Exercise 2.1 (student presents)

Solve the extremal of the functional

$$J = \int_0^1 (t\dot{x} + x^2 + \dot{x}^2)dt$$

where $x(0) = 1$ and $x(1)$ is free.

Solution

Let's use the Euler equation and the transversality condition.

Euler: $2x - 1 - 2\ddot{x} = 0 \Rightarrow x^*(t) = \dfrac{1}{2} + C_1 e^t + C_2 e^{-t}$

Initial point condition: $x^*(0) = \frac{1}{2} + C_1 + C_2 = 1 \;\Rightarrow\; C_2 = \frac{1}{2} - C_1$

Here the end time is fixed $t_f = 1$, but the end state can vary, meaning that $\delta x(t_f)$ is arbitrary. Thus, we use the transversality condition:

$$g_{\dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f) = 0 \;\Rightarrow\; 1 + 2\dot{x}^*(1) = 0 \;\Rightarrow\; \dot{x}^*(1) = -\frac{1}{2}$$

By inserting the solution candidate $x^*(t)$ from above we get:

$$\begin{aligned} -\frac{1}{2} = \dot{x}^*(1) &= C_1 e - \left(\frac{1}{2} - C_1\right)e^{-1} \\ &= C_1(e + e^{-1}) - \frac{1}{2}e^{-1} \end{aligned}$$

We get:

$$C_1 = \frac{1-e}{2(e^2+1)}, \; C_2 = \frac{e^2+e}{2(e^2+1)}$$

And the extremal is:

$$x^*(t) = \frac{1}{2} + \frac{1-e}{2(e^2+1)}e^t + \frac{e^2+e}{2(e^2+1)}e^{-t}$$

Exercise 2.2 (teacher demo)

a) Find the curve, which is an extremal for the functional

$$J(x) = \int_0^{tf} \sqrt{1+\dot{x}^2(t)}dt,$$

when $x(0) = 5$ and the end point has to be on the circle $x^2(t) + (t-5)^2 - 4 = 0$. Check your result geometrically. Hint: The end point condition is of the form $m(x(t_f), t_f) = 0$. Draw a picture to define the dependence of δx_f and δt_f.

b) What if $x(0) = 2$ and the end point is on the curve $\theta(t) = -4t + 5$?

Solution

a) Let's again write the Euler equation:

$$F = (1+\dot{x}^2)^{1/2}, \quad F_x = 0, \quad F_{\dot{x}} = \dot{x}(1+\dot{x}^2)^{-1/2}$$

then

$$\begin{aligned}
0 &= F_x - \frac{d}{dt}F_{\dot{x}} \\
&= \ddot{x}(1+\dot{x}^2)^{-1/2} + \dot{x}(1+\dot{x}^2)^{-3/2} \cdot (1/2) \cdot 2\dot{x} \cdot \ddot{x} \\
&= \ddot{x}\left[(1+\dot{x}^2)^{-1/2} - \dot{x}^2(1+\dot{x}^2)^{-3/2}\right] \\
&= \ddot{x}\left[\frac{1+\dot{x}^{-1/2}}{(1+\dot{x}^2)^{3/2}} - \frac{\dot{x}^2}{(1+\dot{x}^2)^{3/2}}\right] \\
&= \ddot{x}\left[\frac{1}{(1+\dot{x}^2)^{3/2}}\right] \\
\Rightarrow &\quad \ddot{x} = 0.
\end{aligned}$$

The solutions are thus lines

$$x^*(t) = c_1 t + c_2.$$

From the initial condition we get $c_2 = 5$.

Because the end point has to lie on a curve $\theta(t)$ (note that in part a) it is not convenient to calculate $\theta(t)$, because the end point is on a circle), δx_f and δt_f are dependent (see Fig. 1).

We see that

$$x^*(t_f) = \theta(t_f), \tag{1}$$

and also approximately it holds

$$\dot{\theta}(t_f) = \frac{\delta x_f}{\delta t_f}. \tag{2}$$

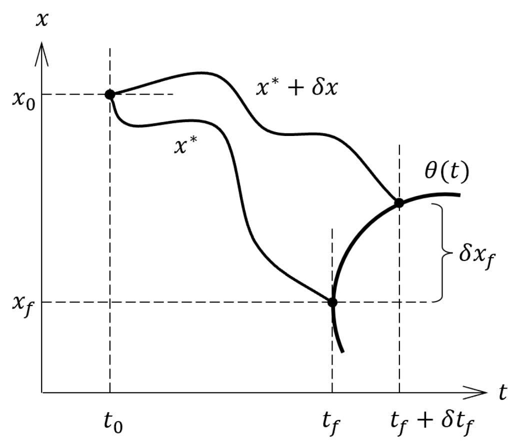

Figure 1: Sketch of a problem, where $x(t_f)$ and t_f are free, but related.

We could insert these geometric interpretations into the following transversality condition that we are using (however, we do it only in part b))

$$F_{\dot{x}}\delta x_f + (F - F_{\dot{x}}\dot{x})\delta t_f = 0,$$

where

$$F_{\dot{x}} = c_1(1 + c_1^2)^{-1/2}$$
$$F - F_{\dot{x}}\dot{x} = (1 + c_1^2)^{1/2} - c_1^2(1 + c_1^2)^{-1/2}.$$

Thus, the transversality condition is

$$\begin{aligned} 0 &= c_1(1 + c_1^2)^{-1/2}\delta x_f + \left[(1 + c_1^2)^{1/2} - c_1^2(1 + c_1^2)^{-1/2}\right]\delta t_f \\ &= c_1\delta x_f + (1 + c_1^2 - c_1^2)\delta t_f \\ &= c_1\delta x_f + \delta t_f. \end{aligned} \tag{3}$$

The end point has to be on the circle $x^2(t) + (t - 5)^2 - 4 = 0$. However, to make it something we can more easily work with, let's do a Taylor approximation of it at the end point $(x(t_f), t_f)$:

$$2x(t_f)\delta x_f + 2(t_f - 5)\delta t_f = 0.$$

We are studying the extremal $x^*(t)$, so we can write

$$2x^*(t_f)\delta x_f + 2(t_f - 5)\delta t_f = 0,$$

so

$$\delta x_f = \frac{5 - t_f}{x^*(t_f)}\delta t_f, \tag{4}$$

when we are at the goal curve. By combining Eqs. (3) and (4), we get

$$\delta t_f\left[\frac{5 - t_f}{x^*(t_f)}c_1 + 1\right] = 0.$$

This has to hold for arbitrary δt_f, thus

$$x^*(t_f) + c_1(5 - t_f) = 0 \quad \Leftrightarrow \quad x^*(t_f) = c_1(t_f - 5),$$

from which follows

$$x^*(t_f) = c_1 t_f + 5 = c_1 t_f - 5c_1 \quad \Rightarrow \quad c_1 = -1$$

so the optimal trajectory is

$$\boxed{x^*(t) = -t + 5.}$$

The solution is depicted in Figure 2. It's the shortest distance from the point $(0,5)$ to the goal set, i.e., the circle. Hence the optimal trajectory cuts the arc of the circle perpendicularly.

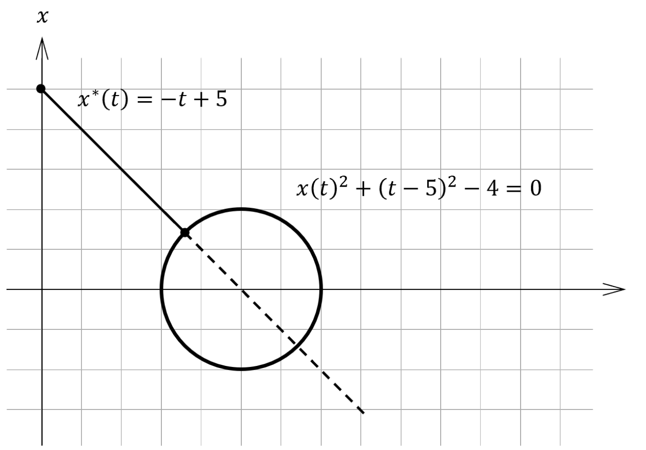

Figure 2: The optimal trajectory for Exercise 2.2 a) is the shortest distance from point $(0,5)$ to the arc of the circle.

b) Now the initial condition is $x(0) = 2$ and the end point has to satisfy the condition

$$x(t_f) = -4t_f + 5 = \theta(t_f).$$

the Euler equation is the same as in part a), so the solutions are lines

$$x^*(t) = c_1 t + 2.$$

The transversality condition is again the same as in part a) (see Kirk p. 151 or Lecture 2), except here we have not inserted the relation from Eq. (2), because we can directly calculate $\dot{\theta}(t)$,

$$\begin{aligned}
0 &= F_{\dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\big(d\dot{\theta}(t_f) - \dot{x}^*(t_f)\big) + F(x^*(t_f), \dot{x}^*(t_f), t_f) \\
&= \frac{\dot{x}^*(t_f)}{(1 + \dot{x}^{*2}(t_f))^{1/2}}\big(-4 - \dot{x}^*(t_f)\big) + (1 + \dot{x}^{*2}(t_f))^{1/2} \\
&= -4\dot{x}^*(t_f) - \dot{x}^{*2}(t_f) + 1 + \dot{x}^{*2}(t_f) \\
&= -4\dot{x}^*(t_f) + 1 \\
&\Rightarrow \; \dot{x}^*(t_f) = \frac{1}{4}.
\end{aligned}$$

Now we can easily conclude that $c_1 = 1/4$, out of which follows

$$\boxed{x^*(t) = \frac{1}{4}t + 2.}$$

Exercise 2.3 (solved in class)

Find the necessary conditions for the extremal $x^*(t)$, $t_0 \leq t \leq t_1$, which maximizes the functional

$$\int_{t_0}^{t_1} F(x(t), \dot{x}(t), t)dt$$

when t_0 and t_1 are only known. Notice, that $x(t_0)$ and $x(t_1)$ can also be chosen optimally.

Solution

We are using same kind of argument as in Kirk pp. 131-132 to derive the necessary conditions.

The variation of the functional is attained as usual

$$\begin{aligned} \delta J(x, \delta x) &= \int_{t_0}^{t_1} \left\{ \frac{\partial F}{\partial x}(x(t), \dot{x}(t), t)\delta x + \frac{\partial F}{\partial \dot{x}}(x(t), \dot{x}(t), t)\delta \dot{x} \right\} dt \\ &= \int_{t_0}^{t_1} \left\{ \frac{\partial F}{\partial x}(x(t), \dot{x}(t), t) - \frac{d}{dt}\left[\frac{\partial F}{\partial \dot{x}}(x(t), \dot{x}(t), t) \right] \right\} \delta x \, dt \\ &\quad + \frac{\partial F}{\partial \dot{x}}(x(t_1), \dot{x}(t_1), t_1)\delta x(t_1) - \frac{\partial F}{\partial \dot{x}}(x(t_0), \dot{x}(t_0), t_0)\delta x(t_0). \end{aligned} \tag{5}$$

Suppose that the curve x^* is an extremal for the free initial and end point problem. We denote the value of $x^*(t_0)$ and $x^*(t_1)$ as x_0 and x_1, respectively. Now, consider a fixed initial and end point problem with the same functional, the same initial and final times, and with specified end points $x(t_0) = x_0$ and $x(t_1) = x_1$. The curve x^* must be an extremal for this fixed initial and end point problem; therefore, x^* must be a solution of the Euler equation, and the integral term in Eq. (5) must be zero on an extremal.

$$\boxed{\frac{\partial F}{\partial x}(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt}\left[\frac{\partial F}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t) \right] = 0, \quad \forall\, t \in [t_0, t_1].}$$

The above reasoning can be used for the transversality conditions. The end points can be varied independently of each other by choosing separately $\delta x(t_0) = 0$ or $\delta x(t_1) = 0$, so both terms have to separately vanish, so we get two transversality conditions:

$$\boxed{\frac{\partial F}{\partial \dot{x}}(x^*(t_0), \dot{x}^*(t_0), t_0) = 0, \quad \frac{\partial F}{\partial \dot{x}}(x^*(t_1), \dot{x}^*(t_1), t_1) = 0.}$$

Exercise 2.4 (self-study)

Find such extremals for the problem

$$\min \int_0^4 [\dot{x}(t)-1]^2[\dot{x}(t)+1]^2 dt$$

$$\begin{cases} x(0)=0 \\ x(4)=2 \end{cases}$$

which have one corner.

Solution

Let the possible corner point be t_1. According to the Weierstrass-Erdmann boundary conditions (WE), the functions

$$F_{\dot{x}}, \quad F - F_{\dot{x}}\dot{x}$$

have to be continuous, but $\dot{x}$ is discontinuous in the corner point t_1 (see Kirk p. 157). Let

$$\begin{cases} a & := & \lim_{t\to t_1^+} \dot{x}(t); \\ b & := & \lim_{t\to t_1^-} \dot{x}(t); \\ a & \neq & b. \end{cases}$$

The aim is to find a, b so that W-E and Euler are both satisfied. From the W-E conditions we get two equations.

Now the functions

$$\begin{aligned} F_{\dot{x}} &= 2\big[\dot{x}(t)-1\big]\big[\dot{x}(t)+1\big]^2 + 2\big[\dot{x}(t)-1\big]^2\big[\dot{x}(t)+1\big] \\ &= 2\big[\dot{x}(t)-1\big]\big[\dot{x}(t)+1\big]\big[\dot{x}(t)-1+\dot{x}(t)+1\big] \\ &= 4\big[\dot{x}(t)-1\big]\big[\dot{x}(t)+1\big]\dot{x}(t) \end{aligned}$$

$$\begin{aligned} F - F_{\dot{x}}\dot{x} &= \big[\dot{x}(t)-1\big]^2\big[\dot{x}(t)+1\big]^2 - 4\big[\dot{x}(t)-1\big]\big[\dot{x}(t)+1\big]\dot{x}^2(t) \\ &= \big[\dot{x}(t)-1\big]\big[\dot{x}(t)+1\big]\big[\dot{x}^2(t)-1-4\dot{x}^2(t)\big] \\ &= -\big[\dot{x}(t)-1\big]\big[\dot{x}(t)+1\big]\big[3\dot{x}^2(t)+1\big] \end{aligned}$$

are continuous, whereas $\dot{x}^*(t)$ is discontinuous in the point t_1.

The aim is to find the possible values for the limits a and b. The first continuity condition gives

$$4(a-1)(a+1)a = 4(b-1)(b+1)b = p(a) = p(b),$$

where p is a polynomial. The other continuity condition gives

$$(a-1)(a+1)(3a^2+1) = (b-1)(b+1)(3b^2+1) = q(a) = q(b),$$

where q is another polynom. Because the conditions are satisfied simultaneously, it also holds

$$p(a)q(b) - p(b)q(a) = 0.$$

By writing this open and grouping some the terms we get

$$(a^2-1)(b^2-1)\big[a(3b^2+1) - b(3a^2+1)\big] = 0.$$

It can be shown, that two possible solution pairs (a,b), for which holds
$a \neq b$, are

$$(a,b) = (1,-1) \quad \vee \quad (a,b) = (-1,1).$$

So we have found the limits for the both sides of $\dot{x}^*(t_1)$ (two different cases).

Lets write the Euler equation:

$$\begin{aligned}
0 &= F_x - \frac{d}{dt}F_{\dot{x}} \\
&= 0 - \big[4\ddot{x}(\dot{x}+1)\dot{x} + 4(\dot{x}-1)\ddot{x}\dot{x} + 4(\dot{x}-1)(\dot{x}+1)\ddot{x}\big] \\
&= -4\ddot{x}\big[\dot{x}^2 + \dot{x} + \dot{x}^2 - \dot{x} + \dot{x}^2 - 1\big] \\
&= -4\ddot{x}[3\dot{x}^2 - 1] \\
&\Rightarrow \ \ddot{x}\big[3\dot{x}^2 - 1\big] = 0.
\end{aligned}$$

We can see that the solution for the differential equation is a line

$$x^*(t) = c_1 t + c_2.$$

There are two solutions, for which we can solve the integration constants from the end point conditions and the continuity condition.

<u>**First solution**</u>, $(a,b) = (1,-1)$

<u>First piece of first solution</u>, $t \in [0, t_1]$

First let's use the initial point condition

$$\begin{aligned}
x_1^*(0) &= 0 \\
c_2 &= 0 \\
&\Rightarrow \ x_1^*(t) = c_1 t.
\end{aligned}$$

Then apply $\dot{x}_1^*(t) = a = 1$:

$$\begin{aligned}
\dot{x}_1^*(t) &= 1 \\
c_1 &= 1 \\
&\Rightarrow \ x_1^*(t) = t.
\end{aligned}$$

<u>Second piece of the first solution</u>, $t \in [t_1, 4]$

Let's use the end point condition

$$
\begin{aligned}
x_1^*(4) &= 2 \\
4c_1 + c_2 &= 2 \\
c_2 &= 2 - 4c_1 \\
&\Rightarrow x_1^*(t) = c_1 t + 2 - 4c_1.
\end{aligned}
$$

Then apply $\dot{x}_1^*(t) = b = -1$:

$$
\begin{aligned}
\dot{x}_1^*(t) &= -1 \\
c_1 &= -1 \\
&\Rightarrow x_1^*(t) = -t + 6.
\end{aligned}
$$

The extremal $x_1^*(t)$ has to be continuous at the corner point t_1:

$$
\begin{aligned}
\lim_{t \to t_1^+} x_1^*(t) &= \lim_{t \to t_1^-} x_1^*(t) \\
t_1 &= -t_1 + 6 \\
t_1 &= 3
\end{aligned}
$$

The second solution, where $(a, b) = (-1, 1)$ is solved the same way, and we attain the two solutions:

$$
x_1^*(t) = \begin{cases} t, & 0 \leq t < 3 \\ -t + 6, & 3 \leq t \leq 4 \end{cases}
$$

$$
x_2^*(t) = \begin{cases} -t, & 0 \leq t < 1 \\ t - 2, & 1 \leq t \leq 4 \end{cases}.
$$

In the figure below the solutions are presented.

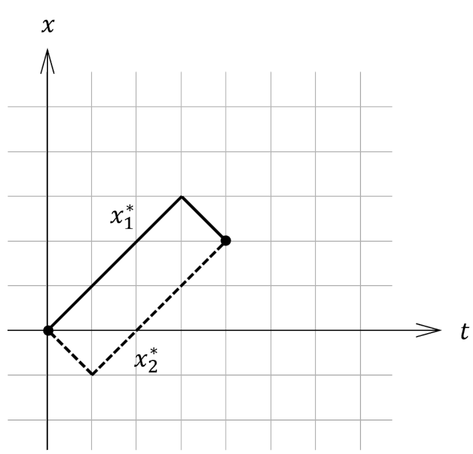

Figure 3: Optimal trajectories for Exercise 2.4, when one corner is allowed in the solution. The possible corner points are $t = 1$ and $t = 3$.

Exercise 2.5 (home assignment)

Find the extremal of the following functional with the help of Euler equation and the appropriate transversality conditions.

$$\int_0^s \left(\frac{1}{2}\dot{x}(t)^2 - x(t) + \frac{3}{2}\right) dt,$$

when $x(0) = 0$, and s and $x(s)$ are free.