MS-E2148 Dynamic optimization

Lecture 2

Contents

- Transversality conditions in calculus of variations
- Solutions with corners
- Material Kirk 4

Recap

- We derived Euler equation for the basic problem using the fundamental theorem of calculus of variations

- Let us find on a closed interval $[t_0, t_f]$ curve x^* that satisfies the boundary conditions $x(t_0) = x_0$ and $x(t_f) = x_f$ and is local extremum for the functional

$$J(x) = \int_{t_0}^{t_f} g(x, \dot{x}, t) dt \tag{1}$$

- In calculus of variations this $x = x^*$ is also called the extremal
- The necessary condition for the extremal x^* is the Euler equation

$$g_x(x^*, \dot{x}^*, t) - \frac{d}{dt} g_{\dot{x}}(x^*, \dot{x}^*, t) = 0 \qquad \forall t \in [t_0, t_f] \tag{2}$$

Transversality conditions

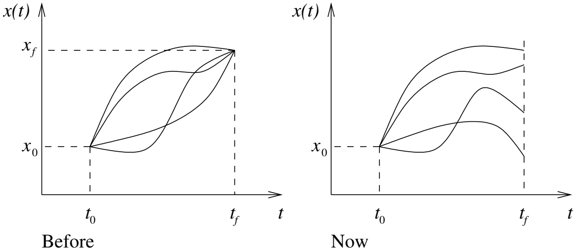
Different problems

- The basic problem is a two-point boundary value problem
- In practical problems the other end point can usually be free, when we get the following problems
  - Free final state
  - Free final time
  - Final state and time are independent
  - Final state and time are free but depend on each other
- Analysis is similar if we have free initial state/time
- The Euler equation is always one of the necessary conditions for the extremal independent of the boundary conditions!

Transversality conditions

Free final state

- Let us find the extremal for the functional (1) so that t_0 and t_f are fixed and $x(t_0) = x_0$, but $x(t_f) =$ is free



Transversality conditions

Free final state

- Variation of J is

$$\begin{aligned}\delta J(x,\delta x) &= g_{\dot{x}}(x,\dot{x},t)\delta x\Big|_{t_0}^{t_f} \\ &\quad + \int_{t_0}^{t_f}\left(g_x(x,\dot{x},t) - \frac{d}{dt}g_{\dot{x}}(x,\dot{x},t)\right)\delta x dt\end{aligned}$$

- Now, the first term is not zero, since $\delta x(t_f)$ is arbitrary
- Beside Euler, we now have the condition:

$$g_{\dot{x}}(x^*(t_f),\dot{x}^*(t_f),t_f) = 0 \tag{3}$$

- Let us find extremal for the functional (1) so that t_0 is fixed, $x(t_0) = x_0$, and $x(t_f) = x_f$, but t_f is free
- Now beside the integration constants, we need to find what is the final time t_f on the extremal, since it is independent variable (not like x_f that depended on fixed t_f)

Transversality conditions

Free final time

- All extremal candidates end up on the horizontal line where $x = x_f$; two curves in comparison end up on (x_f, t_f) and $(x_f, t_f + \delta t_f)$
- The increment of the functional is

$$
\begin{aligned}
\Delta J &= \int_{t_0}^{t_f+\delta t_f} g(x, \dot{x}, t)dt - \int_{t_0}^{t_f} g(x^*, \dot{x}^*, t) \\
&= \int_{t_0}^{t_f} \big(g(x^* + \delta x, \dot{x}^* + \delta \dot{x}, t) - g(x^*, \dot{x}^*, t)\big)dt \\
&\quad + \int_{t_f}^{t_f+\delta t_f} g(x, \dot{x}, t)dt
\end{aligned}
$$

- Let us expand the first term in the integral of ΔJ using Taylor series

$$
\begin{aligned}
\Delta J \;\; &= \;\; \int_{t_0}^{t_f} \left[g_x(x^*, \dot{x}^*, t)\delta x + g_{\dot{x}}(x^*, \dot{x}^*, t)\delta \dot{x} \right] dt \\
& \quad + o(\delta x, \delta \dot{x}) + \int_{t_f}^{t_f+\delta t_f} g(x, \dot{x}, t) dt
\end{aligned}
$$

- The second term in the integral can be written as

$$
\int_{t_f}^{t_f+\delta t_f} g(x, \dot{x}, t) dt = g(x(t_f), \dot{x}(t_f), t_f)\delta t_f + o(\delta t_f) \qquad (4)
$$

- By partial integration and combining the term (4) to the increment:

$$\begin{aligned}\Delta J &= g_{\dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\delta x(t_f) + g(x(t_f), \dot{x}(t_f), t_f)\delta t_f \\ &+ \int_{t_0}^{t_f} \left[g_x(x^*, \dot{x}^*, t) - \frac{d}{dt} g_{\dot{x}}(x^*, \dot{x}^*, t)\right]\delta x dt + o(\cdot)\end{aligned}$$

where we used that $\delta x(t_0) = 0$. Let us use the Taylor expansion on term $g(x(t_f), \dot{x}(t_f), t_f)$:

$$\begin{aligned}\Delta J &= g_{\dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\delta x(t_f) + g(x^*(t_f), \dot{x}^*(t_f), t_f)\delta t_f \\ &+ \int_{t_0}^{t_f} \left[g_x(x^*, \dot{x}^*, t) - \tfrac{d}{dt} g_{\dot{x}}(x^*, \dot{x}^*, t)\right]\delta x dt + o(\cdot)\end{aligned} \tag{5}$$

Transversality conditions

Free final time

- $\delta x(t_f)$ depends on δt_f, so we can approximate (see the figure in two slides with $\delta x_f = 0$):

$$\delta x(t_f) \approx -\dot{x}^*(t_f)\delta t_f \tag{6}$$

- The variation is solved from the increment (5) :

$$\begin{aligned} \delta J(x^*, \delta x) \quad = \quad & \Big(\big(- g_{\dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\big)\dot{x}^*(t_f) \\ & + g(x^*(t_f), \dot{x}^*(t_f), t_f)\Big)\delta t_f \\ & + \int_{t_0}^{t_f} \Big[g_x(x^*, \dot{x}^*, t) - \frac{d}{dt} g_{\dot{x}}(x^*, \dot{x}^*, t)\Big] \delta x dt \end{aligned}$$

Transversality conditions

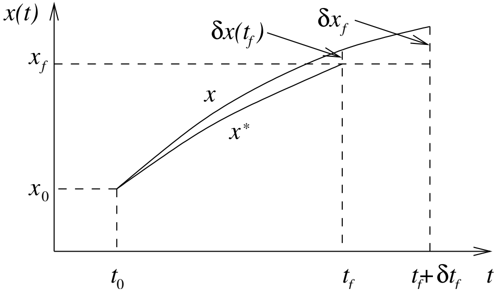
Free final time

- Beside Euler, we have the transversality condition:

$$g(x^*(t_f), \dot{x}^*(t_f), t_f) - g_{\dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\dot{x}^*(t_f) = 0 \qquad (7)$$

- **E.g.**: $J = \int_1^T (2x + \dot{x}^2/2)dt$, $x(1) = 4$, $x(T) = 4$, and $T > 1$
- Euler: $\ddot{x}^* = 2 \Rightarrow x^* = t^2 + c_1 t + c_2$
- Transversality condition: $4x^*(T) - \dot{x}^{*2}(T) = 0$
- Boundary conditions: $x^*(1) = 4 = 1 + c_1 + c_2$;
  $x^*(T) = 4 = T^2 + c_1 T + c_2$;
  $4x^*(T) - \dot{x}^{*2}(T) = 0 = 4c_2 - c_1^2$
- Solving constants c_1, c_2, T (3 constants, 3 equations) we get $x^* = t^2 - 6t + 9$ and $T = 5$

- Let us find the extremal for the functional (1) so that t_0 is fixed, $x(t_0) = x_0$, but $x(t_f)$ and t_f are free



We get: $\delta x_f \approx \delta x(t_f) + \dot{x}^*(t_f)\delta t_f$

- Let us use the increment in (5), and use the above approximation

$$
\begin{aligned}
\delta J(x^*, \delta x) \quad = \quad & g_{\dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\delta x_f \\
& + \Big[g(x^*(t_f), \dot{x}^*(t_f), t_f) \\
& - g_{\dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f) \cdot \dot{x}^*(t_f) \Big] \delta t_f \\
& + \int_{t_0}^{t_f} \left[g_x(x^*, \dot{x}^*, t) - \tfrac{d}{dt} g_{\dot{x}}(x^*, \dot{x}^*, t) \right] \delta x dt
\end{aligned}
\tag{8}
$$

Transversality conditions
Free final state and time

- Variation is zero if Euler holds and the multipliers of δx_f and δt_f are zero
- There are two cases:

1) t_f and $x(t_f)$ are independent, when we get the transversality conditions

$$\begin{aligned} g(x^*(t_f), \dot{x}^*(t_f), t_f) = 0 \\ g_{\dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f) = 0 \end{aligned} \tag{9}$$

2) t_f depends on $x(t_f)$: the final point is on some curve $x(t_f) = \theta(t_f)$, and $\delta x_f \approx \dot{\theta}(t_f)\delta t_f$ with transversality condition:

$$g_{\dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\left[\dot{\theta}(t_f) - \dot{x}^*(t_f)\right] + g(x^*(t_f), \dot{x}^*(t_f), t_f) = 0 \tag{10}$$

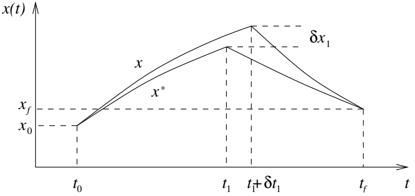
- **E.g.**: find the shortest route from the origin to the curve/line $\theta(t) = -5t + 15$
- We minimize $\int_0^{t_f} \sqrt{1 + \dot{x}^2} dt$ with $x(0) = 0$ and $x(t_f) = -5t_f + 15$
- Euler: $\ddot{x} = 0 \Rightarrow x^* = c_1 t + c_2$
- Boundary condition $x(0) = 0 \Rightarrow c_2 = 0$
- Transversality condition:

$$
\begin{aligned}
0 &= \frac{\dot{x}^*(t_f)}{\sqrt{1+\dot{x}^{*2}(t_f)}}(-5 - \dot{x}^*(t_f)) + \sqrt{1 + \dot{x}^{*2}(t_f)} \\
&= -5\dot{x}^*(t_f) + 1
\end{aligned}
$$

- We get from the transversality condition and the solution candidate $c_1 = 1/5$
- Final condition gives $t_f = 5 \cdot (-5t_f + 15) \Rightarrow t_f = 75/26$

Calculus of variations: solutions with corners

- In the basic problem, the extremal candidates are continuous and continuously differentiable (*smooth*); this is a strong restriction
- Now, we allow extremals that have piecewise continuous first time derivatives, e.g., the $\dot{x}$ is continuous, except for a finite number of points on the interval $[t_0, t_f]$
- Where $\dot{x}$ is discontinuous, we say that x has a *corner*



- Let us assume that g has continuous first and second-order partial differentials with respect to all of its arguments on functional (1) and that $t_0, t_f, x(t_0), x(t_f)$ are fixed
- We assume that $\dot{x}$ has a point of discontinuity (corner) in some point $t_1 \in (t_0, t_f)$ which is not known in advance
- The functional can be represented as

$$\begin{aligned} J(x) &= \int_{t_0}^{t_1} g(x, \dot{x}, t)dt + \int_{t_1}^{t_f} g(x, \dot{x}, t)dt \\ &\equiv J_1(x) + J_2(x) \end{aligned} \tag{11}$$

- We know that if x^* is extremal for J, then $x^*(t)|_{t\in[t_0,t_1]}$ is extremal for J_1 and $x^*(t)|_{t\in[t_1,t_f]}$ is extremal for J_2

- Let us denote t_1^- and t_1^+ as the left and right-hand side of the point of discontinuity
- Since the corner point coordinates are free, the variation $\delta J(x^*, \delta x) =$

$$
\begin{aligned}
& g_{\dot{x}}(x^*(t_1^-), \dot{x}^*(t_1^-), t_1^-)\delta x_1 \\
& + \Big[g(x^*(t_1^-), \dot{x}^*(t_1^-), t_1^-) - g_{\dot{x}}(x^*(t_1^-), \dot{x}^*(t_1^-), t_1^-)\dot{x}^*(t_1^-) \Big] \delta t_1 \\
& + \int_{t_0}^{t_1} \Big[g_x(x^*, \dot{x}^*, t) - \tfrac{d}{dt} g_{\dot{x}}(x^*, \dot{x}^*, t) \Big] \delta x dt \\
& \\
& - g_{\dot{x}}(x^*(t_1^+), \dot{x}^*(t_1^+), t_1^+)\delta x_1 \\
& - \Big[g(x^*(t_1^+), \dot{x}^*(t_1^+), t_1^+) - g_{\dot{x}}(x^*(t_1^+), \dot{x}^*(t_1^+), t_1^+)\dot{x}^*(t_1^+) \Big] \delta t_1 \\
& + \int_{t_1}^{t_f} \Big[g_x(x^*, \dot{x}^*, t) - \tfrac{d}{dt} g_{\dot{x}}(x^*, \dot{x}^*, t) \Big] \delta x dt
\end{aligned}
\tag{12}
$$

- Since δx_1 and δt_1 are arbitrary, the necessary conditions beside Euler for vanishing variation are

$$g_{\dot{x}}(x^*(t_1^-), \dot{x}^*(t_1^-), t_1^-) = g_{\dot{x}}(x^*(t_1^+), \dot{x}^*(t_1^+), t_1^+)$$

  and

$$g(x^*(t_1^-), \dot{x}^*(t_1^-), t_1^-) - \left[g_{\dot{x}}(x^*(t_1^-), \dot{x}^*(t_1^-), t_1^-)\right]\dot{x}^*(t_1^-)$$
$$= g(x^*(t_1^+), \dot{x}^*(t_1^+), t_1^+) - \left[g_{\dot{x}}(x^*(t_1^+), \dot{x}^*(t_1^+), t_1^+)\right]\dot{x}^*(t_1^+)$$

- $\Rightarrow$ functions $g_{\dot{x}}$ and $g - g_{\dot{x}}\dot{x}$ must be continuous over the corner

Calculus of variations: solutions with corners

Example 1

- $J(x) = \int_0^{\pi/2} [\dot{x}^2 - x^2] dt$, $x(0) = 0$ and $x(\pi/2) = 1$
- Euler: $\ddot{x} + x = 0 \Rightarrow x^* = c_3 \cos t + c_4 \sin t$
- W-E corner conditions:

$$2\dot{x}^*(t_1^-) = 2\dot{x}^*(t_1^+)$$

and

$$\begin{aligned} &\dot{x}^{*2}(t_1^-) - x^{*2}(t_1^-) - 2\dot{x}^*(t_1^-)\dot{x}^*(t_1^-) \\ = &\dot{x}^{*2}(t_1^+) - x^{*2}(t_1^+) - 2\dot{x}^*(t_1^+)\dot{x}^*(t_1^+) \end{aligned}$$

which require that $\dot{x}$ is continuous over t_1; i.e., x^* *cannot have a corner*

Calculus of variations: solutions with corners

Example 2

- $J(x) = \int_0^2 [\dot{x}^2 - 1]^2 dt$, $x(0) = 0$, $x(2) = 0$
- Euler: $\dot{x}^3 - \dot{x} = c_1 \Rightarrow x^* = c_2 t + c_3$
- Boundary conditions: $x^* = 0$, where $J(x^*) = 2$
- Corner conditions:

$$\begin{aligned} x^*(t_1^-)[x^{*2}(t_1^-) - 1] &= x^*(t_1^+)[x^{*2}(t_1^+) - 1] \\ \left[-x^{*2}(t_1^-) + 1\right]\left[x^{*2}(t_1^-) + 1\right] &= \left[-x^{*2}(t_1^+) + 1\right]\left[x^{*2}(t_1^+) + 1\right] \end{aligned}$$

- The first equation is satisfied when $\dot{x}^*(t_1^-) = -1, 0, 1$ and $\dot{x}^*(t_1^+) = -1, 0, 1$
- The second is satisfied when $\dot{x}^*(t_1^-) = -1, 1$ and $\dot{x}^*(t_1^+) = -1, 1$

- In the solution, we either have $\dot{x}^*(t_1^-) = 1$ and $\dot{x}^*(t_1^+) = -1$ or $\dot{x}^*(t_1^-) = -1$ and $\dot{x}^*(t_1^+) = 1$
- Putting them together

$$\begin{cases} x^* = t, & t \leq t_1 = 1 \\ x^* = -t + 2, & t \geq t_1 = 1 \end{cases}$$

- This gives solution with $J(x^*) = 0$ which is surely the global minimum (check the integrand)

Summary

- On the extremal, Euler equation must hold
- ... and it is complemented with transversality conditions and Weierstrass-Erdmann corner conditions
- Transversality and corner conditions can be generalized to vector-valued functions