CS:E4830 Kernel Methods in Machine Learning

Lecture 6 : Kernel Support Vector Machines

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6th April, 2022

Some Announcements

- Assignment 2 due today
- No lecture next week,
- Next upcoming session - Lecture 7 (20.04)

Course outline so far

- Introduction to kernel methods
  - Kernel Definition and examples
  - RKHS, Representer Theorem and Kernel Least Squares
- Introductory learning theory
  - Generalization, ERM, Consistency
- Convex optimization Introduction
  - Convexity and Duality
- Algorithms - Supervised
  - *Kernel Support Vector Machines*
  - Logistic regression
  - Kernel methods for large-scale problems
- Algorithms - Unsupervised
  - PCA and k-means
  - Kernel variants
- Kernel methods for Structured output and multi-view data
  - Guest lecture

Supervised learning setup

Binary classification

- Input $\mathcal{X}$, can be in various forms such as images or text documents
- Output $\mathcal{Y} = \{-1, +1\}$ - binary classification. Other formulations include :
  - One-hot encoded binary vector for multi-class classification - Cifar10
  - Multi-label classification - Wikipedia
- Training set $S = (x_i, y_i)_{i=1}^{N}$ consists of samples that are sampled independently and identically from an unknown joint distribution P over $\mathcal{X} \times \mathcal{Y}$
- The goal is to build a classifier f to predict the label $\hat{y}$ for a test instance x.

Maximum margin hyperplane

One good solution is to choose the hyperplane $\mathbf{w}^T\mathbf{x} = 0$ that lies furthest away from the training data (maximizing the minimum margin of the training examples):

$$\begin{aligned} &\textit{Maximize } \gamma \\ &\quad w.r.t. \ \text{ variables } \mathbf{w} \in \mathbb{R}^d \\ &\textit{Subject to } \frac{y_i \mathbf{w}^T \mathbf{x}_i}{\|\mathbf{w}\|} \geq \gamma, \text{for all } i = 1, \ldots, m, \end{aligned}$$

The maximum margin hyperplane has good properties:

- Robustness: small change in the training data will not change the classifications too much
- Theoretically a large margin is tied to a low generalization error
- It can be found efficiently through incremental optimization

Support vector machines (SVM) are based on this principle

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Hinge Loss Function and Kernel SVM

- Kernel SVM solves the following optimization problem :

$$\min_{f \in \mathcal{H}} \frac{1}{N} \sum_{i=1}^{N} \ell_{hinge}(y_i f(x_i)) + \lambda ||f||_{\mathcal{H}}^2$$

  where $\mathcal{H}$ is a reproducing kernel Hilbert space

- Hinge loss is a function $\mathbb{R} \mapsto \mathbb{R}_+$ defined as below :

$$\ell_{hinge}(u) := max(1-u, 0) = \begin{cases} 0 & \text{if } u \geq 1 \\ 1-u & \text{otherwise} \end{cases}$$

Hinge Loss - a convex upper bound

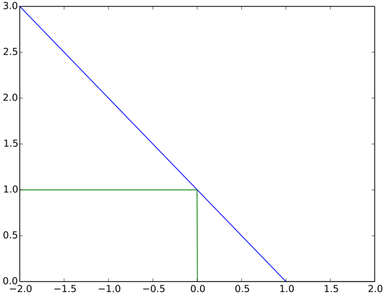


Figure: Hinge and 0-1 loss functions

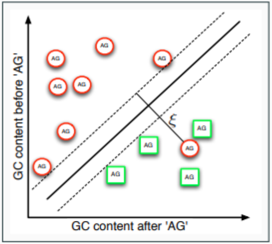
- Hinge Loss (in blue) is given by $\max(1 - yf(x), 0)$
- **Convex Upper bound** on 0-1 loss

Dual representation of the optimal hyperplane

It can be shown theoretically that the **optimal hyperplane** of the soft-margin SVM has a **dual representation** as the linear combination of the training data

$$\mathbf{w} = \sum_{i=1}^{m} \alpha_i y_i \mathbf{x}_i$$

- The coefficients, also called the **dual variables** are non-negative $\alpha_i \geq 0$
- The positive coefficients $\alpha_i > 0$ appear if and only if $\mathbf{x}_i$ is a support vector, for other training points we have $\alpha_i = 0$



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Dual representation of the optimal hyperplane

- Consequently, the functional margin $y\mathbf{w}^T\mathbf{x}$ also can be expressed using the support vectors:

$$y\mathbf{w}^T\mathbf{x} = y\sum_{i=1}^{m}\alpha_i y_i \mathbf{x}_i^T\mathbf{x}$$

- The norm of the weight vector can be expressed as

$$\mathbf{w}^T\mathbf{w} = \sum_{i=1}^{m}\alpha_i y_i \mathbf{x}_i^T \sum_{j=1}^{m}\alpha_j y_j \mathbf{x}_j = \sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\alpha_j y_i y_j \mathbf{x}_i^T\mathbf{x}_j$$

- Note that the training data appears in pairwise inner products: $\mathbf{x}_i^T\mathbf{x}_j$

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Dual representations

- We can replace the explicit inner products with a kernel function

$$\kappa(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{x}_i^T \mathbf{x}_j$$

  which computes an inner product in the space of the arguments, here $\mathbb{R}^d$
- Plug in:
  - Margin:

$$y\mathbf{w}^T\mathbf{x} = y \sum_{i=1}^{m} \alpha_i y_i \kappa(\mathbf{x}_i, \mathbf{x})$$

  - Squared norm:

$$\|\mathbf{w}\|^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \alpha_j y_i y_j \kappa(\mathbf{x}_i, \mathbf{x}_j)$$

From Representer Theorem

- For the following optimization

$$f_{\mathcal{H}} := \arg\min_{f} \frac{1}{N} \sum_{i=1}^{N} \ell_{hinge}(y_i, f(x_i)) + \lambda\theta(||f||^2_{\mathcal{H}})$$

  where $\ell_{hinge}(.,.)$ is the hinge loss function and $\theta : [0, \infty) \mapsto \mathbb{R}$ is non-decreasing function, and $\mathcal{H}$ is an RKHS

- Even though the above problem is potentially an infinite dimensional optimization problem, **Representer Theorem** states its solution can be expressed in the following form

$$f_{\mathcal{H}}(.) = \sum_{j=1}^{N} \boldsymbol{\alpha}_i k(., x_j)$$

  where $\boldsymbol{\alpha}_i \in \mathbb{R}$, i.e. it is linear combination of kernel evaluations at training points

Implications of Representer Theorem

- Representer Theorem allows us to look for the solutions of the following form:

$$f_{\mathcal{H}}(.) = \sum_{j=1}^{N} \boldsymbol{\alpha}_i k(., x_j)$$

- Implications
  - The desired function just involves kernel computation on **training points only** via $k(., x_i)$ in the above solution
  - It reduces the problem of finding $f \in \mathcal{H}$ which could be infinite dimensional to a finite dimensional problem
    - We just need to find the coefficients of the finite linear combination $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N$
  - Also, we can reforumlate the original objective function in the new form (more in next slides)

Reformulating the Objective using Representer Theorem - I

- Recall the original objective :

$$f_{\mathcal{H}} := \arg\min_{f} \frac{1}{n} \sum_{i=1}^{N} \ell(y_i, f(x_i)) + \lambda\theta(||f||_{\mathcal{H}})$$

- Using the implication of representer theorem, for the *i*-th **training point**,

$$f_{\mathcal{H}}(x_i) = \sum_{j=1}^{N} \boldsymbol{\alpha}_j k(x_i, x_j) = [K\boldsymbol{\alpha}]_i$$

which is the *i*-th element of the matrix-vector product $K\boldsymbol{\alpha}$

Reformulating the Objective using Representer Theorem - II

- Similarly, rewriting the regularization term :

$$
\begin{aligned}
||f||^2_{\mathcal{H}} &= \langle f(.), f(.) \rangle \\
&= \left\langle \sum_{i=1}^{N} \boldsymbol{\alpha}_i k(., x_i), \sum_{i=1}^{N} \boldsymbol{\alpha}_i k(., x_i) \right\rangle \text{ (using representer theorem)} \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} \boldsymbol{\alpha}_i \boldsymbol{\alpha}_j k(x_i, x_j) \text{ (evaluation of dot product)} \\
&= \boldsymbol{\alpha}^T K \boldsymbol{\alpha} \text{ (writing in matrix notation)}
\end{aligned}
$$

Reformulating the Objective using Representer Theorem - III

Using the above substitutions, the original (potentially intractable) objective

$$f_{\mathcal{H}} := \arg\min_{f} \frac{1}{N} \sum_{i=1}^{N} \ell(y_i, f(x_i)) + \lambda\theta(||f||_{\mathcal{H}})$$

translates to an equivalent (tractable) form below

$$\arg\min_{\boldsymbol{\alpha} \in \mathbb{R}^N} \frac{1}{N} \sum_{i=1}^{N} \ell(y_i, [K\boldsymbol{\alpha}]_i) + \lambda\theta(\boldsymbol{\alpha}^T K\boldsymbol{\alpha})$$

SVM Problem Refomulation

- Using Representer theorem, the problem can be reformulated as

$$\min_{\boldsymbol{\alpha} \in \mathbb{R}^N} \left\{ \frac{1}{N} \sum_{i=1}^{N} \ell_{hinge}(y_i [K\boldsymbol{\alpha}]_i) + \lambda \boldsymbol{\alpha}^T K \boldsymbol{\alpha} \right\}$$

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- The above optimization problem is convex (why?)
- However, it is non-smooth/non-differentiable optimization problem

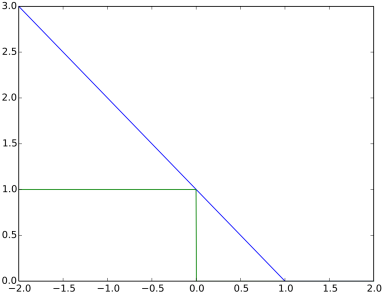

Figure: $\ell_{\text{hinge}}(.)$ is the function in blue, $z = yf(x)$ in the above graph

Another Equivalent Reformulation

- The optimization problem on the previous slide is equivalent (even though not immediately obvious) to the following with an extra optimization variable $\boldsymbol{\xi}$, one for each example in the training set. If we re-write it in terms of slack variables $\boldsymbol{\xi}_i \in \mathbb{R}$ for $i = 1, \ldots, N$

$$\min_{\boldsymbol{\alpha} \in \mathbb{R}^N, \boldsymbol{\xi} \in \mathbb{R}^N} \left\{ \frac{1}{N} \sum_{i=1}^{N} \boldsymbol{\xi}_i + \lambda \boldsymbol{\alpha}^T K \boldsymbol{\alpha} \right\} \text{such that } \boldsymbol{\xi}_i \geq \ell_{hinge}(y_i [K\boldsymbol{\alpha}]_i)$$

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- In the above formulation, the objective is smooth but not the constraints

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- In the above formulation, the objective is smooth but not the constraints
- Recall the definition of hinge loss

$$\ell_{hinge}(u) = max(1 - u, 0) \iff \begin{cases} 0 & \text{if } u \geq 1 \\ 1 - u & \text{otherwise} \end{cases}$$

- Using above, the N constraints ($\boldsymbol{\xi}_i \geq \ell_{hinge}(y_i[K\boldsymbol{\alpha}]_i)$) can be replaced by $2N$ constraints to make the problem smooth as follows :

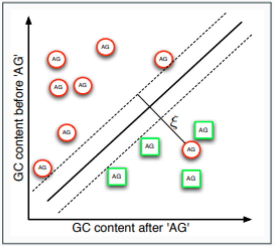
$$\boldsymbol{\xi}_i \geq \ell_{hinge}(y_i[K\boldsymbol{\alpha}]_i) \iff \begin{cases} \boldsymbol{\xi}_i \geq 0 \\ \boldsymbol{\xi}_i \geq 1 - y_i[K\boldsymbol{\alpha}]_i \end{cases}$$

Non-separable data

- To allow non-separable data, we allow the functional margin of some data points to be smaller than 1 by a slack variable $\xi_i \geq 0$
- The relaxed margin constraint will be expressed as

$$y_i \mathbf{w}^T \mathbf{x}_i \geq 1 - \xi_i, \xi_i \geq 0$$

- $\xi_i = 0$ corresponds to having large enough margin > 1
- $\xi_i > 1$ corresponds to negative margin, misclassified point
- The set of support vectors includes all $\mathbf{x}_i$ that have non-zero slack ξ_i (functional margin ≤ 1)



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SVM Primal

SVM Primal Formulation

$$\min_{\boldsymbol{\alpha}\in\mathbb{R}^N,\boldsymbol{\xi}\in\mathbb{R}^N} \left\{ \frac{1}{n}\sum_{i=1}^{N} \boldsymbol{\xi}_i + \lambda\boldsymbol{\alpha}^T K\boldsymbol{\alpha} \right\}$$

such that

$$\begin{cases} 1 - y_i[K\alpha]_i - \boldsymbol{\xi}_i \leq 0 & \text{for } i = 1, \ldots, N \\ -\boldsymbol{\xi}_i \leq 0 & \text{for } i = 1, \ldots, N \end{cases}$$

Putting in the standard convex optimization problem framework where the inequality constraints should in the less than ($\leq$) form

Lagrangian

- The Lagrangian of the problem is :

$$L(\boldsymbol{\alpha}, \boldsymbol{\xi}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{\xi}_i + \lambda \boldsymbol{\alpha}^T K \boldsymbol{\alpha} + \sum_{i=1}^{N} \boldsymbol{\mu}_i [1 - y_i [K\boldsymbol{\alpha}]_i - \boldsymbol{\xi}_i] - \sum_{i=1}^{N} \boldsymbol{\nu}_i \boldsymbol{\xi}_i$$

  - $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathbb{R}^N$, $\boldsymbol{\mu} \geq 0$ and $\boldsymbol{\nu} \geq 0$
- Note that constraints have moved to the Lagrangian.

Computing derivatives w.r.t $\boldsymbol{\xi}$

- The lagrangian of the problem is :

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Lagrangian wrt $\boldsymbol{\xi}$

- $L(\alpha, \xi, \mu, \nu)$ is a linear function in ξ. What is its minimum value

Computing derivatives w.r.t $\boldsymbol{\xi}$

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Lagrangian wrt $\boldsymbol{\xi}$

- $L(\alpha, \xi, \mu, \nu)$ is a linear function in ξ. What is its minimum value
- Its minimum value is $-\infty$, except when it is constant,

$$\nabla_{\boldsymbol{\xi}} L(\boldsymbol{\alpha}, \boldsymbol{\xi}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \frac{\mathbf{1}}{N} - \boldsymbol{\mu} - \boldsymbol{\nu} = \mathbf{0}$$

equivalently,

$$\frac{\mathbf{1}}{N} = \boldsymbol{\mu} + \boldsymbol{\nu}$$

Computing derivatives w.r.t $\boldsymbol{\alpha}$

- The lagrangian of the problem is :

$$L(\boldsymbol{\alpha}, \boldsymbol{\xi}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{\xi}_i + \lambda \boldsymbol{\alpha}^T K \boldsymbol{\alpha} + \sum_{i=1}^{N} \boldsymbol{\mu}_i [1 - y_i [K\boldsymbol{\alpha}]_i - \boldsymbol{\xi}_i] - \sum_{i=1}^{N} \boldsymbol{\nu}_i \boldsymbol{\xi}_i$$

Lagrangian wrt $\boldsymbol{\alpha}$

- $L(\boldsymbol{\alpha}, \boldsymbol{\xi}, \boldsymbol{\mu}, \boldsymbol{\nu})$ is a convex quadratic function in $\boldsymbol{\alpha}$. To find the optimal value, set the gradient to $\mathbf{0}$ (the zero vector) :

$$\nabla_{\boldsymbol{\alpha}} L = \mathbf{0}$$

- The optimal solution $\boldsymbol{\alpha}^*$ is given by

$$\boldsymbol{\alpha}_i^* = \frac{y_i \boldsymbol{\mu}_i}{2\lambda}$$

Lagrange Dual Function and Dual Problem

Lagrange Dual Function

- The Lagrange dual function by substituting the optimal values (with those obtained in previous two slides) is given by :

$$
\begin{aligned}
q(\boldsymbol{\mu}, \boldsymbol{\nu}) &= \min_{\boldsymbol{\alpha} \in \mathbb{R}^N, \boldsymbol{\xi} \in \mathbb{R}^N} L(\boldsymbol{\alpha}, \boldsymbol{\xi}, \boldsymbol{\mu}, \boldsymbol{\nu}) \\
&= \begin{cases} \sum_{i=1}^N \boldsymbol{\mu}_i - \frac{1}{4\lambda} \sum_{i=1}^N \sum_{j=1}^N y_i y_j \boldsymbol{\mu}_i \boldsymbol{\mu}_j K(x_i, x_j) & \text{if } \boldsymbol{\mu} + \boldsymbol{\nu} = \frac{\mathbf{1}}{N} \\ -\infty & \text{otherwise} \end{cases}
\end{aligned}
$$

Lagrange Dual Problem

- The Lagrange dual problem is

$$
\max q(\boldsymbol{\nu}, \boldsymbol{\mu}) \text{ such that } \boldsymbol{\mu} \geq 0, \boldsymbol{\nu} \geq 0
$$

Closer Look At The Dual Problem

- The Lagrange dual problem is

$$\max q(\boldsymbol{\nu}, \boldsymbol{\mu}) \text{ such that } \boldsymbol{\mu} \geq 0, \boldsymbol{\nu} \geq 0$$

- If $0 \leq \boldsymbol{\mu}_i \leq 1/N$ for all i, then the dual function takes finite values. Also, the value of $\boldsymbol{\nu}_i$ is fixed at $\boldsymbol{\nu}_i = 1/N - \boldsymbol{\mu}_i$ in this case.
- The dual problem is therefore given by

$$\max_{\mathbf{0} \leq \boldsymbol{\mu} \leq \mathbf{1}/N} \sum_{i=1}^{N} \boldsymbol{\mu}_i - \frac{1}{4\lambda} \sum_{i=1}^{N} \sum_{j=1}^{N} y_i y_j \boldsymbol{\mu}_i \boldsymbol{\mu}_j K(x_i, x_j)$$

Dual Soft-Margin SVM

A dual optimization problem for the soft-margin SVM with kernels is given by

$$\begin{array}{rc} \textit{Maximize} & OBJ(\boldsymbol{\alpha}) = \sum_{i=1}^{m} \alpha_i - \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \alpha_j y_i y_j \kappa(\mathbf{x}_i, \mathbf{x}_j) \\ w.r.t & \text{variables } \boldsymbol{\alpha} \in \mathbb{R}^m \\ \text{Subject to} & 0 \leq \alpha_i \leq C/m \\ & \text{for all } i = 1, \ldots, m \end{array}$$

- It is a QP with variables α_i, again with a unique optimum
- At optimum, will have implicitly computed the optimal hyperplane $\mathbf{w} = \sum_{i=1}^{m} \alpha_i y_i \mathbf{x}_i$
- The data only appears through the kernel function $\kappa(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{x}_i^T \mathbf{x}_j$
- Full derivation requires techniques of optimization theory, which we will skip here

Rewriting in terms of Primal Variables

Dual problem (from previous slide)

$$\max_{\mathbf{0} \leq \boldsymbol{\mu} \leq \mathbf{1}/N} \sum_{i=1}^{N} \boldsymbol{\mu}_i - \frac{1}{4\lambda} \sum_{i=1}^{N} \sum_{j=1}^{N} y_i y_j \boldsymbol{\mu}_i \boldsymbol{\mu}_j K(x_i, x_j)$$

Since the primal variable $\boldsymbol{\alpha}$ and the dual variable $\boldsymbol{\mu}$ are related by $\boldsymbol{\alpha}_i = \frac{\boldsymbol{\mu}_i y_i}{2\lambda}$, it can be written in the form of primal variables as follows

writing in terms of primal variable $\boldsymbol{\alpha}$

$$\max_{\boldsymbol{\alpha} \in \mathbb{R}^N} 2 \sum_{i=1}^{N} \boldsymbol{\alpha}_i y_i - \sum_{i=1}^{N} \sum_{j=1}^{N} \boldsymbol{\alpha}_i \boldsymbol{\alpha}_j K(x_i, x_j)$$

such that

$$0 \leq y_i \boldsymbol{\alpha}_i \leq \frac{1}{2\lambda N} \text{ for } i = 1, \ldots, N$$

The above can be solved using a standard Quadratic program solver.

Back to Representer Theorem

- Once we have the $\boldsymbol{\alpha}$ from the previous slide, the decision function in RKHS is constructed using representer theorem:

$$f(.) = \sum_{j=1}^{N} \boldsymbol{\alpha}_i k(., x_j)$$

- However, if N is large, this might still be computationally expensive for prediction
- In order to tackle this, the complementarity conditions (next) motivate the idea of *support vectors*, which we see next.

Complementarity conditions at the optimum

(*Check corresponding slide from previous Lecture on Complementary Slackness*)

- These are given by the product of the dual variables and the corresponding constraint as follows :

$$\boldsymbol{\mu}_i[y_i f(x_i) + \boldsymbol{\xi}_i - 1] = 0$$
$$\boldsymbol{\nu}_i \boldsymbol{\xi}_i = 0$$

- Recalling that $\boldsymbol{\mu}_i = \frac{2\lambda\boldsymbol{\alpha}_i}{y_i}$ and $\boldsymbol{\nu}_i = \frac{\mathbf{1}}{N} - \frac{2\lambda\boldsymbol{\alpha}_i}{y_i}$
- In terms of the primal variable $\boldsymbol{\alpha}$, the above can be re-written as

$$\boldsymbol{\alpha}_i[y_i f(x_i) + \boldsymbol{\xi}_i - 1] = 0$$
$$\left(\boldsymbol{\alpha}_i - \frac{y_i}{2\lambda N}\right)\boldsymbol{\xi}_i = 0$$

Interpreting the Complementarity Conditions

Complementarity Conditions

$$\boldsymbol{\alpha}_i[y_i f(x_i) + \boldsymbol{\xi}_i - 1] = 0$$
$$\left(\boldsymbol{\alpha}_i - \frac{y_i}{2\lambda N}\right)\boldsymbol{\xi}_i = 0$$

Interpreting the Complementarity Conditions

Complementarity Conditions

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- If $\boldsymbol{\alpha}_i = 0$, then the second constraint is active : $\boldsymbol{\xi}_i = 0$. This implies $y_i f(x_i) \geq 1$ (Why?). (*These are correctly classified points with some margin*)

Interpreting the Complementarity Conditions

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- If $0 < y_i\boldsymbol{\alpha}_i < \frac{1}{2\lambda N}$, then both the constraints are active, i.e., $\boldsymbol{\xi}_i = 0$ and $y_i f(x_i) + \xi_i - 1 = 0$. This leads to $y_i f(x_i) = 1$. (*These are correctly classified points right at the margin*)

Interpreting the Complementarity Conditions

Complementarity Conditions

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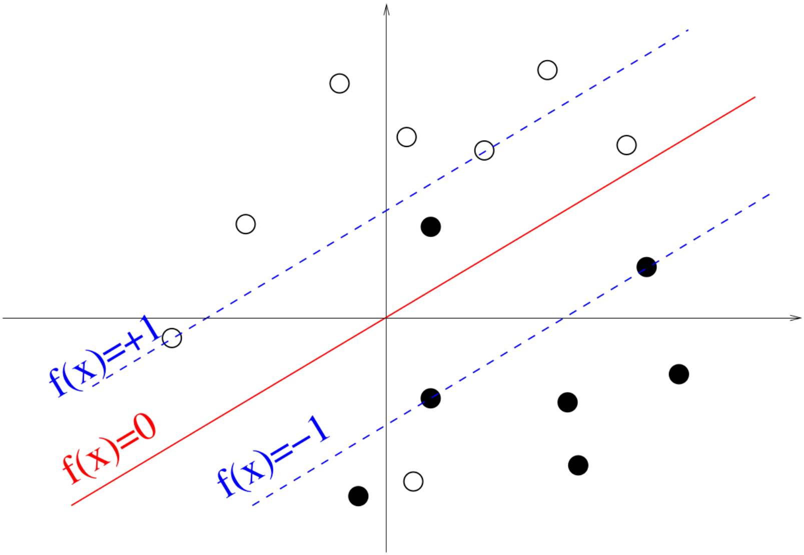
- If $0 < y_i\boldsymbol{\alpha}_i < \frac{1}{2\lambda N}$, then both the constraints are active, i.e., $\boldsymbol{\xi}_i = 0$ and $y_i f(x_i) + \xi_i - 1 = 0$. This leads to $y_i f(x_i) = 1$. (*These are correctly classified points right at the margin*)
- If $\boldsymbol{\alpha}_i = \frac{y_i}{2\lambda N}$, then the second constraint is not active ($\boldsymbol{\xi}_i \geq 0$) but the first one is active : $y_i f(x_i) + \boldsymbol{\xi}_i = 1$. This implies that $y_i f(x_i) \leq 1$. (*These points may be correctly or incorrectly classified points*)

Decision Hyperplanes



Pictorial Depiction for α values

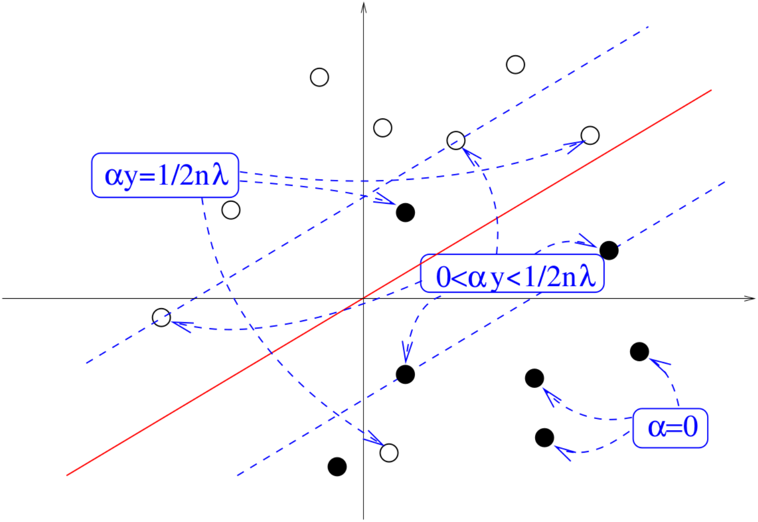


Figure: n in the figure above corresponds to N in our notation

Support Vectors

- From Representer theorem, the function evaluation at any $x \in \mathcal{X}$ (the input space) is given by

$$f(x) = \sum_{i=1}^{N} \boldsymbol{\alpha}_i k(x_i, x) = \sum_{i \in SV} \boldsymbol{\alpha}_i k(x_i, x)$$

  where SV is the set of support vectors i.e. those training points for which $\boldsymbol{\alpha}_i \neq 0$
- Hence the name Support Vector Machines
- The above sparsity of $\boldsymbol{\alpha} \in \mathbb{R}^N$ can be used for
  - Faster prediction since once needs to go over only the support vectors

Another variant - C-SVM

Sometimes, instead of the regularization parameter λ, the SVM problem is written in the following form :

$$\min_{f \in \mathcal{H}} \left\{ C \sum_{i=1}^{N} (\ell_{hinge}(y_i[K\boldsymbol{\alpha}]_i))^2 + \frac{1}{2}||f||_{\mathcal{H}}^2 \right\}$$

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- This is equivalent to the original formulation on the first slide($\min_{f \in \mathcal{H}} \frac{1}{N} \sum_{i=1}^{N} \ell_{hinge}(y_i f(x_i)) + \lambda ||f||_{\mathcal{H}}^2$) with $C = \frac{1}{2N\lambda}$

Another variant - C-SVM

Sometimes, instead of the regularization parameter λ, the SVM problem is written in the following form :

$$\min_{f \in \mathcal{H}} \left\{ C \sum_{i=1}^{N} (\ell_{hinge}(y_i[K\boldsymbol{\alpha}]_i))^2 + \frac{1}{2}||f||_{\mathcal{H}}^2 \right\}$$

- This is equivalent to the original formulation on the first slide($\min_{f \in \mathcal{H}} \frac{1}{N} \sum_{i=1}^{N} \ell_{hinge}(y_i f(x_i)) + \lambda ||f||_{\mathcal{H}}^2$) with $C = \frac{1}{2N\lambda}$
- Using the Lagrangian formulation, the dual can be written as

$$\max_{\boldsymbol{\alpha} \in \mathbb{R}^N} 2 \sum_{i=1}^{N} \boldsymbol{\alpha}_i y_i - \sum_{i=1}^{N} \sum_{j=1}^{N} \boldsymbol{\alpha}_i \boldsymbol{\alpha}_j K(x_i, x_j)$$

such that

$$0 \leq y_i \boldsymbol{\alpha}_i \leq C \text{ for } i = 1, \ldots, N \text{ (also called box constraints)}$$

References

- Most of the material for this lecture is based on a similar course by Julien Mairal's at ENS Paris
- Further details (with somewhat different notation) on SVMs - JST & Christianini book, Chapter 7