Nonlinear Optimization - Homework 2

Christian Segercrantz 481056

November 24, 2021

3.1 FJ and KKT Conditions at Optimal Point

a)

$$\begin{aligned} \text{min.} \quad & -x_1 && (1) \\ \text{subject to:} \quad & x_2 \leq (1-x_1)^3 && (2) \\ & x_1 \geq 0 && (3) \\ & x_2 \geq 0 && (4) \end{aligned}$$

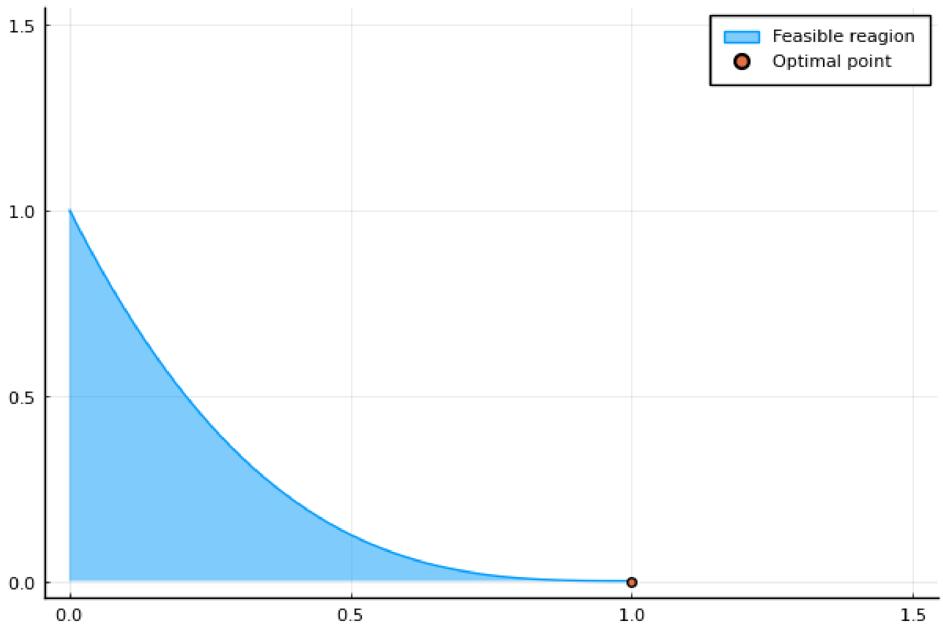

Figure 1: The feasible region of the problem of exercise 3.1. The condition of x_1,$x_2 \geq 0$ is implemented by the limits of the plot.

Figure 1 shows the feasible region for the problem above. Since minimizing $-x_1$ is the same as maximizing x_1, we can identify the optimal point as $\bar{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

b)

We will change around Equation 2 to be $(1-x_1)^3 - x_2 \geq 0$ for it to fit into the FJ conditions. We know that $u_i g_i(\bar{x}) = 0$ for all $i = 1, ..., m$. Hence we can calculate u_i for all $i = 1, ..., m$ as

$$\begin{aligned} u_1 g_1(\bar{x}) = u_1 \cdot (1-1)^3 - 0 = 0 \implies 0 = 0 && (5) \\ u_2 g_2(\bar{x}) = u_2 \cdot 1 = 0 \implies u_2 = 0 && (6) \\ u_3 g_3(\bar{x}) = u_3 \cdot 0 = 0 \implies 0 = 0 && (7) \end{aligned}$$

$$0 = u_0 \nabla f(\bar{x}) + \sum_{i=1}^{m} u_i \nabla g_i(\bar{x}) \tag{8}$$

$$0 = u_0(-1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}) + u_1 \begin{bmatrix} 3(1-x_1)^2 \\ -1 \end{bmatrix} + u_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{9}$$

$$0 = u_0(\begin{bmatrix} -1 \\ 0 \end{bmatrix}) + u_1 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + u_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{10}$$

$$0 = \begin{bmatrix} -u_0 + u_2 \\ -u_1 + u_3 \end{bmatrix} \tag{11}$$

$$\implies \begin{cases} u_0 = u_2 = 0 \\ u_1 = u_3 \end{cases} \tag{12}$$

The point $\bar{x}$ is a FJ point, since we can choose u_1 and u_3 such that FJ conditions are satisfied. U is thus $u = \begin{bmatrix} 0 \\ t \\ 0 \\ t \end{bmatrix}, \quad t > 0.$

c)

The KKT conditions are

$$\nabla f(\bar{x}) + \sum_{i=1}^{m} u_i \nabla g_i(\bar{x}) = 0, \tag{13}$$

$$u_i g_i(x) = 0, \qquad \forall i \tag{14}$$

$$u_i \geq 0, \qquad \forall i \tag{15}$$

which gives us

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix} + u_1 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + u_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{16}$$

$$= \begin{bmatrix} -1 \\ u_3 - u_1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{17}$$

Hence, KKT conditons are not satisfied for any u.

In order for LIQC to hold, the gradient of all active inequality constraints and all equality constraints needs to be linearly indepdendent. We can clearly see that, at $\bar{x}$, $\nabla g_1(\bar{x}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $\nabla g_3(\bar{x}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are linearly dependent.

For Slater's QC to hold, all inequality costraints needs to be convex in the feasible region. Since g_2 and g_3 are linear functions, we know that they are convex. We will examine the Hessian for g_1 to determine it's convexity.

$$H(g_1(x)) = \begin{bmatrix} -6(1-x_1) & 0 \\ 0 & 0 \end{bmatrix} \tag{18}$$

Since the Hessian for g_1 is not positive semi-definite in all of the feasible reagion, e.g. at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, Slater's QC are not satisfied.

3.2 KKT Conditions for a Quadratic Problem

a)

$$\begin{aligned}
\text{min. } & (x_1 + \frac{9}{4})^2 + (x_2 - 2)^2 && (19)\\
\text{subject to: } & x_2 - x_1^2 \geq 0 \iff x_1^2 - x_2 \leq 0 && (20)\\
& x_1 + x_2 \leq 6 \iff x_1 + x_2 - 6 \leq 0 && (21)\\
& x_1 \geq 0 \iff -x_1 \leq 0 && (22)\\
& x_2 \geq 0 \iff -x_2 \leq 0 && (23)
\end{aligned}$$

The KKT conditions for the problem is

$$\begin{aligned}
& \nabla f(\bar{x}) + \sum_{i=1}^{m} u_i \nabla g_i(\bar{x}) = 0 && (24)\\
& u_i g_i(x) = 0, \qquad \forall i && (25)\\
& u_i \geq 0, \qquad \forall i && (26)
\end{aligned}$$

From 25 we get the following at $\bar{x}$:

$$\begin{aligned}
u_1 \left(\left(\frac{3}{2} \right)^2 - 9/4 \right) &= u_1 \cdot 0 = 0 && (27)\\
u_2 \left(\frac{3}{2} + 9/4 - 6 \right) &= -\frac{9}{4} u_2 \implies u_2 = 0 && (28)
\end{aligned}$$

$$\begin{aligned}
0 &= \begin{bmatrix} 2(\bar{x}_1 + \frac{9}{4}) \\ 2(\bar{x}_2 - 2) \end{bmatrix} + u_1 \begin{bmatrix} -2\bar{x}_1 \\ 1 \end{bmatrix} + u_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix} + u_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + u_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} && (29)\\
0 &= \begin{bmatrix} 2(\frac{3}{2} + \frac{9}{4}) \\ 2(\frac{9}{4} - 2) \end{bmatrix} + u_1 \begin{bmatrix} -2\frac{3}{2} \\ 1 \end{bmatrix} + u_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix} + u_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + u_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} && (30)\\
0 &= \begin{bmatrix} 2(\frac{3}{2} + \frac{9}{4}) - \frac{7}{2}u_1 - u_2 + u_3 \\ 2(\frac{9}{4} - 2) + u_1 - u_2 + u_4 \end{bmatrix} && (31)
\end{aligned}$$

b)

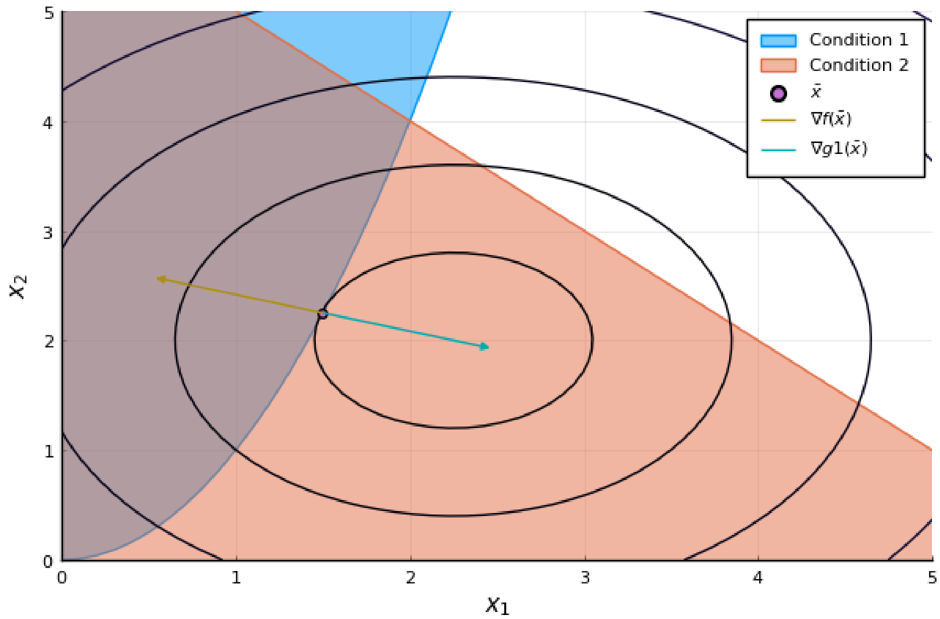

Figure 2

3.3 Lagrangian Dual of a Least-Squares Problem

$$\text{min. } x^\top x \tag{32}$$

$$\text{subject to: } Ax = b \tag{33}$$