# Notes and Comments on Lecture 1

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The *increment* and the *variation* of a functional have the same meaning as the *difference* and the *differential* of a function in ordinary calculus. Moreover, the formal development of these concepts is analogous whenever $X$ is a (normed) vector space.

In (5), in $\Delta J(x, \delta J)$, $x$ is a fixed vector (e.g., can be a minimum, an extremum $x^*$) at which we write down the expansion (at least up to the second order terms of the Taylor series), for any $\delta x \in X$. $\delta J(x, \delta x)$ is a linear function(al) with respect to $\delta x$. Compare with $f'(x)h$ or $\nabla f(x)^\top h$ (in $\mathbb{R}^n$!) in ordinary calculus for which the definitions and proofs are the same. Now relate $\delta x \Leftrightarrow h$, $\delta J(x, \delta x) \Leftrightarrow \nabla f(x)^\top h$; and $\nabla f(x)^\top h$ is linear in $h$.

In fact, above (when $X$ is normed) we speak about the Frechet derivative (denoted $F$ below). If $X$ is a vector space, we may still define the so-called Gateaux differential (denoted $G$), or the directional derivative $f : X \to \mathbb{R}$,

$$\delta f(x; h) = \frac{d}{d\alpha} f(x + \alpha h)\bigg|_{\alpha=0}$$

Example. $f(x) = \int_0^1 g(x, t)\,dt$, $f : C(0,1) \to \mathbb{R}$; $g_x$ continuous in $X$ and $t$. Then,

$$\delta f(x; h) = \frac{d}{d\alpha} \int_0^1 g(x + \alpha h, t)\,dt\bigg|_{\alpha=0}$$

Since $g_x$ is continuous, equate $\frac{d}{d\alpha}\int_0^1 = \int_0^1 \frac{d}{d\alpha}$;

$$\Rightarrow \delta f(x; h) = \int_0^1 g_x(x, t) h(t)\,dt \quad \square$$

Let $f$ be $G$ at $x_0 \in X$. if $x_0$ is an extremum, then $\delta f(x_0; h) = 0$, $\forall h \in X$. The basic difference between $F$ and $G$ is that $\delta J(x, \delta x)$ is linear with respect to $\delta x$, hence easy to handle ☺; whereas $\delta f(x; h)$ doesn't need to be linear, and it's not easy to handle ☹.

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Show that

$$\int_0^1 (\delta x)^2\,dt = \varepsilon(x, \delta x)\,\|\delta x\|$$

Choose

$$\varepsilon(x,\delta x) = \frac{1}{\|x\|}\int_0^1 (\delta x)^2\,dt$$

Then,

$$|\varepsilon(x,\delta x)| = \frac{1}{\|\delta x\|}\left|\int_0^1 (\delta x)^2\,dt\right| \le \int_0^1 \underbrace{\left|(\delta x)^2\right|}_{\text{function of } t!}\,dt$$

$$\le \int_0^1 \underbrace{\|\delta x\|^2}_{\text{a pure number!}}\,dt = \frac{\|\delta x\|^2}{\|\delta x\|} = \|\delta x\| \to 0, \quad \text{for } \|\delta x\| \to 0 \quad \square$$

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The Fundamental theorem, proof by contradiction. Let $x^*$ be a minimum (or a maximum). Assume that there exists $\delta x$ such that $\delta J(x^*,\delta x) < 0$ ($> 0$, respectively). Then,

$$\delta J(x^*, \alpha\delta x) \overset{\text{homog.}}{=} \alpha\delta J(x^*,\delta x) < 0, \text{ for all } \alpha > 0$$

Now, use (5). Divide both sides by $\alpha$ and let $\alpha \to 0^+$ (note: $\varepsilon(x^*,\alpha\delta x)\,\|\alpha\delta x\| = \alpha\,\|\delta x\|\,\varepsilon(x^*,\alpha\delta x)$ and $\varepsilon(x^*,\alpha\delta x) \to 0$, for $\alpha \to 0^+$) to get $\Delta J(x^*,\alpha\delta x)/\alpha < 0$, for all $\alpha > 0$ sufficiently small, so that $x^*$ cannot be a minimum of $J$. Contradiction! A similar proof if $x^*$ is a maximum. $\square$

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Proof of the Fundamental lemma. Assume $h(t)$ is continuous and $h(t) > 0$ for some $t_0 \in [0,T]$, $h(t)$ is continuous, there exists a closed interval $t_1 \le t_0 \le t_2$ so that $h(t) > 0, \forall\, t \in [t_1, t_2]$. Choose

$$\delta x(t) = \begin{cases} (t-t_1)(t_2-t) & t \in [0,T] \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \int_0^T h(t)\delta x(t)\,dt = \int_{t_1}^{t_2} \underbrace{h(t)}_{>0}\underbrace{(t-t_1)(t_2-t)}_{>0}\,dt > 0, \text{ contradiction!} \quad \square$$