Notes and Comments on Lecture 8

Solving an ODE via separation of variables

Let us solve

$$\frac{1}{2}\dot{K} - K^2 + K = 0, \quad K(T) = \frac{1}{2}$$

via separation of variables. Recall that $\dot{K} = \frac{d}{dt}K$. We rewrite the ODE

$$\frac{1}{2}\frac{dK}{dt} = K^2 - K.$$

as

$$\frac{dK}{2(K^2 - K)} = dt.$$

Integrating both sides formally gives

$$\int \frac{dK}{2(K^2 - K)} = t$$

and further

$$\frac{1}{2}(\log(1-K) - \log(K)) + C = \frac{1}{2}\log\left(\frac{1-K}{K}\right) + C = t.$$

Hence

$$\frac{1-K}{K} = \frac{1}{K} - 1 = e^{2(t-C)}.$$

Therefore,

$$K = \frac{1}{e^{2(t-C)} + 1}.$$

$K(T) = \frac{1}{2}$ is satisfied for $C = T$. Therefore,

$$K(t) = \frac{1}{e^{2(t-T)} + 1} = \frac{e^{T-t}}{e^{t-T} + e^{T-t}}.$$