Dynamic Optimization - Homework assignment 9

Christian Segercrantz 481056

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9.4

We know the state equation

$$\dot{x} = -10x(t) + u(t) \tag{1}$$

and we want to minimize the performance measure

$$J = \frac{1}{2}x^2(0.04) + \int_0^{0.04} \left[\frac{1}{4}x^2(t) + \frac{1}{2}u^2(t)\right] dt \tag{2}$$

We will write the Hamiltonian as

$$H(x,u,p,t) = \frac{1}{4}x^2(t) + \frac{1}{2}u^2(t) + \nabla_x J^*(t,x)(-10x(t) + u(t)) \tag{3}$$

The Hamilton-Jacobi-Bellman equation is

$$0 = \min_{u \in U} \left[\nabla_t J^*(t,x) + H(x,u,p,t)\right] \tag{4}$$

with the boundary condition

$$J^*(T,x) = h(x,T). \tag{5}$$

We take the first order derivative of the Hamiltonian w.r.t u and get

$$H_u = u + \nabla_x J^*, \tag{6}$$

which gives us

$$\begin{aligned}
0 =& + u^* + \nabla_x J^* && (7)\\
u^* =& - \nabla_x J^* && (8)\\
\implies H =& \frac{1}{4}x^2 + \frac{1}{2}(-\nabla_x J^*)^2 + \nabla_x J^* \cdot (-10x - \nabla_x J^*) && (9)\\
=& \frac{1}{4}x^2 - \frac{1}{2}(\nabla_x J^*)^2 - 10x\nabla_x J^* && (10)
\end{aligned}$$

We will use the trial function and the following results

$$\begin{aligned}
J(x(t),t) =& a(t)x^2(t) + b(t)x(t) + c(t) && (11)\\
J_x =& 2ax + b && (12)\\
J_t =& \dot{a}x^2 + \dot{b}x + \dot{c} && (13)\\
u =& - \dot{a}x^2 - \dot{b}x - \dot{c} - 2ax - b && (14)
\end{aligned}$$

By inserting this into the HJB equation we get

$$\begin{aligned}
0 =& \dot{a}x^2 + \dot{b}x + \dot{c} + \frac{1}{4}x^2 + \frac{1}{2}(2ax+b)^2 - 10x(2ax+b) && (15)\\
-(\dot{a}x^2 + \dot{b}x + \dot{c}) =& \frac{1}{4}x^2 + \frac{1}{2}a^2x^2 + abx + \frac{1}{2}b^2 - 20ax^2 + 10bx && (16)\\
=& x^2\left(\frac{1}{4} + \frac{1}{2}a^2 - 20a\right) + x(ab + 10b) + \left(\frac{1}{2}b^2\right) && (17)
\end{aligned}$$

And by comparing the terms on both side we get the three ODEs

$$\begin{aligned}
-\dot{a} =& \frac{1}{4} + \frac{1}{2}a^2 - 20a && (18)\\
-\dot{b} =& ab + 10b && (19)\\
-\dot{c} =& \frac{1}{2}b^2 && (20)
\end{aligned}$$