MS-E2148 Dynamic optimization

Notes and Comments on Lecture 5

Minimum time problem. In equation (1), $\dot{x} = a(x,t) + B(x,t)u,\ x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m,\ a(x,t)$ is an n-component vector function of (x,t); $B(x,t) = \left[b_{ij}(x,t)\right]$ is an $n \times m$ matrix function; $B = [b_1, \cdots, b_m]$, with b_i an $n \times 1$ column vector, for all i. The control functions are bounded by constant m-vectors M^-, M^+ and the inequality $M^- \le u(t) \le M^+$ is defined component-wise.

Let's analyse the bang-off-bang control:

$$|u_i| + p^{*\top} b_i(x^*,t)u_i = \begin{cases} \big(1 + s_i(t)\big)u_i, & u_i \ge 0 \qquad (1) \\ \big(-1 + s_i(t)\big)u_i, & u_i \le 0 \qquad (2) \end{cases}$$

where the components of the switching function are $s_i(t) = p^{*\top} b_i(x^*(t),t)$. The optimal control is:

$$u_i^* = \begin{cases} 1, & s_i(t) < -1 & \text{(a)} \\ 0, & -1 < s_i(t) < 1 & \text{(b)} \\ -1, & s_i(t) > 1 & \text{(c)} \\ \text{undefined but } \ge 0, & s_i(t) = -1 & \text{(d)} \\ \text{undefined but } \le 0, & s_i(t) = 1 & \text{(e)} \end{cases}$$

(a) $\left.\begin{array}{l} (1) \Leftrightarrow (\,) < 0, u_i \ge 0 \Rightarrow \text{ min for } u_i = 1 \\ (2) \Leftrightarrow (\,) < 0, u_i \le 0 \Rightarrow \text{ min for } u_i = 0 \end{array}\right\} \Rightarrow u_i = 1 \text{ is minimizing}$

(b) $0 \le s_i(t) < 1:\ \begin{array}{l} (1) \Leftrightarrow (\,) > 0, u_i \ge 0 \Rightarrow \text{ min for } u_i = 0 \\ (2) \Leftrightarrow (\,) < 0, u_i \le 0 \Rightarrow \text{ min for } u_i = 0 \\ (1)\ \&\ (2) \Rightarrow u_i = 0 \text{ minimizing } -1 \le s_i(t), \text{ analogously} \end{array}$

(c) analysis as in (a)

(d) $s_i(t) = -1:\ \begin{array}{l} (1) \Leftrightarrow (\,) = 0, u_i \ge 0 \Rightarrow \text{ min for } u_i \text{ undefined but} \ge 0 \\ (2) \Leftrightarrow (\,) = -2, u_i \le 0 \Rightarrow \text{ min for } u_i = 0 \\ (1)\ \&\ (2) \Rightarrow \text{ minimizing } u_i \text{ undefined but} \ge 0 \end{array}$

(e) analysis as in (d)

On singular solutions, see the example in [Kirk, pages 300-306].