On the annihilator ideal in the bt-algebra of tensor space

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ABSTRACT. We study the representation theory of the *braids and ties* algebra, or the bt-algebra, $\mathcal{E}_n(q)$. Using the cellular basis $\{m_{\mathbb{s}\mathbb{t}}\}$ for $\mathcal{E}_n(q)$ obtained in previous joint work with J. Espinoza we introduce two kinds of permutation modules $M(\boldsymbol{\lambda})$ and $M(\Lambda)$ for $\mathcal{E}_n(q)$. We show that the tensor product module $V^{\otimes n}$ for $\mathcal{E}_n(q)$ is a direct sum of $M(\boldsymbol{\lambda})$'s. We introduce the dual cellular basis $\{n_{\mathbb{s}\mathbb{t}}\}$ for $\mathcal{E}_n(q)$ and study its action on $M(\boldsymbol{\lambda})$ and $M(\Lambda)$. We show that the annihilator ideal $\mathcal{I}$ in $\mathcal{E}_n(q)$ of $V^{\otimes n}$ enjoys a nice compatibility property with respect to $\{n_{\mathbb{s}\mathbb{t}}\}$. We finally study the quotient algebra $\mathcal{E}_n(q)/\mathcal{I}$, showing in particular that it is a simultaneous generalization of Härterich's 'generalized Temperley-Lieb algebra' and Juyumaya's 'partition Temperley-Lieb algebra'.

Keywords: Hecke algebra, cellular algebra, bt-algebra.

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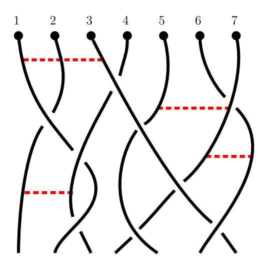
1. INTRODUCTION

In this paper we study the representation theory of the *braids and ties algebra* $\mathcal{E}_n(q)$, or the *bt-algebra* for short. It was introduced by Aicardi and Juyumaya in [Aicar-Juyu1], via an *abstraction* of the presentation for the Yokonuma-Hecke algebra $\mathcal{Y}_{r,n}(q)$ of type A_{n-1}. The bt-algebra carries a diagrammatics consisting of *braids and ties*, as illustrated below where braids are black and ties are dashed red.



(1)

The study of $\mathcal{E}_n(q)$ has attracted much attention recently, both from knot theorists and from representation theorists, see for example [Arcis-Juyu], [Aicar-Juyu2], [Aicar-Juyu3], [Aicar-Juyu4], [Ban], [ChlouPou], [EspRy],[Flo], [JacoAn], [Juyu], [Mar], [Ry] and the references therein. Interesting enough, even though $\mathcal{E}_n(q)$ is derived from $\mathcal{Y}_{r,n}(q)$, both the knot theory and the representation theory of $\mathcal{E}_n(q)$ are almost completely independent from those of $\mathcal{Y}_{r,n}(q)$, in the sense that almost no statement in the literature about $\mathcal{E}_n(q)$ is obtained directly from a corresponding statement about $\mathcal{Y}_{r,n}(q)$.

In our paper [Ry] we constructed a tensor space module $V^{\otimes n}$ for $\mathcal{E}_n(q)$, extending Jimbo's classical tensor space module for the Iwahori-Hecke algebra $\mathcal{H}_n(q)$ of type A_{n-1}, and showed that it is faithful when the dimension of V is large enough, although not in general. Using this we obtained a basis for $\mathcal{E}_n(q)$ which involves *set partitions*

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on $\mathbf{n} := \{1, 2, \ldots, n\}$, in particular we found that the dimension of $\mathcal{E}_n(q)$ is $b_n n!$, where b_n is the n'th Bell number, that is the number of set partitions on $\mathbf{n}$.

In the present paper we develop methods that allow us to treat the $\mathcal{E}_n(q)$-structure on $V^{\otimes n}$ when $V^{\otimes n}$ is not a faithful, and in particular to determine the annihilator ideal $\mathcal{I}$ for the $\mathcal{E}_n(q)$-action on $V^{\otimes n}$ in all cases. Our methods rely on *cellular algebra* theory, in particular on the cellular basis $\{m_{\mathbb{s}\mathbb{t}}\}$ for $\mathcal{E}_n(q)$ that was introduced in [EspRy], in joint work with J. Espinoza, and hence they are independent of the ground ring. We acknowledge that the construction of the $\{m_{\mathbb{s}\mathbb{t}}\}$-basis for $\mathcal{E}_n(q)$ is combinatorially much more involved than the construction of the $\{E_A g_w\}$-basis, or other similar non-cellular bases, but throughout our arguments rely crucially on the compatibility between the multiplicative structure and the order relation on $\mathcal{E}_n(q)$, given by the cell datum. In fact, it appears to be very difficult to obtain the results of our paper without using an appropriate order relation, even for the ground field $\mathbb{C}$ and generic q.

The basis $\{m_{\mathbb{s}\mathbb{t}}\}$ is a generalization of Murphy's *standard basis* $\{x_{\mathfrak{s}\mathfrak{t}}\}$ for $\mathcal{H}_n(q)$. In the classical representation theory of $\mathcal{H}_n(q)$, see for example [DipperJames] and [DipperJamMathas], an important application of $\{x_{\mathfrak{s}\mathfrak{t}}\}$ is to introduce *permutation modules* $M(\lambda)$ for $\mathcal{H}_n(q)$, where λ runs over *integer partitions* of n, and to realize Jimbo's tensor space module $V^{\otimes n}$ as a direct sum of these $M(\lambda)$'s. The $M(\lambda)$'s have many important properties, but for us it is of special relevance that they are endowed with canonical symmetric bilinear forms $(\cdot,\cdot)_\lambda$, that satisfy a certain non-vanishing property with respect to the *dual basis* $\{y_{\mathfrak{s}\mathfrak{t}}\}$ of $\{x_{\mathfrak{s}\mathfrak{t}}\}$.

In the paper we generalize the $M(\lambda)$'s to $\mathcal{E}_n(q)$-permutation modules $M(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda}$ running over *multipartitions*, and show in our Theorem 17 that the $\mathcal{E}_n(q)$-tensor module $V^{\otimes n}$ from [Ry] is a direct sum of these $M(\boldsymbol{\lambda})$'s in analogy with the $\mathcal{H}_n(q)$-case. The $M(\boldsymbol{\lambda})$'s are endowed with bilinear forms, in analogy with the $M(\lambda)$'s, but the parametrizing poset $\mathcal{L}_n$ for $\mathcal{E}_n(q)$ is a combinatorial object which is more complicated than multipartitions, and so the $M(\boldsymbol{\lambda})$'s are not sufficient for our determination of the annihilator ideal $\mathcal{I}$.

To solve this problem we introduce, via $\{m_{\mathbb{s}\mathbb{t}}\}$, another kind of permutation modules that we denote $M(\Lambda)$. We show in a series of Lemmas, culminating with Lemma 14, that the canonical bilinear form $(\cdot,\cdot)_\Lambda$ on $M(\Lambda)$ indeed satisfies the nonvanishing property with respect to the dual basis $\{n_{\mathbb{s}\mathbb{t}}\}$.

The above results constitute the bulk of work of our paper. The situation is not completely analogous to the situation for $\mathcal{H}_n(q)$, but in general there is an inclusion $M(\Lambda) \subseteq M(\boldsymbol{\lambda})$ and this is in fact enough to execute Härterich's wonderful argument, see [Härterich], for the determination of the annihilator ideal of the action of $\mathcal{H}_n(q)$ in Jimbo's tensor space $V^{\otimes n}$, but in the $\mathcal{E}_n(q)$-setting. We do so in Theorem 18.

When working over a ground field $\mathcal{K}$, Theorem 18 gives a $\mathcal{K}$-vector space basis for $\mathcal{I}$ in terms of an explicit subset of $\{n_{\mathbb{s}\mathbb{t}}\}$, thus generalizing Härterich's compatibility result. Note that Härterich's compatibility result is the starting point for the author's joint work with D. Plaza on KLR-gradings on the Temperley-Lieb algebras of type A and B, see [PlaRy]. Note also that other compatibility results for $\{m_{\mathbb{s}\mathbb{t}}\}$ were obtained in [EspRy].

In the final paragraphs of the paper we introduce and study the quotient algebra $\mathcal{ETL}_{n,N}(q) := \mathcal{E}_n(q)/\mathcal{I}$. It is a simultaneous generalization of Härterich's 'generalized Temperley-Lieb' algebra and of Juyumaya's 'partition Temperley-Lieb algebra $\mathcal{PTL}_n(q)$, that themselves are generalizations of the original Temperley-Lieb algebra. Our Theorem 18 allows us to determine the dimension of $\mathcal{ETL}_{n,N}(q)$ as the cardinality of an explicit subset of $\{n_{\mathbb{s}\mathbb{t}}\}$,

Let us now indicate the layout of the paper. In the next section we introduce the basic notation that shall be used throughout the paper. It is mostly concerned with combinatorial notions related to the symmetric group $\mathfrak{S}_n$, that is partitions, Young diagrams, Young tableaux, etc. Throughout the paper, $\mathfrak{S}_n$ is viewed as a Coxeter group and we only consider subgroups of $\mathfrak{S}_n$ that are parabolic subgroups, in the sense of Coxeter groups. It is well known that parabolic subgroups of Coxeter groups give rise to distinguished coset representatives. Throughout we need these distinguished coset representatives and their associated decompositions of the group elements, and we therefore briefly explain why they exist, but from an algorithmic point of view which is possibly less known. In the final paragraphs of this section we recall the definition of $\mathcal{E}_n(q)$, together with its $\{E_A g_w\}$ basis.

In section 3 we first recall from [EspRy] the ideal decomposition

$$\mathcal{E}_n(q) = \bigoplus_{\alpha \in \mathcal{P}ar_n} \mathcal{E}_n^\alpha(q) \tag{2}$$

that reduces the study of $\mathcal{E}_n(q)$ to the study of its summands $\mathcal{E}_n^\alpha(q)$. We next recall from [EspRy] the construction of the cellular basis $\{m_{\mathbb{s}\mathbb{t}}\}$ for $\mathcal{E}_n^\alpha(q)$ and at the same time we introduce the new dual cellular basis $\{n_{\mathbb{s}\mathbb{t}}\}$ by making the appropriate adaptions. It is a generalization of Murphy's dual standard basis $\{y_{\mathfrak{s}\mathfrak{t}}\}$ for $\mathcal{H}_n(q)$. The cell poset $\mathcal{L}_n(\alpha)$ for both bases consists of pairs $(\boldsymbol{\lambda} \mid \boldsymbol{\mu})$ of multipartitions, satisfying certain conditions.

In section 4 we first introduce the permutation module $M(\boldsymbol{\lambda})$. In Lemma 10 we describe a basis $\{x_{\boldsymbol{\mathfrak{s}}}\}$ for $M(\boldsymbol{\lambda})$ together with its $\mathcal{E}_n^\alpha(q)$-action. We next introduce, for $\Lambda \in \mathcal{L}_n(\alpha)$, the permutation module $M(\Lambda)$. We describe in the Lemmas 11 and 12 a basis $\{m_{\boldsymbol{\mathfrak{s}}}\}$ together with the $\mathcal{E}_n^\alpha(q)$-action on it. We then introduce the bilinear form $(\cdot,\cdot)_\Lambda$ on $M(\Lambda)$ and show that it is symmetric and invariant. Finally, in Lemma 14 we prove the following crucial property, alluded to above

$$(m_{\mathbb{t}} n_{\mathbb{t}'\mathbb{s}'}, m_{\mathbb{s}})_\Lambda \neq 0. \tag{3}$$

In section 5 we obtain the main results of our paper. We first give the decomposition of $V^{\otimes n}$ corresponding to 2 and then the decomposition in terms of the $M(\boldsymbol{\lambda})$'s. Finally, in the main Theorem 18, we give the description of $\mathcal{I}$, proving that a subset of $\{n_{\mathbb{s}\mathbb{t}}\}$ induces a basis for it.

It is a great pleasure to thank J. Espinoza for many useful discussions related to the contents of the paper, for sending us comments on preliminary versions of it, and for sharing with us his formula for $\dim \mathcal{PTL}_n(q)$. We also also thank J. Juyumaya for communicating to us his and P. Papi's MAGMA calculations and for pointing out an error in a preliminary version of the paper. Finally, it is a special pleasure to thank M. Härterich for introducing us to Murphy's standard basis and all its deep properties.

2. Notation and Basic Concepts

We fix the ground ring $S := \mathbb{Z}[q, q^{-1}]$, where q is an indeterminate.

Let $\mathfrak{S}_n$ be the symmetric group consisting of bijections of $\mathbf{n} := \{1, 2, \ldots, n\}$. Throughout we consider the natural action of $\mathfrak{S}_n$ on $\mathbf{n}$ as a right action. As is well known, $\mathfrak{S}_n$ is a Coxeter group on $\Sigma_n := \{s_1, \ldots, s_{n-1}\}$ with relations

$$s_i s_j = s_j s_i \qquad \text{for } |i-j| > 1 \tag{4}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad \text{for } i = 1, 2, \ldots, n-2 \tag{5}$$

$$s_i^2 = 1 \qquad \text{for } i = 1, 2, \ldots, n-1. \tag{6}$$

where $s_i := (i, i+1)$. We denote by $<$ the Bruhat-Chevalley order and by $\ell(\cdot)$ the length function on $\mathfrak{S}_n$.

Let $\mathbb{N}$ denote the set of natural numbers. A *composition* $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ of $n \in \mathbb{N}$, written $\mu \models n$, is a sequence in $\mathbb{N}^0$ with sum n. The μ_i's are called the parts of μ. An integer *partition* of n, written $\lambda \vdash n$, or simply a *partition* of n, is a composition whose parts form a non-increasing sequence. If $\mu \models n$ we define $|\mu| := n$ and denote by $\min\{i \mid \mu_k = 0 \text{ for } k > i\}$ the *length* of μ.

Suppose that $\mu = (\mu_1, \mu_2, \dots, \mu_l) \models n$ of length l. The set of all compositions of n is denoted $\mathcal{C}omp_n$ and the set of all partitions of n is denoted $\mathcal{P}ar_n$. We set $\mathcal{C}omp := \bigcup_n \mathcal{C}omp_n$ and $\mathcal{P}ar := \bigcup_n \mathcal{P}ar_n$. If $\mu = (\mu_1, \mu_2, \dots, \mu_l) \models n$ we define the *Young diagram* of μ as the subset of $\mathbb{N}^0 \times \mathbb{N}^0$ given by

$$\mathcal{Y}(\mu) := \left\{(i,j) \mid 1 \le j \le \mu_i \text{ and } i \ge 1\right\}. \tag{7}$$

The elements of $\mathcal{Y}(\mu)$ are the *nodes* of μ. We represent $\mathcal{Y}(\mu)$ by an array of boxes in the plane, using matrix convention for the placement of the nodes. For example, if $\mu = (4,4,0,3)$ then

$$\mathcal{Y}(\mu) = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline \multicolumn{4}{c}{} \\ \hline & & \multicolumn{1}{c|}{} & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array} \tag{8}$$

If $\lambda \in \mathcal{P}ar_n$ we denote by λ' the *conjugate* partition, which is the one obtained from λ by interchanging rows and columns, that is $\mathcal{Y}(\lambda') = \{(i,j) \mid (j,i) \in \mathcal{Y}(\lambda)\}$.

Suppose that $\mu \models n$. Then a μ*-tableau* is a bijection $\mathfrak{t} : \mathcal{Y}(\mu) \to \mathbf{n}$. If $\mathfrak{t}$ is a μ-tableau we identify it with a labelling of the nodes of $\mathcal{Y}(\mu)$, using the elements of $\mathbf{n}$. For example, if $\mu = (4,4,4,1)$ then

$$\mathfrak{t} = \begin{array}{|c|c|c|c|}\hline 1 & 4 & 3 & 13 \\ \hline 2 & 6 & 10 & 8 \\ \hline 5 & 7 & 11 & 9 \\ \hline \multicolumn{1}{|c|}{12} \\ \cline{1-1} \end{array} \tag{9}$$

is the μ-tableau $\mathfrak{t}$, that satisfies $\mathfrak{t}(1,1) = 1, \mathfrak{t}(1,2) = 4, \mathfrak{t}(1,3) = 3$, etc. If $\mathfrak{t}$ is a μ-tableau we define $Shape(\mathfrak{t}) := \mu$. The set of μ-tableaux is denoted $\mathrm{Tab}(\mu)$. A μ-tableau $\mathfrak{t}$ is *row standard* if the entries in $\mathfrak{t}$ increase from left to right in each row and it is *standard* if the entries also increase from top to bottom in each column. The set of row standard λ-tableaux is denoted $\mathrm{RStd}(\lambda)$ and the set of standard λ- tableaux is denoted $\mathrm{Std}(\lambda)$. For $\mu \models n$ we denote by $\mathfrak{t}^\mu$ (resp. $\mathfrak{t}_\mu$) the standard tableau in which the integers $1, 2, \dots, n$ are entered in increasing order from left to right along the rows (resp. columns) of $\mathcal{Y}(\mu)$. For example, if $\mu = (4,3)$ then $\mathfrak{t}^\mu = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}$ (resp. $\mathfrak{t}_\mu = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 7 \\ \hline 2 & 4 & 6 & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}$). If $\lambda \in \mathcal{P}ar_n$ and $\mathfrak{t}$ is a λ-tableau we denote by $\mathfrak{t}'$ the *conjugate* tableau. It is the λ'-tableau defined via $\mathfrak{t}'(i,j) := \mathfrak{t}(j,i)$.

$\mathfrak{S}_n$ acts naturally on the right on $\mathrm{Tab}(\lambda)$, via composition of a function $\mathfrak{t} \in \mathrm{Tab}(\lambda)$ with an element $\sigma \in \mathfrak{S}_n$ viewed as a bijection of $\mathbf{n}$. For $\mathfrak{t} \in \mathrm{Tab}(\lambda)$, we denote by $d(\mathfrak{t})$ the unique element of $\mathfrak{S}_n$ such that $\mathfrak{t} = \mathfrak{t}^\lambda d(\mathfrak{t})$. We set $w_\lambda := d(\mathfrak{t}_\lambda) \in \mathfrak{S}_n$. Then for any $\mathfrak{t} \in \mathrm{Tab}(\lambda)$ we have that

$$d(\mathfrak{t})d(\mathfrak{t}')^{-1} = w_\lambda \text{ and } \ell(d(\mathfrak{t})) + \ell(d(\mathfrak{t}')) = \ell(w_\lambda) \tag{10}$$

see for example the proof of Lemma 2.2 of [Murphy92]. The *Young subgroup* $\mathfrak{S}_\lambda$ associated with λ is the row stabilizer of $\mathfrak{t}^\lambda$. It is a parabolic subgroup of $\mathfrak{S}_n$ in the sense of Coxeter groups. The set $\left\{d(\mathfrak{t}) \,|\, \mathfrak{t} \in \mathrm{RStd}(\lambda)\right\}$ is a set of *distinguished right coset representatives* for $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$, that is $\ell(w_0 d(\mathfrak{t})) = \ell(w_0) + \ell(d(\mathfrak{t}))$ for $w \in \mathfrak{S}_\lambda$. In other words, for any $w \in \mathfrak{S}_n$ there is a unique decomposition

$$w = w_0 d(\mathfrak{t}) \;\text{ where }\; w_0 \in \mathfrak{S}_\lambda \;\text{ and }\; \mathfrak{t} \in \mathrm{RStd}(\lambda) \tag{11}$$

see Proposition 3.3 of [Mathas] and chapter 1.10 of [Hump]. The determination of w_0 and $\mathfrak{t}$ can be realized in an algorithmic way that we now explain. Define first $\mathfrak{s} \in \mathrm{Tab}(\lambda)$ via $w = d(\mathfrak{s})$. Let next $s_{j_1}, s_{j_2}, \dots, s_{j_k}$ be a sequence of elements in S such that when setting $\mathfrak{s}_0 := \mathfrak{s}, \mathfrak{s}_1 := \mathfrak{s}_0 s_{j_1}, \mathfrak{s}_2 := \mathfrak{s}_1 s_{j_2}, \dots, \mathfrak{s}_k := \mathfrak{s}_{k-1} s_{j_k}$ we have that i_j appears strictly below $i_j + 1$ in $\mathfrak{s}_j$ and that $\mathfrak{s}_k$ is equal to $\mathfrak{s}^\lambda$ modulo a permutation of the row elements. Such a sequence always exists. Then $w_0 = d(\mathfrak{s}_k)$ and $\mathfrak{t} = \mathfrak{t}^\lambda s_{j_k} s_{j_{k-1}} \dots s_{j_1}$, that is $d(\mathfrak{t}) = s_{j_k} s_{j_{k-1}} \dots s_{j_1}$. Note that although w_0 and $d(\mathfrak{t})$ are unique, the sequence

$s_{j_1}, s_{j_2}, \ldots, s_{j_k}$ is in general not unique. Here is an example of this algorithm, using $\lambda = (2,3,2)$ and $w = d(\mathfrak{s})$ where

$$\mathfrak{s} = \begin{array}{|c|c|c|} \hline 2 & 6 \\ \hline 3 & 5 & 1 \\ \hline 7 & 4 \\ \hline \end{array}. \tag{12}$$

Then a possible sequence of s_{j_i}'s is given as follows

$$\begin{array}{ccc} 2 & 6 & \\ 3 & 5 & 1 \\ 7 & 4 & \end{array} \xrightarrow{s_1} \begin{array}{ccc} 1 & 6 & \\ 3 & 5 & 2 \\ 7 & 4 & \end{array} \xrightarrow{s_4} \begin{array}{ccc} 1 & 6 & \\ 3 & 4 & 2 \\ 7 & 5 & \end{array} \xrightarrow{s_5} \begin{array}{ccc} 1 & 5 & \\ 3 & 4 & 2 \\ 7 & 6 & \end{array} \xrightarrow{s_4} \begin{array}{ccc} 1 & 4 & \\ 3 & 5 & 2 \\ 7 & 6 & \end{array} \xrightarrow{s_3} \begin{array}{ccc} 1 & 3 & \\ 4 & 5 & 2 \\ 7 & 6 & \end{array} \xrightarrow{s_2} \begin{array}{ccc} 1 & 2 & \\ 4 & 5 & 3 \\ 7 & 6 & \end{array} \tag{13}$$

and so $d(\mathfrak{t}) = s_2 s_3 s_4 s_5 s_4 s_1$ whereas $w_0 = d(\mathfrak{t}) = s_4 s_3 s_6$. Note that

$$\mathfrak{t} = \mathfrak{t}^{\lambda} s_2 s_3 s_4 s_5 s_4 s_1 = \begin{array}{|c|c|c|} \hline 2 & 6 \\ \hline 1 & 3 & 5 \\ \hline 4 & 7 \\ \hline \end{array} \tag{14}$$

which is also the tableau obtained from $\mathfrak{s}$ by ordering the rows.

Let $\mu, \nu \in \mathcal{C}omp$. We write $\mu \trianglelefteq \nu$ if for all $i \geq 1$ we have

$$\sum_{j=1}^{i} \mu_j \leq \sum_{j=1}^{i} \nu_j$$

where, if necessary, we add zero parts at the end of μ and ν so that the sums are defined. This is the dominance order on compositions. It is extended to row standard tableaux as follows. Suppose $\lambda \models n$ and that $\mathfrak{t} \in \mathrm{RStd}(\lambda)$. We then define $\mathfrak{t}_{|\leq m}$ to be the tableau obtained from $\mathfrak{t}$ by deleting all nodes with entries strictly greater than m. Then, for row standard μ-tableaux $\mathfrak{s}$ and $\mathfrak{t}$ we write $\mathfrak{s} \trianglelefteq \mathfrak{t}$ if $Shape(\mathfrak{s}_{|\leq m}) \trianglelefteq Shape(\mathfrak{t}_{|\leq m})$ for all $m = 1, \ldots, n$. We write $\mathfrak{s} \triangleleft \mathfrak{t}$ if $\mathfrak{s} \trianglelefteq \mathfrak{t}$ and $\mathfrak{s} \neq \mathfrak{t}$. This defines the *dominance order* on the set of all row standard tableaux. We have that $\mathfrak{t}^{\lambda}$ (resp. $\mathfrak{t}_{\lambda}$) is the unique maximal (resp. minimal) row standard λ-tableau under the dominance order.

An *r-multicomposition*, or just a multicomposition if confusion is not possible, of n is an r-tuple $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$ of (possibly empty) compositions $\lambda^{(k)}$ such that $\sum_{i=1}^{r} |\lambda^{(i)}| = n$. We call $\lambda^{(k)}$ the k'th component of $\boldsymbol{\lambda}$. An *r-multipartition* is an r-multicomposition whose components are all partitions. The nodes $\mathcal{Y}(\boldsymbol{\lambda})$ of an r-multicomposition $\boldsymbol{\lambda}$ are labelled by triples (x, y, p) with p giving the component number and (x, y) the node of that component. This is the Young diagram for $\boldsymbol{\lambda}$ and is represented graphically as the r-tuple of Young diagrams of the components. For example, the Young diagram of $\boldsymbol{\lambda} = ((3,3),(3,1),(1,1,1))$ is

$$\left(\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline & & \\ \hline \\ \cline{1-1} \end{array}, \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \end{array} \right).$$

The set of r-multicompositions of n is denoted by $Comp_{r,n}$ and the subset of r-multipartitions of n is denoted by $Par_{r,n}$. If $\boldsymbol{\lambda} \in Comp_{r,n}$, then a *$\boldsymbol{\lambda}$-multitableau* $\mathbf{t}$ is a bijection $\mathbf{t} : \mathcal{Y}(\boldsymbol{\lambda}) \to \mathbf{n}$; it is identified with a labelling of $\mathcal{Y}(\boldsymbol{\lambda})$ using the elements of $\mathbf{n}$. The restriction of $\mathbf{t}$ to $\lambda^{(i)}$ is the i'th component $\mathfrak{t}^{(i)}$ of $\mathbf{t}$ and we write $\mathbf{t} = (\mathfrak{t}^{(1)}, \mathfrak{t}^{(2)}, \ldots, \mathfrak{t}^{(r)})$. We say that $\mathbf{t}$ is *row standard* if all its components are row standard, and *standard* if they are all standard. If $\mathbf{t}$ is a $\boldsymbol{\lambda}$-multitableau we write $Shape(\mathbf{t}) = \boldsymbol{\lambda}$. The set of all $\boldsymbol{\lambda}$-tableaux is denoted by $\mathrm{Tab}(\boldsymbol{\lambda})$, the set of all row standard $\boldsymbol{\lambda}$-tableaux is denoted by $\mathrm{RStd}(\boldsymbol{\lambda})$ and the set of all standard $\boldsymbol{\lambda}$-multitableaux by $\mathrm{Std}(\boldsymbol{\lambda})$. In the following examples

$$\mathbf{t} = \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}, \begin{array}{|c|} \hline 6 \\ \hline 7 \\ \hline 8 \\ \hline \end{array} \right), \qquad \boldsymbol{\mathfrak{s}} = \left(\begin{array}{|c|c|c|} \hline 2 & 7 & 8 \\ \hline 1 & 4 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|} \hline 5 & 6 \\ \hline \end{array}, \begin{array}{|c|} \hline 3 \\ \hline 9 \\ \hline \end{array} \right) \tag{15}$$

$\mathbf{t}$ is a standard multitableau whereas $\boldsymbol{\mathfrak{s}}$ is only a row standard multitableau. We denote by $\mathbf{t}^{\boldsymbol{\lambda}}$ (resp. $\mathbf{t}_{\boldsymbol{\lambda}}$) the $\boldsymbol{\lambda}$-multitableau in which $1, 2, \ldots, n$ appear in order along the rows (resp. columns) of the first component, then along the rows of the second component, and so on. For example, in (15) we have $\mathbf{t} = \mathbf{t}^{\boldsymbol{\lambda}}$ with $\boldsymbol{\lambda} = ((3,2),(1,1,1))$. For each multicomposition $\boldsymbol{\lambda}$ we define the Young subgroup $\mathfrak{S}_{\boldsymbol{\lambda}}$ as the row stabilizer of $\mathbf{t}^{\boldsymbol{\lambda}}$. It is a parabolic subgroup in the sense of Coxeter groups. For $\boldsymbol{\mathfrak{s}}$ a row standard $\boldsymbol{\lambda}$-multitableau, we denote by $d(\boldsymbol{\mathfrak{s}})$ the unique element of $\mathfrak{S}_n$ such that $\boldsymbol{\mathfrak{s}} = \mathbf{t}^{\boldsymbol{\lambda}} d(\boldsymbol{\mathfrak{s}})$. The set $\left\{d(\boldsymbol{\mathfrak{s}}) \,|\, \boldsymbol{\mathfrak{s}} \in \mathrm{RStd}(\boldsymbol{\lambda})\right\}$ is a set of *distinguished right coset representatives* for $\mathfrak{S}_{\boldsymbol{\lambda}}$ in $\mathfrak{S}_n$, that is $\ell(w_0 d(\boldsymbol{\mathfrak{s}})) = \ell(w_0) + \ell(d(\boldsymbol{\mathfrak{s}}))$ for $w \in \mathfrak{S}_{\boldsymbol{\lambda}}$. Suppose that $\boldsymbol{\lambda} := (\lambda^{(1)}, \dots, \lambda^{(r)}) \in Par_{r,n}$ and that $\mathbf{t} := (\mathrm{t}^{(1)}, \dots, \mathrm{t}^{(r)}) \in \mathrm{Tab}(\boldsymbol{\lambda})$. Then we define the conjugation of $\boldsymbol{\lambda}$ and $\mathbf{t}$ componentwise, that is

$$\boldsymbol{\lambda}' := ((\lambda^{(1)})', \dots, (\lambda^{(r)})') \text{ and } \mathbf{t}' := ((\mathrm{t}^{(1)})', \dots, (\mathrm{t}^{(r)})') \tag{16}$$

Similarly to 11, there is for any $w \in \mathfrak{S}_n$ a unique decomposition

$$w = w_0 d(\mathbf{t}) \;\text{ where }\; w_0 \in \mathfrak{S}_{\boldsymbol{\lambda}} \;\text{ and }\; \mathbf{t} \in \mathrm{RStd}(\boldsymbol{\lambda}). \tag{17}$$

Suppose that $w = d(\boldsymbol{\mathfrak{s}})$. Then applying the algorithm in 11 on the tableau $\mathfrak{s}$ obtained from $\boldsymbol{\mathfrak{s}} = (\mathfrak{s}^{(1)}, \mathfrak{s}^{(2)}, \dots, \mathfrak{s}^{(r)})$ by concatenating all its components, with $\mathfrak{s}^{(1)}$ on top followed by $\mathfrak{s}^{(2)}$ just below $\mathfrak{s}^{(1)}$ and so on, we obtain w_0 and $\mathbf{t}$ in 17.

Suppose that $\boldsymbol{\lambda} \in Comp_{r,n}$ and that $\mathbf{t} \in \mathrm{Tab}(\boldsymbol{\lambda})$ is row standard. Then for $j = 1, \dots, n$ we set $p_{\mathbf{t}}(j) := k$ if j appears in $\mathrm{t}^{(k)}$. We call $p_{\mathbf{t}}(j)$ the *position* of j in $\mathbf{t}$. When $\mathbf{t} = \mathbf{t}^{\boldsymbol{\lambda}}$, we write $p_{\boldsymbol{\lambda}}(j)$ for $p_{\mathbf{t}^{\boldsymbol{\lambda}}}(j)$. We say that $\mathbf{t}$ is of the *initial kind* if $p_{\mathbf{t}}(j) = p_{\boldsymbol{\lambda}}(j)$ for all $j = 1, \dots, n$.

Suppose that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in Comp_{r,n}$. We write $\boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}$ if $\lambda^{(i)} \trianglelefteq \mu^{(i)}$ for all $i = 1, \dots, n$, this is the dominance order on $Comp_{r,n}$. If $\boldsymbol{\mathfrak{s}}$ and $\mathbf{t}$ are row standard multitableaux and $m = 1, \dots, n$ we define $\boldsymbol{\mathfrak{s}}_{|\leq m}$ and $\mathbf{t}_{|\leq m}$ as for usual tableaux and write $\boldsymbol{\mathfrak{s}} \trianglelefteq \mathbf{t}$ if $Shape(\boldsymbol{\mathfrak{s}}_{|\leq m}) \trianglelefteq Shape(\mathbf{t}_{|\leq m})$ for all m. Note that our dominance order $\trianglelefteq$ is different from the dominance order on multicompositions and multitableaux that is used sometimes in the literature, for example in [DipperJamMathas].

To an r-multicomposition $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ we associate a composition $\|\boldsymbol{\lambda}\|$ of length r in the following way

$$\|\boldsymbol{\lambda}\| := \left(|\lambda^{(1)}|, \dots, |\lambda^{(r)}|\right). \tag{18}$$

Let $\mathfrak{S}_{\|\boldsymbol{\lambda}\|}$ be the associated Young subgroup. Then $w \in \mathfrak{S}_{\|\boldsymbol{\lambda}\|}$ iff $\mathbf{t}^{\boldsymbol{\lambda}} w$ is of the initial kind. For $\mathbf{t} = (\mathrm{t}^{(1)}, \dots, \mathrm{t}^{(r)}) \in \mathrm{Tab}(\boldsymbol{\lambda})$ we define $\|\mathbf{t}\| \in \mathrm{Tab}(\|\boldsymbol{\lambda}\|)$ via

$$\|\mathbf{t}\| = (|\mathrm{t}^{(1)}|, \dots, |\mathrm{t}^{(r)}|) \tag{19}$$

where $|\mathrm{t}^{(i)}|$ is the *row reading* of $\mathrm{t}^{(i)}$. For the multitableaux $\mathbf{t}$ and $\boldsymbol{\mathfrak{s}}$ in 15 we have for example

$$\|\mathbf{t}\| = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 \\ \cline{1-3} \end{array}, \quad \|\boldsymbol{\mathfrak{s}}\| = \begin{array}{|c|c|c|c|c|} \hline 2 & 7 & 8 & 1 & 4 \\ \hline 5 & 6 \\ \cline{1-2} 3 & 9 \\ \cline{1-2} \end{array}. \tag{20}$$

$\mathfrak{S}_{\|\boldsymbol{\lambda}\|}$ is yet another parabolic subgroup of $\mathfrak{S}_n$, with corresponding distinguished right coset representatives $\left\{\mathbf{t} \in \mathrm{Tab}(\boldsymbol{\lambda}) \;\middle|\; \|\mathbf{t}\| \in \mathrm{RStd}(\|\boldsymbol{\lambda}\|)\right\}$. Hence, for any $w \in \mathfrak{S}_n$ there is a unique decomposition

$$w = w_0 d(\mathbf{t}) \;\text{ where }\; w_0 \in \mathfrak{S}_{\|\boldsymbol{\lambda}\|} \;\text{ and }\; \mathbf{t} \in \mathrm{Tab}(\boldsymbol{\lambda}) \;\text{ with }\; \|\mathbf{t}\| \in \mathrm{RStd}(\|\boldsymbol{\lambda}\|). \tag{21}$$

We define $\boldsymbol{\mathfrak{s}}_0 \in \mathrm{Tab}(\boldsymbol{\lambda})$ via $w_0 = \mathbf{t}^{\boldsymbol{\lambda}} d(\boldsymbol{\mathfrak{s}}_0)$; it is of the initial kind. We obtain $\boldsymbol{\mathfrak{s}}_0$ and $\mathbf{t}$ concretely by applying the algorithm given in 13 with respect to w and $\|\boldsymbol{\lambda}\|$. For example, if $w = d(\boldsymbol{\mathfrak{s}})$ where

$$\boldsymbol{\mathfrak{s}} = \left(\begin{array}{|c|} \hline 2 \\ \hline 6 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 \\ \cline{1-1} 5 & 1 \\ \hline \end{array}, \begin{array}{|c|} \hline 7 \\ \hline 4 \\ \hline \end{array} \right) \tag{22}$$

then the calculations in 13 give us

$$\boldsymbol{\mathfrak{s}}_0 = \left(\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 4 \\ \cline{1-1} 5 & 3 \\ \hline \end{array}, \begin{array}{|c|} \hline 7 \\ \hline 6 \\ \hline \end{array} \right), \quad \mathbf{t} = \left(\begin{array}{|c|} \hline 2 \\ \hline 6 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 \\ \cline{1-1} 3 & 5 \\ \hline \end{array}, \begin{array}{|c|} \hline 4 \\ \hline 7 \\ \hline \end{array} \right). \tag{23}$$

It follows from the algorithm that $\mathfrak{s}_0$ is row standard if $\mathfrak{s}$ is row standard.

Suppose that $\mathfrak{s} \in \mathrm{RStd}(\boldsymbol{\lambda}), \mathfrak{s}_1 \in \mathrm{RStd}(\boldsymbol{\lambda}_1)$ with $\|\boldsymbol{\lambda}\| = \|\boldsymbol{\lambda}_1\|$. Let $w = d(\mathfrak{s}), w_1 = d(\mathfrak{s}_1)$ and let $w = w_0 d(\mathfrak{t}), w_1 = (w_1)_0 d(\mathfrak{t}_1)$ be the decompositions of w and w_1 as in 21. Suppose furthermore that $d(\mathfrak{t}) = d(\mathfrak{t}_1)$. Then we have that

$$\mathfrak{s} \trianglelefteq \mathfrak{s}_1 \text{ if and only if } \mathfrak{s}_0 \trianglelefteq (\mathfrak{s}_1)_0. \tag{24}$$

For $\boldsymbol{\lambda}$ any r-multipartition we define $w_{\boldsymbol{\lambda}} \in \mathfrak{S}_n$ via $w_{\boldsymbol{\lambda}} := d(\mathfrak{t}_{\boldsymbol{\lambda}})$. Suppose now that $\mathfrak{s} \in \mathrm{Tab}(\boldsymbol{\lambda})$ and let $\mathfrak{s}'$ be the conjugate multitableau. Set $w = d(\mathfrak{s})$ and $w' = d(\mathfrak{s}')$ with decompositions $w = w_0 d(\mathfrak{t})$ and $w' = (w')_0 d(\mathfrak{t}')$ as in 21. Then it follows from the algorithm described above that $d(\mathfrak{t}) = d(\mathfrak{t}')$ and that $(w')_0 = d(\mathfrak{t}'_0)$ where $w_0 = d(\mathfrak{t}_0)$. Moreover, since $\mathfrak{t}_0$ is of the initial kind we get via 10 that

$$d(\mathfrak{s}) d(\mathfrak{s}')^{-1} = w_{\boldsymbol{\lambda}} \text{ and } \ell(d(\mathfrak{t}_0)) + \ell(d(\mathfrak{t}'_0)) = \ell(w_{\boldsymbol{\lambda}}). \tag{25}$$

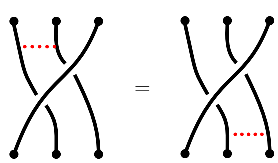
Note that we cannot here replace $d(\mathfrak{t}_0)$ and $d(\mathfrak{t}'_0)$ by $d(\mathfrak{s})$ and $d(\mathfrak{s}')$.

Let us now recall the bt-algebra of the title of the paper. It was originally introduced by Aicardi and Juyumaya in [Aicar-Juyu1].

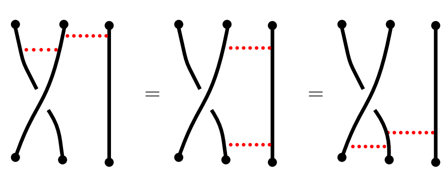
Definition 1 Let n be a positive integer. The braids and ties algebra $\mathcal{E}_n = \mathcal{E}_n(q)$, or the bt-algebra, is the associative S-algebra generated by the elements $g_1, \dots, g_{n-1}$ and $e_1, \dots, e_{n-1}$, subject to the following relations:

$$\begin{aligned}
g_i g_j &= g_j g_i && \text{for } |i-j| > 1 && (26)\\
g_i e_i &= e_i g_i && \text{for all } i && (27)\\
g_i g_j g_i &= g_j g_i g_j && \text{for } |i-j| = 1 && (28)\\
e_i g_j g_i &= g_j g_i e_j && \text{for } |i-j| = 1 && (29)\\
e_i e_j g_j = e_i g_j e_i &= g_j e_i e_j && \text{for } |i-j| = 1 && (30)\\
e_i e_j &= e_j e_i && \text{for all } i, j && (31)\\
g_i e_j &= e_j g_i && \text{for } |i-j| > 1 && (32)\\
e_i^2 &= e_i && \text{for all } i && (33)\\
g_i^2 &= 1 + (q - q^{-1}) e_i g_i && \text{for all } \ i. && (34)
\end{aligned}$$

There is a diagrammatic interpretation of these relations that explains the name of $\mathcal{E}_n(q)$ and that we now briefly indicate even though it only plays a minor role in the present paper. It uses n strands connecting a northern and a southern border. Under the interpretation, the g_i's correspond to usual simple *braids* involving the i'th and the $i+1$'st strand and the e_i's correspond to *ties* involving the i'th and $i+1$'st strands. The relations 29 and 30 are for example visualized as follows.



(35)



(36)

If $\mathcal{K}$ is a commutative ring containing an invertible element, that we also denote q, then we define the specialized algebra $\mathcal{E}_n^{\mathcal{K}}(q)$ via $\mathcal{E}_n^{\mathcal{K}}(q) := \mathcal{E}_n(q) \otimes_S \mathcal{K}$ where $\mathcal{K}$ is made into an S-algebra by mapping $q \in S$ to $q \in \mathcal{K}$.

The concept of *set partitions* is important in the study of $\mathcal{E}_n(q)$. Recall that a set partition $A = \{I_1, I_2, \dots I_r\}$ of $\mathbf{n}$ is a set of nonempty and disjoint subsets of $\mathbf{n}$ whose union is $\mathbf{n}$, in order words, A is the set of classes of an equivalence relation on $\mathbf{n}$. We refer to the I_i's as the *blocks* of A. We denote by $\mathcal{SP}_n$ the set of all set partitions of $\mathbf{n}$. There is a canonical poset structure on $\mathcal{SP}_n$ defined as follows. Suppose that $A = \{I_1, I_2, \dots, I_k\}, B = \{J_1, I_2, \dots, J_l\} \in \mathcal{SP}_n$. Then we say that $A \subseteq B$ if each J_j is a union of some of the I_i's.

Returning to $\mathcal{E}_n(q)$, for $1 \le i < j \le n$ we define $E_{ij} \in \mathcal{E}_n(q)$ by $E_{ij} = e_i$ if $i = j-1$, and recursively downwards on i via

$$E_{ij} := g_i E_{i+1,j} g_i^{-1}. \tag{37}$$

Note that g_i is invertible with inverse $g_i^{-1} = g_i + (q^{-1} - q)e_i$, as follows from 34, so that 37 makes sense. For $A \in \mathcal{SP}_n$ we define $E_A \in \mathcal{E}_n(q)$ as

$$E_A := \prod_{i,j} E_{ij} \tag{38}$$

where the product runs over pairs (i,j), such that $i < j$ and such that i and j belong to the same block of A. One checks that the E_{ij}'s commute and so the product is independent of the order in which it is taken. For $w = s_{i_1} \cdots s_{i_k} \in \mathfrak{S}_n$ a reduced expression for w we define $g_w := g_{i_1} \cdots g_{i_k} \in \mathcal{E}_n(q)$. By the relations it is independent of the choice of reduced expression. We have the following relations in $\mathcal{E}_n(q)$

$$E_A g_w = g_w E_{Aw} \text{ and } E_A E_B = E_C \text{ for } w \in \mathfrak{S}_n, A, B \in \mathcal{SP}_n \tag{39}$$

where $C \in \mathcal{SP}_n$ is minimal with respect to $A \subseteq C, B \subseteq C$. Moreover, it was shown in [Ry] that $\mathcal{E}_n(q)$ has an S-basis of the form $\left\{E_A g_w | A \in \mathcal{SP}_n, w \in \mathfrak{S}_n\right\}$ and so, in particular, the dimension of $\mathcal{E}_n(q)$ is $n! b_n$ where b_n is the Bell number, that is the cardinality of $\mathcal{SP}_n$.

3. Cellular basis for $\mathcal{E}_n(q)$.

In the paper [EspRy], a cellular basis $\{m_{\mathfrak{s}\mathfrak{t}}\}$ for $\mathcal{E}_n(q)$ was constructed. Moreover, it was shown that $\{m_{\mathfrak{s}\mathfrak{t}}\}$ induces cellular bases for interesting subalgebras of $\mathcal{E}_n(q)$ and via this an isomorphism between $\mathcal{E}_n(q)$ and a direct sum of matrix algebras over certain wreath products of Hecke algebras was established.

In this section we first recall the construction of $\{m_{\mathfrak{s}\mathfrak{t}}\}$ and at the same time we introduce a new, *dual*, cellular basis for $\mathcal{E}_n(q)$, that we denote $\{n_{\mathfrak{s}\mathfrak{t}}\}$. We next prove, in a series of Lemmas, certain compatibility properties between the dominance order associated with $\mathcal{E}_n(q)$ and the multiplication of elements from the two bases. These compatibility properties are known in the Hecke algebra setting, and our proofs are essentially reductions to that setting.

Let us first recall the formal definition of cellular algebras, as introduced in [GraLeh].

Definition 2 Let $\mathcal{R}$ be an integral domain. Suppose that A is an $\mathcal{R}$-algebra. Suppose that $(\mathcal{P}, \le)$ is a poset and that for each $\lambda \in \mathcal{P}$ there is a finite indexing set $T(\lambda)$ (the 'λ-tableaux') and elements $c^\lambda_{\mathfrak{s}\mathfrak{t}} \in A$ such that

$$\mathcal{C} = \left\{ c^\lambda_{\mathfrak{s}\mathfrak{t}} \mid \lambda \in \mathcal{P} \text{ and } \mathfrak{s}, \mathfrak{t} \in T(\lambda) \right\} \tag{40}$$

is an $\mathcal{R}$-basis of A. The pair $(\mathcal{C}, \mathcal{P})$ is a *cellular basis* for A if

- **(i)** The $\mathcal{R}$-linear map $*: A \to A$ determined by $(c^\lambda_{\mathfrak{s}\mathfrak{t}})^* = c^\lambda_{\mathfrak{t}\mathfrak{s}}$ for all $\lambda \in \mathcal{P}$ and all $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$ is an algebra anti-automorphism of A.
- **(ii)** For any $\lambda \in \mathcal{P}$, $\mathfrak{t} \in T(\lambda)$ and $a \in A$ there exist elements $r_\mathfrak{v} \in \mathcal{R}$ such that for all $\mathfrak{s} \in T(\lambda)$

$$c^\lambda_{\mathfrak{s}\mathfrak{t}} a \equiv \sum_{\mathfrak{v} \in T(\lambda)} r_\mathfrak{v} c^\lambda_{\mathfrak{s}\mathfrak{v}} \mod A^\lambda$$

where A^λ is the $\mathcal{R}$-submodule of A with basis $\left\{ c^\mu_{\mathfrak{u}\mathfrak{v}} \mid \mu \in \mathcal{P}, \mu > \lambda \text{ and } \mathfrak{u}, \mathfrak{v} \in T(\mu) \right\}$.

If A has a cellular basis we say that A is a *cellular algebra* and $(\mathcal{P}, T, \mathcal{C}, *)$ is called the *cell datum* for A.

We now recall the, somewhat lengthy, construction of the cell datum for $\mathcal{E}_n(q)$. For more details on this construction, the reader should consult [EspRy].

We first consider a certain ideal decomposition of $\mathcal{E}_n(q)$. Let $\mu_{\mathcal{SP}_n}$ be the Möebius function for the lattice $(\mathcal{SP}_n, \subseteq)$. It is given by

$$\mu_{\mathcal{SP}_n}(A,B) = \begin{cases} (-1)^{r-s}\prod_{i=1}^{r-1}(i!)^{r_{i+1}} & \text{if } A \subseteq B \\ 0 & \text{otherwise} \end{cases} \tag{41}$$

where r and s are the number of blocks of A and B respectively, and where r_i is the number of blocks of B containing exactly i blocks of A. Using it, we define for $A \in \mathcal{SP}_n$ orthogonal idempotents $\mathbb{E}_A \in \mathcal{E}_n(q)$ via

$$\mathbb{E}_A := \sum_{B: A\subseteq B} \mu(A,B) E_B. \tag{42}$$

For example $\mathbb{E}_{\{\{1\},\{2\},\{3\}\}} = E_{\{\{1\},\{2\},\{3\}\}} - E_{\{\{1,2\},\{3\}\}} - E_{\{\{1\},\{2,3\}\}} - E_{\{\{1,3\},\{2\}\}} + 2E_{\{\{1,2,3\}\}}$. The following is Proposition 39 in [EspRy].

Proposition 3 The following properties hold.

(1) $\{\mathbb{E}_A | A \in \mathcal{SP}_n\}$ is a set of orthogonal idempotents of $\mathcal{E}_n(q)$.

(2) For all $w \in \mathfrak{S}_n$ and $A \in \mathcal{SP}_n$ we have $\mathbb{E}_A g_w = g_w \mathbb{E}_{Aw}$.

(3) For all $A \in \mathcal{SP}_n$ we have $\mathbb{E}_A E_B = \begin{cases} \mathbb{E}_A & \text{if } B \subseteq A \\ 0 & \text{if } B \not\subseteq A. \end{cases}$

Let $\alpha \in \mathcal{P}ar_n$. A set partition $A = \{I_1, \dots, I_k\}$ of $\mathbf{n}$ is said to be of *type* α if there exists a permutation σ such that $(|I_{i_{1\sigma}}|, \dots, |I_{i_{k\sigma}}|) = \alpha$. We write $|A| = \alpha$ if $A \in \mathcal{SP}_n$ is of type α and we let $\mathcal{SP}_n^\alpha$ be the set of set partitions of type α. For $\alpha \in \mathcal{P}ar_n$ we define the following element $\mathbb{E}_\alpha$ of $\mathcal{E}_n(q)$

$$\mathbb{E}_\alpha := \sum_{A\in\mathcal{SP}_n, |A|=\alpha} \mathbb{E}_A. \tag{43}$$

Then the $\mathbb{E}_\alpha$'s form a family of central orthogonal idempotents in $\mathcal{E}_n(q)$ such that $\sum_{\alpha\in\mathcal{P}ar_n} \mathbb{E}_\alpha = 1$. As a consequence we have the following decomposition of $\mathcal{E}_n(q)$ into a direct sum of two-sided ideals

$$\mathcal{E}_n(q) = \bigoplus_{\alpha\in\mathcal{P}ar_n} \mathcal{E}_n^\alpha(q) \tag{44}$$

where $\mathcal{E}_n^\alpha(q) := \mathbb{E}_\alpha \mathcal{E}_n(q)$. Each ideal $\mathcal{E}_n^\alpha(q)$ is an S-algebra with identity $\mathbb{E}_\alpha$ and the set

$$\{\mathbb{E}_A g_w \mid w \in \mathfrak{S}_n, |A| = \alpha\} \tag{45}$$

is an S-basis for $\mathcal{E}_n^\alpha(q)$. In particular the dimension of $\mathcal{E}_n^\alpha(q)$ is $n!$ times the Faà di Bruno coefficient, that is $b_n(\alpha)n!$, where by definition the Faà di Bruno coefficient $b_n(\alpha)$ is the number of set partitions of $\mathbf{n}$ of type α. For $\mathcal{K}$ an arbitrary field with a nonzero element we have a specialized version of $\mathcal{E}_n^\alpha(q)$

$$\mathcal{E}_n^{\mathcal{K},\alpha}(q) := \mathcal{E}_n^\alpha(q) \otimes_S \mathcal{K}. \tag{46}$$

In view of the decomposition in 44, in order to give the cell datum for $\mathcal{E}_n(q)$ it is enough to give the cell datum for each $\mathcal{E}_n^\alpha(q)$.

The antiautomorphism $*$ is relatively easy to define, since one checks on the relations for $\mathcal{E}_n(q)$ that $\mathcal{E}_n(q)$ is endowed with an S-linear antiautomorphism $*$, satisfying $e_i^* := e_i$ and $g_i^* := g_i$. Using the formula for g_i^{-1}, one checks that $E_A^* = E_A$ and then $\mathbb{E}_A^* = \mathbb{E}_A$. Hence $*$ induces an antiautomorphism on $\mathcal{E}_n^\alpha(q)$.

Next we explain the poset denoted $\mathcal{P}$ in Definition 2. We first describe $\mathcal{P}$ as a set. For $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in Comp_{r,n}$ we define sets I_i via $I_1 := \{1, 2, \dots, |\lambda^{(1)}|\}$, $I_2 := \{|\lambda^{(1)}| + 1, |\lambda^{(1)}| + 2, \dots, |\lambda^{(1)}| + |\lambda^{(2)}|\}$, and so on. Leaving out the empty I_i's we obtain a set partition in $\mathcal{SP}_n$ that we denote $A_{\boldsymbol{\lambda}}$. For $\alpha \in \mathcal{P}ar_n$ we say that $\boldsymbol{\lambda}$ is of type α if $A_{\boldsymbol{\lambda}}$ is of type α.

An r-multipartition $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ is said to be *increasing* if $\lambda^{(i)} < \lambda^{(j)}$ iff $i < j$, where $<$ is any fixed extension of the dominance order $\lhd$ to a total order.

For $\alpha \in \mathcal{P}ar_n$, we now define $\mathcal{L}_n(\alpha)$ to be the set of pairs $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu})$ where $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ is an increasing r-multipartition of n of type α Let $\lambda^{(i_1)} < \dots < \lambda^{(i_s)}$ be the distinct $\lambda^{(i)}$'s and let m_j be the multiplicity of $\lambda^{(i_j)}$ in $(\lambda^{(1)}, \dots, \lambda^{(r)})$. Then we require that $\boldsymbol{\mu}$ be an s-multipartition $\boldsymbol{\mu} = (\mu^{(1)}, \dots, \mu^{(s)})$ of r where $\mu^{(j)} \vdash m_j$. For $\mathcal{P}$ we choose $\mathcal{L}_n(\alpha)$.

We next describe the poset structure on $\mathcal{L}_n(\alpha)$. Suppose that $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu})$ and $\overline{\Lambda} = (\overline{\boldsymbol{\lambda}} \mid \overline{\boldsymbol{\mu}})$ are elements of $\mathcal{L}_n(\alpha)$, such that $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ and $\overline{\boldsymbol{\lambda}} = (\overline{\lambda^{(1)}}, \dots, \overline{\lambda^{(r)}})$. Then we write $\boldsymbol{\lambda} \lhd_1 \overline{\boldsymbol{\lambda}}$ if there exists a permutation σ such that $(\lambda^{(1\sigma)}, \dots, \lambda^{(r\sigma)}) \lhd (\overline{\lambda^{(1)}}, \dots, \overline{\lambda^{(r)}})$ where $\lhd$ is the dominance order on r-multipartitions, introduced above. We then write of $\Lambda \lhd \overline{\Lambda}$ if $\boldsymbol{\lambda} \lhd_1 \overline{\boldsymbol{\lambda}}$ or if $\boldsymbol{\lambda} = \overline{\boldsymbol{\lambda}}$ and $\boldsymbol{\mu} \lhd \overline{\boldsymbol{\mu}}$. As usual we set $\Lambda \unlhd \overline{\Lambda}$ if $\Lambda \lhd \overline{\Lambda}$ or if $\Lambda = \overline{\Lambda}$. This is the description of $\mathcal{L}_n(\alpha)$ as a poset.

For $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n(\alpha)$ as above, we now define the concept of Λ-tableaux. Suppose that $\mathbb{t}$ is a pair $\mathbb{t} = (\mathbf{t} \mid \mathbf{u})$. Then $\mathbb{t}$ is called a Λ-*tableau* if $\mathbf{t}$ is a $\boldsymbol{\lambda}$-multitableau and $\mathbf{u}$ is a $\boldsymbol{\mu}$-multitableau of the initial kind. If $\mathbb{t}$ is a Λ-tableau we define $Shape(\mathbb{t}) := \Lambda$. We let $\mathrm{Tab}(\Lambda)$ denote the set of Λ-tableaux. We say that $\mathbb{t} = (\mathbf{t} \mid \mathbf{u}) \in \mathrm{Tab}(\Lambda)$ is row standard if its ingredients are row standard multitableaux in the sense of the previous section and if moreover $\mathbf{t}$ is an *increasing* multitableau. By increasing we here mean that whenever $\lambda^{(i)} = \lambda^{(j)}$ we have that $i < j$ if and only if $\min(\mathrm{t}^{(i)}) < \min(\mathrm{t}^{(j)})$ where $\min(t)$ is the function that reads off the minimal entry of the tableau t. We say that $\mathbb{t} = (\mathbf{t} \mid \mathbf{u}) \in \mathrm{Tab}(\Lambda)$ is standard if it is row standard and if its ingredients $\mathbf{t}$ and $\mathbf{u}$ are standard multitableaux and we define $\mathrm{Std}(\Lambda)$ to be the set of all standard Λ-tableaux.

Note that our notation is here deviating slightly from the one used in [EspRy] where the condition on $\mathbf{t}$ to be increasing was required for standardness of $\mathbb{t}$, but not for row standardness of $\mathbb{t}$. In the following examples

$$\begin{aligned}
\mathbb{s} &:= \left(\left(\begin{array}{|c|}\hline 1\\\hline 8\\\hline\end{array}, \begin{array}{|c|c|}\hline 5&6\\\hline\end{array}, \begin{array}{|c|c|}\hline 3&9\\\hline\end{array}, \begin{array}{|c|c|}\hline 2&4\\\hline 7\\\cline{1-1}\end{array} \right) \middle| \left(\begin{array}{|c|}\hline 1\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 4\\\hline\end{array} \right) \right) \\
\mathbb{t} &:= \left(\left(\begin{array}{|c|}\hline 1\\\hline 8\\\hline\end{array}, \begin{array}{|c|c|}\hline 3&5\\\hline\end{array}, \begin{array}{|c|c|}\hline 6&9\\\hline\end{array}, \begin{array}{|c|c|}\hline 2&4\\\hline 7\\\cline{1-1}\end{array} \right) \middle| \left(\begin{array}{|c|}\hline 1\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 4\\\hline\end{array} \right) \right)
\end{aligned} \tag{47}$$

$\mathbb{s}$ is not row standard but $\mathbb{t}$ is.

We fix the following combinatorial notation. Suppose that $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n(\alpha)$ with $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ and $\boldsymbol{\mu} = (\mu^{(1)}, \dots, \mu^{(s)})$. By construction, the numbers $m_i := |\mu^{(i)}|$ are the multiplicities of equal $\lambda^{(i)}$'s. Let $\mathfrak{S}_\Lambda \le \mathfrak{S}_n$ be the stabilizer subgroup of the set partition $A_{\boldsymbol{\lambda}} = \{I_1, I_2, \dots, I_m\}$. Then the multiplicities give rise to the subgroup $\mathfrak{S}_\Lambda^m$ of $\mathfrak{S}_\Lambda$ consisting of the order preserving permutations of the blocks of $A_{\boldsymbol{\lambda}}$ that correspond to equal $\lambda^{(i)}$'s. We have $\mathfrak{S}_\Lambda^m \le \mathfrak{S}_\Lambda$, and as an abstract group

$$\mathfrak{S}_\Lambda^m = \mathfrak{S}_{m_1} \times \cdots \times \mathfrak{S}_{m_s}. \tag{48}$$

For example, if $\boldsymbol{\lambda} = ((1^2), (1^2), (1^2), (2), (2), (1^3), (1^3), (1^3), (2,1))$ then

$$\mathbf{t}^{\boldsymbol{\lambda}} = \left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}, \begin{array}{|c|}\hline 5\\\hline 6\\\hline\end{array}, \begin{array}{|c|c|}\hline 7&8\\\hline\end{array}, \begin{array}{|c|c|}\hline 9&10\\\hline\end{array}, \begin{array}{|c|}\hline 11\\\hline 12\\\hline 13\\\hline\end{array}, \begin{array}{|c|}\hline 14\\\hline 15\\\hline 16\\\hline\end{array}, \begin{array}{|c|}\hline 17\\\hline 18\\\hline 19\\\hline\end{array}, \begin{array}{|c|c|}\hline 20&21\\\hline 22\\\cline{1-1}\end{array} \right)$$
$$A_{\boldsymbol{\lambda}} = \big((1,2), (3,4), (5,6), (7,8), (9,10), (11,12,13), (14,15,16), (17,18,19), (20,21,22)\big)$$

and so $\mathfrak{S}_\Lambda^m = \mathfrak{S}_3 \times \mathfrak{S}_2 \times \mathfrak{S}_3 \times \mathfrak{S}_1$. For I_i and I_{i+1} two consecutive blocks of $A_{\boldsymbol{\lambda}}$ such that $\lambda^{(i)} = \lambda^{(i+1)}$ we let $B_i \in \mathfrak{S}_n$ be the element of minimal length, in the sense of

Coxeter groups, that interchanges I_i and I_{i+1}. For example in the above example we have $B_1 = (1,3)(2,4), B_2 = (3,5)(4,6)$, and so on. $\mathfrak{S}_\Lambda$ is generated by the B_i's such that $\lambda^{(i)} = \lambda^{(i+1)}$, in fact it is a Coxeter groups on these B_i's.

It is important that the group algebra $S\mathfrak{S}_\Lambda^m$ can be realized as a subalgebra of $\mathcal{E}_n(q)$. For this we need to describe B_i concretely in terms of the Coxeter generators s_i for $\mathfrak{S}_n$. Let $a := |I_i|$. Then there is a c such that

$$I_i = \{c+1, c+2, \dots, c+a\} \text{ and } I_{i+1} = \{c+a+1, c+a+2, \dots, c+2a\}. \tag{49}$$

Hence B_i is a product of transpositions, as follows

$$B_i = (c+1, c+a+1)(c+2, c+a+2)\cdots(c+a, c+2a). \tag{50}$$

For $i > j$ we set $s_{ij} := s_{i+c}s_{i-1+c}\dots s_{j+c}$ and can then write B_i in terms of the s_{ij}'s, and therefore in terms of simple transpositions s_i, as follows

$$B_i = s_{a,1}s_{a+1,2}\dots s_{2a-1,a}. \tag{51}$$

The complete expansion of B_i in terms of s_i's is a reduced expression for B_i. Inspired by this formula for B_i, we define $\mathbb{B}_i \in \mathcal{E}_n^\alpha(q)$ in the following way

$$\mathbb{B}_i := \mathbb{E}_{\boldsymbol{\lambda}} g_{a,1}g_{a+1,2}\dots g_{2a-1,a} \text{ where } \; g_{ij} := g_{i+c}g_{i-1+c}\dots g_{j+c}. \tag{52}$$

The following Lemma was proved in [EspRy].

Lemma 4 Suppose that $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n(\alpha)$. Then we have an S-algebra embedding $\iota : S\mathfrak{S}_\Lambda^m \hookrightarrow \mathcal{E}_n^\alpha(q)$ given by $B_i \mapsto \mathbb{B}_i$ where $\lambda^{(i)} = \lambda^{(i+1)}$.

We are now ready to recall the construction of the basis elements $\{m_{\mathbb{s}\mathbb{t}}\}$ of the cell datum for $\mathcal{E}_n^\alpha(q)$, and at the same time we introduce the new basis elements $\{n_{\mathbb{s}\mathbb{t}}\}$ for $\mathcal{E}_n^\alpha(q)$. For each $\Lambda \in \mathcal{L}_n(\alpha)$, we first define elements m_Λ and n_Λ that act as starting points for the bases. Suppose that $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu})$ is as above with $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ and $\boldsymbol{\mu} = (\mu^{(1)}, \dots, \mu^{(s)})$. We then define $m_\Lambda \in \mathcal{E}_n^\alpha(q)$ and $n_\Lambda \in \mathcal{E}_n^\alpha(q)$ as follows

$$m_\Lambda := \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}}, \quad n_\Lambda := \mathbb{E}_{\boldsymbol{\lambda}} y_{\boldsymbol{\lambda}} c_{\boldsymbol{\mu}}. \tag{53}$$

Let us explain the factors of these products. The idempotent $\mathbb{E}_{\boldsymbol{\lambda}}$ has already been introduced. The factors $x_{\boldsymbol{\lambda}} \in \mathcal{E}_n^\alpha(q)$ and $y_{\boldsymbol{\lambda}} \in \mathcal{E}_n^\alpha(q)$ are defined as

$$x_{\boldsymbol{\lambda}} := \sum_{w \in \mathfrak{S}_{\boldsymbol{\lambda}}} q^{\ell(w)} g_w \quad \text{and} \quad y_{\boldsymbol{\lambda}} := \sum_{w \in \mathfrak{S}_{\boldsymbol{\lambda}}} (-q)^{-\ell(w)} g_w. \tag{54}$$

Note that $\mathbb{E}_{\boldsymbol{\lambda}}$ commutes with $x_{\boldsymbol{\lambda}}$ and $y_{\boldsymbol{\lambda}}$ and that for $w \in \mathfrak{S}_{\boldsymbol{\lambda}}$, we have that

$$\mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} g_w = g_w \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} = q^{l(w)} \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} \text{ and } \mathbb{E}_{\boldsymbol{\lambda}} y_{\boldsymbol{\lambda}} g_w = g_w \mathbb{E}_{\boldsymbol{\lambda}} y_{\boldsymbol{\lambda}} = (-q)^{-l(w)} \mathbb{E}_{\boldsymbol{\lambda}} y_{\boldsymbol{\lambda}}. \tag{55}$$

In the Hecke algebra case, corresponding to $r = 1$, the elements defined in 54 are the elements denoted $x_{\lambda\lambda}$ and $y_{\lambda\lambda}$ in [Murphy95]. There is an automorphism of the Hecke algebra, denoted # in *loc. cit.*, that interchanges $x_{\lambda\lambda}$ and $y_{\lambda\lambda}$. With our choice of quadratic relation 34, the corresponding automorphism # of $\mathcal{E}_n^\alpha(q)$, interchanging $x_{\boldsymbol{\lambda}}$ and $y_{\boldsymbol{\lambda}}$, is given by $\mathbb{E}_A g_w \mapsto (-1)^{\ell(w)} \mathbb{E}_A g_{w^{-1}}^{-1}$.

In order to define the factors $b_{\boldsymbol{\mu}}$ and $c_{\boldsymbol{\mu}}$ we recall from 48 the decomposition $\mathfrak{S}_\Lambda^m = \mathfrak{S}_{m_1} \times \cdots \times \mathfrak{S}_{m_s}$. Let $x_{\boldsymbol{\mu}}(1) \in \mathfrak{S}_\Lambda^m$ and $y_{\boldsymbol{\mu}}(1) \in \mathfrak{S}_\Lambda^m$ be the $q = 1$ specializations of the elements given in (56) corresponding to the multipartition $\boldsymbol{\mu}$, that is

$$x_{\boldsymbol{\mu}}(1) := \sum_{w \in \mathfrak{S}_{\boldsymbol{\mu}}} g_w \quad \text{and} \quad y_{\boldsymbol{\mu}}(1) := \sum_{w \in \mathfrak{S}_{\boldsymbol{\mu}}} (-1)^{-\ell(w)} g_w. \tag{56}$$

Then $b_{\boldsymbol{\mu}} \in \mathcal{E}_n^\alpha(q)$ and $c_{\boldsymbol{\mu}} \in \mathcal{E}_n^\alpha(q)$ are defined via

$$b_{\boldsymbol{\mu}} := \iota(x_{\boldsymbol{\mu}}(1)) \quad \text{and} \quad c_{\boldsymbol{\mu}} := \iota(y_{\boldsymbol{\mu}}(1)) \tag{57}$$

where $\iota : S\mathfrak{S}_\Lambda^m \hookrightarrow \mathcal{E}_n^\alpha(q)$ is the embedding from Lemma 4.

Let $\mathbb{t}^\Lambda$ (resp. $\mathbb{t}_\Lambda$) be the Λ-tableau given by $\mathbb{t}^\Lambda := (\mathbf{t}^{\boldsymbol{\lambda}} \mid \mathbf{t}^{\boldsymbol{\mu}})$ (resp. $\mathbb{t}_\Lambda := (\mathbf{t}_{\boldsymbol{\lambda}} \mid \mathbf{t}_{\boldsymbol{\mu}})$). Let now $\mathbb{s} = (\mathfrak{s} \mid \mathbf{u})$ be an arbitrary Λ-tableau. Then we set $d(\mathbb{s}) := (d(\mathfrak{s}) \mid \iota(d(\mathbf{u})))$ where $d(\mathfrak{s}) \in \mathfrak{S}_n$ satisfies $\mathbf{t}^{\boldsymbol{\lambda}} d(\mathfrak{s}) = \mathfrak{s}$ and $d(\mathbf{u}) \in \mathfrak{S}_\Lambda^m$ satisfies $\mathbf{t}^{\boldsymbol{\mu}} d(\mathbf{u}) = \mathbf{u}$. For simplicity, we write $(d(\mathfrak{s}) \mid d(\mathbf{u}))$ instead of $(d(\mathfrak{s}) \mid \iota(d(\mathbf{u})))$. Note that since $\mathbf{u} = (\mathfrak{u}_1, \dots, \mathfrak{u}_s)$ is of the initial kind, we have a decomposition $d(\mathbf{u}) = (d(\mathfrak{u}_1), \dots, d(\mathfrak{u}_s))$, according to the decomposition in 48 and so

$$\mathbb{B}_{d(\mathbf{u})} = \mathbb{B}_{d(\mathfrak{u}_1)} \cdots \mathbb{B}_{d(\mathfrak{u}_s)}. \tag{58}$$

We can now finally introduce $m_{\mathbb{s}\mathbb{t}}$ and $n_{\mathbb{s}\mathbb{t}}$. For $\mathbb{s} = (\mathfrak{s} \mid \mathbf{u})$, $\mathbb{t} = (\mathbf{t} \mid \mathbf{v})$ row standard Λ-tableaux we define

$$m_{\mathbb{s}\mathbb{t}} := \mathbb{E}_{\boldsymbol{\lambda}} g^*_{d(\mathfrak{s})} x_{\boldsymbol{\lambda}} \mathbb{B}^*_{d(\mathbf{u})} b_{\boldsymbol{\mu}} \mathbb{B}_{d(\mathbf{v})} g_{d(\mathbf{t})} \tag{59}$$

$$n_{\mathbb{s}\mathbb{t}} := \mathbb{E}_{\boldsymbol{\lambda}} g^*_{d(\mathfrak{s})} y_{\boldsymbol{\lambda}} \mathbb{B}^*_{d(\mathbf{u})} c_{\boldsymbol{\mu}} \mathbb{B}_{d(\mathbf{v})} g_{d(\mathbf{t})}. \tag{60}$$

We set $b_{\mathbf{uv}} := \mathbb{B}^*_{d(\mathbf{u})} b_{\boldsymbol{\mu}} \mathbb{B}_{d(\mathbf{v})}$ and $c_{\mathbf{uv}} := \mathbb{B}^*_{d(\mathbf{u})} c_{\boldsymbol{\mu}} \mathbb{B}_{d(\mathbf{v})}$ and have then also

$$m_{\mathbb{s}\mathbb{t}} = \mathbb{E}_{\boldsymbol{\lambda}} g^*_{d(\mathfrak{s})} x_{\boldsymbol{\lambda}} b_{\mathbf{uv}} g_{d(\mathbf{t})} \tag{61}$$

$$n_{\mathbb{s}\mathbb{t}} = \mathbb{E}_{\boldsymbol{\lambda}} g^*_{d(\mathfrak{s})} y_{\boldsymbol{\lambda}} c_{\mathbf{uv}} g_{d(\mathbf{t})}. \tag{62}$$

In the case where $\mathbb{s} = \mathbb{t}^\Lambda$ we write for simplicity

$$m_{\mathbb{t}} := m_{\mathbb{t}^\Lambda \mathbb{t}} \text{ and } n_{\mathbb{t}} := n_{\mathbb{t}^\Lambda \mathbb{t}}. \tag{63}$$

In [EspRy] the following Theorem was proved.

Theorem 5 $\mathcal{E}_n^\alpha(q)$ is a cellular basis with cell datum $(\mathcal{L}_n(\alpha), \mathrm{Std}(\Lambda), m_{\mathbb{s}\mathbb{t}}, *)$.

Making the straightforward adaptations of the proofs given in [EspRy], we also have the following Theorem.

Theorem 6 $\mathcal{E}_n^\alpha(q)$ is a cellular basis with cell datum $(\mathcal{L}_n(\alpha), \mathrm{Std}(\Lambda), n_{\mathbb{s}\mathbb{t}}, *)$.

By general cellular algebra theory, specialization induces a cell datum on $\mathcal{E}_n^{\mathcal{K},\alpha}(q)$ as well. We shall denote it the same way; in particular $m_{\mathbb{s}\mathbb{t}}$ and $n_{\mathbb{s}\mathbb{t}}$ may refer to elements of both $\mathcal{E}_n^\alpha(q)$ and $\mathcal{E}_n^{\mathcal{K},\alpha}(q)$.

The definition of cellular basis does not stipulate a partial order on the tableaux, but in our setting there is a natural such order that we shall need. It was already introduced in [EspRy]. For $\mathbf{t} = (\mathfrak{t}^{(1)}, \dots, \mathfrak{t}^{(r)})$ and $\bar{\mathbf{t}} = (\overline{\mathfrak{t}^{(1)}}, \dots, \overline{t^{(r)}})$ row standard r-multitableaux, we define $\mathbf{t} \lhd_1 \bar{\mathbf{t}}$ if there exists a permutation σ such that $(\mathfrak{t}^{(1\sigma)}, \dots, \mathfrak{t}^{(r\sigma)}) \lhd (\overline{\mathfrak{t}^{(1)}}, \dots, \overline{\mathfrak{t}^{(r)}})$. Suppose now that $\mathbb{t} = (\mathbf{t} \mid \mathbf{u}) \in \mathrm{Tab}(\Lambda)$ and $\bar{\mathbb{t}} = (\bar{\mathbf{t}} \mid \bar{\mathbf{u}}) \in \mathrm{Tab}(\overline{\Lambda})$. Then we say that $\mathbf{t} \lhd \bar{\mathbf{t}}$ if $\mathbf{t} \lhd_1 \bar{\mathbf{t}}$ or if $\mathbf{t} = \bar{\mathbf{t}}$ and $\mathbf{u} \lhd \bar{\mathbf{u}}$. As usual we set $\mathbb{t} \unlhd \bar{\mathbb{t}}$ if $\mathbb{t} \lhd \bar{\mathbb{t}}$ or $\mathbb{t} = \bar{\mathbb{t}}$.

We now introduce the concept of *conjugation* of the elements of $\mathcal{L}_n(\alpha)$ and of their tableaux; this concept is also not present in general cellular algebras. Suppose that $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n(\alpha)$, with $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ and $\boldsymbol{\mu} = (\mu^{(1)}, \dots, \mu^{(s)})$. Then we define the *conjugate* of Λ via $\Lambda' := (\boldsymbol{\lambda}' \mid \boldsymbol{\mu}')$. Note that $\Lambda' \in \mathcal{L}_n(\alpha)$. Similarly, if $\mathbb{t} = (\mathbf{t} \mid \mathbf{u}) \in \mathrm{Tab}(\Lambda)$ we define the conjugate tableau $\mathbb{t}' \in \mathrm{Tab}(\Lambda')$ by conjugating all the components of $\mathbb{t}$, mimicking what we did for Λ. We have the following compatibility between the dominance orders and conjugation: $\Lambda \unlhd \overline{\Lambda}$ iff $\overline{\Lambda}' \unlhd \Lambda'$ and $\mathbb{s} \unlhd \mathbb{t}$ iff $\mathbb{t}' \unlhd \mathbb{s}'$.

The following simple Lemma shall be used repeatedly.

Lemma 7 Let $\alpha \in \mathcal{P}ar_n$. Suppose that $A \in \mathcal{SP}_n$ is of type α and that i and $i+1$ are in distinct blocks of A. Then in $\mathcal{E}_n^\alpha(q)$ and $\mathcal{E}_n^{\mathcal{K},\alpha}(q)$ the generator g_i verifies the symmetric group quadratic relation when acting on $\mathbb{E}_A$, that is

$$\mathbb{E}_A g_i^2 = \mathbb{E}_A. \tag{64}$$

PROOF. By relation 34 we have that

$$\mathbb{E}_A g_i^2 = \mathbb{E}_A(1+(q-q^{-1})e_i g_i) = \mathbb{E}_A + (q-q^{-1})\mathbb{E}_A e_i g_i = \mathbb{E}_A \tag{65}$$

where we used (3) of Proposition 3 for the last equality. □

For $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n(\alpha)$, and Λ-tableaux $\mathbb{s} = (\mathfrak{s} \mid \mathbf{u})$ and $\mathbb{t} = (\mathbf{t} \mid \mathbf{v})$, we define elements of $\mathcal{E}_n^{\alpha}(q)$ or $\mathcal{E}_n^{\mathcal{K},\alpha}(q)$

$$x_{\mathfrak{s}\mathbf{t}} := g^*_{d(\mathfrak{s})}\mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} g_{d(\mathbf{t})}, \quad y_{\mathfrak{s}\mathbf{t}} := g^*_{d(\mathfrak{s})}\mathbb{E}_{\boldsymbol{\lambda}} y_{\boldsymbol{\lambda}} g_{d(\mathbf{t})}. \tag{66}$$

We also need the following Lemma.

Lemma 8 Suppose that $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n(\alpha)$, that $\Lambda_1 = (\boldsymbol{\lambda}_1 \mid \boldsymbol{\mu}_1) \in \mathcal{L}_n(\alpha)$ and that $\mathfrak{s},\mathbf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ and $\mathfrak{s}_1,\mathbf{t}_1 \in \mathrm{Std}(\boldsymbol{\lambda}_1)$. Suppose moreover that $x_{\mathfrak{s}\mathbf{t}} y_{\mathfrak{s}_1\mathbf{t}_1} \neq 0$. Then $\mathbf{t} \trianglelefteq \mathfrak{s}_1'$.

PROOF. For usual integer partitions this result is well-known, and as we shall see, the proof can be reduced to that case. Without loss of generality we may assume that $\mathfrak{s} = \mathbf{t}^{\boldsymbol{\lambda}}$ and $\mathbf{t}_1 = \mathbf{t}^{\boldsymbol{\lambda}_1}$ and so we have that

$$x_{\mathfrak{s}\mathbf{t}} y_{\mathfrak{s}_1\mathbf{t}_1} = x_{\mathbf{t}^{\lambda}\mathbf{t}} y_{\mathfrak{s}_1\mathbf{t}^{\lambda_1}} = \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} g_{d(\mathbf{t})} g^*_{d(\mathfrak{s}_1)} \mathbb{E}_{\boldsymbol{\lambda}_1} y_{\boldsymbol{\lambda}_1} = x_{\boldsymbol{\lambda}} g_{d(\mathbf{t})} g^*_{d(\mathfrak{s}_1)} \mathbb{E}_{A_{\boldsymbol{\lambda}} d(\mathbf{t})\, d(\mathfrak{s}_1)^{-1}} \mathbb{E}_{\boldsymbol{\lambda}} y_{\boldsymbol{\lambda}_1}. \tag{67}$$

For the last step we used that $x_{\boldsymbol{\lambda}}\mathbb{E}_{\boldsymbol{\lambda}} = \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}}$, together with part (2) of Proposition 3 and the fact that $\mathbb{E}_{\boldsymbol{\lambda}} = \mathbb{E}_{\boldsymbol{\lambda}_1}$ since $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}_1$ are both of type α. Now by hypothesis, 67 is nonzero and so $\mathbb{E}_{A_{\boldsymbol{\lambda}} d(\mathbf{t})\, d(\mathfrak{s}_1)^{-1}} \mathbb{E}_{\boldsymbol{\lambda}} \neq 0$. Hence using the orthogonality of the $\mathbb{E}_A$'s we get that

$$A_{\boldsymbol{\lambda}} d(\mathbf{t}) = A_{\boldsymbol{\lambda}} d(\mathfrak{s}_1). \tag{68}$$

We now choose decompositions with respect to $\mathfrak{S}_{\|\boldsymbol{\lambda}\|} = \mathfrak{S}_\alpha$ as in 21

$$d(\mathbf{t}) = d(\mathbf{t}_0)\, w_{\mathbf{t}} \quad \text{and} \quad d(\mathfrak{s}_1) = d((\mathfrak{s}_1)_0)\, w_{\mathfrak{s}_1}. \tag{69}$$

Clearly $\mathfrak{S}_n$ acts transitively on $\mathcal{SP}_n^{\alpha}$ with $\mathfrak{S}_\alpha$ as the stabilizer group of $A_{\boldsymbol{\lambda}}$ and so we have an identification $\mathcal{SP}_n^{\alpha} = \mathfrak{S}_\alpha \backslash \mathfrak{S}_n$. From 68 we have that $\mathfrak{S}_\alpha w_{\mathbf{t}} = \mathfrak{S}_\alpha w_{\mathfrak{s}_1}$ and so $w_{\mathbf{t}} = w_{\mathfrak{s}_1}$. Hence 67 becomes

$$\mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} g_{d(\mathbf{t}_0)} g_{w_{\mathbf{t}}} g^*_{w_{\mathbf{t}}} g^*_{d((\mathbf{t}_1)_0)} y_{\boldsymbol{\lambda}_1} = x_{\mathbf{t}^{\lambda}\mathbf{t}_0} \mathbb{E}_{\boldsymbol{\lambda}} g_{w_{\mathbf{t}}} g^*_{w_{\mathbf{t}}} y_{(\mathfrak{s}_1)_0\mathbf{t}^{\lambda_1}}. \tag{70}$$

We can now use Lemma 7 repeatedly on the simple transpositions that appear in a reduced expression for $w_{\mathbf{t}}$ and hence 70 becomes

$$\mathbb{E}_{\boldsymbol{\lambda}} x_{\mathbf{t}^{\lambda}\mathbf{t}_0} y_{(\mathfrak{s}_1)_0\mathbf{t}^{\lambda_1}}. \tag{71}$$

In view of 24 and the fact that $\mathbf{t}_0$ and $(\mathfrak{s}_1)_0$ are both of the initial kind, the Lemma now follows from the similar result in the Hecke algebra setting. □

We wish to generalize the Lemma to a statement involving $m_{\mathbb{s}\mathbb{t}}$ and $n_{\mathbb{s}\mathbb{t}}$. For this we need to recall from [EspRy] some of the commutation relations between the various factors of $m_{\mathbb{s}\mathbb{t}}$ and $n_{\mathbb{s}\mathbb{t}}$, as defined in 59 and 60.

The commutation relations involving $\mathbb{E}_{\boldsymbol{\lambda}}$ and $x_{\boldsymbol{\lambda}}$ (resp. $y_{\boldsymbol{\lambda}}$) are easy to describe, since $\mathbb{E}_{\boldsymbol{\lambda}}$ and $x_{\boldsymbol{\lambda}}$ (resp. $y_{\boldsymbol{\lambda}}$) commute with all the factors of 59 (resp. 60), except the ones on the extreme left and right. The commutation relations between $\mathbb{B}_{d(\mathbf{v})}$ and $g_{d(\mathbf{t})}$ are more complicated to describe, but we shall here only need the case when $\mathbf{t}$ is of the initial kind, that is $\mathbf{t} = \mathbf{t}_0$. Combining Lemma 21 and Lemma 22 of [EspRy] we get in that case that

$$\mathbb{E}_{\boldsymbol{\lambda}} \mathbb{B}_{d(\mathbf{v})} g_{d(\mathbf{t}_0)} = \mathbb{E}_{\boldsymbol{\lambda}} g_{d(\mathbf{t}_0)} \mathbb{B}_{d(\mathbf{v})} \tag{72}$$

and similarly for the two factors on the left of 59 and 60. We can now prove the following statement.

Lemma 9 Suppose that $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu})$ and that $\Lambda_1 = (\boldsymbol{\lambda}_1 \mid \boldsymbol{\mu}_1) \in \mathcal{L}_n(\alpha)$ and that $\mathbb{s},\mathbb{t} \in \mathrm{Std}(\Lambda)$ and $\mathbb{s}_1,\mathbb{t}_1 \in \mathrm{Std}(\Lambda_1)$. If in $\mathcal{E}_n^{\alpha}(q)$ or $\mathcal{E}_n^{\mathcal{K},\alpha}(q)$ we have $m_{\mathbb{s}\mathbb{t}} n_{\mathbb{s}_1\mathbb{t}_1} \neq 0$ then $\mathbb{t} \trianglelefteq \mathbb{s}_1'$.

PROOF. Let $\mathbb{s} = (\mathfrak{s} \mid \mathbf{u})$, $\mathbb{t} = (\mathbf{t} \mid \mathbf{v})$, $\mathbb{s}_1 = (\mathfrak{s}_1 \mid \mathbf{u}_1)$ and $\mathbb{t}_1 = (\mathbf{t}_1 \mid \mathbf{v}_1)$. As in the proof of Lemma 8, we can without loss of generality assume that $\mathbb{s} = \mathbb{t}^{\Lambda}$ and $\mathbb{t}_1 = \mathbb{t}^{\Lambda_1}$ and so we have that

$$m_{\mathbb{s}\mathbb{t}}\, n_{\mathbb{s}_1\mathbb{t}_1} = \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathbb{B}_{d(\mathbf{v})}\, g_{d(\mathbf{t})} g^*_{d(\mathfrak{s}_1)} \mathbb{B}^*_{d(\mathbf{u}_1)} c_{\boldsymbol{\mu}_1} y_{\boldsymbol{\lambda}_1} \mathbb{E}_{\boldsymbol{\lambda}_1}. \tag{73}$$

We now argue largely as in Lemma 8. We first observe that $\mathbb{E}_{\boldsymbol{\lambda}_1} = \mathbb{E}_{\boldsymbol{\lambda}}$ since both $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}_1$ are of type α. Using the commutation rules involving $\mathbb{E}_{\boldsymbol{\lambda}}$, see the paragraph prior to 72 and part (3) of Proposition 3, we get that 73 is equal to

$$x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathbb{B}_{d(\mathbf{v})}\, g_{d(\mathbf{t})} g^*_{d(\mathfrak{s}_1)} \mathbb{B}^*_{d(\mathbf{u}_1)} \mathbb{E}_{A_{\boldsymbol{\lambda}} d(\mathbf{t}) d(\mathfrak{s}_1)^{-1}} \mathbb{E}_{\boldsymbol{\lambda}} c_{\boldsymbol{\mu}_1} y_{\boldsymbol{\lambda}_1}. \tag{74}$$

By hypothesis, 73 is nonzero and so $\mathbb{E}_{A_{\boldsymbol{\lambda}} d(\mathbf{t}) d(\mathfrak{s}_1)^{-1}} = \mathbb{E}_{\boldsymbol{\lambda}}$ which implies that $A_{\boldsymbol{\lambda}} d(\mathbf{t}) = A_{\boldsymbol{\lambda}} d(\mathfrak{s}_1)$. As in Lemma 8 we have decompositions $d(\mathbf{t}) = d(\mathbf{t}_0)\, w_{\mathbf{t}}$ and $d(\mathfrak{s}_1) = d((\mathfrak{s}_1)_0)\, w_{\mathfrak{s}_1}$ and so $A_{\boldsymbol{\lambda}} w_{\mathbf{t}} = A_{\boldsymbol{\lambda}} w_{\mathfrak{s}_1}$ and $w_{\mathbf{t}} = w_{\mathfrak{s}_1}$. Hence 74 becomes

$$\begin{aligned} &\mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathbb{B}_{d(\mathbf{v})}\, g_{d(\mathbf{t}_0)} g_{w_{\mathbf{t}}} g^*_{w_{\mathbf{t}}} g^*_{d((\mathfrak{s}_1)_0)} \mathbb{B}^*_{d(\mathbf{u}_1)} c_{\boldsymbol{\mu}_1} y_{\boldsymbol{\lambda}_1} = \\ &\qquad \mathbb{E}_{\boldsymbol{\lambda}} x_{\mathbf{t}^{\lambda}\mathbf{t}_0} b_{\boldsymbol{\mu}} \mathbb{B}_{d(\mathbf{v})}\, g_{w_{\mathbf{t}}} g^*_{w_{\mathbf{t}}} \mathbb{B}^*_{d(\mathbf{u}_1)} c_{\boldsymbol{\mu}_1} y_{(\mathfrak{s}_1)_0\mathbf{t}^{\lambda_1}} = \\ &\qquad\quad x_{\mathbf{t}^{\lambda}\mathbf{t}_0} b_{\boldsymbol{\mu}} \mathbb{B}_{d(\mathbf{v})}\, \mathbb{E}_{\boldsymbol{\lambda}} g_{w_{\mathbf{t}}} g^*_{w_{\mathbf{t}}} \mathbb{B}^*_{d(\mathbf{u}_1)} c_{\boldsymbol{\mu}_1} y_{(\mathfrak{s}_1)_0\mathbf{t}^{\lambda_1}}. \end{aligned} \tag{75}$$

Using Lemma 7 repeatedly on a reduced expression for $w_{\mathbf{t}}$ we may cancel $g_{w_{\mathbf{t}}} g^*_{w_{\mathbf{t}}}$ out and so 75 becomes

$$\mathbb{E}_{\boldsymbol{\lambda}} x_{\mathbf{t}^{\lambda}\mathbf{t}_0} b_{\boldsymbol{\mu}} \mathbb{B}_{d(\mathbf{v})}\, \mathbb{B}^*_{d(\mathbf{u}_1)} c_{\boldsymbol{\mu}_1} y_{(\mathfrak{s}_1)_0\mathbf{t}^{\lambda_1}} = \mathbb{E}_{\boldsymbol{\lambda}} x_{\mathbf{t}^{\lambda}\mathbf{t}_0} y_{(\mathfrak{s}_1)_0\mathbf{t}^{\lambda_1}} b_{\boldsymbol{\mu}} \mathbb{B}_{d(\mathbf{v})}\, \mathbb{B}^*_{d(\mathbf{u}_1)} c_{\boldsymbol{\mu}_1} \tag{76}$$

where we used that $y_{(\mathfrak{s}_1)_0\mathbf{t}^{\lambda_1}}$ is of the initial kind. We now argue once again as in Lemma 8. We know that 76 is nonzero. Since $\mathbf{t}_0$ and $(\mathfrak{s}_1)_0$ are of the initial kind we deduce from Lemma 8 that $\mathbf{t} \trianglelefteq \mathfrak{s}_1'$. But since 76 is nonzero we also get that $b_{\boldsymbol{\mu}} \mathbb{B}_{d(\mathbf{v})}\, \mathbb{B}^*_{d(\mathbf{u}_1)} c_{\boldsymbol{\mu}_1}$ is nonzero. Moreover $\mathbf{v}$ and $\mathbf{u}_1$ are of the initial kind as well, this time by construction, and so we conclude that $\mathbf{v} \trianglelefteq \mathbf{u}_1'$. This proves the Lemma. □

4. PERMUTATION MODULES

Suppose that $\boldsymbol{\lambda} \in Comp_{r,n}$ is of type α. Then we define the *$\mathcal{E}_n^{\alpha}(q)$-permutation module* $M(\boldsymbol{\lambda})$ as the following right ideal of $\mathcal{E}_n^{\alpha}(q)$

$$M(\boldsymbol{\lambda}) := \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} \mathcal{E}_n^{\alpha}(q) \subseteq \mathcal{E}_n^{\alpha}(q). \tag{77}$$

We also have a specialized version of the permutation module that we denote the same way

$$M(\boldsymbol{\lambda}) := \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} \mathcal{E}_n^{\mathcal{K},\alpha}(q) \subseteq \mathcal{E}_n^{\mathcal{K},\alpha}(q). \tag{78}$$

$M(\boldsymbol{\lambda})$ is a generalization of the permutation modules that appear in the representation theory of the symmetric group, the Hecke algebra, the cyclotomic Hecke algebra, and so on, see [DipperJamMathas] and the references therein. For any $\mathfrak{s} \in \mathrm{RStd}(\boldsymbol{\lambda})$, we introduce $x_{\mathfrak{s}} := \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} g_{d(\mathfrak{s})} \in M(\boldsymbol{\lambda})$ and define $r_i = r_i^{\mathfrak{s}}, c_i = c_i^{\mathfrak{s}}, p_i = p_i^{\mathfrak{s}}$ via $\mathfrak{s}(r_i, c_i, p_i) = i$. We then have the following Lemma which describes the $\mathcal{E}_n^{\alpha}(q)$-module (or $\mathcal{E}_n^{\mathcal{K},\alpha}(q)$-module) structure on $M(\boldsymbol{\lambda})$.

Lemma 10 (1) The set $\{x_{\mathfrak{s}} \mid \mathfrak{s} \in \mathrm{RStd}(\boldsymbol{\lambda})\}$ is a basis for $M(\boldsymbol{\lambda})$, over S or $\mathcal{K}$.

(2) Let $\mathfrak{s} \in \mathrm{RStd}(\boldsymbol{\lambda})$. Then

$$x_{\mathfrak{s}} g_i = \begin{cases} x_{\mathfrak{s} s_i} & \text{if } p_i \neq p_{i+1} \\ q x_{\mathfrak{s}} & \text{if } p_i = p_{i+1} \text{ and } r_i = r_{i+1} \\ x_{\mathfrak{s} s_i} & \text{if } p_i = p_{i+1} \text{ and } r_i < r_{i+1} \\ (q - q^{-1}) x_{\mathfrak{s}} + x_{\mathfrak{s} s_i} & \text{if } p_i = p_{i+1} \text{ and } r_i > r_{i+1} \end{cases} \tag{79}$$

$$x_{\mathfrak{s}} e_i = \begin{cases} x_{\mathfrak{s}} & \text{if } p_i = p_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Using 45, we get that the set $\{\mathbb{E}_{\boldsymbol{\lambda}} g_w \,|\, w \in \mathfrak{S}_n\}$ generates $M(\boldsymbol{\lambda})$ over S. On the other hand, for $w \in \mathfrak{S}_n$ we have the decomposition $w = w_0 d(\mathfrak{s})$ with $w_0 \in \mathfrak{S}_{\boldsymbol{\lambda}}$ and $\mathfrak{s} \in \mathrm{RStd}(\boldsymbol{\lambda})$, and from this it follows via 55 that

$$\mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} g_w = q^{\ell(w_0)} \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} g_{\mathfrak{s}} = q^{\ell(w_0)} x_{\mathfrak{s}}. \tag{80}$$

Hence in fact $\{x_{\mathfrak{s}} \,|\, \mathfrak{s} \in \mathrm{RStd}(\boldsymbol{\lambda})\}$ generates $M(\boldsymbol{\lambda})$ over S. In the expansion of $x_{\mathfrak{s}}$ in terms of $\mathbb{E}_{\boldsymbol{\lambda}} g_w$'s we know that $\mathbb{E}_{\boldsymbol{\lambda}} g_{d(\mathfrak{s})}$ appears exactly once, since the $d(\mathfrak{s})$'s are distinguished coset representatives for $\mathfrak{S}_{\boldsymbol{\lambda}}$ in $\mathfrak{S}_n$, whereas the other terms are of the form $\mathbb{E}_{\boldsymbol{\lambda}} g_{d(\mathfrak{s})} g_w$ for $w \in \mathfrak{S}_{\boldsymbol{\lambda}}$. These elements all belong to the basis for $\mathcal{E}_n^{\alpha}(q)$ given in 45, and are all distinct, and so the $x_{\mathfrak{s}}$'s are linearly independent, proving (1).

We next prove (2), where we first consider $x_{\mathfrak{s}} e_i$. Here we have that

$$x_{\mathfrak{s}} e_i = \mathbb{E}_{\boldsymbol{\lambda}} g_{d(\mathfrak{s})} e_i = g_{d(\mathfrak{s})} \mathbb{E}_{A_{\boldsymbol{\lambda}} d(\mathfrak{s})} e_i \tag{81}$$

and so the formula for $x_{\mathfrak{s}} e_i$ follows from Proposition 3. Let us then consider $x_{\mathfrak{s}} g_i$ when $p_i = p_{i+1}$. If $r_i = r_{i+1}$ then we have $d(\mathfrak{s}) s_i = s_j d(\mathfrak{s})$ for some $s_j \in \mathfrak{S}_{\boldsymbol{\lambda}}$, and so $x_{\mathfrak{s}} g_i = q x_{\mathfrak{s}}$ via 80. If $r_i < r_{i+1}$ then $d(\mathfrak{s}) s_i > d(\mathfrak{s})$ and so $x_{\mathfrak{s}} g_i = x_{\mathfrak{s} s_i}$ from the definition of $x_{\mathfrak{s} s_i}$. If $r_i > r_{i+1}$ then $d(\mathfrak{s}) s_i < d(\mathfrak{s})$. We can then choose a reduced expression for $d(\mathfrak{s})$ ending in s_i and so

$$x_{\mathfrak{s}} g_i = \mathbb{E}_{\boldsymbol{\lambda}} g_{d(\mathfrak{s})} g_i = \mathbb{E}_{\boldsymbol{\lambda}} g_{d(\mathfrak{s} s_i)} g_i^2 = \mathbb{E}_{\boldsymbol{\lambda}} g_{d(\mathfrak{s} s_i)} \left(1 + (q - q^{-1}) e_i g_i\right) = x_{\mathfrak{s} s_i} + (q - q^{-1}) x_{\mathfrak{s}} \tag{82}$$

where we used the quadratic relation 34 for the third equality, and the formula for $x_{\mathfrak{s}} e_i$ for the last equality. Finally, if $p_i \neq p_{i+1}$ then either $d(\mathfrak{s}) s_i > d(\mathfrak{s})$ or $d(\mathfrak{s}) s_i < d(\mathfrak{s})$ and so we can repeat the previous arguments, with the only difference that the second term in 82 disappears. These arguments work over $\mathcal{K}$ as well and so the Lemma is proved, □

There is another kind of permutation module that we shall need. Suppose that $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n(\alpha)$. Then we define the permutation module $M(\Lambda)$ as the following right ideal of $\mathcal{E}_n^{\alpha}(q)$

$$M(\Lambda) := \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathcal{E}_n^{\alpha}(q) \subseteq \mathcal{E}_n^{\alpha}(q). \tag{83}$$

Once again there is a specialized version of $M(\Lambda)$ obtained by replacing $\mathcal{E}_n^{\alpha}(q)$ with $\mathcal{E}_n^{\mathcal{K},\alpha}(q)$. The only difference between $M(\Lambda)$ and $M(\boldsymbol{\lambda})$ is the factor $b_{\boldsymbol{\mu}}$ and so we have $M(\Lambda) \subseteq M(\boldsymbol{\lambda})$, with the inclusion being strict in general. For $M(\Lambda)$ we have the following Lemma.

Lemma 11 Let $\Lambda \in \mathcal{L}_n(\alpha)$. Let $m_{\mathbb{t}}$ be as in 63. Then the set $\{m_{\mathbb{t}} \mid \mathbb{t} \in \mathrm{RStd}(\Lambda)\}$ is an S-basis (or a $\mathcal{K}$-basis) for $M(\Lambda)$.

PROOF. For $w \in \mathfrak{S}_n$ we have the decomposition $w = w_0 d(\mathbf{t})$ with $w_0 \in \mathfrak{S}_{\boldsymbol{\lambda}}$ and $\mathbf{t} \in \mathrm{RStd}(\boldsymbol{\lambda})$ and hence, arguing as in 80, we get

$$\mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} g_w = \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} g_{w_0} g_{\mathbf{t}} = \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} g_{w_0} b_{\boldsymbol{\mu}} g_{\mathbf{t}} = q^{\ell(w_0)} \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} g_{\mathbf{t}}. \tag{84}$$

Hence $\left\{\mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} g_{\mathbf{t}} \,\middle|\, \mathbf{t} \in \mathrm{RStd}(\boldsymbol{\lambda})\right\}$ generates $M(\Lambda)$ over S, but the appearing $\mathbf{t}$ may not be increasing in the sense of the definition of $\mathrm{RStd}(\Lambda)$. On the other hand, if $\mathbf{t}$ is not increasing we can find $B_u \in \mathfrak{S}_m$ such that the multitableau $u * \mathbf{t} \in \mathrm{RStd}(\Lambda)$ defined by

$$d(u * \mathbf{t}) = B_u d(\mathbf{t}) \tag{85}$$

becomes increasing; here $\mathfrak{S}_\Lambda^m = \mathfrak{S}_{m_1} \times \ldots \times \mathfrak{S}_{m_s}$ is the subgroup of $\mathfrak{S}_n$ introduced in 48. We then get

$$\mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} g_{\mathbf{t}} = \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathbb{B}_u^{-1} \mathbb{B}_u g_{d(\mathbf{t})} = \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathbb{B}_u^{-1} g_{B_u d(\mathbf{t})} = \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathbb{B}_u^{-1} g_{u * \mathbf{t}} \tag{86}$$

where we for the second equality used Lemma 54 of [EspRy]. Suppose $\boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(s)})$ and denote by $\mathfrak{S}_{\Lambda,\boldsymbol{\mu}}^m := \mathfrak{S}_{\mu^{(1)}} \times \ldots \times \mathfrak{S}_{\mu^{(s)}}$ the corresponding diagonal Young subgroup of $\mathfrak{S}^m_\Lambda$. It gives rise to a diagonal decomposition of B_u^{-1} that is $B_u^{-1} = (B_u)_0 d(\mathbf{u})$ where $(B_u)_0 \in \mathfrak{S}^m_{\Lambda,\boldsymbol{\mu}}$ and $\mathbf{u}$ is a row standard $\boldsymbol{\mu}$-multitableau of the initial kind. Setting $\mathbb{t} = (u * \mathbf{t} \mid \mathbf{u}) \in \mathcal{L}_n(\alpha)$, we can now rewrite 86 as

$$\mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathbb{B}_u g_{u*\mathbf{t}} = \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathbb{B}_{\mathbf{u}} g_{u*\mathbf{t}} = m_{\mathbb{t}}. \tag{87}$$

Thus we have proved that the set $\{m_{\mathbb{t}}\}$ generates $M(\Lambda)$. To show that it is linearly independent we argue as follows. Suppose that $\mathbb{s} = (\mathfrak{s} \mid \mathbf{u})$, that is

$$m_{\mathbb{s}} = \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathbb{B}_{d(\mathbf{u})}\, g_{d(\mathfrak{s})}. \tag{88}$$

We know that $\mathbf{u}$ is row standard of the initial kind and therefore $b_{\boldsymbol{\mu}} \mathbb{B}_{d(\mathbf{u})}$ expands as a sum of terms of the form $\mathbb{B}_{u_0 d(\mathbf{u})}$ where $u_0 \in \mathfrak{S}^m_{\Lambda,\boldsymbol{\mu}}$. But using Lemma 54 of [EspRy] we get that

$$\mathbb{B}_{u_0 d(\mathbf{u})} g_{d(\mathfrak{s})} = g_{B_{u_0} d(\mathbf{u}) d(\mathfrak{s})}. \tag{89}$$

We now note that the multitableau $\mathfrak{v}$ corresponding to $B_{u_0} d(\mathbf{u}) d(\mathfrak{s})$, in other words $\mathfrak{v}$ given by $d(\mathfrak{v}) = B_{u_0} d(\mathbf{u}) d(\mathfrak{s})$, is a row standard $\boldsymbol{\lambda}$-tableau since it is obtained from $\mathbf{t}$ by permuting components with equal $\lambda^{(i)}$'s and so $m_{\mathbb{s}}$ expands as a sum of terms of the form $\mathbb{E}_{\boldsymbol{\lambda}} g_{x B_{u_0} d(\mathbf{u}) d(\mathfrak{s})}$ where $u_0 \in \mathfrak{S}^m_{\Lambda,\boldsymbol{\mu}}$ and $x \in \mathfrak{S}_{\boldsymbol{\lambda}}$. These expansions are distinct for distinct pairs $(\mathfrak{s} \mid \mathbf{u})$ and so the set $\{m_{\mathbb{s}}\}$ is linearly independent, as claimed. Hence it is an S-basis, and therefore also a $\mathcal{K}$-basis, for $M(\Lambda)$.

□

We next give a description of the action of $\mathcal{E}_n^\alpha(q)$ on $\{m_{\mathbb{s}}\}$. We first introduce the following useful notation. Let $\mathfrak{s} \in \mathrm{RStd}(\boldsymbol{\lambda})$ and let let $s_i \in \Sigma_n \subseteq \mathfrak{S}_n$ be a simple transposition. Then we define $\mathfrak{s} \cdot s_i$ via

$$\mathfrak{s} \cdot s_i := \begin{cases} \mathfrak{s} & \text{if } r_i = r_{i+1} \\ \mathfrak{s} s_i & \text{if } r_i \neq r_{i+1}. \end{cases} \tag{90}$$

This extends to an action of $\mathfrak{S}_n$ on $\mathrm{RStd}(\boldsymbol{\lambda})$ that we denote $(\mathfrak{s}, w) \mapsto \mathfrak{s} \cdot w$. We extend this further to $\mathcal{L}_n(\alpha)$ as follows. Let $\mathbb{s} = (\mathfrak{s} \mid \mathbf{u}) \in \mathcal{L}_n(\alpha)$ and suppose that $Shape(\mathfrak{t}^{(p_i)}) = Shape(\mathfrak{t}^{(p_{i+1})})$. Then we define a transposition $\tau_i \in \mathfrak{S}^m_\Lambda$ via $\tau_i = (p_i, p_{i+1})$. For $\mathfrak{s} \in \mathrm{RStd}(\boldsymbol{\lambda})$ we set $m_i := \min(Shape(\mathfrak{t}^{(p_i)})$. Using the notation $\tau_i * \mathfrak{s}$ introduced in 85 we then define

$$\mathbb{s} \cdot s_i := \begin{cases} (\mathfrak{s} \cdot s_i \mid \mathbf{u}) & \text{if } p_i = p_{i+1} \text{ or otherwise} \\ (\mathfrak{s} \cdot s_i \mid \mathbf{u}) & \text{if } Shape(\mathfrak{t}^{(p_i)}) \neq Shape(\mathfrak{t}^{(p_{i+1})}) \\ (\mathfrak{s} \cdot s_i \mid \mathbf{u}) & \text{if } Shape(\mathfrak{t}^{(p_i)}) = Shape(\mathfrak{t}^{(p_{i+1})}) \text{ and } \{m_i, m_{i+1}\} \neq \{i, i+1\} \\ (\tau_i * (\mathfrak{s} \cdot \tau_i) \mid \mathbf{u} \cdot \tau_i) & \text{if } Shape(\mathfrak{t}^{(p_i)}) = Shape(\mathfrak{t}^{(p_{i+1})}) \text{ and } \{m_i, m_{i+1}\} = \{i, i+1\}. \end{cases} \tag{91}$$

One checks that these formulas extend to an action of $\mathfrak{S}_n$ on $\mathrm{RStd}(\Lambda)$. They have their origin in the straightening procedure that carries $\mathfrak{s} s_i$ to an increasing row standard tableaux. Here are two examples, corresponding to the last case in 91:

$$\begin{aligned} &\left(\left(\begin{array}{|c|c|}\hline 1&5\\\hline\end{array}, \begin{array}{|c|c|}\hline 2&6\\\hline\end{array}, \begin{array}{|c|c|}\hline 3&4\\\hline\end{array} \right) \Big| \left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \right) \right) \cdot s_1 = \left(\left(\begin{array}{|c|c|}\hline 1&6\\\hline\end{array}, \begin{array}{|c|c|}\hline 2&5\\\hline\end{array}, \begin{array}{|c|c|}\hline 3&4\\\hline\end{array} \right) \Big| \left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \right) \right) \\ &\left(\left(\begin{array}{|c|c|}\hline 1&5\\\hline\end{array}, \begin{array}{|c|c|}\hline 2&6\\\hline\end{array}, \begin{array}{|c|c|}\hline 3&4\\\hline\end{array} \right) \Big| \left(\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \right) \right) \cdot s_2 = \left(\left(\begin{array}{|c|c|}\hline 1&5\\\hline\end{array}, \begin{array}{|c|c|}\hline 2&4\\\hline\end{array}, \begin{array}{|c|c|}\hline 3&6\\\hline\end{array} \right) \Big| \left(\begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array} \right) \right). \end{aligned} \tag{92}$$

Lemma 12 Suppose that $\mathbb{s} = (\mathfrak{s} \mid \mathbf{u}) \in \mathrm{RStd}(\Lambda)$. Then

$$(1)\ \ m_{\mathbb{s}} g_i = \begin{cases} q m_{\mathbb{s} \cdot s_i} & \text{if } p_i = p_{i+1} \text{ and } r_i = r_{i+1} \\ m_{\mathbb{s} \cdot s_i} & \text{if } p_i = p_{i+1} \text{ and } r_i < r_{i+1} \\ (q - q^{-1}) m_{\mathbb{s}} + m_{\mathbb{s} \cdot s_i} & \text{if } p_i = p_{i+1} \text{ and } r_i > r_{i+1} \\ m_{\mathbb{s} \cdot s_i} & \text{if } p_i \neq p_{i+1} \end{cases} \tag{93}$$

$$(2)\ \ m_{\mathbb{s}} e_i = \begin{cases} m_{\mathbb{s}} & \text{if } p_i = p_{i+1} \\ 0 & \text{if } p_i \neq p_{i+1}. \end{cases}$$

PROOF. The proof of (2) is analogous to the corresponding statement for $x_{\mathfrak{s}} e_i$ in Lemma 10, in detail

$$m_{\mathbb{s}} e_i = \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathbb{B}_{\mathbf{u}} g_{d(\mathfrak{s})} e_i = x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathbb{B}_{\mathbf{u}} g_{d(\mathfrak{s})} \mathbb{E}_{A_{\boldsymbol{\lambda}} d(\mathfrak{s})} e_i \tag{94}$$

and we conclude via Proposition 3.

We therefore consider (1) where we first focus on the case $p_i = p_{i+1}$. If $r_i = r_{i+1}$ we have $d(\mathfrak{s}) s_i = s_j d(\mathfrak{s})$ for some $s_j \in \mathfrak{S}_{\boldsymbol{\lambda}}$, and so

$$m_{\mathbb{s}} g_i = \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathbb{B}_{\mathbf{u}} g_{d(\mathfrak{s})} g_i = \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathbb{B}_{\mathbf{u}} g_j g_{d(\mathfrak{s})}. \tag{95}$$

But s_j is of the form $s_j = d(\mathfrak{t})$ where $\mathfrak{t}$ is a multitableau of the initial kind and so 95 becomes

$$x_{\boldsymbol{\lambda}} g_j b_{\boldsymbol{\mu}} \mathbb{B}_{\mathbf{u}} g_{d(\mathfrak{s})} = q x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathbb{B}_{\mathbf{u}} g_{d(\mathfrak{s})} = q m_{\mathbb{s}}, \tag{96}$$

as claimed. The two other cases when $p_i = p_{i+1}$, that is $r_i > r_{i+1}$ or $r_i < r_{i+1}$, are proved the same way as in Lemma 10.

We then finally consider the case $p_i \neq p_{i+1}$. Arguing as in Lemma 10 we have here

$$m_{\mathbb{s}} g_i = \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathbb{B}_{\mathbf{u}} g_{d(\mathfrak{s})} g_i = \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathbb{B}_{\mathbf{u}} g_{d(\mathfrak{s} s_i)} \tag{97}$$

which is equal to $m_{\mathbb{s} \cdot s_i}$ in all cases except $\{m_i, m_{i+1}\} = \{i, i+1\}$ where $\mathfrak{t}$ is not increasing. But in that case, arguing as we did for 86 and 87 we can rewrite 97 as follows

$$\mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathbb{B}_{\mathbf{u}} g_{d(\mathfrak{s} s_i)} = \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathbb{B}_{\mathbf{u}} \mathbb{B}_{\tau_i} \mathbb{B}_{\tau_i} g_{d(\mathfrak{s} s_i)} = \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathbb{B}_{\mathbf{u} \cdot \tau_i} g_{d(\tau_i * (\mathfrak{s} s_i))} = m_{\mathbb{s} \cdot s_i} \tag{98}$$

as claimed. □

Given the basis for $M(\Lambda)$, we now introduce a bilinear form $(\cdot,\cdot)_\Lambda$ on $M(\Lambda)$ as follows

$$(m_{\mathbb{s}}, m_{\mathbb{t}})_\Lambda := \begin{cases} 1 & \text{if } \mathbb{s} = \mathbb{t} \\ 0 & \text{otherwise.} \end{cases} \tag{99}$$

It is a generalization of the bilinear forms on the classical permutation modules, see for example [DipperJamMathas].

Lemma 13 The following statements hold.

(1) $(\cdot,\cdot)_\Lambda$ is symmetric and nondegenerate.

(2) $(\cdot,\cdot)_\Lambda$ is invariant in the sense that for all $m, m_1 \in M(\Lambda)$ and $a \in \mathcal{E}_n^{\alpha}(q)$ or $a \in \mathcal{E}_n^{\mathcal{K},\alpha}(q)$ we have that $(ma, m_1)_\Lambda = (m, m_1 a^*)_\Lambda$.

PROOF. (1) follows from the fact that $\{m_{\mathbb{s}}\}$ is an orthogonal basis for $M(\Lambda)$. To show (2) we may assume that $m = m_{\mathbb{s}}, m_1 = m_{\mathbb{t}}$ and $a = g_i$ or $a = e_i$ and must check $(m_{\mathbb{s}} a, m_{\mathbb{t}})_\Lambda = (m_{\mathbb{s}}, m_{\mathbb{t}} a)_\Lambda$ for all possibilities of $r_j^{\mathfrak{s}}, c_j^{\mathfrak{s}}, p_j^{\mathfrak{s}}, r_j^{\mathfrak{t}}, c_j^{\mathfrak{t}}, p_j^{\mathfrak{t}}$ and $j = i, i+1$. Here the number of cases is reduced to the half by the symmetry of $(\cdot,\cdot)_\Lambda$. Moreover if $p_i^{\mathfrak{s}} = p_{i+1}^{\mathfrak{s}}$ and $p_i^{\mathfrak{t}} \neq p_{i+1}^{\mathfrak{t}}$ then it follows immediately from Lemma 12 that $(m_{\mathbb{s}} e_i, m_{\mathbb{t}})_\Lambda = (m_{\mathbb{s}}, m_{\mathbb{t}} e_i)_\Lambda = 0$ and that $(m_{\mathbb{s}} g_i, m_{\mathbb{t}})_\Lambda = (m_{\mathbb{s}}, m_{\mathbb{t}} g_i)_\Lambda = 0$ and so we only need consider the cases where $p_i^{\mathfrak{s}} = p_{i+1}^{\mathfrak{s}}$ and $p_i^{\mathfrak{t}} = p_{i+1}^{\mathfrak{t}}$ or $p_i^{\mathfrak{s}} \neq p_{i+1}^{\mathfrak{s}}$ and $p_i^{\mathfrak{t}} \neq p_{i+1}^{\mathfrak{t}}$. In the rest of the proof we shall use the formulas of Lemma 12 repeatedly.

Let us first consider $a = g_i$ and suppose that $p_i^{\mathfrak{s}} = p_{i+1}^{\mathfrak{s}}$ and $p_i^{\mathfrak{t}} = p_{i+1}^{\mathfrak{t}}$. If $r_i^{\mathfrak{s}} < p_{i+1}^{\mathfrak{s}}$ and $r_i^{\mathfrak{t}} < p_{i+1}^{\mathfrak{t}}$ we have $(m_{\mathbb{s}} g_i, m_{\mathbb{t}})_\Lambda = (m_{\mathbb{s} \cdot s_i}, m_{\mathbb{t}})_\Lambda = \delta_{\mathbb{s} \cdot s_i, \mathbb{t}}$ whereas $(m_{\mathbb{s}}, m_{\mathbb{t}} g_i)_\Lambda = (m_{\mathbb{s}}, m_{\mathbb{t}} g_i)_\Lambda = \delta_{\mathbb{s}, \mathbb{t} \cdot s_i}$, where δ is the Kronecker delta, and so $(m_{\mathbb{s}} g_i, m_{\mathbb{t}})_\Lambda = (m_{\mathbb{s}}, m_{\mathbb{t}} g_i)_\Lambda$.

If $r_i^{\mathfrak{s}} < p_{i+1}^{\mathfrak{s}}$ and $r_i^{\mathfrak{t}} > p_{i+1}^{\mathfrak{t}}$ then again $(m_{\mathbb{s}} g_i, m_{\mathbb{t}})_\Lambda = \delta_{\mathbb{s} \cdot s_i, \mathbb{t}}$ whereas $(m_{\mathbb{s}} g_i, m_{\mathbb{t}})_\Lambda = (m_{\mathbb{s}}, m_{\mathbb{t}} g_i)_\Lambda = (m_{\mathbb{s}}, m_{\mathbb{t} \cdot s_i} + (q - q^{-1}) m_{\mathbb{t}})_\Lambda = \delta_{\mathbb{s}, \mathbb{t} \cdot s_i}$, since $\mathbb{s} \neq \mathbb{t}$, and so the two sides coincide.

If $r_i^{\mathfrak{s}} > p_{i+1}^{\mathfrak{s}}$ and $r_i^{\mathfrak{t}} > p_{i+1}^{\mathfrak{t}}$ we have $(m_{\mathbb{s}} g_i, m_{\mathbb{t}})_\Lambda = (m_{\mathbb{s} \cdot s_i} + (q - q^{-1}) m_{\mathbb{s}}, m_{\mathbb{t}})_\Lambda = \delta_{\mathbb{s} \cdot s_i, \mathbb{t}} + (q - q^{-1}) \delta_{\mathbb{s}, \mathbb{t}}$ and $(m_{\mathbb{s}}, m_{\mathbb{t}} g_i)_\Lambda = (m_{\mathbb{s}}, m_{\mathbb{t} \cdot s_i} + (q - q^{-1}) m_{\mathbb{t}})_\Lambda = \delta_{\mathbb{s}, \mathbb{t} \cdot s_i} + (q - q^{-1}) \delta_{\mathbb{s}, \mathbb{t}}$. Once again, the two sides coincide.

We next consider $a = g_i$ but suppose that $p_i^{\mathfrak{s}} \neq p_{i+1}^{\mathfrak{s}}$ and $p_i^{\mathfrak{t}} \neq p_{i+1}^{\mathfrak{t}}$. Here we have $(m_{\mathbb{s}} g_i, m_{\mathbb{t}})_\Lambda = (m_{\mathbb{s} \cdot s_i}, m_{\mathbb{t}})_\Lambda = \delta_{\mathbb{s} \cdot s_i, \mathbb{t}}$ and similarly $(m_{\mathbb{s}}, m_{\mathbb{t}} g_i)_\Lambda = (m_{\mathbb{s}}, m_{\mathbb{t} \cdot s_i})_\Lambda = \delta_{\mathbb{s}, \mathbb{t} \cdot s_i}$ and so the two side also coincide in this case.

Finally we consider the case $a = e_i$ and must check that $(m_{\mathbb{s}} e_i, m_{\mathbb{t}})_\Lambda = (x_{\mathbb{s}}, m_{\mathbb{t}} e_i)_\Lambda$. But if $p_i^{\mathfrak{s}} = p_{i+1}^{\mathfrak{s}}$ and $p_i^{\mathbf{t}} = p_{i+1}^{\mathbf{t}}$ we have $m_{\mathbb{s}} e_i = m_{\mathbb{s}}$ and $m_{\mathbb{t}} e_i = m_{\mathbb{t}}$, and otherwise, if $p_i^{\mathfrak{s}} \neq p_{i+1}^{\mathfrak{s}}$ and $p_i^{\mathbf{t}} \neq p_{i+1}^{\mathbf{t}}$, we have $m_{\mathbb{s}} e_i = 0$ and $m_{\mathbb{t}} e_i = 0$. Hence the two sides also coincide in this case. This finishes the proof of the Lemma. □

We have the following crucial Lemma.

Lemma 14 Suppose that $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu})$ and that $\mathbb{s}, \mathbb{t} \in \mathrm{Std}(\Lambda)$. Then we have

$$(m_{\mathbb{t}} n_{\mathbb{t}'\mathbb{s}'}, m_{\mathbb{s}})_\Lambda \neq 0 \text{ or equivalently } (m_{\mathbb{t}}, m_{\mathbb{s}} n_{\mathbb{s}'\mathbb{t}'})_\Lambda \neq 0. \tag{100}$$

In particular $m_{\mathbb{t}} n_{\mathbb{t}'\mathbb{s}'} \neq 0$ and $m_{\mathbb{s}} n_{\mathbb{s}'\mathbb{t}'} \neq 0$.

PROOF. Suppose that $\mathbb{s} = (\mathfrak{s} \mid \mathbf{u})$ and $\mathbb{t} = (\mathbf{t} \mid \mathbf{v})$. Then the same chain of equalities that took us from 73 to 76 in Lemma 9, but postmultiplied with $\mathbb{B}_{d(\mathbf{u}')} g_{d(\mathfrak{s}')}$, gives us

$$\begin{aligned} & m_{\mathbb{t}} n_{\mathbb{t}'\mathbb{s}'} = \mathbb{E}_{\boldsymbol{\lambda}} x_{\mathbf{t}^{\boldsymbol{\lambda}} \mathbf{t}_0} y_{\mathbf{t}_0' \mathbf{t}^{\boldsymbol{\lambda}'}} b_{\boldsymbol{\mu}} \mathbb{B}_{d(\mathbf{v})} \mathbb{B}^*_{d(\mathbf{v}')} c_{\boldsymbol{\mu}'} \mathbb{B}_{d(\mathbf{u}')} g_{d(\mathfrak{s}')} = \\ & \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} g_{w_{\boldsymbol{\lambda}}} y_{\boldsymbol{\lambda}'} b_{\boldsymbol{\mu}} \mathbb{B}_{w_{\boldsymbol{\mu}}} c_{\boldsymbol{\mu}'} \mathbb{B}_{d(\mathbf{u}')} g_{d(\mathfrak{s}')} \end{aligned} \tag{101}$$

where we used 10 for the last equality: note that $\mathbf{t}_0$ and $\mathbf{v}$ are both of the initial kind. Using the invariance property of the previous Lemma 13 we get from this that

$$\begin{aligned} & (m_{\mathbb{t}} n_{\mathbb{t}'\mathbb{s}'}, m_{\mathbb{s}})_\Lambda = (m_{\mathbb{t}} n_{\mathbb{t}'\mathbb{s}'} g^*_{d(\mathfrak{s})} \mathbb{B}^*_{d(\mathbf{u})}, m_{\mathbb{t}^\Lambda})_\Lambda = \\ & (\mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} g_{w_{\boldsymbol{\lambda}}} y_{\boldsymbol{\lambda}'} b_{\boldsymbol{\mu}} \mathbb{B}_{w_{\boldsymbol{\mu}}} c_{\boldsymbol{\mu}'} \mathbb{B}_{d(\mathbf{u}')} g_{d(\mathfrak{s}')} g^*_{d(\mathfrak{s})} \mathbb{B}^*_{d(\mathbf{u})}, m_{\mathbb{t}^\Lambda})_\Lambda \end{aligned} \tag{102}$$

In view of 25, and the remarks preceding 25, we have decompositions $d(\mathfrak{s}') = d(\mathfrak{s}_0') d(\mathbf{t})$ and $d(\mathfrak{s}) = d(\mathfrak{s}_0) d(\mathbf{t})$ and hence we can use Lemma 7 repeatedly on a reduced expression of $d(\mathbf{t})$ to rewrite $\mathbb{E}_{\boldsymbol{\lambda}} g_{d(\mathfrak{s}')} g^*_{d(\mathfrak{s})} = \mathbb{E}_{\boldsymbol{\lambda}} g_{d(\mathfrak{s}_0')} g^*_{d(\mathfrak{s}_0)} = \mathbb{E}_{\boldsymbol{\lambda}} g^*_{w_{\boldsymbol{\lambda}}}$ and so 102 becomes

$$\begin{aligned} & (\mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} g_{w_{\boldsymbol{\lambda}}} y_{\boldsymbol{\lambda}'} g^*_{w_{\boldsymbol{\lambda}}} b_{\boldsymbol{\mu}} \mathbb{B}_{w_{\boldsymbol{\mu}}} c_{\boldsymbol{\mu}'} \mathbb{B}^*_{w_{\boldsymbol{\mu}}}, m_{\mathbb{t}^\Lambda})_\Lambda = \\ & (\mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} g_{w_{\boldsymbol{\lambda}}} y_{\boldsymbol{\lambda}'} b_{\boldsymbol{\mu}} \mathbb{B}_{w_{\boldsymbol{\mu}}} c_{\boldsymbol{\mu}'}, \mathbb{E}_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} g_{w_{\boldsymbol{\lambda}}} b_{\boldsymbol{\mu}} \mathbb{B}_{w_{\boldsymbol{\mu}}})_\Lambda = (m_{\mathbb{t}^\Lambda} y_{\boldsymbol{\lambda}'} c_{\boldsymbol{\mu}'}, m_{\mathbb{t}_\Lambda})_\Lambda \end{aligned} \tag{103}$$

Note that 103 does not depend on $\mathbb{s}$ and $\mathbb{t}$, only on Λ. The expressions in 103 only involve multitableaux of the initial kind and so the Lemma is reduced to the corresponding Hecke algebra statement. To be precise, for $\lambda \in \mathcal{P}ar_n$, each element w of the double coclass $\mathfrak{S}_\lambda w_\lambda \mathfrak{S}_{\lambda'}$ has a unique decomposition $w = w_1 w_\lambda w_2$ where $w_1 \in \mathfrak{S}_\lambda$ and $w_2 \in \mathfrak{S}_{\lambda'}$ and moreover $\ell(w) = \ell(w_1) + \ell(w_\lambda) + \ell(w_2)$, see Lemma 1.6 of [DipperJames]. Applying this on $\mathfrak{S}_{\boldsymbol{\lambda}}$ and $\mathfrak{S}_{\boldsymbol{\mu}}$ we conclude that $m_{\mathbb{t}_\Lambda}$ occurs exactly once in the expansion of $m_{\mathbb{t}^\Lambda} y_{\boldsymbol{\lambda}'} c_{\boldsymbol{\mu}'}$, corresponding to the '1'-term in both $y_{\boldsymbol{\lambda}'}$ and $c_{\boldsymbol{\mu}'}$, and so 103 is equal to 1. The Lemma is proved.

□

5. THE TENSOR SPACE $V^{\otimes n}$ MODULE FOR $\mathcal{E}_n(q)$

In this section we realize the tensor space $V^{\otimes n}$ module for $\mathcal{E}_n(q)$, introduced in [Ry], as a sum of the permutation modules $M(\boldsymbol{\lambda})$. When the dimension of V is sufficiently large, it is known that $V^{\otimes n}$ is a faithful $\mathcal{E}_n(q)$-module, but here we are interested in the general case where $V^{\otimes n}$ may not be faithful. Using the results from the previous section we determine the annihilator ideal in $\mathcal{E}_n(q)$ of $V^{\otimes n}$. In turns out that is has a nice description in terms of the dual cellular basis $\{n_{\mathbb{s}\mathbb{t}}\}$. It is a main point of our constructions and proofs that they work for arbitrary $\mathcal{K}$.

Let V be the free S-module with basis

$$\mathcal{B} := \{v_i^s \mid 1 \leq i \leq N, 1 \leq s \leq r\} \tag{104}$$

that is V is of dimension rN. We then define linear maps $\mathbf{E}, \mathbf{G} \in \mathrm{End}_S(V^{\otimes 2})$ via

$$(v_i^s \otimes v_j^t)\mathbf{E} := \begin{cases} v_j^s \otimes v_i^t & \text{if } s = t \\ 0 & \text{if } s \neq t \end{cases} \tag{105}$$

and

$$(v_i^s \otimes v_j^t)\mathbf{G} := \begin{cases} v_j^t \otimes v_i^s & \text{if } s \neq t \\ q v_i^s \otimes v_j^t & \text{if } s = t,\ i = j \\ v_j^t \otimes v_i^s & \text{if } s = t,\ i < j \\ (q - q^{-1}) v_i^s \otimes v_j^t + v_j^t \otimes v_i^s & \text{if } s = t,\ i > j. \end{cases} \tag{106}$$

We extend them to linear maps $\mathbf{E}_i$ and $\mathbf{G}_i$ acting in the tensor space $V^{\otimes n}$ by letting $\mathbf{E}$ and $\mathbf{G}$ act in the i'th and $i+1$'st factors. The following is Theorem 1 of [Ry]

Theorem 15 The rules $e_i \mapsto \mathbf{E}_i$ and $g_i \mapsto \mathbf{G}_i$ endow $V^{\otimes n}$ with the structure of an $\mathcal{E}_n(q)$-module.

In the case of $\mathcal{E}_n^{\mathbb{C}}(q)$ it was proved in [Ry] that the specialized tensor module $V^{\mathbb{C},\otimes n}$ is faithful when $r, N \geq n$. In [EspRy] this faithfulness statement was generalized to $V^{\otimes n}$ itself, but still only for $r, N \geq n$; in fact it does not hold otherwise, as we shall see.

Let seq_N be the set of sequences $\underline{i} = (i_1, \dots, i_n)$ of integers where each i_j belongs to $\{1, \dots, N\}$, and let similarly seq_r be the set of sequences $\underline{s} = (s_1, \dots, s_n)$ where each s_j belongs to $\{1, \dots, r\}$. For such $\underline{i}$ and $\underline{s}$ we define

$$v_{\underline{i}}^{\underline{s}} := v_{i_1}^{s_1} \otimes \cdots \otimes v_{i_n}^{s_n} \in V^{\otimes n}. \tag{107}$$

Then the set $\{v_{\underline{i}}^{\underline{s}} \mid \underline{i} \in \text{seq}_N, \underline{s} \in \text{seq}_r\}$ is a basis for $V^{\otimes n}$.

To $\underline{s} \in \text{seq}_r$ we associate sets $I_j := \{i \mid s_i = j\}$, for $j = 1, \dots, r$. The I_j's may be empty, but leaving out the empty I_j's we obtain a set partition in $\mathcal{SP}_n$, that we denote $A_{\underline{s}}$, and we say that $\underline{s}$ is of type α if $A_{\underline{s}}$ is of type α. For example, if $r = 4, n = 13$ and $\underline{s} = (1,2,2,1,2,2,2,1,2,4,1,2,2)$ then $A_{\underline{s}} = \big\{\{1,4,8,11\}, \{2,3,5,6,7,9,12,13\}, \{10\}\big\}$, and so $\underline{s}$ is of type $(8,4,1)$.

Recall now the idempotent decompositions $\sum_{A \in \mathcal{SP}_n} \mathbb{E}_A = 1$ and $\sum_{\alpha \in \mathcal{P}ar_n} \mathbb{E}_\alpha = 1$. They give rise to decompositions of $V^{\otimes n}$

$$V^{\otimes n} = \bigoplus_{A \in \mathcal{SP}_n} \mathbb{E}_A V^{\otimes n}, \qquad V^{\otimes n} = \bigoplus_{\alpha \in \mathcal{P}ar_n} \mathbb{E}_\alpha V^{\otimes n}. \tag{108}$$

Note that $\mathbb{E}_\alpha V^{\otimes n}$ is canonically an $\mathcal{E}_n^\alpha(q)$-module.

We have the following Lemma which gives precise descriptions of $\mathbb{E}_A V^{\otimes n}$ and $\mathbb{E}_\alpha V^{\otimes n}$.

Lemma 16 With the above notations the following statements hold

(1) $E_A v_{\underline{i}}^{\underline{s}} = \begin{cases} v_{\underline{i}}^{\underline{s}} & \text{if } A \subseteq A_{\underline{s}} \\ 0 & \text{otherwise} \end{cases}$

(2) $\mathbb{E}_A V^{\otimes n} = \text{span}_S\{v_{\underline{i}}^{\underline{s}} \mid A_{\underline{s}} = A\}$

(3) $\mathbb{E}_\alpha V^{\otimes n} = \text{span}_S\{v_{\underline{i}}^{\underline{s}} \mid \underline{s} \text{ is of type } \alpha\}$.

Proof. Statement (1) is immediate from the definition of E_A given in 38, so let us prove (2). Suppose that $A = \{I_1, I_2, \dots, I_k\}$. Then by construction, the vector $v_{\underline{i}}^{\underline{s}}$ belongs to the right hand side of (2) exactly when any two terms s_i and s_j of $\underline{s}$ coincide iff i and j are in the same block I_l of A. For example, if $r = 3$ and $A = \{\{1,2\}, \{3,4\}, \{5,6,7\}\}$ then the $\underline{s}$ satisfying $A_{\underline{s}} = A$ are the following ones

$$\begin{gathered}(1,1,2,2,3,3,3), (2,2,1,1,3,3,3), (3,3,1,1,2,2,2), (1,1,2,2,3,3,3), \\ (1,1,3,3,2,2,2), (2,2,3,3,1,1,1).\end{gathered} \tag{109}$$

Hence, in view of 42, that is

$$\mathbb{E}_A = \sum_{A \subseteq B} \mu(A,B) E_B, \tag{110}$$

together with $\mu(A,A)=1$, we conclude from (1) that $\mathbb{E}_A v_{\underline{i}}^{\underline{s}} = v_{\underline{i}}^{\underline{s}}$ whenever $v_{\underline{i}}^{\underline{s}}$ belongs to the right hand side of (2), which proves the inclusion $\supseteq$ of (2). To prove $\subseteq$, we assume that $v \in \mathbb{E}_A V^{\otimes n}$ and consider its expansion $v = \sum \alpha_{\underline{i}}^{\underline{s}} v_{\underline{i}}^{\underline{s}}$ with coefficients $\alpha_{\underline{i}}^{\underline{s}} \in S$. By (1) and 110 we have that $A \subseteq A_{\underline{s}}$ whenever $\alpha_{\underline{i}}^{\underline{s}} \neq 0$ and hence if v does not belong to the right hand side of (2), there are $\underline{i}_0 \in \mathrm{seq}_N$ and $\underline{s}_0 \in \mathrm{seq}_r$ such that $\alpha_{\underline{i}_0}^{\underline{s}_0} \neq 0$ and $A \subsetneqq A_{\underline{s}_0}$. We have that $E_{A_{\underline{s}_0}} v_{\underline{i}_0}^{\underline{s}_0} = v_{\underline{i}_0}^{\underline{s}_0}$ and hence $E_{A_{\underline{s}_0}} v \neq 0$, by (1) once again. On the other hand, by part (3) of Proposition 3 we also have that $E_{A_{\underline{s}_0}} v = 0$, and this gives the desired contradiction. Finally, (3) follows from (2) and the definitions.

□

For N any natural number we define $Comp_{r,n,\le N} \subseteq Comp_{r,n}$ (resp. $Par_{r,n,\le N} \subseteq Par_{r,n}$) as the set of multicompositions (resp. multipartitions) $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(r)})$ such that each $\lambda^{(i)}$ is of length less than N.

Suppose now that $\boldsymbol{\lambda} \in Comp_{r,n,\le N}$ and that $\mathbf{t} \in \mathrm{RStd}(\boldsymbol{\lambda})$. Then we define $\underline{i}^{\mathbf{t}} = (i_1, \dots, i_n) \in \mathrm{seq}_N$ by letting i_j be the row number of the node in $\mathbf{t}$ containing j, and similarly we define $\underline{s}^{\mathbf{t}} = (s_1, \dots, s_n) \in \mathrm{seq}_r$ by letting s_j be the component number of the node in $\mathbf{t}$ containing j. For example if

$$\mathbf{t} = \left(\begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline \end{array}, \begin{array}{|c|}\hline 1 \\ \hline 8 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 6 & 9 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 7 & 10 \\ \hline \end{array}, \begin{array}{|c|}\hline 11 \\ \hline 12 \\ \hline 13 \\ \hline \end{array}, \begin{array}{|c|}\hline 14 \\ \hline 15 \\ \hline 16 \\ \hline \end{array}, \begin{array}{|c|}\hline 20 \\ \hline 18 \\ \hline 19 \\ \hline \end{array}, \begin{array}{|c|c}\hline 23 & \multicolumn{1}{c|}{21} \\ \hline 24 & \\ \cline{1-1} \end{array}, \begin{array}{|c|c}\cline{1-1} 17 & \\ \hline 22 & \multicolumn{1}{c|}{25} \\ \hline \end{array}, \begin{array}{|c|}\hline 5 \\ \hline 2 \\ \hline \end{array} \right) \tag{111}$$

then $\underline{i}^{\mathbf{t}} = (1,2,1,2,1,1,1,2,1,1,1,2,2,1,2,3,1,2,3,1,1,2,1,2,2)$ and $\underline{s}^{\mathbf{t}} = (2,10,1,1,10,3,4,2,3,4,5,5,5,6,6,6,9,7,7,7,8,9,8,9)$. For $\mathbf{t} \in \mathrm{RStd}(\boldsymbol{\lambda})$ we write $v_{\mathbf{t}}$ for $v_{\underline{i}^{\mathbf{t}}}^{\underline{s}^{\mathbf{t}}}$.

The following Theorem relates the tensor space module $V^{\otimes n}$ with the permutation modules $M(\boldsymbol{\lambda})$.

Theorem 17 Let V be the free S-module (or $\mathcal{K}$-vector space) of dimension rN with basis as in 104. Suppose that $\boldsymbol{\lambda} \in Comp_{r,n,\le N}$ is of type $\alpha \in \mathcal{P}ar_n$ and that α is of length r.

(1) The S-linear map $\iota_{\boldsymbol{\lambda}} : M(\boldsymbol{\lambda}) \to V^{\otimes n}$ given by $\iota_{\boldsymbol{\lambda}}(x_{\mathbf{t}}) = v_{\mathbf{t}}$ for $\mathbf{t} \in \mathrm{RStd}(\boldsymbol{\lambda})$, is an embedding of $\mathcal{E}_n^{\alpha}(q)$-modules.

(2) Identifying $M(\boldsymbol{\lambda})$ with $\varphi_{\boldsymbol{\lambda}}(M(\boldsymbol{\lambda}))$ we have $M(\boldsymbol{\lambda}) \subseteq \mathbb{E}_{\alpha} V^{\otimes n}$ and a direct sum decomposition of $\mathcal{E}_n^{\alpha}(q)$-modules

$$\mathbb{E}_{\alpha} V^{\otimes n} \cong \bigoplus_{\substack{\boldsymbol{\lambda} \in Comp_{r,n,\le N} \\ |A_{\boldsymbol{\lambda}}| = \alpha}} M(\boldsymbol{\lambda}). \tag{112}$$

PROOF. (1) follows from Lemma 10 together with 105 and 106 and the definitions and (2) follows from (1) and Lemma 16. □

For any natural number N we define $\mathcal{L}_{n,\le N}(\alpha)$ as the subset of $\mathcal{L}_n(\alpha)$ consisting of the pairs $(\boldsymbol{\lambda} \mid \boldsymbol{\mu})$ such that the components $\lambda^{(k)}$ of $\boldsymbol{\lambda}$ all have less than N columns, and we set $\mathcal{L}_{n,\not\le N}(\alpha) := \mathcal{L}_n(\alpha) \setminus \mathcal{L}_{n,\le N}(\alpha)$. We can now give the main Theorem of our paper.

Theorem 18 Let V be as in 104 with basis $\mathcal{B}$ and suppose that $\alpha \in \mathcal{P}ar_n$ is of length r. Let $\mathcal{I} \subseteq \mathcal{E}_n^{\alpha}(q)$ be the annihilator ideal of the action of $\mathcal{E}_n^{\alpha}(q)$ in $\mathbb{E}_{\alpha} V^{\otimes n}$. Then $\mathcal{I}$ is free over S with basis

$$\left\{ n_{\mathbb{s}\mathbb{t}} \mid \mathbb{s}, \mathbb{t} \in \mathrm{Std}(\Lambda), \Lambda \in \mathcal{L}_{n,\not\le N}(\alpha) \right\}. \tag{113}$$

A similar statement holds over $\mathcal{K}$.

PROOF. Let us focus on the S-case, since the $\mathcal{K}$-case is done the same way. Let $\mathcal{I}_1 \subseteq \mathcal{E}_n^\alpha(q)$ be the S-span of $\left\{ n_{\mathbb{s}\mathbb{t}} \,|\, \mathbb{s}, \mathbb{t} \in \mathrm{Std}(\Lambda), \Lambda \in \mathcal{L}_{n, \nleq N}(\alpha) \right\}$. We first observe that $\mathcal{I}_1$ is a two-sided ideal in $\mathcal{E}_n^\alpha(q)$, as one sees from the $n_{\mathbb{s}\mathbb{t}}$-version of Lemma 56 of [EspRy]: the number of columns of the components never decreases under the straightening procedure of that Lemma. We first prove that $\mathcal{I}_1 \subseteq \mathcal{I}$, that is for a basis element $n_{\mathbb{s}\mathbb{t}}$ of $\mathcal{I}_1$ we prove that $M(\boldsymbol{\lambda}) n_{\mathbb{s}\mathbb{t}} = 0$ for any $M(\boldsymbol{\lambda})$ appearing in 112, or equivalently that $M(\boldsymbol{\lambda}) n_{\mathbb{s}\mathbb{t}} = 0$. But since $\mathcal{I}_1$ is an ideal, in order to show $M(\boldsymbol{\lambda}) n_{\mathbb{s}\mathbb{t}} = 0$ it is enough to show $x_{\boldsymbol{\lambda}} n_{\mathbb{s}\mathbb{t}} = 0$. Note that $\boldsymbol{\lambda}$ may not be a multipartition, only a multicomposition, but in any case $\mathbf{t}^{\boldsymbol{\lambda}}$ is of the initial kind and so we can use Lemma 4.4 of [Murphy95] to rewrite $x_{\boldsymbol{\lambda}}$ as a linear combination of $x_{\mathfrak{s}_1 \mathbf{t}_1}$ where $\mathfrak{s}_1$ and $\mathbf{t}_1$ are $\boldsymbol{\lambda}_1$-multitableaux of the initial kind, for $\boldsymbol{\lambda}_1$ an r-multipartition such that $\boldsymbol{\lambda}_1 \trianglerighteq \boldsymbol{\lambda}$. In general, for any $\sigma \in \mathfrak{S}_r$ there is an $\mathcal{E}_n^\alpha(q)$-isomorphism $M(\boldsymbol{\lambda}) \cong M(\boldsymbol{\lambda}^\sigma)$ where $\boldsymbol{\lambda}^\sigma$ is the r-multicomposition obtained from $\boldsymbol{\lambda}^\sigma$ by permuting the components, as one checks using Lemma 10, and so we may assume that $\boldsymbol{\lambda}_1$ is increasing. Moreover, since $\boldsymbol{\lambda}_1 \trianglerighteq \boldsymbol{\lambda}$ and $\boldsymbol{\lambda} \in Comp_{r,n,\leq N}$ we have $\boldsymbol{\lambda}_1 \in Par_{r,n,\leq N}$. We now observe that $x_{\mathfrak{s}_1 \mathbf{t}_1}$ is also of the form $m_{\mathbb{s}_1 \mathbb{t}_1}$, where $\mathbb{s}_1 = (\mathfrak{s}_1 \,|\, \mathbf{u}_1)$ and $\mathbb{t}_1 = (\mathbf{t}_1 \,|\, \mathbf{v}_1)$ and where the components of $Shape(\mathbf{u}_1)$ and $Shape(\mathbf{v}_1)$ are all one-column partitions, in particular $d(\mathbf{u}_1) = d(\mathbf{v}_1) = 1$. But from Lemma 9 we know that $m_{\mathbb{s}_1 \mathbb{t}_1} n_{\mathbb{s}\mathbb{t}} \neq 0$ implies $\mathbb{t}_1 \trianglelefteq \mathbb{s}'$ which is impossible since all components of $\boldsymbol{\lambda}_1$ have less then N rows, whereas at least one component of $Shape(\mathfrak{s}')$ has more then N rows; here $\mathbb{s} = (\mathfrak{s} \,|\, \mathbf{u})$ for some $\mathbf{u}$. Hence $m_{\mathbb{s}_1 \mathbb{t}_1} n_{\mathbb{s}\mathbb{t}} = 0$ and so the inclusion $\mathcal{I}_1 \subseteq \mathcal{I}$ is proved.

Suppose now that the inclusion $\mathcal{I}_1 \subseteq \mathcal{I}$ is strict. Then there exists $n \in \mathcal{I} \setminus \mathcal{I}_1$. Our plan is then to construct a $\boldsymbol{\lambda} \in Comp_{r,n,\leq N}$ such that $|A_{\boldsymbol{\lambda}}| = \alpha$ and $M(\boldsymbol{\lambda}) n \neq 0$, which is a contradiction since $M(\boldsymbol{\lambda})$ is a direct summand of $\mathbb{E}_\alpha V^{\otimes n}$ by Lemma 17. We shall do so by showing that there exists a $\Lambda := (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n(\alpha)$ with $\boldsymbol{\lambda} \in Comp_{r,n,\leq N}$, such that $M(\Lambda) n \neq 0$. This gives the desired contradiction since $M(\Lambda) \subseteq M(\boldsymbol{\lambda})$.

In order to do so we consider the expansion $n = \sum_{\mathbb{s},\mathbb{t} \in \mathrm{Std}(\Lambda), \Lambda \in \mathcal{L}(\alpha)} \lambda_{\mathbb{s}\mathbb{t}} n_{\mathbb{s}\mathbb{t}}$. We may assume that $Shape(\mathbb{s}) \in \mathcal{L}_{n,\leq N}(\alpha)$ for all $(\mathbb{s}, \mathbb{t})$ occurring in the expansion of n: otherwise if $Shape(\mathbb{s}) \in \mathcal{L}_{n,\nleq N}(\alpha)$ we subtract the corresponding term $\lambda_{\mathbb{s}\mathbb{t}} n_{\mathbb{s}\mathbb{t}}$ from n and still get an element in $\mathcal{I}_1 \subseteq \mathcal{I}$, by the first part of the proof. We now choose $(\mathbb{s}_{min}, \mathbb{t}_{min})$ with $\lambda_{\mathbb{s}_{min}\mathbb{t}_{min}} \neq 0$ and minimal in the sense that if $\mathbb{s} \trianglelefteq \mathbb{s}_{min}$, $\mathbb{t} \trianglelefteq \mathbb{t}_{min}$ and $\lambda_{\mathbb{s}\mathbb{t}} \neq 0$ then $(\mathbb{s}, \mathbb{t}) = (\mathbb{s}_{min}, \mathbb{t}_{min})$. Let $\mathbb{s}_{min} = (\mathfrak{s}_{min} \mid \mathbf{u}_{min})$ and let $\Lambda_{min} = Shape(\mathbb{s}_{min}) = (\boldsymbol{\lambda}_{min} \mid \boldsymbol{\mu}_{min})$. We then have $\boldsymbol{\lambda}'_{min} \in Comp_{r,n,\leq N}$ of type α and so we may consider the permutation module $M(\Lambda'_{min})$. We are interested in the elements $m_{\mathbb{s}'_{min}}$ and $m_{\mathbb{t}'_{min}}$ of $M(\Lambda'_{min})$. Using the minimality of $(\mathbb{s}_{min}, \mathbb{t}_{min})$, together with Lemma 9 and the invariance of the bilinear form, we get

$$(m_{\mathbb{s}'_{min}} n, m_{\mathbb{t}'_{min}})_{\Lambda'} = (m_{\mathbb{s}'_{min}} n_{\mathbb{s}_{min}\mathbb{t}_{min}}, m_{\mathbb{t}'_{min}})_{\Lambda'} \tag{114}$$

which is nonzero by Lemma 14. Hence $M(\Lambda'_{min}) n \neq 0$, as needed. This proves the Theorem. $\square$

In view of the above Theorem, it is natural to consider the following quotient algebra

$$\mathcal{ETL}_{n,N}^\alpha(q) := \mathcal{E}_n^\alpha(q)/\mathcal{I} \tag{115}$$

where $\mathcal{I}$ is the ideal from 113. It is a generalization of the 'generalized Temperley-Lieb' algebras introduced in [Härterich]. For $\mathcal{K}$ a field, there is also a specialized version of $\mathcal{ETL}_{n,N}^\alpha(q)$ that we denote $\mathcal{ETL}_{n,N}^{\mathcal{K},\alpha}(q)$. By construction $\mathcal{ETL}_{n,N}^\alpha(q)$ is the largest quotient of $\mathcal{E}_n^\alpha(q)$ acting faithfully on $\mathbb{E}_\alpha V^{\otimes n}$. We have the following Corollary to Theorem 18.

Corollary 19 $\mathcal{ETL}_{n,N}^\alpha(q)$ is free over S with basis $\{ n_{\mathbb{s}\mathbb{t}} \,|\, \mathbb{s}, \mathbb{t} \in \mathrm{Std}(\Lambda), \Lambda \in \mathcal{L}_{n,\leq N}(\alpha) \}$. A similar statement holds for $\mathcal{ETL}_{n,N}^{\mathcal{K},\alpha}(q)$.

The following Corollary shows that $\mathcal{ETL}_{n,N}^\alpha(q)$ is also a generalization of the *partition Temperley-Lieb algebra* $\mathcal{PTL}_n^\alpha(q)$ that was introduced in [Juyu].

Corollary 20 Let $St_i \in \mathcal{E}_n^\alpha(q)$ be the i'th *Steinberg element* given by

$$St_i := -q^{-3} g_i g_{i+1} g_i + q^{-2} g_i g_{i+1} + q^{-2} g_{i+1} g_i - q^{-1} g_i - q^{-1} g_{i+1} + 1 \tag{116}$$

and define $\mathcal{PTL}_n^\alpha(q)$ as the quotient algebra $\mathcal{PTL}_n^\alpha(q) := \mathcal{E}_n^\alpha(q)/\mathcal{J}_i$ where $\mathcal{J}_i$ is the two-sided ideal of $\mathcal{E}_n^\alpha(q)$ generated by $e_i e_{i+1} St_i$, for some i. Then $\mathcal{PTL}_n^\alpha(q)$ does not depend on the choice of i and, moreover, $\mathcal{PTL}_n^\alpha(q) = \mathcal{ETL}_{n,2}^\alpha(q)$. Similar statements hold over $\mathcal{K}$.

PROOF. In Proposition 4.5 of [Juyu] it is shown that the $e_i e_{i+1} St_i$'s are all conjugate to each other in $\mathcal{E}_n^\alpha(q)$ and so the $\mathcal{J}_i$'s are equal ideals, which proves the first statement. For simplicity we write $\mathcal{J} = \mathcal{J}_1$. We must then show that $\mathcal{J} = \mathcal{I}$ where $\mathcal{I}$ is the ideal described in Theorem 18. The inclusion $\mathcal{J} \subseteq \mathcal{I}$ follows from the fact that $e_{n-1} e_n St_{n-1}$ belongs the basis for $\mathcal{I}$ given in 113. In fact we have $e_{n-1} e_n St_{n-1} = n_{\mathfrak{t}^\Lambda \mathfrak{t}^\Lambda}$ where $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu})$ is chosen such that all components of $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ are one-column partitions, except $\lambda^{(r)}$ which is of the form $\lambda^{(r)} = (3, 1^s)$ for some s, and such that all components of $\boldsymbol{\mu}$ are one-column partitions. For the other inclusion $\mathcal{J} \supseteq \mathcal{I}$ we consider a basis element $n_{\mathfrak{s}\mathfrak{t}}$ for $\mathcal{I}$, as given in 113. Letting $Shape(\mathfrak{s}) = (\boldsymbol{\lambda} \mid \boldsymbol{\mu})$, there is a component of $\boldsymbol{\lambda}$ with more than three columns and so there is an i such that $e_i e_{i+1} St_i$ is a factor of $n_{\mathfrak{s}\mathfrak{t}}$. This shows $\mathcal{J} \supseteq \mathcal{I}$ and concludes the proof of the Corollary. □

Remark. Using Corollary 19 we can determine the dimension of $\mathcal{ETL}_{n,N}^\alpha(q)$, and hence also of $\mathcal{PTL}_n(q) := \bigoplus_{\alpha \in \mathcal{P}ar_n} \mathcal{PTL}_n^\alpha(q)$, since it is the cardinality of

$$\bigcup_{\alpha \in \mathcal{P}ar_n} \left\{ n_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\Lambda), \Lambda \in \mathcal{L}_{n,\leq 2}(\alpha) \right\}. \tag{117}$$

We have for example

$$\dim \mathcal{PTL}_3(q) = 29, \quad \dim \mathcal{PTL}_4(q) = 334, \quad \dim \mathcal{PTL}_5(q) = 5512. \tag{118}$$

These values are confirmed by Juyuamaya and Papi's MAGMA calculations and by Espinoza's preprint, [Esp], that contains a closed general formula for $\dim \mathcal{PTL}_n(q)$.

Remark. There should be a diagrammatic calculus associated with $\mathcal{PTL}_n(q)$ that it would be interesting to study.

References

[Arcis-Juyu] D. Arcis, J. Juyumaya, *Tied Monoids*, arXiv:2001.00625.

[Aicar-Juyu1] F. Aicardi, J. Juyumaya, *An algebra involving braids and ties*, Preprint ICTP IC/2000/179, Trieste.

[Aicar-Juyu2] F. Aicardi, J. Juyumaya, *Markov trace on the algebra of braids and ties*, Moscow Mathematical Journal **16**(3) (2016), 397-431.

[Aicar-Juyu3] F. Aicardi, J. Juyumaya, *Two parameters bt-algebra and invariants for links and tied links*, arXiv:1811.03675

[Aicar-Juyu4] F. Aicardi, J. Juyumaya, *Tied links and invariants for singular links*, Advances in Mathematics **381** (2021), 40 pages.

[Ban] E. O. Banjo, *The Generic Representation Theory of the Juyumaya Algebra of Braids and Ties*, Algebras and Representation Theory **16**(5) (2013), 1385-1395.

[Flo] M. Flores, *A braids and ties algebra of type B*, Journal of Pure and Applied Algebra **224**(1) (2020), 1-32.

[FloJuyuLam] M. Flores, J. Juyumaya, S. Lambropoulou, *A Framization of the Hecke algebra of Type B*, Journal of Pure and Applied Algebra **222**(1) (2018), 778-806.

[ChlouPou] M. Chlouveraki, G. Pouchin, *Representation theory and an isomorphism theorem for the Framisation of the Temperley-Lieb algebra*, Math. Zeitschrift **285**(3-4) (2017), 1357-1380.

[DipperJames] R. Dipper, G. James *Representations of Hecke algebras of general linear groups*, Proc. London Math. Soc. **52**(3) (1986), 20-52.

[DipperJamMathas] R. Dipper, G. James, A. Mathas, *Cyclotomic q-Schur algebras*, Math. Zeitschrift **229** (1998), 385-416.

[Esp] J. Espinoza, *An isomorphism Theorem for the partition Temperley-Lieb algebra*, preprint.

[EspRy] J. Espinoza, S. Ryom-Hansen, *Cell structures for the Yokonuma-Hecke algebra and the algebra of braids and ties*, J. Pure Appl. Algebra **222** (2018), 3675-3720.

[GraLeh] J. J. Graham, G. I. Lehrer, *Cellular algebras*, Inventiones Mathematicae **123** (1996), 1-34.

[Härterich] M. Härterich, *Murphy bases of generalized Temperley-Lieb algebras*, Archiv der Mathematik **72**(5) (1999), 337-345.

[Hump] J. E. Humphreys, *Reflection groups and Coxeter groups*, volume **29** of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1990.

[JacoAn] N. Jacon, L. Poulain d'Andecy, *Clifford theory for Yokonuma-Hecke algebras and deformation of complex reflection groups*, J. London Math. Soc., **96**(3) (2017), 501-523

[Juyu] J. Juyumaya, *A Partition Temperley-Lieb Algebra*, arXiv:1304.5158.

[Mar] I. Marin, *Artin groups and Yokonuma-Hecke algebras*, Int. Math. Res. Not. **13** (2018), 4022-4062

[Martin] P. Martin, *Representation theory of a small ramified partition algebra*, In Okado Masato Boris Feigin, Jimbo Michio. New Trends in Quantum Integrable Systems: Proceedings of the Infinite Analysis 09. World Scientific Publishing Co. Pte. Ltd. (2010).

[Mathas] A. Mathas, *Hecke algebras and Schur algebras of the symmetric group*, Univ. Lecture Notes, 15, A.M.S., Providence, R.I., 1999.

[Murphy92] G. E. Murphy, *On the Representation Theory of the Symmetric Groups and Associated Hecke Algebras*, J. of Algebra **152** (1992), 492–513.

[Murphy95] E. G. Murphy, *The representations of Hecke algebras of type A_n*, J. of Algebra **173** (1995), 97-121.

[PlaRy] D. Plaza, S. Ryom-Hansen *Graded cellular bases for Temperley-Lieb algebras of type A and B*. J. Algebr. Comb. **40**(1) (2014), 137-177.

[Ry] S. Ryom-Hansen, *On the Representation Theory of an Algebra of Braids and Ties*, J. Algebra Comb. **33** (2011), 57-79.

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