Exercise 4.1 (student presents)

Find a control u which minimizes the functional

$$\begin{aligned} \min\ & \tfrac{1}{4}x^2(T) + \int_0^T \tfrac{1}{4}u^2 dt \\ & \dot{x} = x + u. \end{aligned} \tag{1}$$

Assume the initial state is fixed $x(0) = x_0$, end time is fixed T, and the end state $x(T)$ is free.

Solution

Let's form the Hamiltonian

$$H = \frac{1}{4}u^2 + px + pu.$$

Next we form the costate equation, from which we solve $p^*(t)$

$$\begin{aligned} & H_x = -\dot{p}^* \\ \Leftrightarrow \quad & p^* = -\dot{p}^* \\ \Leftrightarrow \quad & \dot{p}^* + p^* = 0 \\ \Rightarrow \quad & p^*(t) = Ae^{-t}. \end{aligned} \tag{2}$$

Now we form the stationarity condition and use (2)

$$\begin{aligned} & H_u = 0 \\ \Leftrightarrow \quad & \tfrac{1}{2}u^* + p^* = 0 \\ \Leftrightarrow \quad & u^* = -2p^* \\ \Rightarrow \quad & u^* = -2Ae^{-t}. \end{aligned} \tag{3}$$

Using the state equation (1) and (3) we get

$$\begin{aligned} & H_p = \dot{x}^* \\ \Leftrightarrow \quad & x^* + u^* = \dot{x}^* \\ \Leftrightarrow \quad & \dot{x}^* - x^* = u^* \\ \Rightarrow \quad & \dot{x}^* - x^* = -2Ae^{-t}. \end{aligned} \tag{4}$$

Let's solve (4), which is a linear non-homogeneous differential equation, and we get (it's left for the student to check that this is correct)

$$x^*(t) = Ae^{-t} + Be^t. \tag{5}$$

Because the end state can be chosen freely, the solution should satisfy

$$\begin{aligned} & h_x(T) - p^*(T) = 0 \\ \Leftrightarrow \quad & \tfrac{1}{2}x^*(T) = p^*(T) \\ \Leftrightarrow \quad & \tfrac{A}{2}e^{-T} + \tfrac{B}{2}e^{T} = Ae^{-T} \\ \Leftrightarrow \quad & A = Be^{2T}. \end{aligned} \tag{6}$$

Using (6) and the initial point condition $x(0) = x_0$ we get

$$\begin{aligned} & A + B = x_0 \\ \Leftrightarrow \quad & Be^{2T} + B = x_0 \\ \Leftrightarrow \quad & B = \tfrac{1}{e^{2T}+1}x_0. \end{aligned} \tag{7}$$

So $A = \frac{e^{2T}}{e^{2T}+1}x_0$. So the optimal trajectory is

$$\begin{aligned} & x^*(t) = \tfrac{e^{2T}}{e^{2T}+1}x_0e^{-t} + \tfrac{1}{e^{2T}+1}x_0e^{t} \\ \Leftrightarrow \quad & x^*(t) = \tfrac{e^{2T-t}+e^{t}}{e^{2T}+1}x_0. \end{aligned} \tag{8}$$

And optimal control

$$\begin{aligned} & u^*(t) = -2\tfrac{e^{2T}}{e^{2T}+1}x_0e^{-t} \\ \Leftrightarrow \quad & u^*(t) = -2\tfrac{e^{2T-t}}{e^{2T}+1}x_0. \end{aligned} \tag{9}$$

With a couple of modifications we can write u^* in feedback-form (notice that the problem was a linear regulator problem). Base your answer on the calculations below or provide an alternative calculation. Also, briefly explain / speculate on the usefulness of the feedback-form.

$$\begin{aligned} u^*(t) &= -2\frac{e^{2T-t}}{e^{2T}+1}x_0 \cdot \frac{e^{-2T+2t}+1}{e^{-2T+2t}+1} \\ &= -2\frac{1}{e^{-2T+2t}+1} \cdot \frac{e^{2T-t}+e^{t}}{e^{2T}+1}x_0 \\ &= -2\frac{e^{T-t}}{e^{T-t}+e^{-T+t}}x^*(t). \end{aligned} \tag{10}$$

Exercise 4.2 (teacher demo)

The fish population in a lake as a function of time t is $x(t)$. The natural growth rate of the population is a concave function $g(x)$, with conditions $g(0) = g(x_m) = 0$ and $g(x) > 0, \ 0 < x < x_m$.

The fish can be caught with the rate $h(t)$. Thus, the change in fish population is defined by the equation

$$\dot{x} = g(x) - h(t).$$

Let p be the price achieved from selling a fish, and $c(x)$ the costs for catching the fish (decreasing), when the population level is x.

a) Show that the present value of the profit is

$$\int_0^\infty e^{-rt}[p - c(x)][g(x) - \dot{x}]dt.$$

b) Find a population, which maximizes the profit (part a). What are the boundaries for $\dot{x}$?

Solution

The model for the lake's fish population:

$x(t)$	fish population at time t
$g(x)$	growth rate function of the population
x_m	carrying capacity of the nature
$h(t)$	fishing rate at time t
p	the market price of fish
$c(x)$	the marginal cost for fishing when the population is x.

Let $g(x)$ be concave, $g(0) = g(x_m) = 0$ and $g(x) > 0$ when $0 < x < x_m$. Also, $c'(x) < 0$ (the marginal cost for fishing decreases the more there are fishes).

The following equation models the dynamics of the fish population

$$\dot{x} = g(x) - h(t). \tag{11}$$

a)

The profit flow at time t:

$$\pi = p \cdot h(t) - c(x) \cdot h(t), \tag{12}$$

and it can be reformulated by substituting with (11)

$$\pi = \big[p - c(x)\big]\big[g(x) - \dot{x}\big]. \tag{13}$$

Let's discount the future profit flows to present value with the factor e^{-rt}, $r > 0$ (what is the intuition behind this?):

$$\int_0^\infty e^{-rt}\big[p - c(x(t))\big]\big[g(x(t)) - \dot{x}(t)\big]\, dt. \tag{14}$$

b)

Notice that after the reformulation from (12) to (13), maximizing (14) is a variational problem (no optimal control problem) that is autonomous and has an infinite time horizon. The integrand depends on time through the discount term, but the Euler equation will be autonomous.

Let's form the Euler equation:

$$e^{-rt}\{-c'(x)\big[g(x)-\dot{x}\big]+\big[p-c(x)\big]g'(x)\}+\frac{d}{dt}\{e^{-rt}\big[p-c(x)\big]\}=0$$

so

$$-c'(x)g(x)+\cancel{c'(x)}\dot{x}+\big[p-c(x)\big]g'(x)-r\big[p-c(x)\big]-\cancel{c'(x)}\dot{x}=0,$$

out of which we get

$$g'(x)-\frac{c'(x)g(x)}{p-c(x)}=r.$$

This is not a differential equation! If this equation has a solution, it is the optimal fish population, which we define $x=x_s$.

In infinite horizon problems, there may be no necessary transversality condition. Even if a transversality condition holds, it may be of little help in determining constants of integration. The needed condition is often obtained from the observation that if such problems depend on time only through the discount term, it may be reasonable to expect the solution $x(t)$ to tend toward a stationary level x_s in the long run. If the solution doesn't level out, the integral can grow unbounded or oscillate. A steady state or stationary state is one in which $\dot{x}=\ddot{x}=0$. Note that limit cycles are also possible. Note that it is just an educated guess that the solution levels out, variational problems with infinite horizons are typically quite hard to solve!

We saw above that the profit functional can be written independent of $\dot{x}$. This is because the profit functional is linear with regards to $\dot{x}$. Thus, the optimal solution is the most rapid approach path to the stable solution. Thus:

$$\begin{aligned}
x_0 > x_s \quad &: \quad \text{Fish with the maximum rate } (h=h_{\max}) \text{ until} \\
&\qquad \text{the population decreases to the state } x_s; \\
x_0 < x_s \quad &: \quad \text{Not fish } (h=0) \text{ until} \\
&\qquad \text{the population grows to the state } x_s.
\end{aligned}$$

When the state x_s is reached, let's fish with the rate $h=g(x_s)$. Then $\dot{x}=0$ and we stay at the steady-state optimum.

If initially $x_0 < x_s$, how fast will we at least reach the steady-state optimum? If the fishing rate is $h=0$, it holds for the change in population

$$\dot{x}=g(x)\geq\min\{g(x_0),g(x_s)\},\quad \forall\, x_0<x<x_s,$$

because g is concave and thus it is also quasi-concave. Similarly, if initially $x_0 > x_s$, it holds for the change in population

$$\dot{x}=g(x)-h_{\max}\leq\max_x\{g(x)\}-h_{\max},\quad \forall\, x_0>x>x_s.$$

Exercise 4.3 (student presents)

A student wants to pass an exam that is held after T time steps. Let's assume that the level of knowledge $k(t)$ increases with a speed that is proportional to the study efficiency $w(t)$. On the other hand, the rate of forgetting is directly proportional to the level of knowledge. The student attempts to minimize the time spent to study. This situation can be explained with the optimization model

$$\min\ J = \int_0^T w(t)dt$$
$$\dot{k}(t) = b\sqrt{w(t)} - ck(t),$$

where b and c are positive coefficients of proportionality, and $w(t) \geq 0, k(0) = k_0, k(T) = k_T > k_0$. Define the optimal study strategy $w(t)$.

Parts 1 and 2 of the solution are presented by different students.

Solution, part 1

Hamiltonian:

$$H \triangleq w(t) + p(t)b\sqrt{w(t)} - p(t)ck(t). \tag{15}$$

From the costate equation:

$$\begin{aligned}
& H_k = -\dot{p}^*(t) \\
\Leftrightarrow \quad & H_k = cp^*(t) \\
\Leftrightarrow \quad & \dot{p}^*(t) + cp^*(t) = 0 \\
\Rightarrow \quad & p^*(t) = \alpha e^{ct}.
\end{aligned} \tag{16}$$

Because the controls are bounded, we use Pontryagin minimization principle. The Hamiltonian should be minimized with the optimal control:

$$H(x^*, u^*, p^*, t) \leq H(x^*, u, p^*, t), \quad \forall\, u \text{ feasible control.}$$

In this problem the condition is:

$$w^*(t) + p^*(t)b\sqrt{w^*(t)} - \cancel{p^*(t)ck^*(t)} \leq \underbrace{w(t) + p^*(t)b\sqrt{w(t)}}_{=f(w(t))} - \cancel{p^*(t)ck^*(t)}.$$

If $\alpha > 0$, then $f(w(t))$ is minimized, when $w(t) = 0$. However, it is not possible to pass the exam without studying, so this does not satisfy the boundary conditions. It has to be that $\alpha \leq 0$.

Lets minimize $f(w)$:

$$\frac{df(w)}{dw} = 0 \quad \Leftrightarrow \quad \sqrt{w^*(t)} = -\frac{1}{2}\alpha b e^{ct}$$

it is the minimum, because

$$\frac{d^2 f(w)}{dw^2} = -\alpha e^{ct} b[w(t)]^{-3/2} \geq 0.$$

Solution, part 2

By inserting back to the state equation we get:

$$\dot{k}(t) = -ck(t) - \frac{1}{2}\alpha b^2 e^{ct}.$$

The solution for the homogenous equation is $k(t) = \beta e^{-ct}$. Lets find the particular solution with the attempt $k(t) = \gamma e^{ct}$:

$$\begin{aligned} \gamma c e^{ct} &= -c\gamma e^{ct} - \frac{1}{2}\alpha b^2 e^{ct} \\ \Rightarrow \quad \gamma &= -\frac{\alpha b^2}{4c}, \end{aligned}$$

so the solution for the general equation is

$$k^*(t) = \beta e^{-ct} - \frac{\alpha b^2}{4c} e^{ct}.$$

The constants α and β is received from the boundary conditions:

$$\begin{aligned} k^*(0) &= \beta - \frac{\alpha b^2}{4c} = k_0 \\ k^*(T) &= \beta e^{-cT} - \frac{\alpha b^2}{4c} e^{cT} = k_T \\ \Rightarrow \quad \alpha &= \frac{4c}{b^2} \cdot \underbrace{\frac{k_T - k_0 e^{-cT}}{e^{-cT} - e^{cT}}}_{=K} < 0 \\ \Rightarrow \quad \beta &= k_0 + K, \end{aligned}$$

so the solution is

$$\boxed{k^*(t) = (k_0 + K)e^{-ct} - Ke^{ct}.}$$

The optimal study strategy is

$$\begin{aligned} b\sqrt{w^*} &= \dot{k}^* + ck^* \\ &= -c(k_0 + K)e^{-ct} - cKe^{ct} + c(k_0 + K)e^{-ct} - cKe^{ct} \\ &= -2cKe^{ct} \end{aligned}$$

so

$$\boxed{w^*(t) = \frac{4c^2}{b^2} K^2 e^{2ct}.}$$

Exercise 4.4 (solved in class)

Find a control u which maximizes the functional

$$\int_0^2 [2x - 3u - u^2]dt,$$

where $\dot{x} = x + u$, $x(0) = 5$, $x(2) =$ free, and the control is bounded: $0 \leq u \leq 2$.

Solution

The Hamiltonian is

$$H = 2x - 3u - u^2 + px + pu$$

We get the following condition from the maximization principle (because now we maximize the functional)

$$\begin{aligned}
2x^* - 3u^* - {u^*}^2 + p^*x^* + p^*u^* &\geq 2x^* - 3u - u^2 + p^*x^* + p^*u \\
\cancel{2x^*} - 3u^* - {u^*}^2 + \cancel{p^*x^*} + p^*u^* &\geq \cancel{2x^*} - 3u - u^2 + \cancel{p^*x^*} + p^*u \\
-3u^* - {u^*}^2 + p^*u^* &\geq -3u - u^2 + p^*u.
\end{aligned}$$

Lets differentiate the right side to find the maximum

$$\frac{d}{du}(-3u - u^2 + p^*u) = -3 - 2u + p^*.$$

For the maximum u^* it holds

$$u^* = \frac{p^* - 3}{2}, \text{ when } \frac{p^* - 3}{2} \in [0, 2].$$

When $p^* < 3$, $u^* = 0$ and when $p^* > 7$, $u^* = 2$, otherwise we follow the equation above. Furthermore, from the costate equation $-\dot{p}^* = 2 + p^*$ and the free final state transversality condition $p^*(2) = 0$ we get $p^*(t) = 2e^{2-t} - 2$.

Initially $p^*(0) \approx 12.78$ and in the end $p^*(2) = 0$. So, initially we use maximum control (because $p^*(0) > 7$) and in the end we have minimal control (because $p^*(2) < 3$). In the interval $[t_1, t_2]$ we use the control $u^* = \frac{p^*-3}{2} = e^{2-t} - \frac{5}{2}$. Lets solve the time points t_1 and t_2

$$\begin{aligned}
p^*(t_1) = 7 &\Rightarrow t_1 = 2 - \ln 4.5 \approx 0.496 \\
p^*(t_2) = 3 &\Rightarrow t_2 = 2 - \ln 2.5 \approx 1.084
\end{aligned}$$

To summarize, the optimal control is

$$u^*(t) = \begin{cases} 2, & \text{if } 0 \leq t \leq 0.496, \\ e^{2-t} - \frac{5}{2}, & \text{if } 0.496 < t \leq 1.084, \\ 0 & \text{if } t > 1.084. \end{cases}$$