Exercise 9.4 (homework)

The first order linear system

$$\dot{x}(t) = -10x(t) + u(t)$$

is to be controlled to minimize the performance measure

$$J = \frac{1}{2}x^2(0.04) + \int_0^{0.04} \left[\frac{1}{4}x^2(t) + \frac{1}{2}u^2(t)\right] dt.$$

The admissible state and control values are not constrained by any boundaries. Find the optimal control law by using the Hamilton-Jacobi-Bellman equation. Use the trial function $J(x(t), t) = a(t)x^2(t) + b(t)x(t) + c(t)$.

Note! In this exercise, you are allowed to use software for solving differential equations and systems of differential equations, but you must clearly state/mark what inputs you used (it's enough to tell the function numbers, for example) and results you got, and give the name of the software you used. One software option is to use the Wolfram Alpha website.

Solution

The Hamiltonian is

$$H = \frac{1}{4}x^2 + \frac{1}{2}u^2 + J_x(-10x + u).$$

The control that minimizes the Hamiltonian is

$$\begin{aligned} &H_u = 0 \\ \Leftrightarrow\ &u^* + J_x = 0 \\ \Leftrightarrow\ &u^* = -J_x. \end{aligned} \tag{1}$$

Let us write the Hamilton-Jacobi-Bellman equation

$$\begin{aligned}
0 &= J_t + \min_u \{H\} \\
&= J_t + \frac{1}{4}x^2 + \frac{1}{2}(-J_x)^2 + J_x(-10x - J_x) \\
&= J_t + \frac{1}{4}x^2 + \frac{1}{2}J_x^2 - 10xJ_x - J_x^2 \\
&= J_t - \frac{1}{2}J_x^2 - 10xJ_x + \frac{1}{4}x^2.
\end{aligned}$$

Because $J(x(T), T) = \frac{1}{2}x^2(0.04)$, let us use a trial function $J(x(t), t) = a(t)x^2(t) + b(t)x(t) + c(t)$.

Now we can calculate the partial derivatives,

$$J_x = 2ax + b$$

$$J_t = \dot{a}x^2 + \dot{b}x + \dot{c}$$

and insert them into the HJB,

$$\begin{aligned}
0 &= \dot{a}x^2 + \dot{b}x + \dot{c} - \frac{1}{2}(2ax + b)^2 - 10x(2ax + b) + \frac{1}{4}x^2 \\
&= \dot{a}x^2 + \dot{b}x + \dot{c} - \frac{1}{2}(4a^2x^2 + 4abx + b^2) - 20ax^2 - 10bx + \frac{1}{4}x^2 \\
&= x^2(\dot{a} - 2a^2 - 20a + \frac{1}{4}) + x(\dot{b} - 2ab - 10b) + \dot{c} - \frac{b^2}{2}.
\end{aligned}$$

The HJB has to be satisfied for all x and all t, thus

$$\begin{cases} \dot{a} - 2a^2 - 20a + \frac{1}{4} & = 0 \\ \dot{b} - 2ab - 10b & = 0 \\ \dot{c} - \frac{b^2}{2} & = 0 \end{cases}$$

or

$$\begin{cases} \dot{a} = & 2a^2 + 20a - \frac{1}{4} \\ \dot{b} = & 2ab + 10b \\ \dot{c} = & \frac{b^2}{2} \end{cases} \tag{2}$$

Additionally, we know that

$$\begin{aligned}
J(x(0.04), 0.04) &= \frac{1}{2}x^2(0.04) \\
&= a(0.04)x^2(0.04) + b(0.04)x(0.04) + c(0.04).
\end{aligned}$$

Thus,

$$a(0.04) = \frac{1}{2},\ b(0.04) = 0,\ c(0.04) = 0. \tag{3}$$

We can solve the differential equation system (2) using the boundary conditions (3) with Mathematica

$$a(t) = \frac{(16 + 3\sqrt{402})e^{\sqrt{402}\,t} + 41(\sqrt{402} - 20)e^{\frac{\sqrt{402}}{25}}}{4\left((22\sqrt{402} - 443)e^{\sqrt{402}\,t} + 41e^{\frac{\sqrt{402}}{25}}\right)}$$

$$= \frac{(\sqrt{402} + 19)e^{\sqrt{402}\,t} + (\sqrt{402} - 19)e^{\frac{\sqrt{402}}{25}}}{2\left((\sqrt{402} - 22)e^{\sqrt{402}\,t} + (\sqrt{402} + 22)e^{\frac{\sqrt{402}}{25}}\right)},$$

and

$$b(t) = c(t) = 0.$$

Now, we can use the first order condition (1) to solve the optimal control

$$\begin{aligned} u^* &= -J_x \\ &= -2ax + b \\ &= -2ax, \end{aligned}$$

where a is defined above.