Assignment 3

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Contents

1 Attribute-specific value functions - value functions

The value functions are as follows:

$$v_1(x_1) = \frac{1}{40}x_1 \tag{1}$$

$$v_2(x_2) = \begin{cases} 0, & 0 \le x_2 \le 2 \\ \frac{1}{12}x_2 - \frac{1}{6}, & 2 \le x_2 < 6 \\ \frac{1}{27}x_2 + \frac{1}{9}, & 6 \le x_2 < 15 \\ \frac{1}{45}x_2 + \frac{1}{3}, & 15 \le x_2 \le 30 \end{cases} \tag{2}$$

$$v_3(x_3) = \begin{cases} \frac{1}{14}x_3 - \frac{1}{7} & 2 \le x < 9 \\ \frac{1}{42}x_3 + \frac{2}{7} & 9 \le x \le 30 \end{cases} \tag{3}$$

$$v_4(x_4) = \frac{1}{18}x_4 - \frac{1}{9} \tag{4}$$

$$v_5(x_5) = \frac{1}{20}x_5 \tag{5}$$

$$v_6(x_6) = \frac{1}{20}x_6 \tag{6}$$

$$v_7(x_7) = \frac{1}{100}x_7 \tag{7}$$

$$v_8(x_8) = \begin{cases} \frac{1024}{1023}\left(1 - 2^{-x_8}\right) & 0 \le x_8 < 10 \\ 1 & 10 \le x_8 \end{cases} \tag{8}$$

$$v_9(x_9) = \begin{cases} \frac{1}{4}(1-\sqrt{5})\left(1 - \left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^{3-x_9}\right) & 0 \le x_9 < 3 \\ 0 & 3 \le x_9 \end{cases} \tag{9}$$

For the derivation for $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ simple linear interpolation was used based on the given information.

The derivation of v_8 made use of the fact that the ratio $\frac{v_8(k+1)-v_8(k)}{v_8(k)-v_8(k-1)} = 0.5$. Using this and the sum of the geometric sum is $\sum_k ar^k = a\left(\frac{1-r^{n+1}}{1-r}\right)$ we can derive the closed form.

The derivation for v_9 can be seen below.

$$B + A(1 - e^{-(3-x_9)/r}) \text{ and } v_9(0) - v_9(1) = v_9(1) - v_9(3) \tag{10}$$

$$B + A(1 - e^{-(3-x_9)/r}) \tag{11}$$

$$v_9(3) = 0 = B + A(1 - e^{-(3-3)/r}) \iff B = 0 \tag{12}$$

$$v_9(0) = 1 = A(1 - e^{-(3-0)/r}) \tag{13}$$

$$\frac{1}{A} = 1 - e^{-3/r} \tag{14}$$

$$A = \frac{1}{1-u^3} \tag{15}$$

$$\tag{16}$$

$$v_9(100) = 0 = A(1 - e^{-(3-100)/r}) \tag{17}$$

$$\tag{18}$$

$$v_9(0) - v_9(1) = v_9(1) - v_9(3) \quad (19)$$
$$2v_9(1) = v_9(0) \quad (20)$$
$$v_9(1) = \frac{1}{2} \quad (21)$$
$$\frac{1}{2} = \frac{1}{1-u^3}(1-u^2) \quad (22)$$
$$1 - u^3 = 2 - 2u^2 \quad (23)$$
$$-u^3 + 2u^2 - 1 = 0 \implies u = \frac{1}{2} + \frac{\sqrt{5}}{2} \left(\vee u = 1 \vee u = \frac{1}{2} - \frac{\sqrt{5}}{2} \right) \quad (24)$$
$$e^{-1/r} = \frac{1}{2} + \frac{\sqrt{5}}{2} \iff r = \frac{1}{ln(2) - ln(1 + sqrt(5))} \quad (25)$$
$$\implies A = \frac{1}{1 - (1/2 + sqrt(5)/2)} \iff A = \frac{1}{4}(1 - \sqrt{5}) \quad (26)$$
$$\implies v_9 = \frac{1}{4}(1-\sqrt{5}) \left(1 - \left(\frac{1}{2} \left(1 + \sqrt{5} \right)^{(3-x_9)} \right) \right) \quad (27)$$

2 Attribute-specific value functions - values

The attribute values for all alternatives can be found from the attached excel.

3 Attribute-specific value functions - value plots

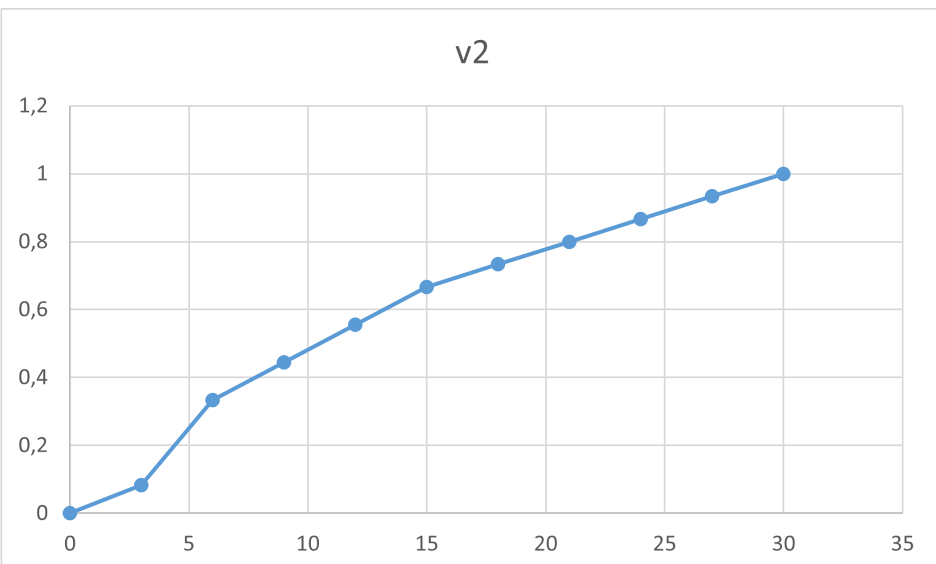

Figure 1: The plot for v_2.

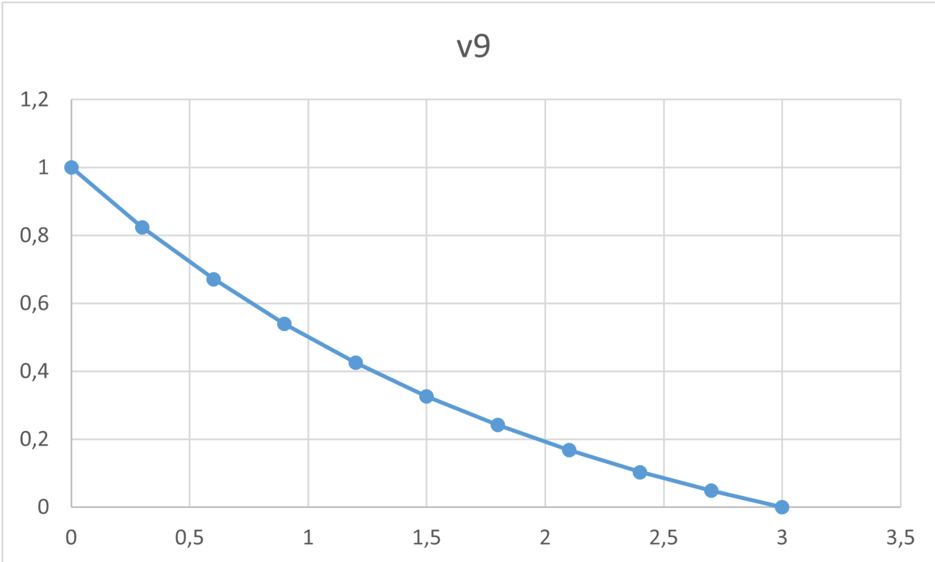

Figure 2: The plot for v_9.

4 Attribute weights

From 1. , and knowing that 20 is the max and 0 the min for both value functions, we get the equation

$$\frac{w_5 v_5(20) - w_5 v_5(0)}{w_6 v_6(20) - w_6 v_6(0)} = \frac{40}{45} \tag{28}$$

$$w_5 = \frac{8}{9} w_6. \tag{29}$$

From 2., and knowing that the max value is 20 and min value is 2, we get

$$\frac{w_6 v_6(20) - w_6 v_6(0)}{w_4 v_4(20) - w_4 v_4(2)} = \frac{45}{30}. \tag{30}$$

$$w_4 = \frac{2}{3} w_6. \tag{31}$$

From 3. we get information about two equally preferred preferences, and again remembering the min value of the value functions,

$$w_8 v_8(1) - w_8 v_8(0) = w_4 v_4(10) - w_4 v_4(2) \tag{32}$$

$$w_8 = \frac{v_4(10)}{v_8(1)} w_4. \tag{33}$$

From 4. we get a similar equation as above

$$w_7 v_7(1) - w_7 v_7(0) = w_4 v_4(3) - w_4 v_4(2) \tag{34}$$

$$w_7 = \frac{v_4(3)}{v_7(1)} w_4. \tag{35}$$

From 5. get multiple the following equality

$$w_1 v_1(40) - w_1 v_1(0) = w_2 v_2(30) - w_2 v_2(0) + w_3 v_3(20) - w_3 v_3(0) \tag{36}$$

$$w_1 = w_2 + w_3 v_3(20). \tag{37}$$

In 6., we know that all but two values do not change, hence we can omit them from the start

$$
\begin{aligned}
w_1 v_1(40) + w_9 v_9(1.2) &= w_1 v_1(10) + w_9 v_9(0) && (38)\\
w_1 + w_9 v_9(1.2) &= w_1 v_1(10) + w_9 && (39)\\
w_1(1 - v_1(10)) &= w_9(1 - v_9(1.2)) && (40)\\
w_9 &= \frac{1 - v_1(10)}{1 - v_9(1.2)} w_1. && (41)
\end{aligned}
$$

In 7., we know that the two changes from the minimum x^0 are equally preferred, i.e

$$
\begin{aligned}
w_4 v_4(10) + w_9 v_9(1.2) - w_4 v_4(2) - w_9 v_9(100) &= w_4 v_4(18) + w_9 v_9(3) - w_4 v_4(2) - w_9 v_9(100) && (42)\\
w_4 v_4(10) + w_9 v_9(1.2) &= w_4 v_4(18) + w_9 v_9(3) && (43)\\
w_4 v_4(10) + w_9 v_9(1.2) &= w_4 v_4(18) && (44)\\
w_9 &= \frac{v_4(18) - v_4(10)}{v_9(1.2)} w_4. && (45)
\end{aligned}
$$

From 8. we get

$$
\begin{aligned}
w_2 v_2(15) - w_2 v_2(0) &= w_3 v_3(30) - w_3 v_3(2) && (46)\\
w_2 v_2(15) &= w_3. && (47)
\end{aligned}
$$

Additionally we know that the sum of weights equals to one

$$\sum_{i=1}^{9} w_i = 1. \tag{48}$$

By solving these equations, we get the following values

$$
\begin{aligned}
w_1 &= 0.062131... \approx 0.06, && (49)\\
w_2 &= 0.0412027... \approx 0.04, && (50)\\
w_3 &= 0.0274685... \approx 0.03, && (51)\\
w_4 &= 0.076779... \approx 0.08, && (52)\\
w_5 &= 0.102372... \approx 0.10, && (53)\\
w_6 &= 0.115168.. \approx 0.12, && (54)\\
w_7 &= 0.42655... \approx 0.43, && (55)\\
w_8 &= 0.0681814... \approx 0.07, && (56)\\
w_9 &= 0.080147... \approx 0.08. && (57)
\end{aligned}
$$

5 Overall values

Table 1 displays the normalized values of the sites times the area.

Table 2: Normalized value functions of the sites times the area.

Site	**Area**	**Normalizied value**
1	1,2	0,34
2	3	1,18
3	2,1	0,95
4	3	0,86
5	0,8	0,34
6	2	0,94
7	3	0,94
8	0,9	0,31
9	1,1	0,50
10	2,4	0,65

Table 1: Normalized vlaue functions of the sites times the area.

Site	**Area**	**Normalizied value**
1	1,2	0,26
2	3	0,88
3	2,1	0,71
4	3	0,64
5	0,8	0,25
6	2	0,70
7	3	0,70
8	0,9	0,23
9	1,1	0,37
10	2,4	0,48

6 Recalculated overall values

Since a_7 most preferred level changes from 100 to 40, the value function becomes $v_7 = \frac{1}{40}x_7$. Thus, the only value in our system of equations for solving the weights that change is $v_7(1)$. The recalculated weights are

$$w_1 = 0.0835016... \approx 0.08, \tag{58}$$
$$w_2 = 0.0553747... \approx 0.06, \tag{59}$$
$$w_3 = 0.0369165... \approx 0.04, \tag{60}$$
$$w_4 = 0.103188... \approx 0.10, \tag{61}$$
$$w_5 = 0.137584... \approx 0.14, \tag{62}$$
$$w_6 = 0.154782.. \approx 0.15, \tag{63}$$
$$w_7 = 0.229306... \approx 0.23, \tag{64}$$
$$w_8 = 0.091633... \approx 0.09, \tag{65}$$
$$w_9 = 0.107714... \approx 0.11. \tag{66}$$

and the scores can be sen in Table 2.

If we calculate $\frac{V_i'(x_i)}{V_i(x_i)}$ we get a constant $\alpha \approx 1.34$ for all i. This suggest that $V_i'(x_i) = \alpha V_i(x_i)$ for all i, i.e. the V' is a affine transformation of V.

7 Site combination selection

The binary linear optimization problem we need to solve is

$$\max_{y} \sum_{j=1}^{10} V(x^j) y_j \tag{67}$$

$$\text{subject to} \sum_{j=1}^{10} c_j y_j \quad \leq 25000 \tag{68}$$

$$y_j \in \mathbb{B}, \tag{69}$$

where b is a binary variable indicating if a site is to be acquired.

The calculations for this part is done in the attached excel. The optimal solution are the following sites: 2, 3, 5, 6, 7, 9. This gives a optimal value of approximately $4,85$ at a cost of 24494€.

8 Multi-objective optimization applied to site combination selection - optimization

The figure produced by the script can be seen in Figure 3.

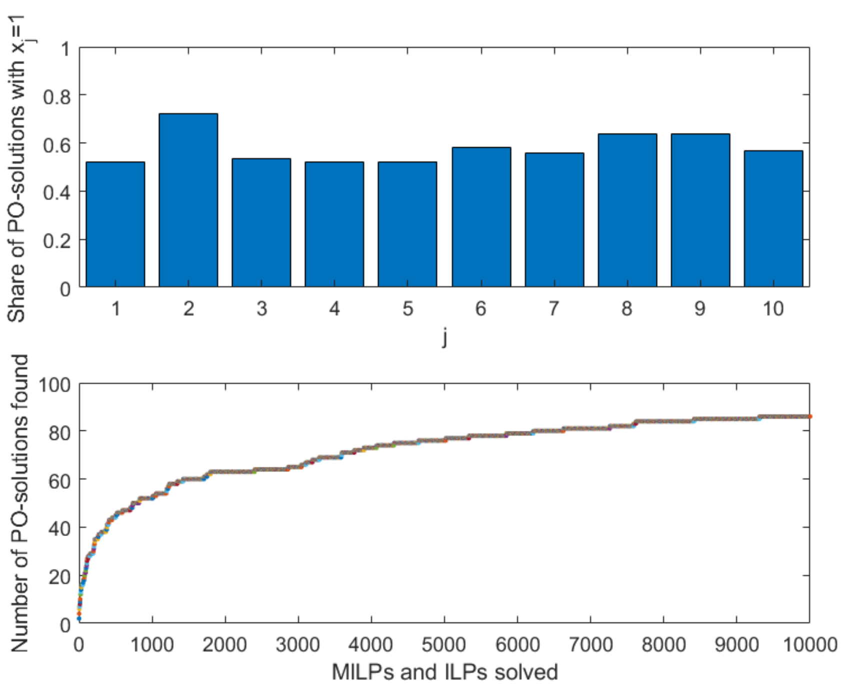

Figure 3: The pareto optimal solutions and core indexes.

9 Multi-objective optimization applied to site combination selection - solutions

86 different pareto optimal solutions were found by the algorithm. The solution is among them, the 7th found pareto optimal solution. The results do not change significantly by running the script multiple

times, all pareto optimal solutions aren't found each time. The core indices are roughly the same however.

10 Multi-objective optimization applied to site combination selection - core indices

The site with the highest core index is site 2 with an approximately index of 72% and the lowest core index are sites 1,4, and 5 with approximately 52%.

11 Robustness analysis of site combination selection - Formulation of feasibility region

The feasible region of S can be defined as

$$S = \{w_i : i = 1...9 | 0.8r_i \leq \frac{w_i}{w_7} \leq 1.25r_i \forall i, \sum_{i=1}^{9} w_i = 1\} \tag{70}$$

$$r_i = \frac{w_i^*}{w_7^*} \tag{71}$$

$$\tag{72}$$

From here we can formulate the linear constraint as

$$0.8\frac{w_i^*}{w_7^*}w_7 - w_i \leq 0 \qquad \forall i \tag{73}$$

$$w_i - 1.25\frac{w_i^*}{w_7^*}w_7 \leq 0 \qquad \forall i \tag{74}$$

12 Robustness analysis of site combination selection - Tables

Table 3 shows the matrix A_w and B_w is simply a 18×1 vector full of zeros.

Table 3: A_w

-1	0	0	0	0	0	0.291319372367055	0	0
0	-1	0	0	0	0	0.193190583761437	0	0
0	0	-1	0	0	0	0.128793838800555	0	0
0	0	0	-1	0	0	0.360001046636372	0	0
0	0	0	0	-1	0	0.480001395515163	0	0
0	0	0	0	0	-1	0.540001569954559	0	0
0	0	0	0	0	0	-0.200000000000000	0	0
0	0	0	0	0	0	0.319688102361037	-1	0
0	0	0	0	0	0	0.375791300707352	0	-1
1	0	0	0	0	0	-0.455186519323524	0	0
0	1	0	0	0	0	-0.301860287127245	0	0
0	0	1	0	0	0	-0.201240373125867	0	0
0	0	0	1	0	0	-0.562501635369332	0	0
0	0	0	0	1	0	-0.750002180492442	0	0
0	0	0	0	0	1	-0.843752453053998	0	0
0	0	0	0	0	0	-0.250000000000000	0	0
0	0	0	0	0	0	-0.499512659939121	1	0
0	0	0	0	0	0	-0.587173907355237	0	1

13 Robustness analysis of site combination selection - Feasible site combinations

We get the number of feasible site combinations as the non-zero values of the vector F: 736.

14 Robustness analysis of site combination selection - Non-dominated feasible combinations

The number of non-dominated, and feasible according to our linear constraints, solutions can be seen from Z_{ND} and is 3.

15 Robustness analysis of site combination selection - Non-dominated sites

The sites that are non-dominated and feasible according to our linear constraints are 2, 3, 6, and 9.

16 Robustness analysis of site combination selection - Previous solution

The reason our previous solution is found among the non-dominated ones is that our original weights are within the feasible region and a optimal solution is always a non-dominated solution.

17 Robustness analysis of site combination selection - Non-dominated combinations

The number of non-dominated solutions can be seen from Z_{ND2} and is 82.

18 Robustness analysis of site combination selection - Too many solutions

Since the amount of solutions found in part 8 is larger than in 17, we can for sure conclude that there are not in the set of non-dominated solutions. This happens since the optimization methods can find solutions that are not actually pareto-optimal solutions.

19 Robustness analysis of site combination selection - Missing solutions

We can find 37 solutions that are non-dominated but not optimal. This is again due to the optimization methods. The randomness used to choose the λ-values can cause the algorithm to never find some optimal solutions.