Introduction to Dynamic Systems

Systems analysis laboratory II

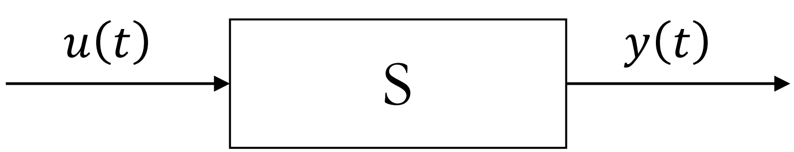
November 8, 2021

Constants, inputs, outputs and disturbances

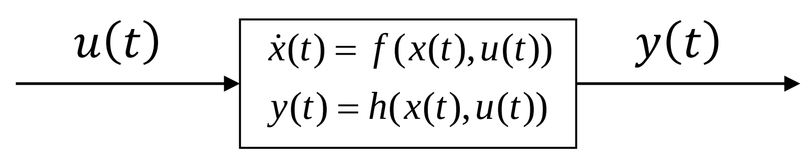
- Constants
  - System parameters
    - constants originating from the system that cannot be changed, e.g., acceleration caused by gravity
  - Design parameters
    - can be varied in practice but constants in the model, e.g., mass of an object
- Variables
  - Outputs $y(t) = \left[y_1(t), \ldots, y_p(t)\right]^T$
  - Inputs/controls $u(t) = [u_1(t), \ldots, u_m(t)]^T$
    - can be selected
  - Disturbances $w(t) = [w_1(t), \ldots, w_r(t)]^T$
    - cannot be selected
- In dynamic systems, $y(t)$ depends not only on $u(t)$ and $w(t)$ but also on all $u(s)$ and $w(s)$, $s < t$
  - The system has a memory

State

- The output of the dynamic system $y(t)$ is affected by $u(s)$ and $w(s)$, $s < t$
  - Would be cumbersome to store every $u(s)$ and $w(s)$
- The state $x(t)$ of the system (or model) contains information which in addition to $u(s)$ and $w(s)$ ($s \in [t, \tau]$) enables the computation of $y(\tau)$ for some $\tau > t$
- In practice the state plays an important role in simulation: it describes the system at each time instant



input-output model
(external model)



state space model
(internal model)

Input-output and state space models

- General (SISO) input-output model of n:th order in continuous time
$$g\left(y^{(n)}(t), y^{(n-1)}(t), \dots, y(t), u^{(\ell)}(t), \dots, u(t)\right) = 0, n \geq \ell,$$
where (a) denotes the a:th time-derivative and g is a nonlinear function
- Transferred to a first order differential equation system by setting $x_i(t) := y^{(i-1)}(t), i = 1, \dots, n$ (not always possible)
- State space model
$$\dot{x}(t) = f\big(x(t), u(t)\big) \qquad \text{State equation}$$
$$y(t) = h\big(x(t), u(t)\big) \qquad \text{Output equation}$$
where $\dim x(t) = n$, $\dim u(t) = m$, $\dim y(t) = p$
- $x(t)$ is the state of the model, n is the order of the model

Linear input-output and state space models

- General linear (SISO) input-output model of n:th order in continuous time
  $a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + y(t) = b_\ell u^{(\ell)}(t) + \ldots + b_0 u(t)$,
  where $n \geq \ell$ and (a) denotes the a:th time-derivative
- Transferred to a first order differential equation by setting $x_i(t) := y^{(i-1)}(t), i = 1, \ldots, n$ and by doing additional tricks if needed
- Linear state space model
  $$\dot{x}(t) = Ax(t) + Bu(t)$$
  $$y(t) = Cx(t) + Du(t)$$
  - $\dim A = n \times n$ (system matrix)
  - $\dim B = n \times m$ (control matrix)
  - $\dim C = p \times n$ (output matrix)
  - $\dim D = p \times m$ (feedforward matrix)

Laplace transform

- The Laplace transform of function $f(t)$ ($f(t) = 0$ when $t < 0$) is $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$
  where s is a complex variable (”frequency”)

Function	L transform
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
…	…

- With dynamic systems, it is usually assumed that
  $$f(0) = f'(0) = f''(0) = f'''(0) = \cdots = 0$$
  – ”initial state of a linearized model = equilibrium point”
  => deviation of the state from the equilibrium = 0
- Simplifies the transform: $f^{(n)}(t) \Rightarrow s^n F(s)$

Transfer function

- General linear input-output model in continuous time
  $a_n y^{(n)}(t) + \cdots + y(t) = b_m u^{(m)}(t) + \ldots + b_0 u(t),\ n \geq m$
- Applying Laplace transform on both sides →
  $$Y(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + 1} U(s)$$
  The quotient is called the transfer function $G(s)$ of the system
- Model type of a dynamic system
- Algebraic equation (cf. differential equation)
- Complex valued Function of a complex variable
  - Frequency domain (Laplace domain) model (cf. time-domain)
- Roots of the denominator in the transfer function are called the poles of the transfer function

Transfer function corresponding to a linear state space model

- Linear state space model

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

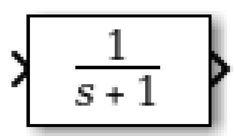
- Using Laplace transform provides

$$G(s) = C(sI - A)^{-1}B + D$$

- Algebraic modifications…

$$G(s) = \frac{(\dots)}{(\dots \det(sI - A) \dots)}$$

- The poles of the transfer function correspond to the eigenvalues of the system matrix A
- Simulink: Transfer Fcn



Equilibrium state and point

- Let $u(t) = u_0$ (constant); where will $x(t)$ and $y(t)$ converge or will they?
- Equilibrium state x_0: $f(x_0, u_o) = 0$
  - one, many, or no solutions
- (x_0, u_0) is an equilibrium point
  - often desirable to get the system into an equilibrium point
- The output of the equilibrium point is $y_0 = h(x_0, u_0)$
- In a linear system
  - origin $(0, 0)$ is always an equilibrium point of the system
  - if (x_0, u_0) is an equilibrium point, then so is (kx_0, ku_0) is for all $k \in \mathbb{R}$
  - if A is invertible, then for every control u_0, there is exactly one equilibrium state $x_0 = -A^{-1}Bu_0$

Linearization

- Consider a nonlinear system (cf. slide "*Input-output and state space models*") in an equilibrium (x_0, u_0) and deviances $\Delta x(t) = x(t) - x_0$, $\Delta y(t) = y(t) - y_0$ and $\Delta u(t) = u(t) - u_0$
- It holds that

$$\frac{d}{dt}\Delta x(t) \approx A'\Delta x(t) + B'\Delta u(t)$$
$$\Delta y(t) \approx C'\Delta x(t) + D'\Delta u(t)$$

where

$$A' = \frac{\partial f}{\partial x}, B' = \frac{\partial f}{\partial u}, C' = \frac{\partial h}{\partial x}, D' = \frac{\partial h}{\partial u}$$

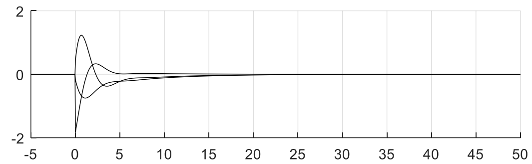
evaluated at (x_0, u_0)

- Linearized model is utilized when examining, e.g., stability or controllability of a nonlinear system

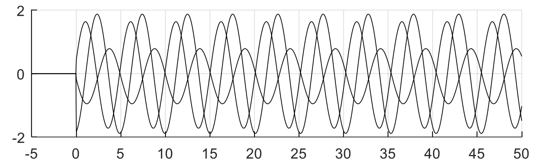
About stability

- Is related to the equilibrium point (x_0, u_0).
- If an equilibrium point is reached, the system will stay in the point regardless of its nature

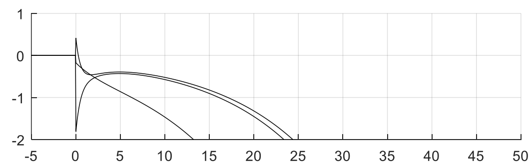
(I) Asymptotically stable



(II) Stable



(III) Unstable



- Local I or II – Stability behavior I or II only when the state is near the equilibrium point
- Global I or II – Stability behavior I or II independent of the current state

About the stability of linear systems 1/2

- Consider a linear dynamic system $\dot{x}(t) = Ax(t) + Bu(t)$, s.t., $\dim x = n$ and assume a constant control u_0 and initial state $x(0)$
- The solution of the system is

$$
\begin{aligned}
x_1(t) &= \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t} + \cdots + \alpha_n e^{\lambda_n t} + k_1 \\
x_2(t) &= \beta_1 e^{\lambda_1 t} + \beta_2 e^{\lambda_2 t} + \cdots + \beta_n e^{\lambda_n t} + k_2 \\
&\vdots \\
x_n(t) &= \nu_1 e^{\lambda_1 t} + \nu_2 e^{\lambda_2 t} + \cdots + \nu_n e^{\lambda_n t} + k_n
\end{aligned}
$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the system matrix A, i.e., $\det(\lambda I - A) = 0$

- $e^{\lambda t} = e^{\mathrm{Re}(\lambda) t}(\cos(\mathrm{Im}(\lambda)\, t) + i \sin(\mathrm{Im}(\lambda)\, t))$

About the stability of linear systems 2/2

- Real parts of the eigenvalues of the system matrix A determine the behavior of the solution and consequently the nature of the equilibrium point $(u_0, x_0 = -A^{-1}Bu_0)$
  - All $\mathrm{Re}(\lambda) < 0$ → Asymptotically stable
  - At least one $\mathrm{Re}(\lambda) > 0$ → Unstable
  - All $\mathrm{Re}(\lambda) \leq 0$ and
    - only unique solutions with $\mathrm{Re}(\lambda) = 0$ → Stable
    - non-unique solutions with $\mathrm{Re}(\lambda) = 0$ → Unstable ($t\cos(\lambda t)$)
- In the linear case, stability is an attribute of the whole system (global), and it does not depend on the values of the states or controls
  - In linear systems, the nature of all the equilibrium points (infinite amount) is same
- In the nonlinear case, stability/unstability/asymptotical stability can be only determined locally for an equilibrium point

Stability of a transfer function

- Applying the Laplace transform for a linear state space model yields
$$G(s) = C(sI - A)^{-1}B + D,$$
i.e., the poles of the transfer function (tf) correspond to the eigenvalues of system matrix A
- The input-output model provided by the tf $G(s)$ is
  - Asymptotically stable, if the roots of the denominator in the tf, i.e., the poles of the tf, lie strictly on the left half of the complex plane
  - Stable, if 1) the poles lie on the left half of the complex plane, and 2) some of the poles are on the imaginary axis and they are unique
  - Unstable, if even one of the poles lie on the right half of the complex plane
  - Unstable, if there are non-unique poles on the imaginary axis

Definition of controllability

System is **controllable**

$\Leftrightarrow$

There exists a control which can drive the system from an arbitrary initial state to any state within a finite time interval

- If a system (open loop) is controllable, a state feedback controller can be constructed and the poles of the resulting feedback system can be selected arbitrarily, e.g., such that the feedback system is asymptotically stable

Testing of controllability

- Difficult for nonlinear systems (linearization!)
- Linear systems: Time-invariant continuous time linear system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

is controllable if and only if the $n \times nm$ matrix

$$Q_c = [B|AB|A^2B|\dots|A^{n-1}B]$$

has a rank of n ($n = \dim x$, $m = \dim u$)

- Rank = number of linearly independent rows/columns
- The matrix Q_c is called the controllability matrix
- Holds also for discrete time systems as well

Interpretation of controllability

- Consider a discrete time system
$$x(t+1) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$
- Assume that initial state x_0 is given
- The state at time n (n = order of the system) is
$$x(n) = A^n x_0 + \sum_{k=0}^{n-1} A^{n-k-1} Bu(k) = A^n x_0 + Q_c \begin{bmatrix} u(n-1) \\ \vdots \\ u(0) \end{bmatrix}$$
- If the rank of the controllability matrix is n, then every vector x of $\mathbb{R}^n$ can be represented in a form
$$x = A^n x_0 + Q_c \begin{bmatrix} u(n-1) \\ \vdots \\ u(0) \end{bmatrix},$$
i.e., with a suitable choice of controls, the system can be driven from its initial state x_0 to a desired state $x(n)$
- The solution (i.e., controls) is not unique, if there is more than one control