Exercise 5.1 (homework)

Find the optimal control to move the following system

$$\begin{aligned}\dot{x}_1 &= -x_1 - u \\ \dot{x}_2 &= -3x_2 - 3u\end{aligned}$$

from an arbitrary initial state x_0 to the origin in minimal time. The control is constrained: $|u| \leq 1$. It is enough to define the optimal control w.r.t. p^*. Remember to check for singular intervals.

Exercise 5.2 (student presents)

We want to move the following system

$$\dot{x} = 2x + u$$

from an arbitrary initial state x_0 to the origin in minimal time, while the control is constrained $|u| \leq 1$. There are some initial states that can not be moved to the origin with any allowed control u. Find those initial states.

Solution

The solution for the state equation is

$$x = \phi(t)[x_0 + \int_0^t \phi(-\tau)u(\tau)d\tau]$$

where $\phi(t) = e^{2t}$. Lets assume we have a control for which $x(T) = 0$; then

$$0 = x_0 + \int_0^T \phi(-\tau)u(\tau)d\tau,$$

which implies that

$$|x_0| = \Big| \int_0^T \phi(-\tau)u(\tau)d\tau \Big|$$

On the other hand

$$\Big| \int_0^T \phi(-\tau)u(\tau)d\tau \Big| \leq \int_0^T |\phi(-\tau)||u(\tau)|d\tau$$

and because $|u| \leq 1$, we get

$$\begin{aligned} |x_0| &\leq \int_0^T |\phi(-\tau)| d\tau \\ &= \frac{1}{2}[1 - e^{-2T}] \end{aligned}$$

i.e.

$$2|x_0| - 1 \leq -e^{-2T} \Rightarrow e^{-2T} \leq 1 - 2|x_0|$$

Because the exponential function is positive for all T, this last condition can only be satisfied if $|x_0| < 1/2$, which is the answer we were looking for.

Exercise 5.3 (solved in class)

We want to move the following system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 + x_2 u \end{aligned}$$

from some initial state to the goal set $S(t)$, while minimizing the functional

$$J(u) = \int_0^{t_f} |u| dt.$$

The controls are constrained: $|u| \leq 1$.

Write the costate equations and form the optimal control as a function of x^* and p^*.

Solution

Hamiltonian: $H = |u| + p_1 x_2 + p_2 x_1 + p_2 x_2 u$

Costate equations:

$$\begin{aligned} \dot{p}_1^* &= -p_2^* \\ \dot{p}_2^* &= -p_1^* - p_2^* u^* \end{aligned}$$

The optimal control is of the form of a minimum control-effort problem ("bang-off-bang"):

$$u^* = \begin{cases} 1, & p_2^* x_2^* < -1 \\ 0, & -1 < p_2^* x_2^* < 1 \\ -1, & 1 < p_2^* x_2^* \\ \text{not defined}\ , & p_2^* x_2^* = \pm 1 \text{ some finite interval} \end{cases}$$

Exercise 5.4 (teacher demo)

The task is to build a road on a terrain. The road should be built in the interval $0 \leq s \leq S$. In this interval, the height of the terrain $y(s)$ is a differentiable function. We want to define the optimal road height $x(s)$ in each point s, when there is a constraint for the ascents and descents of the road

$$\left| \frac{dx}{ds} \right| \leq a$$

The costs for filling or digging one meter are directly proportional to the square of the difference of the height of the terrain and the road. Formulate the problem and analyze the solution.

Solution

The height of the terrain is a differentiable function $y(s)$ and we try to find the profile of the road $x(s)$. The decision variable is the steepness of the road $u(s)$

$$\frac{dx}{ds} = u(s), \quad -a \leq u(s) \leq a$$

and we attempt to minimize the squared error of the difference of the terrain and road height

$$\min \int_0^S [x(s) - y(s)]^2 \, ds,$$

when both the height of the road initially $x(0)$ and in the end $x(S)$ are free.

If there would be no control constraint, the solution would trivially be $x(s) = y(s)$. The Hamiltonian is

$$\mathscr{H} = (x - y)^2 + pu$$

and the costate equation

$$\dot{p}(s) = -2[x(s) - y(s)].$$

From the free initial and end state conditions follows the transversality conditions

$$p(0) = p(S) = 0.$$

Then

$$p(s) = -2 \int_0^s [x(\tau) - y(\tau)] \, d\tau$$

and

$$\int_0^S [x(\tau) - y(\tau)] \, d\tau = 0.$$

We notice three kind of solution intervals:

$$\begin{aligned}
p(s) > 0 \quad &: \quad u(s) = -a, \int_0^s [x(\tau) - y(\tau)] \, d\tau < 0; \\
p(s) = 0 \quad &: \quad u(s) = \dot{y}(s), \int_0^s [x(\tau) - y(\tau)] \, d\tau = 0; \\
p(s) < 0 \quad &: \quad u(s) = +a, \int_0^s [x(\tau) - y(\tau)] \, d\tau > 0.
\end{aligned}$$

Thus on the singular interval $p(s) = 0$ there is the condition

$$(x - y)^2 \hookrightarrow \text{min!} \quad \Rightarrow \quad x(s) = y(s)$$

which fixes the control, when the assumptions that $y(s)$ is differentiable holds. Then the previously presented solution for $u(s)$ is unique and gives the minimum.