Exercise 5.2 (student presents)

We want to move the following system

$$\dot{x} = 2x + u$$

from an arbitrary initial state x_0 to the origin in minimal time, while the control is constrained $|u| \leq 1$. There are some initial states that can not be moved to the origin with any allowed control u. Find those initial states.

$H = |u| + p(2x+u)$

Solution

The solution for the state equation is

$$x = \phi(t)[x_0 + \int_0^t \phi(-\tau)u(\tau)d\tau] \quad | \text{ DE-solution}$$

where $\phi(t) = e^{2t}$. Lets assume we have a control for which $x(T) = 0$; then

$$0 = x_0 + \int_0^T \phi(-\tau)u(\tau)d\tau, \quad \Leftrightarrow x_0 = \int \ldots d\tau$$

which implies that

$$|x_0| = \Big|\int_0^T \phi(-\tau)u(\tau)d\tau\Big|$$

On the other hand

$$\Big|\int_0^T \phi(-\tau)u(\tau)d\tau\Big| \leq \int_0^T |\phi(-\tau)|\overbrace{|u(\tau)|}^{<1}d\tau \quad | \text{ Cauchy-Schwartz}$$

and because $|u| \leq 1$, we get

$$\begin{aligned} |x_0| &\leq \int_0^T |\phi(-\tau)|d\tau = \int_0^T e^{-2t}dt = \Big/_0^T -\tfrac{1}{2}e^{-2t} = -\tfrac{1}{2}(e^{-2T} - e^0) = \tfrac{1}{2}(1 - e^{-2T}) \\ &= \frac{1}{2}[1 - e^{-2T}] \end{aligned}$$

i.e.

$$2|x_0| - 1 \leq -e^{-2T} \Rightarrow e^{-2T} \leq 1 - 2|x_0|$$

Because the exponential function is positive for all T, this last condition can only be satisfied if $|x_0| < 1/2$, which is the answer we were looking for.

$e^{-x} > 0 \Rightarrow 0 < 1 - 2|x_0| \Rightarrow \frac{1}{2} > |x_0|$