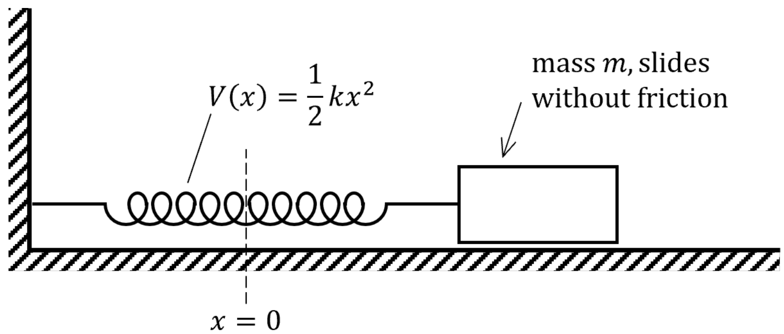
Notes and Comments on Lecture 9

Historical notes

- Much of the calculus of variations was developed by L. Euler around 1750. He also first mentions the Principle of Least Action, later, Hamilton's principle. This principle is the starting point of classical analytical mechanics (AM), developed mainly by J-L. Lagrange around 1780, and W.R. Hamilton around 1830. AM studies involved mechanical problems dealing with balls, springs, rigid rods, and pendulums. In AM, we can easily handle various non-Cartesian coordinate systems and the corresponding momenta using the Lagrangian approach.

- Example. Newtonian spring; potential energy $V = \frac{1}{2}kx^2$, exerted force $F = -V_x = -kx$, kinetic energy $T = \frac{1}{2}m\dot{x}^2$; Newton's equation of motion: $\frac{d}{dt}T = m\ddot{x} = -kx$



- Lagrangian spring. The principle of least action says: the motion of a system between $(t_1, x(t_1)$ and $(t_2, x(t_2))$ is such that the action integral

$$J = \int_{t_1}^{t_2} L(x, \dot{x}, t)\, dt$$

gets the least value, i.e., minimum or maximum, where $L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$ is the Lagrange function of the system.

Now, Euler-Lagrange gives $L_x - \frac{d}{dt}L_{\dot{x}} = 0 \Leftrightarrow m\ddot{=} - kx \Leftrightarrow$ Newton's equations of motion. $\square$

- The Hamiltonian equations of motion are obtained if we write:

$$\min \int_{t_1}^{t_2} L(x, u, t)\, dt$$

$$\text{s.t. } \dot{x} = u$$

The Hamiltonian is: $H = \frac{1}{2}mu^2 - \frac{1}{2}kx^2 + pu$, and the equations of motion (the necessary conditions) are:

$$\left\{ \begin{array}{l} \dot{x} = u, \\ \dot{p} = -H_x = +kx, \\ H_u = 0 \Leftrightarrow p = -mu. \end{array} \right\} \Rightarrow \text{ Newton!}$$

We can now identify $u = v =$ the speed of the system, $-p$ is the mechanical momentum, and $-H$ the total mechanical energy. □

- Hamilton and C. G. Jacobi further developed the theory, the Hamilton-Jacobi partial differential equation. They also applied it to various problems in celestial mechanics.
- Lagrangian and Hamiltonian formulations of classical mechanics (CM) also form the theoretical basis for quantum mechanics (QM) and quantum field theories. In particular, the Schrödinger equation in QM reduces to the Hamilton-Jacobi equation in CM in the limit where the Planck constant $h \to 0$.
- L.S. Pontryagin in the USSR and R. Bellman in the USA independently developed Optimal Control Theory (OCT) around 1956-1957. The theory then was first applied to, e.g., the computation and control of flight path trajectories of airplanes, satellites, rockets, and missiles.

Notes on numerical methods

Except for simple, although often important, special cases, the necessary conditions in OCT are solved numerically. There are two main approaches: either to discretize the original OCT problem and solve it by using non-linear optimization methods (applies also if the OCT problem has originally been written in discrete form); or to use the numerical methods to solve non-linear differential equations. With given boundary conditions (the values for the variables are defined at the same initial and final time points) we can use various two-point-boundary-value problem (TPBVP) solution methods. For discrete dynamic programming (DP) problems, various problem-specific backward and forward DP algorithms have been developed; as well as for the continuous time Hamilton-Jacobi-Bellman (HJB) equation, which is a nonlinear partial differential equation. □