Exercise 3.1 (student presents)

Define the necessary conditions for the extremal, when the functionals and conditions are

a) $J(x) = \int_{t_0}^{t_f} [x_1^2 + x_1 x_2 + x_2^2 + x_3^2] dt$,
s.t. $\boxed{\dot{x}_1 = x_2}$
$\boxed{\dot{x}_2 = -x_1 + (1 - x_1^2)x_2 + x_3}$

b) $J(x) = \int_{t_0}^{t_f} (\lambda + x_3^2) dt$, $\lambda > 0$,
s.t. $\boxed{\dot{x}_1 = x_2}$
$\boxed{\dot{x}_2 = x_3}$

The problems have differential equation constraints (see Lecture 3 or Kirk pp. 169-173).

Solution

In problems with differential equation constraints we form the *augmented integrand function* and write its Euler equation.

$g_a(\vec{w}, \dot{\vec{w}}, \vec{p}, t) = g(\vec{w}, \dot{\vec{w}}, t) + \vec{p}^T f(\vec{w}, t)$

a) The augmented integrand function is:

$$g_a = x_1^2 + x_1 x_2 + x_2^2 + x_3^2 + \boxed{p_1[x_2 - \dot{x}_1]} + \boxed{p_2[-x_1 + (1 - x_1^2)x_2 + x_3 - \dot{x}_2]}$$

The state variable vector is $x = [x_1, x_2, x_3]$, so there are three Euler equations:

$$\begin{aligned} 0 &= \partial_{x_1} g_a - \frac{d}{dt}\partial_{\dot{x}_1} g_a = 2x_1^* + x_2^* + p_2^*[-1 - 2x_1^* x_2^*] + \dot{p}_1^* \\ 0 &= \partial_{x_2} g_a - \frac{d}{dt}\partial_{\dot{x}_2} g_a = x_1^* + 2x_2^* + p_1^* + p_2^*[1 - x_1^{*2}] + \dot{p}_2^* \\ 0 &= \partial_{x_3} g_a - \frac{d}{dt}\partial_{\dot{x}_3} g_a = 2x_3^* + p_2^*. \end{aligned}$$

$\frac{\partial g_a}{\partial \vec{w}} - \frac{d}{dt}\frac{\partial g_a}{\partial \dot{\vec{w}}} = \vec{0}$

These Euler equations together with the differential equation constraints form the necessary conditions for the variation to vanish on the extremal.

b) The augmented integrand is:

$g_a(\vec{w}, \dot{\vec{w}}, \vec{p}, t) = g(\vec{w}, \dot{\vec{w}}, t) + \vec{p}^T f(\vec{w}, t)$

$$g_a = \lambda + x_3^2 + \boxed{p_1[x_2 - \dot{x}_1]} + \boxed{p_2[x_3 - \dot{x}_2]}$$

Euler equations are

$$\begin{aligned} 0 &= \partial_{x_1} g_a - \frac{d}{dt}\partial_{\dot{x}_1} g_a = \dot{p}_1^* \\ 0 &= \partial_{x_2} g_a - \frac{d}{dt}\partial_{\dot{x}_2} g_a = p_1^* + \dot{p}_2^* \\ 0 &= \partial_{x_3} g_a - \frac{d}{dt}\partial_{\dot{x}_3} g_a = 2x_3^* + p_2^* \end{aligned}$$

$\frac{\partial g_a}{\partial \vec{w}} - \frac{d}{dt}\frac{\partial g_a}{\partial \dot{\vec{w}}} = \vec{0}$

These Eulers together with the differential equation constraints form the necessary conditions for the variation to vanish on the extremal.

Exercise 3.2 (solved in class)

In the free end state optimal control problem, show that the Hamiltonian is constant on the optimal trajectory, when the system and the integrand of the cost don't *explicitly* depend on time.

Hint: define $\frac{d}{dt}H$.

Exercise 3.3 (student presents)

Find the control that maximizes the criterion

$$J(u) = \int_1^5 (\boxed{ux - u^2 - x^2})dt$$

with the conditions $\boxed{\dot{x} = x + u}$, and $x(1) = 2$. The end state $x(5)$ is free.

$$H(x,u,p) = g(x,u,t) + p \cdot f(x,u,t)$$
$$\frac{\partial H}{\partial p} = \dot{x}$$
$$-\frac{\partial H}{\partial x} = \dot{p}$$
$$\frac{\partial H}{\partial u} = 0$$

Solution

Hamiltonian

$$H = \boxed{ux - u^2 - x^2} + \boxed{px + pu} \tag{1}$$

Stationarity condition

$$0 = H_u = x^* - \boxed{2u}^* + p^* \tag{2}$$

Costate equation

$$\dot{p}^* = -H_x = -\boxed{u^*} + 2x^* - p^* \tag{3}$$

State equation

$$H_p = \quad \dot{x}^* = x^* + u^* \tag{4}$$

Let's solve $u*$ from (4)

$$\boxed{u^*} = \dot{x}^* - x^*. \tag{5}$$

Insert it in (2) and ($\cancel{2}$ 3), we get two equations and two boundary conditions:

$$
\begin{aligned}
0 &= x^* - 2(\dot{x}^* - x^*) + p^* \\
\dot{p}^* &= -(\dot{x}^* - x^*) + 2x^* - p^* \\
x^*(1) &= 2 \\
p^*(5) &= 0
\end{aligned}
$$

$$p = 2\dot{x} - 3x$$
$$\dot{p} = 2\ddot{x} - 3\dot{x}$$

$$
\begin{aligned}
2\ddot{x} - 3\dot{x} &= -(\dot{x} - x) + 2x - 2\dot{x} + 3x \\
2\ddot{x} - 3\dot{x} &= -\dot{x} + x + 2x - 2\dot{x} + 3x \\
2\ddot{x} - 3\dot{x} &= -3\dot{x} + 6x \\
2\ddot{x} &= 6x \\
\ddot{x} &= 3x
\end{aligned}
$$

Transversality condition $\quad \frac{\partial h}{\partial x}(x, t_f) - p(t_f) = 0, \qquad |\delta x_f = arbitary, h = 0$

the last equation is derived from the free end state condition. From these equations, we get a differential equation for the optimal state x^*.

$$\ddot{x}^* = 3x^*$$

Solve DE

that allows us to solve $x^*(t)$. We can also express the optimal costate as follow:

$$p^* = 2\dot{x}^* - 3x^*$$

Using these two expressions and the end point conditions, we would be able to solve the problem manually (explain briefly why/how, with or without calculations).

(The rest is excluded from student presentation.) However, we use Mathematica to obtain the optimal solutions. They are

$$
\begin{aligned}
p^* &= -\frac{2\sqrt{3}e^{-\sqrt{3}t-\sqrt{3}}\left(e^{2\sqrt{3}t} - e^{10\sqrt{3}}\right)}{-2-\sqrt{3}-2e^{8\sqrt{3}}+\sqrt{3}e^{8\sqrt{3}}}, \\
x^* &= -\frac{2e^{-\sqrt{3}t-\sqrt{3}}\left(2e^{2\sqrt{3}t}+\sqrt{3}e^{2\sqrt{3}t}+2e^{10\sqrt{3}}-\sqrt{3}e^{10\sqrt{3}}\right)}{-2-\sqrt{3}-2e^{8\sqrt{3}}+\sqrt{3}e^{8\sqrt{3}}},
\end{aligned}
$$

and the control variable is solved from the state equation

$$u^* = \frac{2e^{-\sqrt{3}(t+1)}\left(-\left(1+\sqrt{3}\right)e^{2\sqrt{3}t}+\left(\sqrt{3}-1\right)e^{10\sqrt{3}}\right)}{-2-\sqrt{3}+\left(\sqrt{3}-2\right)e^{8\sqrt{3}}}.$$

Here is the Mathematica-code to solve the problem:

```
sol = DSolve[{
    x[t] - 2*(x'[t] - x[t]) + p[t] == 0,
    p'[t] == -(x'[t]+x[t])+2*x[t]-p[t],
    x[1] == 2,
    p[5] == 0}, {x[t], p[t]}, t];
xsol = sol[[1, 2, 2]];
psol = sol[[1, 1, 2]];
u = xsol - D[xsol, t];
```

Exercise 3.4 (teacher demo)

Let's consider the first order linear system

$$\dot{x}(t) = ax(t) + u(t). \tag{6}$$

We want to bring the system from an arbitrary initial state $x(0)$ to origo in time T and minimize

$$J = \int_0^T u^2(t)dt. \tag{7}$$

a) Find the optimal trajectory as a function of $x(0)$, a and T.

b) Find the optimal control as a function of $x(0)$, a and T.

c) Write the optimal control in the form $u^*(x,t) = F(t,T,a)x$.

d) Assume that $a > 0$ and explore how F and u^* behave, when $t \to T$. Explore also how F and u^* behave, when $T \to \infty$.

Exercise 3.5 (homework)

Let's consider the system $\dot{x} = x + u$, $x(0) = 5$. Calculate the optimal control and the optimal state trajectory, when the criterion

$$J(u) = \int_0^2 (2x - 3u - u^2)dt$$

is maximized, and $x(2)$ is free.