MS-E2122 - Nonlinear optimization Homework 2
Prof. Fabricio Oliveira **DEADLINE: Friday 22.10.2021 23:55**

Please read these carefully: please submit your answers to Mycourses before the given deadline as a separate files (.pdf file with the report and Jupyter notebook .ipynb files if there are any). **Do not compress them into a single (zip or equivalent) archive**.

You are required to submit a single .pdf file containing your written answers to both homework problems and the jupyter notebook .ipynb files. For Homework 2.2 and 2.3, download the skeleton notebook from the course homepage and **rename it according to your student number before submitting it**. For example, if you student number is 112233, rename your notebook as 112233.ipynb.

Problem 2.1: Critical (Stationary) Points of a Function

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x) = x_1^3 - x_1 + x_2^3 - x_2$.

(a) Plot the surface of $z = f(x)$ and conclude whether this function is convex or not.

(b) List the 4 points for which the first-order necessary condition holds (critical points).

(c) Use the Hessian $H(x)$ of $f(x)$ to support your argument in part (a). What can you say about the curvature around the four critical points?

Hint: Eigenvalues of $H(x)$ can be computed using the `eigen()` function from the `Julia` package `LinearAlgebra`. First import the package by `using LinearAlgebra` and then type `?eigen` to get detailed information on how to use this function.

Problem 2.2: Optimality Conditions for Unconstrained Problems

Consider the following unconstrained optimization problem

$$\text{min. } f(x_1, x_2) = 2x_1^2 - x_1x_2 + x_2^2 - 3x_1 + e^{2x_1+x_2} \tag{1}$$

(a) Write the first order necessary optimality conditions. Is this condition also sufficient for optimality? Justify your answer.

(b) Is $\overline{x} = (0, 0)$ an optimal solution? If not, provide the conditions that must be satisfied by a direction d along which the function would decrease.

(c) What is the minimum for f in the direction $d = (1, 0)$ from $\overline{x} = (0, 0)$?

Problem 2.3: Unconstrained Optimization Methods

In this exercise, your task is to implement three unconstrained optimization methods: Gradient Descent, Newton's method, and Conjugate Gradient method. You must also implement the Golden Section line search method to compute optimal step sizes at each iteration.

To test these methods, you are to minimize the following two functions:

$$f(x_1, x_2) = 0.26(x_1^2 + x_2^2) - 0.48x_1x_2 \tag{2}$$

$$g(x_1, x_2) = e^{x_1+3x_2-0.1} + e^{x_1-3x_2-0.1} + e^{-x_1-0.1} \tag{3}$$

You are provided with a skeleton notebook that you can download here. This notebook contains detailed information on what parts of the code you need to complete in order to successfully implement these methods and answer the following questions.

(a) Implement all the three methods (Gradient Descent, Newton's method, and Conjugate Gradient method) by following the instructions in the skeleton notebook. **NOTE: Do not change any function names or constants in the code such as tolerance values. The missing parts of the code you are supposed to complete are indicated with the comment TODO**

(b) Apply all the three methods to function (2) with the starting point $(x_1^0, x_2^0) = (7, 3)$. Report in each case the number of iterations it takes to converge to the optimal solution.

(c) Are there any theoretical justifications for the observed number of iterations in Newton's method and the Conjugate Gradient method in part (b)?

(d) Apply all the three methods to function (3) with the starting point $(x_1^0, x_2^0) = (-4, -2)$. Report for each method the number of iterations it takes to converge to the optimal solution.