Exercise 7 – Solutions

16.11.2021

#1 Incomplete information

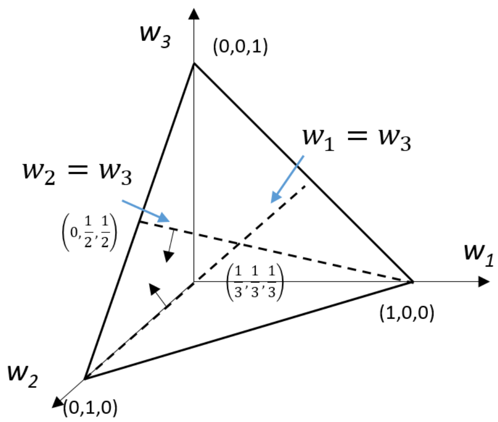
a) The preference statements imply inequalities

$$w_2\left(v_2^N(x_2^*)-v_2^N(x_2^0)\right)\geq w_3\left(v_3^N(x_3^*)-v_3^N(x_3^0)\right)\geq w_1\left(v_1^N(x_1^*)-v_1^N(x_1^0)\right)\Leftrightarrow$$
$$w_2\geq w_3\geq w_1.$$

b) The set of feasible weights:

$$S=\left\{w\in\mathbb{R}^3 | w_2\geq w_3\geq w_1,\sum_{i=1}^{3}w_i=1, w_i\geq 0\ \forall i\right\}.$$

The extreme points of this set are (0,1,0), (0, ½, ½), (1/3, 1/3, 1/3) .



c) Table 1 shows the alternatives' overall values at the extreme points of S.

	w=(0,1,0)	w=(0,1/2,1/2)	w=(1/3,1/3,1/3)
$V(x^1)$	0.50	0.55	0.37
$V(x^2)$	0.40	0.45	0.40
$V(x^3)$	0.50	0.50	0.67

Because the minimum and maximum overall values are obtained at the extreme points, the value intervals become

$$V(x^1)\in[0.37,0.55],\quad V(x^2)\in[0.40,0.45],\quad V(x^3)\in[0.50,0.67].$$

d) Alternative x^k dominates x^j, iff

$\min_w\left(V(x^k,w,v)-V(x^j,w,v)\right)\geq 0$ and

$\max_w\left(V(x^k,w,v)-V(x^j,w,v)\right)>0.$

The alternatives' pairwise value differences at each extreme point are:

	w=(0,1,0)	w=(0,1/2,1/2)	w=(1/3,1/3,1/3)
$V(x^1)-V(x^2)$	0.10	0.10	-0.03
$V(x^2)-V(x^3)$	-0.10	-0.05	-0.27
$V(x^1)-V(x^3)$	0	0.05	-0.30

Because the minimum and maximum value differences are obtained at the extreme points, it is concluded that x^3 dominates x^2 and no other dominance relationships exist.

#2 Sensitivity analysis

a) V(A)=200>V(B)=195>V(C)=185.

b) The normalized value function $V^N(x)$ is a positive affine transformation of $V(x)$:

$$V^N(x) = A * V(x) + B = Ax_1 + Ax_2 + B \tag{1}$$

Now, the condition $V^N(0,0) = 0$ implies that $B = 0$.

Then, substituting $B = 0$ and $V^N(105,105) = 1$ to (1) implies that

$210A = 1 \Leftrightarrow A = 1/210.$

$V^N(x)$ can also be written as $V^N(x) = w_1 v_1^N(x_1) + w_2 v_2^N(x_2)$. Therefore, it applies

$$V^N(x) = w_1 v_1^N(x_1) + w_2 v_2^N(x_2) = \frac{1}{210}x_1 + \frac{1}{210}x_2,$$

based on which with i=1,2, it now holds

$$v_i^N(x_i) = \frac{1}{210w_i}x_i. \tag{2}$$

Moreover, since necessarily now $v_i^N(0) = 0$ and $v_i^N(105) = 1$, one can solve from (2) that $w_1 = w_2 = \frac{1}{2} = 0.5$, and thereby

$$V^N(x) = w_1 v_1^N(x_1) + w_2 v_2^N(x_2) = 0.5 * \frac{x_1}{105} + 0.5 * \frac{x_2}{105}. \tag{3}$$

The weights with which B gets the same value as A are found by solving

$$\begin{cases} w_1 v_1^N(100) + w_2 v_2^N(100) = w_1 v_1^N(90) + w_2 v_2^N(105) \\ w_1 + w_2 = 1 \end{cases}, \tag{4}$$

where $v_1^N(x_1) = \frac{x_1}{105}$ and $v_2^N(x_2) = \frac{x_2}{105}$.

The solution is $w_1 = \frac{1}{3}, w_2 = \frac{2}{3}$. Now B is the most preferred alternative, if $w_2 \geq 2/3$.

Similarly, the weights with which C gets the same value as A are found by solving

$$\begin{cases} w_1 v_1^N(100) + w_2 v_2^N(100) = w_1 v_1^N(105) + w_2 v_2^N(80) \\ w_1 + w_2 = 1 \end{cases} \tag{5}$$

The solution is $w_1 = \frac{4}{5}, w_2 = \frac{1}{5}$. Thus, C is the most preferred one, if $w_1 \geq 4/5$.

B is the closest competitor, because (1/3,2/3) is closer to (0.5,0.5) than (0.8, 0.2):

$$\|(1/3, 2/3) - (0.5, 0.5)\|_2 = \sqrt{(1/6)^2 + (1/6)^2} = \frac{\sqrt{2}}{6} <$$
$$\frac{3\sqrt{2}}{10} = \sqrt{2(3/10)^2} = \|(4/5, 1/5) - (0.5, 0.5)\|_2$$

c) By performing corresponding calculations as in Equations (1) – (3) in part a), one now obtains that

$$A = \frac{1}{1155}, B = 0, w_1 = \frac{10}{11}, w_2 = \frac{1}{11}, v_1^N(x_1) = \frac{x_1}{1050}, v_2^N(x_2) = \frac{x_2}{105}.$$

The weights where B and C get the same score as A are calculated again using Equations (4) and (5). For B the weights are now $w_1 = \frac{5}{6}, w_2 = \frac{1}{6}$. And for C they are $w_1 = \frac{40}{41}, w_2 = \frac{1}{41}$.

C is the closest competitor, because it maximizes V' with $w_1 \geq 40/41$ while B maximizes V' with $w_2 \geq 1/6$ and

$$\|(40/41, 1/41) - (10/11, 1/11)\|_2 = \sqrt{2(30/451)^2} \approx 0.0941 <$$
$$0.1071 \approx \sqrt{2(5/66)^2} = \|(5/6, 1/6) - (10/11, 1/11)\|_2$$