Aalto University
School of Science

Decision making and problem solving – Lecture 2

- *Biases in probability assessment*
- *Expected Utility Theory (EUT)*
- *Assessment of utility functions*

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Last time

- Decision trees are a visual and easy way to model decision-making problems, which involve uncertainties
  - *Paths of decisions and random events*
- Probabilities are used to model uncertainty
  - *Data to estimate probabilities not necessarily available*
- We often need subjective judgements to estimate probabilities

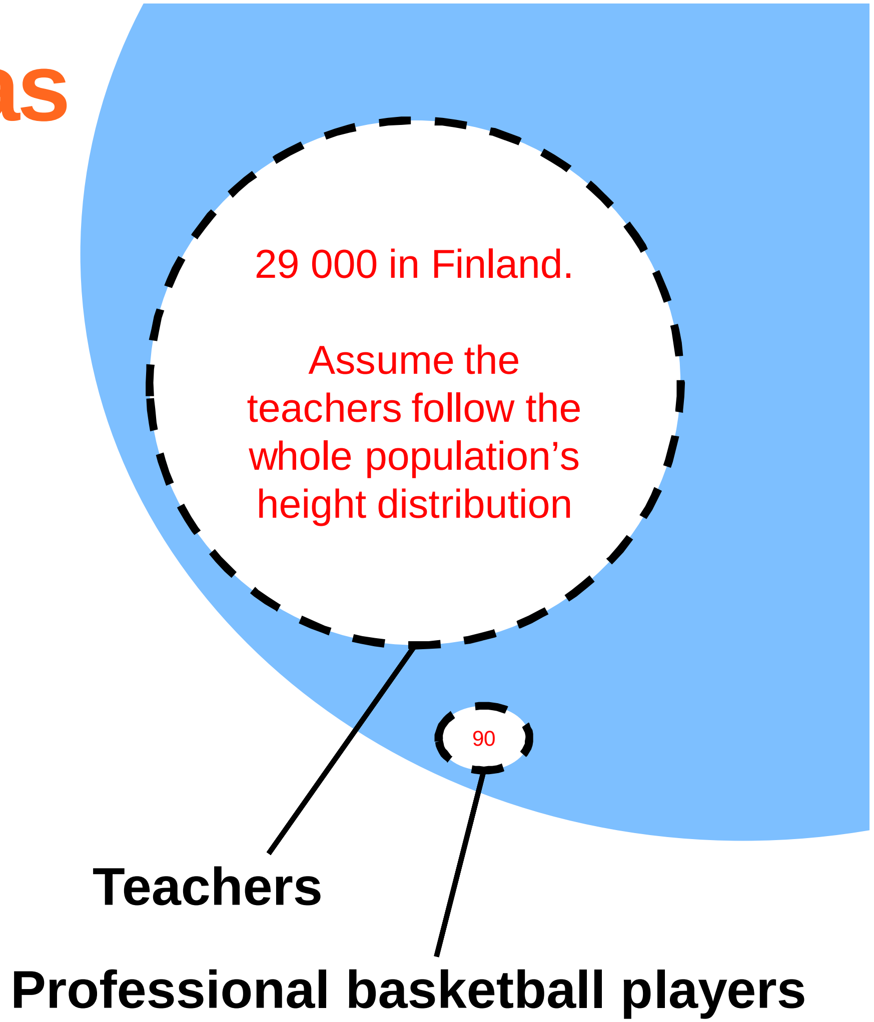
Biases in probability assessment

- Subjective judgements by both ”ordinary people” and ”experts” are prone to numerous biases
  - Cognitive bias: Systematic discrepancy between the ‘correct’ answer and the respondent’s actual answer
    - E.g., assessment of conditional probability differs from the correct value given by Bayes’ rule
  - Motivational biases: judgements are influenced by the desriability or undesirability of events
    - E.g., overoptimism about success probabilities
    - Strategic underestimation of failure probabilities

- Some biases can be very difficult to correct

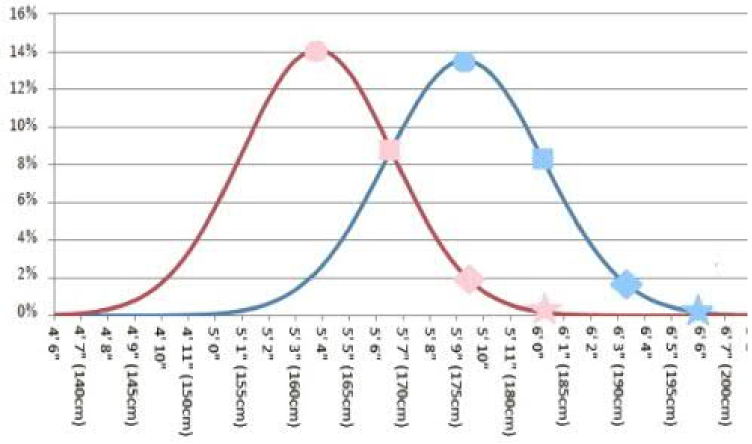
Representativeness bias (cognitive)

- If x fits the description of A well, then P(x∈A) is assumed to be large
- The 'base rate' of A in the population (i.e., the probability of A) is not taken into account
- Example: You see a very tall man in a bar. Is he more likely to be a professional basketball player or a teacher?



Representativeness bias

- ❑ What is ’very tall’?
  - ❑ 195 cm?
  - ❑ Assume all BB players are very tall
- ❑ Based on 30 min of googling[1], the share of Finnish men taller than 195 cm exceeds 0.3 %
- ❑ If BB players go to the bar as often as teachers, **it is more probable that the very tall man is a teacher, if the share of very tall men exceeds 0.31%**
  - – Fall 2020 students’ responses: 76% teacher, 24% basketball player
  - – Your responses: 78% teacher, 22% basketball player

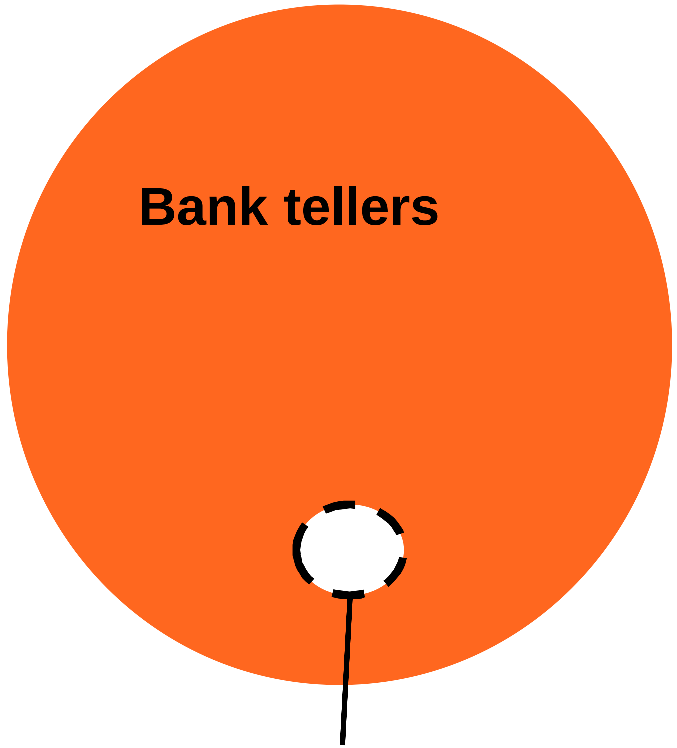


Height	Males					
	20–29 years	30–39 years	40–49 years	50–59 years	60–69 years	70–79 years
Percent under—						
4'10"	–	–	–	(B)	–	–
4'11"	–	–	–	(B)	(B)	–
5'	(B)	–	–	(B)	(B)	–
5'1"	(B)	(B)	(B)	(B)	[1] 0.4	(B)
5'2"	(B)	(B)	(B)	(B)	(B)	(B)
5'3"	(B)	[1] 3.1	[1] 1.9	(B)	[1] 2.3	(B)
5'4"	3.7	[1] 4.4	3.8	[1] 4.3	4.4	5.8
5'5"	7.2	6.7	5.6	7.6	7.8	12.8
5'6"	11.6	13.1	9.8	12.2	14.7	23.0
5'7"	20.6	19.6	19.4	18.6	23.7	35.1
5'8"	33.1	32.2	30.3	30.3	37.7	47.7
5'9"	42.2	45.4	40.4	41.2	50.2	60.3
5'10"	58.6	58.1	54.4	54.3	65.2	75.2
5'11"	70.7	69.4	69.6	70.0	75.0	85.8
6'	79.9	78.5	79.1	81.2	84.3	91.0
6'1"	89.0	89.0	87.4	91.6	93.6	94.9
6'2"	94.1	94.0	92.5	93.7	97.8	98.6
6'3"	98.3	95.8	97.7	96.6	99.9	100.0
6'4"	100.0	97.6	99.0	99.5	100.0	100.0
6'5"	100.0	99.4	99.4	99.6	100.0	100.0
6'6"	100.0	99.5	99.9	100.0	100.0	100.0

[1] note: with the lecturer’s skills

Representativeness bias

- Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in antinuclear demonstrations. Please check the most likely alternative:
  a. Linda is a bank teller.
  b. Linda is a bank teller and active in the feminist movement.
- Many choose b, although $b \subset a$ whereby $P(b)<P(a)$
  - Fall2020 responses: 76% a, 24% b.
  - Your responses: 57% a, 43% b.



Conservativism bias (cognitive)

- When information about some uncertain event is obtained, people typically do not adjust their initial probability estimate about this event as much as they should based on Bayes' theorem.

- Example: Consider two bags X and Y. Bag X contains 30 white balls and 10 black balls, whereas bag Y contains 30 black balls and 10 white balls. Suppose that you select one of these bags at random, and randomly draw five balls one-by-one by replacing them in the bag after each draw. Suppose you get four white balls and one black. What is the probability that you selected bag X with mainly white balls?

- Typically people answer something between 70-80%. Yet, the correct probability is $27/28 \approx 96\%$.
- **Fall2020 responses**: mean response 58%. Many (32%) answered 50%.
- **Your responses**: mean response 55%. Many (20%) answered 50%.

Representativeness and conservativism bias - debiasing

- Demonstrate the logic of joint and conditional probabilities and Bayes' rule
- Split the task into an assessment of
  - The base rates for the event (i.e., prior probability)
    - *E.g., what are the relative shares of teachers and pro basketball players?*
  - The likelihood of the data, given the event (i.e., conditional probabilities)
    - *E.g., what is the relative share of people active in the feminist movement? Is this share roughly the same among bank tellers as it is among the general population or higher/lower?*
    - *What is the likelihood that a male teacher is taller than 195cm? How about a pro basketball player?*

Availability bias (cognitive)

- People assess the probability of an event by the ease with which instances or occurences of this event can be brought to mind.
- Example: In a typical sample of English text, is it more likely that a word starts with the letter K or that K is the third letter?
  - Most (nowadays only many?) people think that words beginning with K are more likely, because it is easier to think of words that begin with "K" than words with "K" as the third letter
  - Yet, there are twice as many words with K as the third letter
  - Fall2020 students' responses: 34% first letter, 66% third letter.
  - Your responses: 45% first letter, 55% third letter.
- Other examples:
  - Due to media coverage, the number of violent crimes such as child murders seems to have increased
  - Yet, compared to 2000's, 18 times as many children were killed per capita in 1950's and twice as many in 1990's

Availability bias - debiasing

- ❑ Conduct probability training
- ❑ Provide counterexamples
- ❑ Provide statistics

- ❑ Based on empirical evidence, availability bias is **difficult to correct**

Anchoring bias (cognitive)

- When assessing probabilities, respondents sometimes consider some reference assessment
- Often, the respondent is *anchored* to the reference assessment
- Example: Is the percentage of African countries in the UN
  A. Greater or less than 65? What is the exact percentage?
     - *'Average' answer: Less, 45%.*
     - *Fall2020 students' responses: Less, median 40%, mean 45%.*
     - *Your responses: Less, median 35%, mean 39%.*
  B. Greater or less than 10? What is the exact percentage?
     - *'Average' answer: Greater, 25%.*
     - *Fall2020 students' responses: Greater, median 20%, mean 37%.*
     - *Your responses: Greater, median 27%, mean 33%.*

Based on Internet sources, the correct share is (maybe) 54/193 = **28%**

Anchoring bias - debiasing

- Avoid providing anchors
- Provide multiple and counteranchors
  - *= if you have to provide an anchor, provide several which differ significantly from each other*
- Use different experts who use different anchors

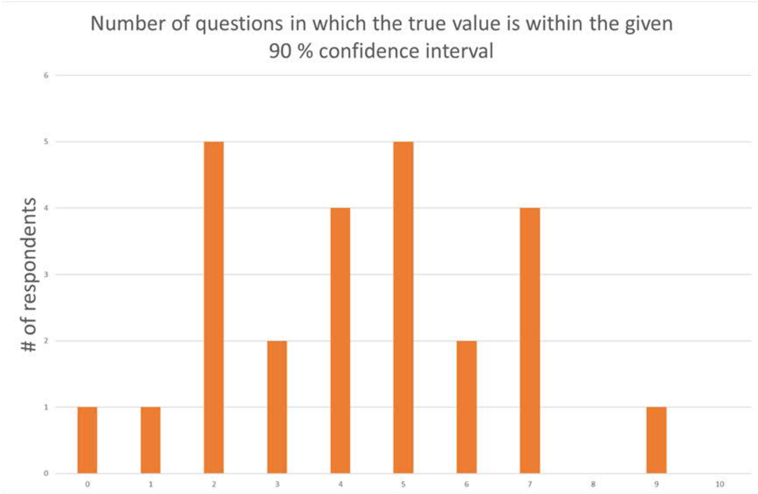
- Based on empirical evidence, anchoring bias is **difficult to correct**

Overconfidence (cognitive)

- People tend to assign overly narrow confidence intervals to their probability estimates

1. Martin Luther King's age at death 39 years
2. Length of the Nile River 6738 km
3. Number of Countries that are members of OPEC 13
4. Number of Books in the Old Testament 39
5. Diameter of the moon 3476 km
6. Weight of an empty Boeing 747 176900 kg
7. Year of Wolfgang Amadeus Mozart's birth 1756
8. Gestation period of an Asian elephant 21.5 months
9. Air distance from London to Tokyo 9590 km
10. Depth of the deepest known point in the oceans 11033 m

Your responses:



- If 3 or more of your intervals missed the correct value, you have demonstrated overconfidence
  - 96% of you did (91% in Fall2020)

Overconfidence - debiasing

- Provide probability training
- Start with extreme estimates (low and high)
- Use fixed values instead of fixed probability in elicitations:
  - Do not say: ”Give a value *x* such that the probability for a change in GDP lower than *x* is 0.05”
  - Do say: ”What is the probability that the change in GDP is lower than -3%?”

- Based on empirical evidence, overconfidence is **difficult to correct**

Desirability / undesirability of events (motivational)

- People tend to believe that there is a less than 50 % probability that **negative outcomes** will occur compared with peers
  - *I am less likely to develop a drinking problem*
  - Your responses: 32% (15 % in Fall2020) more likely, 28% (27 %) less likely, 40% (59 %) equally likely

- People tend to believe that there is a greater than 50 % probability that **positive outcomes** will occur compared with peers
  - *I am more likely to become a homeowner / have a starting salary of more than 3,500€*
  - Your responses on owning a home: 40% (20%) more likely, 12% (12%) less likely, 48% (68%) equally likely
  - Your responses on salary: 23% (20 %) more likely, 10% (10%) less likely, 67% (71%) equally likely

- People tend to underestimate the probability of negative outcomes and overestimate the probability of positive outcomes

Desirability / undesirability of events - debiasing

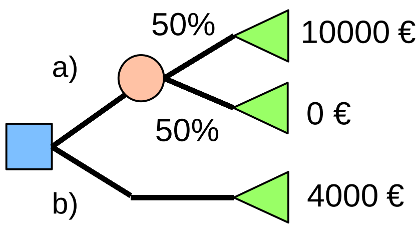
- Use multiple experts with alternative points of view
- Place hypothetical bets against the desired event
  - *“Make the respondent’s money involved”*
- Use decomposition and realistic assessment of partial probabilities
  - *“Split the events”*
- Yet, empirical evidence suggests that **all motivational biases are difficult to correct**

Further reading: **Montibeller, G., and D. von Winterfeldt**, 2015. Cognitive and Motivational Biases in Decision and Risk Analysis, *Risk Analysis*

Risky or not risky?

https://presemo.aalto.fi/riskattitude1/

- Which one would you choose:
  - *a) Participate in a lottery, where you have a 50 % chance of getting nothing and a 50 % chance of getting 10000 €*
  - *b) Take 4000 €*

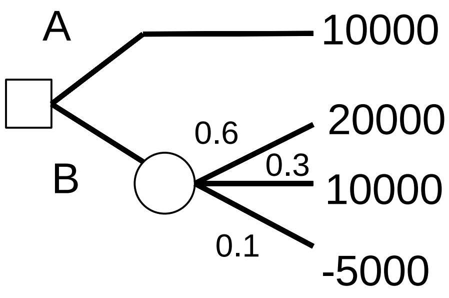


- Many choose the certain outcome of 4000 €, although a)’s expected monetary gain is higher

Option b) involves less risk

How to compare risky alternatives?

- **Last week**
  - We learned how to support decision-making under uncertainty, when the DM's objective is to maximize the expected monetary value
  - Maximizing expected value is rational only if the DM is risk neutral, i.e., indifferent between
    - *obtaining x for sure and*
    - *a gamble with uncertain payoff Y such that x=E[Y]*
  - Usually, DMs are risk averse = they prefer obtaining *x* for sure to a gamble with payoff *Y* such that *x*=E[*Y*]



Expectation = 14500

- **Next:**
  - We learn how to accommodate the DM's risk attitude (=preference over alternatives with uncertain outcomes) in decision models

Expected utility theory (EUT)

- John von Neumann and Oscar Morgenstern (1944) in Theory of Games and Economic Behavior:
  - Axioms of rationality for preferences over alternatives with uncertain outcomes
  - If the DM follows these axioms, she should prefer the alternative with the highest expected utility

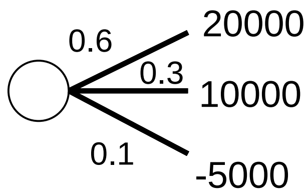
- Elements of EUT
  - Set of outcomes and ”lotteries”
  - Preference relation over the lotteries satisfying four axioms
  - Representation of preference relation with expected utility

EUT: Sets of outcomes and lotteries

- Set of possible outcomes *T*:
  - E.g., revenue T euros / demand T
- Set of all possible lotteries *L*:
  - A lottery $f \in L$ associates a probability $f(t) \in [0,1]$ with each possible outcome $t \in T$
    - *Finite number of outcomes with a positive probability* $f(t) > 0$
    - *Probabilities sum up to one* $\sum_t f(t) = 1$
    - *Lotteries are thus discrete PMFs / decision trees with a single chance node*
- Deterministic outcomes are modeled as degenerate lotteries

Lottery

Decision tree

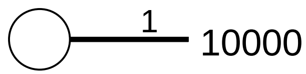


Probability mass function (PMF)

$$f(t) = \begin{cases} 0.6, t = 20000 \\ 0.3, t = 10000 \\ 0.1, t = -5000 \\ 0, elsewhere \end{cases}$$

Degenerate lottery

Decision tree



PDF

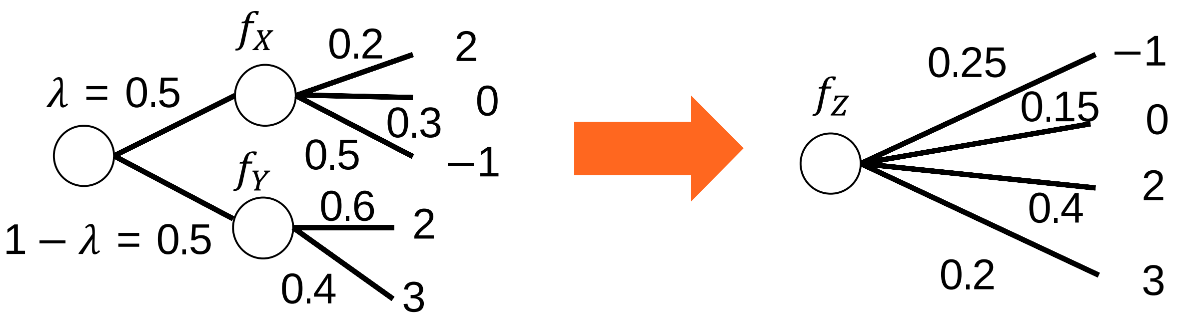
$$f(t) = \begin{cases} 1, t = 10000 \\ 0, elsewhere \end{cases}$$

EUT: Compound lotteries

- Compound lottery:
  - Get lottery $f_X \in L$ with probability λ
  - Get lottery $f_Y \in L$ with probability $1-\lambda$
- Compound lottery can be modeled as lottery $f_Z \in L$**:**

$$f_Z(t) = \lambda f_X(t) + (1-\lambda) f_Y(t) \;\; \forall t \in T \simeq f_Z = \lambda f_X + (1-\lambda) f_Y$$

- Example:
  - You have a 50-50 chance of getting a ticket to lottery $f_X \in L$ or to lottery $f_Y \in L$



Preference relation

- Let $\succcurlyeq$ be preference relation among lotteries in L
  - Preference $f_X \succcurlyeq f_Y$: f_X at least as preferable as f_Y
  - Strict preference $f_X \succ f_Y$ defined as $\neg(f_Y \succcurlyeq f_X)$
  - Indifference $f_X \sim f_Y$ defined as $f_X \succcurlyeq f_Y \land f_Y \succcurlyeq f_X$

EUT axioms A1-A4 for preference relation

- **A1:** $\succcurlyeq$ is complete
  - For any $f_X, f_Y \in L$, either $f_X \succcurlyeq f_Y$ or $f_Y \succcurlyeq f_X$ or both
- **A2:** $\succcurlyeq$ is transitive
  - If $f_X \succcurlyeq f_Y$ and $f_Y \succcurlyeq f_Z$, then $f_X \succcurlyeq f_Z$
- **A3:** Archimedean axiom
  - If $f_X \succ f_Y \succ f_Z$, then $\exists \lambda, \mu \in (0,1)$ such that

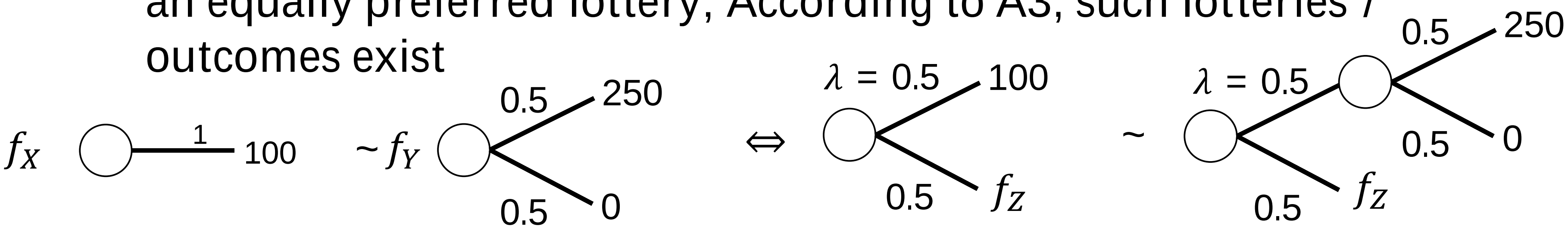
$$\lambda f_X + (1-\lambda) f_Z \succ f_Y \text{ and } f_Y \succ \mu f_X + (1-\mu) f_Z$$

- **A4:** Independence axiom
  - Let $\lambda \in (0,1)$. Then,

$$f_X \succ f_Y \Leftrightarrow \lambda f_X + (1-\lambda) f_Z \succ \lambda f_Y + (1-\lambda) f_Z$$

If the EUT axioms hold for the DM's preferences

- **A3:** Archimedean axiom
  - Let $f_X \succ f_Y \succ f_Z$. Then exists $p \in (0{,}1)$ so that $f_Y \sim pf_X + (1-p)f_Z$
- **A4:** Independence axiom
  - $f_X \sim f_Y \Leftrightarrow \lambda f_X + (1-\lambda)f_Z \sim \lambda f_Y + (1-\lambda)f_Z$
  - Any lottery (or outcome = a degenerate lottery) can be replaced by an equally preferred lottery; According to A3, such lotteries / outcomes exist



  - NOTE: f_Z can be any lottery and can have several possible outcomes

Main result: Preference representation with Expected Utility

- $\succcurlyeq$ satisfies axioms A1-A4 if and only if there exists a real-valued utility function *u(t)* over the set of outcomes *T* such that

$$f_X \succcurlyeq f_Y \Leftrightarrow \sum_{t \in T} f_X(t)u(t) \geq \sum_{t \in T} f_Y(t)u(t)$$

- Implication: **a rational DM** following axioms A1-A4 **selects the alternative with the highest expected utility**

$$E[u(X)] = \sum_{t \in T} f_X(t)u(t)$$

  – A similar result can be obtained for continuous distributions:
    - $f_X \succcurlyeq f_Y \Leftrightarrow E[u(X)] \geq E[u(Y)]$, *where* $E[u(X)] = \int f_X(t)u(t)dt$

Computing expected utility

- Example: Joe's utility function for the number of apples is u(1)=2, u(2)=5, u(3)=7. Would he prefer
  – Two apples for certain (X), or
  – A 50-50 gamble between 1 and 3 apples (Y)?

$$E[u(X)] = u(2) = 5$$

$$E[u(Y)] = 0.5u(1) + 0.5u(3)$$
$$= 0.5 \cdot 2 + 0.5 \cdot 7 = 4.5$$

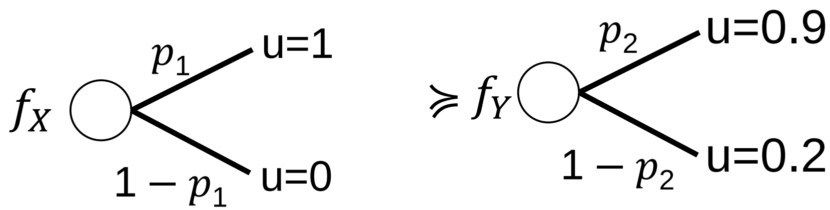
- Example: Jane's utility function for money is $u(t) = t^2$. Which alternative would she prefer?
  – X: 50-50 gamble between 3 and 5M€
  – Y: A random amount of money from Uni(3,5) distribution
  – What if her utility function was $u(t) = \frac{t^2-9}{25-9}$?

$$E[u(X)] = 0.5u(3) + 0.5u(5)$$
$$= 0.5 \cdot 9 + 0.5 \cdot 25 = 17$$

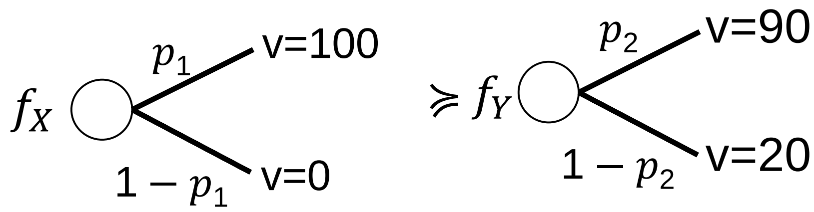
$$E[u(Y)] = \int_3^5 f_Y(t)u(t)dt = \int_3^5 \frac{1}{2}t^2 dt$$
$$= \frac{1}{6}5^3 - \frac{1}{6}3^3 = 16.33333$$

Uniqueness up to positive affine transformations

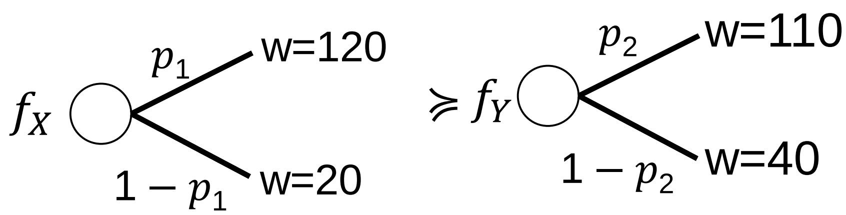
- DM's preferences: $X \succcurlyeq Y$
  - $E[u(X)] = p_1 \geq 0.9p_2 + 0.2(1 - p_2) = E[u(Y)]$



- v: Multiply each utility u by 100
  - $E[v(X)] = 100p_1 = 100E[u(X)] \geq 100E[u(Y)] = 90p_2 + 20(1 - p_2) = E[v(Y)]$

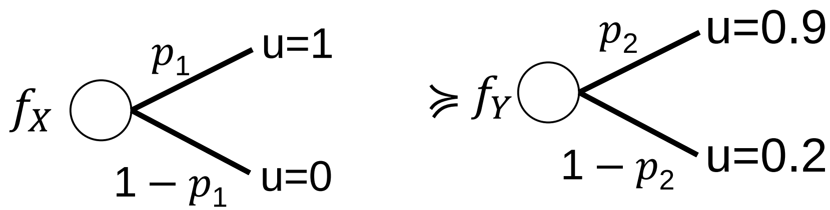


- w: Add 20 to all utilities v
  - $E[w(X)] = 120p_1 + 20(1 - p_1) = 100p_1 + 20 = E[v(X)] + 20 \geq E[v(Y)] + 20 = 90p_2 + 20(1 - p_2) + 20(1 + p_2 - p_2) = 110p_2 + 40(1 - p_2) = E[w(Y)]$

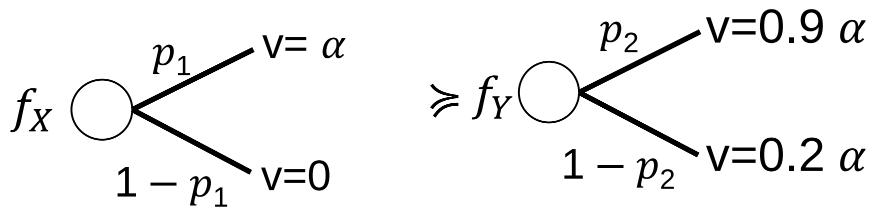


Uniqueness up to positive affine transformations

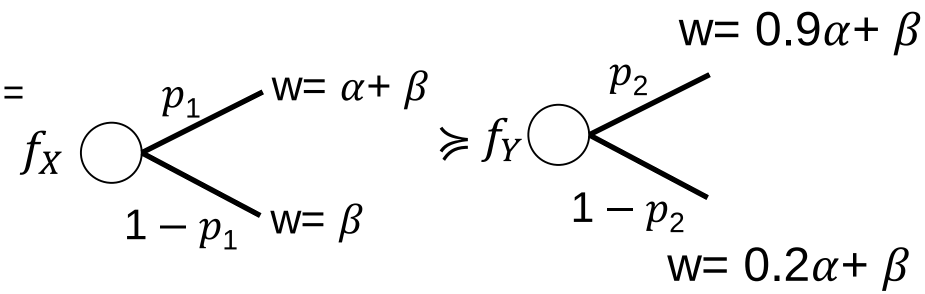
- DM's preferences: $X \succcurlyeq Y$
  - $E[u(X)] = p_1 \geq 0.9p_2 + 0.2(1 - p_2) = E[u(Y)]$



- v: Multiply u by $\alpha > 0$
  - $E[v(X)] = \alpha p_1 = \alpha E[u(X)] \geq \alpha E[u(Y)] = 0.9\alpha p_2 + 0.2\alpha(1 - p_2) = E[v(Y)]$

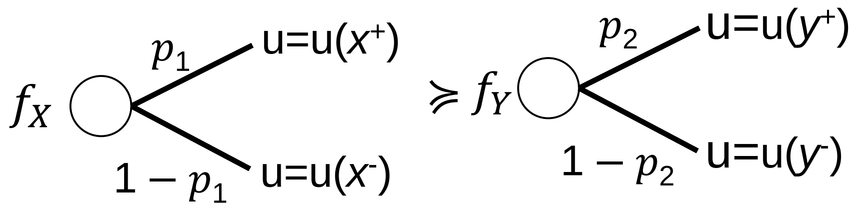


- w: Add β to all utilities v
  - $E[w(X)] = (\alpha + \beta)p_1 + \beta(1 - p_1) = \alpha p_1 + \beta = E[v(X)] + \beta \geq E[v(Y)] + \beta = 0.9\alpha p_2 + 0.2\alpha(1 - p_2) + \beta(1 + p_2 - p_2) = (0.9\alpha + \beta)p_2 + (0.2\alpha + \beta)\,(1 - p_2) = E[w(Y)]$

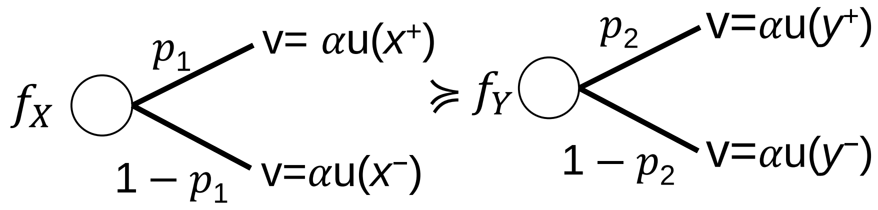


Uniqueness up to positive affine transformations

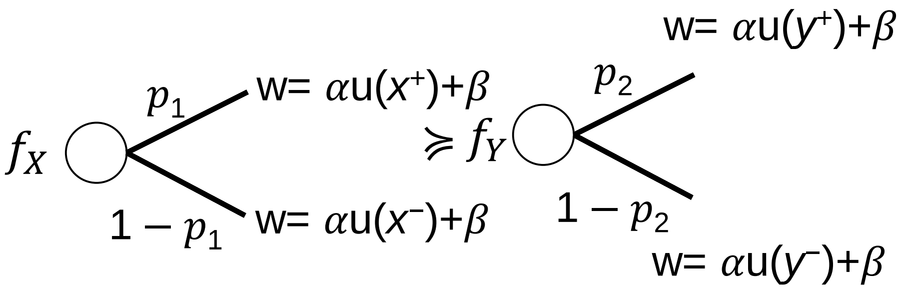
- DM's preferences: $X \succcurlyeq Y$
  - $E[u(X)] = u(x^+)p_1 + u(x^-)(1-p_1) \geq u(y^+)p_2 + u(y^-)(1-p_2) = E[u(Y)]$



- *v*: Multiply *u* by $\alpha > 0$
  - $E[v(X)] = \cdots = \alpha E[u(X)] \geq \alpha E[u(Y)] = \cdots = E[v(Y)]$



- *w*: Add β to all utilities *v*
  - $E[w(X)] = \cdots = E[v(X)] + \beta \geq E[v(Y)] + \beta = \cdots = E[w(Y)]$



Uniqueness up to positive affine transformations

- Let $f_X \succcurlyeq f_Y \Leftrightarrow E[u(X)] \geq E[u(Y)]$. Then $E[\alpha u(X) + \beta] = \alpha E[u(X)] + \beta \geq \alpha E[u(Y)] + \beta = E[\alpha u(Y) + \beta]$ for any $\alpha > 0$
- **Two utility functions $\boldsymbol{u_1(t)}$ and $\boldsymbol{u_2(t) = \alpha u_1(t) + \beta}$, $\boldsymbol{(\alpha > 0)}$ establish the same preference order among any lotteries**:

$$E[u_2(X)] = E[\alpha u_1(X) + \beta] = \alpha E[u_1(X)] + \beta.$$

- Implications:
  - Any linear utility function $u_L(t) = \alpha t + \beta$, $(\alpha > 0)$ is a positive affine transformation of the identity function $u_1(t) = t \Rightarrow u_L(t)$ establishes the same preference order as expected value
  - Utilities for two outcomes can be freely chosen:
    - *E.g., scale utilities represented by u_1 such that and $u_2(t^*) = 1$ and $u_2(t^0) = 0$:*

$$u_2(t) = \frac{u_1(t) - u_1(t^0)}{u_1(t^*) - u_1(t^0)} = \underbrace{\frac{1}{u_1(t^*) - u_1(t^0)}}_{=\alpha>0} u_1(t) - \underbrace{\frac{u_1(t^0)}{u_1(t^*) - u_1(t^0)}}_{=\beta}$$

Let's practice!

https://presemo.aalto.fi/drcuckoo

The utility function of Dr. Cuckoo is $u(t) = \sqrt{t}$. Would he

a) Participate in a lottery A with 50-50 chance of getting either 0 or 400 €?

b) Participate in a lottery B in which the probability of getting 900 € is 30% and getting 0 € is 70%?

$u(0) = 0$, $u(400) = 20$, $u(900) = 30$

a) $E[u(A)] = 0.5 \cdot 0 + 0.5 \cdot 20 = 10$

b) $E[u(B)] = 0.7 \cdot 0 + 0.3 \cdot 30 = 9$

NOTE! the **expectation of lottery A** = 200 € **is smaller** than that of B = 270€

Summary

- Probability elicitation is prone to cognitive and motivational biases
  - Some cognitive biases can be easy to correct, but…
  - Some other cognitive biases and all motivational biases can be difficult to overcome

- The DM's preferences over alternatives with uncertain outcomes can be described by a utility function

- A rational DM (according to the four axioms of rationality) should choose the alternative with the highest expected utility
  - NOT necessarily the alternative with the highest utility of expectation