Dynamic Optimization - Homework assignment 3

Christian Segercrantz 481056

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1.5

a)

We have the functional

$$J(x,t) = \int_0^2 2x - 3u - u^2 dt \tag{1}$$

with the end points $x(0) = 5$ and $x(2)$ is free and the differential equation constraint $\dot{x} = x + u$.

To find the optimal control and optimal state trajectory we need the following costate equations and stationary condition:

$$\begin{align}
H(x,u,p) &= g(x,u,t) + p \cdot f(x,u,t) \tag{2}\\
\frac{\partial H}{\partial p} &= \dot{x} \tag{3}\\
-\frac{\partial H}{\partial x} &= \dot{p} \tag{4}\\
\frac{\partial H}{\partial u} &= 0 \tag{5}\\
\frac{\partial h}{\partial x}(x, t_f) - p(t_f) &= 0, \qquad |\delta x_f = arbitary, h = 0 \tag{6}\\
& \tag{7}
\end{align}$$

Which, by inserting our case, becomes

$$\begin{align}
H(x,u,p) &= 2x - 3u - u^2 + p(x+u) \tag{8}\\
x + u &= \dot{x} \tag{9}\\
-2 - p &= \dot{p} \tag{10}\\
-3 - 2u + p &= 0 \tag{11}\\
p(2) &= 0 \tag{12}
\end{align}$$

We can start by finding the optimal Lagrange multiplier by solving the linear differential equation

$$\begin{align}
\dot{p} + p + 2 = 0&, \qquad | \cdot e^t \tag{13}\\
\frac{d}{dt}\left(pe^t\right) = -2e^t&, \qquad |\int dt \tag{14}\\
pe^t &= -2e^t + c_1 \tag{15}\\
p &= c_1 e^{-t} - 2 \tag{16}\\
p(2) = 0 &= c_1 e^{-2} - 2 \iff c_1 = 2e^2 \tag{17}\\
p^* &= 2e^2 e^{-t} - 2 \tag{18}
\end{align}$$

We can now substitute it into the stationary condition to get u

$$\begin{align}
-3 - 2u + p &= 0 \tag{19}\\
2u &= p - 3 \tag{20}\\
u &= \frac{(2e^2e^{-t} - 2) - 3}{2} \tag{21}\\
u^* &= e^2 e^{-t} - \frac{5}{2} \tag{22}
\end{align}$$

We can now substitute u into the first costate equation

$$x + u = \dot{x} \tag{23}$$

$$\dot{x} - x = e^2 e^{-t} - \frac{5}{2}, \qquad | \cdot e^{-t} \tag{24}$$

$$\frac{d}{dt}\left(xe^{-t}\right) = e^2 e^{-2t} - \frac{5}{2}e^{-t} \qquad | \int dt \tag{25}$$

$$xe^{-t} = -\frac{e^2}{2}e^{-2t} + \frac{5}{2}e^{-t} + c_2 \tag{26}$$

$$x = -\frac{e^2}{2}e^{-t} - \frac{5}{2} + c_2 e^t \tag{27}$$

Using the boundary condition we get that

$$x(0) = 5 = -\frac{e^2}{2}e^0 - \frac{5}{2} + c_2 e^0 \iff c_2 = \frac{15 + e^2}{2} \tag{28}$$

and thus have the solutions

$$x^* = -\frac{1}{2}e^{-t+2} + \frac{15 + e^2}{2}e^t - \frac{5}{2} \tag{29}$$

$$u^* = e^{-t+2} - \frac{5}{2} \tag{30}$$

$$p^* = 2e^{-t+2} - 2 \tag{31}$$