Dynamic Optimization - Homework assignment 4

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4.5

a)

We denote the following quantities:

1. t = Time
2. $x(t)$ = Brand equity at time t
3. $R(x)$ = The maximum revenue a firm can earn by selling the product with brand equity x.
4. $I(t)$ = Advertising cost at time t, max at $\bar{I}$

Additionally, we know that $\dot{R} > 0$, $\ddot{R} < 0$ which gives the function R a concave shape, i.e. it will have a maximum at least locally. Furthermore, we know that the change in brand equity can be written as $\dot{x} = I(t) - bx(t)$, since advertising increases the rate and it decays at a proportional rate.

The profit flow π can thus be modeled as

$$\begin{aligned} \pi &= R(x) - I(t) && (1) \\ \implies \pi &= R(x) - \dot{x}(t) - bx(t) && (2) \\ & && (3) \end{aligned}$$

which does not explicitly depend on the time t. Here we have assumed all other costs and incomes are neglectable. With this information we can model this as a infinite horizon autonomous problem as

$$\int_0^\infty e^{-rt}(R(x) - \dot{x}(t) - bx(t))dt \tag{4}$$

by adding the discounting factor e^{-rt} we get the present value, and the functional is explicitly independent of t.

b)

Since our problem is an infinite horizon autonomous problem, and we know that the function is concave, we can assume that it can be optimized to a steady state $\lim_{t_f \to \infty} x \implies x_s$. This results in the stationary conditions $\dot{x} = 0$, $\ddot{x} = 0$. Using this we can solve the problem using the most rapid approach path. Since our profit flow function is a linear function of $\dot{x}$ dependent of x, i.e. has the form $\int_0^\infty e^{-rt}\left(M(x) - N(x)\dot{x}\right)dt$. From the lecture slides we thus get

$$\dot{x} = \begin{cases} B(x), & x_0 < x_s \\ 0, & x_0 = x_s \\ A(x), & x_0 > x_s \end{cases}, \tag{5}$$

where $A(x)$ and $B(x)$ represent the boundaries of the problem. We can find the problem, knowing that the advertisement $I(t)$ is $0 \leq I \leq \bar{I}$. This leads, by substitution to $-bx \leq \dot{x} \leq \bar{I} - bx$ and we get the solution for $\dot{x}^*$ to

$$\dot{x} = \begin{cases} \bar{I} - bx, & x_0 < x_s \\ 0, & x_0 = x_s \\ -bx, & x_0 > x_s \end{cases} \tag{6}$$

and by realigning the formula for the change in brand equity to $I(t) = \dot{x}(t) + bx(t)$, we get the optimal solution for the advertisement to

$$I^* = \begin{cases} 0, & x_0 < x_s \\ bx_s, & x_0 = x_s \\ \bar{I}, & x_0 > x_s \end{cases} \tag{7}$$