MS-E2148 Dynamic optimization

Lecture 4

Contents

- Minimum principle in control problems
- Infinite horizon calculus of variations
- Material Kirk 5

Recap

- We derived the necessary conditions to the optimal control problem

Control problem

- We derived the necessary conditions for the extremal using the Hamiltonian

$$H(x, u, p, t) = g(x, u, t) + p^T [f(x, u, t)]$$

from the *costate equations*, *stationary condition*, and *state equation*

$$\begin{aligned} \dot{p}^* &= -H_x(x^*, u^*, p^*, t) \\ 0 &= H_u(x^*, u^*, p^*, t) \\ \dot{x}^* &= H_p(x^*, u^*, p^*, t) \end{aligned} \tag{1}$$

and we need suitable transversality conditions

Constrained control problems

- Pontryagin 1956
- The necessary conditions were derived for the control problem using calculus of variations by assuming that the control can be varied freely

- The control can be varied freely *if and only if* u^* is strictly within the admissible controls (not on the boundary)
- If the control is constrained, the control extremal can be on the boundary of the admissible controls, and the control u cannot be varied freely

$\Rightarrow$ Variation of δJ does not vanish!

Constrained control problems

- Let us choose the candidate $u^* \in U$ within $[t_0, t_f]$ and vary it in its neighbourhood: $u^* \pm \delta u \in U$
- If u^* is between $[t_1, t_2] \subset [t_0, t_f]$ on the boundary ∂U of admissible controls U, then $u^* - \delta u \in U$ but $u^* + \delta u \notin U$ (or vice versa)
- Let us denote $\delta \hat{u} =$ those variations whose negatives $(-\delta \hat{u})$ produce inadmissible control $u^* - \delta \hat{u}$; the necessary condition for these variations $\delta \hat{u}$ is,

$$\delta J(u^*, \delta \hat{u}) \geq 0;$$

  as the necessary condition for other variations is $\delta J(u^*, \delta u) = 0$.

Constrained control problems

- The necessary conditions for the constrained control problem are

$$\begin{aligned} \delta J(u^*, \delta u) &\geq 0 \quad \text{if } u^* \in \partial U \text{ for some time interval} \\ \delta J(u^*, \delta u) &= 0 \quad \text{if } u^* \notin \partial U \text{ never.} \end{aligned} \tag{2}$$

- The variation of J can be represented with the Hamiltonian:

$$\begin{aligned} \delta J(u^*, \delta u) \;=\; & \left[h_x(x^*(t_f), t_f) - p^*(t_f)\right]^T \delta x_f \\ & + \Big[H(x^*(t_f), u^*(t_f), p^*(t_f), t_f) + h_t(x^*(t_f), t_f)\Big] \delta t_f \\ & + \int_{t_0}^{t_f} \Big[\Big(\dot{p}^* + H_x(x^*, u^*, p^*, t)\Big)^T \delta x \\ & + \Big(H_u(x^*, u^*, p^*, t)\Big)^T \delta u \\ & + \Big(H_p(x^*, u^*, p^*, t) - \dot{x}^*\Big)^T \delta p \Big] dt \end{aligned}$$

Constrained control problems

- If the state equation is satisfied, the boundary points are fixed and p^* chosen so that the multiplier of δx is zero, then the variation can be simplified as

$$\delta J(u^*, \delta u) = \int_{t_0}^{t_f} H_u(x^*, u^*, p^*, t)^T \delta u(t) dt$$

- Since the integrand was the first-order Taylor approximation, we write:

$$\delta J(u^*, \delta u) = \int_{t_0}^{t_f} \Big[H(x^*, u^* + \delta u, p^*, t) - H(x^*, u^*, p^*, t) \Big] dt$$

(+higher order terms)

Constrained control problems

Pontryagin minimum principle

- From condition (2) we can derive using δJ

$$H(x^*, u^* + \delta u, p^*, t) \geq H(x^*, u^*, p^*, t)$$

- Since the control can be on the boundary ∂U at any time, we must require

$$H(x^*, u^*, p^*, t) \leq H(x^*, u, p^*, t), \quad \forall t \in [t_0, t_f] \quad \forall u \in U \quad (3)$$

- (3) is the *Pontryagin minimum principle*, and it is necessary condition for u^* to minimize the minimum for the Hamiltonian
- Similary, we have the *maximum principle* if the problem is maximization problem
- Conditions holds only on the optimal trajectory (not like HJB later will hold everywhere)

- Stationary condition $H_u = 0$ can be replaced using the minimum principle, and the necessary conditions are:

$$
\begin{aligned}
\dot{p}^* &= -H_x(x^*, u^*, p^*, t) && \text{costate equation} \\
H(x^*, u^*, p^*, t) &\leq H(x^*, u, p^*, t) && \text{minimum principle} \\
\dot{x}^* &= H_p(x^*, u^*, p^*, t) && \text{state equation}
\end{aligned}
$$

and transversality conditions from the past:

$$
\begin{aligned}
&\Big[h_x(x^*(t_f), t_f) - p^*(t_f)\Big]^T \delta x_f \\
&+ \Big[H(x^*(t_f), u^*(t_f), p^*(t_f), t_f) + h_t(x^*(t_f), t_f)\Big]\delta t_f = 0
\end{aligned}
$$

Constrained control problems

Example

$$\min J(u) = \int_{t_0}^{t_f} \frac{1}{2}\left[x_1^2 + u^2\right] dt$$

s.t.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 + u, \end{aligned}$$

and $x(t_0) = x_0$, t_f is known and $x(t_f)$ is free, and

a) controls are not bounded

b) controls are bounded between $[-1, 1]$

a)

- Hamiltonian: $H = \frac{1}{2}x_1^2 + \frac{1}{2}u^2 + p_1 x_2 - p_2 x_2 + p_2 u$
- Costate equations: $\dot{p}_1^* = -x_1^*$, $\dot{p}_2^* = -p_1^* + p_2^*$
- Stationary condition: $0 = H_u = u^* + p_2^* \Rightarrow u^* = -p_2^*$
- Transversality condition (free end state): $p_1^*(t_f) = p_2^*(t_f) = 0$

b)

- We replace the stationary condition with the minimum principle; The terms in Hamiltonian that depend on u are

$$\frac{1}{2}u^2 + p_2^* u$$

- Based on a) the candidate is $u^* = -p_2^*$ is good if $|u^*| \leq 1$, which is satisfied when $|p_2^*| \leq 1$

Constrained control problems

Example

- The costates are not, however, *bounded*, so $u^* = -p_2^*$ is not admissible control when $|p_2^*| > 1$
- Let us choose $u^* = -1$ when $p_2^* > 1$ and $u^* = 1$ when $p_2^* < -1$ we can satisfy the minimum principle, so the optimal control is

$$u^* = \begin{cases} -1, & p_2^* > 1 \\ -p_2^*, & -1 \leq p_2^* \leq 1 \\ +1, & p_2^* < -1 \end{cases}$$

- This control satisfies the minimum principle since

$$\frac{1}{2}(u^*)^2 + p_2^* u^* \leq \frac{1}{2}u^2 + p_2^* u$$

for all p_2^* and $u \in [-1, 1]$

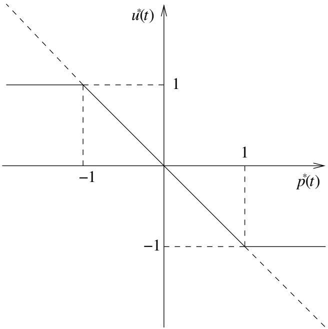
- Check: if $u^* = -1$ and $p_2^* > 1$, the minimum principle is

$$\begin{aligned}\frac{1}{2}(-1)^2 + p_2^*(-1) &\leq \frac{1}{2}u^2 + p_2^* u \\ \Rightarrow -u^2 - 2p_2^* u + 1 - 2p_2^* &\leq 0\end{aligned}$$

  the left-hand side is a downward opening parabola in u that is non-positive between $u \in [-1, 1]$
- If $u^* = 1$ and $p_2^* < -1$, the minimum principle is

$$\begin{aligned}\frac{1}{2}1^2 + p_2^* \cdot 1 &\leq \frac{1}{2}u^2 + p_2^* u \\ \Rightarrow u^2 + 2p_2^* u - (1 + 2p_2^*) &\geq 0\end{aligned}$$

  the left-hand side is upward opening parabola in u that is non-negative between $u \in [-1, 1]$



The optimal control can be given as a function of p_2 in b)

Constrained control problems

Observations

- Like in the example, we have to figure out the control using the minimum principle; then it is enough to consider only the terms that depend on the control in the Hamiltonian

Infinite horizon calculus of variations

- Basic problem in calculus of variations with $t_f \to \infty$:

$$J(x) = \int_{t_0}^{\infty} g(x, \dot{x}, t)dt, \qquad x(t_0) = x_0 \tag{4}$$

- Variation:

$$\begin{aligned} \delta J(x, \delta x) &= g_{\dot{x}}(x, \dot{x}, t)\delta x \Big|_{t_0}^{t_f} \\ &\quad + \int_{t_0}^{t_f} \left[g_x(x, \dot{x}, t) - \frac{d}{dt} g_{\dot{x}}(x, \dot{x}, t) \right]^T \delta x dt \end{aligned}$$

- The initial state is fixed (so $\delta x(t_0) = 0$), and we need some boundary condition beside Euler

Infinite horizon calculus of variations

- Since the final state is not fixed (but the final time is ”fixed” to value $t_f = \infty$), we either require that
  1. $\lim_{t_f \to \infty} \delta x(t_f) = 0$, or
  2. $\lim_{t_f \to \infty} g_{\dot{x}}(x(t_f), \dot{x}(t_f), t) = 0$
- These transversality conditions may not help finding the integration constants, when the infinite horizon problems have to be examined as *special cases* of the basic problem

- We examine two special cases:
  1. Autonomous problem
  2. Most rapid approach path

- The problem (4) is *autonomous* if integrand has no explicit time dependency; often autonomous problems maximize a functional of a form

$$J(x) = \int_0^\infty e^{-rt} g(x, \dot{x}) dt, \qquad x(0) = x_0$$

  where e^{-rt} is a discount factor
- Then we can assume that the solution x^* levels out to some x_s ("steady state") by the time:

$$\lim_{t_f \to \infty} x = x_s$$

- *Stationary condition*: $\dot{x} = \ddot{x} = 0$, which can be used in solving x_s (note limit cycles are possible)
- For example, optimal replacement or epidemic control (how much funds to use for medication)

- **E.g.**: $\int_0^\infty e^{-rt}[x^2 + ax + b\dot{x} + c\dot{x}^2]dt$, $x(0) = x_0$
- Euler: $\ddot{x} - r\dot{x} - \frac{x}{c} = \frac{a+rb}{2c}$
- General solution to Euler: $x_{yl} = K_1 e^{w_1 t} + K_2 e^{w_2 t}$, particular solution $x_{er} = -\frac{a+rb}{2}$, which gives

$$x^* = K_1 e^{w_1 t} + K_2 e^{w_2 t} - \frac{a+rb}{2}$$

  where $w_1, w_2 = r/2 \pm \sqrt{(r/2)^2 + 1/c}$, and K_1, K_2 are integration constants
- The initial condition can be used in solving one of the integration constants but what about the other?

$\Rightarrow$ The objective is autonomous, so we can use the stationary condition $\ddot{x} = \dot{x} = 0$ when $t_f \to \infty$, $x^*(t_f) = x_s$ and from Euler $x_s = -(a + rb)/2$; we get the initial end final conditions

$$\begin{aligned} x^*(0) &= K_1 + K_2 + x_s = x_0 \\ x^*(t_f) &= K_1 e^{w_1 t_f} + K_2 e^{w_2 t_f} + x_s = x_s \end{aligned}$$

Extremal:

$$x^* = (x_0 - x_s)e^{w_2 t} + x_s$$

- Constant $w_2 < 0$, so x^* approaches x_s asymptotically

- The objective is *autonomous* and *linear* in $\dot{x}$:

$$J(x) = \int_0^\infty e^{-rt}[M(x) + N(x)\dot{x}]dt, \qquad x(0) = x_0 \qquad (5)$$

  so that the derivative is bounded with some functions $A(x) \leq \dot{x} \leq B(x)$ for all $t \geq 0$
- Euler equation is $M'(x) + rN(x) = 0$ that is not a differential equation
- Let us assume that there is a unique solution $x^* = x_s$ that is "steady state" solution since the objective function is autonomous, and then use stationary condition!

- The problem starts at x_0. So if we don't have $x_0 = x_s$, the unique solution should be reached *as rapidly as possible*
- Reasoning: we can formulate (5) so that it is independent of $\dot{x}$, which means that the change of x does not cause any cost $\Rightarrow$ go to the steady state as quickly as possible
- Let us define a function $S(x) = \int_0^x N(y)dy$, and we have

$$J(x) = \int_0^\infty e^{-rt}\Big[M(x) + S'(x)\dot{x}\Big]dt \tag{6}$$

- By partial integration we get the second term of (6) as

$$\begin{aligned}\int_0^\infty e^{-rt} S'(x)\dot{x}dt &= e^{-rt}S(x)\Big|_0^\infty - \int_0^\infty -re^{-rt}S(x)dt \\ &= \int_0^\infty re^{-rt}S(x)dt\end{aligned}$$

  and thus the objective functional is

$$J(x) = \int_0^\infty e^{-rt}\Big[M(x) + rS(x)\Big]dt$$

  that is independent of $\dot{x}$

- The solution of the most rapid approach path is then

$$\begin{cases} \dot{x}^* = B(x) & \text{if } x_0 < x_s \\ x^* = x_s & \text{if } x_0 = x_s \\ \dot{x}^* = A(x) & \text{if } x_0 > x_s \end{cases}$$

Summary

- Constrained control problems and Pontryagin minimum principle
- ∞ horizon calculus of variation: autonomous and most rapid approach path