Problem 9.1: ADMM and Scaled Form ADMM

In this exercise, we derive a scaled form for the Alternating Direction Method of Multipliers (ADMM). Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^m \to \mathbb{R}$ be convex functions. Consider the following optimization problem

$$\min_{x,z}.\ f(x) + g(z) \tag{1}$$

$$\text{subject to: } Ax + Bz = c \tag{2}$$

with variables $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$. Assume that the problem data is $A \in \mathbb{R}^{p\times n}$, $B \in \mathbb{R}^{p\times m}$, and $c \in \mathbb{R}^p$. Notice that the objective function has two independent sets of variables x and z. Let us define the augmented Lagrangian of (1) – (2) as

$$L_\rho(x, z, y) = f(x) + g(z) + y^\top(Ax + Bz - c) + \frac{\rho}{2}||Ax + Bz - c||_2^2. \tag{3}$$

with dual variables $y \in \mathbb{R}^p$ and penalty parameter $\rho > 0$. The augmented Lagrangian (3) can be seen as the (unaugmented) Lagrangian of the problem

$$\min_{x,z}.\ f(x) + g(z) + \frac{\rho}{2}||Ax + Bz - c||_2^2 \tag{4}$$

$$\text{subject to: } Ax + Bz = c \tag{5}$$

The problem (4) – (5) is equivalent to the problem (1) – (2): for any feasible solution (x, z), the additional term in the objective (4) evaluates to zero. Solving the augmented Lagrangian (3) by ADMM consists of the following iterations

$$x^{k+1} = \operatorname*{argmin}_x L_\rho(x, z^k, y^k) \tag{6}$$

$$z^{k+1} = \operatorname*{argmin}_z L_\rho(x^{k+1}, z, y^k) \tag{7}$$

$$y^{k+1} = y^k + \rho(Ax^{k+1} + Bz^{k+1} - c) \tag{8}$$

(a) Motivate a suitable stopping criterion for the ADMM iterations (6) – (8).

(b) Derive the *scaled form* for the ADMM iterations (6) – (8) by defining the *primal residual* r and the *scaled dual variables* u as

$$r = Ax + Bz - c \quad \text{and} \quad u = \frac{y}{\rho} \tag{9}$$

Hint: Apply the definitions or r and u to (3) and rewrite the ADMM iterations (6) – (8) by replacing the original dual variables y by their scaled counterparts u.

Problem 9.2: ADMM for Quadratic Optimization Problems

Consider the following standard form quadratic optimization problem

$$\min_{x}.\ \frac{1}{2}x^\top Px + q^\top x \tag{10}$$
$$\text{subject to: } Ax = b \tag{11}$$
$$x \geq 0 \tag{12}$$

with variables $x \in \mathbb{R}^n$. Assume that $P \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, $q \in \mathbb{R}^n$, $A \in \mathbb{R}^{p\times n}$, and $b \in \mathbb{R}^p$. We can express the problem (10) – (12) in ADMM form as

$$\min_{x,z}.\ f(x) + g(z) \tag{13}$$
$$\text{subject to: } x = z \tag{14}$$

where

$$f(x) = \frac{1}{2}x^\top Px + q^\top x \text{ with } \mathbf{dom}\, f = \{x \in \mathbb{R}^n : Ax = b\}$$

is the original objective with a restricted domain, and $g : \mathbb{R}^n \to \{0, \infty\}$ is the indicator function of the nonnegative orthant $\mathbb{R}^n_+$ corresponding to the constraint $x \geq 0$. Write the augmented Lagrangian for (13) – (14) using the scaled dual variables, and write the corresponding scaled form ADMM iterations using the results of Exercise 9.1.