Please read these carefully: please submit your answers to Mycourses before the given deadline as a separate files (.pdf file with the report and Jupyter notebook .ipynb files if there are any). **Do not compress them into a single (zip or equivalent) archive**.

Problem 3.1: FJ and KKT Conditions at Optimal Point

Consider the following optimization problem:

$$\begin{aligned} \text{min. } & -x_1 && (1)\\ \text{subject to: } & x_2 \leq (1-x_1)^3 && (2)\\ & x_1 \geq 0 && (3)\\ & x_2 \geq 0 && (4) \end{aligned}$$

(a) Draw the feasible region of the problem (1) – (4) and identify its optimal point $\overline{x}$.

(b) Is the optimal point $\overline{x}$ a FJ point? Justify your answer.

(c) Are KKT conditions valid at the optimal point $\overline{x}$? Does $\overline{x}$ satisfy either the Linear Independence Constraint Qualification (LIQC) or Slater's CQ?

Problem 3.2: KKT Conditions for a Quadratic Problem

Consider the following optimization problem:

$$\begin{aligned} \text{min. } & (x_1 - \frac{9}{4})^2 + (x_2 - 2)^2 && (5)\\ \text{subject to: } & x_2 - x_1^2 \geq 0 && (6)\\ & x_1 + x_2 \leq 6 && (7)\\ & x_1 \geq 0 && (8)\\ & x_2 \geq 0 && (9) \end{aligned}$$

(a) Write the KKT optimality conditions for the problem (5) – (9) and verify that these conditions hold at the point $\overline{x} = (\frac{3}{2}, \frac{9}{4})$

(b) Draw the feasible region (6) – (9) and verify graphically that the KKT conditions hold at $\overline{x} = (\frac{3}{2}, \frac{9}{4})$.

(c) Justify why the point $\overline{x} = (\frac{3}{2}, \frac{9}{4})$ is a unique global optimal solution.

Problem 3.3: Lagrangian Dual of a Least-Squares Problem

Consider the following least-squares optimization problem with equality constraints:

$$\begin{aligned} \text{min. } & x^\top x && (10)\\ \text{subject to: } & Ax = b && (11) \end{aligned}$$

with decision variables $x \in \mathbb{R}^n$ and problem data $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

(a) Write the Lagrangian dual problem of the primal (10) – (11) by using dual variables $v \in \mathbb{R}^m$.

(b) Derive the optimal dual and primal solutions $\overline{v}$ and $\overline{x}$, respectively, by solving the Lagrangian dual problem of part (a). Does strong duality hold between the primal and the dual problem? *Hint:* consider the Slater's Constraint Qualification.

Problem 3.4: Concavity of Lagrangian Dual Functions

Let $X \subset \mathbb{R}^n$ be a nonempty compact set, and let $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$, and $h : \mathbb{R}^n \to \mathbb{R}^l$ be continuous functions. Consider the following optimization problem.

$$\begin{aligned} \text{min. } & f(x) && (12)\\ \text{subject to: } & g(x) \leq 0 && (13)\\ & h(x) = 0 && (14)\\ & x \in X && (15) \end{aligned}$$

Consider the Lagrangian dual function of the problem (12) – (15):

$$\theta(u, v) := \inf \{f(x) + u^\top g(x) + v^\top h(x) : x \in X\} \tag{16}$$

where $u \in \mathbb{R}^m$ with $u \geq 0$ and $v \in \mathbb{R}^l$. For ease of notation, let us define $w \in \mathbb{R}^{m+l}$ and $\beta : \mathbb{R}^n \to \mathbb{R}^{m+l}$ as follows:

$$w = \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{and} \quad \beta(x) = \begin{pmatrix} g(x) \\ h(x) \end{pmatrix}$$

Using this notation, $w^\top \beta(x) = u^\top g(x) + v^\top h(x)$, and the Lagrangian dual function (16) becomes

$$\theta(w) := \inf \{f(x) + w^\top \beta(x) : x \in X\} \tag{17}$$

Show that the Lagrangian dual function (17) is concave in $\mathbb{R}^{m+l}$.