

Advanced probabilistic methods - Sketch

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$$\mathcal{L}(q) = \mathbb{E}_q[\log p(\mathbf{X}, \mathbf{Z})] - \mathbb{E}_q[\log q(\mathbf{Z})] \quad (1)$$

$$\mathbb{E}_q \log p(\mathbf{X}, \mathbf{Z}) = \mathbb{E}_q [\log p(\tau) + \log p\theta + \log p(\mathbf{z} \mid \tau) + \log p(\mathbf{x} \mid \mathbf{z}, \theta)] \quad (2)$$

$$= \mathbb{E}_{q(\tau)} [\log p(\tau)] + \mathbb{E}_{q(\theta)} [\log p\theta] + \mathbb{E}_{q(\mathbf{z})q(\tau)} [\log p(\mathbf{z} \mid \tau)] + \mathbb{E}_{q(\mathbf{z})q(\theta)} [\log p(\mathbf{x} \mid \mathbf{z}, \theta)] \quad (3)$$

$$\mathbb{E}_q[\log q(\mathbf{Z})] = \mathbb{E}_q [-\log q(\mathbf{z}) - \log q(\tau) - \log q(\theta)] \quad (4)$$

$$= -\mathbb{E}_{q(\mathbf{z})} [\log q(\mathbf{z})] - \mathbb{E}_{q(\tau)} [\log q(\tau)] - \mathbb{E}_{q(\theta)} [\log q(\theta)] \quad (5)$$

$$\mathcal{L}(q) = \mathbb{E}_{q(\tau)} [\log p(\tau)] + \mathbb{E}_{q(\theta)} [\log p(\theta)] + \mathbb{E}_{q(\mathbf{z})q(\tau)} [\log p(\mathbf{z} \mid \tau)] + \mathbb{E}_{q(\mathbf{z})q(\theta)} [\log p(\mathbf{x} \mid \mathbf{z}, \theta)] \quad (6)$$

$$- \mathbb{E}_{q(\mathbf{z})} [\log q(\mathbf{z})] - \mathbb{E}_{q(\tau)} [\log q(\tau)] - \mathbb{E}_{q(\theta)} [\log q(\theta)] \quad (7)$$

$$p(\theta) = \mathcal{N}(\theta \mid 0, \beta_0^{-1}) \quad (8)$$

$$\mathbb{E}_{q(\theta)} [\log p(\theta)] = \mathbb{E}_{q(\theta)} [\log \mathcal{N}(\theta \mid 0, \beta_0^{-1})] \quad (9)$$

$$\propto \mathbb{E}_{q(\theta)} \left[-\frac{1}{2} \theta^2 \beta_0 \right] \quad (10)$$

$$= -\frac{1}{2} \mathbb{E}_{q(\theta)} [\theta^2] \beta_0 \quad (11)$$

$$\mathbb{E}_{q(\theta)} [\theta^2] = \text{var}(\theta) + \mathbb{E}_{q(\theta)} [\theta]^2 \quad (12)$$

$$= \beta_2^{-1} + m_2^2 \quad (13)$$

$$\mathbb{E}_{q(\theta)} [\log p(\theta)] \propto -\frac{\beta_0}{2} (\beta_2^{-1} + m_2^2) \quad (14)$$

$$q(\theta) = \mathcal{N}(\theta \mid m_2, \beta_2^{-1}) \quad (15)$$

$$\mathbb{E}_{q(\theta)} [\log q(\theta)] = \mathbb{E}_{q(\theta)} [\log \mathcal{N}(\theta \mid m_2, \beta_2^{-1})] \quad (16)$$

$$\propto \mathbb{E}_{q(\theta)} \left[-\frac{1}{2} (\theta - m_2)^2 \beta_2 \right] \quad (17)$$

$$= -\frac{1}{2} \mathbb{E}_{q(\theta)} [(\theta - m_2)^2] \beta_2 \quad (18)$$

$$= -\frac{1}{2} \mathbb{E}_{q(\theta)} [(\theta^2 - 2\theta m_2 + m_2^2)] \beta_2 \quad (19)$$

$$\propto -\frac{1}{2} \mathbb{E}_{q(\theta)} [\theta^2 - 2\theta m_2] \beta_2 \quad (20)$$

$$= -\frac{\beta_2}{2} (\mathbb{E}_{q(\theta)} [\theta^2] - \mathbb{E}_{q(\theta)} [2\theta m_2]) \quad (21)$$

$$= -\frac{\beta_2}{2} (\beta_2^{-1} + m_2^2 - 2m_2^2) \quad (22)$$

$$= -\frac{\beta_2}{2} (\beta_2^{-1} - m_2^2) \quad (23)$$

$$= \frac{m_2^2 \beta_2}{2} - \frac{1}{2} \quad (24)$$

$$(25)$$

$$p(x \mid z, \theta) = \prod_{n=1}^N \mathcal{N}(x_n \mid 0, 1)^{z_{n1}} \mathcal{N}(x_n \mid \theta, 1)^{z_{n2}} \quad (26)$$

$$E_{q(z)q(\theta)} [\log p(x \mid z, \theta)] = E_{q(z)q(\theta)} \left[\sum_{n=1}^N z_{n1} \log \mathcal{N}(x_n \mid 0, 1) + \sum_{n=1}^N z_{n2} \log \mathcal{N}(x_n \mid \theta, 1) \right] \quad (27)$$

$$\propto E_{q(z)q(\theta)} \left[\sum_{n=1}^N z_{n1} \left(-\frac{1}{2} x_n^2 \right) + \sum_{n=1}^N z_{n2} \left(-\frac{1}{2} \right) (x_n - \theta)^2 \right] \quad (28)$$

$$= E_{q(z)q(\theta)} \left[\sum_{n=1}^N z_{n1} \left(-\frac{1}{2} x_n^2 \right) + \sum_{n=1}^N z_{n2} \left(-\frac{1}{2} \right) (x_n^2 - 2x_n\theta + \theta^2) \right] \quad (29)$$

$$= \sum_{n=1}^N E_{q(z)} [z_{n1}] \left(-\frac{1}{2} x_n^2 \right) + \sum_{n=1}^N E_{q(z)} [z_{n2}] \left(-\frac{1}{2} \right) (x_n^2 - 2x_n E_{q(\theta)} [\theta] + E_{q(\theta)} [\theta^2]) \quad (30)$$

$$E_{q(z)} [z_{nk}] = r_{nk} \quad (31)$$

$$E_{q(\theta)} [\theta] = m_2 \quad (32)$$

$$E_{q(\theta)} [\theta^2] = \beta_2^{-1} + m_2^2 \quad (33)$$

$$E_{q(z)q(\theta)} [\log p(x \mid z, \theta)] \propto \sum_{n=1}^N r_{n1} \left(-\frac{1}{2} x_n^2 \right) + \sum_{n=1}^N r_{n2} \left(-\frac{1}{2} \right) (x_n^2 - 2x_n m_2 + \beta_2^{-1} + m_2^2) \quad (34)$$

$$= \sum_{n=1}^N r_{n1} \left(-\frac{1}{2} x_n^2 \right) + \sum_{n=1}^N r_{n2} \left(-\frac{1}{2} \right) ((x_n - m_2)^2 + \beta_2^{-1}) \quad (35)$$

$$p(x \mid M_1) = \int p(x, \theta \mid M_1) d\theta = \int p(x \mid \theta, M_1) p(\theta \mid M_1) d\theta \quad (36)$$

$$p(\theta \mid M_1) = \text{Beta}(1, 1) \quad (37)$$

$$p(x \mid \theta, M_1) = \theta^k (1 - \theta)^{N-k}, \quad |k = \# \text{black pebbles}| \quad (38)$$

$$p(x \mid M_1) = \int \theta^k (1 - \theta)^{N-k} \text{Beta}(1, 1) d\theta \quad (39)$$