MS-E2148 Dynamic optimization

Lecture 5

Contents

Recap

- Infinite horizon problems
- Minimum principle in solving control problems

Minimum-time problem

- The task is to get the state into a set $G(t)$ in minimum time:

  $$\min J = \int_{t_0}^{t_f} dt, \quad \dot{x} = f(x, u, t), \quad x(t_0) = x_0, \quad x(t_f) \in G(t_f)$$

  when the controls are bounded: $|u| \leq 1$
- Due to bounded controls, it may be that there is no minimum time for all initial states (the goal set runs away)
- *Reachable states* $R(t)$ are the states that can be reached at time t from the initial state x_0 using admissible controls
- See controllability and observability of (linear) systems

Minimum-time problem

Example of reachable states

- The system $\dot{x} = u$, with constraint $|u| \leq 1$
- The solution candidates are of the form $x = x_0 + \int_{t_0}^{t} u(\tau) d\tau$
- With admissible controls

$$x_0 - (t - t_0) \leq x \leq x_0 + (t - t_0)$$

so the set of reachable states increases as a function of t

- If the goal set and the reachable states do not have a common state, there is no optimal control for the minimum-time problem

Minimum-time problem

- Let us examine the problem

$$\min \int_{t_0}^{t_f} dt, \qquad \dot{x} = a(x,t) + B(x,t)u \tag{1}$$

  where a is a vector and B is a suitable size matrix that depend explicitly on state and time
- The controls are bounded: $M^- \leq u \leq M^+$
- The Hamiltonian is

$$H(x,u,p,t) = 1 + p^T[a(x,t) + B(x,t)u] \tag{2}$$

Minimum-time problem

- Minimum principle:

$$\begin{aligned} 1 + {p^*}^T[a(x^*, t) + B(x^*, t)u^*] &\leq 1 + {p^*}^T[a(x^*, t) + B(x^*, t)u] \\ \Rightarrow {p^*}^T B(x^*, t)u^* &\leq {p^*}^T B(x^*, t)u \end{aligned}$$

- Since $B = [b_1, b_2, ..., b_m]$ is a matrix and u is a vector, the term ${p^*}^T B(x^*, t)u$ can be written

$$ {p^*}^T B(x^*, t)u = \sum_{i=1}^{m} {p^*}^T b_i(x^*, t)u_i \tag{3}$$

Minimum-time problem

- We can solve the optimal control from the minimum principle:

$$u_i^* = \begin{cases} M_i^+, & s_i(t) < 0 \\ \text{undefined}, & s_i(t) = 0 \\ M_i^-, & s_i(t) > 0 \end{cases}$$

- This is called the *bang-bang* control

Minimum-time problem

Swithcing function and singular interval

- We denote the multiplier of the linear term of the control, $s(t) = {p^*}^T B(x^*, t)$, in the Hamiltonian and it is called the *switching function* (multiplier of u in Hamiltonian)
- The sign of the switching function tells which end of the bang-bang control is used
- Time interval $t \in [t_1, t_2]$, $t_2 > t_1$, where $s(t) \equiv 0$ is a *singular interval* – this is an interval where the necessary condition does not give a unique solution to u^*

Minimum-time problem

Example

- Let us examine a rocket landing on the surface of Mars:

$$m(t)\ddot{x} = -gm(t) + T(t) = -gm(t) - k\dot{m}(t)$$

- The thrust $T(t)$ is proportional to the usage of fuel (”mass flow”)
- $x =$ distance from the surface; $x(t_0) =$ distance initially
- We can write this as a first-order system, where $x_1 = x$ and $u = \dot{m}$:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -g - ku/x_3 \\ \dot{x}_3 &= u \end{aligned}$$

Minimum-time problem

Example

- Objective: land in minimum time $\int_{t_0}^{t_f} dt$, where t_f is free
- Hamiltonian:

$$H(x, u, p) = 1 + p_1 x_2 - g p_2 - p_2 \frac{ku}{x_3} + p_3 u$$

- Minimum principle:

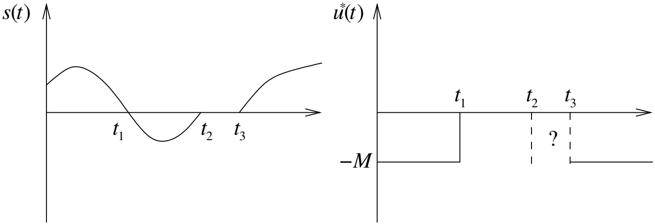
$$\left(- p_2^* \frac{k}{x_3^*} + p_3^* \right) u^* \leq \left(- p_2^* \frac{k}{x_3^*} + p_3^* \right) u$$

- Switching function: $s(t) = -p_2^* \frac{k}{x_3^*} + p_3^*$

- Control $u = \dot{m}$ is bounded between $[-M, 0]$ and by the bang-bang principle

$$u^* = \begin{cases} 0, & s(t) < 0 \\ \text{undefined}, & s(t) = 0 \\ -M, & s(t) > 0 \end{cases}$$

- The roots of the switching function $s(t)$ give when the control jumps between the extreme controls



- If $s(t) = 0$ for some time interval $[t_2, t_3]$, the control is undefined and we say it is a singular interval; we should always check the possibility of singular solutions

Minimum control-effort problems

- Let us examine the problem

$$\min \int_{t_0}^{t_f} \left[\sum_{i=1}^{m} |u_i| \right] dt, \qquad \dot{x} = a(x,t) + B(x,t)u$$

  where the system is the same as in the minimum-time problem
- The controls are bounded $|u| \leq 1$
- The components of the control vector are also independent; i.e., the choice of u_i does not influence u_j, $i \neq j$

Minimum control-effort problems

- Hamiltonian:

$$H(x, u, p, t) = \sum_{i=1}^{m} |u_i| + p^T [a(x, t) + B(x, t)u] \qquad (4)$$

- Minimum principle:

$$\sum_{i=1}^{m} |u_i^*| + {p^*}^T B(x^*, t)u^* \leq \sum_{i=1}^{m} |u_i| + {p^*}^T B(x^*, t)u$$

which can be written in form (see equation (3))

$$|u_i^*| + {p^*}^T b_i(x^*, t)u_i^* \leq |u_i| + {p^*}^T b_i(x^*, t)u_i \qquad i = 1, 2, ..., m$$

Minimum control-effort problems

- It is not yet possible to derive the optimal control from the minimum principle because of the nonlinear absolute value
- By the properties of absolute value, we can write

$$|u_i| + {p^*}^T b_i(x^*, t) u_i = \begin{cases} (1 + s_i(t)) u_i, & u_i \geq 0 \\ (-1 + s_i(t)) u_i, & u_i \leq 0 \end{cases}$$

  where the components of the switching function are $s_i(t) = {p^*}^T b_i(x^*, t)$

Minimum control-effort problems

- The optimal control is

$$u_i^* = \begin{cases} 1, & s_i(t) < -1 \\ 0, & -1 < s_i(t) < 1 \\ -1, & s_i(t) > 1 \\ \text{undefined but } \geq 0, & s_i(t) = -1 \\ \text{undefined but } \leq 0, & s_i(t) = 1 \end{cases}$$

- This is called *bang-off-bang* control

Minimum control-effort problems

Example

- System is $\dot{x} = -ax + u$, where $a > 0$
- Controls are bounded $|u| \leq 1$
- The objective is

$$\min J(u) = \int_{t_0}^{t_f} |u| dt,$$

where t_f is free and the initial and final states are fixed: $x(t_0) = x_0$ and $x(t_f) = 0$

- Hamiltonian: $H(x, u, p) = |u| - pax + pu$
- In (4) we have $b_i = B = 1$, and the optimal control is of the form

$$u^* = \begin{cases} 1, & p^* < -1 \\ \text{undefined but } \geq 0, & p^* = -1 \\ 0, & -1 < p^* < 1 \\ \text{undefined but } \leq 0, & p^* = 1 \\ -1, & p^* > 1 \end{cases}$$

- Can there be singular intervals, i.e., $[t_1, t_2]$ where $p^* = -1$ or $p^* = 1$?
- By the costate equations $\dot{p}^* = -H_x$ the costate is of the form $p^* = c_1 e^{at}$, where c_1 is an integration constant
- The exponent function is strictly monotone and $a > 0$, so there cannot be any singular intervals $\Rightarrow$ the control can be undefined only momentarily (pointwise), that does not matter.

Singular solutions

- In the previous, the singularity in the minimum-time and minimum control-effort problems was due to the fact that the Hamiltonian was linear in control
- Then we define the switching function that gives the multiplier for the control in the Hamiltonian
- If the switching function is zero *more than in a single point*, we call it singular interval and the corresponding solution is the singular solution
- Note! The possibility of singular solutions is not, of course, restricted to minimum-time or minimum control-effort problems!

- Let us examine the problem

$$\min \int_0^{t_f} \frac{1}{2}\left[x_1^2 + x_2^2\right] dt$$

when t_f and $x(t_f)$ are free, and

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned}$$

and the control is bounded: $|u| \leq 1$

Singular solutions
Example

- Hamiltonian: $H = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + p_1 x_2 + p_2 u$ is linear in u and the switching function is $s(t) = p_2$
- The control that minimizes Hamiltonian is

$$u^* = \begin{cases} -1, & \text{when } p_2 > 0 \\ 1, & \text{when } p_2 < 0 \end{cases}$$

- The costate equations are

$$\begin{aligned} \dot{p}_1^* &= -H_{x_1} = -x_1^* \\ \dot{p}_2^* &= -H_{x_2} = -x_2^* - p_1^* \end{aligned}$$

- On a singular interval we have $s(t) = p_2^* = 0$, and hence also $\dot{p}_2^* = 0$; from the second costate equation we get

$$\dot{p}_2^* = -x_2^* - p_1^* = 0$$

- We use the condition $H = 0$, and on the singular interval it holds

$$\begin{aligned} \frac{1}{2}x_1^{*2} + \frac{1}{2}x_2^{*2} + p_1^* x_2^* + p_2^* u^* &= 0 \\ \Rightarrow x_1^{*2} + x_2^{*2} - 2x_2^{*2} &= 0 \\ \Rightarrow x_1^{*2} - x_2^{*2} &= 0 \\ \Rightarrow (x_1^* - x_2^*)(x_1^* + x_2^*) &= 0 \end{aligned}$$

Singular solutions

Example

- The singularity condition holds on $t \in [t_1, t_2]$

$$x_1^* - x_2^* = 0 \quad \text{or} \quad x_1^* + x_2^* = 0$$

- (Solve the problem and draw the figures of x and u)

Summary

- Minimum-time problems
- Minimum control-effort problems
- Switching function and singular intervals