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The Supply of Energy from and to Atmospheric Eddies

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sodium hydrate. Acids were also used for dissolving the metallic splash, with negative results.

If the xenon is present as a metallic compound, it must therefore go into solution as a compound on dissolution. Silicon hydrides, when treated with sodium hydrate, gives sodium silicate; it may be that xenon behaves in a similar manner, and that a xenate of sodium is formed.

The Supply of Energy from and to Atmospheric Eddies.

By Lewis F. Richardson.

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Introduction.

This paper extends Osborne Reynold's theory of the Criterion of Turbulence, to make it apply to the case in which work is done by the eddies, acting as thermodynamic engines in a gravitating atmosphere. For simplicity, the air is supposed to be dry. The gist of the solution is contained in equations (4·15) and (5·3) below.

Some analogies to thermodynamics are then introduced, including the conception of an eddy-heat-per-mass, and its diminution when the air expands. These conceptions are helpful in discussing the manner in which eddies diffuse. I am indebted to Mr. W. H. Dines, F.R.S., for his valuable advice at many stages of this work.

While it was being finished off, a treatment of the Criterion of Turbulence, by Wilhelm Schmidt, of Vienna,* has come to hand. For the supply of energy from the wind, he gives much fuller observational data than I have done. But, for the thermodynamic loss, he concludes that the activity per volume has the average value $b\partial^2\theta/\partial h^2$ where b is the gas constant in erg units, θ is the absolute temperature, and h the height. This is incompatible with the result arrived at in equation (4·15) below, and is, moreover, difficult to reconcile with observation. I am inclined to think that Prof. Schmidt has accidentally admitted a factor, $p^{0\cdot29}$, in the diffusion equation. On integrating partially, and taking account of boundary conditions, this change would explain the discrepancy, if the turbulence were independent of height, as he assumes.

* 'Ann. der Hydrog. u. Mar. Meteorol.,' November-December, 1918.

[*Added May* 25, 1920.—At the meeting at which this paper was discussed Mr. G. I. Taylor mentioned some studies of stability which formed part of his Adams Prize Essay in 1914. The oscillations of superposed laminæ of different density became unstable when a criterion was reached which resembled (5·4) below, except that the numerical coefficient contained a factor of 2 or of 4 depending on the number of the laminæ. By a rough energy-method he had arrived at a numerical coefficient agreeing with that of (5·4). It is to be hoped that these investigations will appear in print.]

I. *The Available Forms of Energy.*

Since energy is conserved, we might find the supply to eddies by counting up the losses from all the other forms.

In order to relieve ourselves of the trouble of considering effects at vertical walls, let us consider a large volume of atmosphere, say that standing on a land area measuring 10 km. × 10 km. or larger. Let this area be A cm.^{2}.

Let $\bar{p}_G$ be the mean pressure at the earth's surface. Then the mass standing upon A is $A\bar{p}_G/g = M$, say. Let dM be an element of this mass. The changes of the energy present in this mass may be classified as

(i) Change of intrinsic thermal energy ΔI,

$$\Delta I = \int \gamma_v \,.\, \Delta\theta \,.\, dM, \tag{1·1}$$

where γ_v is the thermal capacity per mass at constant volume, and θ is the temperature.

(ii) Change of gravitational energy $\Delta\Gamma$,

$$\Delta\Gamma = \Delta \int ghdM, \tag{1·2}$$

where h is the height of dM and g is the acceleration of gravity.

(iii) Imagine the wind velocity to be smoothed, so as to remove the gusts. Let v_X, v_Y, v_H be the actual winds; let $\bar{v}_X$, $\bar{v}_Y$, $\bar{v}_H$ be the means of the same. The means may be supposed to be taken over a time-interval, long compared with gusts, but short compared with the passage of barometric depressions.

Denote deviations from the mean by dashes, so that

$$v_X = \bar{v}_X + v_X'\,; \qquad v_Y = \bar{v}_Y + v_Y'\,; \qquad v_H = \bar{v}_H + v_H', \tag{1·3}$$

Square, add and multiply by half the density ρ, and again take the mean, denoted by a bar, over a similar interval of time. Then, as is well known, the terms containing products of means and deviations vanish, leaving

$$\tfrac{1}{2}\rho\,(v_X^2 + v_Y^2 + v_H^2) = \tfrac{1}{2}\rho\,(\bar{v}_X^2 + \bar{v}_Y^2 + \bar{v}_H^2) + \tfrac{1}{2}\rho\,(v_X'^2 + v_Y'^2 + v_H'^2), \tag{1·4}$$

so that the kinetic energy may be divided into that associated with the mean and that associated with the deviations. Denote these parts, for our large atmospheric block, by $\bar{E}$ and E' respectively, and their increases by $\Delta\bar{E}$ and $\Delta E'$. The object of this paper is to study $\Delta E'$.

By the continuity of energy

$$(\Delta I+\Delta\Gamma+\Delta\bar{E}+\Delta E')/\Delta t = G+R+W, \tag{1·5}$$

where G is the rate at which heat-energy is flowing into the large portion of atmosphere from the vegetation and ground or from the sea. And R is the excess of the radiation flowing in over that flowing out.

And W is the rate at which work is being transmitted across the vertical walls in the form of (pressure) × (velocity). As the size of the area A increases, W dwindles relatively to the other terms. Let us neglect W.

The immediate effect of the radiation R is to alter the temperature of the air, vegetation, land or sea. Afterwards, these temperature-changes may cause eddies. But in this paper we suppose that the temperature distribution is known. Therefore we may ignore radiation, provided that we also ignore the changes of temperature which it is at the instant producing. Let $\Delta_0 I$ be the change in the intrinsic energy *not* due to radiation. Accordingly (1·5), becomes

$$(\Delta_0 I+\Delta\Gamma+\Delta\bar{E}+\Delta E')/\Delta t = G. \tag{1·6}$$

This equation may be further simplified, for Mr. W. H. Dines has shown that there is a connection between the changes of gravitational and intrinsic energy.* On inserting the appropriate constants, γ_v the thermal capacity per mass at constant volume, and b the "gas constant" defined by the characteristic equation

$$p = b\rho\theta, \tag{1·7}$$

where ρ is density, Mr. Dines' theorem takes the following form: "*In a column of clear air bounded by vertical walls, and extending from the ground upwards to a height at which the pressure is negligible, the change in total gravitational energy, reckoned from the ground, is b/γ_v times the corresponding change in the total intrinsic energy.*" (1·8)

Now, $\gamma_v+b = \gamma_p$, where γ_p is the thermal capacity per mass at constant pressure. The theorem (1·8) may therefore be interpreted thus: a column of air, of unit mass, in cooling, gives out energy as if it had a specific heat γ_p. Of this energy, a part, b/γ_p, is due to the sinking of the column. (1·9)

So equation (1·6) simplifies to

$$\frac{\gamma_p}{\gamma_v}\frac{\partial_0 I}{\partial t}+\frac{\partial\bar{E}}{\partial t}+\frac{\partial E'}{\partial t} = G. \tag{1·10}$$

The whole energy is sometimes correctly classified in ways involving the

* 'Q. J. Roy. Met. Soc.,' p. 188, July, 1913.

pressure p. Margules,* for instance, calculates the energy associated with local irregularities of pressure. In the present scheme this energy comes in (i) and (ii), and, to include pressural energy in addition, would be to count some of the energy twice over. It seems to be most natural to regard pressure only as concerned in the transfer across surfaces. Once across, energy appears in some other form—gravitational, intrinsic, or molar kinetic.

II. *The Supply from the Mean Wind.*

Osborne Reynolds† showed that the kinetic energy of eddying was increased by the work done by the eddy-stresses upon the corresponding rates of mean strain.

In the atmosphere, high above trees and houses, many of the rates of mean strain are negligible, and the activity may be abbreviated to

$$\text{activity per volume} = \widehat{xh}\,\frac{\partial \bar{v}_{\text{X}}}{\partial h} + \widehat{yh}\,\frac{\partial \bar{v}_{\text{Y}}}{\partial h}, \tag{2·1}$$

where $\widehat{xh}$, $\widehat{yh}$, are the shearing stress on horizontal planes and where h is the upward co-ordinate.

If μ be the "eddy viscosity" defined as

$$\mu = \frac{\text{eddy shearing stress}}{\text{corresponding rate of mean shear}}. \tag{2·2}$$

Then equation (2·1) may be written

$$\text{activity per volume} = \mu_{\text{XH}}\left(\frac{\partial \bar{v}_{\text{X}}}{\partial h}\right)^2 + \mu_{\text{YH}}\left(\frac{\partial \bar{v}_{\text{Y}}}{\partial h}\right)^2, \tag{2·3}$$

where μ_{XH} and μ_{YH} are possibly not equal, as suggested by the author,‡ expressing the rate at which the mean-wind gives up its energy to eddies. Numerical data, based on observation, are collected by H. U. Sverdrup.§

III. *The Loss by Molecular Viscosity.*

Reynolds|| balances the above gain of energy against the energy converted into heat at a rate per volume equal to

$$\mu_c \left\{ 2\left(\frac{\partial v_{\text{X}}'}{\partial x}\right)^2 + 2\left(\frac{\partial v_{\text{Y}}'}{\partial y}\right)^2 + 2\left(\frac{\partial v_{\text{H}}'}{\partial h}\right)^2 + \left(\frac{\partial v_{\text{H}}'}{\partial y} + \frac{\partial v_{\text{Y}}'}{\partial h}\right)^2 \right.$$
$$\left. + \left(\frac{\partial v_{\text{X}}'}{\partial h} + \frac{\partial v_{\text{H}}'}{\partial x}\right)^2 + \left(\frac{\partial v_{\text{Y}}'}{\partial x} + \frac{\partial v_{\text{X}}'}{\partial y}\right)^2 \right\}, \tag{3·1}$$

* "The Mechanical Equivalent of Pressure," 'Abbe's Translations,' 3rd series, Smithsonian Publications.

† Lamb, 'Hydrodynamics,' 4th ed., §369.

‡ 'Phil. Trans.,' A, p. 1 (1920).

§ 'Geo. Inst., Leipzig,' 2nd ser., Band II, Heft 4.

|| Lamb, 'Hydrodynamics,' 4th ed., p. 369.

where μ_c is the molecular viscosity and dashes denote deviations from the mean.

This rate of loss will vanish if the eddying energy vanishes, for then v_X', v_Y', v_H' will be zero everywhere. Further, it is clear that the loss by molecular viscosity increases with the eddying velocities and with the smallness of the eddies.

The expression (3·1) when integrated over the volume standing on the large area A will be denoted by $-\left(\frac{\partial E'}{\partial t}\right)_1$.

The suffix is necessary because it is not the only part of $\frac{\partial E'}{\partial t}$.

To gain some rough idea of the numerical value of the dissipation, let us imagine that the complicated irregular motion is analysed into a series of regular motions superposed upon one another according to the method of Fourier. In other words, let each component of the instantaneous velocity be expanded in sines and cosines of each of the co-ordinates. Let us pick out for examination the terms

$$\left.\begin{aligned} v_X' &= B\sin\frac{\pi x}{n_X}\cos\frac{\pi y}{n_Y}\sin\frac{\pi h}{n_H} \\ & \qquad\qquad\qquad\qquad ; \; v_Y' = 0 \\ v_H' &= B\cos\frac{\pi x}{n_X}\cos\frac{\pi y}{n_Y}\cos\frac{\pi h}{n_H} \end{aligned}\right\} \tag{3·2}$$

which signifies a motion divided up into rectangular cells, so that the eddies would look rather like papers curled up in the pigeonholes of an office desk. The size of the cell is n_X by n_H centimetres in the plane of the motion, and n_Y at right angles to the motion.

The mean kinetic energy associated with this distribution is got by squaring the trigonometrical terms and integrating over a volume of atmosphere very large compared with the size of the cell. Taking the mean with respect to any one co-ordinate replaces the square of a sine or cosine by $\frac{1}{2}$. So the three successive operations introduce the factor $(\frac{1}{2})^3$, and the mean kinetic energy is therefore

$$\tfrac{1}{2}\rho\,(v_X'^2+v_Y'^2+v_H'^2) = \tfrac{1}{2}\rho\,\{\tfrac{1}{8}+0+\tfrac{1}{8}\} = \tfrac{1}{8}\rho B^2 = E'/(\text{the large volume}). \tag{3·3}$$

Next to find the rate of dissipation we have to insert v_X', v_Y', v_H' in the dissipation function quoted above. The differentiations introduce factors π/n_X, π/n_Y, π/n_H. The term $\partial v_X/\partial h + \partial v_H/\partial x$ conveniently vanishes. Squaring and taking the mean over a large volume again introduces the factor $\frac{1}{8}$. So that the mean rate of dissipation comes to

$$2\mu_c\pi^2\left\{\frac{1}{n_X^2}+\frac{1}{n_Y^2}+\frac{1}{n_H^2}\right\}\tfrac{1}{8}B^2 = \frac{\partial E'}{\partial t}\,\frac{1}{(\text{the large volume})}. \tag{3·4}$$

The quantity $(n_X^{-2}+n_Y^{-2}+n_H^{-2})$ is a measure of the smallness of the eddy. Let us denote it by l^{-2}, so that we may speak of l as the linear size of an eddy. If the cell which encloses the eddy were cubical, then l would be 0·577 of the edge of the cube. If the cube were drawn out into a very long square prism, then l would be 0·707 of the side of the square. If the prism were stretched into a plate, l would be the thickness of the plate.

Then from (3·3) and (3·4)

$$\frac{\partial \mathrm{E}'}{\partial t} = \frac{2\mu_c\pi^2}{\rho l^2}\mathrm{E}'. \tag{3·5}$$

Now $\mu_c = 1·7\times 10^{-4}$ for air at 273° A.

So if l be say 10 metres

$$\frac{1}{\mathrm{E}'}\frac{\partial \mathrm{E}'}{\partial t} = 2·6\times 10^{-6}\,\mathrm{sec}^{-1}.$$

So that after 24 hours the eddying energy would have fallen to 0·8 of its initial value. The changes which we observe near the ground are much more rapid, so that either the eddies must be smaller than the size represented by $l = 10$ metres, or else other causes than molecular viscosity must cooperate to destroy the eddying energy.

We will not now stop to inquire what would be the interaction on the average between eddies of different sizes and positions represented by the different terms of the Fourier expansion. The equation (3·5) is to be understood as a type, not as a complete formulation.

IV. *Convection.*

The rising of cumuli during the daytime and the diminution of turbulence in the surface wind on clear nights both shows us that convection has an important influence.

The eddies are themselves the engines by which the available heat is converted into their visible motion, so that the rate of conversion must depend somehow upon the rate of eddying. Let us explore this question.

But the eddies are imperfect engines, for they lead to mixing of portions of air at different temperatures, and mixing is an irreversible process. If, however, we can by any means find the diminution of the intrinsic heat energy of the atmosphere, we shall be able to find the energy supplied to the eddies, at least if the air be dry, by the aid of the theorem of Section 1 given by Mr. W. H. Dines.*

Now we can find the rate of change of intrinsic energy in terms of the

* 'Q. J. Roy. Met. Soc.,' p. 188, July, 1913.

rate of eddying, by means of the equation for the diffusion of potential temperature, denoted by Φ,

$$\frac{\bar{D}\Phi}{\bar{D}t} = \frac{\partial}{\partial p}\left(\xi \frac{\partial \Phi}{\partial p}\right), \tag{4·1}$$

where p is pressure used as a measure of depth, and where ξ is a measure of turbulence, for which I have suggested the name "turbulivity." In quoting this equation from a previous paper it is necessary to make clear three points which I then* left confused. For we are now examining small effects, and precision in details is essential to success.

(i) When there are no eddies we are accustomed to compute the flow of heat or of water vapour across a plane from the flow of mass across the plane. As the effect of eddies is to be treated as additional, we should not include in it any flow due to the mean motion of mass across a plane, and accordingly $\bar{D}$ is here a differentiator which follows the *mean* motion of the fluid, in the sense that the effect of eddies is to cause equal masses to cross the co-ordinate surface from opposite sides. In the case in which there is no convergence of air in the general circulation, $\bar{D}$ accordingly refers to a surface of fixed pressure, not to one of fixed height. Therefore, as p is the other independent, $\bar{D}/\bar{D}t$ may be replaced by curly $\partial/\partial t$, indicating differentiation at constant pressure.

(ii) Secondly, the equation would not hold if entropy per mass, σ, were substituted in place of Φ, because entropy is increased by mixing. In fact, as for dry air,

$$d\sigma = \gamma_p d \log \Phi \tag{4·2}$$

the equation for the diffusion of σ takes the different form

$$\frac{\partial e^{\sigma/\gamma_p}}{\partial t} = \frac{\partial}{\partial p}\left(\xi \frac{\partial e^{\sigma/\gamma_p}}{\partial p}\right). \tag{4·3}$$

(iii) The eddy motion produces an atmosphere with an intricate structure; but as the diversity is smoothed out by molecular diffusion at constant pressure, the mean value of Φ for a secluded portion of air remains unchanged. Therefore in equation (4·1) we may regard Φ as a mean value over a portion of an isobaric surface.

With this explanation let us return to (4·1), and deduce from it the rate of change of temperature.

For dry air

$$\Phi = \theta\left(\frac{p_i}{p}\right)^{0\cdot29} \tag{4·4}$$

where p_i is the standard pressure adopted in defining Φ.

* 'Roy. Soc. Proc.,' A, London, vol. 96, p. 10 (1919).

So, at constant pressure

$$\frac{\partial \Phi}{\partial t} = \left(\frac{p_i}{p}\right)^{0\cdot 29} \cdot \frac{\partial \theta}{\partial t}. \tag{4·5}$$

Insert this in (4·1), transfer the factor in $p^{0\cdot 29}$ to the other side of the equation, multiply by γ_p and integrate from the top to the bottom of the atmosphere, with respect to the element of mass per unit horizontal area dp/g

$$\frac{\gamma_p}{g}\int_0^G \frac{\partial \theta}{\partial t} dp = \frac{1}{p_i^{0\cdot 29}} \frac{\gamma_p}{g}\int_0^G p^{0\cdot 29} \frac{\partial}{\partial p}\left(\xi \frac{\partial \Phi}{\partial p}\right) dp. \tag{4·6}$$

Integrate the second member by parts

$$\frac{\gamma_p}{g}\int_0^G \frac{\partial \theta}{\partial t} dp = \gamma_p \,{}_0^G\!\left[\frac{\theta}{\Phi}\frac{\xi}{g}\frac{\partial \Phi}{\partial p}\right] - \frac{1}{p_i^{0\cdot 29}}\frac{\gamma_p}{g}\int_0^G \xi \frac{\partial \Phi}{\partial p} d\left(p^{0\cdot 29}\right) \tag{4·7}$$

The two terms on the right are both activities due to convection per unit horizontal area. They may be distinguished as the "boundary activity" and the "body activity" respectively. The boundary activity must vanish at the top of the atmosphere. The sign of the second integral in (4·7) determines whether, in the body of the atmosphere, heat is being transformed to eddying energy or *vice versâ*.

Now ξ is essentially positive and so is $d(p^{0\cdot 29})$, since the integration is made downwards. Therefore (4·7) implies that heat becomes eddying energy if the potential temperature increases downwards. That is what we should expect. If there were no eddying, ξ would vanish and equation (4·7) would correctly indicate no activity. The atmosphere might then be unstable, but if so, it would be at rest in equilibrium—a hypothetical case merely.

Consider the boundary activity in equation (4·7). Now, in the paper* referred to it is shown that $\xi/g \,.\, \partial\Phi/\partial p$ is the amount of Z rising across surface per area per time, when Z is defined by the statement that Φ is the amount of Z per unit mass of atmosphere, and the surface ought to have been specified as one crossed by equal masses from opposite sides. Thus equation (4·7) shows that the boundary convective activity is simply the rate at which eddies remove heat energy from the ground, vegetation, or sea per horizontal square centimetre.

To correspond with the equation of continuity of energy in the form (1·10) of Section I, we must integrate (4·7) over the large horizontal area A. When so integrated, the boundary activity per area becomes what we have formerly denoted by G. And the first member is γ_p/γ_v times the whole rate of increase in intrinsic energy, not arising from radiation, nor from dissipation of eddying

* 'Roy. Soc. Proc.,' A, London, vol. 96, p. 11 (1919) equation (16).

kinetic energy by molecular viscosity. So we obtain for the integral of the body activity, the following expression:—

$$\frac{\gamma_p}{\gamma_v}\frac{\partial_0 \mathrm{I}}{\partial t} - \mathrm{G} = -\iint \left[p_i^{-0\cdot 29} \frac{\gamma_p}{g} \int_0^{\mathrm{G}} \xi \frac{\partial \Phi}{\partial p} \, dp^{0\cdot 29} \right] d\mathrm{A}. \tag{4·8}$$

This expression can be much simplified by using the following relations

$$\xi = g^2 \rho c, \text{ where } c \text{ is eddy conductivity} \tag{4·9}$$

$$p = b\rho\theta \text{ (gas equation)}, \tag{4·10}$$

$$\frac{\partial p}{\partial h} = -g\rho \text{ (hydrostatic equation)}, \tag{4·11}$$

$$d\sigma = \gamma_p d \log \Phi = \gamma_p d \log \theta - b d \log p, \tag{4·12}$$

$$\left(\frac{p_i}{p}\right)^{0\cdot 29} = \frac{\theta}{\Phi} \text{ (definition of potential temp.)}. \tag{4·13}$$

The stages by which the transformation takes place may be as follows:—

$$-\frac{1}{p_i^{0\cdot 29}} \frac{\gamma_p}{g} \int_0^{\mathrm{G}} \xi \frac{\partial \Phi}{\partial p} \, dp^{0\cdot 29} = -\frac{\gamma_p}{g} \int_0^{\mathrm{G}} g^2 \rho c \frac{\partial \Phi}{\partial p} d\left(\frac{\theta}{\Phi}\right) = \gamma_p \int_0^{\mathrm{G}} c \frac{\partial \Phi}{\partial h} \left\{ \frac{d\theta}{\Phi} - \frac{\theta}{\Phi^2} d\Phi \right\}$$

$$= \int_0^{\mathrm{G}} c \frac{\partial \sigma}{\partial h} \left\{ d\theta - \theta d \log \Phi \right\} = \int_0^{\mathrm{G}} c \frac{\partial \sigma}{\partial h} \frac{b\theta}{\gamma_p} \frac{dp}{p} = -\frac{g}{\gamma_p} \int_0^{\mathrm{G}} c \frac{\partial \sigma}{\partial h} \, dh. \tag{4·14}$$

The last form is the simplest. When inserted in (4·8) the latter becomes

$$\frac{\gamma_p}{\gamma_v}\frac{\partial_0 \mathrm{I}}{\partial t} - \mathrm{G} = +\iint \left[\frac{g}{\gamma_p} \int_{\mathrm{G}}^{0} c \frac{\partial \sigma}{\partial h} \, dh \right] d\mathrm{A}. \tag{4·15}$$

In words this may be stated as follows:—*The average rate at which intrinsic and gravitional energy jointly are being transformed into the energy of eddies is equal per volume to* $\partial\sigma/\partial h \times cg/\gamma_p$, *that is to say to the product of the eddy conductivity into the acceleration of gravity into the gradient of entropy per mass upward, divided by the thermal capacity per mass at constant pressure.*

Now there exist observations of c and of $\partial\sigma/\partial h$. So let us divide the activity per volume $cg/\gamma_p \times \partial\sigma/\partial h$ into the factors c and $g/\gamma_p \times \partial\sigma/\partial h$ and make a table of the latter under average conditions. For computing $\partial\sigma/\partial h$ from observations, a convenient formula is

$$\frac{\partial \sigma}{\partial h} = \frac{1}{\theta} \left\{ \gamma_p \frac{\partial \theta}{\partial h} + g \right\}, \tag{4·16}$$

it applies accurately to dry air, and with quite small errors to clear moist air when γ_p has its value for dry air. So I have taken data for clear days where such data were available. It is not worth considering cloudy days until the theory can be arranged *throughout* for that purpose, and that would very greatly add to its complexity.

Table I.

(Computed from Josef Reger's Analysis of the Lindenberg Observations,* clear days only.)

Hour (0 = midnight).	2.	8.	14.	20.
Layer, metres above M.S.L.	$g/\gamma_p\ \partial\sigma/\partial h$ expressed in sec.$^{-2}$ equals $10^{-4}\times$ the following			
122 to 500	6·52	3·02	−0·31	2·54
500 1000	1·50	1·95	0·76	1·07
1000 1500	1·83	1·70	1·46	0·88
1500 2000	2·51	1·20	1·47	0·20

Table II.†

(England, S.E., days both clear and cloudy.)

Height.	$g/\gamma_p\ \partial\sigma/\partial h$ in sec.$^{-2}$	Height.	$g/\gamma p\ \partial\sigma/\partial h$ in sec.$^{-2}$.
km.	$10^{-4}\times$	km.	$10^{-4}\times$
13·5	4·42	6 5	1·07
12·5	4·74	5·5	1·20
11·5	4·07	4·5	1·28 ‡
10·5	3·03	3·5	1·47 ‡
9·5	2·59	2·5	1·66 ‡
8·5	1·49	1·5	1·92 ‡
7·5	1·01		

‡ Doubtful, owing to formula for dry air having been used.

These Tables show that on clear days in the troposphere $g/\gamma_p.\partial\sigma/\partial h$ is of the order of 10^{-4} sec.$^{-2}$ on the average.

Now the eddy conductivity at the height of a kilometre has been found by numerous observers to be of the order of 100 grm. sec.$^{-1}$ cm.$^{-1}$.

Taking these numbers it follows that $cg/\gamma_p.\partial\sigma/\partial h = 10^{-2}$ erg. sec.$^{-1}$ cm^{-3}. This is the convective activity per volume lost by the eddies.

V. *Joint Effects of the Various Supplies.*

In the equation of continuity in the form (1·10) let us now insert the convective activity given by (4·15) together with Reynold's expression for the supply of energy from the mean wind (2·3). The result is

$$\frac{\partial E'}{\partial t} = \iint\left\{\mu\left(\frac{\partial\bar{v}_X}{\partial h}\right)^2+\mu\left(\frac{\partial\bar{v}_Y}{\partial h}\right)^2\right\}dh\,dA-\int\left[\frac{g}{\gamma_p}\int_G^0 c\frac{\partial\sigma}{\partial h}\,dh\right]dA. \qquad (5·1)$$

* Vieweg und Sohn, Braunschweig, 'Arbeiten des K. P. Aeronautischen Observatoriums bei Lindenberg,' vol. 8, p. 247 (1912).

† Computed from 'Characteristics of the Free Atmosphere,' by W. H. Dines, p. 62.

In reckoning the change of temperature due to convection, we took no account of the heat produced by the dissipation of eddy motion by molecular viscosity. So that $\partial \mathrm{E}'/\partial t$, as just given, ought to be decreased by the quantity (3·5) and a completer formulation is therefore :—

$$\frac{\partial \mathrm{E}'}{\partial t}-\frac{2\mu_c\pi^2}{\rho l^2}\mathrm{E}' = \iint\left\{\mu\left(\frac{\partial \bar{v}_{\mathrm{X}}}{\partial h}\right)^2+\mu\left(\frac{\partial \bar{v}_{\mathrm{Y}}}{\partial h}\right)^2\right\} dh\, d\mathrm{A} - \int\left[\frac{g}{\gamma_p}\int_{\mathrm{G}}^{0} c\,\frac{\partial \sigma}{\partial h}\,dh\right] d\mathrm{A}. \tag{5·2}$$

Now G. I. Taylor* put forward the hypothesis that the eddy viscosity $\mu =$ eddy conductivity c, and observation has so far tended to confirm this. Let us make this simplification. Then (5·2) becomes

$$\frac{\partial \mathrm{E}'}{\partial t}-\frac{2\mu_c\pi^2}{\rho l^2}\mathrm{E}' = \iint c\left\{\left(\frac{\partial \bar{v}_{\mathrm{X}}}{\partial h}\right)^2+\left(\frac{\partial \bar{v}_{\mathrm{Y}}}{dh}\right)^2-\frac{g}{\gamma_p}\frac{\partial \sigma}{\partial h}\right\} dh\, d\mathrm{A}. \tag{5·3}$$

This is a very interesting equation. To take a special application of it, suppose that the large piece of atmosphere, throughout which the integral is taken, is initially at rest. Suppose next that an approaching depression sets it in motion. Will the winds be turbulent or not? Suppose that they are very slightly turbulent, so that c and E' are just not zero, but so that $2\mu_c\pi^2\mathrm{E}'/(\rho l^2)$ is very small. The question is: will E' increase? It will tend to increase if

$$\left(\frac{\partial \bar{v}_{\mathrm{X}}}{\partial h}\right)^2+\left(\frac{\partial \bar{v}_{\mathrm{Y}}}{\partial h}\right)^2 > \frac{g}{\gamma_p}\frac{\partial \sigma}{\partial h}. \tag{5·4}$$

If the atmosphere has the ordinary value of $g/\gamma_p \cdot \partial\sigma/\partial h$ equal to say $+10^{-4}\,\mathrm{sec.}^{-2}$, then the wind will become turbulent if the change of velocity with height exceeds $\pm\sqrt{10^{-4}\,\mathrm{sec.}^{-2}}$. That is to say, a change of about 1 metre $\mathrm{sec.}^{-1}$ in 100 metres of height would be the critical value on an average clear day in the troposphere. Now pilot balloon ascents† show such up-grades generally in the first 100 metres above ground and frequently up to a kilometre. But above a kilometre such up-grades are rather uncommon until the base of the stratosphere is reached, where again there is a rapid change.‡ But at this height, as Table II shows, $g/\gamma_p \cdot \partial\sigma/\partial h$ has increased considerably, so that a more rapid variation of wind, say 18 metres $\mathrm{sec.}^{-1}$ per kilometre of height, would be required to produce turbulence. The stability of the stratosphere is well shown by the large value of $g/\gamma_p \cdot \partial\sigma/\partial h$. We conclude then that the usual source of atmospheric eddies is in the first few hundred metres, and that they may sometimes be formed at the base of the stratosphere.

The limit to E' will be set by the dissipation term $2\mu_c\pi^2\mathrm{E}'/\rho l^2$.

* 'Phil. Trans.,' A, 1914.

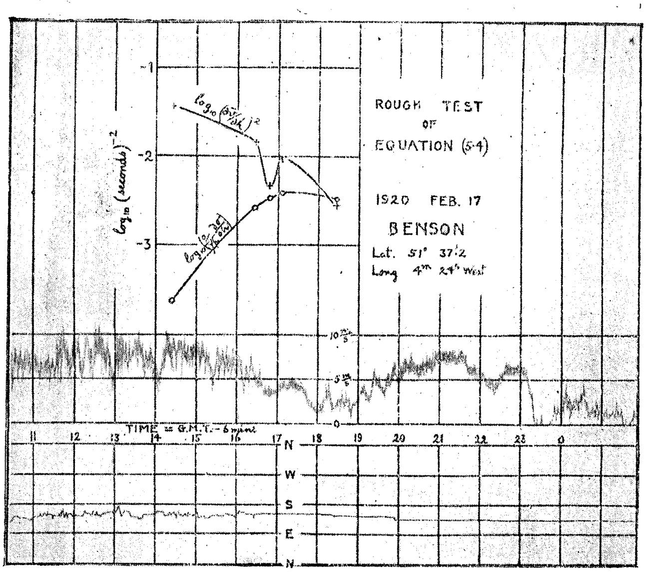
† 'Manual of Meteorology,' by Sir Napier Shaw, Part IV, Ch. VI.

‡ G. M. B. Dobson, 'Q. J. Met. Soc.,' Jan., 1920.

Stirring a dry atmosphere tends to bring the entropy-gradient, $\partial\sigma/\partial h$, towards zero. So if the atmosphere is originally stable, equation (5·3) indicates that the more it is stirred the easier does it become to stir.

The dying away of the surface wind at night has for some time been explained as an effect of radiation, which by increasing $\partial\sigma/\partial h$ makes the lower layers too stable for eddies to exist in them. Equation (5·3) conforms to this explanation and gives to it an expression which is quantitative. Equation (5·3) also shows that, when the day wind is sufficiently strong $(\partial\bar{v}/\partial h)^2$ will overpower the term in $\partial\sigma/\partial h$, so that eddies will continue to exist and to supply momentum to the surface air, even on a clear night. That fits with observation. At Benson, for example, where the anemometer head is 26 metres above the plain, the day wind usually dies away on a clear winter night if it is 5 metres/sec. but not if it is 11 metres/sec., provided that evenings on which the barometric gradient is changing rapidly are excluded.

A theory like this one, which supposes the mean velocities to be in horizontal straight lines, can only fit with observation at a height above the ground which is large compared with the irregularities of the surface.



To make a proper test of equation (5·4) the following installation would be suitable: Two anemometers, one above the other, to determine $\partial\bar{v}/\partial h$; the lower should be at a height large compared with the local irregularities on the ground; a differential thermometer to read the difference of temperature between these two anemometer heads; and a pressure tube anemometer placed at an intermediate level to record the range of the gusts. I am not aware of the existence of such an installation.

Choosing an occasion at Benson, when trees interfered as little as possible with the wind currents, I obtained the results shown in the figure. The temperature readings were taken with an aspirated thermometer. *It is seen that the approach of the curves* for $\log_{10}\left(\frac{g}{\gamma_p}\frac{\partial\sigma}{\partial h}\right)$ and $\log_{10}\left(\frac{\partial v}{\partial h}\right)^2$ *coincides in time at* 16 h. 40 m. *with a marked decrease in the range of the gusts shown by the anemometer.* A perfect agreement could not be expected because $\partial v/\partial h$ was measured over the interval 1 m. to 26 m., while $\partial\sigma/\partial h$ only over the interval 1 m. to 16 m. Again it would have been better if gustiness could have been measured at the average height, but trees prevented this. On this night the wind did not die away after sunset, but the current from 19 h. to 23 h. is probably the katabatic wind from the Chiltern Hills described by Mr. E. V. Newham in M. O. Professional Notes No. 2.

VI. *The Irreversibility of Convection.*

We may think of this by the aid of the resulting increase of the total entropy S in an atmospheric column standing upon a horizontal unit area. Since $\partial p/g$ is the mass in the short length of the column

$$\frac{dS}{dt} = \frac{1}{g}\int_0^G \frac{\partial\sigma}{\partial t}\,dp. \tag{6·1}$$

But (4·1) with (4·2) gives as an alternative form of (4·3)

$$\frac{\partial\sigma}{\partial t} = \frac{1}{\Phi}\left(\xi\Phi\frac{\partial\sigma}{\partial p}\right). \tag{6·2}$$

Inserting this in (6·1) and integrating by parts

$$\frac{dS}{dt} = \frac{1}{g}\int_0^G \frac{1}{\Phi}\frac{\partial}{\partial p}\left(\xi\,\Phi\frac{\partial\sigma}{\partial p}\right)dp = {}_0^G\left[\frac{\xi}{g}\frac{\partial\sigma}{\partial p}\right] + \frac{1}{g}\int_0^G \xi\,\Phi\frac{\partial\sigma}{\partial p}\frac{\partial\Phi}{\Phi^2\partial p}\,dp. \tag{6·3}$$

The integrated term is simply the rate at which entropy is entering the column from the earth's surface. Remembering that $d\sigma = \gamma_p d\log\Phi$ the remaining integral may be written in alternative forms as follows:—

$$\frac{dS}{dt} - \left[\frac{\xi}{g}\frac{\partial\sigma}{\partial p}\right]_G = \frac{1}{g\gamma_p}\int_0^G \xi\left(\frac{\partial\sigma}{\partial p}\right)dp = \gamma_p\int_G^0 c\left(\frac{\partial\log\Phi}{\partial h}\right)^2 dh = \frac{1}{\gamma_p}\int_G^0 c\left(\frac{\partial\sigma}{\partial h}\right)^2 dh. \tag{6·4}$$

This quantity is essentially positive as we should expect, since the entropy cannot decrease. It shows that the rate of increase of entropy depends upon the square of $\partial\sigma/\partial h$, whereas the rate at which eddying energy is converted into heat depends upon the first power of the same, see (4·15). Thus the nearer the gradient of entropy approaches to zero the more nearly will the eddies approach to perfectly reversible engines. If the eddy conductivity c were confined to a range of height in which $\partial\sigma/\partial h$ were constant, so that we might remove the integral signs, then dividing the last form of (6·4) by the last of (4·14) it would follow that

$$\frac{\text{increase of entropy}}{\text{energy supplied from eddies}} = \frac{1}{g}\frac{\partial\sigma}{\partial h}. \tag{6·5}$$

Multiplying by the temperature θ of the layer we should find

$$\frac{\text{energy made unavailable}}{\text{energy supplied from eddies}} = \frac{\theta}{g}\frac{\partial\sigma}{\partial h}. \tag{6·6}$$

This fraction is a pure number, which has an average value of about one-third in the troposphere. We see from (4·16) that in an isothermal atmosphere it would have the value unity. In an adiabatic atmosphere it is zero.

VII. *Eddy Thermodynamics.* (Revised May 11, 1920.)

The direct eddy stresses are, according to O. Reynolds

$$\widehat{xx} = -\rho\overline{v_X{}'v_X{}'}; \qquad \widehat{yy} = -\rho\overline{v_Y{}'v_Y{}'}; \qquad \widehat{hh} = -\rho\overline{v_H{}'v_H{}'}, \tag{7·1}$$

when tractions are reckoned positive, and bars denote mean values, and dashes denote deviations from the mean.

Following custom in the theory of viscous fluids, we could define an "eddy pressure," $\wp$, as the negative mean of the direct tractions

$$\wp = -\tfrac{1}{3}(\widehat{xx}+\widehat{yy}+\widehat{hh}). \tag{7·2}$$

From (7·1) and (7·2) it follows that

$$\frac{\wp}{\rho} = \tfrac{2}{3}\,.\,\overline{\tfrac{1}{2}(v_X{}'^2+v_Y{}'^2+v_H{}'^2)}. \tag{7·3}$$

This resembles the "gas equation" $\dfrac{p}{\rho} = b\theta$ and suggests that we might tentatively introduce the idea of "eddy-heat per mass," defined thus

$$\text{“eddy-heat per mass”} = \overline{\tfrac{1}{2}(v_X{}'^2+v_Y{}'^2+v_H{}'^2)} = \Theta \text{ say}, \tag{7·4}$$

so that

$$\wp = \tfrac{2}{3}\rho\Theta. \tag{7·5}$$

Next taking a suggestion from the adiabatic cooling of gases, *let us enquire whether the eddy-heat-per-mass would diminish if the air containing the eddies were suddenly expanded.* In this connection we must note that the energy per mass Θ, with which we are concerned, is not acoustical, and so moves with, not through, the fluid (Helmholtz, Kelvin). It is true that Θ as defined by (7·4) would include some acoustical energy, but let us suppose that sufficient time has elapsed to allow this to disperse itself with the speed of sound.

To obtain some light on the question, let us imagine a region containing a lively eddy motion accompanied by a mean-expansion. Reynolds' investigation relates to an incompressible fluid, so we must make a fresh start.

Consider a plane moving so that on the average as much mass crosses it from one side as from the other. For simplicity let the plane be set at right angles to $\bar{v}_X$. The molar kinetic energy in the fluid at any point is $\frac{1}{2}\rho(v_X{}^2+v_Y{}^2+v_H{}^2)$ per volume. And not being acoustical, it is moving with the velocity of the fluid v_X. As the plane is moving at the speed $\bar{v}_X$, the volume of fluid crossing an area dA of the plane in unit time is $(v_X - \bar{v}_X)\,dA$. So the mean rate at which molar kinetic energy crosses the plane is

$$\frac{1}{A}\int\tfrac{1}{2}\rho\,(v_X{}^2+v_Y{}^2+v_H{}^2)(v_X-\bar{v}_X)\,dA. \tag{7·6}$$

Now introduce dashes for variations from the mean so that

$$v_X = \bar{v}_X + v_X{}'; \qquad \rho = \bar{\rho}+\rho'; \qquad \text{and the like.} \tag{7·7}$$

Then (7·6) becomes, on squaring out,

$$\frac{1}{A}\int\tfrac{1}{2}(\bar{\rho}+\rho')\{\bar{v}_X{}^2+\bar{v}_Y{}^2+\bar{v}_H{}^2+2\bar{v}_X v_X{}'+2\bar{v}_Y v_Y{}'+2\bar{v}_H v_H{}' + v_X{}'^2+v_Y{}'^2+v_H{}'^2\}v_X{}'\,dA. \tag{7·8}$$

Now the mean values of the rapidly varying quantities, which are indicated by dashes, will approximately vanish when taken over a sufficiently large area. So also will vanish the mean of any dashed into any barred quantity, since the barred quantities vary slowly from point to point. Thus we may pick out from (7·8) the following terms which vanish:—

$$\frac{1}{A}\int\tfrac{1}{2}\bar{\rho}\,\{\bar{v}_X{}^2+\bar{v}_Y{}^2+\bar{v}_H{}^2\}v_X{}'\,dA = 0. \tag{7·9}$$

Again, since an increase of density is as likely to be accompanied by a positive as by a negative $v_X{}'$, so the following terms in $\rho' v_X{}'$ vanish:—

$$\frac{1}{A}\int\tfrac{1}{2}\rho'\,\{\bar{v}_X{}^2+\bar{v}_Y{}^2+\bar{v}_H{}^2\}v_X{}'\,dA = 0, \tag{7·10}$$

which is equivalent to assuming the absence of convection of mass.

In order to divide our difficulties, let us at this stage assume that

$$\frac{1}{A}\int\tfrac{1}{2}\rho\,(v_X{}'^2+v_Y{}'^2+v_H{}'^2)\,v_X{}'\,dA = 0. \tag{7·11}$$

This assumption implies that the eddying kinetic energy per volume is not correllated with $v_X{}'$, so that portions of high energy are not moving specially to either side of the plane. That is to say, we have assumed no diffusion of eddying energy, from where it is intense to where it is feeble. We shall return to the question of diffusion in the next section.

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Let (7·9), (7·10), (7·11) be assumed, then (7·8) simplifies to

$$\frac{1}{A}\int(\bar{v}_X \rho v_X' v_X' + \bar{v}_Y \rho v_Y' v_X' + \bar{v}_H \rho v_H' v_X')\, dA. \tag{7·12}$$

Now, in conformity with Reynolds' statement of the eddy stresses, when the bar replaces $\frac{1}{A}\int(\quad)\,dA$ as an averaging operator, put to complete (7·1).

$$\widehat{xy} = -\overline{\rho v_X' v_Y'}; \qquad \widehat{yh} = -\overline{\rho v_Y' v_H'}; \qquad \widehat{hx} = -\overline{\rho v_H' v_X'}. \tag{7·13}$$

Then the mean rate at which molar kinetic energy is crossing unit area of a plane set at right angles to $\bar{v}_X$ and moving with the fluid at its centre, becomes from (7·12)

$$-\bar{v}_X \,.\, \widehat{xx} - \bar{v}_Y \,.\, \widehat{xy} - \bar{v}_H \,.\, \widehat{xh}. \tag{7·14}$$

Now, if we added to the plane, which we may imagine as square, five other planes, also moving with the fluid to form at one instant a unit cube; then we see that the rate of increase of molar kinetic energy in the moving, distorting, swelling cube will instantaneously be

$$\frac{\partial}{\partial x}(\bar{v}_X \,.\, \widehat{xx} + \bar{v}_Y \,.\, \widehat{xy} + \bar{v}_H \,.\, \widehat{xh}) + \frac{\partial}{\partial y}(\bar{v}_X \,.\, \widehat{yx} + \bar{v}_Y \,\widehat{yy} + \bar{v}_H \widehat{yh})$$

$$+ \frac{\partial}{\partial h}(\bar{v}_X \,.\, \widehat{hx} + \bar{v}_Y \,\widehat{hy} + \bar{v}_H \widehat{hh}). \tag{7·15}$$

Since the mass of the cube is ρ and does not change as the cube swells, the expression (7·16) must be equal to

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}\{\tfrac{1}{2}(v_X{}^2 + v_Y{}^2 + v_H{}^2)\}, \tag{7·16}$$

where D denotes a differentiation following the mean motion. This gives the decrease in the total molar kinetic energy. We want the part of it associated with the eddies, separated from that associated with the mean wind. But the changes which we have been considering are between the cube and fluid touching it, so that they concern only the relative velocity of the cube and its surroundings. Therefore without loss of generality, we may take $\bar{v}_X$, $\bar{v}_Y$, $\bar{v}_H$ as vanishing at the centre of the cube. Then the energy associated with the mean motion vanishes, leaving only the part in Θ. So (7·15) when equated to (7·16) yields

$$\left.\begin{array}{l} \widehat{xx}\dfrac{\partial \bar{v}_X}{\partial x} + \widehat{xy}\dfrac{\partial \bar{v}_Y}{\partial x} + \widehat{xh}\dfrac{\partial \bar{v}_H}{\partial x} \\[2ex] \widehat{yx}\dfrac{\partial \bar{v}_X}{\partial y} + \widehat{yy}\dfrac{\partial \bar{v}_Y}{\partial y} + \widehat{yh}\dfrac{\partial \bar{v}_H}{\partial y} \\[2ex] \widehat{hx}\dfrac{\partial \bar{v}_X}{\partial h} + \widehat{hy}\dfrac{\partial \bar{v}_Y}{\partial h} + \widehat{hh}\dfrac{\partial \bar{v}_H}{\partial h} \end{array}\right\} = \rho \frac{\mathrm{D}}{\mathrm{D}t}\Theta. \tag{7·17}$$

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The first member is Reynolds' Activity, formed of the products of the eddy-stresses into the corresponding components of mean strain. We have arrived at it by a proof which differs from Reynolds' in that we have:—

(i) Supposed the fluid to be compressible.

(ii) Done away with the rigid envelope at which he made all the velocities to vanish, and have replaced it by certain assumptions as to the approximate vanishing of various means. Just as ordinary thermodynamics is correct because the number of molecules is enormous, so eddy-thermodynamics is an approximation which improves as the number of eddies increases.

In so far as Reynolds' Activity depends upon distortion, we have already dealt with it in Sections II and V above. Let us now suppose that the rates-of-mean-shearing are zero, and let us examine the remaining part due to the expansion. Accordingly, (7·17) will be limited here to

$$\widehat{xx}\frac{\partial\bar{v}_{\mathrm{X}}}{\partial x}+\widehat{yy}\frac{\partial\bar{v}_{\mathrm{Y}}}{\partial y}+\widehat{hh}\frac{\partial\bar{v}_{\mathrm{H}}}{\partial h}=\rho\frac{\mathrm{D}\Theta}{\mathrm{D}t}. \tag{7·18}$$

Now observations by G. I. Taylor and by the author* have shown that, except in the first few metres above the surface, $\widehat{xx}=\widehat{yy}=\widehat{hh}$, at least roughly. In other words, there is a tendency to equi-partition of energy between the three components of velocity. For simplicity, let us assume this to be the case, so that from (7·2)

$$\widehat{xx}=\widehat{yy}=\widehat{hh}=-\wp. \tag{7·19}$$

Then from (7·18)

$$\wp\left\{\frac{\partial\bar{v}_{\mathrm{X}}}{\partial x}+\frac{\partial\bar{v}_{\mathrm{Y}}}{\partial y}+\frac{\partial\bar{v}_{\mathrm{H}}}{\partial h}\right\}=-\rho\frac{\mathrm{D}\Theta}{\mathrm{D}t}. \tag{7·20}$$

But by the continuity of mass

$$\frac{\partial\bar{v}_{\mathrm{X}}}{\partial x}+\frac{\partial\bar{v}_{\mathrm{Y}}}{\partial y}+\frac{\partial\bar{v}_{\mathrm{H}}}{\partial h}=-\frac{\mathrm{D}}{\mathrm{D}t}(\log\rho), \tag{7·21}$$

therefore from (7·20) and (7·21)

$$\wp\frac{\mathrm{D}}{\mathrm{D}t}\log\rho=\rho\frac{\mathrm{D}\Theta}{\mathrm{D}t}, \tag{7·22}$$

or, using (7·5) and transposing

$$\frac{\mathrm{D}}{\mathrm{D}t}\{\tfrac{2}{3}\log\rho-\log\Theta\}=0, \tag{7·23}$$

or $\dfrac{\rho^{2/3}}{\Theta}$ is constant following the mean motion.

That is exactly like the adiabatic equation for a gas having a tem-

* 'Phil. Trans.,' A, 1920.

perature Θ and a ratio of specific heats equal to 5/3—like mercury vapour, krypton, and other monatomic gases.

One is tempted to call Θ the "eddy-temperature," but it is perhaps best to avoid this term, for Θ has different dimensions, and so cannot be expressed in degrees on a thermometer. It is correctly called "eddy-heat-per-mass."

We are thus led to the conception of a "potential eddy-heat-per-mass," defined as

$$\Theta\,(\rho_i/\rho)^{2/3}, \tag{7·25}$$

where ρ_i is some standard density.

In arriving at (7·14), we ignored the fact that the molecules do not all move with the velocity v_X, v_Y, v_H, which we attribute to the fluid at a point. But these molecular motions are the cause of the ordinary pressure p. By a method, closely analogous to that by which (7·14) and (7·15) are obtained, it can be shown that (7·14) should be increased by $-v_X p$ and (7·15) by

$$\frac{\partial}{\partial x}(v_X p)+\frac{\partial}{\partial y}(v_Y p)+\frac{\partial}{\partial h}(v_H p)+\text{terms depending on molecular viscosity.}$$

These in turn lead to the lowering of the *ordinary* temperature at a point. But all that is well understood, and our simplest plan is to leave out p from all the equations, and to expect no result relating to ordinary cooling.

Similarly, with regard to gravity. That has been left out. If it had been included, we should have had to consider the mean flux of gravitational energy over a plane moving with the mean motion. The flux will vanish if

$$\overline{\rho' v_H'} = 0, \tag{7·26}$$

that is to say, if convection can be neglected. But having already dealt with convection in Sections IV and V, we may omit it here. Thus the increase of molar-kinetic and of gravitational energy in the cube is still given by (7·15). But, in place of (7·16), we shall have

$$\rho\,\frac{\bar{\mathrm{D}}}{\bar{\mathrm{D}}t}\left\{\psi+\tfrac{1}{2}\left(v_X{}^2+v_Y{}^2+v_H{}^2\right)\right\}, \tag{7·27}$$

where ψ is the gravity potential.

It is not now permissible to suppose v_H zero at the centre of the moving cube, since v_H must be relative to the attracting earth. So the first member of (7·17) must be increased by

$$\bar{v}_H\left\{\frac{\partial\widehat{xh}}{\partial x}+\frac{\partial\widehat{yh}}{\partial y}+\frac{\partial\widehat{hh}}{\partial h}\right\}, \tag{7·28}$$

while the second member of the same equation is increased by

$$\rho\,\frac{\bar{\mathrm{D}}}{\bar{\mathrm{D}}t}\left\{\psi+\tfrac{1}{2}\bar{v}_H{}^2\right\}. \tag{7·29}$$

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But these two increases balance one another, on account of the dynamical equation. Therefore (7·17) is correct also when gravity is present.

The foregoing theory, which states that the eddy-heat-per-mass is diminished by a mean expansion, may be expected to have some application to the following natural phenomena: (i) turbulent winds flowing up mountain sides; (ii) numerous small eddies carried together upwards in some larger circulation, such as that made visible by cumulus or thunder cloud.

But on account of the assumption (7·11) the theory does not apply to the case in which the small eddies wander individually from crowded to roomy regions.

VIII. *Diffusion of Eddying Energy.* (Revised May 11, 1920.)

Lieut.-Col. A. Ogilvie* describes flying over the Nile in a wind of 13 metres/sec.: "towards 10 o'clock or so the river appeared like a sheet of glass, but a little later, when the sun began to heat the ground, one saw ripples coursing over the surface of the river," and after half-an-hour or so the gusts reached the aviators at a height of about 2/3 km.

Capt. C. K. M. Douglas† also writes of "rising turbulence" as of something commonly experienced. These observations fit in with the theory of Section V to the effect that turbulence usually originates near the ground. According to what law then does it diffuse? Two different processes may go on at the same time. Eddies may wander individually from places where they are crowded to places where they are rare. We may speak of this as a *true* diffusion of eddy-heat-per-mass. Or large whirls, such as those indicated by cumuli, may lift several cubic kilometres of air containing thousands of smaller eddies. This might be called a large-eddy-diffusion of small-eddy-heat-per-mass. Let us discuss this first.

Small-scale-Eddy-heat Carried up by Larger Eddies.

Can it, for instance, be fitted into the following form of diffusion equation?

$$\frac{\partial \chi}{\partial t} = -\frac{\partial}{\partial p}\left(\xi_1 \frac{\partial \chi}{\partial p}\right), \tag{8·1}$$

where ξ_1 is some unknown quantity which is not zero when $\partial\chi/\partial p = 0$. For this to be a possible form χ must satisfy the following conditions.‡

1. χ must be carried along with the moving air. (8·2)
2. χ must be unaltered by the expansion or contraction of the air with which it moves. (8·3)

* 'Aeronautical Journal,' p. 415, July, 1919.

† 'J. Scott. Met. Soc.,' vol. 17.

‡ Taken with revision from 'Roy. Soc. Proc.,' A, vol. 96, pp. 10, 11 (1919).

And if dm be an element of mass and $\int \chi dm$ be taken over a definite portion of air, then

3. $\int \chi dm$ must be unchanged by internal rearrangement of the portion at constant pressure. (8·4)
4. $\int \chi dm$ must be unchanged by delay. (8·5)

May we put for χ the "potential-eddy-heat-per-mass" discussed in Section VII? That would satisfy (8·2), (8·3), (8·4). Condition (8·5) is not at all satisfied unless we treat the diffusion-process as supplemental to the changes indicated in (5·2) but with this understanding (8·5), and therefore also (8·1), would be satisfied if

$$\chi = \text{potential-eddy-heat-per-mass} = \Theta \left(\frac{\rho i}{\rho}\right)^{2/3},$$

$$= \left(\frac{\rho i}{\rho}\right)^{2/3} \frac{1}{2} \overline{(v_X{}'^2 + v_Y{}'^2 + v_H{}'^2)},$$

where Θ is the "eddy-heat-per-mass" and ρi is the standard density adopted for the purposes of definition.

The *true diffusion of eddy-heat* bears the same kind of relation to the convection of small eddies with cumuli as molecular conduction of ordinary heat bears to eddy-conduction of the same. We have already encountered in (7·11) an expression giving the mean rate at which energy diffuses across unit area of a plane perpendicular to the x—axis, in the form

$$\frac{1}{A} \int \frac{1}{2} \rho (v_X{}'^2 + v_Y{}'^2 + v_H{}'^2) v_X{}' \, dA,$$

but it is required to transform this into another applicable to observations.

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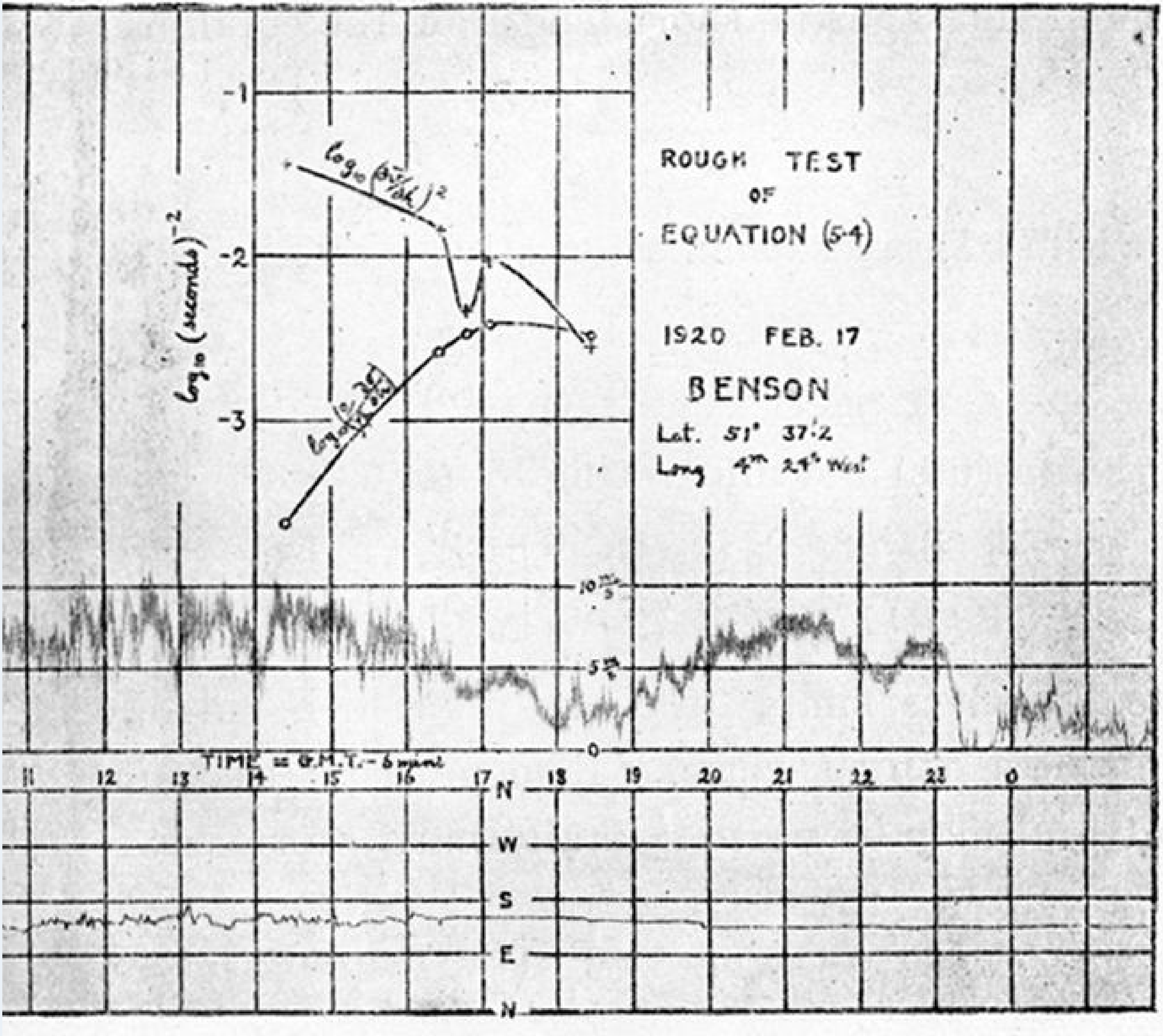
ROUGH TEST
OF
EQUATION (5·4)
1920 FEB. 17
BENSON
Lat. 51° 37'·2
Long 4m 24s West
$\log_{10}$ (seconds)$^{-2}$
-1
-2
-3
$\log_{10}(\partial \bar{v}/\partial h)^2$
10 m/s
5 m/s
0
TIME = G.M.T. - 6 mins
11
12
13
14
15
16
17
18
19
20
21
22
23
0
N
W
S
E
N