Exercise 4.1 (student presents)

Find a control u which minimizes the functional

$$\min\ \tfrac{1}{4}x^2(T) + \int_0^T \tfrac{1}{4}u^2 dt$$
$$\dot{x} = x + u.$$

Assume the initial state is fixed $x(0) = x_0$, end time is fixed T, and the end state $x(T)$ is free.

Exercise 4.2 (teacher demo)

The fish population in a lake as a function of time t is $x(t)$. The natural growth rate of the population is a concave function $g(x)$, with conditions $g(0) = g(x_m) = 0$ and $g(x) > 0,\ 0 < x < x_m$.

The fish can be caught with the rate $h(t)$. Thus, the change in fish population is defined by the equation

$$\dot{x} = g(x) - h(t).$$

Let p be the price achieved from selling a fish, and $c(x)$ the costs for catching the fish (decreasing), when the population level is x.

a) Show that the present value of the profit is

$$\int_0^\infty e^{-rt}[p - c(x)][g(x) - \dot{x}]dt.$$

b) Find a population, which maximizes the profit (part a). What are the boundaries for $\dot{x}$?

Exercise 4.3 (student presents)

A student wants to pass an exam that is held after T time steps. Let's assume that the level of knowledge $k(t)$ increases with a speed that is proportional to the study efficiency $w(t)$. On the other hand, the rate of forgetting is directly proportional to the level of knowledge. The student attempts to minimize the time spent to study. This situation can be explained with the optimization model

$$\min \; J = \int_0^T w(t)dt$$

$$\dot{k}(t) = b\sqrt{w(t)} - ck(t),$$

where b and c are positive coefficients of proportionality, and $w(t) \geq 0, k(0) = k_0, k(T) = k_T > k_0$. Define the optimal study strategy $w(t)$.

Exercise 4.4 (solved in class)

Find a control u which maximizes the functional

$$\int_0^2 [2x - 3u - u^2]dt,$$

where $\dot{x} = x + u$, $x(0) = 5$, $x(2) =$ free, and the control is bounded: $0 \leq u \leq 2$.

Exercise 4.5 (homework)

You are the marketing manager of a luxury sports car company. The brand equity at time t for the sports cars is $x(t)$. Let $R(x)$ be the maximum revenue a firm can earn by selling the product with brand equity x. Assume $\dot{R}(0) > 0$ and $\ddot{R} < 0$.

The brand equity increases with advertising $I(t)$ and decays at constant proportional rate $bx(t)$, i.e., $\dot{x}(t) = I(t) - bx(t)$. The upper bound for advertising is $\bar{I}$ (reflects a maximum permitted spending in advertising). Your task is to choose an advertising spending $I(t)$ to maximize the present value of the flow of profits.

a) Formulate the problem as an infinite horizon autonomous problem.

b) Solve the most rapid approach path for the problem.