



Aalto University  
School of Science

# Decision making and problem solving – Lecture 1

- *Decision trees*
- *Elicitation of probabilities*

# Motivation

- ❑ You have just revised some key concepts of probability calculus
  - Conditional probability
  - Law of total probability
  - Bayes' rule
  
- ❑ This time:
  - How to build a probability-based model to support decision-making under uncertainty?
  - How to elicitate the probabilities needed for these models?

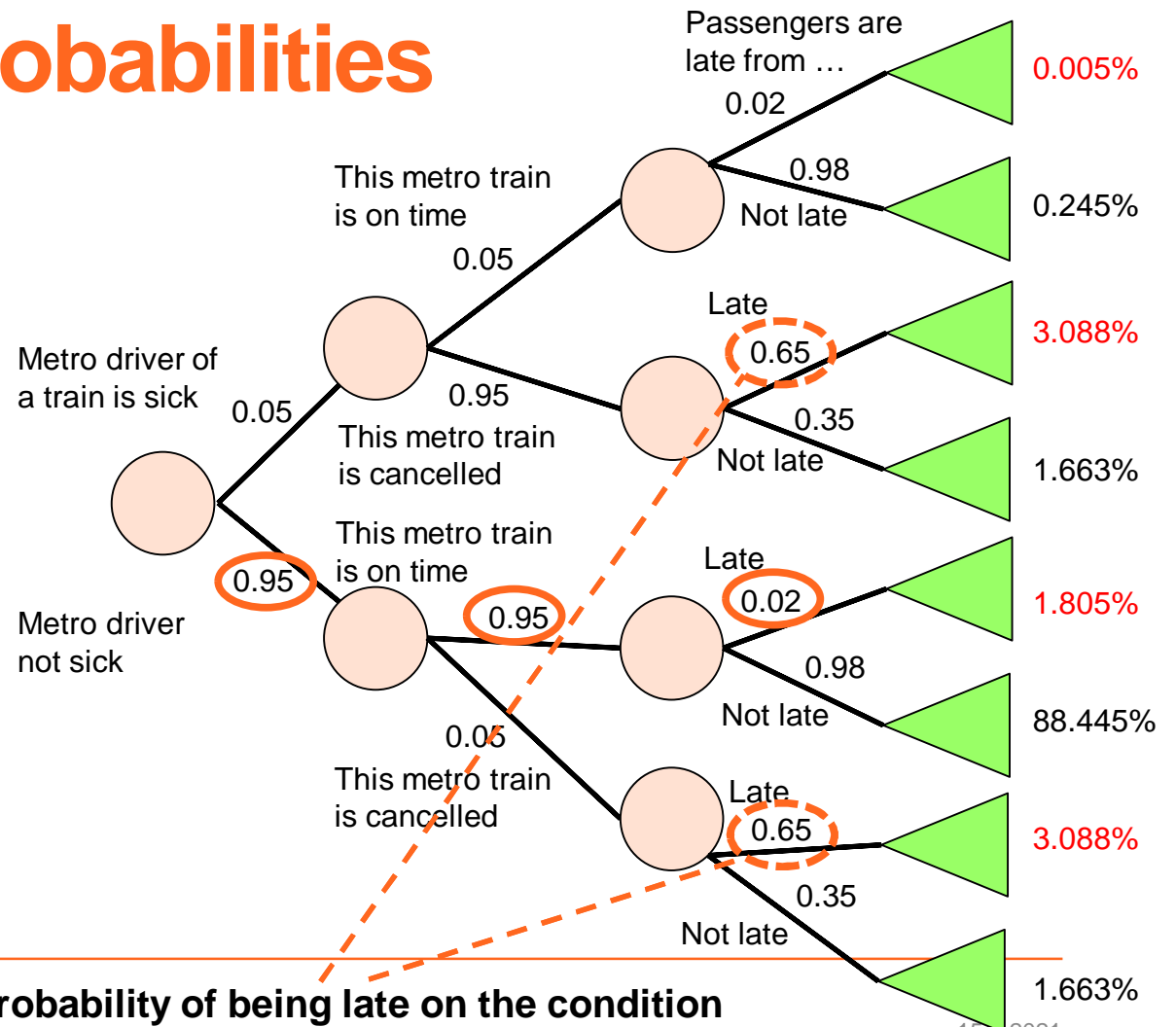
# Why probabilities for modeling uncertainty?

- ❑ Decisions are often made under uncertainty
  - ❑ *“How many train drivers should be trained, when future traffic is uncertain?”*
  - ❑ *“Should I buy an old or a new car, given that I only need an operational one and want to minimize costs = purchase price, maintenance & repair costs, selling price, etc.?”*
  - ❑ *“Should I buy my first my apartment now or postpone the decision, given that future interest rates, mortgage costs, personal income and apartment prices are uncertain?”*
- ❑ Probability theory dominates the modeling of uncertainty in decision analysis
  - Well established rules for computations, understandable
  - Other models (e.g., evidence theory, fuzzy sets) exist, too

# Conditional probabilities

- ❑ The probabilities of sequential, mutually exclusive and collectively exhaustive events can be represented in form of a tree
- ❑ The probability of a sequence of events is obtained by multiplying the probabilities on the path  

$$0.95 \times 0.95 \times 0.02 = 1.805 \%$$
- ❑ The total probability of being late is **7.985 %**



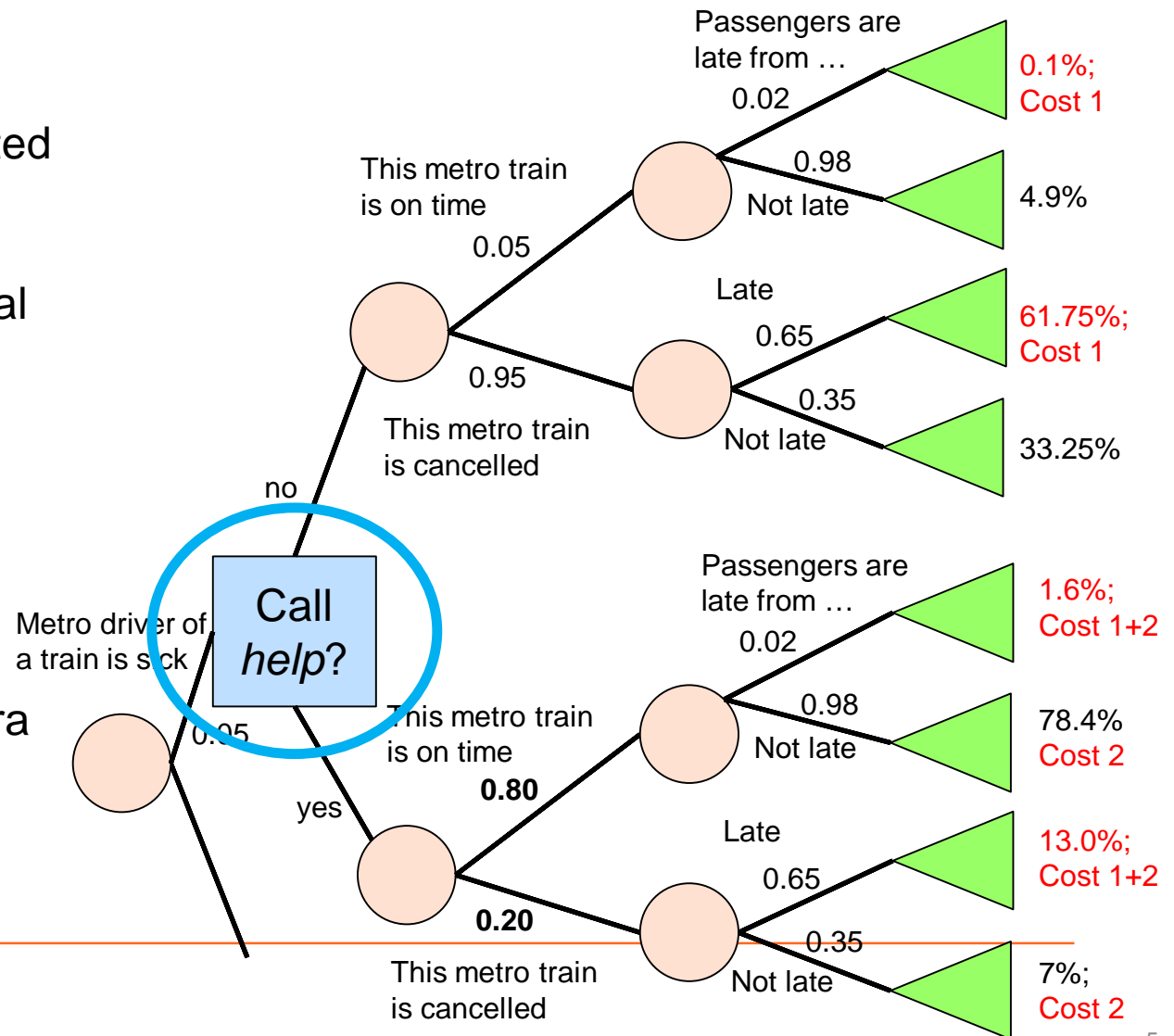
# What if....

- ❑ We, the operator is interested in financial aspects and assume that being late results in unwanted financial consequences (**Cost 1**)?

- ❑ numerical *outcomes* for states (consequences)

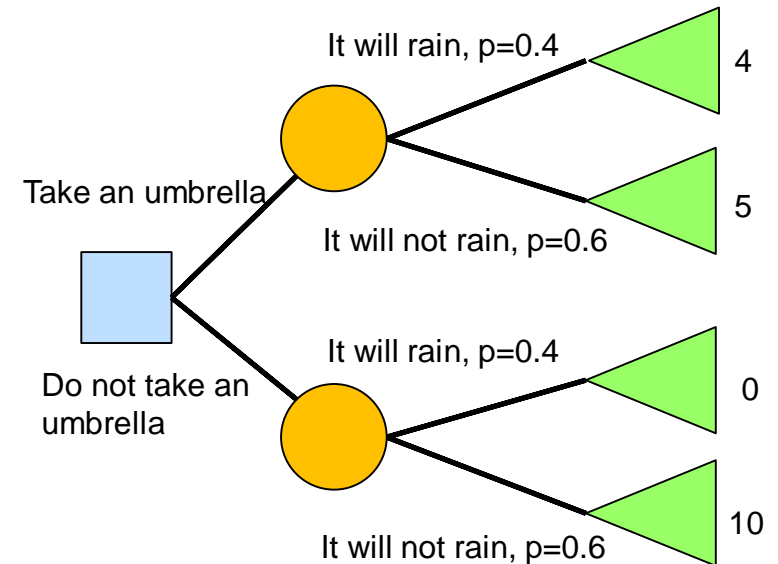
- ❑ We had a possibility to influence the probability **p(this metro train is on time | metro driver of this train is sick)** by use of extra personnel (*help*) at a cost (**Cost 2**)?

- ❑ Now the event probabilities depend on our **decision**



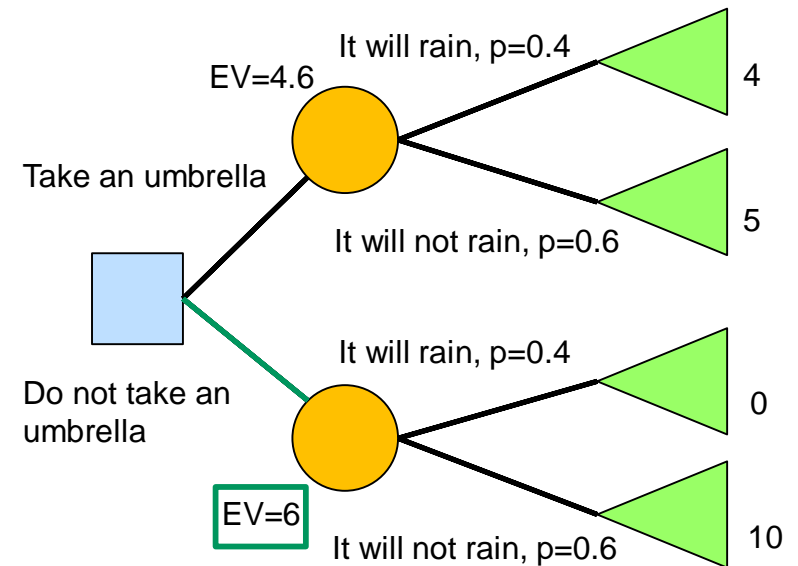
# Decision trees

- ❑ Decision-making under uncertainty can be modeled by a decision tree
- ❑ Decision trees consist of
  - **Decision nodes** (squares) – DM can choose which arc to follow
  - **Chance nodes** (circles; cf. states of nature) – chance represented by probabilities dictates which arc will be followed (states of nature). The probabilities following a chance node must sum up to 1
  - **Consequence nodes** (triangles; resulting consequences) – at the end of the tree; describe the consequence (e.g., profit, cost, revenue, utility) of following the path leading to this node
- ❑ Decisions and chance events are displayed in a **logical temporal sequence from left to right**
  - ❑ Only chance nodes whose results are known can precede a decision node
- ❑ Each chain of decisions and chance events represents a possible outcome



# Solving a decision tree

- ❑ A decision tree is solved by starting from the leaves (**consequence nodes**) and going backward toward the root:
  - At each **chance node**: compute the expected value at the node
  - At each **decision node**: select the arc with the highest expected value
- ❑ The optimal strategy consists of the arcs selected at **decision nodes**



# Example: Decision tree (1/12)

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- Your uncle is going to buy a tractor. He has two alternatives:
  1. A new tractor (17 000 €)
  2. A used tractor (14 000 €)
- The engine of the old tractor may be defect, which is hard to ascertain. Your uncle estimates a 15 % probability for the defect.
- If the engine is defect, he has to buy a new tractor and gets 2000 € for the old one.
- Before buying the tractor, your uncle can take the old tractor to a garage for an evaluation, which costs 1 500 €
  - If the engine is OK, the garage can confirm it without exception.
  - If the engine is defect, there is a 20 % chance that the garage does not notice it.
- Your uncle maximizes expected monetary value

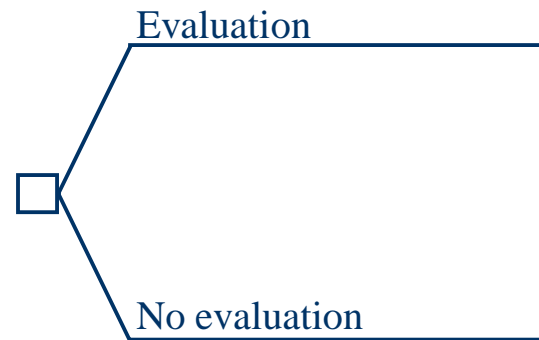




# Example: Decision tree (2/12)

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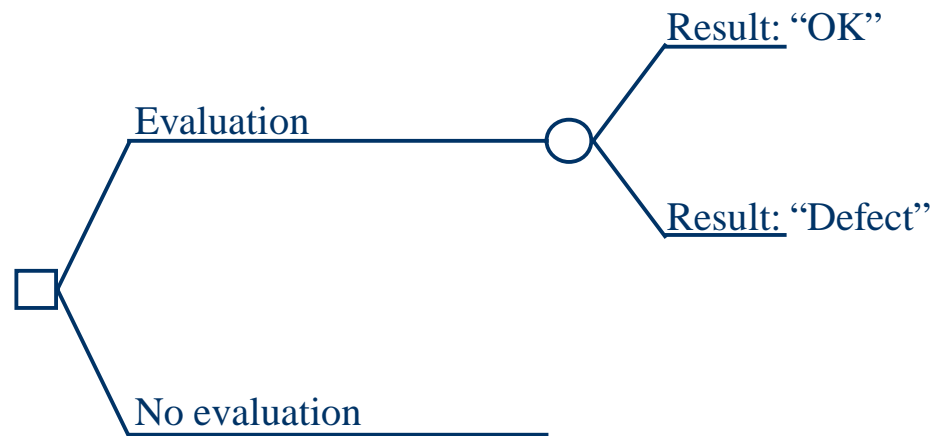
- Before making the buying decision and before you get to know the result of any uncertain event, you must **decide** upon taking the old tractor to a garage for an **evaluation**.
- The decision node 'evaluation' is placed leftmost in the tree



# Example: Decision tree (3/12)

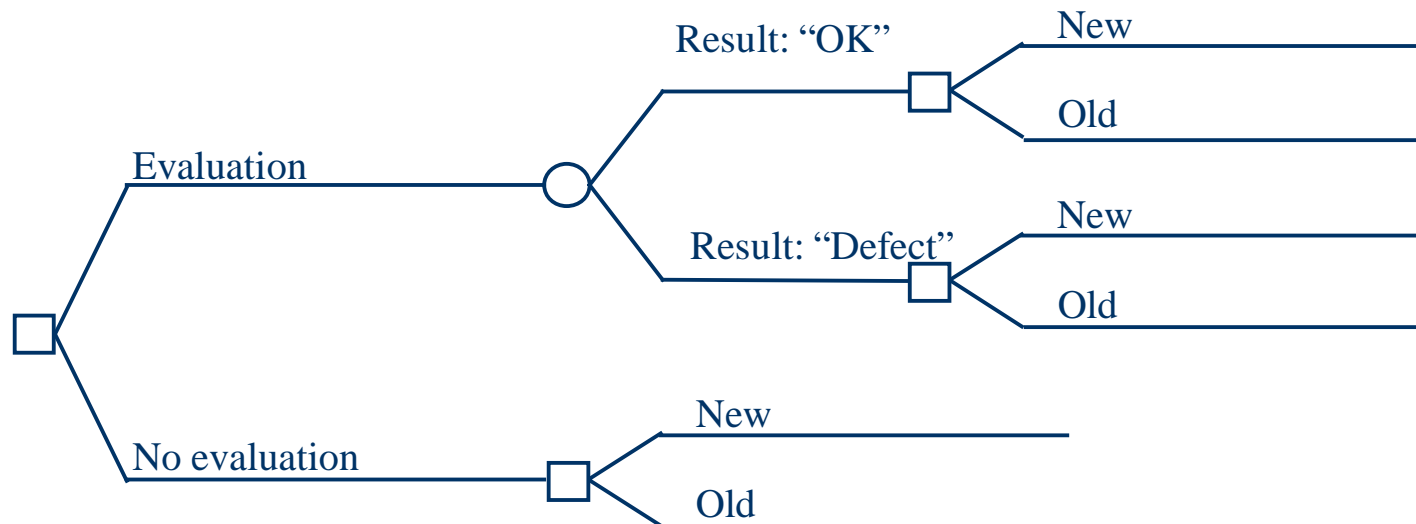
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- If the old tractor is evaluated, your uncle receives the results of the evaluation



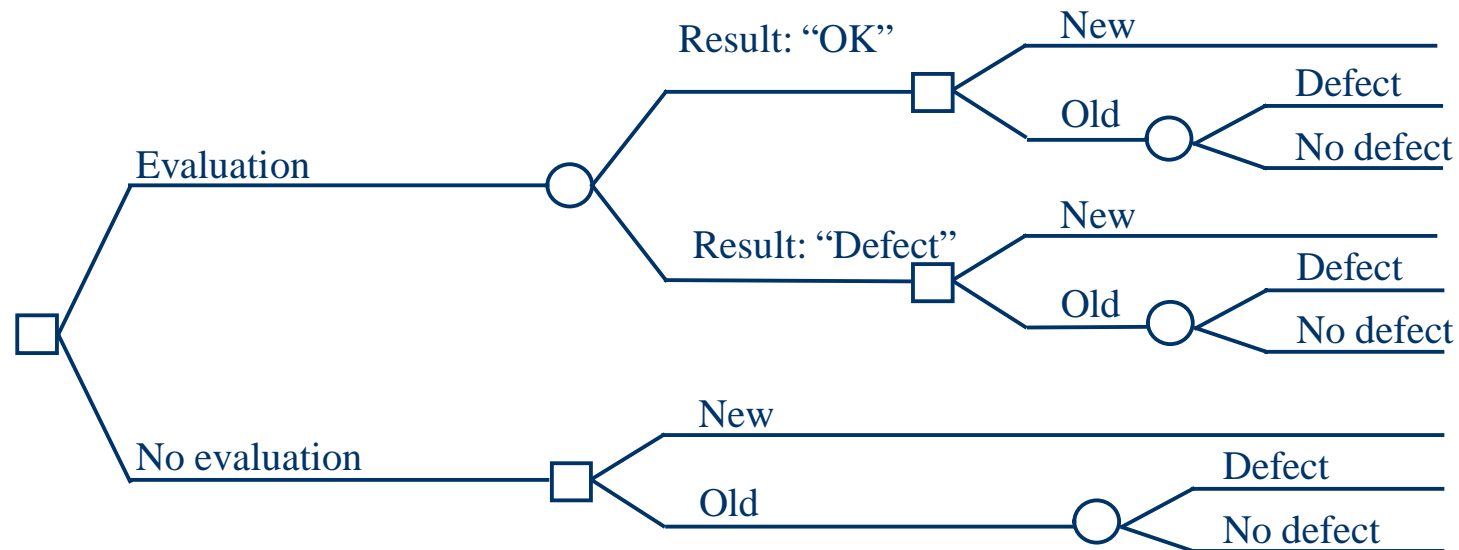
# Example: Decision tree (4/12)

- The next step is to decide which tractor to buy



# Example: Decision tree (5/12)

- ...But the engine of the old tractor can be defect

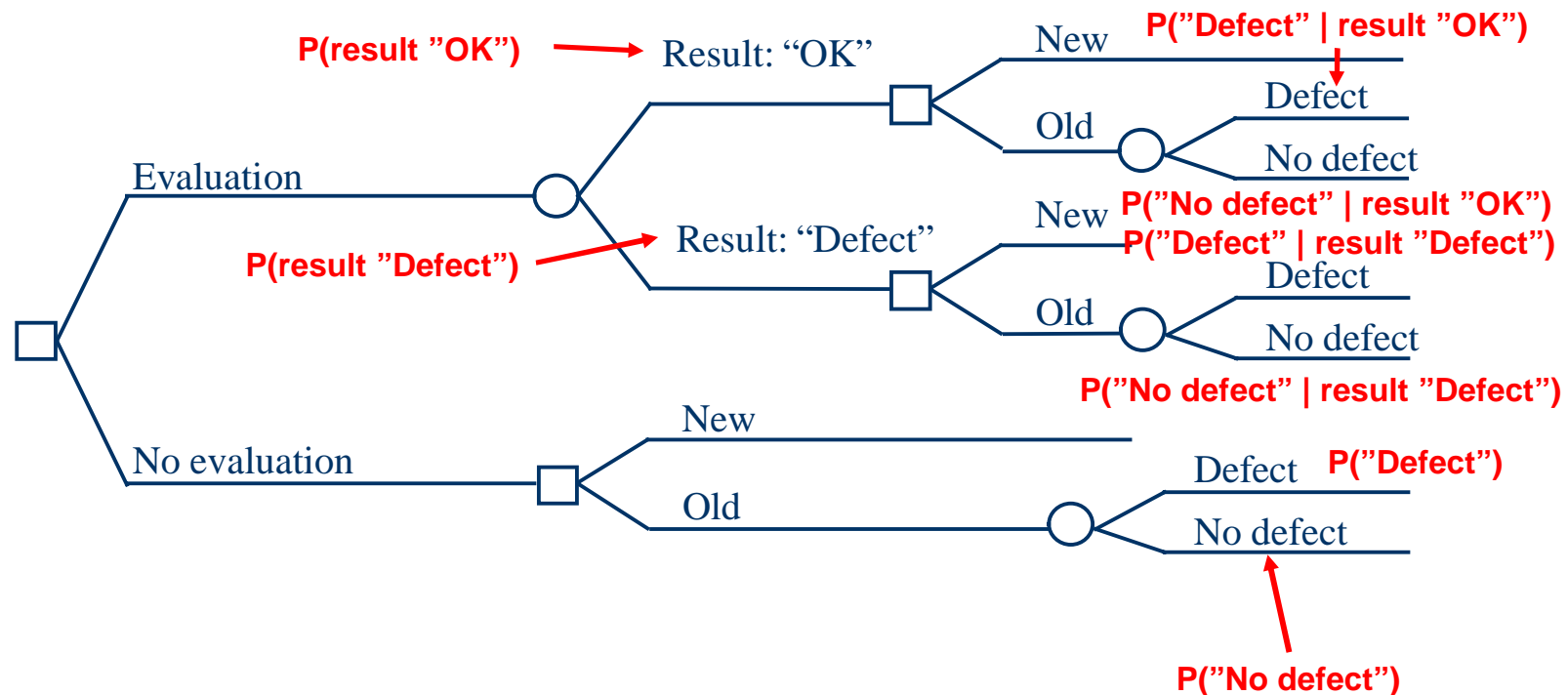


- Now all chance nodes and decisions are in chronological order such that in each node, we can follow the path to the left to find out what we know



# Example: Decision tree (6/12)

- We next need the probabilities for all outcomes of the chance nodes



# Remember: Law of total probability

□ If  $E_1, \dots, E_n$  are mutually exclusive and  $A = \bigcup_i E_i$ , then

$$P(A) = P(A|E_1)P(E_1) + \dots + P(A|E_n)P(E_n)$$

□ Most frequent use of this law:

- Probabilities  $P(A|B)$ ,  $P(A|B^c)$ , and  $P(B)$  are known
- These can be used to compute  $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$

# Remember: Bayes' rule

□ **Bayes' rule:**  $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

□ Follows from

1. The definition of conditional probability:  $P(A|B) = \frac{P(A \cap B)}{P(B)}$ ,  $P(B|A) = \frac{P(B \cap A)}{P(A)}$ ,
2. Commutative laws:  $P(B \cap A) = P(A \cap B)$ .

# Example: Bayes' rule

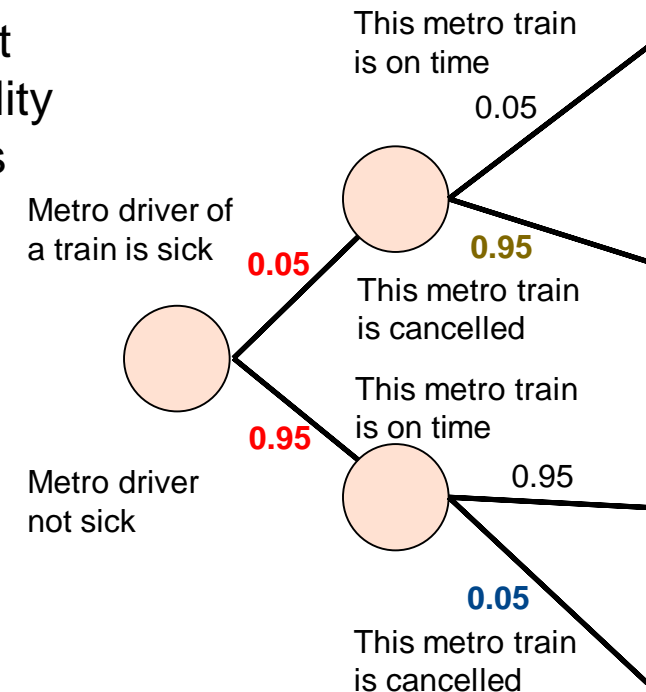
A metro train is cancelled (event C) and we have not had the opportunity to *call help*. What is the probability that the driver originally allocated to drive the train is sick (event S)? = What is  $P(S|C)$ ?

## Solution:

□  $P(S)=0.05$ ,  $P(S^c)=0.95$ ,  $P(C|S)=0.95$ ,  $P(C|S^c)=0.05$

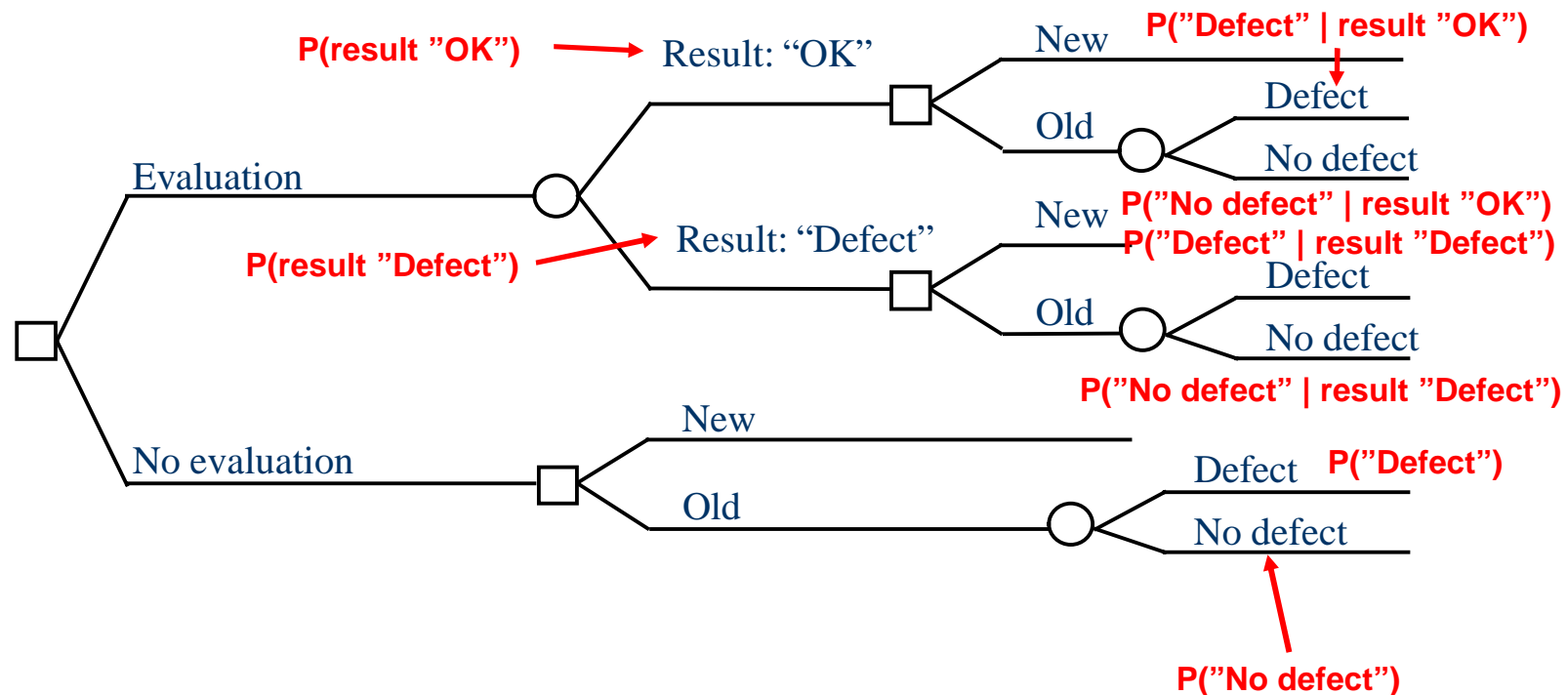
Law of total probability:  $P(C)=P(C|S)P(S)+P(C|S^c)P(S^c)=0.95 \times 0.05 + 0.05 \times 0.95 = 0.095$

Bayes' rule:  $P(S|C) = \frac{P(C|S)P(S)}{P(C)} = \frac{0.95 \cdot 0.05}{0.095} = 50\%$





# Example: Decision tree (6/12)



# Example: Decision tree (7/12)

- Solve all probabilities. You know that
  - "Your uncle estimates a 15 % probability for the defect." =>  $P(\text{Defect})=0.15$
  - "If the engine is OK, the garage can confirm it without exception." =>  $P(\text{result "OK"} \mid \text{No defect})=1$
  - "If the engine is defect, there is a 20 % chance that the garage does not notice it." =>  $P(\text{result "OK"} \mid \text{Defect})=0.20$

$$P(\text{result "OK"}) = P(\text{result "OK"} \mid \text{No defect}) \cdot P(\text{No defect}) + P(\text{result "OK"} \mid \text{Defect}) \cdot P(\text{Defect}) \\ = 1.0 \cdot 0.85 + 0.20 \cdot 0.15 = 0.88$$

$$P(\text{result "defect"}) = 1 - P(\text{result "OK"}) = 0.12$$

$$P(\text{Defect} \mid \text{result "OK"}) = \frac{P(\text{result "OK"} \mid \text{Defect}) \cdot P(\text{Defect})}{P(\text{result "OK"})} = \frac{0.20 \cdot 0.15}{0.88} \approx 0.034$$

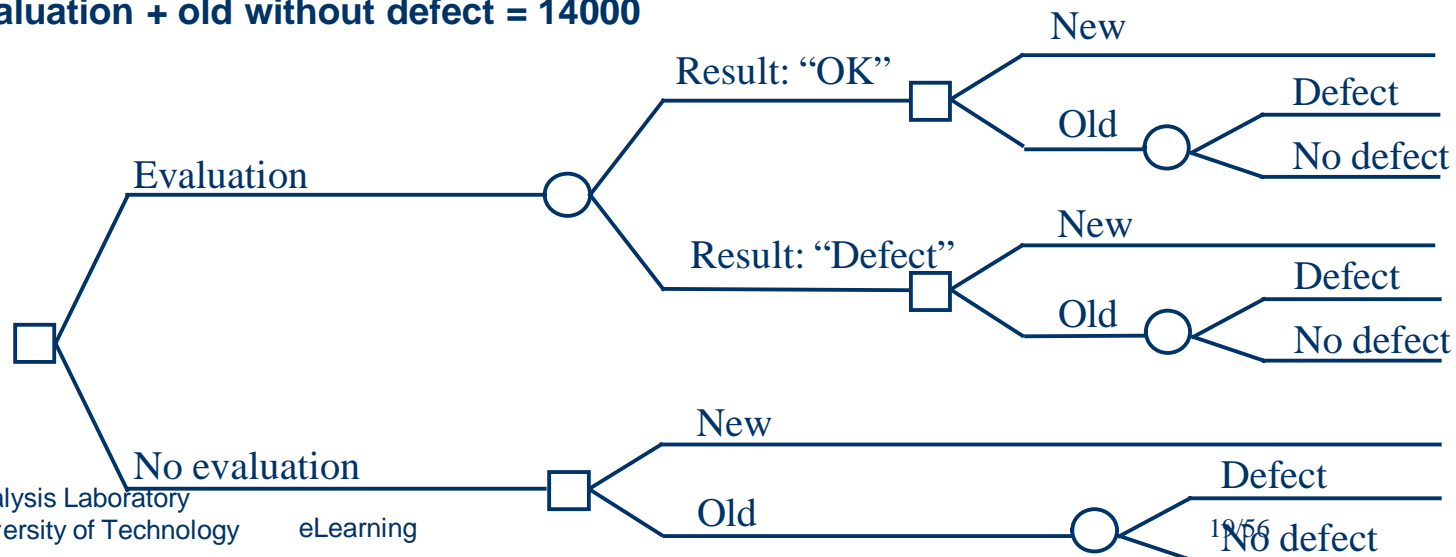
$$P(\text{No defect} \mid \text{result "OK"}) = 1 - 0.034 = 0.966$$

$$P(\text{Defect} \mid \text{result "defect"}) = \frac{P(\text{result "defect"} \mid \text{Defect}) \cdot P(\text{Defect})}{P(\text{result "defect"})} = \frac{0.80 \cdot 0.15}{0.12} = 1.00$$

$$P(\text{No Defect} \mid \text{result "defect"}) = 1 - 1 = 0$$

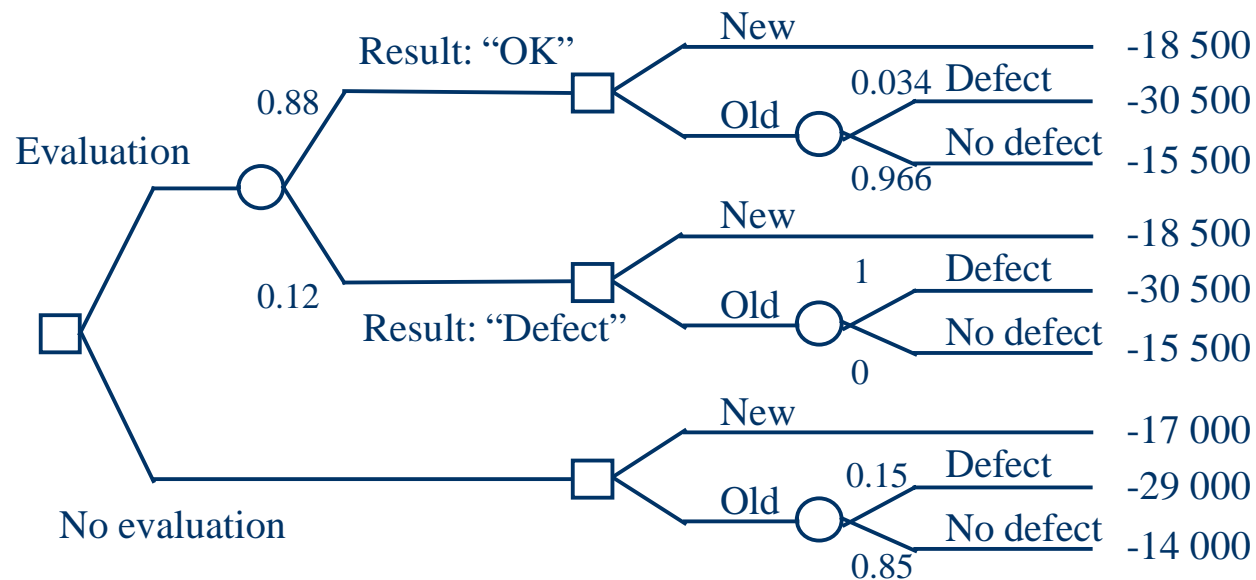
# Example: Decision tree (8/12)

- Compute monetary values for each **end node**
  - **Evaluation + new** =  $1500 + 17000 = 18500$
  - **Evaluation + old with defect** =  $1500 + 14000 - 2000 + 17000 = 30500$
  - **Evaluation + old without defect** =  $1500 + 14000 = 15500$
  - **No evaluation + new** =  $17000$
  - **No evaluation + old with defect** =  $14000 - 2000 + 17000 = 29000$
  - **No evaluation + old without defect** =  $14000$



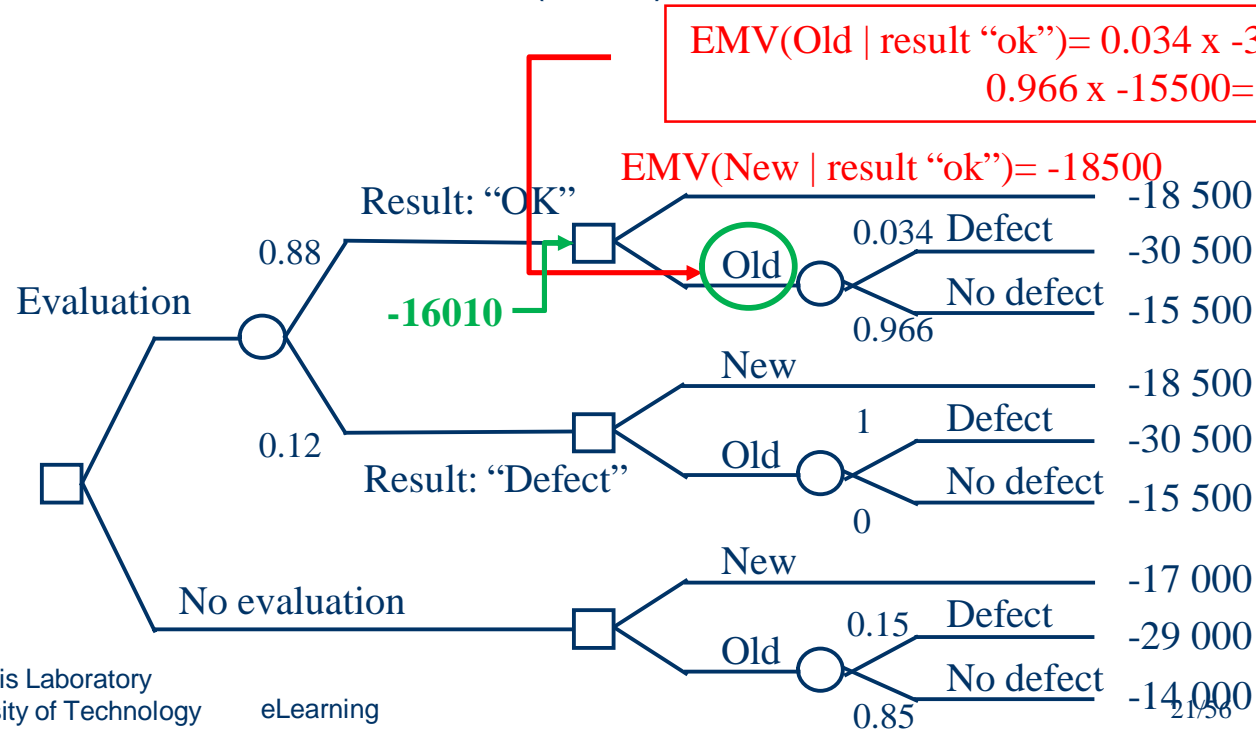
# Example: Decision tree (9/12)

- We now have a decision tree presentation of the problem



# Example: Decision tree (10/12)

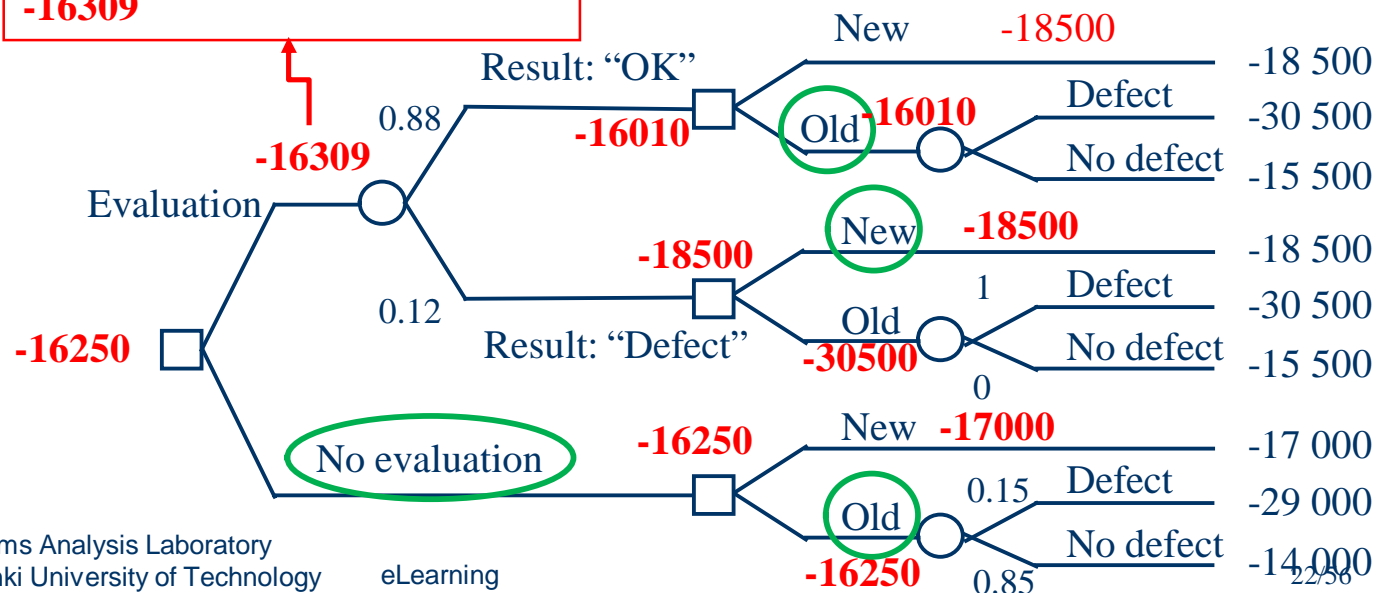
- Starting from the right, compute expected monetary values for each decision
- Place the value of the **better** (=best) decision to the decision node



# Example: Decision tree (11/12)

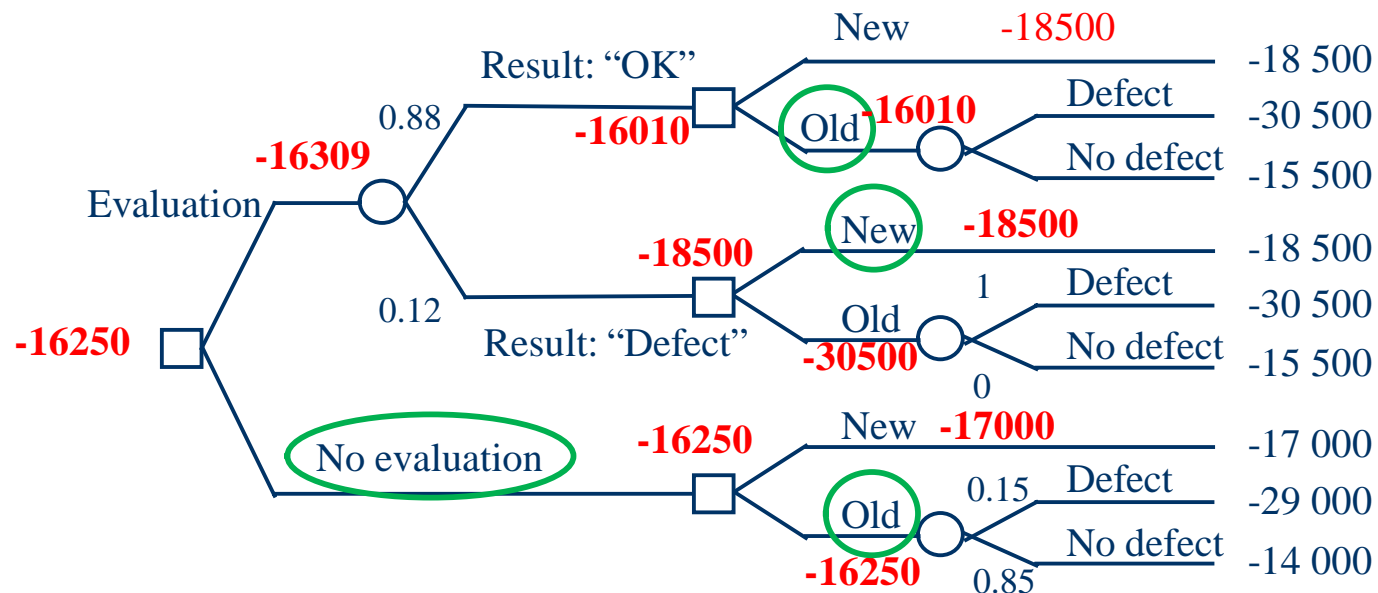
- Starting from the right, compute expected monetary values for each decision
- Place the value of the **better** decision to the decision node

$$0.88 \times -16010 + 0.12 \times -18500 = -16309$$



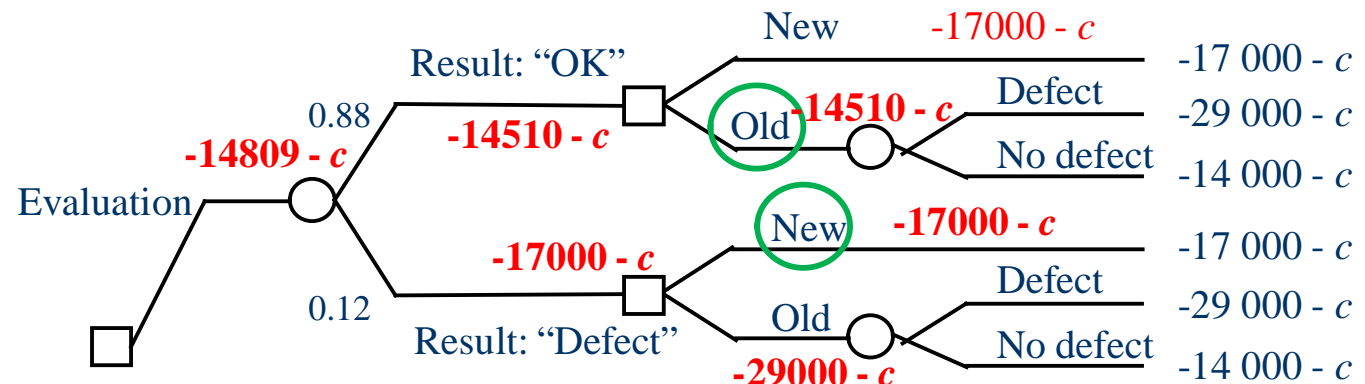
# Example: Decision tree (12/12)

- The optimal solution is to buy the old tractor without evaluating it



## ... How much should we pay for the sample information by the garage?

- ❑ The expected monetary value was higher without evaluating the old tractor
- ❑ Determine evaluation cost  $c$  so that you are **indifferent** between
  1. Not taking the old tractor for an evaluation (EMV = -16250€)
  2. Taking the old tractor for an evaluation



- ❑ Indifference, when EMVs equal:  $-16250 = -14809 - c \Rightarrow c = 1441\text{€}$ 
  - ❑ Expected value of sample information = Expected value with sample information – Expected value without sample information =  $-14809\text{€} - (-16250\text{€}) = 1441\text{€}$



# Expected value of perfect information

You are participating in a gambling game, where Mia has tossed a coin. Mia knows the result of the toss.

You can choose between *heads* or *tails*. If you choose correctly, Bernard will pay you 50€ Otherwise you pay Bernard 50€ You are maximizing EMV.

Mia suggests that you could offer her money and she'd tell you the correct choice. How much would you at most pay her?

<https://presemio.aalto.fi/coin>



# Expected value of perfect information

You can choose between *heads* or *tails*. If you choose correctly, Bernard will pay you 50€ Otherwise you pay Bernard 50€ You are maximizing EMV.

EMV without perfect information =  $0.5 \times 50\text{€} + 0.5 \times (-50\text{€}) = 0\text{€}$

Mia suggests that you could offer her money and she'd tell you the correct choice. How much should you at most pay her?

EMV with perfect information =  $1 \times 50\text{€} + 0 \times (-50\text{€}) = 50\text{€}$

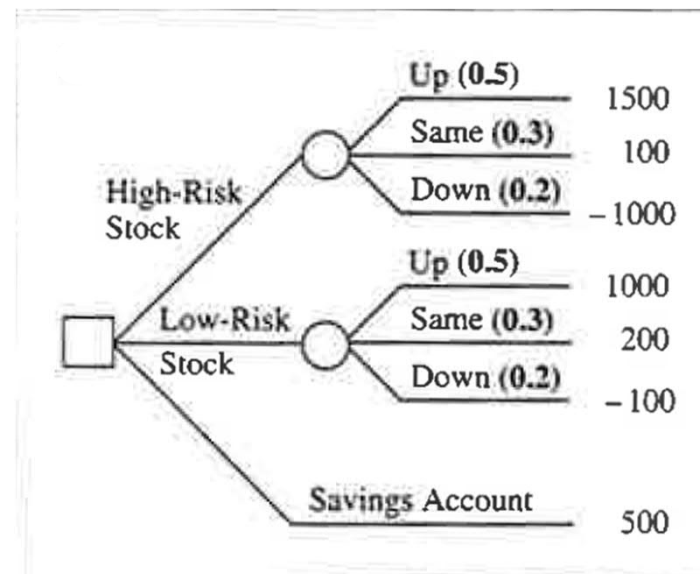
You should pay at most  $50\text{€} - 0\text{€} = 50\text{€}$

# Example: expected value of perfect information

- ❑ You are considering between three investment alternatives: high-risk stock, low-risk stock, and savings account
- ❑ Savings account: certain payoff of 500€
- ❑ Stocks:
  - 200€ brokerage fee
  - Payoffs depend on market conditions

	Up	Same	Down
<b>High-risk</b>	1700	300	-800
<b>Low-risk</b>	1200	400	100
Probability	0.5	0.3	0.1

## Decision tree



Source: Clemen, R.T. (1996): Making Hard Decisions: An Introduction to Decision Analysis, 2nd edition, Duxbury Press, Belmont.

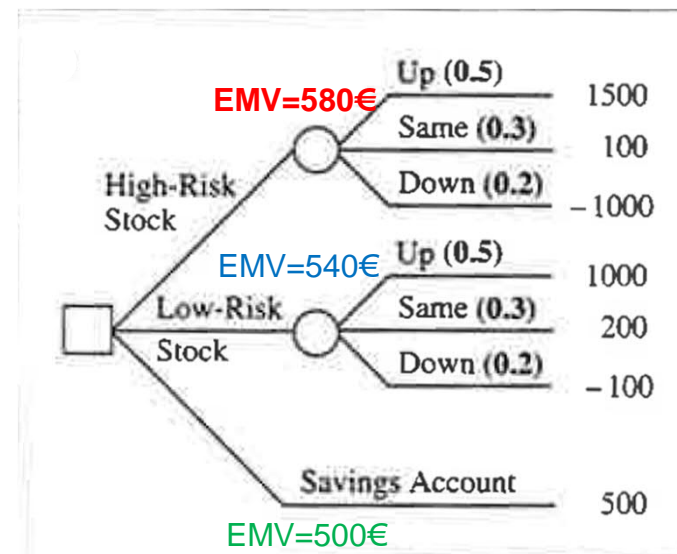
# Example: investing in the stock market

- The expected monetary values (EMVs) for the different alternatives are

- **HRS**:  $0.5 \cdot 1500 + 0.3 \cdot 100 - 0.2 \cdot 1000 = 580$
- **LRS**:  $0.5 \cdot 1000 + 0.3 \cdot 200 - 0.2 \cdot 100 = 540$
- **Savings Account**: 500

→ It is optimal\* to invest in **high-risk** stock

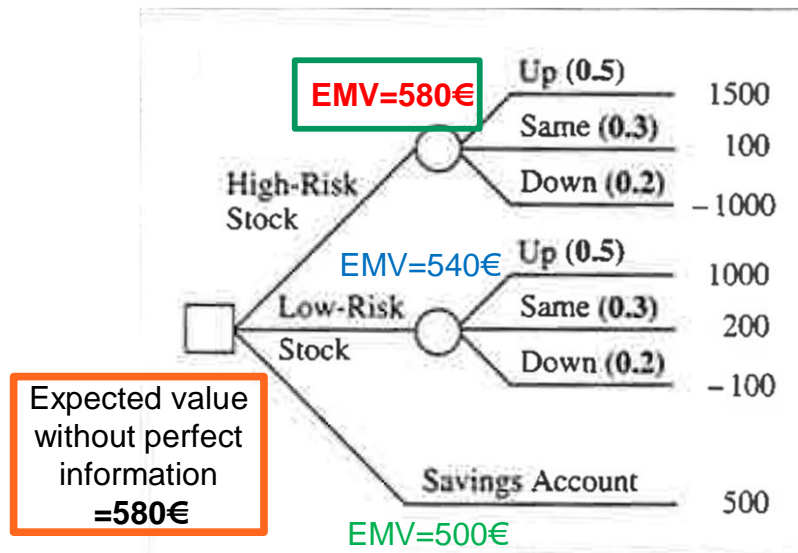
## Decision tree



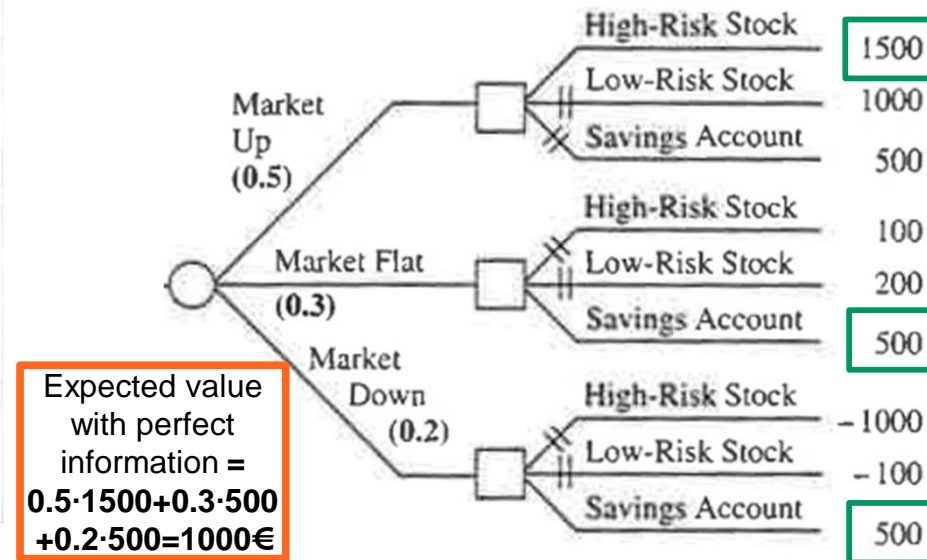
Source: Clemen, R.T. (1996): Making Hard Decisions: An Introduction to Decision Analysis, 2nd edition, Duxbury Press, Belmont.

# Expected value of perfect information

Decision tree



Reversed decision tree: you know the state of the world when making the decision(s)



Expected value of perfect information  
 $= 1000€ - 580€ = 420€$

# Expected value of perfect information

- ❑ How much could the expected value be expected to increase, if
  - Additional information about the uncertainties was received before the decision
  - The decision would be made according to this information?
    - *Note: this analysis is done before any information is obtained*
- ❑ Perfect information: certain information about how the uncertainties are resolved – “if we could choose after we know the state of the world”
  - ❑ Expected value of perfect information = *Expected value with perfect information – Expected value without perfect information*
- ❑ **Expected value of perfect information** is computed through a reversed decision tree in which all chance nodes precede all decision nodes

# Probability assessment

- ☐ Go to the course's MyCourses site
  - ☐ <https://mycourses.aalto.fi/course/view.php?id=29666>
- ☐ Use a few minutes to answer
  - 9 probability assessment questions
    - *Either questionnaire sheet A or B*
  - To the Fractile assessment task
- ☐ Do not communicate with others
- ☐ Do not look up the answers on the internet

## Question 2 should read (the text is too long for the MyCourses module)

Consider two bags X and Y. Bag X contains 30 white balls and 10 black balls, whereas bag Y contains 30 black balls and 10 white balls. Suppose that you select one of these bags at random, and randomly draw five balls one-by-one by replacing them in the bag after each draw. Suppose you get four white balls and one black. What is the probability that you selected bag X with mainly white balls?



# Estimation of probabilities

## ❑ How to obtain the probabilities needed in decision models?

1. If possible, use *objective* data
2. If objective data is not available, obtain *subjective* probability estimates from experts through
  - Betting approach
  - Reference lottery
  - Direct judgement

# Estimation of probabilities: Betting approach

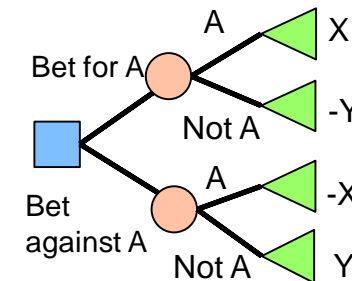
## ❑ Goal: to estimate the probability of event A

- E.g., A="GDP growth is above 3% next year" or A="Sweden will join NATO within the next five years"

## ❑ Betting approach:

- Bet for A: win X € if A happens, lose Y € if not
  - Expected monetary value  $X \cdot P(A) - Y \cdot [1 - P(A)]$
- Bet against A: lose X € if A happens, win Y € if not
  - Expected monetary value  $-X \cdot P(A) + Y \cdot [1 - P(A)]$
- Adjust X and Y until the respondent is indifferent between betting for or against A
- Assuming risk-neutrality<sup>(\*)</sup>, the expected monetary values of betting for or against A must be equal:

$$X \cdot P(A) - Y \cdot [1 - P(A)] = -X \cdot P(A) + Y \cdot [1 - P(A)] \Rightarrow P(A) = \frac{Y}{X + Y}$$



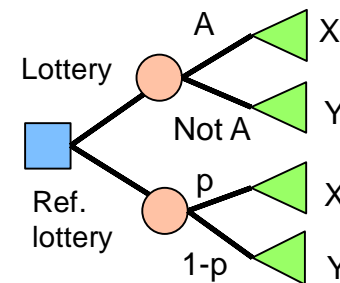
# Estimation of probabilities: Reference lottery

## ❑ Lottery:

- Win X if A happens
- Win Y if A does not happen
- X is preferred to Y

## ❑ Reference lottery:

- Win X with (known) probability  $p$
- Win Y with (known) probability  $(1-p)$
- Probability  $p$  can be visualized with, e.g., a wheel of fortune



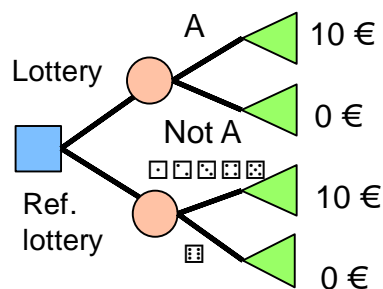
## ❑ Adjust $p$ until the respondent is **indifferent** between the **two lotteries**:

$$X \cdot P(A) + Y \cdot [1 - P(A)] = X \cdot p + Y \cdot [1 - p] \Rightarrow P(A) = p$$

## ❑ Here, the respondent's risk attitude does not affect the results (shown later)

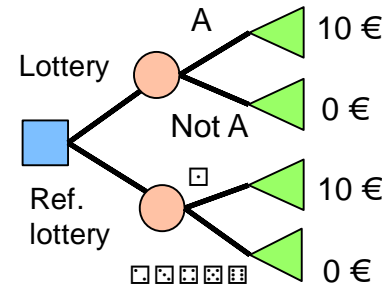
# Reference lottery: example

□ **Event A:** "HIFK wins Jokerit"



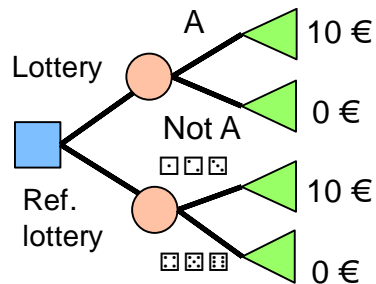
The respondent chooses the reference lottery:

$$10 \cdot P(A) < 10 \cdot \frac{5}{6}$$



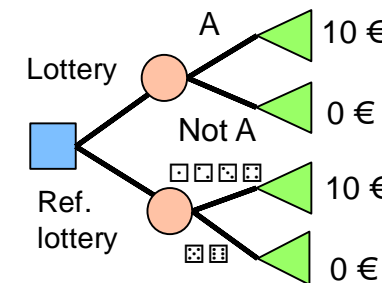
The respondent chooses the lottery:

$$10 \cdot P(A) > 10 \cdot \frac{1}{6}$$



Chooses the lottery:

$$P(A) > \frac{1}{2}$$



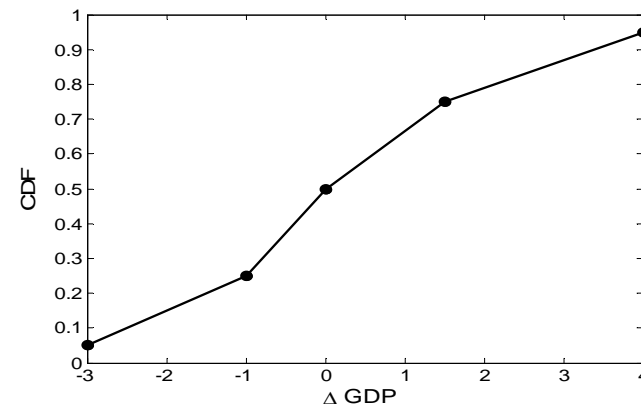
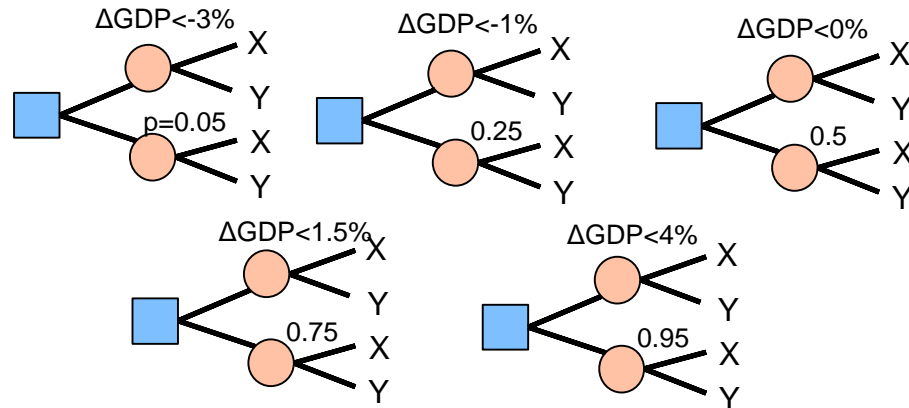
Chooses the reference lottery:

$$P(A) < \frac{2}{3}$$

These four answers revealed the **probability estimate of A** to be **in (0.5, 0.67)**. Further questions should reveal the respondent's estimate for  $P(A)$

# Estimation of continuous probability distributions

- ❑ A continuous distribution can be approximated by estimating several event probabilities (X is preferred to Y)
- ❑ Example:
  - Goal: to assess the distribution of the change in GDP ( $\Delta\text{GDP}$ ) in Finland next year
  - Means: elicitation of probability  $p$  for five different reference lotteries



# Estimation of continuous probability distributions

- ❑ Often experts assess the descriptive statistics of the distribution directly, e.g.,
  - The feasible range (min, max)
  - Median  $f_{50}$  (i.e.,  $P(X < f_{50}) = 0.5$ )
  - Other quantiles (e.g., 5%, 25%, 75%, 95%)
- ❑ In the previous example:
  - "The 5% and 95% quantiles are  $f_5 = -3\%$  and  $f_{95} = 4\%$ "
  - "The change in GDP is just as likely to be positive as it is to be negative"
  - "There is a 25% chance that the change in GDP is below  $-1\%$ "
  - "There is a 25% chance that the change in GDP is above  $1.5\%$ "

# Summary

- ❑ Decision trees are probability-based models to support decision-making under uncertainty
  - Which decision alternative should I choose?
  - How much would I be willing to pay for perfect information or (imperfect) sample information about how the uncertainties are resolved?
  
- ❑ Subjective probability assessments often required
  - Probability elicitation techniques require some effort

# Next time

- **Are ‘experts’ capable of giving good (probability) estimates?**
  - What kind of *elicitation biases* should we be aware of?
- **How to model the DM’s preferences over risky alternatives?**
  - What if the DM does not maximize expected monetary value, but has an attitude towards risk?
    - *Would you*
      1. choose 5000 € for sure OR
      2. participate in 50-50 lottery between 0 € and 12000 €?
  - What were the magic numbers in the umbrella example?