{x}, y) < |\mathbf{x}|. \quad (39)$$

The largest element of an encoded list:

$$\begin{aligned} \text{biggest}(\mathbf{x}) &:= \text{biggest}'(\mathbf{x}, |\mathbf{x}| - 1), \\ \text{biggest}'(\mathbf{x}, 0) &:= (\mathbf{x})_0, \\ \text{biggest}'(\mathbf{x}, n + 1) &:= \max((\mathbf{x})_{n+1}, \text{biggest}'(\mathbf{x}, n)). \end{aligned} \quad (40)$$

To obtain the sub-list of \mathbf{x} starting at position s and ending at position e :

$$\mathbf{x}[s \dots e] \equiv \text{slice}(\mathbf{x}, s, e) := \prod_{i=s'}^{e'} (p_{i-s'})^{(\mathbf{x})_i+1}, \quad (41)$$

$$\begin{aligned} e' &:= \max(0, \min(e, |\mathbf{x}| - 1)), \\ s' &:= \max(0, \min(s, e')). \end{aligned}$$

To append a single value to the end of an encoded list:

$$\text{splice}(\mathbf{x}, n) := \mathbf{x} \cdot p_{|\mathbf{x}|}^{n+1}. \quad (42)$$

To concatenate two encoded lists:

$$\mathbf{x} \mathbin{++} \mathbf{y} := \text{splice}(\mathbf{x}, (\mathbf{y})_0) \mathbin{++} \mathbf{y}[1 \dots |\mathbf{y}| - 1] \quad (43)$$

$x \mathbin{++} y$ is bounded by $p_{|\mathbf{x}|+|\mathbf{y}|}^{(|\mathbf{x}|+|\mathbf{y}|)(\text{biggest}(\mathbf{x})+\text{biggest}(\mathbf{y}))}$, so is \mathcal{E}^3 -computable.

3 Graphs

Definition 3.1. A graph is a set of vertices V and a set of directed edges $E = E^+ \cup E^-$, with two associated maps. The first,

$$e \rightarrow (\iota(e), \tau(e)),$$

associates each edge with its initial and terminal vertices. The second,

$$e \rightarrow \bar{e},$$

is an involution on the edges, which must satisfy the conditions that $\bar{\bar{e}} = e$ and $\iota(\bar{e}) = \tau(e)$, $\forall e \in E$. We will write e_{ij} to mean the edge with $\iota(e_{ij}) = i$ and $\tau(e_{ij}) = j$. For any pair of edges e and \bar{e} , one is ‘positive’ and belonging to E^+ , and one ‘negative’ and belonging to E^- .

Definition 3.2. A *path* in a graph is a sequence of edges (e_1, e_2, \dots, e_n) such that $\tau(e_i) = \iota(e_{i+1})$, $1 \leq i < n$. A path is *reduced* if it does not contain an edge followed by its inverse, that is, if $e_{i+1} \neq \bar{e}_i$, $1 \leq i < n$.

Definition 3.3. A *circuit* is a path (e_1, \dots, e_n) with $\iota(e_1) = \tau(e_n)$.

Definition 3.4. A graph is *connected* if there exists a path between each pair of its vertices.

4 Computable Trees

Definition 4.1. A tree is a connected, non-empty graph with no reduced circuits.

A *rooted* tree has a particular vertex v_0 said to be at the ‘top’. In an *ordered* tree, the set of children of each vertex is ordered, so there is a leftmost child, and so on.

Definition 4.2. A *plane recursive tree* is a rooted, ordered tree obtained by a process which begins with the root node and recursively adds vertices below existing ones.

AJD Why “plane”? Every tree is planar. Where is the particular planar embedding used?

We can assign a labelling to a plane recursive tree by giving vertices labels corresponding to the order in which they were added to the tree, beginning

with 0 for the root. This way, the sequence of labels encountered on a path heading downwards is always increasing.

If we say that the positive edges point downwards, every positive edge can also be given a unique labelling by saying that the edge e_{ij} has label $\max(i, j)$. For the purposes of the computation, negative edges will have the same label as their positive counterpart.

Definition 4.3. Suppose we have a tree $T = (V, E)$ with root vertex v_0 . T is \mathcal{E}^n -computable if there is an \mathcal{E}^n -decidable labelling

$$i : V \rightarrow \mathbb{N},$$

(we can assume $i(v_0) = 0$) and an \mathcal{E}^n -computable ‘parent function’

$$\phi_T : i(V) \rightarrow i(V),$$

which gives the label of the parent of a vertex. For completeness, we also require that $\phi_T(v_0) = i(v_0)$.

Definition 4.4. A finite plane recursive tree can be entirely described in a canonical way by a *depth-first walk*. Note that every node has a single parent and a finite ordered set of children. The process on ‘visiting’ a vertex v_i goes as follows: for each child node v_j , travel along e_{ij} and ‘visit’ v_j , then travel backwards along \bar{e}_{ij} . The sequence of labelled edges travelled along by beginning this process at v_0 is the depth-first walk of the tree.

Note that the root vertex’s label will not appear in this sequence because it has no positive edge leading into it.

Because an edge and its inverse have the same label, the first occurrence of a label in the sequence represents the positive edge, and the second occurrence represents the trip back up the negative edge.

The depth-first walk can be encoded as a Gödel number by the method in Section 2.6. We will define a function $\phi_{\mathbf{t}}(n)$ which takes an encoded depth-first walk \mathbf{t} of a tree and gives the label of the parent of the vertex with label n .

4.1 Some functions to work with encoded trees

Let \mathbf{t} be an encoded depth-first walk of a tree.

To find the first and second occurrences of the edge with label n in the walk:

$$\begin{aligned} \text{fst}(\mathbf{t}, n) &:= \text{find}(\mathbf{t}, n), \\ \text{snd}(\mathbf{t}, n) &:= \text{find}(\mathbf{t}[\text{fst}(\mathbf{t}, n) + 1 \dots |\mathbf{t}| - 1], n). \end{aligned} \tag{44}$$

find is \mathcal{E}^3 -computable, and snd is bounded by $|\mathbf{t}|$ so is also \mathcal{E}^3 -computable.

To get the label of the n^{th} child of the root:

$$\begin{aligned}\text{child}(\mathbf{t}, 0) &:= (\mathbf{t})_0, \\ \text{child}(\mathbf{t}, n+1) &:= (\mathbf{t})_{\text{snd}(\mathbf{t}, \text{child}(\mathbf{t}, n)) + 1}.\end{aligned}\tag{45}$$

child is constructed from \mathcal{E}^3 functions and is bounded by the number of vertices in the tree. As the tree is finite, child is \mathcal{E}^3 -computable.

The depth-first walk of the subtree descended from a vertex v_n is the subsequence found between the two occurrences of n in \mathbf{t} .

To find the subtree descended from the n^{th} child node of the root:

$$\text{subtree}(\mathbf{t}, n) := \mathbf{t}[\text{fst}(\text{child}(\mathbf{t}, n)) + 1 \dots \text{snd}(\text{child}(\mathbf{t}, n)) - 1].\tag{46}$$

subtree is constructed from \mathcal{E}^3 operations so is \mathcal{E}^3 -computable.

The number of children of the root node:

$$\text{kids}(\mathbf{t}) := \min_{i < \mathbf{t}} (\text{child}(\mathbf{t}, i) = 0).\tag{47}$$

kids is constructed by bounded minimisation on \mathcal{E}^3 operations so is \mathcal{E}^3 -computable.

Whether the root has a child with label n :

$$\text{haschild}(\mathbf{t}, n) := \left(\mu_{i < \text{kids}(\mathbf{t})} (\text{child}(\mathbf{t}, i) = n) \right) < \text{kids}(\mathbf{t}).\tag{48}$$

Finally, the parent function $\phi_{\mathbf{t}}$. The idea is basically to do the depth-first walk, remembering which vertex you just came from. If you reach a subtree which doesn't contain n then turn back, or if you reach the required vertex then return the label of the vertex you just came from.

$$\begin{aligned}\phi_{\mathbf{t}}(0) &:= 0, \\ \phi_{\mathbf{t}}(n) &:= \text{p}'(x, n, 0).\end{aligned}\tag{49}$$

$$\text{p}'(x, n, p) := \begin{cases} 0 & n \notin \mathbf{t}, \\ p & \text{haschild}(\mathbf{t}, n), \\ \text{p}'(\text{subtree}(\mathbf{t}, z), n, \text{child}(\mathbf{t}, z)) & \text{otherwise.} \end{cases}$$

$$z := \min_{i < \text{kids}(\mathbf{t})} (n \in \text{subtree}(\mathbf{t}, i)).$$

We will also need to know if the tree contains an edge e_{ij} :

$$e_{ij} \in \mathbf{t} \Leftrightarrow \phi_{\mathbf{t}}(i) = j \vee \phi_{\mathbf{t}}(j) = i\tag{50}$$

Lemma 4.5. A finite ordered tree is \mathcal{E}^3 computable.

Proof. As the tree is finite, of course its labelling is an \mathcal{E}^3 -decidable subset of \mathbb{N} , and ϕ_t as defined above is \mathcal{E}^3 -computable. \square

5 Computable Groups

Following [Can66], we say a group G is \mathcal{E}^n -computable if it has a triple of \mathcal{E}^n -computable functions (i, m, j) , such that

AJD i is not computable: its image has computable characteristic function.

- $i : G \rightarrow \mathbb{N}$ is an injection of G onto an \mathcal{E}^n -decidable subset of \mathbb{N} ;
- $m : i(G) \times i(G) \rightarrow i(G)$ computes the product of two group elements, i.e. $m(i(g_1), i(g_2)) = i(g_1 g_2)$, $\forall g_1, g_2 \in G$;
- $j : i(G) \rightarrow i(G)$ computes the inverse of any element of G .

Lemma 5.1. [CG73, 3.1] A free group $F = \langle a_1, \dots \rangle$ on finitely or countably many generators is \mathcal{E}^3 -computable.

The proof of the above lemma is not important for the purposes of this paper, but the index defined on the free group will be used later in this paper. The index $i(w)$ of a word $w = a_{i_0}^{\alpha_0} \dots a_{i_r}^{\alpha_r}$ is

$$i(w) = \prod_{k=0}^r p_k e^{J(i_k, \alpha_k)}.$$

$J(x, y)$ is an \mathcal{E}^3 -computable integer pairing function defined in [CG73].

Definition 5.2. [CG73, 2.3] Let $A \subset \mathbb{N}$. Then the class of functions $\mathcal{E}^n(A)$ for $n \geq 2$ (with domain \mathbb{N}^k for arbitrary k and range \mathbb{N}) is the smallest class of functions, closed under composition and bounded recursion, containing all of \mathcal{E}^n in addition to

$$E(x) := x - \lfloor x^{\frac{1}{2}} \rfloor, \tag{51}$$

$$c_A(x) := \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \notin A. \end{cases} \tag{52}$$

AJD $E(x) = \min_{i < x} (i^2 \leq x)$, so is in \mathcal{E}^2 .

Lemma 5.3. Let $A \subset \mathbb{N}$. If c_A is \mathcal{E}^m -computable, then any $\mathcal{E}^n(A)$ -computable function, $n \leq m$, is \mathcal{E}^m -computable.

AJD In this lemma and the next theorem, $n \geq 2$.

Theorem 5.4. [CG73, 3.2] For every countable group G there exists $a \in \mathbb{N}$ such that G is $\mathcal{E}^3(A)$ -computable. In particular, the word problem of G is one such A .

AJD What does it mean for a group to be $\mathcal{E}^n A$ -computable? Define the word problem of a group G and explain the notation $\text{WP}(G)$.

I think that a corollary should be given here, as an analogue to Rabin's result, to which it should be compared. Namely, $\text{WP}(G)$ is \mathcal{E}^n -decidable if and only if G is \mathcal{E}^n -computable.

6 Computability of Bass-Serre groups

Let $\Gamma = (V, E)$ be a connected graph.

Definition 6.1. Associate with each vertex v a vertex group $G_v = \langle X_v \mid R_v \rangle$, with the X_v all pairwise disjoint. Call the set of all vertex groups \mathbf{G} .

For each edge $e \in E$ associate two isomorphic groups $A_e \leq G_{\iota(e)}$ and $B_e \leq G_{\tau(e)}$, with $A_e = B_{\bar{e}}$, and an isomorphism $\phi_e : A_e \rightarrow B_e$.

(\mathbf{G}, Γ) is called a *graph of groups*.

AJD Also, it is required that $\phi_e^{-1} = \phi_{\bar{e}}$. It makes life easier if A_e and B_e are defined as the *edge groups* of e and \bar{e} , respectively.

Definition 6.2. Let T be a spanning tree of Γ (a subtree of Γ containing every vertex), with a root vertex v_0 . Then $G = \pi_1(\mathbf{G}, \Gamma, T, v_0)$ is called the *fundamental group* of (\mathbf{G}, Γ) , defined by

$$G = \pi_1(\mathbf{G}, \Gamma, T, v_0) = (*_{v \in V} G_v) *_{\phi_e} F(E) \quad (53)$$

The generators of G are

$$X = \left(\bigcup_{v \in V} X_v \right) \cup \left(\bigcup_{e \in E} \{e\} \right),$$

and the relations are

$$R = \left(\bigcup_{v \in V} R_v \right) \cup \{e^{-1} A_e e = B_e, \forall e \in E\} \cup \{e^{-1} = \bar{e}, \forall e \in E\} \cup \{e = 1, \forall e \in T\}.$$

AJD “Then the *fundamental group* of (\mathbf{G}, Γ) , with respect to T and v_0 , is the group $G = \pi_1(\mathbf{G}, \Gamma, T, v_0)$ defined by ...”.

The free part of the amalgam above should be $F(E(\Gamma \setminus T))$.

I think (53) should be given as a consequence of the definition of G by generators and relations.

The definition of the relation involving edge groups needs to be made more explicit: $e^{-1}ae = \phi_e^{-1}(b) \dots$

Definition 6.3. A word $w \in X^*$ is *admissible* if it is in the form

$$w = a_0 e_1 a_1 e_2 \dots a_{n-1} e_n a_n$$

where (e_1, \dots, e_n) is a circuit in Γ with vertex sequence (v_0, \dots, v_n) , such that $v_0 = v_n$, $v_i = \iota(e_{i+1}) = \tau(e_i)$, $0 < i < n$, and $a_i \in G_{v_i}$, $i = 0, \dots, n$.

AJD I think what you mean is that a word is admissible if it *can* be factored in this way: that is there is a sequence $a_0, e_1, a_1, e_2, \dots, a_{n-1}, e_n, a_n$, satisfying the conditions that you give, such that $w \equiv a_0 e_1 a_1 e_2 \dots a_{n-1} e_n a_n$.

It will be more convenient from now on to refer to G_{v_i} by G_i .

Lemma 6.4. (Higgins?) Let $w \in X^*$ be an admissible word, and suppose $w =_G 1$. Then either:

- $n = 0$ and $a_0 = 1$ in G_{v_0} , or
- $n > 0$ and there exists $0 < i < n$ such that $e_{i+1} = \bar{e}_i$ and $a_i \in B_{e_i}$.

Definition 6.5. Let $\phi_T(n)$ be the parent function for T . The path in T from v_i to v_j can be found by freely reducing the path $\psi(i, j)$, defined by:

$$\begin{aligned} \psi(0, 0) &:= \epsilon, \\ \psi(i, 0) &:= (e_{i, p_T(i)}) \cdot \psi(p_T(i)), \\ \psi(0, i) &:= \psi(i, 0)^{-1}, \\ \psi(i, j) &:= \psi(i, 0) \psi(0, j). \end{aligned} \tag{54}$$

The free reduction of $\psi(i, j)$ will be implicit from now on. AJD Should there be an \mathcal{E}^3 function which takes an encoded tree and produces an encoding of the freely reduced path from v_i to v_j ? If so perhaps this should all go in Section 4.

]On the other hand perhaps all of this can be done by using the free group on the edges and reducing words there: which is known to be \mathcal{E}^3 .

What we really want is a function $\pi(g) : G \rightarrow X^*$ that gives an admissible word beginning and ending at v_0 and containing a representation of g as a contiguous subword. AJD So is $\pi(g)$ ideally an admissible word w which represents $g \in G$?

If g belongs to some G_{v_i} , then $\pi(g) = \psi(0, i) \cdot g \cdot \psi(i, 0)$. Otherwise, if $g = e_{ij}$, some edge letter, then $\pi(e_{ij}) = \psi(0, i) \cdot e_{ij} \cdot \psi(j, 0)$.

Lemma 6.6. Let w be a word on G . Then by replacing each generator g by $\pi(g)$ and freely reducing the resulting word, we obtain an admissible word starting at v_0 which is equivalent to w . Call this word $\pi(w)$, abusing notation a bit.

AJD w is a word on X (or in X^*). w should represent the element g of G , to make the statement of the lemma coherent.

Is π working with generators or with their images under i ?

At some point presumably it's necessary to show this whole process corresponds to a function on an encoded word.

Do products of generators in G_v get replaced by elements of G_v they represent or are they left as words in the generators.

The index of a word in the generators will have to be computed using the multiplication function m_v for G_v .

Proof. The new word is equivalent to w because the only letters we are adding are edge letters from the spanning tree, which are all trivial in the presentation of G . AJD Also, it is admissible. This follows since the unreduced word is admissible and free reduction preserves admissibility. (However, the sequence of the definition may not be immediately apparent.) \square

AJD Before stating the next theorem the precise definition of \mathcal{E}^n -computable homomorphism and \mathcal{E}^n -decidable subgroup, should be given and discussed (see e.g. Cannonito&Gatterdam '73).

Theorem 6.7. Suppose we have $G = \pi_1(\mathbf{G}, \Gamma, T, v_0)$, where Γ has ν vertices. Suppose all the G_i are f.g. AJD (“finitely generated”: weed out abbreviations) \mathcal{E}^n -computable groups for $n \geq 3$. Assume all the identified subgroups (AJD edge groups) are \mathcal{E}^n -decidable, and all the isomorphisms ϕ_e are \mathcal{E}^n -computable. Then G is \mathcal{E}^{n+1} -computable.

Proof. We will show that the word problem of G is \mathcal{E}^{n+1} -decidable. We can assume that the generating sets of all the G_v are disjoint.

AJD Also there is an \mathcal{E}^n -computable structure (i_v, m_v, j_v) , for G_v , for all vertices v .

Encode the depth-first walk of the spanning tree T as a Gödel number \mathbf{t} per Section 4.1. Then the parent function $\phi_{\mathbf{t}}$ is \mathcal{E}^3 -computable.

To define $i(G)$, we will begin by using the first ν^2 numbers to represent potential edges. For any $i = a \cdot \nu + b$ and $i \leq \nu^2$, $i \in i(G) \Leftrightarrow e_{ab} \in T$. AJD I think you mean

“For $0 \leq a, b \leq \nu - 1$ define $\iota(a, b) = a \cdot \nu + b$. Then define $i(e_{a,b}) = \iota(a, b)$, for all $e_{a,b} \in E$.”

(This map ι is a pairing function $I \times I \rightarrow J$, where $I = \{0, \dots, \nu - 1\}$ and $J = \{0, \dots, \nu^2 - 1\}$.)

We can assume the generators of the vertex groups have indices greater than ν^2 . The rest of $i(G)$ works the same way as the standard free group index in Cannonito-Gatterdam. ((Not explained well: C-G assign a number to each

generator, and then words are encoded as sequences of those numbers paired with a power, and the index of an element is the least index of an equivalent word. I want the numbers assigned to the group generators to be greater than ν^2 .)

AJD You've described indices for elements of E . To give indices for elements of X_v you presumably use i_v and then some pairing function, to make indices of elements of distinct X_v and X_u different. This needs to be done so that the images of G_v and G_u don't intersect, but using a pairing function this is no problem.

Given a word w , compute $w' = \pi(w)$. We now need to split w' into an admissible sequence $(a_0, e_1, \dots, e_n, a_n)$.

AJD So the definition of admissible should include the definition of "admissible sequence".

It's not clear whether you mean to do this for group elements or their images under i .

Are the a 's going to be words in the generators or group elements?

We must decide for each letter in w' either which vertex group it is from or if it is an edge letter. Assign to each letter w_i of w' a code as follows: If w_i belongs to G_i , then its code is i . If it is an edge letter, then its code is $\text{biggest}(\mathbf{t}) + 1$. We can then define an \mathcal{E}^n -decidable equivalence relation \approx on the indices of the elements of G , whose equivalence classes correspond to the vertex groups plus one more for each edge letter. For completeness, say that $g \not\approx n$ whenever $n \notin i(G)$.

AJD Is this equivalence relation defined on X or on $i(X)$? (It's easy to lift it from X to $i(X)$.)

As there are finitely many equivalence classes of \approx , and they are all \mathcal{E}^n -decidable, \approx is \mathcal{E}^n -decidable.

The next task is to split w' into 'syllables', or contiguous subwords. A syllable is either a single edge letter or a word from one of the vertex groups.

From now on, let \mathbf{w} be an encoded admissible word.

AJD So Lemma 6.6 needs to be strengthened to show this can be done at an appropriate level of the hierarchy.

I think at this point you are going to explain how to construct an admissible sequence from an admissible word.

Define a function which gives the position of the start of the syllable to which $(\mathbf{w})_i$ belongs:

$$\begin{aligned} \text{backtrack}(\mathbf{w}, 0) &:= 0, \\ \text{backtrack}(\mathbf{w}, i+1) &:= \begin{cases} i+1 & (\mathbf{w})_{i+1} \leq \nu^2, \\ i+1 & (\mathbf{w})_i \not\approx (\mathbf{w})_{i+1}, \\ \text{backtrack}(\mathbf{w}, i) & (\mathbf{w})_i \approx (\mathbf{w})_{i+1}. \end{cases} \end{aligned} \quad (55)$$

AJD The function `backtrack` is constructed from \mathcal{E}^n operations and is bounded by $|\mathbf{w}|$, so is \mathcal{E}^n -computable.

Now, the start of the n^{th} syllable is given by:

$$\begin{aligned} \text{start}(\mathbf{w}, 0) &:= 0, \\ \text{start}(\mathbf{w}, n+1) &:= \min_{\text{start}(\mathbf{w}, n)+1 \leq i < |\mathbf{w}|} (\text{backtrack}(\mathbf{w}, i) \neq \text{start}(\mathbf{w}, n)). \end{aligned} \quad (56)$$

The number of syllables can be computed like so:

$$\text{numsyllables}(\mathbf{w}) = \min_{i < |\mathbf{w}|} (\text{start}(\mathbf{w}, i) = |\mathbf{w}|). \quad (57)$$

And now the n^{th} syllable itself can be found:

$$\text{syllable}(\mathbf{w}, n) = \mathbf{w}[\text{start}(\mathbf{w}, n) \dots \text{start}(\mathbf{w}, n+1) - 1]. \quad (58)$$

AJD The function `syllable` is constructed from \mathcal{E}^n -computable functions and is bounded by \mathbf{w} so is \mathcal{E}^n -computable. **AJD** For a non-edge syllable you obtain a syllable consisting of a sequence $i(x_1), \dots, i(x_n)$, where the x_i are in some fixed X_v . The exact procedure for testing to see if this belongs to an edge group needs more explanation.

We can then use the conditions of Lemma 6.4 to determine if $w' = (a_0, e_1, \dots, e_n, a_n)$, encoded as \mathbf{w}' , is trivial.

If $n = 0$, then $w' = 1 \Leftrightarrow a_0 = 1$, which is an \mathcal{E}^n -decidable question.

If $n > 0$, then we need to find a sequence of the form $e^{-1}A_e e$ or $eB_e e^{-1}$. The first of these is given by

$$\begin{aligned} \min_{i < \text{numsyllables}(\mathbf{w}') - 2} & ((\text{syllable}(\mathbf{w}', i) = e \in E) \wedge \\ & (\text{syllable}(\mathbf{w}', i+1) \in B_e) \wedge \\ & (\text{syllable}(\mathbf{w}', i+2) = \text{syllable}(\mathbf{w}', i)^{-1})) \end{aligned} \quad (59)$$

(and the same the other way round for A_e .) Note that since the first syllable must be an edge letter, we can say the last syllable is the inverse of the first by checking its length is 1, then computing its inverse. We don't need to know how to compute the whole multiplication table to do this.

If the result of that calculation is $\text{numsyllables}(\mathbf{w}') - 2$ then w is not trivial. Otherwise, we can replace the found sequence $eB_e e^{-1}$ with $\phi_e(\text{syllable}(\mathbf{w}', i+1))$, and try again. The new word is still admissible and has fewer syllables than the original one, so repeated applications of this process will eventually lead to a word of one syllable or a negative answer. Because ϕ_e might increase the index of the word, this recursion raises us up a level on the Grzegorzcyk hierarchy.

So the word problem $WP(G)$ is \mathcal{E}^{n+1} -decidable, and hence G is \mathcal{E}^{n+1} -computable by Lemma 5.3 and Theorem 5.4. \square

Theorems [CG73, 4.6] and [CG73, 5.3] follow as corollaries of the above result,

as free products with amalgamation and HNN extensions can be considered as the fundamental groups of graphs with one edge connecting two vertices and one vertex, respectively.

Lemma 6.8. The n th prime $p_n \approx n \log n$.

Lemma 6.9. $\exp(x, y) := x^y \in \mathcal{E}^3$.

Lemma 6.10. The Gödel encoding of any word of length n on a finite alphabet is bounded above by $p_n^{J(n,n)}$, an \mathcal{E}^3 function.

Corollary 6.11. *Same assumptions as Theorem 6.7, but also assume that the edge groups are finitely generated. So each edge homomorphism $\phi_{e_{ij}}$ sends a finitely generated subgroup $E_{ij} \leq G_i$ to another finitely generated subgroup $E_{ji} \leq G_j$.*

Then $G = \pi_1(\mathbf{G}, \Gamma, T, v_0)$ is \mathcal{E}^n -computable.

Proof. In the algorithm from Theorem 6.7, the potential for unbounded recursion comes from applying the edge homomorphisms repeatedly. We will show that in this case the result of applying the edge homomorphisms repeatedly is bounded by an \mathcal{E}^3 -computable function.

Let w be a word on G of length n . An upper bound on the number of syllables in w is n , so let's say there are n applications of edge homomorphisms.

A homomorphism $\phi_{e_{ij}}$ rewrites generators $x \in X_i$ as words $\phi_{e_{ij}}(x) \in X_j^*$. As $G - i$ is finitely generated, there is a word $\phi_{e_{ij}}(x)$ of maximum, finite, length k_{ij} . So a word $w_i \in E_{ij}$ of length m is rewritten to a word $w_j \in E_j$ of length at most $k_{ij}m$.

Because the graph is finite, we can find a greatest $k \in \{k_{ij}\}$. So after reducing all the syllables (assuming that can be done for the input word), the resulting word in G_0 has length at most $k^n n = L$.

Hence the index of the resulting word is bounded by

$$p_{k^n n}^{J(k^n n, k^n n)},$$

an \mathcal{E}^3 function, by Lemma 6.10. So recursion on the edge homomorphisms is bounded by an \mathcal{E}^n function, and so G is \mathcal{E}^n -computable. □

Corollary 6.12. *There exists a graph of \mathcal{E}^3 -computable groups whose fundamental group is \mathcal{E}^4 -computable.*

Proof. Let $\mathbf{G} = \{G_0, G_1\}$ be two copies of the free group on two generators, so $G_0 = \langle x, y \rangle$ and $G_1 = \langle a, b \rangle$. AJD Let $G_0 = \langle x, y \rangle$ and $G_1 = \langle a, b \rangle$ be two copies of the free group on two generators. Let (\mathbf{G}, Γ) be the graph of groups with $\mathbf{G} = \{G_0, G_1\}$ and Γ the graph with a single edge e from v_0 to v_1 ; so $\Gamma = T = (\{v_0, v_1\}, \{e, \bar{e}\})$.

Let $E_{01} = \langle \{x^{-n}yx^n, n \in \mathbb{N}\} \rangle$, $E_{10} = \langle \{a^{-n}ba^n, n \in \mathbb{N}\} \rangle$ be the (infinitely generated) identified subgroups of G_0 and G_1 , respectively. AJD In fact the given generators of the edge subgroups freely generate these subgroups. This means that a homomorphism can be defined merely by defining the images of the generators.

Define

$$\begin{aligned}\phi_e(x^{-2n}yx^{2n}) &:= a^{(-2n)!}ba^{(2n)!}, \\ \phi_{\bar{e}}(a^{-2n-1}ba^{2n+1}) &:= x^{-(2n)!-1}yx^{(2n)!+1}.\end{aligned}$$

AJD ϕ_e and $\phi_{\bar{e}}$ must be isomorphisms with $\phi_e^{-1} = \phi_{\bar{e}}$. This can be arranged by mapping $\{x^{-(2n+1)}yx^{2n+1} : n \in \mathbb{Z}\}$ bijectively onto $\{a^{-m}ba^m : m \in \mathbb{Z}, m \neq (2n)!, \forall n \in \mathbb{Z}\}$. I haven't checked if this is still \mathcal{E}^3 but it does seem to give the right sort of growth. First of all, the homomorphisms are \mathcal{E}^3 -computable because words of the form $a^{-n}ba^n$ can be recognised by an \mathcal{E}^3 -computable function. Hence all the ingredients of the graph of groups are \mathcal{E}^3 -computable.

A word of the form

$$ea^{-1}e^{-1} \dots x^{-1}ea^{-1}e^{-1}x^{-n}yx^neae^{-1}x \dots eae^{-1}$$

will be reduced to a word on G_0 whose length is proportional to $\text{factorial}^{(m)}(n)$, where m is the number of syllables. AJD More detail is needed here. As m depends on the input, the length of the reduced word can only be computed by unbounded recursion on an \mathcal{E}^3 function, so is only \mathcal{E}^4 -computable.

Therefore, G is \mathcal{E}^4 -computable. □

7 Pregroups

A pregroup (P, D) is \mathcal{E}^n -computable if:

- there is an \mathcal{E}^n -computable indexing $i : P \rightarrow \mathbb{N}$,
- D is \mathcal{E}^n -decidable as a subset of $i(P) \times i(P)$ (from now on we will consider D as consisting of pairs of indices instead of pairs of elements),
- $m : D \rightarrow i(P)$ is \mathcal{E}^n -computable.

Lemma 7.1. Let (P, D) be a countable, \mathcal{E}^n -computable pregroup. The universal group $\mathbf{U}(P)$ is \mathcal{E}^{n+1} -computable.

Proof. The index (i', m') of $\mathbf{U}(P)$ will be on reduced P -words, encoded as Gödel numbers. So it is only necessary to define the multiplication function.

Because of Theorem 4 in Rimlinger, any P -words X and Y represent the same element of $\mathbf{U}(P)$ if and only if $X \sim Y$, that is, they are the same length and there is an interleaving of X equal to Y . The upshot is, when we have a reduced P -word X , we can find the $Y \sim X$ of minimal index by enumerating all the P -words of length $l_P(x)$. This step needs to be performed after any multiplication, so take that as read from here on.

Say we have two reduced P -words, $x = x_1 \dots x_m$ and $y = y_1 \dots y_n$. If $(x_m, y_1) \notin D$, then the product $m'(\mathbf{x}, \mathbf{y})$ is the concatenation $\mathbf{x} \uplus \mathbf{y}$. (There can't be any other P -word of the same length equivalent to xy and with lower index, can there?)

If $x_m = y_1^{-1}$, then $m'(\mathbf{x}, \mathbf{y}) = m'([x_1 \dots x_{m-1}], [y_2 \dots y_n])$.

If $xy = z \in P$, then $m'(\mathbf{x}, \mathbf{y}) = m'(m'([x_1 \dots x_{n-1}], [z]), [y_2 \dots y_n])$. As z may have an index as big as any \mathcal{E}^n function, the recursion becomes unbounded at this point, so m' is \mathcal{E}^{n+1} computable. \square

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