

This week's homework [Homework 3](#) is due no later than **Friday 26.11.2021 23:55**.

Problem 8.1: Robust LP with Polyhedral Uncertainty

Consider the following *robust* linear optimization problem with *polyhedral uncertainty*:

$$\min_x c^\top x \quad (1)$$

$$\text{subject to: } \max_{a_i \in \mathcal{P}_i} a_i^\top x \leq b_i, \quad i = 1, \dots, m \quad (2)$$

with decision variables $x \in \mathbb{R}^n$ and polyhedral sets

$$\mathcal{P}_i = \{a_i : C_i a_i \leq d_i\}, \text{ for all } i = 1, \dots, m.$$

The problem data are $c \in \mathbb{R}^n$, $C_i \in \mathbb{R}^{m_i \times n}$, $a_i \in \mathbb{R}^n$, $d_i \in \mathbb{R}^{m_i}$, and $b \in \mathbb{R}^m$. We assume that the polyhedral sets \mathcal{P}_i are nonempty for all $i = 1, \dots, m$. Notice that the problem (1) – (2) is an example of a bilevel optimization problem that we studied in Exercise 7.

Show that the problem (1) – (2) is equivalent to the following linear optimization problem:

$$\min_{x, u} c^\top x \quad (3)$$

$$\text{subject to: } d_i^\top u_i \leq b_i, \quad i = 1, \dots, m \quad (4)$$

$$C_i^\top u_i = x, \quad i = 1, \dots, m \quad (5)$$

$$u_i \geq 0, \quad i = 1, \dots, m \quad (6)$$

with variables $x \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^{m_i}$ for all $i = 1, \dots, m$.

Hint: Replace the inner optimization problems in the constraints (2):

$$\max_{a_i \in \mathcal{P}_i} a_i^\top x, \quad i = 1, \dots, m \quad (7)$$

by writing their Lagrangian dual problems with dual variables u_i for all $i = 1, \dots, m$.

Solution.

We can express the problem (1) – (2) as

$$\min_x c^\top x \quad (8)$$

$$\text{subject to: } g_i(x) \leq b_i, \quad i = 1, \dots, m, \quad (9)$$

where $g_i(x)$ is the optimal value of the linear optimization problem

$$g_i(x) = \max_{a_i} a_i^\top x \quad (10)$$

$$\text{subject to: } C_i a_i \leq d_i, \quad (11)$$

for all $i = 1, \dots, m$. The Lagrangian dual function of (10) – (11) is

$$\theta(u_i) = \max_{a_i} \{a_i^\top x + u_i^\top (d_i - C_i a_i)\} \quad (12)$$

and the corresponding Lagrangian dual problem is

$$\min_{u_i} \theta(u_i) \quad (13)$$

$$\text{subject to: } u_i \geq 0. \quad (14)$$

We can write the problem (13) – (14) as

$$\min_{u_i} \{ \max_{a_i} \{ x^\top a_i + u_i^\top (d_i - C_i a_i) \} \}$$

$$\text{subject to: } u_i \geq 0,$$

which can be rewritten as

$$\begin{aligned} & \min_{u_i} \{ \max_{a_i} \{ x^\top a_i + u_i^\top d_i - u_i^\top C_i a_i \} \} \\ & \text{subject to: } u_i \geq 0 \end{aligned}$$

or

$$\min_{u_i} \{ d_i^\top u_i + \{ \max_{a_i} (x^\top - u_i^\top C_i) a_i \} \} \quad (15)$$

$$\text{subject to: } u_i \geq 0. \quad (16)$$

Now, since the values of a_i are not restricted, the value of the inner maximization problem in (15) becomes ∞ unless $x^\top - u_i^\top C_i = 0$ or $C_i^\top u_i = x$. Thus, we can write (15) – (16) as the following linear optimization problem

$$\min_{u_i} d_i^\top u_i \quad (17)$$

$$\text{subject to: } C_i^\top u_i = x \quad (18)$$

$$u_i \geq 0, \quad (19)$$

which is the dual of (10) – (11). As strong duality holds between linear optimization problems, the optimal value of (17) – (19) is equal to $g_i(x)$ in (10) – (11). Therefore, we have $g_i(x) \leq b_i$ in (9) if and only if there exists a u_i with

$$d_i^\top u_i \leq b_i, \quad C_i^\top u_i = x, \quad u_i \geq 0 \quad (20)$$

for all $i = 1, \dots, m$. Now, replacing $g_i(x)$ in (8) – (9) with the constraints (20) for all $i = 1, \dots, m$, we finally get

$$\begin{aligned} & \min_{x, u} c^\top x \\ & \text{subject to: } d_i^\top u_i \leq b_i, \quad i = 1, \dots, m \\ & \quad C_i^\top u_i = x, \quad i = 1, \dots, m \\ & \quad u_i \geq 0, \quad i = 1, \dots, m. \end{aligned}$$

Problem 8.2: Lagrangian of a Quadratic Optimization Problem

Consider the following quadratic optimization problem with inequality constraints:

$$\min_x \frac{1}{2} x^\top H x + d^\top x \quad (21)$$

$$\text{subject to: } A x \leq b \quad (22)$$

with decision variables $x \in \mathbb{R}^n$. The problem data are $d \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $H \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix. The objective function

$$f(x) = \frac{1}{2} x^\top H x + d^\top x$$

is thus a strictly convex function (why?). Write the Lagrangian dual problem of (21) – (22) and derive its dual explicitly.

Solution

The objective function is strictly convex since its Hessian $\nabla^2 f(x) = H > 0$ for all $x \in \mathbb{R}^n$, which is a necessary and sufficient condition for strict convexity. The Lagrangian dual function of (21) – (22) can be written as

$$\theta(u) = \min_x \left\{ \frac{1}{2} x^\top H x + d^\top x + u^\top (A x - b) \right\}$$

and the corresponding Lagrangian dual problem becomes

$$\max_u. \{ \theta(u) : u \geq 0 \}$$

or

$$\max_u. \left\{ -u^\top b + \min_x. \left\{ \frac{1}{2} x^\top H x + d^\top x + u^\top A x \right\} \right\} \quad (23)$$

$$\text{subject to: } u \geq 0 \quad (24)$$

Let us define

$$g(x) = \frac{1}{2} x^\top H x + d^\top x + u^\top A x$$

which allows us to write the inner minimization problem of (23) as

$$\min_x. g(x) \quad (25)$$

We can solve the inner minimization problem (25) by taking the gradient of $g(x)$ and setting it to zero:

$$\begin{aligned} \nabla g(x) &= \nabla \left(\frac{1}{2} x^\top H x + d^\top x + u^\top A x \right) \\ &= Hx + d + A^\top u = 0. \end{aligned} \quad (26)$$

Since $H > 0$ is invertible, we get the unique optimal primal solution from (26):

$$x = -H^{-1}(d + A^\top u) \quad (27)$$

Moreover, by multiplying (26) with x^\top we get

$$x^\top H x + x^\top d + x^\top A^\top u = 0$$

which can be written as

$$x^\top H x + d^\top x + u^\top A x = 0 \quad (28)$$

Now, by first substituting (28) to the Lagrangian dual problem (23) – (24), we get

$$\max_u. \left\{ -u^\top b + \min_x. \left\{ -\frac{1}{2} x^\top H x \right\} \right\} \quad (29)$$

$$\text{subject to: } u \geq 0 \quad (30)$$

and by further substituting (27) to (29) – (30), the Lagrangian dual becomes

$$\max_u. \left\{ -u^\top b - \frac{1}{2} (-H^{-1}(d + A^\top u))^\top H (-H^{-1}(d + A^\top u)) \right\} \quad (31)$$

$$\text{subject to: } u \geq 0 \quad (32)$$

Let us define the function inside (31) as

$$h(u) = -u^\top b - \frac{1}{2} (-H^{-1}(d + A^\top u))^\top H (-H^{-1}(d + A^\top u))$$

Noticing that $(H^{-1})^\top = H^{-1}$ since H is symmetric, we can further simplify $h(u)$ as:

$$\begin{aligned}
 h(u) &= -u^\top b - \frac{1}{2}(d + A^\top u)^\top H^{-1}(d + A^\top u) \\
 &= -u^\top b - \frac{1}{2}(d^\top + u^\top A)(H^{-1}d + H^{-1}A^\top u) \\
 &= -u^\top b - \frac{1}{2}(d^\top H^{-1}d + d^\top H^{-1}A^\top u + u^\top AH^{-1}d + u^\top AH^{-1}A^\top u) \\
 &= -u^\top b - \frac{1}{2}(d^\top H^{-1}d + u^\top AH^{-1}d + u^\top AH^{-1}d + u^\top AH^{-1}A^\top u) \\
 &= -u^\top b - \frac{1}{2}(d^\top H^{-1}d + 2u^\top AH^{-1}d + u^\top AH^{-1}A^\top u) \\
 &= -u^\top b - u^\top AH^{-1}d - \frac{1}{2}d^\top H^{-1}d - \frac{1}{2}u^\top AH^{-1}A^\top u \\
 &= -u^\top (b + AH^{-1}d) - \frac{1}{2}u^\top AH^{-1}A^\top u - \frac{1}{2}d^\top H^{-1}d
 \end{aligned} \tag{33}$$

Now, by defining

$$D = AH^{-1}A^\top \quad \text{and} \quad c = b + AH^{-1}d$$

we can write (33) as

$$h(u) = -c^\top u - \frac{1}{2}u^\top Du - \frac{1}{2}d^\top H^{-1}d \tag{34}$$

and substituting (34) back to the Lagrangian dual problem (31), it becomes

$$\max_u \left\{ -c^\top u - \frac{1}{2}u^\top Du - \frac{1}{2}d^\top H^{-1}d : u \geq 0 \right\} \tag{35}$$

Now, we can further notice that $(-1/2)d^\top H^{-1}d$ is a constant, so it has no effect on the optimization problem (35). We can thus finally write the dual (35) as

$$\begin{aligned}
 &\max_u \quad -c^\top u - \frac{1}{2}u^\top Du \\
 &\text{subject to: } u \geq 0
 \end{aligned}$$

or, as a minimization problem:

$$\begin{aligned}
 &\min_u \quad \frac{1}{2}u^\top Du + c^\top u \\
 &\text{subject to: } u \geq 0
 \end{aligned}$$

which is of similar form as the primal problem but simpler constraint structure.

Problem 8.3: Duality in Linear Optimization

Consider the following linear optimization problem:

$$\min_x c^\top x \tag{36}$$

$$\text{subject to: } Ax = b \tag{37}$$

$$x \geq 0 \tag{38}$$

with decision variables $x \in \mathbb{R}^n$. The problem data are $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$. We will call (36) – (38) the *primal* problem.

(a) Derive the dual problem of the primal (36) – (38) by using Lagrangian duality.

- (b) Show that the dual of the dual problem derived in part (a) is equivalent to the primal problem (36) – (38). *Hint:* Use Lagrangian duality.
- (c) Show that weak duality holds between the primal (36) – (38) and its dual problem.

Solution.

- (a) The Lagrangian dual function of (36) – (38) can be written as

$$\theta(v) = \min_{x \geq 0} \{c^\top x - v^\top (Ax - b)\} \quad (39)$$

and the corresponding Lagrangian dual problem becomes

$$\max_v \theta(v) = \max_v \left\{ v^\top b + \min_{x \geq 0} \{(c - A^\top v)^\top x\} \right\} \quad (40)$$

In (40), the inner term must be non-negative: $c - A^\top v \geq 0$, because otherwise the value of the inner minimization problem becomes $-\infty$. Thus, the dual of the LP problem (36) – (38) becomes

$$\max_v v^\top b \quad (41)$$

$$\text{subject to: } A^\top v \leq c \quad (42)$$

- (b) The Lagrangian dual function of the dual (41) – (42) can be written as

$$\theta(x) = \max_v \{v^\top b + x^\top (c - A^\top v)\} \quad (43)$$

and the corresponding dual problem becomes

$$\min_{x \geq 0} \theta(x) = \min_{x \geq 0} \left\{ c^\top x + \max_v \{(b - Ax)^\top v\} \right\} \quad (44)$$

Since the variable vector v is unrestricted, the inner maximization problem becomes ∞ unless $b - Ax = 0$. Thus, the dual of the dual problem becomes

$$\min_x c^\top x$$

$$\text{subject to: } Ax = b$$

$$x \geq 0$$

which is exactly the original primal problem.

- (c) For any pair of feasible primal and dual solutions x and v , respectively, we have

$$v^\top b = v^\top Ax = x^\top A^\top v \leq x^\top c = c^\top x$$

which is exactly the definition of weak duality.