Exercise 1.1 (student presents)

Vary the following functionals

a) $J(x) = \int_{t_0}^{t_f} [x^3(t) - x^2(t)\dot{x}(t)]dt$.

b) $J(x) = \int_{t_0}^{t_f} [x_1^2(t) + x_1(t)x_2(t) + x_2^2(t) + 2\dot{x}_1(t)\dot{x}_2(t)]dt$.

c) $J(x) = \int_{t_0}^{t_f} e^{x(t)}dt$.

Assume that the end points are fixed.

Solution

Let the given functional be

$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\, dt$$

and we want to observe how the value of J changes, when $x(t)$ is changed a bit, i.e., what is $J(x + \delta x)$. The functional J is differentiable, so its increment can be written

$$\begin{aligned} \Delta J(x, \delta x) &= J(x + \delta x) - J(x) \\ &= \delta J(x, \delta x) + \varepsilon(x, \delta x)||\delta x||, \end{aligned}$$

where $\delta J(x, \delta x)$ is some linear functional w.r.t. δx and the error term

$$\lim_{||\delta x|| \to 0} \varepsilon(x, \delta x) = 0.$$

Then the functional $\delta J(x, \delta x)$ is called the variation of J for function x. The connection to the differentiability of functions is obvious. It is important to learn the variation technique, because later on we will calculate variations for many different functionals.

a) The functional

$$J(x) = \int_{t_0}^{t_f} [x^3(t) - x^2(t)\dot{x}(t)]\, dt$$

is varied, when the end points are fixed. We attain the variation δJ by making a Taylor approximation of the integrand w.r.t. the variables x and $\dot{x}$, and we only choose the linear terms:

$$
\begin{aligned}
\delta J(x,\delta x) &= \int_{t_0}^{t_f} [\frac{\partial g}{\partial x}\delta x + \frac{\partial g}{\partial \dot{x}}\delta \dot{x}]\, dt \\
&= \int_{t_0}^{t_f} \left\{(3x^2 - 2x\dot{x})\delta x - x^2 \delta \dot{x}\right\} dt.
\end{aligned}
$$

By partial integration

$$
\int_{t_0}^{t_f} x^2 \delta \dot{x}\, dt = \underbrace{x^2(t_f)\delta x(t_f) - x^2(t_0)\delta x(t_0)}_{=0,\ \text{fixed end points}} - \int_{t_0}^{t_f} 2x\dot{x}\delta x,
$$

and inserting

$$
\begin{aligned}
\delta J(x,\delta x) &= \int_{t_0}^{t_f} \left\{(3x^2 - \cancel{2x\dot{x}})\delta x + \cancel{2x\dot{x}}\delta x\right\} dt \\
&= \boxed{\int_{t_0}^{t_f} 3x^2 \delta x\, dt.}
\end{aligned}
$$

b)

$$
J(x) = \int_{t_0}^{t_f} [x_1^2(t) + x_1(t)x_2(t) + x_2^2(t) + 2\dot{x}_1(t)\dot{x}_2(t)\, dt
$$

$$
\begin{aligned}
\Rightarrow \delta J(x,\delta x) &= \int_{t_0}^{t_f} \left\{[2x_1 + x_2]\delta x_1 + [x_1 + 2x_2]\delta x_2 \right. \\
&\quad \left. + [2\dot{x}_2]\delta \dot{x}_1 + [2\dot{x}_1]\delta \dot{x}_2\right\} dt.
\end{aligned}
$$

Partial integration:

$$
\int_{t_0}^{t_f} [2\dot{x}_2]\delta \dot{x}_1\, dt = 2\Big(\underbrace{\dot{x}_2(t_f)\delta x_1(t_f) - \dot{x}_2(t_0)\delta x_1(t_0)}_{=0,\ \text{fixed end points}}\Big) - \int_{t_0}^{t_f} 2\ddot{x}_2 \delta x_1\, dt.
$$

$$
\Rightarrow \delta J(x,\delta x) = \boxed{\int_{t_0}^{t_f} \left\{[2x_1 + x_2 - 2\ddot{x}_2]\delta x_1 + [x_1 + 2x_2 - 2\ddot{x}_1]\delta x_2\right\} dt.}
$$

c)

$$
J(x) = \int_{t_0}^{t_f} e^{x(t)}\, dt
$$

$$
\Rightarrow \delta J(x,\delta x) = \boxed{\int_{t_0}^{t_f} e^{x} \delta x\, dt.}
$$

Exercise 1.2 (teacher demo)

Show that the trajectory $x^*(t)$, along which the mass point slides (frictionless) in the shortest time from a to b, when it starts from rest and it is affected by the gravitational force ($mgh = 1/2mv^2$), is a cycloid. Its parametric representation is $y(\theta) = c\sin^2(\theta/2)$, $x(\theta) = (c/2(\theta - \sin(\theta))$, where c is a constant and θ represents the angle of the trajectory with respect to the vertical axis.

Exercise 1.3 (student presents)

Find the extremals of the following functionals:

a) $J(x) = \int_0^1 [x^2 + \dot{x}^2]dt$, $x(0) = 0, x(1) = 1$

b) $J(x) = \int_0^2 [x^2 + 2\dot{x}x + \dot{x}^2]dt$, $x(0) = 1$, $x(2) = -3$

c) $J(x) = \int_0^{\pi/2} [\dot{x}_1^2 + \dot{x}_2^2 + 2x_1x_2]dt$, $x(0) = 0, x(\pi/2) = 1$

Solution

Euler's equation is $g_x - \frac{d}{dt}g_{\dot{x}} = 0$. Let's calculate the terms g_x and $\frac{d}{dt}g_{\dot{x}}$, and then write the DE where they are set to be equal. In part c) there are two Euler's equations, $g_{x_1} - \frac{d}{dt}g_{\dot{x}_1} = 0$ and $g_{x_2} - \frac{d}{dt}g_{\dot{x}_2} = 0$, and they have to be satisfied simultaneously.

a) Euler: $x = \ddot{x} \Leftrightarrow x - \ddot{x} = 0$.

The characteristic equation is $1 - z^2 = 0$. The roots are $z_1 = 1$ and $z_2 = -1$. The solution is $x(t) = c_1e^{z_1t} + c_2e^{z_2t} = c_1e^t + c_2e^{-t}$. The constants are solved from the end point conditions. $x(0) = 0 \Rightarrow c_1 + c_2 = 0$ and $x(1) = 1 \Rightarrow c_1e + c_2e^{-1} = 1$. The constants are

$$c_1 = \frac{1}{e - e^{-1}} \quad \text{and} \quad c_2 = -\frac{1}{e - e^{-1}}.$$

The final solution is

$$x^*(t) = \frac{1}{e - e^{-1}}(e^t - e^{-t}) = \frac{1}{e^2 - 1}(e^{2t} - 1)e^{1-t}$$

b) Euler: $x = \ddot{x}$, the solution: $x^* = -\big(e^{-t}(3e^2 - e^4 + e^{2t} - 3e^{2+2t})\big)/\big(e^4 - 1\big)$

c) Note! Student presentation for part (c) is in the exercise session for round 2!

Euler's equations: $\ddot{x}_1 = x_2, \ddot{x}_2 = x_1$. Let's solve Euler's equation by inserting $x_1 = \ddot{x}_2$ in the first Euler, and then we get the homogeneous equation $x_2^{(4)} - x = 0$. The solution of its characteristic equation $z^4 - 1 = 0$ are $1, -1, i, -i$. Because $\cos t + \sin t$ is a linear combination of e^{it} and e^{-it} it can be written in the form

$$x_2(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t. \tag{1}$$

Let's differentiate twice and we get $\ddot{x}_2(t) = c_1 e^t + c_2 e^{-t} - c_3 \cos t - c_4 \sin t = x_1(t)$. From the end point conditions $x_1(0) = 0$ and $x_2(0) = 0$ we get the equations

$$\begin{aligned} c_1 + c_2 - c_3 &= 0 \\ c_1 + c_2 + c_3 &= 0. \end{aligned}$$

So $c_3 = 0$ and $c_1 + c_2 = 0$. From the end point conditions $x_1(\pi/2) = x_2(\pi/2) = 1$ we get

$$\begin{aligned} c_1 e^{\pi/2} + c_2 e^{-\pi/2} + c_4 &= 1 \\ c_1 e^{\pi/2} + c_2 e^{-\pi/2} - c_4 &= 1, \end{aligned}$$

so also $c_4 = 0$. As the solution of the above equations we get $c_1 = \frac{1}{2\sinh(\pi/2)}$ and $c_2 = -\frac{1}{2\sinh(\pi/2)}$. Now we can write the solutions:

$$\begin{aligned} x_1^* &= \frac{\sinh(t)}{\sinh(\pi/2)}, \\ x_2^* &= \frac{\sinh(t)}{\sinh(\pi/2)}, \end{aligned}$$

Because $\sinh(t) = \frac{1}{2}\left(e^t - e^{-t}\right)$

Exercise 1.4 (solved in class)

Find the trajectory x^* that minimizes the functional

$$J(x) = \int_0^1 \left[\frac{1}{2}\dot{x}^2 + 3x\dot{x} + 2x^2 + 4x\right] dt$$

and passes the points $x(0) = 1$ and $x(1) = 4$.

Exercise 1.5 (home assignment)

Find the extremals of the following functionals with the help of Euler's equation. The end points are $x(0) = A$ and $x(T) = B$. The intermediate solutions have to be displayed.

a)

$$\int_0^T [\dot{x}^2/t^3] dt$$

b)

$$\int_0^T [\dot{x}^2 - 8xt + t] dt$$