Exercise 8.1 (student presents)

A system without a disturbance is

$$x_{k+1} = x_k + u_k, \qquad x_k, u_k \in \mathbb{R}$$

and its initial state x_0. The cost to be minimized over all periods is

$$x_2^2 + u_0^2 + u_1^2.$$

There are no constraints for the control

a) What is the cost function $g_k(x_k, u_k)$ in periods $k = 0, 1, 2$?

b) Find the optimal control policy $u_k^* = \mu_k^*(x_k)$, $k = 0, 1$ with the DP-algorithm.

Solution

a) The cost function is piecewise:

$$g_2(x_2) = x_2^2, \qquad g_1(x_1, u_1) = u_1^2, \qquad g_0(x_0, u_0) = u_0^2$$

Explain why g_2 is independent of u_2!

b) $k = 2$:

$J_3(x_3) = 0$ $\qquad J_k = \min_{u_k} [g_k + J_{k+1}]$

$$J_2(x_2) = g_2(x_2) = x_2^2$$

$k = 1$:

$x_1 + u_1$

$$J_1(x_1) = \min_{u_1} \Big[g_1(x_1, u_1) + J_2(x_2) \Big] = \min_{u_1} \Big[u_1^2 + J_2(x_2) \Big] = \min_{u_1} \Big[u_1^2 + (x_1 + u_1)^2 \Big]$$

Lets denote $L_1(x_1, u_1) = u_1^2 + (x_1 + u_1)^2$, which is continuous with regard to u_1 and x_2. The control that minimizes the cost-to-go function:

$$0 = \frac{\partial L_1}{\partial u_1} = 2u_1 + 2(x_1 + u_1)$$

out of which $\boxed{u_1 = -x_1/2.}$ Because $\partial_{u_1}^2 L_1 > 0$, it is the minimum. Thus $\mu_1^*(x_1) = -x_1/2$ and $J_1(x_1) = x_1^2/2$.

$$J_1(x_1) = \left(-\frac{x_1}{2}\right)^2 + \left(x_1 - \frac{x_1}{2}\right)^2 = x_1^2 - \frac{2x_1^2}{2} + \left(-\frac{x_1}{2}\right)^2 + \left(-\frac{x_1}{2}\right)^2$$

$$= \frac{x_1^2}{4} + \frac{x_1^2}{4} = \frac{x_1^2}{2}$$

$k = 0$:

$$J_0(x_0) = \min_{u_0} \left[u_0^2 + J_1(x_1) \right] = \min_{u_0} \left[u_0^2 + 0.5(x_0 + u_0)^2 \right].$$

$$0 = \frac{\partial L_0}{\partial u_0} = 2u_0 + (x_0 + u_0)$$

out of which $u_0 = -x_0/3$. It is the minimum, because $\partial^2_{u_0} L_0 > 0$. Thus $\mu_0^*(x_0) = -x_0/3$ and $J_0(x_0) = x_0^2/3$.

Now we can also solve u_1^* and x_2^* as functions of the initial state:

$$x_1^* = x_0 + u_0^* = x_0 - \frac{1}{3}x_0 = \frac{2}{3}x_0$$
$$u_1^* = -\frac{1}{2}x_1^* = -\frac{1}{2}\frac{2}{3}x_0 = -\frac{1}{3}x_0$$
$$x_2^* = x_1^* + u_1^* = \frac{2}{3}x_0 - \frac{1}{3}x_0 = \frac{1}{3}x_0$$

The cost of the optimal solution with the initial state x_0 is:

$$x_2^2 + u_0^2 + u_1^2 = \left(\frac{1}{3}x_0\right)^2 + \left(-\frac{1}{3}x_0\right)^2 + \left(-\frac{1}{3}x_0\right)^2 = \frac{1}{3}x_0^2$$

which is the same as $J_0(x_0)$.