

Note! Problem 1.3 c) from the previous round is also presented by students on this round!

### Exercise 2.1 (student presents)

Solve the extremal of the functional

$$J = \int_0^1 (t\dot{x} + x^2 + \dot{x}^2)dt$$

where  $x(0) = 1$  and  $x(1)$  is free.

### Solution

Let's use the Euler equation and the transversality condition.

$$\text{Euler: } 2x - 1 - 2\ddot{x} = 0 \Rightarrow x^*(t) = \frac{1}{2} + C_1 e^t + C_2 e^{-t}$$

$$\text{Initial point condition: } x^*(0) = \frac{1}{2} + C_1 + C_2 = 1 \Rightarrow C_2 = \frac{1}{2} - C_1$$

Here the end time is fixed  $t_f = 1$ , but the end state can vary, meaning that  $\delta x(t_f)$  is arbitrary. Thus, we use the transversality condition  $g_{\dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f) = 0 \Rightarrow 1 + 2\dot{x}^*(1) = 0 \Rightarrow \dot{x}^*(1) = -1/2$ . By inserting the solution candidate  $x^*(t)$  from above we get

$$\begin{aligned} -\frac{1}{2} = \dot{x}^*(1) &= C_1 e - \left(\frac{1}{2} - C_1\right)e^{-1} \\ &= C_1(e + e^{-1}) - \frac{1}{2}e^{-1}. \end{aligned}$$

$$\text{We get } C_1 = \frac{1-e}{2(e^2+1)}, C_2 = \frac{e^2+e}{2(e^2+1)}, \text{ and the extremal is } x^*(t) = \frac{1}{2} + \frac{1-e}{2(e^2+1)}e^t + \frac{e^2+e}{2(e^2+1)}e^{-t}.$$

### Exercise 2.2 (teacher demo)

a) Find the curve, which is an extremal for the functional

$$J(x) = \int_0^{t_f} \sqrt{1 + \dot{x}^2(t)} dt,$$

when  $x(0) = 5$  and the end point has to be on the circle  $x^2(t) + (t - 5)^2 - 4 = 0$ . Check your result geometrically. Hint: The end point condition is of the form  $m(x(t_f), t_f) = 0$ . Draw a picture to define the dependence of  $\delta x_f$  and  $\delta t_f$ .

b) What if  $x(0) = 2$  and the end point is on the curve  $\theta(t) = -4t + 5$  ?

### Exercise 2.3 (solved in class)

Find the necessary conditions for the extremal  $x^*(t)$ ,  $t_0 \leq t \leq t_1$ , which maximizes the functional

$$\int_{t_0}^{t_1} F(x(t), \dot{x}(t), t) dt$$

when  $t_0$  and  $t_1$  are only known. Notice, that  $x(t_0)$  and  $x(t_1)$  can also be chosen optimally.

### Exercise 2.4 (self-study)

Find such extremals for the problem

$$\min \int_0^4 [\dot{x}(t) - 1]^2 [\dot{x}(t) + 1]^2 dt$$

$$\begin{cases} x(0) = 0 \\ x(4) = 2 \end{cases}$$

which have one corner.

### Exercise 2.5 (home assignment)

Find the extremal of the following functional with the help of Euler equation and the appropriate transversality conditions.

$$\int_0^s \left( \frac{1}{2} \dot{x}(t)^2 - x(t) + \frac{3}{2} \right) dt,$$

when  $x(0) = 0$ , and  $s$  and  $x(s)$  are free.