MS-E2122 - Nonlinear Optimization
Lecture 3

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Outline of this lecture

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Convexity of functions

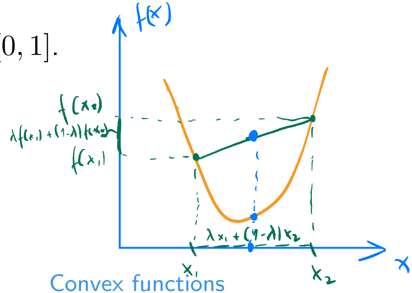
Convexity is a key feature in optimisation. In convex optimisation problems, local optimality always implies global optimality.

Definition 1 (Convexity of a function)

Let $f : S \to \mathbb{R}$ where $S \subseteq \mathbb{R}^n$ is a nonempty convex set. The function f is said to be convex on S if

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$$

for each $x_1, x_2 \in S$ and for each $\lambda \in [0, 1]$.



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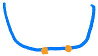

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for each $x_1, x_2 \in S$ and for each $\lambda \in [0, 1]$.

$$\boxed{\begin{array}{ll} \min. & f(x) \\ s.t.: & x \in S \end{array}}$$

$\Downarrow$

convex if
f is convex (min) or
f is concave (max) and
S is a convex set

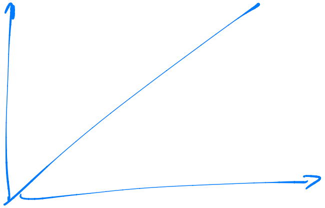
Some remarks:

- f is concave if $-f$ is convex;
- if strict inequality holds, f is strictly convex.

Convexity of functions

Examples of convex functions:

- $f(x) = a^\top x + b$;



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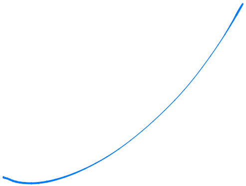

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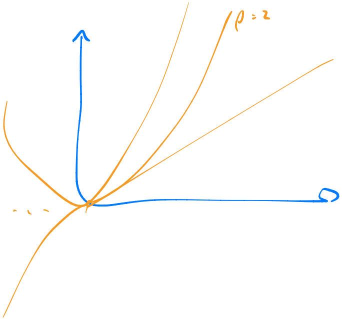

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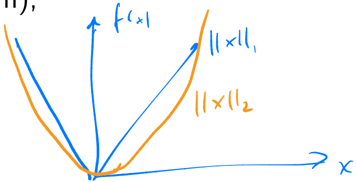

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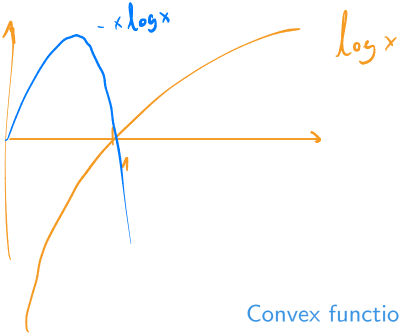

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- $f(x) = ||x||_p$ (p-norm);
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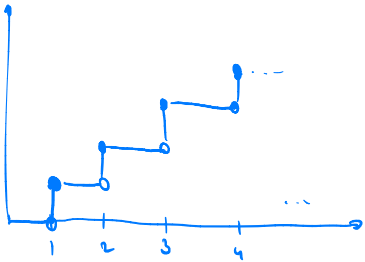
- $f(x) = \max\{x_1, \dots, x_n\}$.



Convexity of functions

Convexity preserving operations:

- let $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ be convex. Then these are convex:

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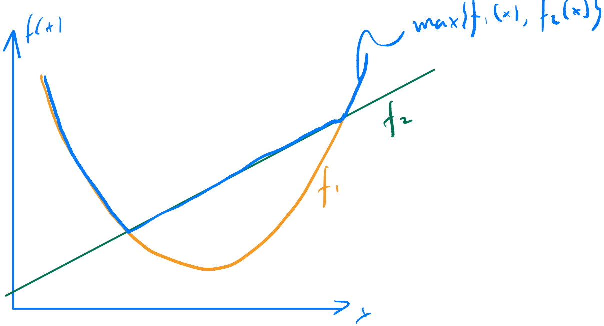
$3x^2 + 4x^3 \quad (x \geq 0)$



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- $f(x) = \frac{1}{g(x)}$ on S, where $g : \mathbb{R}^n \to \mathbb{R}$ is concave and $S = \{x : g(x) > 0\}$;

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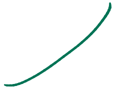

- $f(x) = g(h(x))$, where $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing convex function and $h : \mathbb{R}^n \to \mathbb{R}$ is convex.

$$f'(x) = g'(h(x)) \cdot h'(x)$$

$$\underbrace{f''(x)}_{\substack{\geq 0 \\ \leq 0}} = \underbrace{g''(h(x))}_{\substack{\geq 0 \\ \leq 0}} \cdot \underbrace{h'(x)^2}_{\substack{\geq 0 \\ \geq 0}} + \underbrace{h''(x)}_{\substack{\geq 0 \\ \leq 0}} \cdot \underbrace{g'(h(x))}_{\substack{\geq 0 \\ \leq 0}}$$

g noninc. concave

h concave

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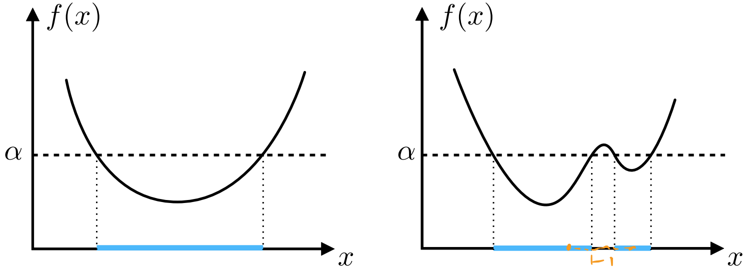
- $f(x) = g(h(x))$, where $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing convex function and $h : \mathbb{R}^n \to \mathbb{R}$ is convex.
- $f(x) = g(h(x))$, where $g : \mathbb{R}^m \to \mathbb{R}$ is convex and $h : \mathbb{R}^n \to \mathbb{R}^m$ is affine: $h(x) = Ax + b$ with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Lower level sets

Definition 2 (Lower level set)

Let $S \subseteq \mathbb{R}^n$ be a nonempty set. The lower level set of $f : \mathbb{R}^n \mapsto \mathbb{R}$ for $\alpha \in \mathbb{R}$ is given by $S_\alpha = \{x \in S : f(x) \leq \alpha\}$.

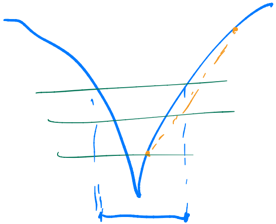


The lower level sets S_α (in blue) of two functions, given a value of α. Notice the nonconvexity of the level set of the nonconvex function (on the right)

Convex functions and lower level sets

Convex level sets are necessary for one to be able to state global optimality conditions.

Convex functions always present convex lower level sets. The converse is not necessarily true.



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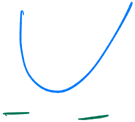
Lemma 3

Let $S \subseteq \mathbb{R}^n$ be a nonempty convex set and $f : S \to \mathbb{R}$ a convex function. Then, any level set S_α with $\alpha \in \mathbb{R}$ is convex.



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f convex $\Rightarrow$ S_α is convex
$\not\Leftarrow$

Proof.

Let $x_1, x_2 \in S_\alpha$. Thus, $x_1, x_2 \in S$ with $f(x_1) \leq \alpha$ and $f(x_2) \leq \alpha$. Let $\lambda \in (0, 1)$ and $x = \lambda x_1 + (1 - \lambda)x_2$. Since S is convex, we have $x \in S$. Now, by the convexity of f, we have

convexity

$$f(x) \leq \lambda f(x_1) + (1 - \lambda)f(x_2) \leq \lambda\alpha + (1 - \lambda)\alpha = \alpha$$

and thus $x \in S_\alpha$. $\square$

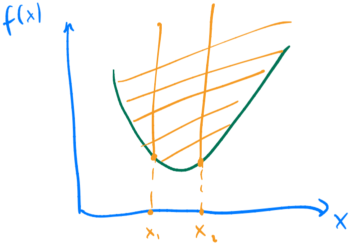
Epigraphs and convex functions

Convex functions can be characterised by examining supporting hyperplanes of their epigraphs.

Definition 4 (Ephigraph)

Let $S \subseteq \mathbb{R}^n$ be a nonempty set and $f : S \to \mathbb{R}$. The epigraph of f is

$$\mathbf{epi}(f) = \{(x, y) : x \in S, y \in \mathbb{R}, y \geq f(x)\} \subseteq \mathbb{R}^{n+1}$$



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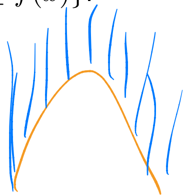

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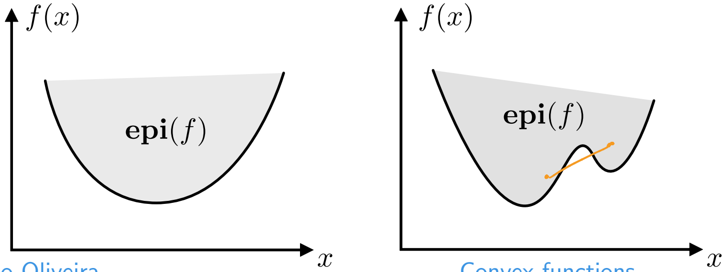

The epigraph $\mathbf{epi}()f$ of a convex function is a convex set (in grey on the left).

Epigraphs and convex functions

Epigraphs can be used to infer the convexity of functions.

Theorem 5

Let $S \subseteq \mathbb{R}^n$ *be a nonempty convex set and* $f : S \to \mathbb{R}$*. Then* f *is convex if and only if* $\mathbf{epi}(f)$ *is a convex set.*

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Proof.

First, suppose f is convex and let $(x_1, y_1), (x_2, y_2) \in \mathbf{epi}(f)$. Then, for $\lambda \in (0, 1)$ we have

f convex $\Rightarrow$ $\mathrm{epi}(f)$ convex.

$$\overset{\text{epi}}{\lambda y_1 + (1-\lambda) y_2 \geq \lambda f(x_1) + (1-\lambda) f(x_2)} \overset{\text{convex.}}{\geq} f(\lambda x_1 + (1-\lambda) x_2).$$

As $\lambda x_1 + (1-\lambda) x_2 \in S$, $(\underbrace{\lambda x_1 + (1-\lambda) x_2}_{x}, \underbrace{\lambda y_1 + (1-\lambda) y_2}_{y}) \in \mathbf{epi}(f)$.

$f(x) \leq y$

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$$\lambda y_1 + (1 - \lambda)y_2 \geq \lambda f(x_1) + (1 - \lambda)f(x_2) \geq f(\lambda x_1 + (1 - \lambda)x_2).$$

As $\lambda x_1 + (1 - \lambda)x_2 \in S$, $(\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2) \in \mathbf{epi}(f)$.

Conversely, suppose $\mathbf{epi}(f)$ is convex. For $x_1, x_2 \in S$. Then $(x_1, f(x_1)) \in \mathbf{epi}(f)$, $(x_2, f(x_2)) \in \mathbf{epi}(f)$ and thus $(\lambda x_1 + (1 - \lambda)x_2, \lambda f(x_1) + (1 - \lambda)f(x_2)) \in \mathbf{epi}(f)$ for $\lambda \in (0, 1)$. These imply that $\lambda f(x_1) + (1 - \lambda)f(x_2) \geq f(\lambda x_1 + (1 - \lambda)x_2)$. $\square$

$f(a)$ y

epi f convex => f convex. $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$

Epigraphs and convex functions

As epigraphs are convex sets, they have supporting hyperplanes at their boundary points, which leads to the notion of subgradients.

Definition 6 (Subgradients)

Let $S \subseteq \mathbb{R}^n$ be a nonempty convex set and $f : S \to \mathbb{R}$ a convex function. Then $\xi \in \mathbb{R}^n$ is a subgradient of f at $\overline{x} \in S$ if

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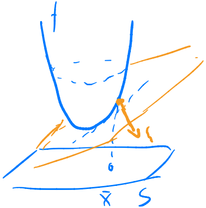
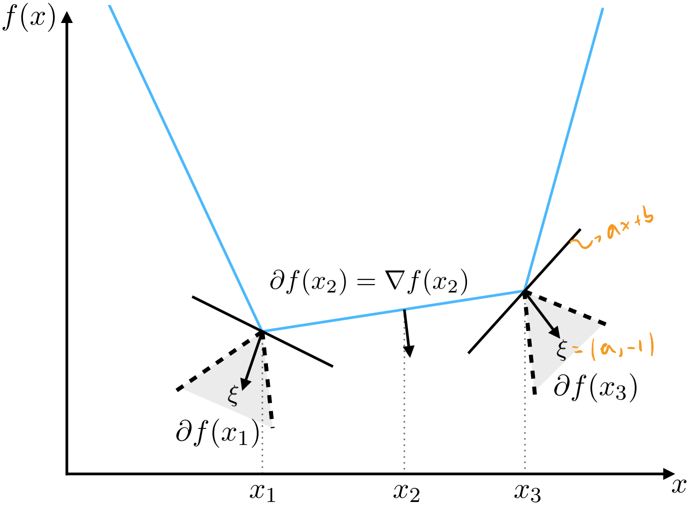


Remark: $\partial f(\overline{x}) = \left\{\xi : f(x) \geq f(\overline{x}) + \xi^\top (x - \overline{x})\right\}$ is the subdifferential of f at $\overline{x}$, which is convex.



A representation of the subdifferential (in grey) for nondifferentiable (x_1 and x_3) and differentiable (x_2) points

Subgradients and supporting hyperplanes

Every convex function has at least one subgradient in the interior of its domain, which can also be used to show convexity of f.

Theorem 7

Let $S \subseteq \mathbb{R}^n$ be a nonempty convex set and $f : S \to \mathbb{R}$ a convex function. Then for all $\overline{x} \in \mathbf{int}(S)$, there exists $\xi \in \mathbb{R}^n$ such that

$$H = \left\{(x, y) : y = f(\overline{x}) + \xi^\top (x - \overline{x})\right\}$$

supports $\mathbf{epi}(f)$ at $(\overline{x}, f(\overline{x}))$. In particular,

$$f(x) \geq f(\overline{x}) + \xi^\top (x - \overline{x}), \forall x \in S.$$

subgradient ineq.

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Proof sketch: use Theorem 5 and the support of convex sets to show that the subgradient inequality holds.

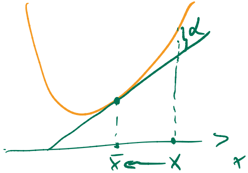
Differentiability and gradients

Let us first define differentiability.

Definition 8

Let $S \subseteq \mathbb{R}^n$ be a nonempty set. $f : S \to \mathbb{R}$ is differentiable at $\overline{x} \in \mathbf{int}(S)$ if there exists, for all $\overline{x} \in \mathbf{int}(S)$, a vector $\nabla f(\overline{x})$, called a gradient vector, and a function $\alpha : \mathbb{R}^n \to \mathbb{R}$ such that

$$f(x) = \underbrace{f(\overline{x}) + \nabla f(\overline{x})^\top (x - \overline{x})}_{\text{1st order}} + \underbrace{||x - \overline{x}||\alpha(\overline{x}; x - \overline{x})}_{\text{error}}$$



where $\lim_{x \to \overline{x}} \alpha(\overline{x}; x - \overline{x}) = 0$

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$$f(x) = f(\overline{x}) + \nabla f(\overline{x})^\top (x - \overline{x}) + ||x - \overline{x}||\alpha(\overline{x}; x - \overline{x})$$

where $\lim_{x \to \overline{x}} \alpha(\overline{x}; x - \overline{x}) = 0$

Remarks:

- This is called the first-order (Taylor series) expansion of f. Without the α term, it is the first-order approximation.
- $\nabla f(\overline{x}) = \left(\frac{\partial f(\overline{x})}{\partial x_1}, \dots, \frac{\partial f(\overline{x})}{\partial x_n}\right)$ is unique.

Differentiability and gradients

If f is convex and differentiable, $\partial(x)$ is the singleton $\{\nabla f(x)\}$.

Lemma 9

Let $S \subseteq \mathbb{R}^n$ be a nonempty convex set and $f : S \to \mathbb{R}$ a convex function. Suppose that f is differentiable at $\overline{x} \in \mathbf{int}(S)$. Then $\partial f(\overline{x}) = \{\nabla f(\overline{x})\}$ is a singleton.

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Proof.

From Theorem 7, $\partial f(\overline{x}) \neq \emptyset$. Moreover, combining the existence of a subgradient ξ and differentiability of f at $\overline{x}$, we obtain:

$d = (x - \overline{x})$

sub. ineq.

$$f(\overline{x} + \lambda d) \geq f(\overline{x}) + \lambda \xi^\top d \quad (1)$$

$$f(\overline{x} + \lambda d) = f(\overline{x}) + \lambda \nabla f(\overline{x})^\top d + \lambda ||d|| \alpha(\overline{x}; \lambda d) \quad (2)$$

diff.

$(\xi - \nabla f(x))^\top (\xi - \nabla f(x)) \leq 0$

$\|\xi - \nabla f(\overline{x})\|_2^2 \not\leq 0$

$\xi = \nabla f(\overline{x})$

Subtracting (2) from (1), we get $0 \geq \lambda(\xi - \nabla f(\overline{x}))^\top d - \lambda ||d|| \alpha(\overline{x}; \lambda d)$. Dividing by $\lambda > 0$ and letting $\lambda \to 0^+$, we obtain $(\xi - \nabla f(\overline{x}))^\top d \leq 0$. Now, by setting $d = \xi - \nabla f(\overline{x})$, it becomes clear that $\xi = \nabla f(\overline{x})$. $\square$

Differentiability and gradients

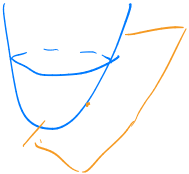
Theorem 10 gives an important characterisation of convex differentiable functions.

(Differentiable) functions are convex if and only if they are always underestimated by affine (first-order) approximations at $\overline{x}$.

Theorem 10

Let $S \subseteq \mathbb{R}^n$ be a nonempty convex open set, and let $f : S \to \mathbb{R}$ be differentiable on S. f is convex if and only if for any $\overline{x} \in S$, we have

$$f(x) \geq f(\overline{x}) + \nabla f(\overline{x})^\top (x - \overline{x}), \; \forall x \in S.$$



Differentiability and gradients

Theorem 10 gives an important characterisation of convex differentiable functions.

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Proof sketch: combine Theorem 7 and Lemma 9.

Remark: this powerful tool, known as affine bounding, is a part of many optimisation algorithms.

Second-order differentiability

A function is twice differentiable if it has a second order expansion.

Definition 11 (second-order differentiability)

Let $S \subseteq \mathbb{R}^n$ be a nonempty set, and let $f : S \to \mathbb{R}$. Then f is twice differentiable at $\overline{x} \in \mathbf{int}(S)$ if there exists, for each $\overline{x} \in S$, a vector $\nabla f(\overline{x}) \in \mathbb{R}^n$, an $n \times n$ symmetric matrix $H(\overline{x})$ (the Hessian), and a function $\alpha : \mathbb{R}^n \to \mathbb{R}$ such that

$$f(x) = \underbrace{f(\overline{x})+\nabla f(\overline{x})^\top(x-\overline{x})+\frac{1}{2}(x-\overline{x})^\top \overset{\nabla^2 f(\overline{x})}{H(\overline{x})}(x-\overline{x})}_{\text{2nd order}}+\underbrace{||x-\overline{x}||^2\alpha(\overline{x};x-\overline{x})}_{\text{error}}$$

where $\lim_{x\to\overline{x}} \alpha(\overline{x};x-\overline{x}) = 0$.

Second-order differentiability

Remarks:

1. Let $f_{ij}(\overline{x}) = \frac{\partial^2 f(\overline{x})}{\partial x_i \partial x_j}$. The Hessian $H(\overline{x})$ at $\overline{x}$ is given by

$$H(\overline{x}) = \begin{bmatrix} f_{11}(\overline{x}) & \dots & f_{1n}(\overline{x}) \\ \vdots & \ddots & \vdots \\ f_{n1}(\overline{x}) & \dots & f_{nn}(\overline{x}) \end{bmatrix}$$

2. The Hessian is positive semi-definite at $\overline{x}$ if $x^\top H(\overline{x})x \geq 0$ for $x \in \mathbb{R}^n$.

$\cup \quad \cap$

↳ $\lambda \geq 0$

↳ ...

$\alpha = f(\bar{x}) + \nabla f(\bar{x})^T (x - \bar{x}) + \frac{1}{2} (x - \bar{x})^T H(\bar{x}) (x - \bar{x})$

Second-order differentiability

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$\mathbb{R}^n \to \mathbb{R}^{m=1}$

Example: $f(x_1, x_2) = 2x_1 + 6x_2 - 2x_1^2 - 3x_2^2 + 4x_1x_2$.

$$\nabla f(\overline{x}) = \begin{bmatrix} 2 - 4\overline{x}_1 + 4\overline{x}_2 \\ 6 - 6\overline{x}_2 + 4\overline{x}_1 \end{bmatrix} \text{ and } H(\overline{x}) = \begin{bmatrix} -4 & \overset{4}{\cancel{-6}} \\ +4 & -6 \end{bmatrix}.$$

Second-order differentiability

Positive semidefinite Hessians are used to infer convexity.

Theorem 12

Let $S \subseteq \mathbb{R}^n$ be a nonempty convex open set, and let $f : S \to \mathbb{R}$ be twice differentiable on S. Then f is convex if and only if the Hessian matrix is positive semidefinite (PSD) at each point in S.

Second-order differentiability

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Remarks:

- Checking for PSD of Hessians can be done efficiently using Gauss-Jordan eliminations;
- Positive definite Hessians ($x^\top H(\overline{x})x > 0$ for $x \in \mathbb{R}^n$) imply strict convexity.

Quasiconvexity almost

Some of the results for convex functions can be generalised to nonconvex functions that possess convex lower level sets.

Definition 13 (quasiconvex functions)

Let $S \subseteq \mathbb{R}^n$ be a nonempty convex set and $f : S \to \mathbb{R}$. Function f is quasiconvex if, for each $x_1, x_2 \in S$ and $\lambda \in (0, 1)$, we have

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \max\{f(x_1), f(x_2)\}.$$

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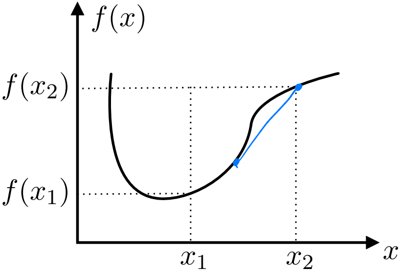
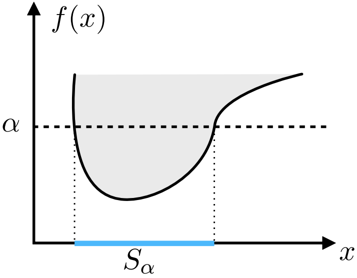
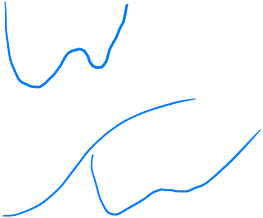






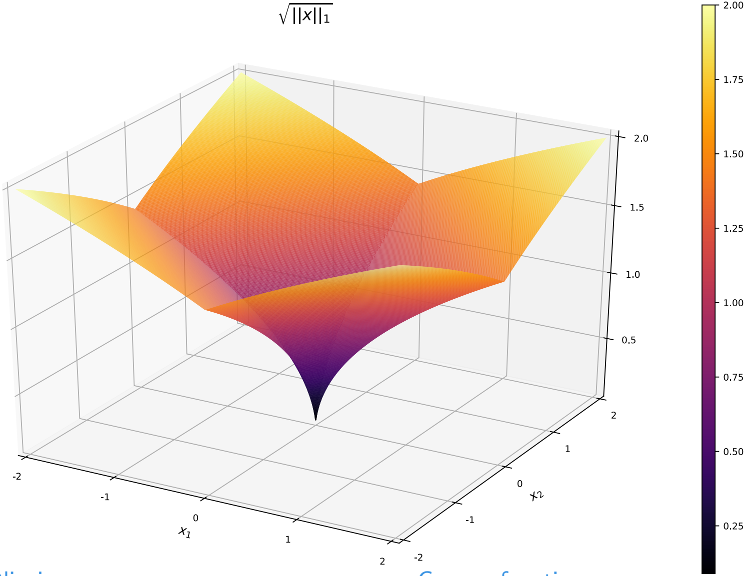
Quasiconvexity

Examples of quasi-convex functions:

- $f(x) = \sqrt{||x||_1}$
- $f(x) = \log x$ is quasilinear for $x > 0$
- $f(x) = \inf \{z \in \mathbb{Z} : z \geq x\}$ is quasilinear
- $f(x_1, x_2) = x_1 x_2$ is quasiconcave on $S = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 > 0\right\}$
- $f(x_1, x_2) = \log(x_1^2 + x_2^2)$

Remark: f is quasiconcave if $-f$ is quasiconvex. Quasilinear functions are both quasiconvex and quasiconcave.

Quasiconvexity



Quasiconvexity

Quasiconvex functions do not necessarily have convex epigraphs.
However, their lower level sets are convex.

Theorem 14

Let $S \subseteq \mathbb{R}^n$ be a nonempty convex set and $f : S \to \mathbb{R}$. Function f is quasiconvex if and only if $S_\alpha = \{x \in S : f(x) \leq \alpha\}$ is convex for all $\alpha \in \mathbb{R}$.

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Proof.

Suppose f is quasiconvex and let $x_1, x_2 \in S_\alpha$. Thus, $x_1, x_2 \in S$ and $\max\{f(x_1), f(x_2)\} \leq \alpha$. Let $x = \lambda x_1 + (1-\lambda)x_2$ for $\lambda \in (0,1)$. As S is convex, $x \in S$. By quasiconvexity of f,

$$f(x) \leq \max\{f(x_1), f(x_2)\} \leq \alpha.$$

Hence, $x \in S_\alpha$ and S_α is convex.

Quasiconvexity

Proof (cont.)

Conversely, assume that S_α is convex for $\alpha \in \mathbb{R}$. Let $x_1, x_2 \in S$, and let $x = \lambda x_1 + (1-\lambda)x_2$ for $\lambda \in (0,1)$. Thus, for $\alpha = \max\{f(x_1), f(x_2)\}$, we have $x_1, x_2 \in S_\alpha$. The convexity of S_α implies that $x \in S_\alpha$, and thus $f(x) \leq \alpha = \max\{f(x_1), f(x_2)\}$, which implies that f is quasiconvex.

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Proof (cont.)

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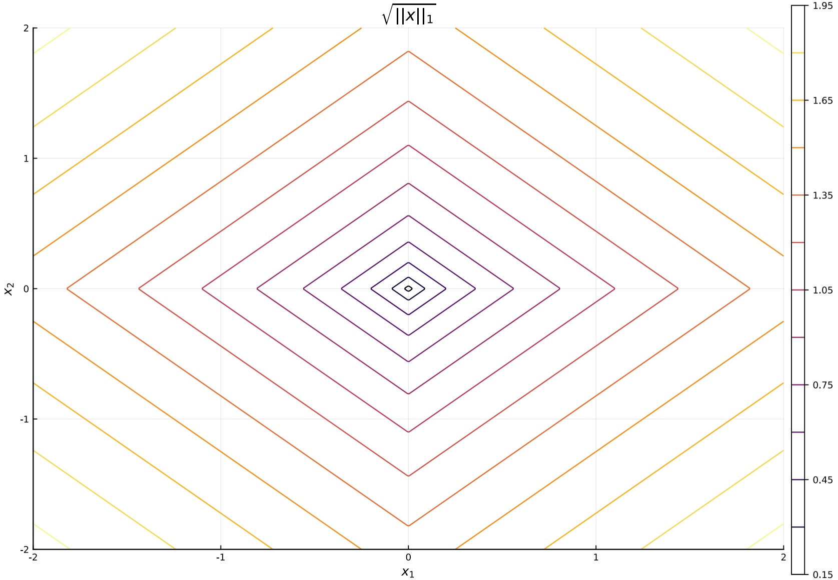
One important property of continuous quasiconvex functions is the following first-order condition.

Theorem 15

Let $S \subseteq \mathbb{R}^n$ be a nonempty open convex set, and let $f : S \to \mathbb{R}$ be differentiable on S. Then f is quasiconvex if and only if, for $x_1, x_2 \in S$ and $f(x_1) \leq f(x_2)$, $\nabla f(x_2)^\top (x_1 - x_2) \leq 0$.

Remark: geometrically, this means that $\nabla f(x_2)$ defines a supporting hyperplane to the (convex) lower level set $S_{f(x_2)}$ at x_2.


$\sqrt{||x||_1}$
2
1
0
-1
-2
-2
-1
0
1
2
x_1
x_2
1.95
1.65
1.35
1.05
0.75
0.45
0.15

Quasiconvexity and pseudoconvexity

For a quasiconvex function, local optimality does not generalise to the entire domain, unless the function is strictly quasiconvex.

Definition 16 (strict quasiconvexity)

Let $S \subseteq \mathbb{R}^n$ be a nonempty convex set and $f : S \to \mathbb{R}$. Function f is strictly quasiconvex if, for each $x_1, x_2 \in S$ with $f(x_1) \neq f(x_2)$ and $\lambda \in (0, 1)$, we have

$$f(\lambda x_1 + (1 - \lambda)x_2) < \max \{f(x_1), f(x_2)\}.$$

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$$f(\lambda x_1 + (1-\lambda)x_2) < \max\{f(x_1), f(x_2)\}.$$

Remark:

1. Notice that the definition precludes the existence of flat spots anywhere else than at extreme points.
2. Pseudoconvex functions are those for which the first-order conditions in Theorem 15 are sufficient for global optimality.