Annotated examples in problems 1.1 a) and in 1.4 to inspire student presentations

Exercise 1.1 (student presents)

Vary the following functionals

a) $J(x) = \int_{t_0}^{t_f} [x^3(t) - x^2(t)\dot{x}(t)]dt.$ Note! Part (a) was done in class >> no student presentation

b) $J(x) = \int_{t_0}^{t_f} [x_1^2(t) + x_1(t)x_2(t) + x_2^2(t) + 2\dot{x}_1(t)\dot{x}_2(t)]dt.$

c) $J(x) = \int_{t_0}^{t_f} e^{x(t)} dt.$

Assume that the end points are fixed.

Solution

Let the given functional be

$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\, dt$$

and we want to observe how the value of J changes, when $x(t)$ is changed a bit, i.e., what is $J(x + \delta x)$. The functional J is differentiable, so its increment can be written

$$\begin{aligned} \Delta J(x, \delta x) &= J(x + \delta x) - J(x) \\ &= \delta J(x, \delta x) + \underbrace{\varepsilon(x, \delta x)||\delta x||}_{\text{Higher order terms}}, \end{aligned}$$

where $\delta J(x, \delta x)$ is some linear functional w.r.t. δx and the error term

$$\lim_{||\delta x|| \to 0} \varepsilon(x, \delta x) = 0.$$

Then the functional $\delta J(x, \delta x)$ is called the variation of J for function x. The connection to the differentiability of functions is obvious. It is important to learn the variation technique, because later on we will calculate variations for many different functionals.

a) The functional

$$J(x) = \int_{t_0}^{t_f} [x^3(t) - x^2(t)\dot{x}(t)]\, dt$$

is varied, when the end points are fixed. We attain the variation δJ by making a Taylor approximation of the integrand w.r.t. the variables x and $\dot{x}$, and we only choose the linear terms:

$$\delta J(x,\delta x) = \int_{t_0}^{t_f} [\frac{\partial g}{\partial x}\delta x + \frac{\partial g}{\partial \dot{x}}\delta \dot{x}]\, dt$$

$$= \int_{t_0}^{t_f} \{(3x^2 - 2x\dot{x})\delta x - x^2\delta\dot{x}\}\, dt.$$

Chain rule! x is x(t)

$$\frac{\partial x^2}{\partial t} = 2x\dot{x}$$

By partial integration

The integrand is:
(x(t))^2 delta x'(t)

$$\int_{t_0}^{t_f} x^2\delta\dot{x}\, dt = \underbrace{x^2(t_f)\delta x(t_f) - x^2(t_0)\delta x(t_0)}_{=0,\text{ fixed end points}} - \int_{t_0}^{t_f} 2x\dot{x}\delta x,$$

with fixed end points, variation is zero

$$\int \delta\dot{x}\, dt = \delta x$$

because end points are fixed!

and inserting

$$\delta J(x,\delta x) = \int_{t_0}^{t_f} \{(3x^2 - \cancel{2x\dot{x}})\delta x + \cancel{2x\dot{x}}\delta x\}\, dt$$

$$= \boxed{\int_{t_0}^{t_f} 3x^2\delta x\, dt.}$$

b)

$$J(x) = \int_{t_0}^{t_f} [x_1^2(t) + x_1(t)x_2(t) + x_2^2(t) + 2\dot{x}_1(t)\dot{x}_2(t)\, dt$$

$$\Rightarrow \delta J(x,\delta x) = \int_{t_0}^{t_f} \{[2x_1 + x_2]\delta x_1 + [x_1 + 2x_2]\delta x_2$$
$$+[2\dot{x}_2]\delta\dot{x}_1 + [2\dot{x}_1]\delta\dot{x}_2\}\, dt.$$

Partial integration:

$$\int_{t_0}^{t_f} [2\dot{x}_2]\delta\dot{x}_1\, dt = 2\big(\underbrace{\dot{x}_2(t_f)\delta x_1(t_f) - \dot{x}_2(t_0)\delta x_1(t_0)}_{=0,\text{ fixed end points}}\big) - \int_{t_0}^{t_f} 2\ddot{x}_2\delta x_1\, dt.$$

$$\Rightarrow \delta J(x,\delta x) = \boxed{\int_{t_0}^{t_f} \{[2x_1 + x_2 - 2\ddot{x}_2]\delta x_1 + [x_1 + 2x_2 - 2\ddot{x}_1]\delta x_2\}\, dt.}$$

c)

$$J(x) = \int_{t_0}^{t_f} e^{x(t)}\, dt$$

$$\Rightarrow \delta J(x,\delta x) = \boxed{\int_{t_0}^{t_f} e^{x}\delta x\, dt.}$$

Exercise 1.2 (teacher demo)

Show that the trajectory $x^*(t)$, along which the mass point slides (frictionless) in the shortest time from a to b, when it starts from rest and it is affected by the gravitational force ($mgh = 1/2mv^2$), is a cycloid. Its parametric representation is $y(\theta) = c\sin^2(\theta/2)$, $x(\theta) = (c/2(\theta - \sin(\theta))$, where c is a constant and θ represents the angle of the trajectory with respect to the vertical axis.

Solution

The Brachistochrone problem (from greek $\beta\rho\alpha\chi\iota\sigma\tau o\zeta$ *brachistos* - shortest, $\chi\rho o\nu o\zeta$ *chronos* - time): A body moves in the xy-plane from origin to point b. When the only force affecting the body is gravity, what trajectory $y(x)$ should the body take, so that it arrives to b in shortest time.

Time spent:

$$J = \int_0^{t_f} dt = \int_0^b \frac{ds}{v}, \quad v \text{ is the speed of the body.}$$

The travelled distance can be solved from Pythagora's Theorem $ds^2 = dx^2 + dy^2 = [(dx/dx)^2 + (dy/dx)^2]dx^2$:

$$ds = \sqrt{1 + (dy/dx)^2}dx.$$

The potential energy turns into kinetic energy:

$$mgy = \frac{1}{2}mv^2 \quad \Longrightarrow \quad v = \sqrt{2gy}.$$

Insert:

$$J = \int_0^b \frac{\sqrt{1+[dy/dx]^2}}{\sqrt{2gy}}\,dx = \frac{1}{\sqrt{2g}}\int_0^b \frac{\sqrt{1+[dy/dx]^2}}{\sqrt{y}}\,dx = \frac{1}{\sqrt{2g}}\int_0^b F(y,y')\,dx$$

Now the integrand $F(y, y')$ doesn't explicitly depend on variable x, and thus we can use the special case of Euler's equation

$$g - \dot{x}g_{\dot{x}} = A,$$

which fits the exercise, when $x \triangleq t$ and $y \triangleq x$. In other words

$$F - y'\frac{\partial F}{\partial y'} = A \Leftrightarrow y'\frac{\partial F}{\partial y'} - F = D,$$

where D is a constant.

Notice that we could have straight away used Euler

$$F_y - \frac{d}{dx}F_{y'} = 0 \Rightarrow y'[F_y - \frac{d}{dx}F_{y'}] = 0.$$

By adding some terms that cancel each other out

$$y''F_{y'} + y'\frac{d}{dx}F_{y'} - F_y y' - y''F_{y'} - F_x = \frac{d}{dx}[y'F_{y'} - F],$$

which should also equal 0. From this we get that the expression to be differentiated $y'F_{y'} - F$ has to equal a constant.
Let's calculate:

$$y'\frac{\partial F}{\partial y'} - F = \frac{(y')^2}{\sqrt{y(1+(y')^2)}} - \frac{1+(y')^2}{\sqrt{y(1+(y')^2)}} = -\frac{1}{\sqrt{y(1+(y')^2)}} = D.$$

Then:

$$y' = \sqrt{\frac{1}{D^2 y} - 1} = \sqrt{\frac{C}{y} - 1}, \tag{1}$$

where we choose $C = 1/D^2$. Let's make a change of variables (for how the parametrization is attained, see `https://en.wikipedia.org/wiki/Tautochrone_curve`)

$$y = C\sin^2\frac{\theta}{2} = -\frac{C}{2}(\cos\theta - 1), \quad \left(\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}\right)$$

$$\Rightarrow \quad \frac{dy}{d\theta} = \frac{C}{2}\sin\theta = C\sin\frac{\theta}{2}\cos\frac{\theta}{2}. \quad \left(\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$$

Then let's use the chain rule $\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx}$ to solve the Euler $y' = \sqrt{\frac{C}{y} - 1}$, and insert the parametric representation of y.

$$C\sin\frac{\theta}{2}\cos\frac{\theta}{2}\frac{d\theta}{dx} = \sqrt{\frac{1}{\sin^2(\theta/2)} - 1} = \sqrt{\frac{1-\sin^2(\theta/2)}{\sin^2(\theta/2)}} = \sqrt{\frac{cos^2(\theta/2)}{\sin^2(\theta/2)}} = \frac{\cos(\theta/2)}{\sin(\theta/2)}$$

so

$$C\sin^2\frac{\theta}{2} = \frac{dx}{d\theta}.$$

By integrating both sides of the equation we get

$$x(\theta) = \int C\sin^2\frac{\theta}{2}d\theta = \int -\frac{C}{2}(\cos\theta - 1)d\theta = \frac{C}{2}(\theta - \sin\theta) + k$$

.

We know that initially, when the body is at the origin, the body has only potential energy, and thus can only fall downwards, and thus for the parametric representation, we have the initial point condition $x(0) = 0$. Hence,

$$x(\theta) = \frac{C}{2}(\theta - \sin\theta)$$

and thus we have arrived to the parametric representation

$$\begin{cases} x(\theta) &= \frac{C}{2}(\theta - \sin\theta) \\ y(\theta) &= C\sin^2\frac{\theta}{2} \end{cases},$$

which is the equation of a cycloid.

Trivia: A cycloid is a tautochrone, i.e., bodies placed in different positions on the the cycloid slide down to the bottom in equal amount of time.

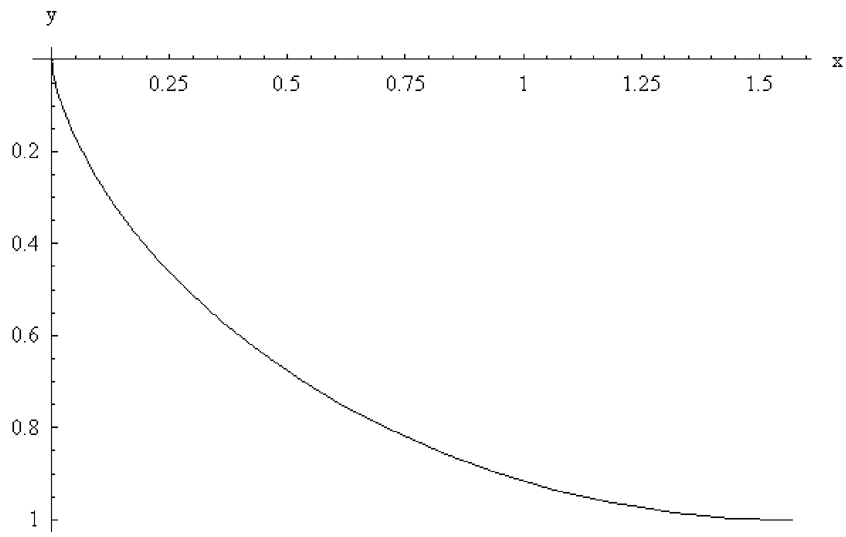

Figure 1: Cycloid curve.

Exercise 1.3 (student presents)

Find the extremals of the following functionals:

a) $J(x) = \int_0^1 [x^2 + \dot{x}^2]dt$, $x(0) = 0, x(1) = 1$

b) $J(x) = \int_0^2 [x^2 + 2\dot{x}x + \dot{x}^2]dt$, $x(0) = 1$, $x(2) = -3$

Note! Student presentation for part (c) is in the exercise session for round 2!

c) $J(x) = \int_0^{\pi/2} [\dot{x}_1^2 + \dot{x}_2^2 + 2x_1x_2]dt$, $x(0) = 0, x(\pi/2) = 1$

Solution

Euler's equation is $g_x - \frac{d}{dt}g_{\dot{x}} = 0$. Let's calculate the terms g_x and $\frac{d}{dt}g_{\dot{x}}$, and then write the DE where they are set to be equal. In part c) there are two Euler's equations, $g_{x_1} - \frac{d}{dt}g_{\dot{x}_1} = 0$ and $g_{x_2} - \frac{d}{dt}g_{\dot{x}_2} = 0$, and they have to be satisfied simultaneously.

a) Euler: $x = \ddot{x} \Leftrightarrow x - \ddot{x} = 0$.

The characteristic equation is $1 - z^2 = 0$. The roots are $z_1 = 1$ and $z_2 = -1$. The solution is $x(t) = c_1e^{z_1t} + c_2e^{z_2t} = c_1e^t + c_2e^{-t}$. The constants are solved from the end point conditions. $x(0) = 0 \Rightarrow c_1 + c_2 = 0$ and $x(1) = 1 \Rightarrow c_1e + c_2e^{-1} = 1$. The constants are

$$c_1 = \frac{1}{e - e^{-1}} \quad \text{and} \quad c_2 = -\frac{1}{e - e^{-1}}.$$

The final solution is

$$x^*(t) = \frac{1}{e - e^{-1}}(e^t - e^{-t}) = \frac{1}{e^2 - 1}(e^{2t} - 1)e^{1-t}$$

b) Euler: $x = \ddot{x}$, the solution: $x^* = -\big(e^{-t}(3e^2 - e^4 + e^{2t} - 3e^{2+2t})\big)/\big(e^4 - 1\big)$

c) Note! Student presentation for part (c) is in the exercise session for round 2!

Euler's equations: $\ddot{x}_1 = x_2, \ddot{x}_2 = x_1$. Let's solve Euler's equation by inserting $x_1 = \ddot{x}_2$ in the first Euler, and then we get the homogeneous equation $x_2^{(4)} - x = 0$. The solution of its characteristic equation $z^4 - 1 = 0$ are $1, -1, i, -i$. Because $\cos t + \sin t$ is a linear combination of e^{it} and e^{-it} it can be written in the form

$$x_2(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t. \tag{2}$$

Let's differentiate twice and we get $\ddot{x}_2(t) = c_1 e^t + c_2 e^{-t} - c_3 \cos t - c_4 \sin t = x_1(t)$. From the end point conditions $x_1(0) = 0$ and $x_2(0) = 0$ we get the equations

$$\begin{aligned} c_1 + c_2 - c_3 &= 0 \\ c_1 + c_2 + c_3 &= 0. \end{aligned}$$

So $c_3 = 0$ and $c_1 + c_2 = 0$. From the end point conditions $x_1(\pi/2) = x_2(\pi/2) = 1$ we get

$$\begin{aligned} c_1 e^{\pi/2} + c_2 e^{-\pi/2} + c_4 &= 1 \\ c_1 e^{\pi/2} + c_2 e^{-\pi/2} - c_4 &= 1, \end{aligned}$$

so also $c_4 = 0$. As the solution of the above equations we get $c_1 = \frac{1}{2\sinh(\pi/2)}$ and $c_2 = -\frac{1}{2\sinh(\pi/2)}$. Now we can write the solutions:

$$x_1^* = \frac{\sinh(t)}{\sinh(\pi/2)},$$

$$x_2^* = \frac{\sinh(t)}{\sinh(\pi/2)},$$

$$\text{Because} \sinh(t) = \frac{1}{2}\left(e^t - e^{-t}\right)$$

Exercise 1.4 (solved in class)

Find the trajectory x^* that minimizes the functional

$$J(x) = \int_0^1 \left[\underline{\frac{1}{2}}\dot{x}^2 + 3x\dot{x} + \underline{2x^2} + 4x\right] dt$$

and passes the points $x(0) = 1$ and $x(1) = 4$.

Solution

Write Euler's equation. It is a linear second order DE. You can write the characteristic equation of its homogeneous part. The result is a equation, which has three integration constants, which all can be solved from three equations: the original DE and the two end point conditions.

Euler:

For the particular solution x_p = -1, we have:
x"_p = 0 and -4x_p = 4
Thus, x"_p - 4x = 4 holds and x_p is a particular solution.

$$g_x = \cancel{3\dot{x}} + 4x + 4$$

$$\frac{d}{dt} g_{\dot{x}} = \frac{d}{dt}(\dot{x} + 3x) = \ddot{x} + \cancel{3\dot{x}}$$

$$g_x - \frac{d}{dt} g_{\dot{x}} = -\ddot{x} + 4x + 4 = 0 \Leftrightarrow \boxed{\ddot{x} - 4x = 4}$$

0

Homogeneous solution means the right side is set to zero instead of 4

This is a non-homogeneous second order differential equation. Let's solve the particular solution for the DE. $x_P(t) = -1$ clearly solves the DE. Next solve the homogeneous equation by forming its characteristic equation. The final solution for the DE is the solution for the homogeneous part + the particular solution. Thus $x(t) = c_1 e^{2t} + c_2 e^{-2t} - 1$. By inserting the end point conditions

x(t)=x*(t)

$$c_1 = \frac{2e^{-2} - 5}{e^{-2} - e^2}$$

$$c_2 = \frac{5 - 2e^2}{e^{-2} - e^2}.$$

Without the end point conditions,
the coefficients c_1 and c_2 can't be solved,
and thus remain unknown.