MS-E2122 - Nonlinear Optimization
Lecture 11

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Outline of this lecture

Methods of feasible directions

Feasible direction methods

Conditional gradient: the Frank-Wolfe method

Sequential quadratic programming - SQP

The concept of feasible direction

Algorithms of this type progress taking into account two aspects:

1. $x_k + \lambda d$ is feasible
2. $f(x_k + \lambda d_k) \leq f(x_k)$.

Since primal feasibility is observed, these methods are also called primal methods.

However, some variants do not necessarily retain feasibility during the iterations.

We will discuss 2 main types:

1. Conditional gradient: Frank-Wolfe;
2. Sequential quadratic programming - SQP.

Obtaining improving feasible directions

Let us first revisit the definition of an improving feasible direction.

Definition 1

Consider the problem min. $\{f(x) : x \in S\}$ with $f : \mathbb{R}^n \to \mathbb{R}$ and $\emptyset \neq S \subseteq \mathbb{R}^n$. A vector d is a feasible direction at $x \in S$ if exists $\delta > 0$ such that $x + \lambda d \in S$ for all $\lambda \in (0, \delta)$. Moreover, d is an improving feasible direction at $x \in S$ if there exists a $\delta > 0$ such that $f(x + \lambda d) < f(x)$ and $x + \lambda d \in S$ for $\lambda \in (0, \delta)$.

Feasible direction methods work as follows. Given $x^k \in S$

1. Obtain an improving feasible direction d^k and a step size λ^k;
2. Make $x^{k+1} = x^k + \lambda^k d^k$.

Remark: Obtaining d^k and λ^k have to be easier to solve than the original problem for the method to make sense.

The Frank-Wolfe method

Recall that, if $\nabla f(x^k)$ is a feasible descent direction, then

$$\nabla f(x^k)^\top (x - x^k) < 0 \text{ for } x \in S.$$

A straightforward way to obtain improving feasible directions $d = (x - x^k)$ is by solving the direction search problem DS.

$$(DS): \text{ min. } \left\{ \nabla f(x^k)^\top (x - x^k) \mid x \in S \right\}.$$

Letting $\overline{x}^k = \arg\min_{x \in S} \left\{ \nabla f(x^k)^\top (x - x^k) \right\}$ and obtaining $\overline{\lambda}^k \in (0, 1]$, the method iterates making

$$x^{k+1} = x^k + \lambda^k (\overline{x}^k - x^k).$$

Remark: for convex S, $\lambda^k \in (0, 1]$ guarantees feasibility.

The Frank-Wolfe method

Algorithm Frank-Wolfe method

initialise. $\epsilon > 0, x^0 \in S, k = 0.$
while $|\nabla f(x)^\top d^k| > \epsilon$ **do**
  $\overline{x}^k = \arg\min \left\{\nabla f(x^k)^\top d : x \in S\right\}.$
  $d^k = \overline{x}^k - x^k$
  $\lambda^k = \arg\min_\lambda \left\{f(x^k + \lambda d^k) : 0 \leq \lambda \leq \overline{\lambda}\right\}.$
  $x^{k+1} = x^k + \lambda^k d^k; k = k + 1.$
end while
return x^k.

Remarks:

1. For $f(x)$ nonlinear and a polyhedral feasible region S, the subproblems DS are linear programming problems.
2. Can be employed with Armijo to ease line search for complicated $f(x)$.

The Frank-Wolfe method

Example: min. $\{e^{-(x_1-3)/2} + e^{(4x_2+x_1-20)/10} + e^{(-4x_2+x_1)/10} :$ $2x_1 + 3x_2 \leq 8, x_1 + 4x_2 \leq 6\}$. The first iteration...

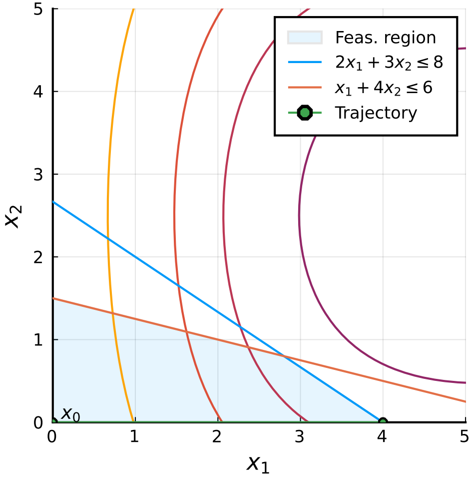


Figure: The FW method with exact line search.

The Frank-Wolfe method

Example: min. $\{e^{-(x_1-3)/2} + e^{(4x_2+x_1-20)/10} + e^{(-4x_2+x_1)/10} : 2x_1 + 3x_2 \leq 8, x_1 + 4x_2 \leq 6\}$. All iterations.

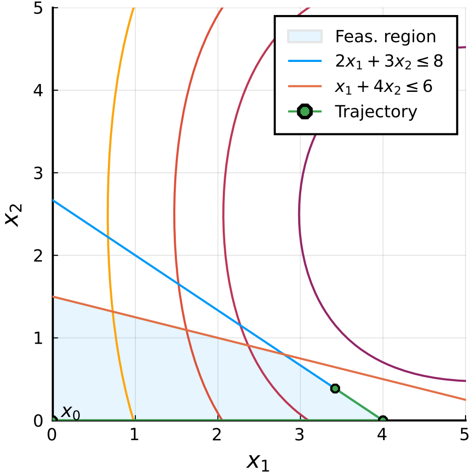


Figure: Total of 2 iterations are required for $e = 10^{-4}$.

The Frank-Wolfe method

Example: min. $\{e^{-(x_1-3)/2} + e^{(4x_2+x_1-20)/10} + e^{(-4x_2+x_1)/10} : 2x_1 + 3x_2 \leq 8, x_1 + 4x_2 \leq 6\}$. All iterations with Armijo.

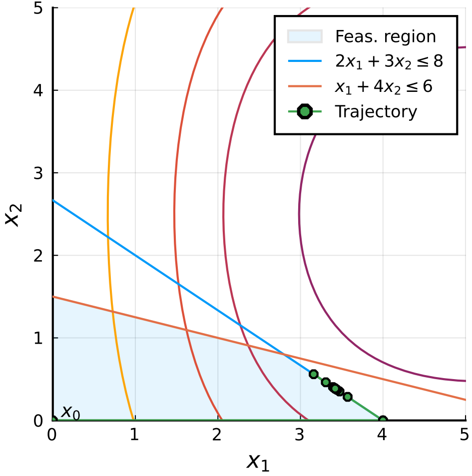


Figure: Total of 15 iterations are required for $e = 10^{-4}$.

Sequential quadratic programming - SQP

SQP is inspired on the idea of employing Newton's method to solve the KKT system directly.

Let $P = \text{min. } \{f(x) : h_i(x) = 0, i = 1, \dots, l\}$. The KKT conditions for P are

$$W(x, v) = \begin{cases} \nabla f(x) + \sum_{i=1}^{l} v_i \nabla h_i(x) = 0 \\ h_i(x) = 0, i = 1, \dots, l. \end{cases}$$

Using Newton(-Raphson) to solve $W(x, v)$ at (x^k, v^k), we obtain

$$W(x^k, v^k) + \nabla W(x^k, v^k) \begin{bmatrix} x - x^k \\ v - v^k \end{bmatrix} = 0. \tag{1}$$

Sequential quadratic programming - SQP

Let $\nabla^2 L(x^k, v^k) = \nabla^2 f(x^k) + \sum_{i=1}^{l} v_i^k \nabla^2 h_i(x^k)$ be the Hessian of the Lagrangian function

$$L(x, v) = f(x) + v^\top h(x)$$

at x^k. Thus

$$\nabla W(x^k, v^k) = \begin{bmatrix} \nabla^2 L(x^k, v^k) & \nabla h(x^k)^\top \\ \nabla h(x^k) & 0 \end{bmatrix}.$$

Setting $d = (x - x^k)$, we can rewrite (1) as

$$\nabla^2 L(x^k, v^k) d + \nabla h(x^k)^\top v = -\nabla f(x^k) \quad (2)$$

$$\nabla h(x^k) d = -h(x^k), \quad (3)$$

which can be repeatedly solved until

$$||(x^k, v^k)^\top - (x^{k-1}, v^{k-1})^\top|| = 0,$$

i.e., convergence, is observed. Then, (x^k, v^k) is a KKT point.

Sequential quadratic programming - SQP

Instead of solving a Newton system, SQP relies on successively solving the problem:

$$QP(x^k, v^k) : \text{min. } \; f(x^k) + \nabla f(x^k)^\top d + \frac{1}{2} d^\top \nabla^2 L(x^k, v^k) d \quad (4)$$

$$\text{subject to: } \; h_i(x^k) + \nabla h_i(x^k)^\top d = 0, i = 1, \dots, l, \quad (5)$$

to which optimality conditions are given by (2) and (3).

Two alternative ways of interpreting this objective function:

1. a second-order approximation of $f(x)$, also considering a term $(1/2) \sum_{i=1}^{l} v_i^k d^\top \nabla^2 h_i(x^k) d$ representing constraint curvature;
2. Let $L(x, v) = f(x) + \sum_{i=1}^{l} v_i h_i(x)$. Then, (4) can be seen as the second-order approximation of $L(x, v)$,

$$L(x^k, v^k) + \nabla_x L(x^k, v^k)^\top d + \frac{1}{2} d^\top \nabla^2 L(x^k, v^k) d$$

which explains its alternative name: projected Lagrangian.

Sequential quadratic programming - SQP

To see (2), notice that

$$L(x,v) \approx L(x^k,v^k) + \nabla_x L(x^k,v^k)^\top d + \frac{1}{2} d^\top \nabla^2 L(x^k,v^k) d =$$

$$f(x_k) + {v^k}^\top h(x^k) + (\nabla f(x^k) + {v^k}^\top \nabla h(x^k))^\top d$$

$$+ \frac{1}{2} d^\top (\nabla^2 f(x^k) + \sum_{i=1}^{l} v_i^k \nabla^2 h_i(x^k)) d$$

and that $\nabla h(x^k)^\top (x - x^k) = 0$ (from (5), as $h(x^k) = 0$). For the general case, we have

$$\begin{aligned} QP(x^k,u^k,v^k) : \text{min. } & \nabla f(x^k)^\top d + \frac{1}{2} d^\top \nabla^2 L(x^k,u^k,v^k) d \\ \text{subject to: } & g_i(x^k) + \nabla g_i(x^k)^\top d \leq 0, i = 1,\dots,m \\ & h_i(x^k) + \nabla h_i(x^k)^\top d = 0, i = 1,\dots,l, \end{aligned}$$

where $L(x,u,v) = f(x) + \sum_{i=1}^m u_i g_i(x) + \sum_{i=1}^l v_i h_i(x)$.

Sequential quadratic programming - SQP

A pseudocode for the standard SQP method is presented in 2.

Algorithm SQP method

1: **initialise.** $\epsilon > 0, x^0 \in S, u^0 \geq 0, v^0, k = 0.$
2: **while** $||d^k|| > \epsilon$ **do**
3: $\quad d^k = \arg\min QP(x^k, u^k, v^k)$
4: $\quad$ obtain u^{k+1}, v^{k+1} from $QP(x^k, u^k, v^k)$
5: $\quad x^{k+1} = x^k + d^k, k = k + 1.$
6: **end while**
7: **return** x^k.

Remark: notice that the step in Line 5 requires dual variable values, which can be trivially recovered from simplex-based solvers.

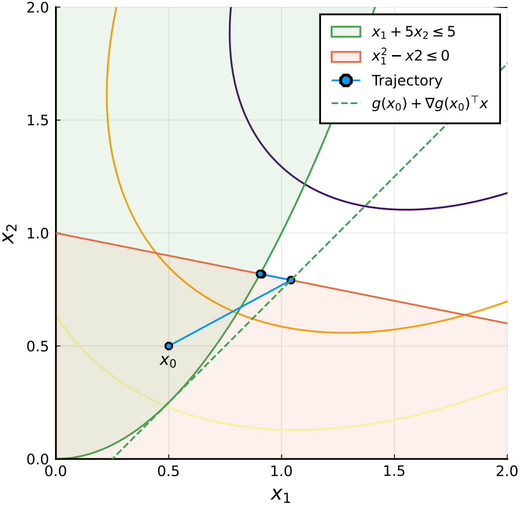
Sequential quadratic programming - SQP

Some relevant aspects:

1. Can be used in conjunction with quasi-Newton (BFGS) to approximate $\nabla^2 L(x^k, v^k)$.
2. Closely mimics convergence properties of Newton's method, i.e., under appropriate conditions, quadratic (superlinear) convergence is observed.
3. Can exploit efficient (dual) simplex solvers.
4. Can consider general nonlinear constraints, using first-order approximations.
5. Line searches cannot be easily performed, because feasibility is only implicitly considered in $QP(x^k, v^k)$
6. Might present divergence, in a similar way than Newton's method, if started too far from the optimum.

Sequential quadratic programming - SQP

Example: min. $\{2x_1^2 + 2x_2^2 - 2x_1x_2 - 4x_1 - 6x_2 : x_1^2 - x_2 \leq 0,$
$x_1 + 5x_2 \leq 5, x_1 \geq 0, x_2 \geq 0\}$



Sequential quadratic programming - SQP

The l_1-SQP is a variant that addresses divergence issues while presenting superior computational performance.

- Relies on a similar principle of penalty methods, encoding penalisation for infeasibility in the objective function.
- This allows for considering line searches or trust regions, which in turn can guarantee convergence.

Let us consider the trust-region l_1-penalty QP subproblem:

$$
\begin{aligned}
&l_1 - QP(x^k, v^k): \\
&\quad \text{min. } \nabla f(x^k)^\top d + \frac{1}{2} d^\top \nabla^2 L(x^k, v^k) d \\
&\qquad + \mu \left[\sum_{i=1}^{m} [g_i(x^k) + \nabla g_i(x^k)^\top d]^+ + \sum_{i=1}^{l} |h_i(x^k) + \nabla h_i(x^k)^\top d| \right] \\
&\text{subject to: } -\Delta^k \leq d \leq \Delta^k,
\end{aligned}
$$

where μ is a penalty term, $[\,\cdot\,] = \max\{0, \cdot\}$, and Δ^k is a trust region term.

Sequential quadratic programming - SQP

$l_1 - QP(x^k, v^k)$ can be recast as a QP with linear constraints:

$$l_1 - QP(x^k, v^k):$$

$$\text{min. } \nabla f(x^k)^\top d + \frac{1}{2} d^\top \nabla^2 L(x^k, v^k) d + \mu \left[\sum_{i=1}^{m} y_i + \sum_{i=1}^{l} (z_i^+ - z_i^-) \right]$$

$$\begin{aligned} \text{subject to: } & -\Delta^k \leq d \leq \Delta^k \\ & y_i \geq g_i(x^k) + \nabla g_i(x^k)^\top d, i = 1 \dots, m \\ & z_i^+ - z_i^- = h_i(x^k) + \nabla h_i(x^k)^\top d, i = 1, \dots, l \\ & y, z^+, z^- \geq 0 \end{aligned}$$

Remarks:

1. l_1-SQP is globally convergent (does not diverge) and enjoys superlinear convergence rate.
2. the l_1 term is often called a merit function in the literature.