Dynamic Optimization - Homework assignment 1

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1.5

a)

We have the functional

$$J(x,t) = \int_0^T \frac{\dot{x}^2}{t^3} dt \tag{1}$$

with the end points $x(0) = A$ and $x(t) = B$.

To find the extremals fo the functional, we use Eulers equation

$$g_x - \frac{d}{dt} g_{\dot{x}} = 0. \tag{2}$$

where

$$g(x, \dot{x}, t) = g = \frac{\dot{x}^2}{t^3} \tag{3}$$

$$\frac{\partial}{\partial x} g = g_x = 0 \tag{4}$$

$$\frac{\partial}{\partial \dot{x}} g = g_{\dot{x}} = \frac{2\dot{x}}{t^3} \tag{5}$$

which gives us the equation

$$0 - \frac{d}{dt} \frac{2\dot{x}}{t^3} = 0. \tag{6}$$

The derivative can be calculated as

$$\frac{d}{dt} \frac{2\dot{x}}{t^3} = \frac{\frac{d}{dt}(2\dot{x})t^3 - \frac{d}{dt}(t^3)2\dot{x}}{(t^3)^2} = \frac{2\ddot{x}t^3 - 6t^2\dot{x}}{t^6} = \frac{2\ddot{x}t - 6\dot{x}}{t^4} \tag{7}$$

from which our already modified eulers equation becomes

$$\frac{2\ddot{x}t - 6\dot{x}}{t^4} = 0 \tag{8}$$

$$\frac{\ddot{x}}{t^3} - \frac{3\dot{x}}{t^4} = 0 \tag{9}$$

$$\frac{\ddot{x}}{t^3} = \frac{3\dot{x}}{t^4} \tag{10}$$

$$\frac{\ddot{x}}{\dot{x}} = \frac{3}{t}. \tag{11}$$

We can begin solving the ODE by integrating

$$\int \ddot{x} t dt = \int 3\dot{x} dt \tag{12}$$

$$t\dot{x} - \int \dot{x} dt = 3x + c \tag{13}$$

$$t\dot{x} - x = 3x + c \tag{14}$$

$$t\dot{x} - 4x - c = 0 \tag{15}$$

which yields us the linear first order ODE which can be solved as

$$\dot{x} - \frac{4}{t}x - \frac{c}{t} = 0, \qquad | \cdot e^{P(x)}, P(x) = \int -\frac{4}{t}dt = -4\ln(t) \tag{16}$$

$$\dot{x}e^{-4\ln t} - \frac{4}{t}xe^{-4\ln t} - \frac{c}{t}e^{-4\ln t} = 0, \qquad |e^{-4\ln t} = t^{-4} \tag{17}$$

$$\frac{d}{dt}\left(xt^{-4}\right) = \frac{c}{t}t^{-4} = ct^{-5}, \qquad | \int dt \tag{18}$$

$$xt^{-4} = -ct^{-4} + b, \qquad | \cdot t^4 \tag{19}$$

$$x = -c + bt^4, \qquad c_1 = b, c_2 = -c \tag{20}$$

$$x(t) = c_1 t^4 + c_2. \tag{21}$$

We can solve the constants from the en point conditions: $x(0) = A$ and $x(t) = B$

$$x(0) = c_1 0^4 + c_2 = A \implies c_2 = A \tag{22}$$

$$x(T) = c_1 T^4 + A = B \implies c_1 = \frac{B-A}{T^4} \tag{23}$$

$$\implies x(t) = \frac{B-A}{T^4}t^4 + A. \tag{24}$$

b)

We have the functional

$$J(x,t) = \int_0^T (\dot{x} - 8xt + t)dt \tag{25}$$

with the end points $x(0) = A$ and $x(t) = B$.

To find the extremals fo the functional, we use Eulers equation

$$g_x - \frac{d}{dt}g_{\dot{x}} = 0. \tag{26}$$

where

$$g(x,\dot{x},t) = g = \dot{x}^2 - 8xt + t \tag{27}$$

$$\frac{\partial}{\partial x}g = g_x = -8t \tag{28}$$

$$\frac{\partial}{\partial \dot{x}}g = g_{\dot{x}} = 2\dot{x} \tag{29}$$

which gives us the equation

$$-8t - \frac{d}{dt}2\dot{x} = 0 \tag{30}$$

$$8t + 2\ddot{x} = 0 \tag{31}$$

$$\ddot{x} = -4t. \tag{32}$$

We can solve the DE by integrating twice

$$\int \ddot{x}dt = \int -4tdt \tag{33}$$

$$\dot{x} = -2t^2 + c, \qquad | \int dt \tag{34}$$

$$\int \dot{x} = \int -2t^2 + cdt \tag{35}$$

$$x = -\frac{2}{3}t^3 + ct + b, \qquad c_1 = c, c_2 = b \tag{36}$$

Solving the linear ODE we get the solution

$$x(t) = -\frac{2t^3}{3} + c_1 t + c_2. \tag{37}$$

We can solve the constants from the en point conditions: $x(0) = A$ and $x(t) = B$

$$x(0) = -\frac{2 \cdot 0^3}{3} + c_1 0 + c_2 = A \implies c_2 = A \tag{38}$$

$$x(T) = -\frac{2T^3}{3} + c_1 T + A = B \implies c_1 = \frac{B - A + \frac{2}{3}T^3}{T} \tag{39}$$

$$\implies x(t) = -\frac{2}{3}t^3 + \frac{B - A + \frac{2}{3}T^3}{T}t + A. \tag{40}$$