Exercise 9.1 (solved in class)

The problem is

$$\begin{aligned} \dot{x} &= -x + u - 1, \qquad x(0) = 1, \\ J &= x^2(1) + \int_0^1 u^2 dt. \end{aligned}$$

Solve the optimal control using the HJB equation. Trial function: $V(t, x) = a(t)x^2 + 2b(t)x + c(t)$.

Note! In this exercise, you are allowed to use software for solving differential equations and systems of differential equations, but you must clearly state/mark what inputs you used (it's enough to tell the function numbers, for example) and results you got, and give the name of the software you used. One software option is to use the Wolfram Alpha website.

Exercise 9.2 (selft-study)

a) Form the Hamilton-Jacobi Bellman equation for the problem

$$\min \int_0^T e^{-rt} g(x(t), u(t)) dt \qquad \text{s.t.} \quad \dot{x} = f(x, u) \tag{1}$$

when instead of the function J the function $V(x, t) = e^{rt} J(x, t)$ is used.

b) What role has the factor e^{-rt} in the cost functional? Characterize its role when t varies between zero and infinity. The parameter $r > 0$.

c) Solve the problem

$$\min J = \int_0^\infty e^{-rt} [x^2(t) + u^2(t)] dt$$

that has the scalar system

$$\begin{aligned} \dot{x}(t) &= x(t) + u(t), \qquad r > 0 \\ x(0) &= x_0 > 0 \end{aligned}$$

by using the trial function $V(x, t) = Ax^2$ for the present value of the optimal cost. Apply the HJB equation of part a).

Exercise 9.3 (teacher demo)

The amount of savings in the beginning is S. All income is interest income (interest is constant). The change in capital x is modeled with the equation $\dot{x}(t) = \alpha x(t) - r(t)$, where $\alpha > 0$ is the interest and r is the consumption rate. The utility experienced from momentary consumption r is $U(r)$, where U is the utility; let $U(r) = r^{1/2}$.

The utility that will be achieved in the future is discounted: the utility achieved today ($t = 0$) is more valuable now than in the future. The discount factor is $e^{-\beta t}$ ($\beta > \alpha/2$), thus the present value of the maximized total utility is

$$J = \int_0^T e^{-\beta t} U(r) dt \tag{2}$$

with the end condition $x(T) = 0$ (the capital is fully used).

a) Write the Hamilton-Jacobi-Bellman equation.

b) Solve the HJB equation by using the trial function $J(x, t) = f(t)g(x)$
(hint: attempt $g(x) = Ax^{1/2}$).

c) What is the optimal trajectory of capital?

Exercise 9.4 (homework)

The first order linear system

$$\dot{x}(t) = -10x(t) + u(t)$$

is to be controlled to minimize the performance measure

$$J = \frac{1}{2}x^2(0.04) + \int_0^{0.04} \left[\frac{1}{4}x^2(t) + \frac{1}{2}u^2(t)\right] dt.$$

The admissible state and control values are not constrained by any boundaries. Find the optimal control law by using the Hamilton-Jacobi-Bellman equation. The admissible state and control values are not constrained by any boundaries. Find the optimal control law by using the Hamilton-Jacobi-Bellman equation. Use the trial function $J(x(t), t) = a(t)x^2(t) + b(t)x(t) + c(t)$.

Note! In this exercise, you are allowed to use software for solving differential equations and systems of differential equations, but you must clearly state/mark what inputs you used (it's enough to tell the function numbers, for example) and results you got, and give the name of the software you used. One software option is to use the Wolfram Alpha website.