CS-E4830: Kernel methods of machine learning - Assignment 1

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Task 1

$$K_1(x, y) = (\langle x, y\rangle + c)^m \tag{1}$$

We know that the inner product $\langle x, y\rangle$ is a kernel. By the binomial theorem we get

$$\sum_{k=0}^{m} \binom{m}{k} \langle x, y\rangle^{m-k} c^k \tag{2}$$

Since we know that

1. A conic combination of kernels is a kernel
2. $c \geq 0$ and $\binom{m}{k} \geq 0$
3. The product of a kernel is a kernel

We can see that the polynomial kernel is a conic combination of the inner product to a power and thus a kernel:

$$\begin{aligned}
&\sum_{k=0}^{m} \binom{m}{k} \langle x, y\rangle^{m-k} c^k &(3)\\
=&\sum_{k=0}^{m} C \langle x, y\rangle^{m-k}, \qquad C = \binom{m}{k} c^k &(4)
\end{aligned}$$

Task 2

$$\begin{aligned}
h(x) =& \mathrm{sgn}\left(\sum_{i=1}^{n} \alpha_i k(x, x_i) + b\right), \quad k(x, x_i) = \langle \phi(x), \phi(x_i)\rangle &(5)\\
=& \mathrm{sgn}\left(\sum_{i=1}^{n} \alpha_i \langle \phi(x), \phi(x_i)\rangle + b\right) &(6)\\
=& \mathrm{sgn}\left(\sum_{i=1}^{n} \langle \alpha_i \phi(x) + b, \alpha_i \phi(x_i) + b\rangle\right) &(7)
\end{aligned}$$

We note that

$$\begin{aligned}
||\phi(x) - c_-||^2 &= \langle \phi(x) - c_-, \phi(x) - c_-\rangle &(8)\\
||\phi(x) - c_+||^2 &= \langle \phi(x) - c_+, \phi(x) - c_+\rangle. &(9)
\end{aligned}$$

$$\begin{aligned}
||\phi(x) - c_-||^2 =& \langle \phi(x) - c_-, \phi(x) - c_-\rangle &(10)\\
=& \langle \phi(x), \phi(x)\rangle - c_- &(11)
\end{aligned}$$

Task 3

$$K_2(x, y) = \cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y) \tag{12}$$

The feature map would thus be $\phi(x) = \begin{bmatrix} \cos(x) \\ i\sin(x) \end{bmatrix}$