CS:E4830 Kernel Methods in Machine Learning

Lecture 3 : Representer Theorem and Polynomial Kernel Example

16th March, 2022

Some Announcements

- Lecture slides of this (3rd) lecture - uploaded to Mycourses
- Assignment 1 has been released, deadline 21st March
- Tomorrow (Thursday, 17th March) - tutorial session for assignment 1

Outline For Today

1. Recap and Motivation
2. Representer Theorem
   - Proof
   - Kernel Least Square Regression
3. Explicit Feature Maps
   - Polynomial kernel with example
   - LibShorttext

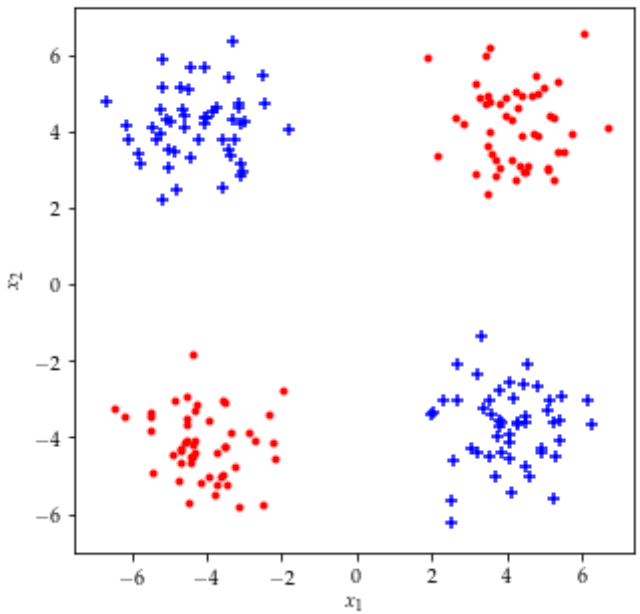
Recap and Motivation

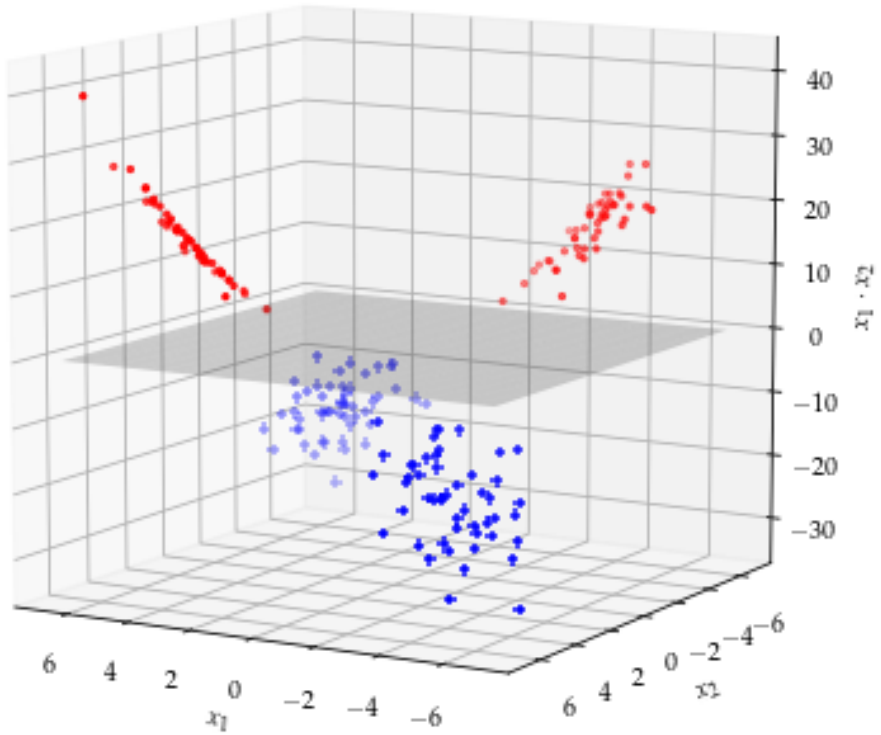
So far...

- Lecture 1
  - Linear and Non-linear classification
  - Kernel functions $k(x, x') = \langle \phi(x), \phi(x') \rangle_{\mathcal{H}}$
  - Properties of kernel functions
- Lecture 2
  - Positive definite functions
  - Equivalence between kernel and positive definite functions
  - Moore-Aronszjan theorem and Reproducing Kernel Hilbert Space
- This lecture - Representer Theorem

Non-linear Classification Example

- Dataset in 2-D (left), which is not linearly separable
- can be separated by a plane in 3-D (third feature is the product $x_1 x_2$)





Non-linear classification

Prediction function can involve non-linear combination of features

- For the classification function f_2 below, which is linear in weights and **non-linear in input features**

$$f_2(x) = w^{(1)}x^{(1)} + w^{(2)}x^{(2)} + w^{(3)}x^{(1)}x^{(2)} + w^{(4)}x^{(2)}x^{(1)}$$

- The decision function $f_2(x)$ captures **non-linear combination** of the input components as well such as $x^{(1)}x^{(2)}, x^{(2)}x^{(1)}$
- Non-linear feature map $\phi_2 : \mathbb{R}^2 \mapsto \mathbb{R}^4$, and is given by $\phi_2(x) = (x^{(1)}, x^{(2)}, x^{(1)}x^{(2)}, x^{(2)}x^{(1)})^T$
  - $\phi_2(x) \in \mathcal{H}$, which is referred to as the feature space
- Note that the decision function $f_2(x)$ **is still linear in the weight vector co-efficients** $w^{(j)}$'s and is parameterized by $(w^{(1)}, w^{(2)}, w^{(3)}, w^{(4)})^T$

Implcit vs Explicit Computation

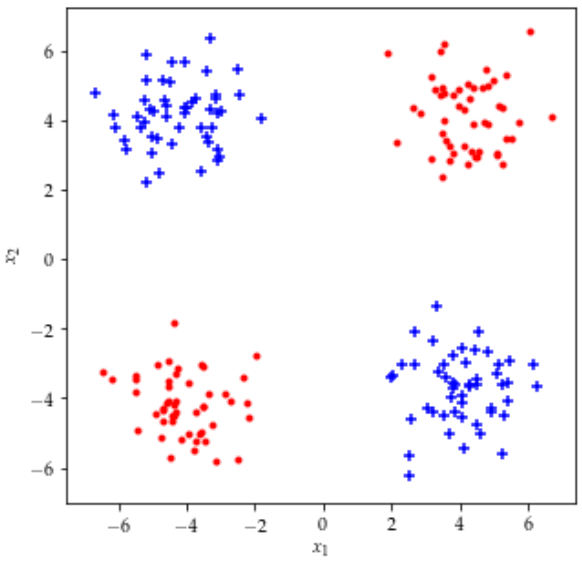
- Example - Polynomial kernel of degree 2. Assuming inputs $x, z \in \mathbb{R}^d$, i.e. $x = (x^{(1)}, x^{(2)}, \ldots, x^{(d)})$

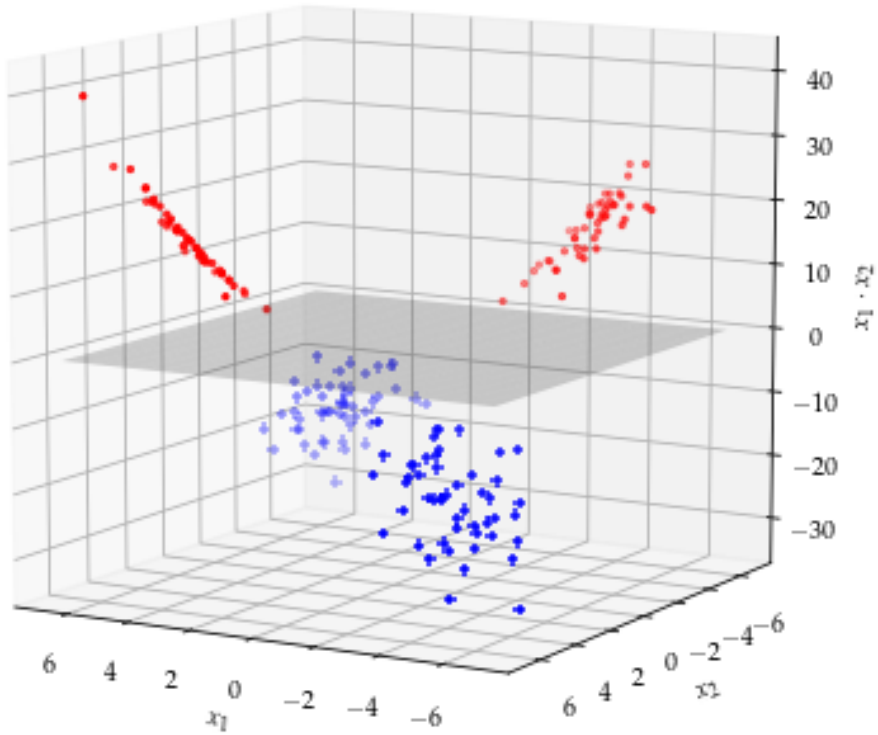
$$k_{poly}(x, z) = (\langle x, z \rangle)^2 = \langle \phi(x), \phi(z) \rangle$$

- What is the dimensionality of the input space and feature space, $\phi(.)$ (keeping order into account. i.e., $x^{(1)}x^{(2)}$ is different from $x^{(2)}x^{(1)}$)?
  - $\phi : \mathbb{R}^d \mapsto \mathbb{R}^{d^2}$
  - $\phi(x) = \{x^{(1)}x^{(1)}, x^{(1)}x^{(2)}, \ldots x^{(1)}x^{(d)}, \ldots, x^{(d)}x^{(1)}, x^{(d)}x^{(2)}, \ldots, x^{(d)}x^{(d)}\}$
- Computational complexity of
  - $\langle \phi(x), \phi(z) \rangle$? $O(d^2)$
  - $k_{poly}(x, z)$? $O(d)$
- What if we consider another kernel with a higher degree such as $k_{poly}(x, z) = (\langle x, z \rangle)^{10}$

Decision Boundary in Non-linear Case

How to represent the decision boundary when using Gaussian kernel - feature space is infinite dimensional





Representer Theorem

Empirical Loss

- For a loss function L, and dataset D of finite size N, Empirical loss $\hat{\mathcal{R}}_{L,D}$ of a classifier f is defined as

$$\hat{\mathcal{R}}_{L,D} := \frac{1}{N} \sum_{i=1}^{N} L(y_i, f(x_i))$$

- The classifier obtained by minimizing the empirical loss is given by

$$\hat{f}_D := \arg \min_{f:\mathcal{X} \mapsto \mathbb{R}} \hat{\mathcal{R}}_{L,D}$$

- Minimizing Empirical risk over an arbitrary function class ($f : \mathcal{X} \mapsto \mathbb{R}$) can lead to overfitting (as seen earlier in the lecture)
- Empirical risk minimization (ERM) is, therefore, performed over a smaller function class of function, which are typically smooth functions. This is given by

$$\hat{f}_{\mathcal{H}} := \arg \min_{f \in \mathcal{H}} \hat{\mathcal{R}}_{L,D}$$

- Pick a function class with bounded RKHS norm i.e. $\{f : ||f||_{\mathcal{H}} \leq \lambda\}$

Constrained versus Penalized Problem

- Contrained formulation (from previous slide)

$$\hat{f}_{\mathcal{H}} := \arg \min_{\{f : ||f||_{\mathcal{H}} \leq \lambda\}} \frac{1}{N} \sum_{i=1}^{N} L(y_i, f(x_i))$$

- Lagrangian formulation

$$\hat{f}_{\mathcal{H}} := \arg \min_{f} \frac{1}{N} \sum_{i=1}^{N} L(y_i, f(x_i)) + \lambda ||f||_{\mathcal{H}}^2$$

  for $\lambda > 0$

- The above is optimization problem over potentially an infinite dimensional space, since $\mathcal{H}$ can be an infinite dimensional as in the case of Gaussian kernel.

Representer Theorem

- For the following objective function

$$\hat{f}_{\mathcal{H}} := \arg\min_{f} \frac{1}{N} \sum_{i=1}^{N} L(y_i, f(x_i)) + \lambda\theta(||f||_{\mathcal{H}}) \tag{1}$$

  where $\theta : [0, \infty) \mapsto \mathbb{R}$ is non-decreasing function

- Even though the above problem is potentially an infinite dimensional optimization problem, **Representer Theorem** states its solution can be expressed in the following form

$$\hat{f}_{\mathcal{H}}(.) = \sum_{i=1}^{N} \alpha_i k(., x_i)$$

  where $\alpha_i \in \mathbb{R}$

- Infinite to finite dimensional problem - instead of finding potentially infinite coefficients of $\hat{f}$, we need to find N coefficients of the above linear combination α_i

Representer Theorem - Proof

- Decompose the RKHS $\mathcal{H}$ into the following :

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}^{\perp}$$

  - where $\mathcal{H}_0 = \{f \in \mathcal{H} : f(x) = \sum_{i=1}^{N} \alpha_i k(x, x_i), (\alpha_i)_{i=1}^{N}\}$. It can also be seen as a *finite dimensional sub-space* spanned by the kernel evaluations at the points x_i, i.e. $\mathcal{H}_0 = \text{span}\{k(., x_1), k(., x_2), \ldots, k(., x_N)\}$
  - $\mathcal{H}^{\perp}$ is the component of $\mathcal{H}$, which is orthogonal to $\mathcal{H}_0$.
- Therefore, the function $f \in \mathcal{H}$ is decomposed as

$$f = f_0 + f^{\perp}$$

- For the loss term in the objective function (in previous slide)

$$f(x_i) = \langle f, k(., x_i) \rangle_{\mathcal{H}} = \langle f_0, k(., x_i) \rangle_{\mathcal{H}_0} + \langle f^{\perp}, k(., x_i) \rangle_{\mathcal{H}^{\perp}}$$

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The second term above involving $f^{\perp}$ in the loss function ($L(.,.)$ on previous slide) computation is 0.

Representer Theorem - Proof

- For the regularization term use the Pythagoras theorem, i.e.

$$||f||^2_{\mathcal{H}} = ||f^{\perp}||^2_{\mathcal{H}^{\perp}} + ||f_0||^2_{\mathcal{H}_0}$$

  This implies that $||f_0||_{\mathcal{H}_0} \leq ||f||_{\mathcal{H}}$. Since $\theta(.)$ is a non-decreasing function $\theta(||f_0||_{\mathcal{H}_0}) \leq \theta(||f||_{\mathcal{H}})$
- Therefore, the optimal solution has no component in $\mathcal{H}^{\perp}$, and has the form

$$f_{\mathcal{H}}(.) = \sum_{i=1}^{N} \alpha_i k(., x_i)$$

Representer Theorem - Wikipedia

Formal statement [edit]

The following Representer Theorem and its proof are due to Schölkopf, Herbrich, and Smola:[citatic

Theorem: Consider a positive-definite real-valued kernel $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ on a non-empty set $\mathcal{X}$

- a training sample $(x_1, y_1), \ldots, (x_n, y_n) \in \mathcal{X} \times \mathbb{R}$,
- a strictly increasing real-valued function g: $[0, \infty) \to \mathbb{R}$, and
- an arbitrary error function E: $(\mathcal{X} \times \mathbb{R}^2)^n \to \mathbb{R} \cup \{\infty\}$,

which together define the following regularized empirical risk functional on H_k:

$$f \mapsto E\left((x_1, y_1, f(x_1)), \ldots, (x_n, y_n, f(x_n))\right) + g\left(\|f\|\right).$$

Then, any minimizer of the empirical risk

$$f^* = \operatorname{argmin}_{f \in H_k} \{E\left((x_1, y_1, f(x_1)), \ldots, (x_n, y_n, f(x_n))\right) + g\left(\|f\|\right)\}, \quad (*)$$

admits a representation of the form:

$$f^*(\cdot) = \sum_{i=1}^{n} \alpha_i k(\cdot, x_i),$$

where $\alpha_i \in \mathbb{R}$ for all $1 \leq i \leq n$.

Figure

Slightly out of context

French President Emmanuel Macron and Bernhard Schölkopf at the Élysée Palace in Paris.

Bernhard Schölkopf meets French President Emmanuel Macron

30 October 2019

Empirical Inference

The Machine Learning scientist is in Paris to attend the Global Forum on AI for Humanity which brings together many of the world's leading AI experts who together work on promoting and protecting a human-centric and ethical approach to AI, grounded in human rights.

Practical Implications of Representer Theorem

- It allows us to look for the solutions of the following form :

$$f_{\mathcal{H}}(.) = \sum_{i=1}^{N} \alpha_i k(., x_i)$$

  which is a finite dimensional optimization problem as against the original infinite dimensional problem.

- For $j = 1 \ldots N$

$$f_{\mathcal{H}}(x_j) = \sum_{i=1}^{N} \alpha_i k(x_i, x_j) = [K\boldsymbol{\alpha}]_j$$

  which is the j-th element of the matrix-vector product $K\boldsymbol{\alpha}$

- Also,

$$||f||^2_{\mathcal{H}} = \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j k(x_i, x_j) = \boldsymbol{\alpha}^T K \boldsymbol{\alpha}$$

Least Square Regression

- Let f be the prediction function, then squared error is given by

$$\ell(f(x), y) = (y - f(x))^2$$

- Let $\mathcal{F}$ be a function class (not necessarily an RKHS) from which we are choosing our function
- Least Square regresion find a function with smallest sqaured error

$$\hat{f} \in \arg\min_{f \in \mathcal{F}} \frac{1}{N} \sum_{i=1}^{N} (y_i - f(x_i))^2$$

- Possible problems :
  - Can be unstable in high dimensions
  - Overfit if the function space $\mathcal{F}$ is too large

Kernel Ridge Regression

- Finding f in an RKHS with a kernel $k(x, x')$

$$\hat{f} \in \arg\min_{f \in \mathcal{H}} \frac{1}{N} \sum_{i=1}^{N} (y_i - f(x_i))^2 + \lambda ||f||_{\mathcal{H}}$$

- The above formulation has two advantages :
  - Prevents overfitting
  - Representer theorem enables an efficient solution of the form

$$\hat{f}(.) = \sum_{i=1}^{N} \alpha_i k(., x_i)$$

Solving Kernel Ridge Regression

- Lets denote by
  - The label vector $\mathbf{y} \in \mathbb{R}^N$ denoting the true values for the inputs
  - The kernel matrix K, where $K_{i,j} = K(x_i, x_j)$
  - $\boldsymbol{\alpha} \in \mathbb{R}^N$, the co-efficients we want to find
- For the input instance, the prediction by the desired function can be written as follows :

$$(\hat{f}(x_1), \ldots, \hat{f}(x_N))^T = K\boldsymbol{\alpha}$$

- We also know that

$$||f||^2_{\mathcal{H}} = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \alpha_j k(x_i, x_j) = \boldsymbol{\alpha}^T K \boldsymbol{\alpha}$$

- Solving Kernel Ridge Regression involves solving

$$\arg\min_{\boldsymbol{\alpha} \in \mathbb{R}^N} \frac{1}{N}(K\boldsymbol{\alpha} - \mathbf{y})^T(K\boldsymbol{\alpha} - \mathbf{y}) + \lambda \boldsymbol{\alpha}^T K \boldsymbol{\alpha}$$

for $\lambda > 0$

Kernel Ridge Regression - Solution

- Desired optimization problem

$$\arg\min_{\boldsymbol{\alpha}\in\mathbb{R}^N} \frac{1}{N}(K\boldsymbol{\alpha}-\mathbf{y})^T(K\boldsymbol{\alpha}-\mathbf{y})+\lambda\boldsymbol{\alpha}^T K\boldsymbol{\alpha}$$

- The above is convex and differntiable w.r.t to $\boldsymbol{\alpha}$, and can be analytically found by setting the gradient

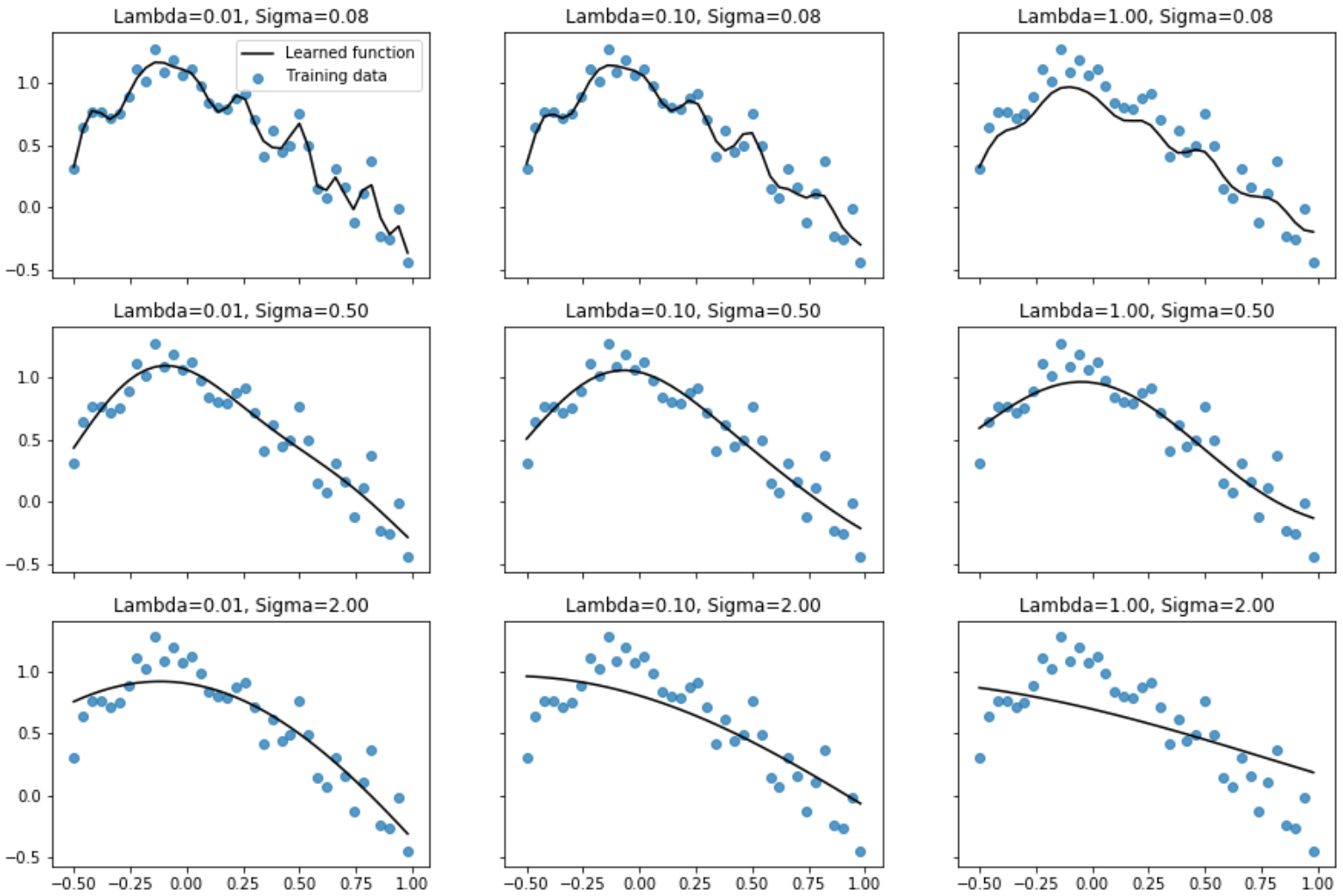
$$\frac{2}{N}K(K\alpha-\mathbf{y})+2\lambda K\alpha=\mathbf{0}$$

- Since K is positive semi-definite (from previous lecture), we can invert $K+\lambda NI$, and hence the solution is given by

$$\boldsymbol{\alpha}=(K+\lambda NI)^{-1}\mathbf{y}$$

where I is the identity matrix

Kernel Ridge Regression with Gaussian Kernel



Comparison with Deep learning

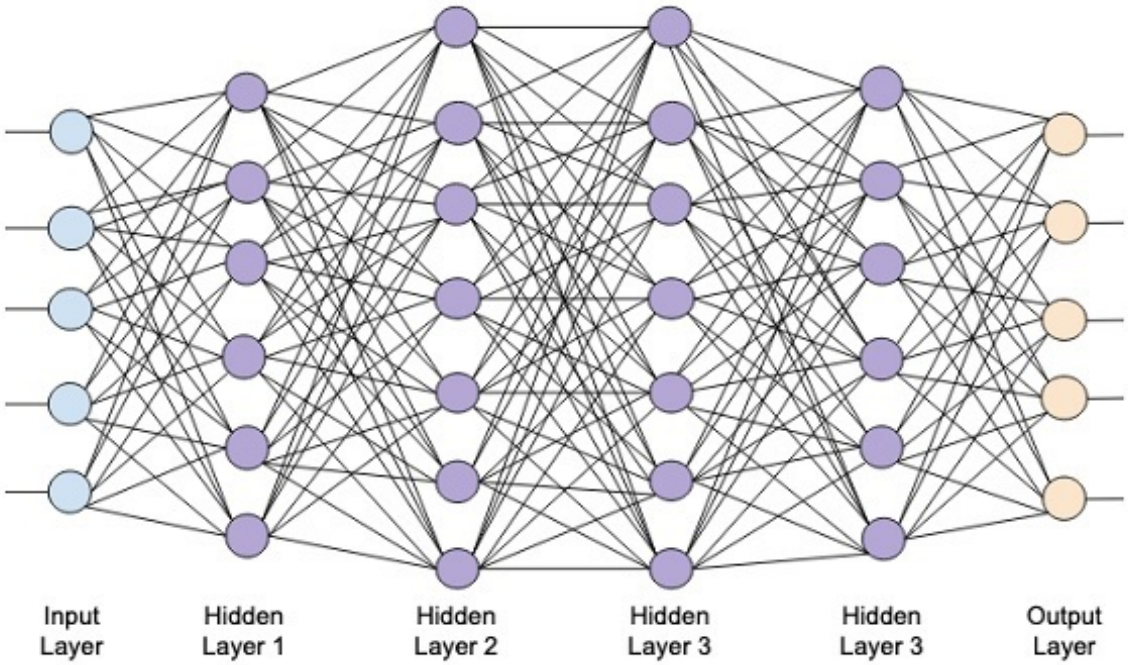


Figure: Representation learning in deep networks

In deep learning

- Feature mapping $\phi(x)$ is learnt explicitly
- Deep learning is also called representation learning

In kernel methods

- Feature mapping $\phi(x)$ is also a representation but that is not really learnt
- Given a kernel it is a fixed representation

Implicit vs Explicit Feature Maps

Recall - Linear classification

Assuming the input data x is 2-dimensional (i.e. in $\mathbb{R}^2$)

Linear classification

- Consider the classification function f_1 below, which is linear in both the input features and weights

$$f_1(x) = w^{(1)}x^{(1)} + w^{(2)}x^{(2)}$$

- In this case, the decision function $f_1(x)$ is trying to capture only **linear combination** of the input components x_1, x_2
- Linear feature map $\phi : \mathbb{R}^2 \mapsto \mathbb{R}^2$, and is given by, $\phi_1(x) = (x_1, x_2)^T$
- $f_1(.)$ is parameterized by two co-efficients $(w_1, w_2)^T$

Recall - Non-linear classification

Prediction function can involve non-linear combination of features

- For the classification function f_2 below, which is linear in weights and **non-linear in input features**

$$f_2(x) = w^{(1)}x^{(1)} + w^{(2)}x^{(2)} + w^{(3)}x^{(1)}x^{(2)} + w^{(4)}x^{(2)}x^{(1)}$$

- Here, the decision function $f_2(x)$ is trying to capture **non-linear combination** of the input components as well such as $x^{(1)}x^{(2)}, x^{(2)}x^{(1)}$
- Non-linear feature map $\phi_2 : \mathbb{R}^2 \mapsto \mathbb{R}^4$, and is given by $\phi_2(x) = (x^{(1)}, x^{(2)}, x^{(1)}x^{(2)}, x^{(2)}x^{(1)})^T$
  - $\phi_2(x) \in \mathcal{H}$, which is referred to as the feature space
- Note that the decision function $f_2(x)$ is still linear in the weight vector co-efficients $w^{(j)}$'s and is parameterized by $(w^{(1)}, w^{(2)}, w^{(3)}, w^{(4)})^T$

Kernel Matrix Bottleneck

In many problem scenarios in modern day big data setup, it is not difficult to get millions (or even bigger) of training samples :

- Computing a million $\times$ million kernel matrix is not trivial
  - For instance, one needs to invert the kernel matrix for kernel ridge regression (as we will see later) - complexity $O(N^3)$ for N training data points
- The computational bottle-neck also limits hyper-parameter tuning

Linear Classification with Linear kernel

- In some cases, we may have **high dimensional input features** such as classification problems involving **textual data** :
  - Classification of data as in Wikipedia, News Articles, Research articles
  - Ranking of queries/Documents in Searching Engines
  - Web-advertising

In these cases, data may exhibit *linear separability*, it might just be enough to use linear classifiers, i.e., no feature mapping $\phi(x)$ is required

- $f(x) = \mathbf{w}^T\mathbf{x}$ is the decision function, i.e. $\phi(\mathbf{x}) = \mathbf{x}$ (aka linear kernel)

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- How we might check linear separability of the given dataset in high dimensions ?

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Linear SVM

Soft-margin SVM as a regularised learning problem

- We can rewrite the soft-margin SVM problem

$$\begin{aligned} \textit{Minimize} \quad & \tfrac{1}{2}||\mathbf{w}||^2 + \tfrac{C}{m}\sum_{i=1}^{m}\xi_i \\ \text{Subject to} \quad & \xi_i \geq \max(1 - y_i\mathbf{w}^T\mathbf{x}_i, 0) \\ & \text{for all } i = 1, \ldots, N. \\ & \xi_i \geq 0 \end{aligned}$$

  equivalently in terms of Hinge loss as

$$\min_{\mathbf{w}} \frac{1}{m}\sum_{i=1}^{m}\mathcal{L}_{Hinge}(\mathbf{w}^T\mathbf{x}_i, y_i) + \frac{\lambda}{2}||\mathbf{w}||^2$$

- This is a so called **regularized learning problem**
  - First term minimizes a loss function on training data
  - Second term, called the **regularizer**, controls the complexity of the model
  - The parameter $\lambda = \frac{1}{C}$ controls the balance between the two terms

Figure: Linear SVM from Machine Learning Supervised Methods

A scalable C++ implementation (upto millions of training samples) of linear classification and regression comes in Liblinear solver [1]

[1] https://www.csie.ntu.edu.tw/ cjlin/liblinear/, which is also the underlying solver in scikit-learn for Linear classification problems

Short text classification - Example from eBay

```
Tickets calgary flames vancouver canucks 2 tickets sept 24 2012 row 2
Stamps  stamps italy kingdom used high value f 9608
Music   brahms j hungarian dances cd new
Jewelry & Watches       5 colors optional hotaru fashion silicone date calendar jelly candy sport watch
Tickets free legoland ticket for friend w legoland pass member expires 8 31 12
Books   specimen book of monotype printing types vol 2 c 1971 garamond to othello
Tickets 4 new york jets vs new england patriots tickets 11 22 sec 336 row 17 10 yd
Art     lowe 1858 antique fern botanical print polypodium lycopodioides
Art     m c escher black white print house of stairs
Books   days in the lives of social workers 54 professionals tell real life stories
Music   bizet g carmen sung in english cd new
Stamps  parliament house cent fdc perth fdi pmk 2
Jewelry & Watches       faceted multigem sterling silver bracelet by silverrushstyle
Tickets michael jackson the immortal tour 8 15 12 staples 2 lower level
```

- Each row is training example in the form of LABEL $\langle$ TAB $\rangle$ INPUT-VECTOR
- In the example above, *Tickets*, *Stamps*, *Music* etc are labels or classes
- Consider thousands or hundreds of thousand such examples

Polynomial kernel and n-grams

In short text classification (as in previous slide), it might be useful to consider bigrams/trigrams also as features along with normal uni-gram features (similar to projecting data to high dimensions)

- Movie review example - *Not bad, one-time watch* **vs** pretty bad, not worth the time
- bigrams - pairs of words (such as 'calgary flames', 'flames vancouver') in a window of size 2
- trigrams - triplets of words in a window of size 3

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Explicit feature map for Polynomial kernel with unigram + bi-grams features

Non-linear classification

For the classification function f_3 below :

$$\begin{aligned} f_3(x) = & w^{(1)}x^{(1)} + w^{(2)}x^{(2)} + w^{(3)}x^{(3)} \\ & + w^{(4)}x^{(1)}x^{(2)} + w^{(5)}x^{(1)}x^{(3)} \\ & + w^{(6)}x^{(2)}x^{(1)} + w^{(7)}x^{(2)}x^{(3)} \\ & + w^{(8)}x^{(3)}x^{(1)} + w^{(9)}x^{(3)}x^{(2)} \end{aligned}$$

- Here, the decision function $f_3(x)$ maps a vector in $\mathbb{R}^3$ to $\mathbb{R}$
- Non-linear feature map $\phi_3 : \mathbb{R}^3 \mapsto \mathbb{R}^9$, and is given by $\phi_3(x) = (x^{(1)}, x^{(2)}, x^{(3)}, x^{(1)}x^{(2)}, x^{(1)}x^{(3)}, x^{(2)}x^{(1)}, x^{(2)}x^{(3)}, x^{(3)}x^{(1)}, x^{(3)}x^{(2)})^T$
- The learning process is to learn a linear decision boundary in $\mathbb{R}^9$, namely the 9 co-efficients $(w_1, \ldots, w_9)^T$
- When sample size N is large, it may be easier to solve a linear system without having to do pairwise computation as required for the kernel matrix

How about Large N, and Large D

Two challenges :

- Using kernels i.e. implicit feature mapping - there might be hundreds of thousand training examples N, so computing the kernel matrix of size N^2 becomes challenging.
- Without kernel i.e. explicit feature mapping - there will be D^2 bigrams and D^3 trigrams. Therefore, one has to learn co-efficients which are quadratic/cubic in the data dimensionality

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Solution -

- Exploit input data sparsity - create bigrams/trigrams on-demand
- Main idea - create bigram features for only those pairs of words which appear in the training data
- As an example for ebay data, for D=1,000, and sparsity $s = 10$, computation $O(s^2)$ is much smaller than $O(D^2)$
- Then, *learn a linear classifier which is a much easier problem* than non-linear method involving kernel especially with large sample size N.

How about Large N, and Large D

Two challenges :

- Using kernels i.e. implicit feature mapping - there might be hundreds of thousand training examples N, so computing the kernel matrix of size N^2 becomes challenging.
- Without kernel i.e. explicit feature mapping - there will be D^2 bigrams and D^3 trigrams. Therefore, one has to learn co-efficients which are quadratic/cubic in the data dimensionality

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Libshorttext (`https://www.csie.ntu.edu.tw/~cjlin/libshorttext/`) python wrapper over Liblinear provides a scalable implemetation.

Term Frequency and Inverse Document Frequency

TfIDF based data representation for sparse text data

- Bag of Words
  - Converts **sequence of words** representation of a document to a **set of words**
- Tf-IDF weighting of every term in the document
  - Term-Frequency $\times$ Inverse Document Frequency
  - Term-Frequency (log normalized) of term t in document d

$$tf(t, d) = \log(1 + f_{t,d})$$

where $f_{t,d}$ is the number of times t appears in document d

  - Inverse Document Frequency - How frequent is the term across documents

$$idf(t) = \log(1 + \frac{N}{n_t})$$

where n_t is the number of times term t appears from the training set of N documents

Linear vs Kernel Methods

Linear Classification	**Kernel Classification**
Linear Regression	Kernel Regression with Linear Kernel
Example - LinearSVM using LibLinear	Example - KernelSVM such as LibSVM [2]
Linear SVM + Bigram feature	LibSVM + Polynomial Kernel
Con - Requires new explicit features	Pros - Just change the kernel
Pros - Suitable for large sample size	Con - Requires some work to scale

Table: Linear vs Non-linear (Kernel) Classification

- With appropriate explicit features for Linear classification, both are statistically equivalent

[2] https://www.csie.ntu.edu.tw/ cjlin/libsvm/

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Table: Linear vs Non-linear (Kernel) Classification

- With appropriate explicit features for Linear classification, both are statistically equivalent
- To verify, try a linear SVM using LibLinear, and LibSVM with Linear Kernel on the same problem for increasing sample sizes

[2] https://www.csie.ntu.edu.tw/ cjlin/libsvm/

Conclusion

- Revisited the idea of linear and non-linear classification
- Representer theorem
  - Proof
  - Implications
- Kernel Ridge Regression
- Kernel matrix bottleneck for large sample size
  - Explicit feature maps for polynomial kernel
  - Sparse data representation

References

Books for further study

- Learning with kernels - Schoelkopf and Smola
- Kernel Methods for Pattern Analysis - Shawe-Taylor and Christianini