Advanced probabilistic methods - Sketch

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March 17, 2022

$$\mathcal{L}(q) = \mathbb{E}_q[\log p(\mathbf{X}, \mathbf{Z})] - \mathbb{E}_q[\log q(\mathbf{Z})] \tag{1}$$

$$\mathbb{E}_q \log p(\mathbf{X}, \mathbf{Z}) = \mathbb{E}_q \left[\log p(\tau) + \log p\theta + \log p(\mathbf{z} \mid \tau) + \log p(\mathbf{x} \mid \mathbf{z}\theta)\right] \tag{2}$$

$$= \mathbb{E}_{q(\tau)} \left[\log p(\tau)\right] + \mathbb{E}_{q(\theta)} \left[\log p\theta\right] + \mathbb{E}_{q(\mathbf{z})q(\tau)} \left[\log p(\mathbf{z} \mid \tau)\right] + \mathbb{E}_{q(\mathbf{z})q(\theta)} \left[\log p(\mathbf{x} \mid \mathbf{z}, \theta)\right] \tag{3}$$

$$\mathbb{E}_q[\log q(\mathbf{Z})] = \mathbb{E}_q \left[-\log q(\mathbf{z}) - \log q(\tau) - \log q(\theta)\right] \tag{4}$$

$$= -\mathbb{E}_{q(\mathbf{z})} \left[\log q(\mathbf{z})\right] - \mathbb{E}_{q(\tau)} \left[\log q(\tau)\right] - \mathbb{E}_{q(\theta)} \left[\log q(\theta)\right] \tag{5}$$

$$\mathcal{L}(q) = \mathbb{E}_{q(\tau)} \left[\log p(\tau)\right] + \mathbb{E}_{q(\theta)} \left[\log p(\theta)\right] + \mathbb{E}_{q(\mathbf{z})q(\tau)} \left[\log p(\mathbf{z} \mid \tau)\right] + \mathbb{E}_{q(\mathbf{z})q(\theta)} \left[\log p(\mathbf{x} \mid \mathbf{z}, \theta)\right] \tag{6}$$

$$- \mathbb{E}_{q(\mathbf{z})} \left[\log q(\mathbf{z})\right] - \mathbb{E}_{q(\tau)} \left[\log q(\tau)\right] - \mathbb{E}_{q(\theta)} \left[\log q(\theta)\right] \tag{7}$$

$$p(\theta) = \mathcal{N}(\theta \mid 0, \beta_0^{-1}) \tag{8}$$

$$\mathbb{E}_{q(\theta)} \left[\log p(\theta)\right] = \mathbb{E}_{q(\theta)} \left[\log \mathcal{N}(\theta \mid 0, \beta_0^{-1})\right] \tag{9}$$

$$\propto \mathbb{E}_{q(\theta)} \left[-\frac{1}{2}\theta^2 \beta_0\right] \tag{10}$$

$$= -\frac{1}{2}\mathbb{E}_{q(\theta)} \left[\theta^2\right] \beta_0 \tag{11}$$

$$\mathbb{E}_{q(\theta)} \left[\theta^2\right] = \mathrm{var}(\theta) + \mathbb{E}_{q(\theta)} \left[\theta\right]^2 \tag{12}$$

$$= \beta_2^{-1} + m_2^2 \tag{13}$$

$$\mathbb{E}_{q(\theta)} \left[\log p(\theta)\right] \propto -\frac{\beta_0}{2}(\beta_2^{-1} + m_2^2) \tag{14}$$

$$q(\theta) = \mathcal{N}(\theta \mid m_2, \beta_2^{-1}) \tag{15}$$

$$\mathbb{E}_{q(\theta)} \left[\log q(\theta)\right] = \mathbb{E}_{q(\theta)} \left[\log \mathcal{N}(\theta \mid m_2, \beta_2^{-1})\right] \tag{16}$$

$$\propto \mathbb{E}_{q(\theta)} \left[-\frac{1}{2}(\theta - m_2)^2 \beta_2\right] \tag{17}$$

$$= -\frac{1}{2}\mathbb{E}_{q(\theta)} \left[(\theta - m_2)^2\right] \beta_2 \tag{18}$$

$$= -\frac{1}{2}\mathbb{E}_{q(\theta)} \left[(\theta^2 - 2\theta m_2 + m_2^2\right] \beta_2 \tag{19}$$

$$\propto -\frac{1}{2}\mathbb{E}_{q(\theta)} \left[\theta^2 - 2\theta m_2\right] \beta_2 \tag{20}$$

$$= -\frac{\beta_2}{2}(\mathbb{E}_{q(\theta)} \left[\theta^2\right] - \mathbb{E}_{q(\theta)} \left[2\theta m_2\right]) \tag{21}$$

$$= -\frac{\beta_2}{2}(\beta_2^{-1} + m_2^2 - 2m_2^2) \tag{22}$$

$$= -\frac{\beta_2}{2}(\beta_2^{-1} - m_2^2) \tag{23}$$

$$= \frac{m_2^2 \beta_2}{2} - \frac{1}{2} \tag{24}$$

$$\tag{25}$$

$$p(x \mid z, \theta) = \prod_{n=1}^{N} \mathcal{N}(x_n \mid 0, 1)^{z_{n1}} \mathcal{N}(x_n \mid \theta, 1)^{z_{n2}} \tag{26}$$

$$E_{q(z)q(\theta)}\left[\log p(x \mid z, \theta)\right] = E_{q(z)q(\theta)}\left[\sum_{n=1}^{N} z_{n1} \log \mathcal{N}(x_n \mid 0, 1) + \sum_{n=1}^{N} z_{n2} \log \mathcal{N}(x_n \mid \theta, 1)\right] \tag{27}$$

$$\propto E_{q(z)q(\theta)}\left[\sum_{n=1}^{N} z_{n1}\left(-\frac{1}{2}x_n^2\right) + \sum_{n=1}^{N} z_{n2}\left(-\frac{1}{2}\right)(x_n - \theta)^2\right] \tag{28}$$

$$= E_{q(z)q(\theta)}\left[\sum_{n=1}^{N} z_{n1}\left(-\frac{1}{2}x_n^2\right) + \sum_{n=1}^{N} z_{n2}\left(-\frac{1}{2}\right)(x_n^2 - 2x_n\theta + \theta^2)\right] \tag{29}$$

$$= \sum_{n=1}^{N} E_{q(z)}\left[z_{n1}\right]\left(-\frac{1}{2}x_n^2\right) + \sum_{n=1}^{N} E_{q(z)}\left[z_{n2}\right]\left(-\frac{1}{2}\right)(x_n^2 - 2x_n E_{q(\theta)}\left[\theta\right] + E_{q(\theta)}\left[\theta^2\right]) \tag{30}$$

$$E_{q(z)}\left[z_{nk}\right] = r_{nk} \tag{31}$$
$$E_{q(\theta)}\left[\theta\right] = m_2 \tag{32}$$
$$E_{q(\theta)}\left[\theta^2\right] = \beta_2^{-1} + m_2^2 \tag{33}$$

$$E_{q(z)q(\theta)}\left[\log p(x \mid z, \theta)\right] \propto \sum_{n=1}^{N} r_{n1}\left(-\frac{1}{2}x_n^2\right) + \sum_{n=1}^{N} r_{n2}\left(-\frac{1}{2}\right)(x_n^2 - 2x_n m_2 + \beta_2^{-1} + m_2^2) \tag{34}$$

$$= \sum_{n=1}^{N} r_{n1}\left(-\frac{1}{2}x_n^2\right) + \sum_{n=1}^{N} r_{n2}\left(-\frac{1}{2}\right)((x_n - m_2)^2 + \beta_2^{-1}) \tag{35}$$

We denote x_i as the number of black pebbles in the ith bag and n the total amount of pebbels.

$$x_i \sim \text{Bin}(x_i \mid n, a) = \binom{n}{x_i} a^{x_i}(1 - a)^{x_i} \tag{36}$$

$$a \sim \text{Beta}(a \mid 1, 1) = 1 \tag{37}$$

$$p(x \mid M_1) = \int_0^1 \text{Bin}(x_1 \mid n, a)\text{Bin}(x_2 \mid n, a)\text{Beta}(a \mid 1, 1) \tag{38}$$

$$= \int_0^1 \binom{n}{x_1} a^{x_1}(1 - a)^{n - x_1} \binom{n}{x_2} a^{x_2}(1 - a)^{n - x_2} da \tag{39}$$

$$= \binom{n}{x_1}\binom{n}{x_2} \int_0^1 a^{x_1 + x_2}(1 - a)^{2n - x_1 - x_2} da \tag{40}$$

$$= \binom{n}{x_1}\binom{n}{x_2} \int_0^1 a^{x_1 + x_2 - 1 + 1}(1 - a)^{x_1 + x_2 - 1 + 1} da \tag{41}$$

$$= \binom{n}{x_1}\binom{n}{x_2} \text{B}(x_1 + x_2 + 1, 2n - x_1 - x_2 + 1) \tag{42}$$

For the second model we have

$$x_i \sim \text{Bin}(n, a_i) = \binom{n}{x_i} a_i^{x_i}(1 - a_i)^{n - x_i} \tag{43}$$

$$b \sim \text{Beta}(1, 1) = 1 \tag{44}$$

$$\tag{45}$$

We again note $i = b$ for black marbles and $i = w$ for the white ones.

$$p(x \mid M_2) = \int_0^1 \int_0^1 \mathrm{Bin}(x_1 \mid n, a_1)\mathrm{Bin}(x_2 \mid n, a_2)\mathrm{Beta}(a_1 \mid 1, 1)\mathrm{Beta}(a_2 \mid 1, 1)da_1 da_2 \tag{46}$$

$$= \int_0^1 \int_0^1 \binom{n}{x_1} a_1^{x_1}(1 - a_1)^{n - x_1} \binom{n}{x_2} a_2^{x_2}(1 - a_2)^{n - x_2} da_1 da_2 \tag{47}$$

$$= \binom{n}{x_1}\binom{n}{x_2} \int_0^1 a_1^{x_1}(1 - a_1)^{n - x_1} da_1 \int_0^1 a_2^{x_2}(1 - a_2)^{n - x_2} da_2 \tag{48}$$

$$= \binom{n}{x_1}\binom{n}{x_2} \int_0^1 a_1^{x_1 - 1 + 1}(1 - a_1)^{n - x_1 - 1 + 1} da_1 \int_0^1 a_2^{x_2 - 1 + 1}(1 - a_2)^{n - x_2 - 1 + 1} da_2 \tag{49}$$

$$= \binom{n}{x_1}\binom{n}{x_2} \mathrm{B}(x_1 + 1, n - x_1 + 1)\mathrm{B}(x_2 + 1, n - x_2 + 1) \tag{50}$$

$$\tag{51}$$

We can now formulate the posterior odds as

$$\frac{p(x \mid M_1)}{p(x \mid M_2)} = \frac{\binom{n}{x_1}\binom{n}{x_2}\mathrm{B}(x_1 + x_2 + 1, 2n - x_1 - x_2 + 1)}{\binom{n}{x_1}\binom{n}{x_2}\mathrm{B}(x_1 + 1, n - x_1 + 1)\mathrm{B}(x_2 + 1, n - x_2 + 1)} \tag{52}$$

$$= \frac{\mathrm{B}(x_1 + x_2 + 1, 2n - x_1 - x_2 + 1)}{\mathrm{B}(x_1 + 1, n - x_1 + 1)\mathrm{B}(x_2 + 1, n - x_2 + 1)} \tag{53}$$