MS-E2148 Dynamic optimization

Recap

Summary

- The course considered two kinds of problems:
  1. Discrete time problems
  2. Continuous time problems
- These were solved with two methods:
  1. Dynamic programming (DP)
  2. Calculus of variations
  3. Pontryagin's minimum principle
- HJB equation is DP algorithm in continuous time problems
- Bellman equation is DP algorithm for ∞ horizon discounted, stationary problem
- Calculus of variations was also used in deriving the necessary conditions for the control problem
- Stochastic was only in the discrete time problems

- The increment of a functional is

$$\Delta J(x, \delta x) \equiv J(x + \delta x) - J(x) \tag{1}$$

- If the increment can be expressed with linear functional $\delta J(x, \delta x)$ in δx:

$$\Delta J(x, \delta x) = \delta J(x, \delta x) + g(x, \delta x)\|\delta x\| \tag{2}$$

  where $\lim_{\|\delta x\| \to 0}\{g(x, \delta x)\} = 0$, then J is differentiable in x and δJ is the variation of J with function/variation x
- On the course, we assume that J is differentiable in x, so the variation is linear approximation to the increment

- On the extremal x^* the variation vanishes; the necessary condition is the Euler equation

$$g_x(x^*, \dot{x}^*, t) - \frac{d}{dt} g_{\dot{x}}(x^*, \dot{x}^*, t) = 0 \qquad \forall t \in [t_0, t_f] \tag{3}$$

- If the boundary points are fixed, i.e.,

$$x(t_0) = x_0 \qquad \text{and} \qquad x(t_f) = x_f \tag{4}$$

  there are no additional conditions to Euler

- $x(t_f)$ free, t_f fixed:

$$g_{\dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f) = 0 \tag{5}$$

- t_f free, $x(t_f)$ fixed:

$$g(x^*(t_f), \dot{x}^*(t_f), t_f) - g_{\dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\dot{x}^*(t_f) = 0 \tag{6}$$

- $x(t_f)$ and t_f free but independent:

(5) and (6)

- $x(t_f)$ and t_f free and $x(t_f) = \theta(t_f)$:

$$g_{\dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\left[\dot{\theta}(t_f) - \dot{x}^*(t_f)\right] + g(x^*(t_f), \dot{x}^*(t_f), t_f) = 0 \tag{7}$$

$$g_{\dot{x}}(x^*(t_1^-), \dot{x}^*(t_1^-), t_1^-) = g_{\dot{x}}(x^*(t_1^+), \dot{x}^*(t_1^+), t_1^+)$$

and

$$\begin{aligned} &g(x^*(t_1^-), \dot{x}^*(t_1^-), t_1^-) - \left[g_{\dot{x}}(x^*(t_1^-), \dot{x}^*(t_1^-), t_1^-)\right]\dot{x}^*(t_1^-) \\ &= g(x^*(t_1^+), \dot{x}^*(t_1^+), t_1^+) - \left[g_{\dot{x}}(x^*(t_1^+), \dot{x}^*(t_1^+), t_1^+)\right]\dot{x}^*(t_1^+) \end{aligned}$$

but

$$\dot{x}^*(t_1^-) \neq \dot{x}^*(t_1^+)$$

where the corner is at $t = t_1$

- Hamiltonian: $H(x,u,p,t) = g(x,u,t) + p[f(x,u,t)]$
- Necessary conditions:

$$
\begin{array}{rcl}
\dot{p}^* & = & -H_x(x^*,u^*,p^*,t) \\
0 & = & H_u(x^*,u^*,p^*,t) \\
\dot{x}^* & = & H_p(x^*,u^*,p^*,t)
\end{array}
\tag{8}
$$

and

$$
\begin{aligned}
&\Big[h_x(x^*(t_f),t_f) - p^*(t_f)\Big]\delta x_f \\
&+\Big[H(x^*(t_f),u^*(t_f),p^*(t_f),t_f) + h_t(x^*(t_f),t_f)\Big]\delta t_f \quad = \quad 0
\end{aligned}
\tag{9}
$$

- The boundary conditions have to be derived from (9) case by case

- Necessary conditions:

$$\begin{aligned}\dot{p}^* &= -H_x(x^*, u^*, p^*, t)\\ H(x^*, u^*, p^*, t) &\leq H(x^*, u, p^*, t)\\ \dot{x}^* &= H_p(x^*, u^*, p^*, t)\end{aligned}$$

  and complemented with (9)

Special cases

- Control problems where the final time is free and the Hamiltonian has no explicit time dependency: $H = 0$ for all t
- ∞ horizon calculus of variations: require stationary condition $\dot{x} = \ddot{x} = 0$
- Minimum time problem: use bang-bang control
- Minimum control-effort problem: use bang-off-bang control
- Singular solutions: examine if the switching function can have root for finite length time interval
- ∞ horizon, discrete time, stationary discounted problem: Bellman equation gives the optimal J^*:

$$J^*(x) = \min_u E_w \Big\{ g(x, u, w) + \alpha J^*(f(x, u, w)) \Big\} \qquad (10)$$

Hamilton-Jacobi-Bellman equation

Cost-go-go for the continuous-time problem

$$0 = \min_{u \in U} \Big[g(x,u) + \nabla_t J^*(t,x) + \nabla_x J^*(t,x)^T f(x,u) \Big], \quad \forall t, x, \tag{11}$$

with boundary condition $J^*(T, x) = h(x)$

DP algorithm

The cost-to-go for discrete-time problem

- For each initial state x_0 the optimal cost $J^*(x_0)$ follows from the next state:

$$J_N(x_N) = g_N(x_N), \tag{12}$$

$$J_k(x_k) = \min_{u_k \in U_k(x_k)} E_{w_k}\{g_k(x_k, u_k, w_k) + J_{k+1}(f_k(x_k, u_k, w_k))\}, \tag{13}$$

$k = 0, 1, ..., N-1$

- If $u_k^* = \mu_k^*(x_k)$ minimizes equation (13) right-hand side for each x_k and k, then the control law $\pi^* = \{\mu_0^*, ..., \mu_{N-1}^*\}$ is optimal

DP algorithm

Shortest path problem

- Deterministic problem, finite states
- Optimal cost from i to t in $N-k$ steps is

$$J_k(i) = \min_{j=1,\dots,N} \left[a_{ij} + J_{k+1}(j)\right], \; k = 0, 1, \dots, N-2,$$

  where $J_{N-1}(i) = a_{it}, i = 1, 2, \dots, N.$
- Note that in above $J_{k+1}(j) =$ optimal cost from j to t in $N-k-1$ steps

APPENDIX:

HJB: $0 = J_t + \min_{u(t)} \left\{ g + J_x^T f \right\}$

E-L: $0 = g_x - \frac{d}{dt}(g_{\dot{x}})$

Hamiltonian: $H = g + p^T(t) f(x(t), u(t), t)$

costate: $\dot{p}(t) = -\frac{\partial H}{\partial x}$

free final state: $0 = g_{\dot{x}}$ or $h_x - p = 0$

free final time: $0 = g - g_{\dot{x}}\dot{x}$ or $H + h_t = 0$

free final state and time: $g = g_{\dot{x}} = 0$ or $h_x - p = 0 = H + h_t$

goal: $0 = g + \left[\frac{\partial g}{\partial \dot{x}}\right]^T \left[\frac{d\theta}{dt} - \dot{x}\right]$ or $H + h_t + (h_x - p)^T \frac{d\theta}{dt} = 0$

W-E: $g_{\dot{x}}$ and $g - g_{\dot{x}}\dot{x}$ continuous