Please submit your answers to Mycourses before the given deadline as a separate files (.pdf file with the report and Jupyter notebook .ipynb files if there are any). **Do not compress them into a single (zip or equivalent) archive**.

Problem 1.1: Optimality conditions

Consider the problem

$$\begin{aligned}(P): \quad & \text{min. } (x_1-4)^2+(x_2-6)^2 \\ \text{subject to: } & x_2 \geq x_1^2 \\ & x_2 \leq 4.\end{aligned}$$

Write a necessary condition for optimality and verify that it is satisfied by the point (2,4). Is this point optimal? Justify your answer citing any results from the lectures that you may require.

Problem 1.2: Projections onto a convex set

Assume that $S \subset \mathbb{R}^n$ is a closed convex set. Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ be two points with $x \notin S$ and $y \notin S$. Let $\overline{x} \in S$ and $\overline{y} \in S$ be the unique minimum distance points (i.e., *projections*) of x and y, respectively, in the set S. Show that

$$||\overline{x}-\overline{y}|| \leq ||x-y|| \tag{1}$$

Hint: Use the Closest-point Theorem (Theorem 9) presented in Lecture 2. You may also want to draw a picture by hand to clearly understand the meaning of (1).

Problem 1.3: Convexity of Sets

Are the following sets convex? Justify your answer in each case.

(a) $S = \{x \in \mathbb{R}^n : \alpha \leq a^\top x \leq \beta\}$, where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ are scalars and $a \in \mathbb{R}^n$ is a normal vector.

(b) $S = \{x \in \mathbb{R}^n : \alpha_i \leq x_i \leq \beta_i,\ i = 1, \dots, n\}$, where $\alpha_i \in \mathbb{R}$ and $\beta_i \in \mathbb{R}$ are scalars for all $i = 1, \dots, n$.

(c) $S = \{x \in \mathbb{R}^n : a_1^\top x \leq b_1,\ a_2^\top x \leq b_2\}$, where $a_1 \in \mathbb{R}^n$ and $a_2 \in \mathbb{R}^n$ are normal vectors, and $b_1 \in \mathbb{R}$ and $b_2 \in \mathbb{R}$ are scalars.

(d) $S = \{x \in \mathbb{R}^n : x = A^T y,\ y \in \mathbb{R}^m,\ y \geq 0\}$, where $A \in \mathbb{R}^{m \times n}$.

Problem 1.4: Convexity of Functions

(a) Let $g : \mathbb{R}^m \to \mathbb{R}$ be a convex function, A an $m \times n$ matrix, and b a vector in $\mathbb{R}^m$. Show that

$$f(x) = g(Ax + b)$$

is a convex function.

(b) Let $h : \mathbb{R}^n \to \mathbb{R}$ be a convex function, and let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ be scalars with $\alpha \geq 0$. Show that

$$f(x) = \alpha h(x) + \beta$$

is a convex function. *Hint:* Make use of composition (see Exercise 3.2).

(c) Let A be a positive semidefinite symmetric $n \times n$ matrix and $\beta \in \mathbb{R}$ a scalar with $\beta > 0$. Show that

$$f(x) = e^{\beta x^\top A x}$$

is a convex function. *Hint:* Make use of composition.

In addition, plot the contours of $f : S \to \mathbb{R}$ of $f(x) = e^{\beta x^\top A x}$ in the domain

$$S = \{x = (x_1, x_2) : -3 \leq x_1 \leq 3,\ -3 \leq x_2 \leq 3\}$$

by using values of $\beta = 2$ and $A = \begin{pmatrix} 0.02 & 0 \\ 0 & 0.02 \end{pmatrix}$. *Hint:* You can use the `Plots` function `contour`