Problem 7.1: KKT Conditions for Equality Constrained Problems

Let $X \subset \mathbb{R}^n$ be a nonempty open set, and let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Moreover, let $g_i : \mathbb{R}^n \to \mathbb{R}$ be differentiable for all $i = 1, \dots, m$, and let $h_i : \mathbb{R}^n \to \mathbb{R}$ be differentiable for all $i = 1, \dots, l$. Consider the following optimization problem P:

$$\begin{aligned} (P): \quad & \text{min. } f(x) \\ \text{subject to: } & g_i(x) \leq 0, && i = 1, \dots, m \\ & h_i(x) = 0, && i = 1, \dots, l \\ & x \in X \end{aligned}$$

Let $\overline{x}$ be a feasible solution to P, and let $I = \{i : g_i(\overline{x}) = 0\}$ be the index set of *active* inequality constraints. Also, let $\nabla g_i(\overline{x})$ for $i \in I$ and $\nabla h_i(\overline{x})$ for $i = 1, \dots, l$ be linearly independent (to enforce constraint qualification). Derive KKT conditions for the problem P.

Hint: Notice that $h_i(x) = 0$ can be equivalently replaced by the two inequalities

$$h_i(x) \leq 0 \text{ and } -h_i(x) \leq 0.$$

Problem 7.2: KKT Transformation of a Bilevel Optimization Problem

Consider the following *bilevel* optimization problem:

$$\min_x . \; c_1^\top x + c_2^\top y \tag{1}$$
$$\text{subject to: } Ax + By \leq \alpha \tag{2}$$
$$y \in \underset{y}{\text{argmin}} \;\; c_3^\top y \tag{3}$$
$$\text{subject to: } \;\; Dx + Ey \leq \beta \tag{4}$$

In problem (1) − (4), we seek an optimal value of x knowing that y, which minimizes another optimization problem, depends on the value of x. This is a way of modeling hierarchical decision problems such as Stackelberg competition.

Reformulate the problem (1) − (4) by replacing the constraints (3) − (4) that form the inner optimization problem:

$$\begin{aligned} & y \in \underset{y}{\text{argmin}} \;\; c_3^\top y \\ & \text{subject to: } \;\; Dx + Ey \leq \beta \end{aligned}$$

with the KKT optimality conditions of this problem. You can assume that $\beta \in \mathbb{R}^m$, which implies that (4) has $i = 1, \dots, m$ inequality constraints. Is the resulting problem convex? Justify your answer.

Hint: You can write the constraint (4) as

$$d_i x + e_i y \leq \beta_i, \quad i = 1, \dots, m$$

where d_i and e_i correspond to the ith rows of the matrices D and E, respectively, and β_i is the ith element of the vector $\beta \in \mathbb{R}^m$.

Problem 7.3: Example of a Bilevel Transformation

Consider the following bilevel optimization problem:

$$
\begin{align}
\min_{x}.\ & x - 4y \tag{5}\\
\text{subject to: } & x \geq 0 \tag{6}\\
& y \in \operatorname*{argmin}_{y} y \tag{7}\\
& \quad \text{subject to: } -x - y \leq -3 \tag{8}\\
& \quad\quad -2x + y \leq 0 \tag{9}\\
& \quad\quad 2x + y \leq 12 \tag{10}\\
& \quad\quad -3x + 2y \leq -4 \tag{11}\\
& \quad\quad y \geq 0 \tag{12}
\end{align}
$$

Reformulate the problem (5) – (12) by replacing the constraints (7) – (12) that form the inner optimization problem:

$$
\begin{aligned}
y \in \operatorname*{argmin}_{y} \ & y\\
\text{subject to: } & -x - y \leq -3\\
& -2x + y \leq 0\\
& 2x + y \leq 12\\
& -3x + 2y \leq -4\\
& y \geq 0
\end{aligned}
$$

with the KKT conditions of this problem. Try to model and solve the reformulated problem with `Julia` using `JuMP`. One locally optimal solution for the problem is $(x, y) = (2, 1)$ with objective value $f(x, y) = x - 4y = -2$. Can you find this local optimum by trying different initial (starting) values for the different variables? Is the reformulated problem convex? Justify your answer.