

# Nonlinear Optimization - Homework 2

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November 23, 2021

### 3.1 FJ and KKT Conditions at Optimal Point

a)

$$\min. \quad -x_1 \quad (1)$$

$$\text{subject to: } x_2 \leq (1 - x_1)^3 \quad (2)$$

$$x_1 \geq 0 \quad (3)$$

$$x_2 \geq 0 \quad (4)$$

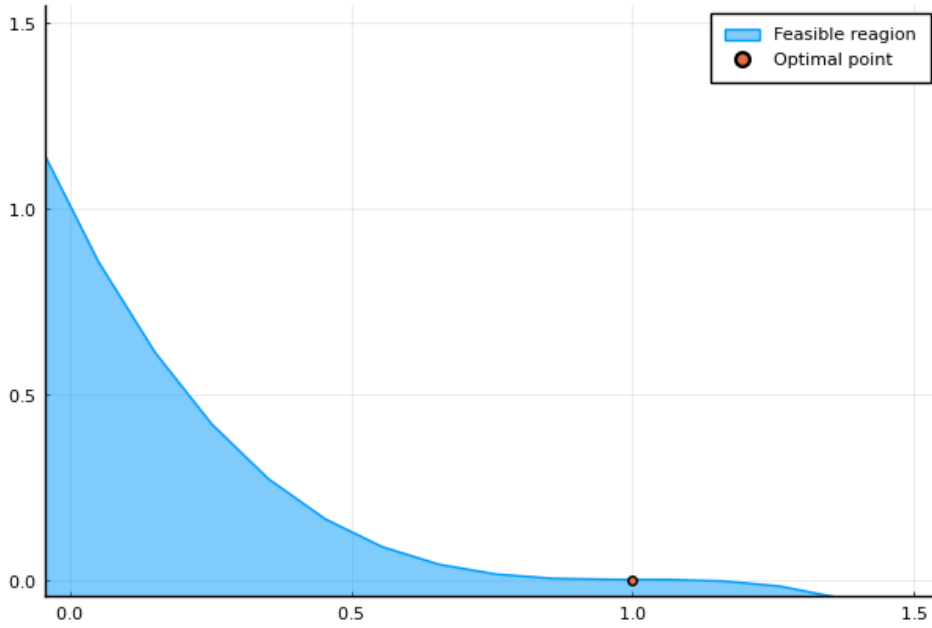


Figure 1: The feasible region of the problem of exercise 3.1. The condition of  $x_1, x_2 \geq 0$  is implemented by the limits of the plot.

Figure 1 shows the feasible region for the problem above. Since minimizing  $-x_1$  is the same as maximizing  $x_1$ , we can identify the optimal point as  $\bar{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .

b)

We will change around Equation 2 to be  $(1 - x_1)^3 - x_2 \geq 0$  for it to fit into the FJ conditions. We know that  $u_i g_i(\bar{x}) = 0$  for all  $i = 1, \dots, m$ . Hence we can calculate  $u_i$  for all  $i = 1, \dots, m$  as

$$u_1 g_1(\bar{x}) = u_1 \cdot (1 - 1)^3 - 0 = 0 \implies 0 = 0 \quad (5)$$

$$u_2 g_2(\bar{x}) = u_2 \cdot 1 = 0 \implies u_2 = 0 \quad (6)$$

$$u_3 g_3(\bar{x}) = u_3 \cdot 0 = 0 \implies 0 = 0 \quad (7)$$

$$0 = u_0 \nabla f(\bar{x}) + \sum_{i=1}^m u_i \nabla g_i(\bar{x}) \quad (8)$$

$$0 = u_0 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + u_1 \begin{bmatrix} 3(1-x_1)^2 \\ -1 \end{bmatrix} + u_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (9)$$

$$0 = u_0 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + u_1 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + u_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (10)$$

$$0 = \begin{bmatrix} -u_0 + u_2 \\ -u_1 + u_3 \end{bmatrix} \quad (11)$$

$$\implies \begin{cases} u_0 = u_2 = 0 \\ u_1 = u_3 \end{cases} \quad (12)$$

The point  $\bar{x}$  is a FJ point, since we can choose  $u_1$  and  $u_3$  such that FJ conditions are satisfied. U is

$$\text{thus } u = \begin{bmatrix} 0 \\ t \\ 0 \\ t \end{bmatrix}, \quad t > 0.$$

c)

The KKT conditions are

$$\nabla f(\bar{x}) + \sum_{i=1}^m u_i \nabla g_i(\bar{x}) = 0, \quad (13)$$

which gives us

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix} + u_1 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + u_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (14)$$

$$= \begin{bmatrix} -1 \\ u_3 - u_1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (15)$$

Hence, KKT conditons are not satisfied for any  $u$ .

In order for LIQC to hold, the gradient of all active inequality constraints and all equality constraints needs to be linearly independent. We can clearly see that, at  $\bar{x}$ ,  $\nabla g_1(\bar{x}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$  and

$\nabla g_3(\bar{x}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  are linearly dependent.

For Slater's QC to hold, all inequality constraints needs to be convex in the feasible region. Since  $g_2$  and  $g_3$  are linear functions, we know that they are convex. We will examine the Hessian for  $g_1$  to determine it's convexity.

$$H(g_1(x)) = \begin{bmatrix} -6(1-x_1) & 0 \\ 0 & 0 \end{bmatrix} \quad (16)$$

Since the Hessian for  $g_1$  is not positive semi-definite in all of the feasible reagon, e.g. at  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ , Slater's QC are not satisfied.

### 3.2 KKT Conditions for a Quadratic Problem

a)

$$\min. (x_1 + \frac{9}{4})^2 + (x_2 - 2)^2 \quad (17)$$

$$\text{subject to: } x_2 - x_1^2 \geq 0 \quad (18)$$

$$x_1 + x_2 \leq 6 \quad (19)$$

$$x_1 \geq 0 \quad (20)$$

$$x_2 \geq 0 \quad (21)$$

The KKT conditions for the problem is

$$0 = \nabla f(\bar{x}) + \sum_{i=1}^m u_i \nabla g_i(\bar{x}) \quad (22)$$

$$0 = \begin{bmatrix} 2(\bar{x}_1 + \frac{9}{4}) \\ 2(\bar{x}_2 - 2) \end{bmatrix} + u_1 \begin{bmatrix} -2\bar{x}_1 \\ 1 \end{bmatrix} + u_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix} + u_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + u_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (23)$$

$$0 = \begin{bmatrix} 2(\frac{3}{2} + \frac{9}{4}) \\ 2(\frac{9}{4} - 2) \end{bmatrix} + u_1 \begin{bmatrix} -2\frac{3}{2} \\ 1 \end{bmatrix} + u_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix} + u_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + u_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (24)$$

$$0 = \begin{bmatrix} 2(\frac{3}{2} + \frac{9}{4}) - \frac{7}{2}u_1 - u_2 + u_3 \\ 2(\frac{9}{4} - 2) + u_1 - u_2 + u_4 \end{bmatrix} \quad (25)$$

b)

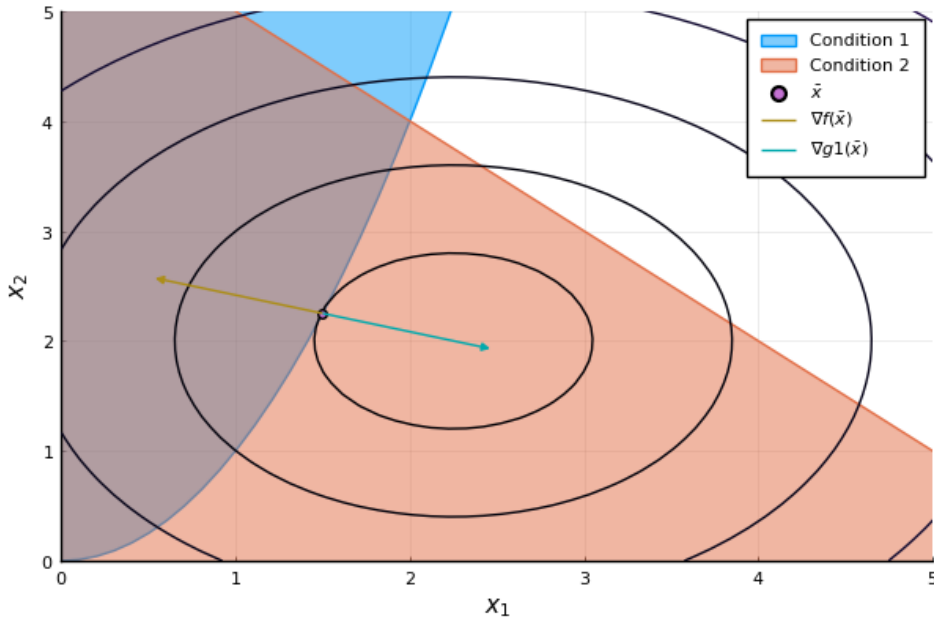


Figure 2

### 3.3 Lagrangian Dual of a Least-Squares Problem

$$\begin{aligned} \min. \quad & x^\top x & (26) \\ \text{subject to: } & Ax = b & (27) \end{aligned}$$