Exercise 5.2 (student presents)

We want to move the following system

$$\dot{x} = 2x + u$$

from an arbitrary initial state x_0 to the origin in minimal time, while the control is constrained $|u| \leq 1$. There are some initial states that can not be moved to the origin with any allowed control u. Find those initial states.

Solution

The solution for the state equation is

$$x = \phi(t)[x_0 + \int_0^t \phi(-\tau)u(\tau)d\tau]$$

where $\phi(t) = e^{2t}$. Lets assume we have a control for which $x(T) = 0$; then

$$0 = x_0 + \int_0^T \phi(-\tau)u(\tau)d\tau,$$

which implies that

$$|x_0| = \left| \int_0^T \phi(-\tau)u(\tau)d\tau \right|$$

On the other hand

$$\left| \int_0^T \phi(-\tau)u(\tau)d\tau \right| \leq \int_0^T |\phi(-\tau)||u(\tau)|d\tau$$

and because $|u| \leq 1$, we get

$$
\begin{aligned}
|x_0| &\leq \int_0^T |\phi(-\tau)|d\tau \\
&= \frac{1}{2}[1 - e^{-2T}]
\end{aligned}
$$

i.e.

$$2|x_0| - 1 \leq -e^{-2T} \Rightarrow e^{-2T} \leq 1 - 2|x_0|$$

Because the exponential function is positive for all T, this last condition can only be satisfied if $|x_0| < 1/2$, which is the answer we were looking for.