Dynamic Optimization - Homework assignment 7

Christian Segercrantz 481056

February 15, 2022

7.6

We know that on the 0th and 1st day the probabilities of the demand being 0, 1, 2, or 3 is $20\%, 40\%, 30\%$ and 10% and after that $10\%, 20\%, 30\%,$ and 40% respectively. We can sell a package at 5€á piece. We additionally know that we can manufacture 0-3 packages per day at the cost of 0€, 4€, 7€, or 10€. We also assume that unsold packages can transfer to the following day. The mean of the demand of the 0th and 1st day is 1.3 and for the remaining days 2. We assume that $x_0 = 0$ and that since the demand is 0 in the end $J_4(x_4) = 0$

We can model the packages per day as

$$x_{k+1} = x_k + u_k - w_k, \qquad x_{k+1} \geq 0 \tag{1}$$

where x is the amount of items per day, u is the amount of new packages fabricated and w is the stochastic demand. Here we assume that the packs can be stored from one day to the next.

The cost can be modeled as

$$g_k(x_k, u_k, w_k) = cu_k - 5\min(w_k, x_k) = \begin{cases} cu_k - 5x_k, & w_k \geq x_k \\ cu_k - 5w_k, & w_k < x_k \end{cases} \tag{2}$$

where c describes the cost per fabricated package.

The control policy $u_k = \mu_k^*(x_k)$ has the following constrains

$$u \in \begin{cases} \{0,1,2,3\}, & x_k = 0 \\ \{0,1,2\}, & x_k = 1 \\ \{0,1\}, & x_k = 2 \\ \{0\}, & x_k = 3 \end{cases} \tag{3}$$

We want to minimize the cost according to the formula

$$\sum_{n=1}^{3} g_k(x_k, u_k, w_k) \tag{4}$$

where c is the cost of production.

$$J_k(x_k) = \max_{u_k} \mathbb{E}_{w_k}\{g_k(x_k, u_k, w_k) + J_{k+1}(x_{k+1})\} \tag{5}$$

$$= \max_{u_k} \mathbb{E}_{w_k}\{g_k(x_k, u_k, w_k) + J_{k+1}(x_k + u_k - w_k)\} \tag{6}$$

a)

We start with k=3

$$J_3(x_3) = \begin{cases} cu_3 - 5x_3 + J_4(x_4), & w_3 \geq x_3 \\ cu_3 - 5w_3 + J_4(x_4), & w_3 < x_3 \end{cases} \tag{7}$$

$$\mathbb{E}_{w_k}\{J_3(x_3)\} = cu_3 - 5(0.1\min(0, \max(x_3 + u_3, 3)) + 0.2\min(1, \max(x_3 + u_3, 3)) + \tag{8}$$

$$0.3\min(2, \max(x_3 + u_3, 3)) + 0.4\min(3, \max(x_3 + u_3, 3))) \tag{9}$$

We can write a table to find the optimal control policy which can be seen in Table 1. An example of the calculation is e.g. for $x_3 = 2, u_3 = 1$: $4 - 5(0.1 \cdot 0 + 0.2 \cdot 1 + 0.3 \cdot 2 + 0.4 \cdot 3) = -6$

Table 1: $k = 3$, the minimi per row is marked with bold.

k=3	$u_3 = 0$	$u_3 = 1$	$u_3 = 2$	$u_3 = 3$	μ_3^*
$x_3 = 0$	0	-0.5	**-1**	0	2
$x_3 = 1$	**-4.5**	-4	-3	-	0
$x_3 = 2$	**-8**	-6	-	-	0
$x_3 = 3$	**-10**	-	-	-	0

For k=2 we then get

$$J_2(x_2) = \begin{cases} cu_2 - 5x_2 + J_3(x_3), & w_2 \geq x_2 \\ cu_2 - 5w_2 + J_3(x_3), & w_2 < x_2 \end{cases} \tag{10}$$

$$= \begin{cases} cu_2 - 5x_2 + J_2(x_2 + u_2 - w_2), & w_2 \geq x_2 \\ cu_2 - 5w_2 + J_2(x_2 + u_2 - w_2), & w_2 < x_2 \end{cases} \tag{11}$$

$$J_2(x_2) = \mathbb{E}_{w_k}\{g_2(x_2, u_2, w_2)\} + \mathbb{E}_{w_k}\{J_3(x_2 + u_2 - w_2)\}. \tag{12}$$

The first term is calculated the same way as earlier, but the second term is now the expected value of the optimal values from earlier. I.e. for $x_2 = 2, u_2 = 1$ we get that

$$\mathbb{E}_{w_k}\{J_3(x_2+u_2-w_2)\} = 0.1J_3(x_2+u_2-0)+0.2J_3(x_2+u_2-1)+0.2J_3(x_2+u_2-2)+0.3J_3(x_2+u_2-3) \tag{13}$$

which will result equal $-6 - 4.35 = -10.35$. We get the values for J_3 from Table 1. The values for $J_2(x_2)$ can be seen in Table 2 For $k = 1$ we only have to count the row for when $x_1 = 0$ sine we have no packages in the beginning. We will however count all, since we will need it in the b) part. The result can be seen from Table 3

Table 2: k=2, the minimi per row is marked with bold

k=2	$u_2 = 0$	$u_2 = 1$	$u_2 = 2$	$u_2 = 3$	μ_2^*
$x_2 = 0$	0	-1.85	-3.4	**-4.35**	3
$x_2 = 1$	-5.85	-6.4	**-7.35**	-	2
$x_2 = 2$	**-10.4**	-10.35	-	-	0
$x_2 = 3$	**-14.35**	-	-	-	0

Table 3: k=1, the minimi per row is marked with bold

k=1	$u_1 = 0$	$u_1 = 1$	$u_1 = 2$	$u_1 = 3$	μ_1^*
$x_1 = 0$	-4.35	-4.95	-5.76	**-6.17**	3
$x_1 = 1$	-8.95	-8.76	**-9.17**	-	2
$x_1 = 2$	**-12.76**	-12.17	-	-	0
$x_1 = 3$	**-16.17**	-	-	-	0

We can see that the optimal control policy is $\mu_1^*(x_1) = 3$ since $x_1 = 0$. Which gives us $J_1(0) = -6.17$€loss i.e. 6.17€profit. The optimal control policies and expected profits for specific days are seen in each table.

b)

We now have an additional day which leads to another iteration. Since the probability distribution is the same as for $k = 1$ we can do the exactly same calculation except using J_1 isntead of J_2. Since we only have one possibility for x_0 which is 0 we get the result seen in Table 4. The control policy is $\mu_0^*(0) = 3)$, which gives us $J_0(0) = -8.21$€loss i.e. 8.21€profit.

Table 4: k=0, the minimi per row is marked with bold

k=0	$u_0 = 0$	$u_0 = 1$	$u_0 = 2$	$u_0 = 3$	μ_0^*
$x_0 = 0$	-6.17	-6.77	-7.688	**-8.206**	3