Exercise 9.1 (solved in class)

The problem is

$$\begin{aligned} \dot{x} &= -x + u - 1, \qquad x(0) = 1, \\ J &= x^2(1) + \int_0^1 u^2 dt. \end{aligned}$$

Solve the optimal control using the HJB equation. Trial function: $V(t, x) = a(t)x^2 + 2b(t)x + c(t)$.

Note! In this exercise, you are allowed to use software for solving differential equations and systems of differential equations, but you must clearly state/mark what inputs you used (it's enough to tell the function numbers, for example) and results you got, and give the name of the software you used. One software option is to use the Wolfram Alpha website.

Solution

Hamiltonian: $H = u^2 - J_x x + J_x u - J_x$
First order condition: $H_u = 0 \Rightarrow u^* = -J_x^*/2$
HJB: $0 = J_t^* - (J_x^*)^2/4 - J_x^* - xJ_x^*$

Trial function:

$$\begin{aligned} J^* &= a(t)x^2 + 2b(t)x + c(t), \\ J_x^* &= 2a(t)x + 2b(t), \\ J_t^* &= \dot{a}x^2 + 2\dot{b}x + \dot{c}. \end{aligned}$$

Let us insert the trial function in the HJB equation, which gives

$$x^2(2a + a^2) + 2x(a + b + ab) + b^2 + 2b = x^2\dot{a} + 2x\dot{b} + \dot{c}.$$

Because the HJB has to be satisfied for all x and all t, we get three DEs:

$$\begin{aligned} \dot{a} =& 2a + a^2, \\ \dot{b} =& a + b + ab, \\ \dot{c} =& b^2 + 2b. \end{aligned}$$

From the boundary condition, we know that $J^*(x(1)) = x^2(1)$. Thus,

$$a(1)x(1)^2 + 2b(1)x(1) + c(1) = x^2(1).$$

By comparing terms, we get $a(1) = 1$, $b(1) = 0$ and $c(1) = 0$. By solving the three DEs using these boundary conditions (for example using Mathematica), we get

$$
\begin{aligned}
a(t) =& \frac{2e^{2t}}{3e^2 - e^{2t}}, \\
b(t) =& \frac{2e^t(e^t - e)}{3e^2 - e^{2t}}, \\
c(t) =& \frac{2(e - e^t)^2}{3e^2 - e^{2t}}.
\end{aligned}
$$

The optimal feedback control is obtained by combining the 1. order condition with the trial function,

$$u^* = -J_x^*/2 = -(ax + b) = \frac{1}{e^{2t} - 3e^2}\big(2e^{2t}x(t) + 2e^t(e^t - e)\big).$$

Exercise 9.2 (self-study)

a) Form the Hamilton-Jacobi Bellman equation for the problem

$$\min \int_0^T e^{-rt} g(x(t), u(t))dt \qquad \text{s.t.} \quad \dot{x} = f(x, u) \tag{1}$$

when instead of the function J the function $V(x, t) = e^{rt}J(x, t)$ is used.

b) What role has the factor e^{-rt} in the cost functional? Characterize its role when t varies between zero and infinity. The parameter $r > 0$.

c) Solve the problem

$$\min J = \int_0^\infty e^{-rt}[x^2(t) + u^2(t)]dt$$

that has the scalar system

$$
\begin{aligned}
\dot{x}(t) &= x(t) + u(t), \qquad r > 0 \\
x(0) &= x_0 > 0
\end{aligned}
$$

by using the trial function $V(x, t) = Ax^2$ for the present value of the optimal cost. Apply the HJB equation of part a).

Solution

a) From $V(x,t) = e^{rt}J(x,t)$, we get $J(x,t) = e^{-rt}V(x,t)$, and further $J_t(x,t) = e^{-rt}V_t(x,t) - re^{-rt}V(x,t)$ and $J_x(x,t) = e^{-rt}V_x(x,t)$. Inserting into the HJB equation, we get

$$0 = \min_u \left[e^{-rt}g + e^{-rt}(V_t - rV) + e^{-rt}V_x f\right],$$

and further

$$e^{-rt}\left[rV - V_t\right] = \min_u \left[e^{-rt}g + e^{-rt}V_x f\right]$$

We can divide the equation with e^{-rt}

$$rV - V_t = \min_u \left[g + V_x f\right].$$

b) It is a discount factor, which approaches one, when t goes to zero, and the discount factor goes to zero when t grows to infinity. It weights the instantaneous costs, more the earlier than the later. The discount factor is also strictly decreasing since $\partial_t e^{-rt} < 0$ and convex since $\partial_{tt} e^{-rt} > 0$ for all t.

c) Now, because the time horizon of the problem is infinite, it is natural to search for a V that only depends on the state. Let us use the trial $V(x,t) = Ax^2$. On the right-side of the HJB we have to minimize $x^2 + u^2 + 2Ax(x+u)$. Let us find the u^* that minimizes the expression,

$$0 = \left.\frac{d}{du}\right|_{u=u^*} \left[x^2 + u^2 + 2Ax(x+u)\right],$$

which gives us

$$u^* = -Ax.$$

Let us insert this back to the HJB equation

$$rAx^2 = x^2 + (-Ax)^2 + 2Ax(x - Ax),$$

or

$$rAx^2 = x^2 + A^2x^2 + 2Ax^2(1 - A).$$

The coefficients of the x^2 terms have to be equal for both sides of the equation,

$$rA = 1 + A^2 + 2A(1 - A),$$

or

$$A^2 + (r-2)A - 1 = 0.$$

The solutions are

$$A = 1 - \frac{r}{2} \pm \sqrt{(1 - \frac{r}{2})^2 + 1}.$$

The integrand always attains positive values, thus $V(x,t) \geq 0$. Thus, the positive root is the solution for the problem.

Exercise 9.3 (teacher demo)

The amount of savings in the beginning is S. All income is interest income (interest is constant). The change in capital x is modeled with the equation $\dot{x}(t) = \alpha x(t) - r(t)$, where $\alpha > 0$ is the interest and r is the consumption rate. The utility experienced from momentary consumption r is $U(r)$, where U is the utility; let $U(r) = r^{1/2}$.

The utility that will be achieved in the future is discounted: the utility achieved today ($t = 0$) is more valuable now than in the future. The discount factor is $e^{-\beta t}$ ($\beta > \alpha/2$), thus the present value of the maximized total utility is

$$J = \int_0^T e^{-\beta t} U(r) dt \tag{2}$$

with the end condition $x(T) = 0$ (the capital is fully used).

a) Write the Hamilton-Jacobi-Bellman equation.

b) Solve the HJB equation by using the trial function $J(x,t) = f(t)g(x)$
(hint: attempt $g(x) = Ax^{1/2}$).

c) What is the optimal trajectory of capital?

Solution

Initially the amount of savings is S, the equation for change in capital

$$\dot{x}(t) = \alpha x(t) - r(t)$$

where $\alpha > 0$ is the interest and $r(t)$ is the consumption rate which acts as a control variable. The natural constraint is

$$r(t) \geq 0, \quad \forall\, t \in [0, T].$$

The utility experienced from consumption is

$$U(r(t)) = \sqrt{r(t)},$$

which is discounted into present value with the discount factor $\beta > \alpha/2$, then the present value of the total utility is

$$J = \int_0^T e^{-\beta t} U(r(t))\, dt.$$

The aim is to maximize J, so that in the end all capital is used.

a) Let us form the HJB equation. The Hamiltonian for this problem is

$$H(x(t), r(t), t) = e^{-\beta t}\sqrt{r(t)} + J_x^{*T}(x(t), t)\big[\alpha x(t) - r(t)\big].$$

By writing the partial derivative with regard to r and setting it equal to zero, the following equation is obtained

$$\frac{\partial H}{\partial r} = \frac{1}{2}\frac{e^{-\beta t}}{\sqrt{r^*(t)}} - J_x^*(x(t), t) = 0,$$

from which we get the maximum

$$r^*(t) = \frac{e^{-2\beta t}}{4\big[J_x^*(x(t), t)\big]^2} \geq 0,$$

because the second partial derivative is negative. Now the HJB equation can be formed:

$$0 = J_t^*(x(t), t) + \frac{e^{-2\beta t}}{2J_x^*(x(t), t)} + \alpha J_x^*(x(t), t)x(t) - \frac{e^{-2\beta t}}{4J_x^*(x(t), t)}$$

thus

$$\boxed{J_t^*(x(t), t) + \frac{e^{-2\beta t}}{4J_x^*(x(t), t)} + \alpha J_x^*(x(t), t)x(t) = 0.}$$

b) Lets attempt the following solution

$$J^*(x(t), t) = A\sqrt{x(t)}f(t),$$

where $A > 0$ and $f(t)$ is some differentiable function (note that we are calculating the partial derivative):

$$J_t^*(x(t), t) = A\sqrt{x(t)}f'(t), \quad J_x^*(x(t), t) = \frac{Af(t)}{2\sqrt{x(t)}}.$$

Inserting this into the HJB equation the following equation is attained

$$\big[Af'(t) + \frac{e^{-2\beta t}}{2Af(t)} + \frac{1}{2}\alpha A f(t)\big]\sqrt{x(t)} = 0.$$

This is fulfilled for all $t \in [0, T]$ if only $f(t)$ satisfies the ordinary differential equation

$$Af'(t) + \frac{e^{-2\beta t}}{2Af(t)} + \frac{1}{2}\alpha A f(t) = 0.$$

By multiplying both sides with $e^{\alpha t}$ and organizing the terms the equation can be written in the following form

$$\underbrace{\alpha e^{\alpha t} f^2 + 2e^{\alpha t} f f'}_{\frac{d}{dt}(e^{\alpha t} f^2)} = -\frac{1}{A^2} e^{(\alpha-2\beta)t}.$$

By integrating both sides, we get

$$e^{\alpha t} f^2 = \frac{1}{A^2(2\beta-\alpha)} e^{(\alpha-2\beta)t} + C.$$

If the right-side is positive, we find the solutions

$$f(t) = \pm\sqrt{\frac{1}{A^2(2\beta-\alpha)} e^{-2\beta t} + Ce^{-\alpha t}}.$$

From these only the positive sign is possible, otherwise $J^* < 0$ (in that case the total utility achieved by consumption would be negative). We have achieved a candidate for the cost-to-go -function

$$J^*(x(t),t) = A\sqrt{x(t)}\sqrt{\frac{1}{A^2(2\beta-\alpha)} e^{-2\beta t} + Ce^{-\alpha t}}.$$

The integration constant C should be chosen so that the boundary condition of the HJB equation

$$J^*(x(T),T)) = 0$$

is satisfied. Hence, we get the condition for C:

$$C = \frac{1}{A^2(\alpha-2\beta)} e^{(\alpha-2\beta)T}.$$

Inserting this into the cost-to-function expression, we get

$$\boxed{J^*(x(t),t)) = \sqrt{\frac{x(t)}{2\beta-\alpha}}\sqrt{e^{-2\beta t} - e^{\alpha(T-t)-2\beta T}}.}$$

The exponential expression in the root expression is always non-negative, when $t \in [0,T]$. Thus, the function is well-defined and satisfies the boundary condition of the HJB equation.

Inserting J^* into the optimal control expression previously calculated, we get the optimal feedback control

$$r^*(t) = \frac{2\beta-\alpha}{1-e^{(2\beta-\alpha)(t-T)}} x(t).$$

c) Inserting the optimal feedback control into the state equation, we get the differential equation for the optimal trajectory

$$\dot{x}(t) = \big(\alpha - \frac{2\beta-\alpha}{1-e^{(2\beta-\alpha)(t-T)}}\big)x(t).$$

By separating the equation and integrating both sides, we get the form

$$\int_S^{x(t)} \frac{1}{x}\,dx = \int_0^t \big(\alpha - \frac{2\beta-\alpha}{1-e^{(2\beta-\alpha)(t-T)}}\big)\,dt$$

hence

$$\log \frac{x(t)}{S} = \alpha t - (2\beta - \alpha) \int_0^t \frac{1}{1 - e^{(2\beta-\alpha)(t-T)}}\, dt.$$

We find the following formula from the integral cookbook

$$\int_0^t \frac{1}{a + be^{kt}}\, dt = \frac{1}{ak}\big(kt - \log(a + be^{kt})\big).$$

Check that this holds! By using the above formula and simplifying (again, check this by yourself), we get the final result

$$x(t) = Se^{2(\alpha-\beta)t}\big(1 - e^{(2\beta-\alpha)(t-T)}\big).$$

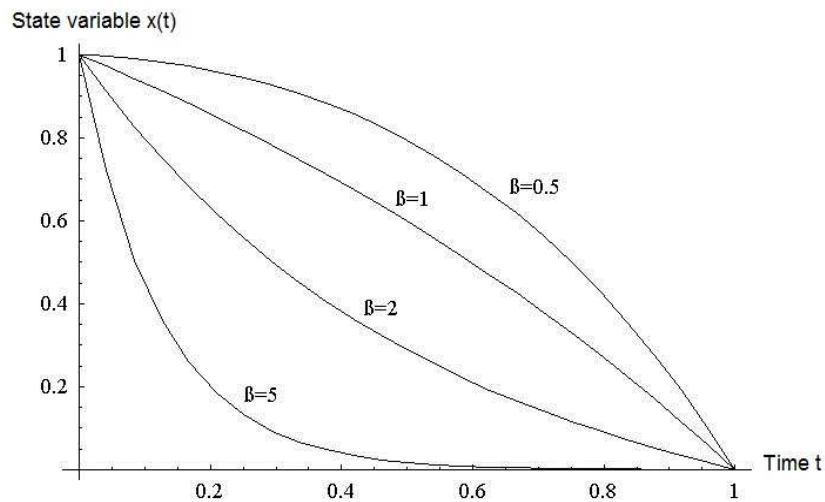

Figure 1: The optimal consumption trajectories for Exercise 9.3, with different values for the discount factor β when $T = 1$, $S = 1$ and $\alpha = 0.95$.

Note that $x(T) = 0$ so the end condition is also satisfied. This is natural, because if there would be unused capital in the end, the total utility is not maximized.