Note! Problem 1.3 c) from the previous round is also presented by students on this round!

Exercise 2.1 (student presents)

Solve the extremal of the functional

$$J = \int_0^1 (t\dot{x} + x^2 + \dot{x}^2)dt$$

where $\boxed{x(0) = 1}$ and $x(1)$ is free.

$\frac{\partial}{\partial \dot{x}} g(x(t_f), \dot{x}(t_f), t_f) = 0$

Solution

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Let's use the Euler equation and the transversality condition.

Euler: $2x - 1 - 2\ddot{x} = 0 \Rightarrow x^*(t) = \dfrac{1}{2} + C_1 e^t + C_2 e^{-t}$

Initial point condition: $\boxed{x^*(0) = \frac{1}{2} + C_1 + C_2 = 1 \Rightarrow C_2 = \boxed{\frac{1}{2} - C_1}}$

Here the end time is fixed $t_f = 1$, but the end state can vary, meaning that $\delta x(t_f)$ is arbitrary. Thus, we use the transversality condition $g_{\dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f) = 0 \Rightarrow 1 + 2\dot{x}^*(1) = 0 \Rightarrow \dot{x}^*(1) = -1/2$. By inserting the solution candidate $x^*(t)$ from above we get

$$\begin{aligned} -\tfrac{1}{2} = \dot{x}^*(1) &= C_1 e - \left(\boxed{\tfrac{1}{2} - C_1}\right) e^{-1} \\ &= C_1(e + e^{-1}) - \tfrac{1}{2} e^{-1}. \end{aligned}$$

$\frac{\partial}{\partial \dot{x}}$

We get $C_1 = \dfrac{1-e}{2(e^2+1)}$, $C_2 = \dfrac{e^2+e}{2(e^2+1)}$, and the extremal is $x^*(t) = \dfrac{1}{2} + \dfrac{1-e}{2(e^2+1)} e^t + \dfrac{e^2+e}{2(e^2+1)} e^{-t}$.

Exercise 2.2 (teacher demo)

a) Find the curve, which is an extremal for the functional

$$J(x) = \int_0^{tf} \sqrt{1 + \dot{x}^2(t)}dt,$$

when $x(0) = 5$ and the end point has to be on the circle $x^2(t) + (t - 5)^2 - 4 = 0$. Check your result geometrically. Hint: The end point condition is of the form $m(x(t_f), t_f) = 0$. Draw a picture to define the dependence of δx_f and δt_f.

b) What if $x(0) = 2$ and the end point is on the curve $\theta(t) = -4t + 5$?

Exercise 2.3 (solved in class)

Find the necessary conditions for the extremal $x^*(t)$, $t_0 \leq t \leq t_1$, which maximizes the functional

$$\int_{t_0}^{t_1} F(x(t), \dot{x}(t), t)dt$$

when t_0 and t_1 are only known. Notice, that $x(t_0)$ and $x(t_1)$ can also be chosen optimally.

Exercise 2.4 (self-study)

Find such extremals for the problem

$$\min \int_0^4 [\dot{x}(t) - 1]^2[\dot{x}(t) + 1]^2 dt$$

$$\begin{cases} x(0) = 0 \\ x(4) = 2 \end{cases}$$

which have one corner.

Exercise 2.5 (home assignment)

Find the extremal of the following functional with the help of Euler equation and the appropriate transversality conditions.

$$\int_0^s \left(\frac{1}{2}\dot{x}(t)^2 - x(t) + \frac{3}{2}\right) dt,$$

when $x(0) = 0$, and s and $x(s)$ are free.