Dynamic Optimization - Homework assignment 9

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9.4

We know the state equation

$$\dot{x} = -10x(t) + u(t) \tag{1}$$

and we want to minimize the performance measure

$$J = \frac{1}{2}x^2(0.04) + \int_0^{0.04} \left[\frac{1}{4}x^2(t) + \frac{1}{2}u^2(t)\right] dt \tag{2}$$

We will write the Hamiltonian as

$$H(x, u, p, t) = \frac{1}{4}x^2(t) + \frac{1}{2}u^2(t) + \nabla_x J^*(t, x)(-10x(t) + u(t)) \tag{3}$$

The Hamilton-Jacobi-Bellman equation is

$$0 = \min_{u \in U} \left[\nabla_t J^*(t, x) + H(x, u, p, t)\right]. \tag{4}$$

We take the first order derivative of the Hamiltonian w.r.t u and get

$$H_u = u + \nabla_x J^*, \tag{5}$$

which we then minimize by setting the left hand side to 0. This gives us

$$\begin{align} 0 =& u^* + \nabla_x J^* \tag{6}\\ u^* =& -\nabla_x J^* \tag{7}\\ \implies H =& \frac{1}{4}x^2 + \frac{1}{2}(-\nabla_x J^*)^2 + \nabla_x J^* \cdot (-10x - \nabla_x J^*) \tag{8}\\ =& \frac{1}{4}x^2 - \frac{1}{2}(\nabla_x J^*)^2 - 10x\nabla_x J^* \tag{9}\\ \implies \text{HJB: } 0 =& J_t^*(t, x) + \frac{1}{4}x^2 - \frac{1}{2}(J_x^*)^2 - 10xJ_x^*, \qquad |(-J_x^*)^2 = (J_x^*)^2 \tag{10} \end{align}$$

We will use the trial function and the following results

$$\begin{align} J(x(t), t) =& a(t)x^2(t) + b(t)x(t) + c(t) \tag{11}\\ J_x =& 2ax + b \tag{12}\\ J_t =& \dot{a}x^2 + \dot{b}x + \dot{c} \tag{13}\\ u =& -2ax - b \tag{14} \end{align}$$

By inserting this into the HJB equation we get

$$\begin{align} 0 =& \dot{a}x^2 + \dot{b}x + \dot{c} + \frac{1}{4}x^2 - \frac{1}{2}(2ax + b)^2 - 10x(2ax + b) \tag{15}\\ -(\dot{a}x^2 + \dot{b}x + \dot{c}) =& \frac{1}{4}x^2 - 2a^2x^2 - 2abx - \frac{1}{2}b^2 - 20ax^2 + 10bx \tag{16}\\ -(\dot{a}x^2 + \dot{b}x + \dot{c}) =& x^2\left(\frac{1}{4} - 2a^2 - 20a\right) + x(-2ab - 10b) - \frac{1}{2}b^2 \tag{17} \end{align}$$

And by comparing the terms on both side we get the three DEs

$$\begin{align} -\dot{a} =& \frac{1}{4} - 2a^2 - 20a \tag{18}\\ -\dot{b} =& -2ab - 10b \tag{19}\\ -\dot{c} =& -\frac{1}{2}b^2 \tag{20} \end{align}$$

Additionally we know from the final state that

$$a(0.04)x^2(0.04) + b(0.04)x(0.04) + c(0.04) = \frac{1}{2}x^2(0.04) \tag{21}$$

$$\iff \tag{22}$$

$$a(0.04) = \frac{1}{2} \tag{23}$$

$$b(0.04) = 0 \tag{24}$$

$$c(0.04) = 0 \tag{25}$$

We insert equations 18-20 into Mathematica (in the form seen below) to solve

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DSolve[{-a'[t] ==  1/4 - 2 a[t]^2 - 20 a[t],
        -b'[t] == -2 a[t]*b[t] -  10 b[t],
        -c'[t] == -1/2 b[t]^2},
       {a, b, c}, t]
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and get the following results.

$$a^* = -\frac{\sqrt{402}e^{\sqrt{402}t+4\sqrt{402}c_1} + 20e^{\sqrt{402}t+4\sqrt{402}c_1} - \sqrt{402} + 20}{4\left(1 + e^{\sqrt{402}t+4\sqrt{402}c_1}\right)} \tag{26}$$

$$b^* = c_2 \exp\left(-\sqrt{\frac{201}{2}}\left(-t + \sqrt{\frac{2}{201}}\log\left(1 + e^{\sqrt{402}(t+4c_1)}\right)\right)\right) \tag{27}$$

$$c^* = c_3 - \frac{e^{-4\sqrt{402}c_1}{c_2}^2}{2\sqrt{402}\left(1 + e^{\sqrt{402}(t+4c_1)}\right)} \tag{28}$$

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The optimal control law is then

$$u^* = -\nabla_x J^* = 2ax + b \tag{29}$$

$$u^* = \frac{x\left(\sqrt{402}e^{\sqrt{402}x+4\sqrt{402}c_1} + 20e^{\sqrt{402}x+4\sqrt{402}c_1} - \sqrt{402} + 20\right)}{2\left(1 + e^{\sqrt{402}x+4\sqrt{402}c_1}\right)} - \tag{30}$$

$$c_2 \exp\left(-\sqrt{\frac{201}{2}}\left(-x + \sqrt{\frac{2}{201}}\log\left(1 + e^{\sqrt{402}(x+4c_1)}\right)\right)\right) \tag{31}$$

Using Equations 23-25 we get the following for for the answers:

$$a^* = -\frac{-2.40131 + (40.0499 - 4.904702795618173\text{*^-15}i)\left(-e^{\sqrt{402}t}\right)}{4\left(48.0861 - (1. - 1.2246467991473532\text{*^-16}i)e^{\sqrt{402}t}\right)} \tag{32}$$

$$b^* = 0 \tag{33}$$

$$c^* = 0 \tag{34}$$

$$u^* = \frac{x\left(-2.40131 + (40.0499 - 4.904702795618173 \cdot 10^{-15}i)\left(-e^{\sqrt{402}x}\right)\right)}{2\left(48.0861 - (1. - 1.2246467991473532 \cdot 10^{-16}i)e^{\sqrt{402}x}\right)} \tag{35}$$

$$\approx \frac{\left(-40.0499e^{\sqrt{402}x} - 2.40131\right)x}{2\left(48.0861 - 1.e^{\sqrt{402}x}\right)} \tag{36}$$