MS-E2122 - Nonlinear Optimization
Lecture 10

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November 25, 2021

Outline of this lecture

Barrier functions

Barrier method

Interior point method for LP/ QP

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Barrier functions

Barrier method

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Barrier functions

Same idea as in penalty methods: turn constrained optimisation unconstrained and solve them iteratively.

Main difference: barrier functions prevent the search from leaving the feasible region. Consider the primal problem P

$$
\begin{aligned}
(P): \quad \text{min.} \quad & f(x) \\
\text{subject to:} \quad & g(x) \leq 0 \\
& x \in X.
\end{aligned}
$$

$f(x) = \frac{1}{x}$

$\hookrightarrow x > 0$

Barrier functions

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Main difference: barrier functions prevent the search from leaving the feasible region. Consider the primal problem P

$$\begin{aligned}(P):\ \text{min.}\ & f(x)\\ \text{subject to:}\ & g(x) \leq 0\\ & x \in X.\end{aligned}$$

We define the barrier problem BP as

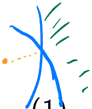
$$\begin{aligned}(BP):\ & \inf_{\mu}\ \theta(\mu)\\ \text{subject to:}\ & \mu > 0,\end{aligned}$$

where $\theta(\mu) = \inf_x \{f(x) + \mu B(x) : g(x) < 0, x \in X\}$ and $B(x)$ is a barrier function.

Barrier functions

The barrier function $B : \mathbb{R}^m \to \mathbb{R}$ is such that

$g(x) \leq 0$



$$B(x) = \sum_{i=1}^{m} \phi(g_i(x)), \text{ where } \begin{cases} \phi(y) \geq 0, & \text{if } y < 0; \\ \phi(y) = \infty, & \text{when } y \to 0^-. \end{cases} \tag{1}$$

Barrier functions

The barrier function $B : \mathbb{R}^m \to \mathbb{R}$ is such that

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Some common alternatives include

- $B(x) = -\sum_{i=1}^{m} \frac{1}{g_i(x)}$ $g_i(x) < 0$
- $B(x) = -\sum_{i=1}^{m} \ln(\min\{1, -g_i(x)\})$.

Perhaps the most important is Frisch's log barrier function

$$B(x) = -\sum_{i=1}^{m} \ln(-g_i(x)).$$

Barrier functions

Ideally, $B(x)$ would serve as an indicator function

$$B(x) = \begin{cases} \infty, & \text{if } g(x) \geq 0 \\ 0, & \text{if } g(x) < 0. \end{cases}$$

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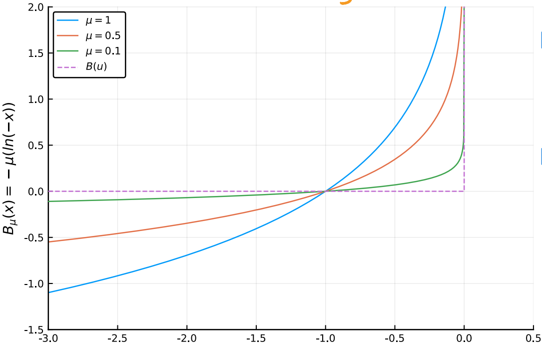
$g(x) = 0$

To avoid numerical issues, the shape of $B(x)$ is controlled by μ.

$g(x) < 0$



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- As $\mu \to 0^+$, $B(x)$ becomes closer to an indicator function.
- Notice that $B(x) > 0$ is not required for all feasible points, but must be so for those near $g(x) \leq 0$.

Barrier functions

We will proceed by repeatedly solving $\theta(\mu)$ and iteratively reducing the value of μ. For that to work, we need the following result.

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$\ell \leq x \leq u$

Theorem 1 (Convergence of barrier methods)

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be continuous functions and $X \in \mathbb{R}^n$ a nonempty closed set in problem P. Suppose $\{x : g(x) < 0, x \in X\}$ is not empty. Let $\overline{x}$ be the optimal solution of P such that, for any neighbourhood $N_\epsilon(\overline{x}) = \{x : ||x - \overline{x}|| \leq \epsilon\}$, there exists $x \in X \cap N_\epsilon$ for which $g(x) < 0$. Then

BP

$\lambda < \mu$

$$\underbrace{\min \{f(x) : g(x) \leq 0, x \in X\}}_{P} = \lim_{\mu \to 0^+} \theta(\mu) = \inf_{\mu > 0} \theta(\mu).$$

1) $B(x_\lambda) \geq B(x_\mu)$

Letting $\theta(\mu) = f(x_\mu) + \mu B(x_\mu)$, where $B(x)$ is a barrier function observing (1), $x_\mu \in X$, and $g(x_\mu) < 0$, the limit of $\{x_\mu\}$ is optimal to P and $\mu B(x_\mu) \to 0$ as $\mu \to 0^+$.

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Barrier method

Interior point method for LP/ QP

Barrier method

Algorithm Barrier method

1: **initialise.** $\epsilon > 0, x^0 \in X$ with $g(x^k) < 0, \mu^k, \beta \in (0,1), k = 0.$
2: **while** $\mu^k B(x^k) > \epsilon$ **do**
3: $\quad \overline{x}^{k+1} = \arg\min \left\{ f(x) + \mu^k B(x) : x \in X \right\}$ -> unc. optimisation.
4: $\quad \mu^{k+1} = \beta \mu^k,\ k = k + 1$
5: **end while**
6: **return** x^k.

Barrier method

Algorithm Barrier method

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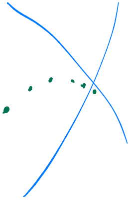
while $\mu^k B(x^k) > \epsilon$ **do**
    $\overline{x}^{k+1} = \arg\min \{f(x) + \mu^k B(x) : x \in X\}$
    $\mu^{k+1} = \beta\mu^k$, $k = k + 1$
end while
return x^k.

$\bar{x}^{k+1} \Rightarrow d^{k+1} = \bar{x}^{k+1} - \bar{x}^k$

$\bar{x}^{k+1} = \bar{x}^k + d^{k+1}$

$\lambda = 1$

Remarks:



1. Notice that, starting with $x^0 \in X$ with $g(x^0) < 0$, x^k for all $k > 1$ satisfies $g(x^k) < 0$ due to the barrier function.
2. However, applying the barrier method using a fixed step size may cause infeasibility issues.
3. Due to 1., these methods are called interior point methods.

Barrier method

Example 1: $P = \text{min.}\ \{\underbrace{(x+1)^2}_{f(x)} : \underbrace{x \geq 0}_{g(x) = -x}\}$ with $B(x) = -\ln(x)$

$$BP: (x+1)^2 - \mu \ln(-x) = f_\mu(x)$$

$$f'_\mu(x) = 0 \Rightarrow 2(x+1) - \frac{\mu}{x} =$$

$$2x^2 + x - \mu = 0 \Rightarrow x_\mu = -\frac{1}{2} + \frac{\sqrt{4+8\mu}}{4}$$

$$\mu \to 0^+ \Rightarrow x_\mu = 0$$

Barrier method

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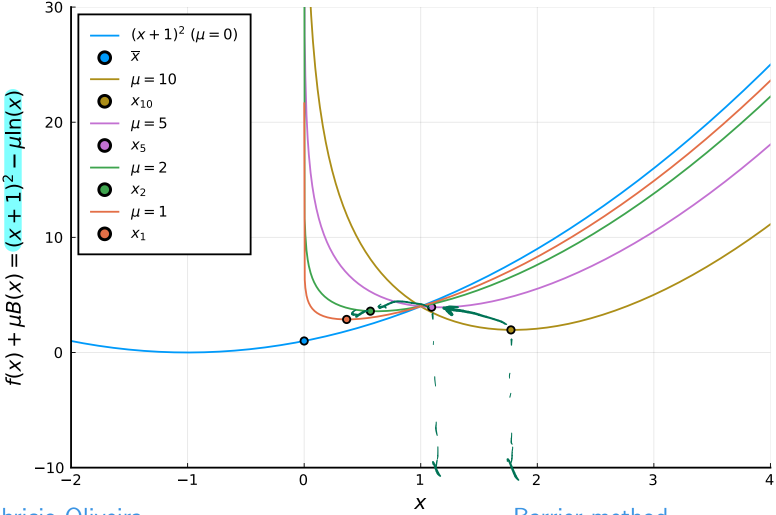

Barrier method

Example 2: $P = \text{min.}\ \{\underbrace{(x_1-2)^4 + (x_1-2x_2)^2}_{f(x)} : x_1^2 - x_2 \leq 0\}$
with $B(x) = -\frac{1}{x_1^2 - x_2}$.

$$f_\mu(x) = f(x) - \mu . \left(\frac{1}{(x_1^2 - x_2)} \right)$$

Barrier method

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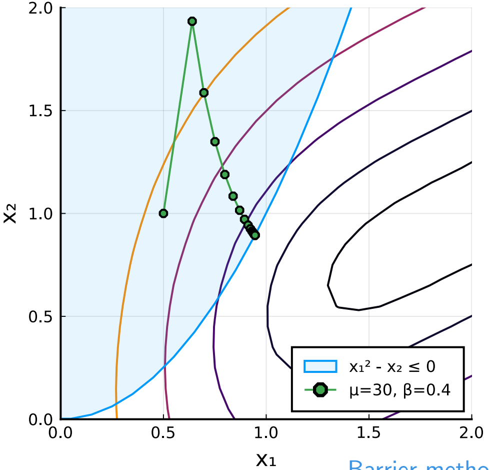

$f_\mu(x) = f(x) + \mu B(x)$

Barrier method

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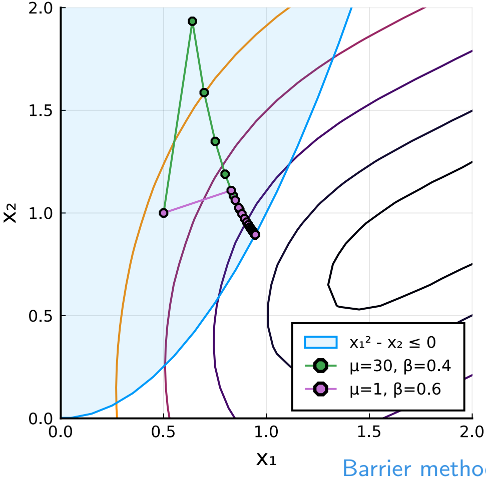

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Interior point methods for LP/QP

Consider the following LP and its dual

$$
\begin{aligned}
(P):\ \ \text{min.}\ \ & c^\top x \\
\text{subject to:}\ \ & Ax = b\ \ : v \\
& x \geq 0\ \ : u
\end{aligned}
\qquad\qquad
\begin{aligned}
(D):\ \ \text{max.}\ \ & b^\top v \\
\text{subject to:}\ \ & A^\top v + u = c \\
& u \geq 0, v \in \mathbb{R}^m.
\end{aligned}
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Interior point methods for LP/QP

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$$

The optimal solution $(\overline{x}, \overline{v}, \overline{u}) = \overline{w}$ satisfies KKT conditions of P:

$$
\begin{aligned}
& Ax = b,\ x \geq 0 \\
& A^\top v + u = c,\ u \geq 0,\ v \in \mathbb{R}^m \\
& u^\top x = 0.
\end{aligned}
$$

$c^T x + (b - Ax)^T v - u^T x$

Let us consider the barrier problem for P by using the logarithmic barrier function.

Interior point methods for LP/QP

The barrier problem is given by:

$g(x) = -x$

$$(BP): \ \text{min.} \ c^\top x - \mu \sum_{i=1}^{n} \ln(x_j)$$

$$\text{subject to: } Ax = b.$$

$\min \ c^T x$

$\bar{A}x \le \bar{b}$

$Ax \le b$

$x \ge 0$

$\Downarrow$

$\bar{A}x + s = \bar{b}$

$Ax = b$

$x, s \ge 0$

$\Downarrow$

$\bar{\bar{A}}\bar{x} = \bar{\bar{b}}$

$\bar{x} \ge 0$

$\begin{bmatrix} \bar{A} \\ A \end{bmatrix}$ $\begin{bmatrix} \bar{b} \\ b \end{bmatrix}$

Interior point methods for LP/QP

The barrier problem is given by:

$$(BP): \text{ min. } c^\top x - \mu \sum_{i=1}^{n} \ln(x_j)$$
$$\text{subject to: } Ax = b.$$

The KKT conditions of BP are

$$Ax = b, \ x > 0$$
$$A^\top v = c - \mu \left(\frac{1}{x_1}, \dots, \frac{1}{x_n}\right).$$

$A^\top v = c - u$

Notice that since $\mu > 0$ and $x > 0$, $u = \mu\left(\frac{1}{x_1}, \dots, \frac{1}{x_n}\right)$ serve as an estimate for the Lagrangian dual variables.

Interior point methods (IPM) for LP/QP

Let $X \in \mathbb{R}^{n \times n}$ and $U \in \mathbb{R}^{n \times n}$ be defined as

$$X = \mathbf{diag}(x) = \begin{bmatrix} \ddots & & \\ & x_i & \\ & & \ddots \end{bmatrix} \text{ and } U = \mathbf{diag}(u) = \begin{bmatrix} \ddots & & \\ & u_i & \\ & & \ddots \end{bmatrix}$$

and let $e = [1, \dots, 1]^\top$ be a vector of ones of suitable dimension.

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and let $e = [1, \dots, 1]^\top$ be a vector of ones of suitable dimension.
We can rewrite the KKT conditions of BP as

$$X^{-1} = \begin{bmatrix} 1/x_1 & & \\ & \ddots & \\ & & 1/x_n \end{bmatrix}$$

$$Ax = b, \; x > 0 \tag{2}$$

$$A^\top v + u = c \tag{3}$$

$$u = \mu X^{-1} e \;\Rightarrow\; XUe = \mu e \tag{4}$$

$x_1 u_1 = \mu$
$x_2 u_2 = \mu$
$\vdots$

$Ax = b$
$A^T v + u = c$
$x^T u = 0$

Remark: condition (3) is called relaxed complementarity condition with 0 replaced by μ. This is known as the perturbed KKT system.

Interior point methods (IPM) for LP/QP

According to Theorem 1, $w_\mu = (x_\mu, v_\mu, u_\mu)$ approaches the optimal primal-dual solution of P as $\mu \to 0^+$.

Remarks:

1. The trajectory formed by successive solutions $\{w_\mu\}$ is called a central path due to its interiority forced by the barrier function.

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2. Notice that $XUe = \mu e \Rightarrow u_i x_i = \mu$ for all $i = 1, \dots, n$.
3. $c^\top x - b^\top v = u^\top x$ measures the duality gap for the current μ.

$$c^T x = (A^T v + u)^T x = (A^T v)^T x + u^T x = v^T (\underbrace{Ax}_{b}) + u^T x$$

$$c^T x - v^T b = u^T x = \sum_{i=1}^{n} u_i x_i = \sum_{i=1}^{n} \frac{\mu}{x_i} \cdot x_i \Rightarrow n \cdot \mu$$

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According to Theorem 1, $w_\mu = (x_\mu, v_\mu, u_\mu)$ approaches the optimal primal-dual solution of P as $\mu \to 0^+$.

Remarks:

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2. Notice that $XUe = \mu e \Rightarrow u_i x_i = \mu$ for all $i = 1, \dots, n$.
3. $c^\top x - b^\top v = u^\top x$ measures the duality gap for the current μ.
4. Also, $u^\top x = \sum_{i=1}^n u_i x_i = n\mu$ is equal to the total slack violation and can be used as a stopping condition.

The notion of interiority

For large enough μ, the solution of the barrier problem is close to the analytic centre of the feasibility set.

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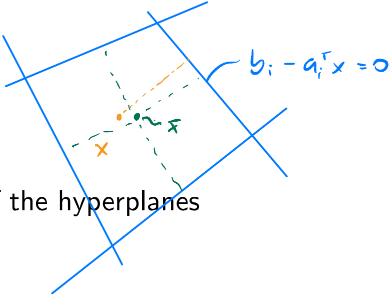
The analytic centre of a polyhedral set $S = \{x \in \mathbb{R}^n : Ax \leq b\}$ is given by the solution of

$$\max_x. \prod_{i=1}^{m} (b_i - a_i^\top x)$$

$$\text{subject to: } x \in X,$$



i.e., finding $\overline{x} \in S$ of maximum distance to each of the hyperplanes $a_i^\top x = b_i$.

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subject to: $x \in X$,

$a_i^T x + s = b_i$

$\bar{\bar{a}}_i^T \bar{\bar{x}} = \bar{\bar{b}}_i$

i.e., finding $\overline{x} \in S$ of maximum distance to each of the hyperplanes $a_i^\top x = b_i$. This is equivalent to

$\cancel{c^T x} - \mu \ln (x)$

$$\min_x . \sum_{i=1}^{m} -\ln(b_i - a_i^\top x)$$

subject to: $x \in X$.

IPM for LP/QP: primal/dual method

Primal/dual path following method is a specialisation of IPM to linear and quadratic problems.

It combines BP with one "additional trick": instead of solving BP to optimality, perform a single Newton step for each μ.

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Suppose we start with a $\overline{\mu} > 0$ and a $w^k = (x^k, v^k, u^k)$ sufficiently close to $w_{\overline{\mu}}$. Then, for a sufficiently small $\beta \in (0,1)$, $\beta\overline{\mu}$ will lead to a w^{k+1} sufficiently close to $w_{\beta\overline{\mu}}$.

Remark: β is typically related to convergence results. Values like $\mu^0 = (x^\top u)/n$ and $\beta \in [0.1, 0.5]$ are often used in practice.

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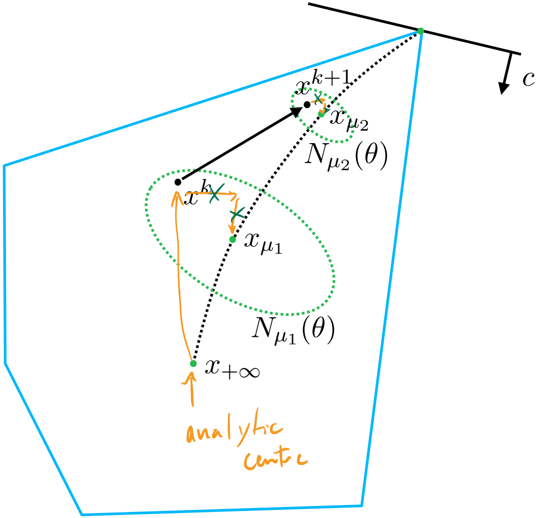
For example, let $N_\mu(\theta) = ||X_\mu U_\mu e - \mu e|| \leq \theta\mu$. Then, by selecting $\beta = 1 - \frac{\sigma}{\sqrt{n}}$, $\sigma = \theta = 0.1$, and $\mu^0 = (x^\top u)/n$, successive Newton steps are guaranteed to remain within $N_\mu(\theta)$.

Further reading: see this reference (link) for more details.

IPM for LP/QP: primal/dual method



IPM for LP/QP: primal/dual method

Let the perturbed KKT system (2) – (4) for each $\hat{\mu}$ be denoted as $H(w) = 0$. Let $J(\overline{w})$ be the Jacobian of $H(w)$ at $\overline{w}$.

Applying Newton's method to solve $H(w) = 0$ for $\overline{w}$, we obtain

$$J(\overline{w})d_w = -H(\overline{w}) \tag{5}$$

where $d_w = (w - \overline{w})$.

$$H(w) = \left| \begin{array}{l} Ax - b \\ Av + u - c \\ XUe - \mu e \end{array} \right.$$

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where $d_w = (w - \overline{w})$. By rewriting $d_w = (d_x, d_v, d_u)$, (5) can be equivalently stated as

$$\begin{aligned} &Ad_x = 0 \\ &A^\top d_v + d_u = 0 \\ &\overline{U}d_x + \overline{X}d_u = \hat{\mu}e - \overline{X}\,\overline{U}e. \end{aligned}$$

$\beta\hat{\mu}$

$\bar{u}$ $\bar{x}$

The algorithm proceeds by iteratively solving the above system with $\mu^{k+1} = \beta\mu^k$ with $\beta \in (0, 1)$ until $n\mu^k$ is small enough.

IPM for LP/QP: primal/dual method

Remark: Notice that primal feasibility conditions are included in the system $H(w) = 0$. This is typically referred to as the equality constrained Newton's method with "Newton system"

$$J(w) = \begin{bmatrix} A & 0^\top & 0 \\ 0 & A^\top & I \\ \overline{U} & 0^\top & \overline{X} \end{bmatrix} \begin{bmatrix} d_x \\ d_v \\ d_u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \hat{\mu}e - \overline{X}\,\overline{U}e \end{bmatrix}. \quad (6)$$

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In practice, updates incorporate primal and dual infeasibility, which can be shown to vanish as the algorithm progress. Updates become

$$\begin{bmatrix} A & 0^\top & 0 \\ 0 & A^\top & I \\ U^k & 0^\top & X^k \end{bmatrix} \begin{bmatrix} d_x^{k+1} \\ d_v^{k+1} \\ d_u^{k+1} \end{bmatrix} = - \begin{bmatrix} \overbrace{Ax^k - b}^{\text{primal inf}} \\ Av + u - c \;\} \text{ dual inf.} \\ X^k U^k e - \mu^{k+1} e \end{bmatrix}, \tag{7}$$

where $\mu^{k+1} = \beta \mu^k$.

IPM for LP/QP: primal/dual method

Let the residuals (i.e., the amount of infeasibility) be

$$r_p(x, u, v) = Ax - b \;{}^{=0}\text{ (primal)}; \;\; r_d(x, u, v) = A^\top v + u - c \;{}^{=0}\text{ (dual)}.$$

Let $r(w) = r(x, u, v) = (r_p(x, u, v), r_d(x, u, v))$. The optimality conditions (6) require the residuals to vanish, that is $r(\overline{w}) = 0$.

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Consider the first-order approximation for r at w for a step d_w

$$r(w + d_w) \approx r(w) + Dr(w)d_w,$$

where $Dr(w)$ is the derivative of r evaluated at w. The step d_w for which the residue vanishes is $Dr(w)d_w = -r(w)$ (cf. (7)).

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where $Dr(w)$ is the derivative of r evaluated at w. The step d_w for which the residue vanishes is $Dr(w)d_w = -r(w)$ (cf. (7)).

The directional derivative of $||r(w + td_w)||_2^2$ in the direction d_w is

$$\frac{d}{dt}||r(w + td_w)||_2^2\Big|_{t\to 0^+} = 2r(w)^\top Dr(w)d_w = -2\overbrace{r(w)^\top r(w)}^{\geq 0}, \leq 0$$

$$= 2r(w + \cancel{td_w}) \cdot Dr(w + \cancel{td}) \cdot d_w$$

which is strictly decreasing.

IPM for LP/QP: primal/dual method

Algorithm Interior point method for LP

1: **initialise.** primal-dual feasible w^k, $\epsilon > 0$, μ^k, $\beta \in (0, 1)$, $k = 0$.
2: **while** $n\mu = c^\top x^k - b^\top v^k > \epsilon$ **do**
3: compute $d_{w^{k+1}} = (d_{x^{k+1}}, d_{v^{k+1}}, d_{u^{k+1}})$ using (7) and w^k.
4: $w^{k+1} = w^k + d_{w^{k+1}}$
5: $\mu^{k+1} = \beta\mu^k$, $k = k + 1$
6: **end while**
7: **return** w^k.

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5: $\mu^{k+1} = \beta\mu^k$, $k = k + 1$
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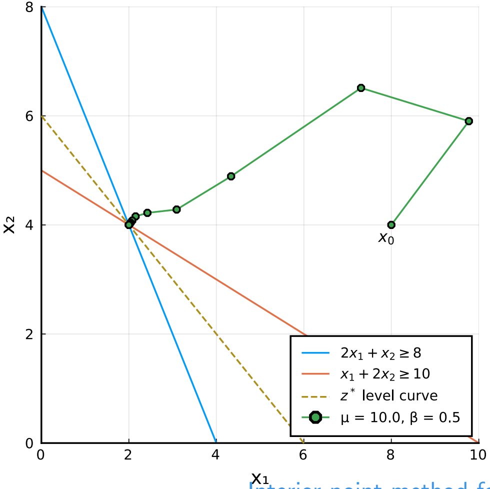
Remarks:

1. Notice that the step size is set to one. A line search could be performed between Lines 3 and 4.
2. This method has polynomial complexity which, under specific conditions, can be shown to be $O(\sqrt{n}\ln(1/\epsilon))$.

The primal-dual interior point (IP) method

Example:

min. $z = x_1 + x_2 : 2x_1 + x_2 \geq 8,\ x_1 + 2x_2 \geq 10,\ x_1, x_2 \geq 0.$



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