

# Dynamical Objects for Cohomologically Expanding Maps.

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## Abstract

The goal of this paper is to construct invariant dynamical objects for a (not necessarily invertible) smooth self map of a compact manifold. We prove a result that takes advantage of differences in rates of expansion in the terms of a sheaf cohomological long exact sequence to create unique lifts of finite dimensional invariant subspaces of one term of the sequence to invariant subspaces of the preceding term. This allows us to take invariant cohomological classes and under the right circumstances construct unique currents of a given type, including unique measures of a given type, that represent those classes and are invariant under pullback. A dynamically interesting self map may have a plethora of invariant measures, so the uniqueness of the constructed currents is important. It means that if local growth is not too big compared to the growth rate of the cohomological class then the expanding cohomological class gives sufficient “marching orders” to the system to prohibit the formation of any other such invariant

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current of the same type (say from some local dynamical subsystem). Because we use subsheaves of the sheaf of currents we give conditions under which a subsheaf will have the same cohomology as the sheaf containing it. Using a smoothing argument this allows us to show that the sheaf cohomology of the currents under consideration can be canonically identified with the deRham cohomology groups. Our main theorem can be applied in both the smooth and holomorphic setting.

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## 1 Introduction

Our purpose is to construct invariant dynamical objects for a self map  $f: X \rightarrow X$  of a compact topological space. We make use of sheaf cohomology and differences in rates of expansion in different terms of a long exact sequence to construct invariant sections of a sheaf. We will show that there are invariant degree 1 currents (or eigencurrents) corresponding to each expanding eigenvector of  $H^1(X, \mathbb{R})$ . We also show that successive preimages of sufficiently regular degree one currents converge to one of these eigencurrents. We show that if most of the expansion  $f: X \rightarrow X$  is "along" an invariant cohomological class  $v \in H^k(X, \mathbb{R})$  then there is an invariant current  $c$  in that cohomology class and other sufficiently regular currents in the same class converge to  $c$  under successive pullback.

The group cohomology of  $\mathbb{Z}$  acting on a space of functions on  $X$  via pullback has been studied in the context of dynamical systems [Kat03]. This work seems related to ours, but to be pursued in an essentially different direction. Our map  $f$  is not assumed to be invertible, so there is not necessarily a  $\mathbb{Z}$  action, only an  $\mathbb{N}$  action. Also, we use sheaves rather than functions and make substantial use of cohomological tools. Most importantly, we are particularly interested in the construction of invariant currents, especially when the current is some sense unique.

Our results are actually motivated by results in higher dimensional holomorphic dynamics showing the existence of a unique closed positive  $(1, 1)$  current under a variety of circumstances (just about any recent paper on higher dimensional holomorphic dynamics either proves such results or makes essential use of such results, see e.g. [FS92], [HOV94], [HOV95], [BS91a], [BS91b], [BS92], [BLS93], [BS98a], [BS98b], [BS99], [Can01], [McM02], [FS94a], [FS94b],

[FS95b], [FS01], [FS95a], [JW00], [FJ03], [Ued94], [Ued98], [Ued97], and [DS05]).

While invariant measures have been a focal point in dynamics, it seems that invariant currents also have an important role to play. We will show under mild conditions that if some degree one cohomological class of a smooth self map  $f$  of a compact manifold is invariant and expanded there is necessarily a invariant degree one current of a certain type representing that class. We obtain analogous results for higher degree currents given bounds on the local growth rates of  $f$ . The uniqueness of these classes is significant. It seems clear that one could modify a map locally near a fixed point to obtain other invariant currents of the same type without affecting the topology. Thus our results also say that any local modification that created an invariant current of the given type *must violate the local growth conditions*. In other words, as long as things do not grow too fast compared to the growth rate of the cohomology class, the expansion of the cohomology class gives sufficient “marching orders” to points that no other invariant cohomological class of the given type can be created by purely local dynamical behavior. Our results give explicit conditions under which uniqueness is guaranteed. For degree one currents, no restriction on local growth rates is necessary for our results.

## 2 Cohomomorphisms

We will make use of sheaves in this paper. There are two standard definitions of sheaves on a topological space  $X$ , one as a topological space ([Bre97],[GR84]), and one as a functor on the category  $\text{Top}_X$  satisfying various axioms ([Har77],[Wei97]). Since we will often want to make use of a topology on sections of a sheaf  $\mathcal{A}$  that differs from the topology these inherit using the topological definition of a sheaf, we will instead use the functor definition of a sheaf.

Our sheaves will always be sheaves of  $\mathbb{K}$  modules over some fixed field  $\mathbb{K}$ . We will require that  $\mathbb{K}$  have an absolute value for which  $\mathbb{K}$  is complete.

Given a continuous map  $f: X \rightarrow Y$  and sheaves  $\mathcal{A}$  and  $\mathcal{B}$  on  $X$  and  $Y$  respectively, an  $f$ -cohomomorphism is a generalized notion of a pullback from  $\mathcal{B}$  to  $\mathcal{A}$  through  $f$ . Different types of geometric objects pull back differently, and this allows us to handle all cases at once.

We take the following facts from [Bre97] page 14–15.

**Definition 1.** If  $\mathcal{A}$  and  $\mathcal{B}$  are sheaves on  $X$  and  $Y$  then an “ $f$ -cohomomorphism”

$k: \mathcal{B} \rightarrow \mathcal{A}$  is a collection of homomorphisms  $k_U: \mathcal{B}(U) \rightarrow \mathcal{A}(f^{-1}(U))$ , for  $U$  open in  $Y$ , compatible with restrictions.

Note that if  $\mathcal{A}$  is a sheaf on  $X$  and  $f: X \rightarrow Y$  is continuous then there is a canonical cohomomorphism  $f_*\mathcal{A} \rightsquigarrow \mathcal{A}$  where  $f_*\mathcal{A}$  is the direct image of  $\mathcal{A}$ , i.e. given an open  $U \subset Y$ ,  $f_*\mathcal{A}(U) = \mathcal{A}(f^{-1}(U))$ .

*Remark.* Given a continuous map  $f: X \rightarrow Y$  of topological spaces  $X$  and  $Y$  and sheaves  $\mathcal{A}$  and  $\mathcal{B}$  on  $X$  and  $Y$  respectively, all  $f$ -cohomomorphisms  $f: \mathcal{B} \rightsquigarrow \mathcal{A}$  are given by a composition of the form

$$\mathcal{B} \xrightarrow{j} f_*\mathcal{A} \xrightarrow{f_*} \mathcal{A}$$

where  $j: \mathcal{B} \rightarrow f_*\mathcal{A}$  is a sheaf homomorphism, and each such composition is seen to give an  $f$ -cohomomorphism.

The usual notion of “a morphism of sheaves on  $X$ ” is the same as an  $\text{id}_X$  cohomomorphism of sheaves on  $X$ .

## 2.1 Cohomomorphisms and $\Gamma$ .

The functor  $\Gamma$  returns the global sections of that sheaf. Given a morphism  $\phi: \mathcal{A} \rightarrow \mathcal{A}'$  of sheaves on  $X$ ,  $\Gamma\phi$  is just the homomorphism  $\mathcal{A}(X) \rightarrow \mathcal{A}'(X)$ . Given sheaves  $\mathcal{A}$  and  $\mathcal{B}$  on  $X$  and  $Y$  and given  $f: X \rightarrow Y$  continuous then for a sheaf cohomomorphism  $F: \mathcal{B} \rightarrow \mathcal{A}$  one defines  $\Gamma F$  to be the homomorphism  $\mathcal{B}(Y) \rightarrow \mathcal{A}(X)$ . This extends  $\Gamma$  to be a functor on the category of topological spaces with an associated sheaf where morphisms are given by cohomomorphisms.

## 3 Invariant Global Sections

Fix a continuous self map  $f: X \rightarrow X$  of a topological space  $X$ . We will be interested in  $f$  self cohomomorphisms of sheaves  $\mathcal{A}$  on  $X$ . As we will typically have several sheaves of interest on  $X$ , each with a corresponding  $f$  self cohomomorphism, we let  $f_{\mathcal{A}}: \mathcal{A} \rightsquigarrow \mathcal{A}$  be the default notation for an  $f$ -cohomomorphism of  $\mathcal{A}$ .

Assume that  $X$  is a manifold and that

$$\mathcal{A} \xrightarrow{p} \mathcal{B} \xrightarrow{q} \mathcal{C}$$

is a short exact sequence of sheaves on  $X$ . Let  $f: X \rightarrow X$  be a continuous self map of  $X$  and assume further that we are given  $f$  self cohomomorphisms of each of these sheaves and that

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{p} & \mathcal{B} & \xrightarrow{q} & \mathcal{C} \\ \downarrow f_{\mathcal{A}} & & \downarrow f_{\mathcal{B}} & & \downarrow f_{\mathcal{C}} \\ \mathcal{A} & \xrightarrow{p} & \mathcal{B} & \xrightarrow{q} & \mathcal{C} \end{array} \quad (1)$$

commutes. We will say that a commutative diagram as in (1) is an  $f$  self-cohomomorphism of the sequence  $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$ .

Applying the functor  $\Gamma$  to this diagram, the rows can be extended in the usual long exact sequence. The resulting diagram is commutative ([Bre97] page 62).

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{A}(X) & \xrightarrow{\Gamma_p} & \mathcal{B}(X) & \xrightarrow{\Gamma_q} & \mathcal{C}(X) & \xrightarrow{\delta} H^1(X, \mathcal{A})_{H^1_p} \longrightarrow \cdots \\ & \downarrow \Gamma f_{\mathcal{A}} & & \downarrow \Gamma f_{\mathcal{B}} & & \downarrow \Gamma f_{\mathcal{C}} & \downarrow H^1 f_{\mathcal{A}} \\ 0 \longrightarrow & \mathcal{A}(X) & \xrightarrow{\Gamma_p} & \mathcal{B}(X) & \xrightarrow{\Gamma_q} & \mathcal{C}(X) & \xrightarrow{\delta} H^1(X, \mathcal{A})_{H^1_p} \longrightarrow \cdots \end{array} \quad (2)$$

One can think of  $\mathcal{B}$  as providing local potentials for members of  $\mathcal{C}$  and of  $\mathcal{A}$  as being those potentials which give rise to the zero member of  $\mathcal{C}$ . It will be assumed that the reader is familiar with interpreting  $H^1(X, \mathcal{A})$  as classifying equivalence classes of bundles with transition functions in  $\mathcal{A}$ . We will frequently refer to members of  $H^1(X, \mathcal{A})$  as bundles. Sections of such bundles will be assumed to be given locally by local sections of  $\mathcal{B}$ , so that every member  $c$  of  $\Gamma(\mathcal{C})$  is given locally by potentials in  $\mathcal{B}$ , and these potentials, taken together, are a section of the corresponding bundle  $\delta(c) \in H^1(X, \mathcal{A})$ .

**Convention 1.** We will frequently refer to a member  $v$  of  $H^1(X, \mathcal{A})$  as a bundle, to a member  $c \in \Gamma(\mathcal{C})$  as a divisor and if  $\delta(c) = v$  we will call  $c$  a divisor of the bundle  $v$ . We think this substantially adds to the readability of the paper.

**Definition 2.** The *support* of a divisor  $c \in \Gamma(\mathcal{C})$  is defined to be the complement of the union of all open sets  $U$  such that  $c|_U = 0$ .

**Lemma 3.** *If an open set  $U$  lies outside the support of some  $c \in \Gamma(\mathcal{C})$  then  $f^{-1}(U)$  lies outside the support of  $f_{\mathcal{C}}(c)$*

*Proof.* We note that by the definition of an  $f$ -cohomomorphism  $f_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}$ , since the cohomomorphism  $f_{\mathcal{C}}$  on  $\mathcal{C}(U)$  is a homomorphism from  $\mathcal{C}(U)$  to  $\mathcal{C}(f^{-1}(U))$  and the induced action of  $f_{\mathcal{C}}$  on  $\Gamma(\mathcal{C})$  restricted to  $U$  must agree with its action  $\mathcal{C}(U) \rightarrow \mathcal{C}(f^{-1}(U))$ , then if an open set  $U$  is outside the support of  $c$  then  $f^{-1}(U)$  is outside the support of  $f_{\mathcal{C}}(c)$ .  $\square$

The following conditions for a given  $v \in H^1(X, \mathcal{A})$  will be of interest:

**Definition 4.** We will refer to a bundle  $v \in H^1(X, \mathcal{A})$  for which  $(H^1 p)(v) = 0$  as being *closed*.

Note that this notion depends upon the exact sequence  $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$ , and not just on  $v$ . If  $\mathcal{B}$  is  $\gamma$  acyclic then every member of  $H^1(X, \mathcal{A})$  is closed.

**Definition 5.** We will call a bundle  $v \in H^1(X, \mathcal{A})$  *base point free* if for every  $x \in X$  there is some divisor  $c \in \Gamma(\mathcal{C})$  associated to  $v$  whose support does not contain  $x$ .

**Lemma 6.** *If  $\mathcal{B}$  is soft,  $X$  is a regular topological space, and  $a \in H^1(X, \mathcal{A})$  is a closed bundle then  $a$  is base point free.*

*Proof.* From the long exact sequence there is some  $c' \in \Gamma(\mathcal{C})$  with  $\delta(c') = a$  and given any point  $x \in X$ , from the fact that  $\mathcal{B} \xrightarrow{q} \mathcal{C}$  the germ  $c'_x$  of  $c'$  at  $x$  is the image under  $q_x$  of some germ  $b''_x$  of  $\Gamma(\mathcal{B})$  at  $x$ . Choose an open neighborhood  $U$  of  $x$  on which there is some  $b' \in \mathcal{B}(U)$  with  $b'_x = b''_x$ . The topological assumption on  $X$  implies that there is a neighborhood  $V \subseteq U$  of  $x$ . The fact that  $\mathcal{B}$  is soft implies there is some  $b \in \Gamma(\mathcal{B})$  such that  $b|_V = b'|_V$ . Then  $c = c' - b \in \Gamma(\mathcal{C})$  has  $\delta(c) = a$  and  $x \notin \text{Supp}(c)$ .  $\square$

**Definition 7.** We will refer to a bundle  $a \in H^1(X, \mathcal{A})$  such that  $f_{\mathcal{A}}(a) = \lambda \cdot a$  for some  $\lambda \in \mathbb{C}$  as a  $\lambda$  *eigenbundle*.

We also find it useful to introduce a relevant notion of expansiveness of a map  $f: X \rightarrow X$  relative to a base point free closed eigenbundle  $v \in H^1(X, \mathcal{A})$ .

**Definition 8.** Given a base point free closed eigenbundle  $v \in H^1(X, \mathcal{A})$  then we say that  $f$  is cohomologically expansive at  $x$  for  $v$  if for any open neighborhood  $U$  of  $x$  and any divisor  $c \in \Gamma(\mathcal{C})$  of  $v$ , the set  $U$  intersects the support of  $f_{\mathcal{C}}^k(c)$  for all sufficiently large  $k$ .

*Remark.* It is a corollary of the definition that the set of points at which  $f$  is cohomologically expansive for  $v$  is closed and forward invariant. If  $\text{Supp } f_{\mathcal{C}}^k(c) = f^{-k}(\text{Supp}(c))$  for each  $c \in \Gamma(\mathcal{C})$  then the set of cohomologically expansive points is totally invariant.

The notion of being cohomologically expansive at  $x$  for  $v$  means roughly that under iteration by  $f$  small neighborhoods  $U$  of  $x$  always grow to cover enough of  $X$  that the pullback of the bundle  $v$  to the set  $f^k(U)$  is a nontrivial bundle on  $f^k(U)$  whenever  $k$  is large.

We show that if  $\mathcal{B}$  is soft and  $X$  is a compact metric space then some minimal expansion takes place at points where  $f$  is cohomologically expansive for a closed eigenbundle  $a \in H^1(X, \mathcal{A})$ .

We use  $B_\epsilon(x)$  to denote the ball of radius  $\epsilon$  about  $x$ .

**Lemma 9.** *Let  $X$  be a compact metric space. If  $\mathcal{B}$  is soft and  $v$  is a closed eigenbundle then there exists  $\delta > 0$  such that for every  $\epsilon > 0$  there exists some  $K > 0$  such that if  $f$  is cohomologically expanding at  $x$  then for every  $k > K$ ,  $\text{diam } f^k(B_\epsilon(x)) > \delta$ .*

*Proof.* The bundle  $v$  is base point free by Lemma 6. Using compactness we can conclude that there is a finite open cover  $U_1, \dots, U_\ell$  of  $X$  such that for each  $j$ ,  $U_j$  is disjoint from  $\text{Supp } c_j$  for some  $c_j \in \Gamma(\mathcal{C})$  with  $\delta(c_j) = v$ . We will prove the lemma by contradiction. Let  $\delta$  be the Lebesgue number of the cover  $U_1, \dots, U_\ell$ . If the lemma is false there is some  $\epsilon > 0$  and some increasing sequence  $k_n$  and points  $x_n$  at which  $f$  is cohomologically expansive such that  $\text{diam } f^{k_n}(B_\epsilon(x_n)) \leq \delta$  for each  $n$ . By going to a subsequence if necessary we can assume  $x_n$  converges to a point  $x_\infty$ . Letting  $U = B_{\frac{1}{2}\epsilon}(x_\infty)$  we see that  $U \subset B_\epsilon(x_n)$  for all large  $n$  and thus there is some one  $c_j$  of  $c_1, \dots, c_\ell$  such that  $f^{k_n}(U)$  is disjoint from  $\text{Supp } c_j$  for infinitely many values of  $n$ . Consequently  $U$  is disjoint from  $\text{Supp } f_{\mathcal{C}}^{k_n}(c_j)$  for infinitely many  $n$ , contrary to  $x_\infty$  being a point at which  $f$  is cohomologically expansive for  $v$ .  $\square$

We included Lemma 9 to show that our notion of cohomological expansion is genuinely expansive. However, depending on the nature of  $\mathcal{A}$ , being cohomologically expansive can imply that neighborhoods grow a great deal under iteration indeed. In Lemma 10 we show that given any closed set  $K$  such that the pullback of a fixed point free closed eigenbundle  $a \in H^1(X, \mathcal{A})$  to  $K$  is a trivial bundle then any neighborhood  $U$  of a point at which  $f$  is cohomologically expanding for  $a$  is so expanded under iteration that  $f^k(U) \not\subset \text{int } K$  for

all sufficiently large  $k$ . The collection of such sets  $K$  typically contains very large sets about every point so no matter where  $f^k(x)$  is the conclusion that  $f^k(U)$  does not lie in any  $\text{int } K$  implies some points of  $f^k(U)$  must lie far away from  $f^k(x)$ . The point is roughly that large iterates of any neighborhood of  $x$  can not be homotopically contracted to a point in  $X$ .

**Lemma 10.** *If  $\mathcal{B}$  is soft, then for any closed set  $K \subset X$  such that the image of  $H^1(X, \mathcal{A}) \rightarrow H^1(K, \mathcal{A}|_K)$  is zero, given any divisor  $c \in \Gamma(\mathcal{C})$ , there is another divisor  $c' \in \Gamma(\mathcal{C})$  associated to the same bundle and  $c'$  is supported outside the interior of  $K$ . Consequently, if  $f$  is cohomologically expansive at  $x \in X$  for some base point free closed eigenbundle  $a \in H^1(X, \mathcal{A})$  then necessarily for any neighborhood  $U$  of  $x$ ,  $f^k(U) \not\subset \text{int } K$  for all large  $k$ , where  $\text{int } K$  is the interior of  $K$ .*

*Proof.* We use the commutative diagram

$$\begin{array}{ccccc} H^0(X, \mathcal{B}) & \xrightarrow{\Gamma q} & H^0(X, \mathcal{C}) & \xrightarrow{\delta} & H^1(X, \mathcal{A}) \\ \downarrow & & \downarrow & & \downarrow \\ H^0(K, \mathcal{B}|_K) & \xrightarrow{\Gamma q} & H^0(K, \mathcal{C}|_K) & \xrightarrow{\delta} & 0 \end{array}$$

which we have written using  $H^0$  instead of  $\Gamma$  so it is clear what the ambient space is in each case. From exactness there exists some  $\beta \in H^0(K, \mathcal{B}|_K)$  such that  $\delta(\beta) = c|_K$ . Then since  $\mathcal{B}$  is soft the map  $H^0(X, \mathcal{B}) \rightarrow H^0(K, \mathcal{B}|_K)$  is surjective so there is some  $b \in \Gamma(\mathcal{B}) = H^0(X, \mathcal{B})$  such that  $b|_K = \beta$ . Then  $c' = c - (\Gamma q)(b)$  has  $\delta(c') = \delta(c)$  and  $c'|_K = 0$  so  $\text{Supp}(c')$  is disjoint from the interior of  $K$ .

It is easy to see that if  $f$  is cohomologically expansive at  $x \in X$  for some fixed point free closed eigenbundle  $a \in H^1(X, \mathcal{A})$  then necessarily for any neighborhood  $U$  of  $x$ ,  $f^k(U) \cap \text{Supp } c \neq \emptyset$  for all large  $k$  for any  $c \in \Gamma(\mathcal{C})$  such that  $\delta(c) = a$ . Hence  $f^k(U)$  can not lie in the interior of  $K$  for any large  $k$ .  $\square$

**Convention 2.** *We let  $\mathbb{K}$  be either  $\mathbb{R}$  or  $\mathbb{C}$ , although our central theorems only require  $\mathbb{K}$  to be a complete field with an absolute value.*

The following Theorem takes advantage of the fact that in an exact sequence the eigenvalues of members of nonadjacent members of the sequence do not have to agree to give conditions under which one can uniquely “lift”



fixed members of one term of the exact sequence to a fixed member of the preceding term. Interpreted as a statement in the context of sheaf cohomology we will be able to use this Theorem to make dynamical conclusions.

The theorem shows that each closed eigenbundle of the induced map  $f_{\mathcal{A}}: H^1(X, \mathcal{A}) \rightarrow H^1(X, \mathcal{A})$  with sufficiently large eigenvalue has a unique associated invariant divisor  $c \in \Gamma(\mathcal{C})$ .

**Definition 11.** Given any finite dimensional  $\mathbb{K}$  vector space  $V$  along with a linear map  $g: V \rightarrow V$  and any positive real number  $r$ , we let the  $r$  chronically expanding subspace of  $V$  be the span of the subspaces associated<sup>2</sup> to eigenvalues of absolute value greater than  $r$ . We refer to the 1 chronically expanding subspace simply as the chronically expanding subspace.

**Theorem 12** (Unique Invariant Subspace Theorem). *We will assume the following:*

- $f: X \rightarrow X$  is a continuous self map of a topological space  $X$ .
- We are given an  $f$  self cohomomorphism of a short exact sequence of sheaves on  $X$ ,

$$\mathcal{A} \xrightarrow{p} \mathcal{B} \xrightarrow{q} \mathcal{C}$$

- $\Gamma(\mathcal{B})$  is a Banach space over  $\mathbb{K}$ , and there exists some  $\alpha, d \in \mathbb{R}_{>0}$  such that  $\|\Gamma f_{\mathcal{B}}^k(\mathbf{B})\| \leq d \cdot \alpha^k \|\mathbf{B}\|$  for  $k \in \mathbb{N}$ ,  $\mathbf{B} \in \Gamma(\mathcal{B})$ ,
- $\Gamma(\mathcal{C})$  is a topological vector space over  $\mathbb{K}$ .
- If a sequence  $\mathbf{C}_i \in \Gamma(\mathcal{C})$  of divisors converges to another divisor  $\mathbf{C}_{\infty}$  then the support of  $\mathbf{C}_{\infty}$  is contained in the closure of the union of the supports of  $\mathbf{C}_i$ .
- The maps  $\Gamma f_{\mathcal{C}}$  and  $\Gamma q$  are continuous.
- We are given a finite dimensional  $H^1(f_{\mathcal{A}})$  invariant subspace  $W$  of the  $\alpha$  chronically expanding subspace of  $H^1(X, \mathcal{A})$ . We also require  $W$  to be comprised only of closed bundles.

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<sup>2</sup>Meaning for each eigenvector  $\lambda$  we include not just the  $\lambda$  eigenspace, but also every  $v \in V$  such that  $(g - \lambda \cdot \text{id}_V)^n(v) = 0$  for some positive integer  $n$ .

Then given any  $\mathbb{K}$  linear map  $s: W \rightarrow \Gamma(\mathcal{C})$  such that  $\delta s = \text{id}_W$  there is a  $\mathbb{K}$  linear map  $\tau: W \rightarrow \Gamma(\mathcal{B})$  satisfying

$$\kappa := \lim_{k \rightarrow \infty} (\Gamma f_{\mathcal{C}})^k s g^k = s + (\Gamma q) \tau \quad (3)$$

where  $g: W \rightarrow W$  is the inverse of  $H^1 f_{\mathcal{A}}|_W$ . Under iterated pullback the rescaled pullbacks of any divisor  $\mathbf{C} \in \Gamma(\mathcal{C})$  of a bundle  $w \in W$  converge toward the invariant plane of divisors  $\kappa(W) \subset \Gamma(\mathcal{C})$ . The map  $\kappa: W \rightarrow \Gamma(\mathcal{C})$  is the unique map making the diagram

$$\begin{array}{ccccc} & & W & & \\ & \swarrow \kappa & \downarrow & \searrow \iota & \\ \Gamma(\mathcal{B}) & \xrightarrow{\Gamma q} & \Gamma(\mathcal{C}) & \xrightarrow{\delta} & H^1(X, \mathcal{A}) \\ \downarrow \Gamma f_{\mathcal{B}} & & \downarrow \Gamma f_{\mathcal{C}} & & \downarrow H^1 f_{\mathcal{A}} \\ & & W & & \\ & \swarrow \kappa & \downarrow & \searrow \iota & \\ \Gamma(\mathcal{B}) & \xrightarrow{\Gamma q} & \Gamma(\mathcal{C}) & \xrightarrow{\delta} & H^1(X, \mathcal{A}) \end{array}$$

commute. Finally, for any basepoint free eigenbundle  $v \in W$  the support of the corresponding invariant divisor  $\kappa(v) \in \Gamma(\mathcal{C})$  is contained in the set of points on which  $f$  is cohomologically expansive for  $v$ .

*Proof.* We note that  $\delta((\Gamma f_{\mathcal{C}})sg - s) = 0$  and so there is a map  $\sigma: W \rightarrow \Gamma(\mathcal{B})$  such that  $(\Gamma q)\sigma = (\Gamma f_{\mathcal{C}})sg - s$ .

Define  $\Phi: \text{Hom}(W, \Gamma(\mathcal{B})) \rightarrow \text{Hom}(W, \Gamma(\mathcal{B}))$  by  $\Phi(\sigma) = (\Gamma f_{\mathcal{B}})\sigma g^{-1}$ . We will show that the sequence of maps  $\Phi^k$  is exponentially contracting on  $\text{Hom}(W, \Gamma(\mathcal{B}))$ . Fix a norm  $\|\cdot\|$  on  $W$ . The assumption that  $W$  lies in the  $\alpha$  chronically expanding subspace of  $H^1(X, \mathcal{A})$  implies that there exists some  $\beta > \alpha$  and some  $c > 0$  such that  $\|g^{-k}(w)\| \leq c\beta^{-k}\|w\|$  for  $k \in \mathbb{N}$ ,  $w \in W$ . This with the assumption on the rate of expansion of  $\Gamma f_{\mathcal{B}}$  easily implies that

$$\|\Phi^k(\phi)(w)\| = \|(\Gamma f_{\mathcal{B}})^k(\phi(g^{-k}(w)))\| \leq cd\left(\frac{\alpha}{\beta}\right)^k \|\phi\| \cdot \|w\|$$

Thus  $\Phi^k$  is an operator of norm no more than  $cd\left(\frac{\alpha}{\beta}\right)^k$ , where  $\alpha < \beta$ .

Letting  $\tau_k = \sigma + \Phi(\sigma) + \Phi^2(\sigma) + \dots + \Phi^k(\sigma)$  then  $\lim_{k \rightarrow \infty} \tau_k$  converges to some map  $\tau$ . It is easily confirmed that  $(\Gamma q)\tau_k = (\Gamma f_{\mathcal{C}})^k s g^{-k} - s$ . Equation (3) then follows by continuity of  $\Gamma q$ .

The conclusions about the map  $\kappa$  are easy consequences of its definition.

For the final conclusion note that if we just let  $W$  be the span of  $v$  then we have already shown that if  $\mathbf{C}$  is the unique invariant member of  $\Gamma(\mathcal{C})$  associated to  $v$  then for any divisor  $c' \in \Gamma(\mathcal{C})$  satisfying  $\delta(c') = v$  letting  $\lambda$  be the eigenvalue of  $v$  we can write  $c' = \kappa(v) + (\Gamma q)(b)$  and equation 3 becomes  $(\Gamma f_{\mathcal{C}})^k c' / \lambda^k = \kappa(v) + (\Gamma q)(\Gamma f_{\mathcal{B}})^k b \lambda^k$  where the final term goes to zero as  $k \rightarrow \infty$  (by our assumptions on growth rates of  $g^{-1}$  and  $\Gamma f_{\mathcal{B}}$ ). Hence  $(\Gamma f_{\mathcal{C}})^k(c') / \lambda^k$  converges to  $c = \kappa(v)$ . If  $U$  is any open subset of  $X$  and if the support of  $c'$  is disjoint from  $f^n(U)$  for arbitrarily large values of  $n$ , then the support of  $(\Gamma f_{\mathcal{C}})^n(c')$  must be disjoint from  $U$  for arbitrarily large values of  $n$ . Since, rescaled, these converge to  $c$  then  $U$  must lie outside the support of  $c$ .  $\square$

*Remark.* While we have not formally required  $X$  to be compact, the requirement that  $\Gamma(\mathcal{B})$  be a Banach space makes this the main case in which Theorem 12 is apt to have interesting applications.

Theorem 12 shows that among all members of  $\Gamma(\mathcal{C})$  representing a cohomology class in  $W$  there is a unique invariant linear subspace which can be identified with  $W$  and all other such members of  $\Gamma(\mathcal{C})$  are contracted to this invariant copy of  $W$  in  $\Gamma(\mathcal{C})$  under (rescaled) pullback.

**Corollary 13.** *Assume that the hypothesis of Theorem 12 are satisfied, and that  $g: W \rightarrow W$  is dominated by a single simple real eigenvalue  $r > 0$  with eigenvector  $v$ . Let  $\mathbf{C} \equiv \kappa(v)$  be the unique invariant divisor of  $v$ . Then given a divisor  $\mathbf{C}' \in \Gamma(\mathcal{C})$  of any  $w \in W$  the successive rescaled pullbacks  $f_{\mathcal{C}}^k(\mathbf{C}')/r^k$  converge to a multiple (possibly zero) of  $\mathbf{C}$ .*

*Proof.* This is a direct consequence of equation (3).  $\square$

The assumption that  $g: W \rightarrow W$  is dominated by a single simple real eigenvalue is meant to handle the most typical situation, and is not an essential restriction.

*Remark.* Given that for a fixed  $f: X \rightarrow X$  the category of  $\mathcal{SC}$  sheaves  $\mathcal{A}$  on  $X$  endowed with an  $f$  self cohomomorphism  $F$  is an abelian category with enough injectives, then the functor  $\text{Fixed } \Gamma$  which gives the fixed global sections of  $\mathcal{A}$  under  $F$  will be left exact and its right derived functors should be of dynamical interest. In the case where  $\mathcal{A}$  is a sheaf of functions and  $f$  is invertible this is just group cohomology with the group  $\mathbb{Z}$  acting on  $\Gamma(\mathcal{A})$  and has been an object of study for some time (see, e.g. [Kat03]). We

anticipate studying the case of more general sheaves  $\mathcal{A}$  and the right derived functors of the composition **Fixed**  $\Gamma$  in a future paper, including the case of endomorphisms.

### 3.1 Regularity and Positivity

Typically our regularity results for the members invariant plane  $\kappa(W)$  will be most easily described in terms of  $\mathcal{B}$  rather than  $\mathcal{C}$ . We therefore make the following definition.

**Definition 14.** Given a subsheaf  $\mathcal{B}' \subset \mathcal{B}$  we will say a divisor  $\mathbf{C} \in \Gamma(\mathcal{C})$  has local  $\mathcal{B}'$  potentials if  $\mathbf{C} \in \Gamma(q(\mathcal{B}'))$ . This is equivalent to requiring that about each point  $x \in X$  there is an open neighborhood  $U$  and some  $\mathbf{B}' \in \mathcal{B}'(U)$  such that  $q(\mathbf{B}') = \mathbf{C}|_U$ .

The proof of Theorem 12 implicitly provides a method to prove regularity results for members of the invariant plane  $\kappa(W)$ . We make this explicit as a corollary (of the proof).

**Corollary 15.** *Assume we are given  $f: X \rightarrow X$  and a short exact sequence of sheaves  $\mathcal{A} \xrightarrow{p} \mathcal{B} \xrightarrow{q} \mathcal{C}$  satisfying the hypothesis of Theorem 12. Assume that  $\mathcal{B}'$  is a subsheaf of  $\mathcal{B}$  and that  $\Gamma f_{\mathcal{B}}(\mathcal{B}') \subset \mathcal{B}'$ . Let  $\mathcal{C}'$  be the image of  $\mathcal{B}'$  under  $q: \mathcal{B} \rightarrow \mathcal{C}$ . Let  $\mathcal{A}' \subset \mathcal{A}$  be the kernel of  $q: \mathcal{B} \rightarrow \mathcal{C}'$ . Assume that the canonical map  $H^1(X, \mathcal{A}') \rightarrow H^1(X, \mathcal{A})$  is injective. Assume that there are basis members  $w_1, \dots, w_k$  of  $W$  with divisors each of which has local potentials in  $\mathcal{B}'$ . Let  $r$  be the the inverse of the absolute value of the largest eigenvalue of  $g^{-1}$  (so for all  $j \geq 0$ ,  $g^{-j}$  is an operator of norm no more than  $cr^{-j}$  for some  $c > 0$ ) Finally assume that for any sequence of numbers  $a_j, j = 0, 1, 2, \dots$  such that  $|a_j|$  is no more than a constant times  $r^{-j}$  as  $j \rightarrow \infty$  then for  $\mathbf{B} \in \Gamma(\mathcal{B}')$  the exponentially decaying sequence*

$$a_0 \mathbf{B} + a_1 (\Gamma f_{\mathcal{B}})(\mathbf{B}) + a_2 (\Gamma f_{\mathcal{B}})^2(\mathbf{B}) + \dots \quad (4)$$

*converges in the Banach space structure on  $\Gamma(\mathcal{B})$  to a member of  $\Gamma(\mathcal{B}')$ . Then the map  $\kappa: W \rightarrow \Gamma(\mathcal{C})$  lands in  $\Gamma(\mathcal{C}')$ .*

*Proof.* Since  $W$  lies in the  $\alpha$  chronically expanding subspace of  $W$  then necessarily  $\alpha/r < 1$ . Thus the terms of equation (4) have exponentially decreasing norms and the series is exponentially decaying.

By the assumption of a divisor in  $\Gamma(\mathcal{C}')$  for each member  $w_j$  of a basis then the map  $s: W \rightarrow \Gamma(\mathcal{C})$  in Theorem 12 can be assumed to land in  $\Gamma(\mathcal{C}')$ . Then  $(\Gamma f_{\mathcal{C}})sg^{-1} - s$  lands in  $\Gamma(\mathcal{C}')$  and satisfies  $\delta((\Gamma f_{\mathcal{C}})sg^{-1} - s) = 0$ . Since  $H^1(X, \mathcal{A}') \rightarrow H^1(X, \mathcal{A})$  injects it easily follows that for each  $w_j$  one can choose  $\sigma(w_j)$  to be a member  $B_j$  of  $\Gamma(\mathcal{B})'$ . Using the basis  $w_1, \dots, w_k$  to write  $g^{-1}$  as a matrix  $A$ , and letting  $a_{ij,\ell}$  be the  $ij$  entry of  $A^\ell$  (so for each  $ij$ ,  $a_{ij,\ell}$  is bounded by a constant times  $r^{-\ell}$ ) we see that  $\tau_\ell(w_j) = B_j + (\Gamma f_{\mathcal{B}})(a_{1j,1}B_1 + \dots + a_{kj,1}B_k) + (\Gamma f_{\mathcal{B}})^2(a_{1j,2}B_2 + \dots + a_{kj,2}B_k) + \dots + (\Gamma f_{\mathcal{B}})^\ell(a_{1j,\ell}B_1 + \dots + a_{kj,\ell}B_k)$ . Gathering all the  $B_1$  terms,  $B_2$  terms, etc... from the right hand side we see that  $\tau = \lim_{k \rightarrow \infty} \tau_k$  is a member of  $\Gamma(\mathcal{B}')$  and thus that  $\kappa$  lands in  $\Gamma(\mathcal{C}')$  by equation (3).  $\square$

The following trivial observation will suffice for our needed positivity conclusions.

*Observation.* Assume we have an  $f$  self cohomomorphism of a short exact sequence of sheaves  $\mathcal{A} \xrightarrow{p} \mathcal{B} \xrightarrow{q} \mathcal{C}$  satisfying the hypothesis of Theorem 12, and also a subsheaf  $\mathcal{C}' \subset \mathcal{C}$  such that

1.  $\mathcal{C}'$  is closed under multiplication by  $\mathbb{R}_{>0}$ . Note that  $\mathcal{C}'$  is not necessarily a sheaf of  $\mathbb{K}$  modules, or even of groups.
2.  $f_{\mathcal{C}}(\mathcal{C}') \subset \mathcal{C}'$
3.  $\Gamma(\mathcal{C}')$  is closed in  $\Gamma(\mathcal{C})$ .

Then for any closed eigenbundle  $v \in H^1(X, \mathcal{A})$  with eigenvalue in  $\mathbb{K}_0$  and at least one divisor  $C' \in \Gamma(\mathcal{C}')$  the unique invariant divisor  $C \in \Gamma(\mathcal{C})$  of  $v$  also lies in  $\Gamma(\mathcal{C}')$ .

*Proof.* The proof is trivial since  $C = \lim_{k \rightarrow \infty} (\Gamma f_{\mathcal{C}})^k(C')/\lambda^k$  where  $\lambda \in \mathbb{R}_{>0}$  is the eigenvalue of  $v$ .  $\square$

## 4 Subsheaf Cohomology

In applications of Theorem 12 it is common that there is a well understood exact sequence of sheaves

$$\mathcal{S}_0 \xrightarrow{d_0} \mathcal{S}_1 \xrightarrow{d_1} \mathcal{S}_2 \xrightarrow{d_2} \dots \quad (5)$$

and that  $\mathcal{B}$  is a subsheaf of  $\mathcal{S}_k$  for some  $k$ ,  $\mathcal{A}$  is the kernel of  $d_k|_{\mathcal{B}}: \mathcal{B} \rightarrow \mathcal{S}_{k+1}$  and  $\mathcal{C}$  is the image of  $\mathcal{B}$  in  $\mathcal{S}_{k+1}$ . Moreover, in these cases the self cohomomorphism  $f$  on  $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$  is induced by an  $f$  self cohomomorphism of the sequence (5). In order to apply Theorem 12 to these cases we need to understand the  $R$  module  $H^1(X, \mathcal{A})$  and its induced self map.

There does not seem to be a computationally useful way to extract an injective resolution of  $\mathcal{A}$  using subsheaves of  $\mathcal{S}_0 \xrightarrow{d_0} \mathcal{S}_1 \xrightarrow{d_1} \dots$  even if this last sequence is acyclic. Consider for example the case where for each  $n$ ,  $\mathcal{S}_n$  is the sheaf of currents of degree  $n$  and  $\mathcal{B} \subset \mathcal{S}_k$  is a subsheaf of mildly regular currents. It is not clear one could make the regularization method of [dR84] work to compare  $H^1(X, \mathcal{A})$  to deRham cohomology groups because his chain homotopy operator  $A$  does not restrict well to  $\mathcal{B}$  since  $dA$  does not preserve regularity. We use a standard sheaf cohomological trick, which we include here as a proposition which we will need and which we expect to be commonly used in conjunction with Theorem 12 because of the requirement that  $\Gamma(\mathcal{B})$  be a Banach space.

**Theorem 16** (Subsheaf Cohomology). *Assume we are given an exact sequence of sheaves  $\mathcal{S}_0 \xrightarrow{d_0} \mathcal{S}_1 \xrightarrow{d_1} \mathcal{S}_2 \xrightarrow{d_2} \dots$  and that  $\mathcal{B}$  is a subsheaf of  $\mathcal{S}_k$  for some  $k \geq 1$ . Let  $\mathcal{A} = \ker d_k|_{\mathcal{B}}$ , and  $\mathcal{B}'$  be the preimage of  $\mathcal{B}$  under  $d_{k-1}$ . Further assume that for each  $j \geq 1$  we have  $H^j(X, \mathcal{B}') = 0$ ,  $H^j(X, \mathcal{B}) = 0$  and for any  $m$  satisfying  $0 \leq m \leq k-1$  we have  $H^j(X, \mathcal{S}_m) = 0$  for  $j \geq 1$ . Then for each  $n \geq 1$  there is a canonical isomorphism*

$$H^n(X, \mathcal{A}) \cong H^{n+k}(X, \ker d_0).$$

*Proof.* While this result is essential for us, its proof is a standard cohomological trick. First one notes that  $\ker d_{k-1}|_{\mathcal{B}'} = \ker d_{k-1}$  by the definition of  $\mathcal{B}'$ . One has the short exact sequences of sheaves:

$$\ker d_{k-1} \rightarrow \mathcal{B}' \rightarrow (d_k(\mathcal{B}') = \mathcal{A})$$

and

$$\ker d_j \rightarrow \mathcal{S}_j \rightarrow \ker d_{j+1}, \quad j = 0, \dots, k-2.$$

Considering the long exact sequences for these shows that the induced maps  $H^n(X, \mathcal{A}) \rightarrow H^{n+1}(X, \ker d_{k-1})$  and  $H^{n+j}(X, \ker d_{k-j}) \rightarrow H^{n+j-1}(X, \ker d_{k-j-1})$  are isomorphisms for  $j = 1, \dots, k-1$ . Composing each of these canonical isomorphisms gives a canonical isomorphism from  $H^n(X, \mathcal{A}) \rightarrow H^{n+k}(X, \ker d_0)$ .  $\square$

*Remark.* We take it as clear from the functorality of the  $\delta$  map in the long exact sequence that given an  $f$ -self cohomomorphism of  $\mathcal{S}_0 \xrightarrow{d_0} \mathcal{S}_1 \xrightarrow{d_1} \mathcal{S}_2 \xrightarrow{d_2} \cdots$  which maps  $\mathcal{B}$  to itself that the induced map of  $H^1(X, \mathcal{A})$  is identified with the induced map of  $H^{k+1}(X, \ker d_0)$  via the above isomorphism.

We will need one more tool be able to make effective use of Theorem 16 for calculating sheaf cohomology of subsheaves of sheaves of currents.

**Definition 17.** By an *interval flow*  $h$  on a bounded open interval  $I \subset \mathbb{R}$  we will mean the flow obtained by integrating a vector field of the form  $\sigma(t) \frac{\partial}{\partial t}$  where  $\sigma$  is positive exactly on  $I$  and zero elsewhere. We use  $h(x, t)$  to denote the location of  $x \in R$  after following the flow for time  $t$ .

**Definition 18.** By an  $n$ -box in  $\mathbb{R}^n$  we will mean an open subset which is a product of  $n$  bounded open intervals  $I_1, \dots, I_n$ . By an  $n$ -box in an  $n$  dimensional manifold we will mean an  $n$ -box which is compactly supported in some coordinate patch. By an  $n$ -subbox of an  $n$  box  $U = I_1 \times \cdots \times I_n$  we will mean an  $n$  box of the form  $I'_1 \times \cdots \times I'_n$  where  $I'_k$  is a subinterval of  $I_k$  for each  $k \in 1, \dots, n$ .

**Definition 19.** By an  $n$ -box flow we will mean the  $\mathbb{R}^n$  action  $h$  on  $\mathbb{R}^n$  which is the product of  $n$  interval flows  $h_1(t_1), \dots, h_n(t_n)$  on  $\mathbb{R}^n$ . That is  $h(x, t) = (h_1(x_1, t_1), \dots, h_n(x_n, t_n))$  where  $x = (x_1, \dots, x_n)$ ,  $t = (t_1, \dots, t_n)$  and  $h_1, \dots, h_n$  are interval flows on  $I_1, \dots, I_n$  respectively. We refer to the  $n$ -box  $I_1 \times \cdots \times I_n$  as the open support of the  $n$ -box flow. We will often  $h_t$  to denote the diffeomorphism  $h(\cdot, t): \mathbb{R}^n \rightarrow \mathbb{R}^n$ .

**Definition 20.** Let  $h$  be an  $n$ -box flow on an  $n$ -box  $B$ . Let  $\rho$  be a compactly supported smooth volume form on  $\mathbb{R}^n$ . With this data we define an operator  $\mathcal{S}_{h,\rho}$  on smooth  $k$  forms on any  $n$  box  $U$  containing  $B$  by

$$\mathcal{S}_{h,\rho}(\phi) = \int_{\mathbb{R}^n} h_t^*(\phi) \rho(t) \quad (6)$$

We say  $\mathcal{S}_{h,\rho}$  defines a *box smear* on  $U$ , or *smears*  $U$ . We will omit the subscript from  $\mathcal{S}_{h,\rho}$  when the meaning is clear from context. It is clear  $\mathcal{S}(\phi)$  is compactly supported in  $U$  if  $\phi$  is.

It is clear from the definition of  $\mathcal{S}$  that if  $\psi$  is an  $n - k$  form on  $U$  then

$$\int_U \mathcal{S}_{H,\rho}(\phi) \wedge \psi = \int_U \phi \wedge \mathcal{S}_{-H,\rho}(\psi)$$

where  $-H$  is the family  $H_t$  with the parameter negated. From this motivation we define a smear of a current.

**Definition 21.** Given  $h, \rho$  defining a smear on an  $n$  box  $U$  we define the smear  $\mathcal{S}_{h,\rho}$  on currents on  $U$  via

$$\langle \mathcal{S}_{h,\rho}(\mathbf{C}), \phi \rangle \equiv \langle \mathbf{C}, \mathcal{S}_{-h,\rho}(\phi) \rangle .$$

**Lemma 22.** *Given  $h, \rho$  defining a smear  $\mathcal{S}$  on an  $n$  box  $U$  then  $\mathbf{d}(\mathcal{S}(\mathbf{C})) = \mathcal{S}(\mathbf{d}\mathbf{C})$  for currents  $\mathbf{C}$  on any open subset of  $U$  containing the open support of the smear. Also, restricted to the open support of the smear,  $\mathcal{S}(\mathbf{C})$  is a smooth form on  $V$ .*

*Proof.* We remark that it is clear that  $\mathbf{d}(\mathcal{S}(\phi)) = \mathcal{S}(\mathbf{d}\phi)$  for forms  $\phi$ , and consequently for currents  $\phi$  via the definition.

Because on the open support of the smear, a smear is just convolution with a smooth function, then we see that if  $V$  is an open subset of the open support of smear  $\mathcal{S}$  on  $U$  then for any current  $\mathbf{C}$  on  $U$ ,  $\mathcal{S}(\mathbf{C})|_V$  is a smooth form on  $V$ .  $\square$

**Proposition 23.** *Let  $\mathcal{B}$  be a sheaf of degree  $k$  currents. Assume that  $\mathcal{B}$  contains the sheaf of smooth  $k$  forms on  $X$ , and that  $\mathcal{B}(U)$  is closed under smears on any  $n$ -box  $U \subset X$ . Let  $\mathcal{B}'$  be the preimage under  $\mathbf{d}$  of  $\mathcal{B}$  in the sheaf of degree  $k-1$  currents. Then  $\mathcal{B}'$  is soft, and therefore,  $\Gamma$ -acyclic.*

*Proof.* To show that  $\mathcal{B}'$  is soft it is sufficient to show that  $\mathcal{B}'$  is locally soft ([Bre97] page 69). Given an  $n$ -box  $U$  in  $X$  we therefore only need to show that if  $K$  is a closed subset of  $X$  in  $U$  and if  $W$  is an open neighborhood of  $K$  then given any member  $\mathbf{B}'_0$  of  $\mathcal{B}'(W)$  there is an open neighborhood  $W_0 \subset W$  of  $K$  and a member  $\mathbf{B}' \in \mathcal{B}'(U)$  such that  $\mathbf{B}'|_{W_0} = \mathbf{B}'_0|_{W_0}$ .

Choose any pair of open sets  $V_1, V_2$  such that  $K \Subset V_1 \Subset V_2 \Subset W$ . Then  $\overline{V_2} \setminus V_1$  is compact and can therefore be covered by finitely many (open)  $n$ -subboxes  $Y_1, \dots, Y_N$  of  $U$ . Moreover these subboxes can all be chosen to be disjoint from  $K$  and to lie inside  $W$ . Letting  $\mathcal{S}_1, \dots, \mathcal{S}_n$  be smears on  $U$  with open support  $Y_1, \dots, Y_N$  respectively then let  $\mathbf{B} = \mathcal{S}_1(\mathcal{S}_2(\dots(\mathcal{S}_N(\mathbf{B}'_0))\dots))$ . Then on each  $Y_j$ ,  $\mathbf{B}$  is given by a smooth  $k$  form. Also,  $\mathbf{B}|_W = \mathbf{B}'_0|_W$ . Finally, we choose a smooth function  $\psi: U \rightarrow [0, 1]$  which is one on a neighborhood of  $\overline{V_1}$  and zero on a neighborhood of  $U \setminus V_2$ . Then the current  $\mathbf{B}' \equiv \psi\mathbf{B}$  extends (by zero) to a current on all of  $U$ . Then for each  $Y_j$ ,  $\mathbf{B}'|_{Y_j}$  is a smooth function times a smooth form. Thus  $\mathbf{d}(\mathbf{B}'|_{Y_j})$  is a smooth form and



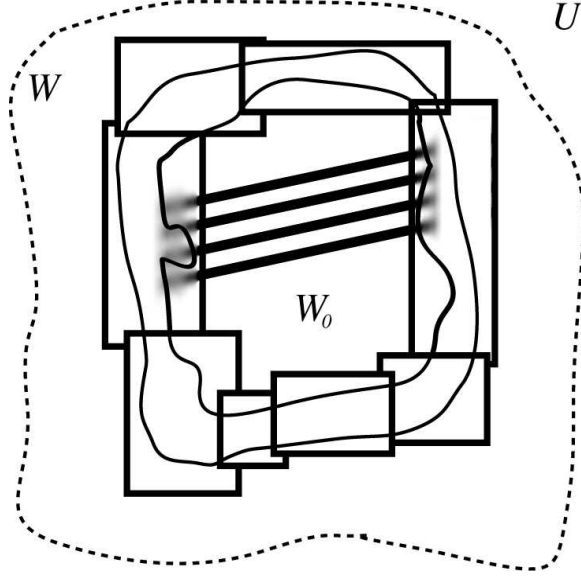


Figure 1: A current comprised of parallel submanifolds smeared and cropped.

lies in  $\mathcal{B}(Y_j)$ . The boxes  $Y_j$  cover  $\overline{V_2} \setminus V_1$ . Outside  $V_2$ ,  $\mathbf{B}'$  is identically zero. We know that  $\mathbf{dB} \in \mathcal{B}(W)$  by Lemma 22. We also know that  $\psi \equiv 1$  on an open neighborhood  $W_1$  of  $\overline{V_1}$ . Thus  $\mathbf{d}(\mathbf{B}'|_{W_1}) = \mathbf{d}(\mathbf{B}|_{W_1}) \in \mathcal{B}(W_1)$ . We thus conclude that  $\mathbf{B}' \in \mathcal{B}'(U)$  since its restriction to each  $Y_j$ , to  $W_1$  and to  $U \setminus \overline{V_2}$  is a section of  $\mathcal{B}'$ . Letting  $W_0 = V_2 \setminus (\overline{Y_1 \cup Y_2 \cup \dots \cup Y_N})$  then  $W_0$  is an open neighborhood of  $K$ , then  $W_0 \subset W_1$  so  $\mathbf{B}'|_{W_0} = \mathbf{B}|_{W_0} = \mathbf{B}'_0$  since  $W_0$  is disjoint from the open support of each of the smears  $\mathcal{S}_1, \dots, \mathcal{S}_N$ . This completes the proof that  $\mathcal{B}'$  is soft.  $\square$

The following gives a broad generalization of the equalivalence of the cohomology of currents with the deRham cohomology groups. To the author's knowledge, this result is new.

**Corollary 24.** *Let  $\mathcal{B}$  be a sheaf of degree  $k$  currents. Assume that  $\mathcal{B}$  contains the sheaf of smooth  $k$  forms on  $X$ , and that  $\mathcal{B}(U)$  is closed under smears on any  $n$ -box  $U \subset X$ . Letting  $\mathcal{A}$  be the subsheaf of  $\mathbf{d}$  closed members of  $\mathcal{B}$ , then*

$$H^m(X, \mathcal{A}) = H^{m+k}(X, \mathbb{K}),$$

where  $\mathbb{K}$  is  $\mathbb{R}$  or  $\mathbb{C}$  depending on whether or not we allow complex valued currents and forms.

*Proof.* This is an immediate consequence of Proposition 23 and Theorem 16.  $\square$

## 5 Invariant Currents

**Notation 1.** If  $\mathcal{G}$  is some sheaf of functions on a smooth orientable manifold  $X$  we will use  $\mathcal{F}^k(\mathcal{G})$  to denote the sheaf of  $k$  forms on  $X$  with coefficients in  $\mathcal{G}$ . We will let  $\mathcal{F}_c^k(\mathcal{G})$  be the subsheaf of closed (in the sense of currents) members of  $\mathcal{F}^k(\mathcal{G})$ .

It will be convenient to use either degree or dimension of a current depending on the context (just as dimension and codimension are useful for discussing manifolds), so we will not stick to just one of these terms. We will let  $\mathcal{C}^k$  denote the sheaf of degree  $k$  currents with the index written above as is typical for cohomology since  $d$  increases the degree. We will similarly write  $\mathcal{C}_k$  for the sheaf of dimension  $k$  currents with the index written below since  $d$  decreases dimension as is common for homology. We use the following convention to realize a form  $\alpha$  as a current so that if  $\alpha$  is  $C^1$  then  $d\alpha$  is the same whether computed as a current or a form.

**Definition 25.** Given an  $k$  form  $\alpha$  with  $L^1$  coefficients on an  $n$  manifold  $X$  we realize  $\alpha$  as a degree  $k$  current via

$$\beta \mapsto (-1)^{\binom{k+1}{2}} \int_X \alpha \wedge \beta$$

**Definition 26.** Given a (possibly complex) nonzero deRham cohomology class  $c \in H_{\text{deRham}}^k(X)$  with  $f^*(c) = \alpha \cdot c$  for some scalar  $\alpha \in \mathbb{C}$  we will refer to a current  $C$  in the same cohomology class as  $\alpha$  as an eigencurrent for  $f$  if  $f^*(C) = \alpha C$ .

Currents naturally pushforward, rather than pullback. Because we are considering maps which are not necessarily invertible we need to address how this pullback is performed. If  $f$  has critical points it is impossible to define a continuous pullback operation  $f^*$  on all currents in a way that agrees with expected cases. For instance, consider  $f(x) = x^2$  and let  $C_a$  be the dimension one current on  $\mathbb{R}$  with  $C_a(h(x)dx) = h(a)$ , i.e.  $C_a$  is a unit mass vector. Then the pullback  $f^*(C_a)$  should be the sum of weighted unit masses at the two preimages of this vector (just like the pullback of a point mass

is a sum of point masses each weighted by multiplicity), that is,  $f^*(C_a) = \frac{1}{2\sqrt{a}}(C_{\sqrt{a}} - C_{-\sqrt{a}})$ . However, these pullbacks do not converge to a current as  $a \rightarrow 0$  so  $f^*(C_0)$  is not defined. Since we want  $f^*$  to be continuous, we are forced to work with currents that have some extremely mild regularity. We address this in the next section.

## 5.1 Nimble Forms and Lenient Currents

Finding a good set of currents to use to study smooth finite self maps (not necessarily invertible) of compact manifolds turns out to be rather delicate. Our solution is to first expand our class of forms to include pushforwards (in the sense of currents) of forms through an appropriate class of smooth maps. Then we restrict our attention to currents which act on this extended class of forms.

This solution has the very nice property that it can potentially be adapted directly to study the dynamics of other various other categories of smooth maps (by simply changing which forms are considered nimble, according to the class of maps used). It will be convenient to first define the natural pushforward operator on forms:

**Definition 27.** Given a compact orientable manifold  $X$  we let  $\mathcal{S}_X$  be the category of smooth maps  $f: X \rightarrow X$  of nonzero degree and having the property that the critical set has measure zero. We use critical set here to mean the points at which  $Df$  is not invertible.

It follows from our definition that the image of any set of positive measure under some  $f \in \mathcal{S}_X$  has positive measure.

**Definition 28.** Given a compact orientable manifold  $X$  we define  $\mathcal{N}^k$  to be those currents  $\varphi$  which are a finite sum of currents of the form  $p_*(\sigma)$  where  $p: X \rightarrow X$  is a map in  $\mathcal{S}_X$  and  $\sigma$  is a form of degree  $k$ . The pushforward  $p_*(\sigma)$  is computed in the sense of currents.

We will later show that nimble forms are also, in fact, bona fide forms.

**Definition 29.** We topologize  $\mathcal{N}^k$  by saying  $\varphi_j \rightarrow \varphi$  in  $\mathcal{N}^k$  if for sufficiently large  $j$  there are maps  $f_1, \dots, f_k$  and  $k$  forms  $\sigma_{1j}, \dots, \sigma_{kj}$  as well as forms  $\sigma_1, \dots, \sigma_k$  such that  $\sum_i f_{i*}(\sigma_{ij}) = \varphi_j$  and  $\sum_i f_{i*}(\sigma_i) = \varphi$  (where pushforwards are taken in the sense of currents) and for each  $i \in 1, \dots, k$ , the forms  $\sigma_{ij}$  converge to  $\sigma_i$  in the strong sense (i.e. all derivatives converge uniformly).

**Lemma 30.** *Given a compact orientable manifold  $Y$ ,  $\mathcal{N}^k(Y)$  is a topological vector space.*

*Proof.* This follows easily from our definition of the topology.  $\square$

We now define the corresponding space of currents.

**Definition 31.** We define the dimension  $k$  *lenient currents*  $\mathcal{L}_k(Y)$  to be the topological dual of  $\mathcal{N}^k(Y)$ . Every member of  $\mathcal{L}_k(Y)$  is a dimension  $k$  current, but with the added structure of its action on all nimble  $k$  forms. We give  $\mathcal{L}_k$  the weak topology, i.e.  $C_i \rightarrow C$  in  $\mathcal{L}_k$  iff  $\langle C_i, \varphi \rangle \rightarrow \langle C, \varphi \rangle$  for every  $\varphi \in \mathcal{N}^k$ . We write  $\mathcal{L}^k$  for the lenient currents of degree  $k$ .

We define operations of wedge products with smooth forms as is usual for currents. It is clear that the lenient dimension  $k$  currents give a sheaf on  $X$ .

The following properties of nimble forms are also immediately clear.

**Lemma 32.** *Let  $f: X \rightarrow X$  be a member of  $\mathcal{S}_X$ . The pushforward (as a current) of a nimble  $k$  form by  $f$  is again a nimble form. Moreover  $f_*: \mathcal{N}^k(X) \rightarrow \mathcal{N}^k(X)$  is continuous (in the topology of nimble forms). Also the exterior derivative of a nimble form (as a current) is a nimble form and  $d: \mathcal{N}^k(X) \rightarrow \mathcal{N}^{k+1}(X)$  is continuous.*

The basic necessary facts about pulling back lenient currents are then immediate. We state them here:

**Lemma 33.** *Given  $f: X \rightarrow X$  a member of  $\mathcal{S}_X$  the induced map  $f^*$  on the sheaf of lenient degree  $k$  currents is an  $f$  cohomomorphism of sheaves. Both  $f^*: \mathcal{L}^k(X) \rightarrow \mathcal{L}^k(X)$  and  $d: \mathcal{L}^k(X) \rightarrow \mathcal{L}^{k+1}(X)$  are continuous. Lastly,  $f^*d = df^*: \mathcal{L}^k(Y) \rightarrow \mathcal{L}^{k+1}(X)$ .*

**Proposition 34.** *Assume that  $f: X \rightarrow X$  is a member of  $\mathcal{S}_X$ . Let  $R$  be the regular set of  $f$ . By Sard's theorem  $R$  has full measure. Since the critical set is compact then  $R$  is an open subset of  $X$ . Since the preimage of a measure zero set has measure zero for  $\mathcal{S}_X$  maps then  $f^{-1}(R)$  is also a full measure open set in  $X$ . There is a well defined operation  $f_*$  which maps  $k$  forms on  $f^{-1}(R)$  to  $k$  forms on  $R$ . Given a  $k$  form  $\beta$  on  $X$ ,  $f_*(\beta)$  is defined on any open subset  $V \subset R$  such that each component  $U_1, \dots, U_m$  of  $f^{-1}(V)$  maps diffeomorphically onto  $V$  by the formula*

$$f_*(\beta)|_V \equiv \frac{1}{\deg f} \sum_i \left( (f|_{U_i})^{-1} \right)^* (\beta) \cdot \sigma_i \quad (7)$$

where  $\sigma_i \in \{\pm 1\}$  is the oriented degree of  $f|_{U_i}: U_i \rightarrow V$ . The pushforward  $f_\star$  satisfies:

- $f_\star d = df_\star$  (keeping in mind that  $f_\star$  returns a current on  $R$ )
- $f_\star(1) = 1$
- $f_\star(f^*(\beta) \wedge \alpha) = \beta \wedge f_\star(\alpha)$
- $(f_\star)^n = (f^n)_\star$
- The formula

$$\int_X f^*(\beta) \wedge \alpha = \int_X \beta \wedge f_\star(\alpha) \quad (8)$$

holds for any  $k$  form  $\beta$  with  $L_{\text{loc}}^\infty$  coefficients on  $Y$  and any smooth  $n-k$  form  $\alpha$  on  $X$ . This justifies using  $f_\star$  to pullback currents. (Part of the conclusion is that both sides are integrable.)

*Proof.* Each statement is a consequence of formula (7) except the integrability conclusion for equation (8). Local charts can be given which are bounded subsets of  $\mathbb{R}^n$  and for which  $Df$  remains uniformly bounded (over each of the charts) and thus  $f^*(\beta)$  will be a form with  $L_{\text{loc}}^\infty$  coefficients in these charts. Thus the left hand side of (8) is the integral of a bounded function over a finite union of bounded charts and is therefore absolutely integrable. Since  $f_\star(f^*(\beta) \wedge \alpha) = \beta \wedge f_\star(\alpha)$  it is sufficient to show that if  $\gamma$  is an  $n$  form with  $L_{\text{loc}}^\infty$  coefficients then

$$\int_{f^{-1}(R)} \gamma = \int_R f_\star(\gamma). \quad (9)$$

Typically  $f_\star(\gamma)$  is unbounded so we need to show that the right hand side of (9) is integrable. About any point  $x \in R$  we can find an open  $V$  such that each of the preimages  $U_1, \dots, U_k$  of  $V$  is mapped diffeomorphically onto  $V$ . Since  $X$  is orientable and  $n$  dimensional there is a well defined notion of the absolute value of an  $n$  form. Then

$$\int_V |f_\star(\gamma)| \leq \frac{1}{\deg f} \sum_i \int_V \left| \left( (f|_{U_i})^{-1} \right)^* (\gamma) \right| = \sum_i \int_{U_i} |\gamma| = \int_{f^{-1}(V)} |\gamma|.$$

Now  $R$  is covered by countably many such sets  $V$  and listing them as  $V_0, V_1, V_2, \dots$ , we can let  $V'_0 = V_0, V'_1 = V_1 \setminus V_0, V'_2 = V_2 \setminus (V_0 \cup V_1), \dots$ . Then  $R$  is the union

of the countable collection of disjoint measurable sets  $V'_j$  and

$$\int_R |f_\star(\gamma)| = \sum_j \int_{V_j} |f_\star(\gamma)| \leq \sum_j \int_{f^{-1}(V_j)} |\gamma| = \int_{f^{-1}(R)} |\gamma|.$$

Since  $\int_{f^{-1}(R)} |\gamma|$  is finite then  $f_\star(\gamma)$  is an  $L^1$  form. Using precisely the same argument but with the absolute values removed and the inequalities replaced with equalities then shows  $\int_R f_\star(\gamma) = \int_{f^{-1}(R)} \gamma$ .  $\square$

Since  $R$  and  $f^{-1}(R)$  are open and full measure then  $f_\star$  is an operator which takes in forms on  $X$  and returns forms defined almost everywhere on  $X$ .

We now show that nimble forms are bona fide forms.

**Lemma 35.** *If  $g: X \rightarrow X$  is a map in  $\mathcal{S}_X$  and  $\sigma$  is a smooth  $k$  form on  $X$  then the current  $g_\star(\sigma)$  is the current of integration against the form  $g_\star(\sigma)$ .*

*Proof.* If  $\varphi$  is a smooth  $n - k$  form then by definition  $\langle g_\star(\sigma), \varphi \rangle = \langle \sigma, g^*(\varphi) \rangle = (-1)^{\binom{k+1}{2}} \int_X \sigma \wedge g^*(\varphi) = (-1)^{\binom{k+1}{2}} \int_X g_\star(\sigma) \wedge \varphi = \langle g_\star(\sigma), \varphi \rangle$  by formula (8) of Proposition 34  $\square$

As described in [Fed69], an inner product on a vector space  $V$  can be viewed as an isomorphism  $\ell: V \rightarrow V^*$  satisfying certain properties. The inverse of  $\ell$  gives the induced inner product on  $V^*$ . The fact that  $\langle v, w \rangle \leq \|v\| \cdot \|w\|$  with equality iff  $v$  and  $w$  are scalar multiples implies that the inner product norm on  $V^*$  is the same as the operator norm of  $V^*$  acting on  $V$ .

The induced map  $\bigwedge^k \ell: \bigwedge^k V \rightarrow \bigwedge^k V^*$  gives an inner product on  $\bigwedge^k V$ . We call this the canonical inner product on  $\bigwedge^k V$  induced by the inner product on  $V$ . Hence, given a Riemannian metric on  $X$ , there are canonical smoothly varying inner products on  $\bigwedge^k T_x X$  and  $\bigwedge^k T_x^* X$  for each  $x \in X$ . At any point  $x \in X$  we define  $\|\bigwedge^k D_x f\|$  to be the operator norm of the linear function  $\bigwedge^k D_x f: \bigwedge^k T_x X \rightarrow \bigwedge^k T_{f(x)}^* X$ . We define  $\|\bigwedge^k Df\|$  to be the  $L^\infty_{\text{loc}}$  norm of the map  $x \mapsto \|\bigwedge^k D_x f\|$ . Also, given a  $k$  form  $\varphi$  we define the *comass*  $\|\varphi\|_{L^\infty_{\text{loc}}}$  of  $\varphi$  to be the  $L^\infty_{\text{loc}}$  norm of the function  $x \mapsto \|\bigwedge^k \varphi_x\|$ . It is clear that the  $k$  forms with the comass norm is a Banach space. We now show that the  $k$  forms with  $L^\infty_{\text{loc}}$  coefficients are naturally lenient currents. We start by defining the action on nimble forms.

**Definition 36.** Given an  $n - k$  form  $\mathbb{C}$  with  $L_{\text{loc}}^\infty$  coefficients we define

$$\langle \mathbb{C}, p_*(\sigma) \rangle = (-1)^{\binom{n-k+1}{2}} \int_X \mathbb{C} \wedge p_*(\sigma)$$

**Lemma 37.** The space  $\mathcal{F}^{n-k}(L_{\text{loc}}^\infty)$  of  $n - k$  forms with  $L_{\text{loc}}^\infty$  coefficients under the comass norm includes continuously into  $\mathcal{L}_k(X)$  where the action of  $\mathbb{C} \in \mathcal{F}^{n-k}(L_{\text{loc}}^\infty)$  on some  $\varphi = \sum_i f_{i*}(\sigma_i) \in \mathcal{N}^k(X)$ , with each  $f_i \in \mathcal{S}_X$  and each  $\sigma_i \in \mathcal{F}^k(C^\infty)$  is given by

$$\langle \mathbb{C}, \varphi \rangle \equiv \sum_i \int_X f_i^*(\mathbb{C}) \wedge \sigma_i.$$

*Proof.* The assumption that  $X$  is compact means that any two Riemannian metrics on  $X$  are comparable. Choose one so the notion of the comass norm makes sense. The result is then a straightforward consequence of equation (8), Lemma 35, and our definitions.  $\square$

*Remark.* It follows that a current with local  $\mathcal{F}^k(L_{\text{loc}}^\infty)$  potentials is also a lenient current.

*Remark.* Given a member  $\mathbb{C}$  of  $\mathcal{F}^k(L_{\text{loc}}^\infty)$  then  $f^*(\mathbb{C})$  is the same whether done as a lenient current or as a form. This, along with the fact that  $\mathbf{d}f^* = f^*\mathbf{d}$  justifies the ad hoc pullback of closed positive  $(1, 1)$  currents used so successfully in holomorphic dynamics. Similarly  $\mathbf{d}\mathbb{C}$  gives the same result whether calculated as a lenient current or a form if  $\mathbb{C} \in \mathcal{F}^k(C^1)$ .

## 5.2 Hölder Lemmas

We will want to apply Corollary 15 to show that each eigencurrent we construct has local  $\mathbf{d}$  potentials (or  $\mathbf{d}\mathbf{d}^c$  potentials in the holomorphic case) which are forms with Hölder continuous coefficients. In order to do this we will need a few facts which we include here in order to avoid having to include regularization results as afterthoughts to our main theorems.

*Observation.* Let  $H_\alpha$  be the functions with coefficients that are Hölder of exponent at least equal to some fixed  $\alpha > 0$ . Since diffeomorphisms preserve Hölder exponents and averages of Hölder functions are Hölder then we take it as clear that Corollary 24 applies to show that  $H^1(X, \mathcal{A}') = H^1(X, \mathcal{A})$  where  $\mathcal{A}'$  is the closed members of  $\mathcal{F}^k(H_\alpha)$  and  $\mathcal{A}$  is the closed degree  $k$  currents.

**Lemma 38.** *Let  $X$  be a compact manifold (real or complex) with a Riemannian metric and of real dimension  $n$ . Let  $f: X \rightarrow X$  be a smooth map. Then local coordinate charts  $U_i$  can be chosen on  $X$  (each representing a convex open subset of  $\mathbb{R}^n$ ) so that there is a positive constant  $1 < M$  so that for any  $k$  form  $\varphi$ , there exist constants  $c, C > 0$  such that writing each  $f^{k*}(\varphi)$  in any of the charts  $U_i$  as*

$$f^{k*}(\varphi) = \sum a_{ki} dx^{\wedge i}$$

*then each function  $a_{ki}$  satisfies*

$$\sup_{x \in U_i} |a_{ki}| \leq c \cdot \|f^{k*}(\varphi)\|_{comass} \quad (10)$$

*and for each  $j \in 1, \dots, n$ ,*

$$\sup_{x \in U_i} \left| \frac{\partial a_{ki}}{\partial x_j} \right| \leq C \cdot M^k.$$

*Proof.* Equation (10) is a basic fact.

The rest is a straightforward consequence of realizing a self map of a manifold as being made up of a bunch of maps between different coordinate patches in  $\mathbb{R}^n$ . That is, one chooses an open cover of patches  $U_i$  of  $X$ . Each patch is realized in  $\mathbb{R}^n$  as a round ball. Thinking of each patch as lying in  $\mathbb{R}^n$  then we can find explicit maps from between open subsets of  $\mathbb{R}^n$  of the form  $p_{ij}: U_i \cap f^{-1}(U_j) \rightarrow U_j$ . By shrinking each open ball  $U_i$  a small amount the resulting patches still cover  $X$  but the derivatives of the maps  $p_{ij}$  are all now bounded (since we are working on relatively compact subsets of the previous maps  $p_{ij}$ ).

Then given any  $x$  we can keep track of which patch  $f^k(x)$  is in at each time and can then realize the map  $f^k(x)$  as a composition  $p_{i_1 i_2} \circ p_{i_2 i_3} \circ \dots \circ p_{i_{k-1} i_k}$ . Since each partial derivative of each  $p_{ij}$  is uniformly bounded then any partial derivative of the composition grows at most exponentially with  $k$  and we are done.  $\square$

The following observation will also be useful:

**Lemma 39.** *If there are positive constants  $c, C, m, M$  with  $m < 1 < M$  such that a sequence of smooth functions  $h_k$  on an open convex set  $U \subset \mathbb{R}^n$  satisfies*

$$\|h_k\|_{sup} < c \cdot m^k$$



and

$$\left\| \frac{\partial h_k}{\partial x_j} \right\|_{sup} < C \cdot M^k$$

for all  $k \in 0, 1, 2, \dots$  then  $h_1 + h_2 + h_3 + \dots$  converges to a bounded continuous function which is Hölder of any exponent  $\alpha < \frac{\log(m)}{\log(m/M)}$ .

*Proof.* The proof is elementary.  $\square$

### 5.3 Eigencurrents for Cohomologically Expanding Smooth Maps

We will call a section  $V$  of  $\bigwedge^k TX$  a  $k$ -vector field. We define  $\|V\|_{L_{loc}^\infty}$  to be the  $L_{loc}^\infty$  norm of the function  $x \mapsto \|V_x\|$ . Whether Theorem 12 applies to a map will depend the size of  $\|\bigwedge^k Df\|$ . Replacing  $f$  with an iterate does not affect the needed estimate so we make the following definition.

**Definition 40.** We define  $\Upsilon_k$  to be the limit supremum as  $j \rightarrow \infty$  of  $\|\bigwedge^k D(f^j)\|_{\frac{1}{j}}$ . It follows that  $\Upsilon_1 \geq e^\lambda$  for any Lyapunov exponent  $\lambda$  and that  $\Upsilon_k \leq \Upsilon_1^k$  ([Fed69] page 33).

We let  $\mathcal{B}$  be the sheaf  $\mathcal{F}^{k-1}(L_{loc}^\infty)$ . The norm  $\|\cdot\|_\infty$  clearly makes  $\Gamma(\mathcal{B})$  into a Banach space. Given a member  $B \in \Gamma(\mathcal{B})$ , since the operator norm on each  $\bigwedge^k T_x X$  is equal to the norm already defined on  $\bigwedge^k T_x^* X$  for each  $x \in X$  then  $\|B\|_\infty$  is equal to supremum of the  $L_{loc}^\infty$  norm of the function  $x \mapsto B(V_x)$  as  $V$  varies over all  $L_{loc}^\infty$   $k$ -vector fields of norm no more than one.

**Theorem 41.** Given  $f: X \rightarrow X$  an a map in  $\mathcal{S}_X$  for the compact orientable manifold  $X$ , assume that  $c \in H_{deRham}^k(X)$  is a cohomology class (using either real or complex deRham cohomology) which is an eigenvector for  $f^*$  with eigenvalue  $\beta$ . Assume also that  $|\beta| > \Upsilon^{k-1}$ . Then there exists a unique eigencurrent  $C$  with local  $\mathcal{F}^{k-1}(L_{loc}^\infty)$  potentials representing the class  $c$ . Moreover  $C$  has local  $\mathcal{F}^{k-1}(H)$  potentials.

Also, given any neighborhood  $U \subset X$  of any point in the support of  $C$ , then for every lenient current  $C'$  with local  $\mathcal{F}^{k-1}(L_{loc}^\infty)$  potentials and which represents the cohomology class  $c$  then  $f^k(U) \cap \text{Supp } C' \neq \emptyset$  for all large  $k$ .

Assume that the linear map  $f^*: H_{deRham}^k(X) \rightarrow H_{deRham}^k(X)$  is dominated by a single simple real eigenvalue  $r$ . Given  $C'$  any current which has local  $\mathcal{F}^{k-1}(L_{loc}^\infty)$  potentials and which represents a cohomology class in the  $\Upsilon^{k-1}$  chronically expanding subspace of  $H_{deRham}^k(X)$ , then the successive rescaled

pullbacks  $f^{k*}(C')/r^k$  of  $C'$  converge to a multiple of  $C$  in the sense of lenient currents (and thus also in the sense of currents).

*Proof.* We let  $\mathcal{B} = \mathcal{F}^{k-1}(L_{\text{loc}}^\infty)$ ,  $\mathcal{A}$  and  $\mathcal{C}$  be the kernel and image respectively of  $\mathcal{B} \xrightarrow{d} \mathcal{L}^k$ . By Theorem 24,  $H^1(X, \mathcal{A})$  can be canonically identified with  $H^k(X, \mathbb{K})$ . Since  $\mathcal{B}$  is  $\Gamma$ -acyclic then every member of  $H^1(X, \mathcal{A})$  is a closed bundle with respect to the short exact sequence  $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$ .

From Lemma 33 there is an induced  $f$  cohomorphism of the short exact sequence  $\mathcal{A} \xrightarrow{t} \mathcal{B} \xrightarrow{d} \mathcal{C}$ . Also  $\Gamma(\mathcal{C})$  is a space of lenient currents by Lemma 33 and thus has a natural structure as a topological vector space. If a sequence  $\mathbf{B}_i \in \Gamma(\mathcal{B})$  converges to  $\mathbf{B} \in \Gamma(\mathcal{B})$  then  $\langle d\mathbf{B}_i, \varphi \rangle = \int_X B_i \wedge d\varphi = \int_X B \wedge d\varphi = \langle d\mathbf{B}, \varphi \rangle$  so the map  $d: \Gamma(\mathcal{B}) \rightarrow \Gamma(\mathcal{C})$  is continuous.

The cohomomorphism  $\Gamma f_{\mathcal{B}}$  is pullback  $f^*$  of differential forms. Fixing any real  $\alpha$  satisfying  $\Upsilon_{k-1} < \alpha < |\beta|$  it is clear from the definition of  $\Upsilon_{k-1}$  that one can choose a real  $d > 0$  such that  $\|\bigwedge^{k-1} D(f^\ell)\| \leq d \cdot \alpha^\ell$  for all  $\ell \in \mathbb{N}$ . The  $\ell^{\text{th}}$  pullback  $f^{\ell*}(\mathbf{B})$  of  $\mathbf{B} \in \Gamma(\mathcal{B})$  satisfies  $\|f^{\ell*}(\mathbf{B})\|_\infty = \sup_V \|\mathbf{B}(\bigwedge^k D(f^\ell)(V))\|_\infty$  where the supremum is taken over all  $k$ -covector fields  $V$  with  $\|V\|_\infty \leq 1$ . However  $\bigwedge^k D(f^\ell)(V)$  is a  $k$ -covector field of norm no more than  $\|\bigwedge^k D(f^\ell)\|$ , so  $\|f^{\ell*}(\mathbf{B})\|_\infty \leq \|\mathbf{B}\|_\infty \cdot \|\bigwedge^k D(f^\ell)\|_\infty \leq d \cdot \alpha^\ell \|\mathbf{B}\|_\infty$ .

Given any  $W$  in the  $\Upsilon_{k-1}$  chronically expanding subspace of  $H^k(X, \mathbb{K})$ , we can alter our choice of  $\alpha > \Upsilon_{k-1}$  so that  $W$  also lies in the  $\alpha$  chronically expanding subspace of  $H^k(X, \mathbb{K})$ .

We can therefore apply Theorem 12 to conclude that there is a (unique) map  $\kappa: W \rightarrow \Gamma(\mathcal{C})$  such that  $f^*\kappa = \kappa f^*$ , where the first  $f^*$  is pullback of currents and the second is pullback on  $H^k(X, \mathbb{K})$ .

In fact  $\kappa(W)$  lies in the space of currents with locally Hölder potentials (meaning  $\mathcal{F}^{k-1}(H)$  potentials) by applying Corollary 15 in conjunction with Observation 5.2, Lemma 38 and Lemma 39. The second half of the Theorem is a consequence of equation (3).  $\square$

*Remark.* Theorem 41 gives regular degree one eigencurrents for every eigenvalue of  $f^*: H^1(X, \mathbb{K}) \rightarrow H^1(X, \mathbb{K})$  of norm greater one without requiring any constraints on the local behavior of  $f$ . The degree one eigencurrents seem to be, in some sense, more robust than currents of lower dimension, including invariant measures. Moreover since codimension one closed submanifolds are closed currents with local  $\mathcal{F}^0(L_{\text{loc}}^\infty)$  potentials then successive rescaled preimages of such manifolds in the right cohomological class will converge to the eigencurrent.

*Remark.* The fact that eigencurrents constructed via Theorem 41 have local potentials which are forms does not imply their support has positive Lebesgue measure as the classical example of a monotonic nonconstant function which is constant on a set of full measure shows.

*Remark.* The assumption that  $f^*: H_{\text{deRham}}^1(X) \rightarrow H_{\text{deRham}}^1(X)$  is dominated by a single simple real eigenvalue  $r$  is not essential, but just meant to handle the simplest case. In fact the proof actually shows that if  $W$  lies in the  $\Upsilon_{k-1}$  chronically expanding subspace of  $H^k(X, \mathbb{K})$  then every current in the invariant plane  $\kappa(W) \subset \Gamma(\mathcal{C})$  of currents has local  $\mathcal{F}^{k-1}(H)$  potentials and any current with cohomological class in  $W$  with local  $\mathcal{F}^{k-1}(L_{\text{loc}}^\infty)$  potentials is attracted to  $\kappa(W)$  under successive rescaled pullback.

Since measures are of particular interest in dynamics, we note that  $H^1(X, \mathcal{F}^{n-1}(L_{\text{loc}}^\infty)) = H^n(X, \mathbb{K}) = \mathbb{K}$  by Corollary 24 so there is a unique  $f^*$  eigenvalue and it is precisely the topological degree of  $f$ . We thus obtain:

**Corollary 42.** *Given that  $\Upsilon_{n-1} < \deg f$  then there is a unique dimension zero eigencurrent  $\mathbf{C}$  with  $\mathcal{F}^{k-1}(L_{\text{loc}}^\infty)$  potentials (and in fact it has  $\mathcal{F}^{k-1}(H)$  potentials) and the successive rescaled preimages of any  $\mathbf{C}'$  with  $\mathcal{F}^{k-1}(L_{\text{loc}}^\infty)$  potentials converge to  $\mathbf{C}$ . If additionally there is no point  $x \in X$  about which  $f$  is locally an orientation reversing diffeomorphism then  $\mathbf{C}$  (and every other member of  $\kappa(W)$ ) is a positive distribution and is therefore a Radon measure.*

*Proof.* Since  $f^*$  pulls back dimension zero currents (i.e. distributions) which are positive to distributions which are positive then by Corollary 3.1 the distribution  $\mathbf{C}$  is positive. It is therefore a Radon measure (see e.g. [HL99] page 270).  $\square$

*Remark.* In the case where  $f$  is orientation reversing on some parts of  $X$  (but not on all of  $X$ ) some special remarks apply. If it happens that successive rescaled images of some point converge to a dimension zero eigencurrent then since preimages of points are counted with multiplicity then when pulled back through a portion of  $X$  on which  $f$  reverses orientation the sign of a point is flipped. Thus in this case the eigencurrent may not describe so much the distribution of preimages as the *relative density* of preimages counted negatively as compared to those counted positively. The number of actual preimages of a point may grow exponentially faster than the degree of the map in such cases so that dividing by the degree does not yield a measure in the limit unless some such “cancellation” takes place in the limit. One would

expect that the corresponding eigencurrents have local potentials which are not of bounded variation in such a case.

## 5.4 Eigencurrents for Smooth Covering Maps

We will call a covering map which is locally a diffeomorphism a *smooth covering map*. We now consider the special case of smooth self covering maps  $f: X \rightarrow X$  of a compact smooth orientable manifold  $X$ . We show that in this case we have a substantially broader collection of currents whose successive pullbacks converge to an eigencurrent, albeit we need different estimates for Theorem 12 to apply. We will pull back currents by pushing forward forms with  $f_*$ . Since the regular set of  $f$  is all of  $X$  then  $f_*$  is a well defined operator from smooth forms to smooth forms.

**Definition 43.** For a map satisfying the hypothesis of Proposition 34 we define the operation  $f^*$  from currents on  $X$  to currents on  $Y$  by

$$\langle f^*(C), \alpha \rangle \equiv \langle C, f_*(\alpha) \rangle .$$

Clearly  $f^*$  preserves the dimension of a current.

Let  $\mathcal{M}_{k-1}$  be the sheaf for which  $\mathcal{M}_{k-1}(U)$  is the Banach space of bounded linear operations on the topological vector space comprised of  $\mathcal{F}^{k-1}(C^\infty)(U)$  with the  $\|\cdot\|_\infty$  norm. Equivalently,  $\mathcal{M}_{k-1}$  is the sheaf of dimension  $k-1$  currents of finite mass.

Choose a Riemannian metric on  $X$ . If  $f: X \rightarrow X$  is a smooth cover then for each  $x \in X$  and each  $\ell \in \mathbb{N}$ ,  $D_x(f^\ell): T_x X \rightarrow T_{f^\ell(x)} X$  is invertible. We let  $\nu_k(x, \ell)$  be the operator norm of the inverse of  $\bigwedge^k D_x(f^\ell): \bigwedge^k T_x X \rightarrow \bigwedge^k T_{f^\ell(x)} X$ . We define  $\nu_k(\ell) = \sup_{x \in X} \nu_k(x, \ell)^{1/\ell}$ . We define  $\nu_k = \limsup_{\ell \rightarrow \infty} \nu_k(\ell)$ . The iterated pushforward operation  $f_*^\ell: \mathcal{F}^{k-1}(C^\infty)(X) \rightarrow \mathcal{F}^{k-1}(C^\infty)(X)$  satisfies  $\|f_*^\ell(\varphi)\|_\infty \leq \nu_k(\ell) \cdot \|\varphi\|_\infty$  as is straightforward to verify. If  $f$  is invertible then  $\nu_k$  is a bound on the growth of the  $k^{\text{th}}$  wedge product of the derivative under  $f^{-1}$ . For non-invertible  $f$ ,  $\nu_k$  represents a bound on the growth of the  $k^{\text{th}}$  wedge product of the derivative under any sequence of successive branches of  $f^{-1}$ .

**Theorem 44.** Given  $f: X \rightarrow X$  a smooth self covering map and that  $c \in H_{deRham}^k(X)$  is a cohomology class (using either real or complex deRham cohomology) which is an eigenvector for  $f^*$  with eigenvalue  $\beta$ . Assume

also that  $|\beta| > \nu_{k-1}$ . Then there exists a unique eigencurrent  $\mathbf{C}$  with local  $\mathcal{M}_{k-1}$  potentials representing the class  $c$ . Moreover  $\mathbf{C}$  has local  $\mathcal{F}^{k-1}(C^0)$  potentials. Consequently  $\mathbf{C}$  is a current of order one.

Also, given any neighborhood  $U \subset X$  of any point in the support of  $\mathbf{C}$ , then for every lenient current  $\mathbf{C}'$  with local  $\mathcal{M}_{k-1}$  potentials and which represents the cohomology class  $c$  then  $f^k(U) \cap \text{Supp } \mathbf{C}' \neq \emptyset$  for all large  $k$ .

Assume that the linear map  $f^*: H_{deRham}^k(X) \rightarrow H_{deRham}^k(X)$  is dominated by a single simple real eigenvalue  $r$ . Given  $\mathbf{C}'$  any current which has local  $\mathcal{M}_{k-1}$  potentials and which represents a cohomology class in the  $\nu^{k-1}$  chronically expanding subspace of  $H_{deRham}^k(X)$ , then the successive rescaled pullbacks  $f^{k*}(\mathbf{C}')/r^k$  of  $\mathbf{C}'$  converge a multiple of  $\mathbf{C}$ .

*Proof.* We let  $\mathcal{A}$  and  $\mathcal{C}$  be the kernel and image respectively of  $\mathbf{d}: \mathcal{M}_{k-1} \rightarrow \mathcal{C}^k$ . Since  $\mathbf{d}f_* = f_*\mathbf{d}$  then pullback of currents gives an  $f$  cohomomorphism of the short exact sequence of sheaves  $\mathcal{A} \rightarrow \mathcal{M}_{k-1} \rightarrow \mathcal{C}$ .

Since  $\Gamma\mathcal{M}_{k-1}$  is the continuous linear operators on a normed vector space then it is a Banach space. From the observations previous to the statement of Theorem 44 one concludes that for any  $\alpha > \nu_{k-1}$  there is a constant  $d > 0$  such that  $\|f^{\ell*}(\mathbf{B})\| \leq d \cdot \alpha^k \|\mathbf{B}\|$  for all  $\ell \in \mathbb{N}$ .

Since  $\Gamma(\mathcal{C})$  is a space of currents it is naturally a topological vector space over  $\mathbb{K}$ .

The map  $f^*: \Gamma(\mathcal{C}) \rightarrow \Gamma(\mathcal{C})$  is continuous since if  $\mathbf{C}_i \rightarrow \mathbf{C}$  in  $\Gamma(\mathcal{C})$  then  $\langle f^*(\mathbf{C}_i), \varphi \rangle = \langle \mathbf{C}_i, f_*(\varphi) \rangle \rightarrow \langle \mathbf{C}, f_*(\varphi) \rangle = \langle f^*(\mathbf{C}), \varphi \rangle$ .

If  $\mathbf{P}_i \rightarrow \mathbf{P}$  in  $\Gamma\mathcal{M}_{k-1}$  (using the Banach space structure) then  $\|\mathbf{P}_i - \mathbf{P}\| \rightarrow 0$  by assumption then  $\|\mathbf{P}(\mathbf{d}\varphi) - \mathbf{P}_i(\mathbf{d}\varphi)\| \leq \|\mathbf{P} - \mathbf{P}_i\| \cdot \|\mathbf{d}\varphi\| \rightarrow 0$ . Hence  $\langle \mathbf{d}\mathbf{P}_i, \varphi \rangle = \mathbf{P}_i(\mathbf{d}\varphi) \rightarrow \mathbf{P}(\mathbf{d}\varphi) = \langle \mathbf{d}\mathbf{P}, \varphi \rangle$  and so we conclude that the map  $\mathbf{d}: \Gamma\mathcal{M}_{k-1} \rightarrow \Gamma(\mathcal{C})$  is continuous.

Given any  $W$  in the  $\nu_{k-1}$  chronically expanding subspace of  $H^k(X, \mathbb{K})$ , we can alter our choice of  $\alpha > \nu_{k-1}$  so that  $W$  also lies in the  $\alpha$  chronically expanding subspace of  $H^k(X, \mathbb{K})$ .

We can therefore apply Theorem 12 to conclude that there is a (unique) map  $\kappa: W \rightarrow \Gamma(\mathcal{C})$  such that  $f^*\kappa = \kappa f^*$ , where the first  $f^*$  is pullback of currents and the second is pullback on  $H^k(X, \mathbb{K})$ .

In fact  $\kappa(W)$  in the currents with locally continuous potentials by applying Corollary 15 in conjunction with Observation 5.2, Lemma 38 and Lemma 39. The second half of the Theorem is a consequence of equation (3).  $\square$

**Proposition 45.** *Let  $Y$  be an oriented codimension  $k$  submanifold of  $X$ . If the cohomological class of  $Y$  (as a current) lies in the  $\nu_{k-1}$  chronically expanding subspace of  $H^k(X, \mathbb{K})$  then the successive rescaled preimages of  $Y$  converge to the invariant plane of currents  $\kappa(W)$ . If  $f^*: H^k(X, \mathbb{K}) \rightarrow H^k(X, \mathbb{K})$  is dominated by a single real eigenvalue  $r > \nu_{k-1}$  then the successive rescaled preimages of  $Y$  converge to a multiple (possibly zero) of the  $r$  eigencurrent. In particular, if  $\nu_{n-1} < \deg f$  then the successive rescaled preimages of any point converge to the unique invariant measure with  $\mathcal{M}_{n-1}$  potentials.*

*Proof.* This follows immediately from Theorem 44 if we show that  $Y$  has local potentials in  $\mathcal{M}_{k-1}$ . This is equivalent to showing that locally  $Y = dP$  where  $\langle P, \varphi \rangle \leq a \cdot \|\varphi\|_\infty$  for some  $a > 0$ . Let  $B$  be a ball in  $\mathbb{R}^n$  and  $Y_0$  a  $k$ -plane in  $\mathbb{R}^n$ . Then there is a  $k+1$  half plane  $P$  such that, as currents in  $U$ ,  $\partial P = Y_0$ . Moreover it is clear that  $\langle P, \varphi \rangle \leq a \|\varphi\|_\infty$  for some real  $a > 0$ . (There are also local potentials for  $Y$  which are given by forms with  $L^1_{\text{loc}}$  coefficients. These can be constructed by choosing a projection  $\pi$  from  $U \setminus Y_0$  to a codimension one cylinder  $C$  with axis  $Y_0$ , and choosing a volume form  $\sigma$  on  $C$ . The local potential is the pullback  $\pi^*(\sigma)$ .)  $\square$

*Remark.* As with Theorem 41, Theorem 44 gives regular degree one eigencurrents for every eigenvalue of  $f^*: H^1(X, \mathbb{K}) \rightarrow H^1(X, \mathbb{K})$  of norm greater one without requiring any constraints on the local behavior of  $f$ . In holomorphic dynamics much progress has been made in constructing degree one eigencurrents and then constructing dynamically important invariant measures via a generalized wedge product (see the references cited at the beginning of Section 6).

*Remark.* The proof of Proposition 45 could clearly be modified to apply to many singular manifolds as well.

## 6 Holomorphic Endomorphisms

We now restrict our interest to holomorphic dynamics. Thus all manifolds are assumed to be complex manifolds and all maps are assumed to be holomorphic unless stated otherwise.

Holomorphic endomorphisms of the Riemann sphere have been studied in great detail. For endomorphisms much of the theory is still in its beginnings. Much attention has been paid to holomorphic automorphisms of  $\mathbb{C}^2$  [FM89], [FS92], [HOV94], [HOV95], [BS91a], [BS91b], [BS92], [BLS93],

[BS98a], [BS98b], [BS99] or K3 surfaces [Can01], [McM02], the major developments for endomorphisms have been on  $\mathbb{P}^n$ , [FS94a], [FS94b], [FS95b], [FS01], [FS95a], [JW00], [FJ03], [Ued94], [Ued98], [Ued97]. Recent significant developments have been made for endomorphisms of Kahler manifolds in [DS05]. The paper [DS05] shows existence of eigencurrents (or Green's currents) for endomorphisms of Kahler manifolds under a simple condition on the comparative rates of growth of volume in two different dimensions. They also show that a specific weighted sum of an arbitrary closed positive smooth current will converge to the Green's current, and that the Green's current has a Hölder continuous potential. In this setting our theorem shows that arbitrary (rescaled) preimages of a broader class of currents will converge to the Green's current. A wide variety of results have been proven in these various circumstances either showing the existence of invariant currents, showing convergence of currents to invariant currents, or studying the properties of these invariant currents. We include here results that follow from the method of this paper, which we are sure substantially overlap with existing results. Presumably our cohomologically lifting theorem could be used in conjunction with Theorem 12 to show existence of higher degree  $(k, k)$  currents given certain bounds on local growth rates.

## 6.1 $dd^c$ Cohomology

Let  $Z$  be a complex manifold and let  $f: Z \rightarrow Z$  be a holomorphic self map of  $Z$ . Let  $\mathcal{H}$  be the sheaf of pluriharmonic functions, let  $L_{\text{loc}}^\infty$  be the sheaf of locally bounded functions, and let  $\mathcal{C}$  be the sheaf of currents with local potentials in  $L_{\text{loc}}^\infty$ , i.e. currents locally of the form  $dd^c b$ , for  $b$  a locally bounded function. The members of  $\mathcal{C}$  are closed  $(1, 1)$  currents on  $Z$ .

Using the usual pullback on functions, and the induced pullback on currents with function potentials (i.e. pullback the current by pulling back its local potentials), then we get a self cohomomorphism of the exact sequence of sheaves

$$\mathcal{H} \rightarrow L_{\text{loc}}^\infty \xrightarrow{dd^c} \mathcal{C}. \quad (11)$$

We note that  $H^1(Z, \mathcal{H})$  is a finite dimensional  $\mathbb{R}$  vector space as can be seen from the long exact sequence for the short exact sequence  $\mathbb{R} \rightarrow \mathcal{O} \rightarrow \mathcal{H}$  where the first map is inclusion and the second takes the imaginary part. The terms  $H^1(Z, \mathcal{O}) \rightarrow H^1(Z, \mathcal{H}) \rightarrow H^2(Z, \mathbb{R})$  give the finite dimensionality since  $\mathcal{O}$  is a coherent analytic sheaf (see e.g. [Tay02] page 302).

Then from Theorem 12 we obtain:

**Corollary 46.** *Given  $v$  any closed eigenbundle of  $H^1(Z, \mathcal{H})$  for  $f^*$  with eigenvalue  $r > 1$ , there is a unique closed  $(1, 1)$  current  $C$  such that  $\lim_{k \rightarrow \infty} f^{k*}(C')/r^k$  converges to  $C$  for any divisor  $C'$  of  $v$ .*

*Remark.* We note that the terms “closed eigenbundle” and “divisor” in Corollary 46 are understood using the long exact sequence for (11).

We can apply Corollary 15 to show that

**Corollary 47.** *Any such invariant current  $C$  so obtained has Hölder continuous local potentials.*

*Proof.* The result follows from Lemma 5.2, Lemma 38, the fact that the  $\text{dd}^c$  closed Hölder continuous functions are the same as the  $\text{dd}^c$  closed  $L_{\text{loc}}^\infty$  functions and from Corollary 15.  $\square$

Also from Observation 3.1,

**Corollary 48.** *If  $v$  has a plurisubharmonic section the current  $C$  is positive.*

## 7 Result via Invariant Sections

We stated early on that our construction of invariant members of  $H^0(\mathcal{C})$  for a self cohomomorphism of a short exact sequence  $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$  of sheaves could be done in terms of finding invariant sections of bundles. We illustrate this here in a specific case where we can take advantage of geometry to make further conclusions. Finding an invariant section of a bundle is equivalent to finding an invariant trivialization of the bundle, and we will make our initial statement in terms of a trivialization.

Let  $Z$  be a compact complex manifold. Let  $f: Z \rightarrow Z$  be a holomorphic endomorphism. Let  $p \in H^1(Z, \mathcal{H})$  be an eigenvector for  $f^*$  with real eigenvalue  $\lambda$  of norm greater than one. If  $f^*$  were to have complex eigenvalues of interest, an analogous construction can be made to the one that follows.

We note that there is a canonical bundle map  $\tilde{f}: f^*(p) \rightarrow p$  which gives the map  $f$  on the base space. It is easy to show that there is a map  $\sigma: p \rightarrow \lambda p$  which is the identity on the base space and takes the form  $r \mapsto \lambda r + b$  on the fibers, where  $b$  is a constant. What is more, the map  $\tau_\lambda$  is easily seen to be



unique up to the addition of a constant. Then define the map  $\check{f}: p \rightarrow p$  to be the composition of

$$p \xrightarrow{\tau_\lambda} \lambda p = f^*(p) \xrightarrow{\check{f}} p.$$

Then  $\check{f}$  is the map  $f$  on the base space and takes the form  $r \mapsto \lambda r + b$  on the fibers.

Since every pluriharmonic bundle is trivial as a smooth bundle, then we can choose a smooth trivialization  $t: p \rightarrow \mathbb{R}$ , i.e.  $t(a + r) = \sigma(a) + r$  for any  $a \in p$ ,  $r \in \mathbb{R}$ , where  $a + r$  is computed in the fiber containing  $a$ .

**Theorem 49.** *There is a unique continuous trivialization  $\mathbf{g}: p \rightarrow \mathbb{R}$  such that:*

$$\mathbf{g}(a + r) = \mathbf{g}(a) + r \quad \text{for } a \in p \text{ and } r \in \mathbb{R},$$

$$\mathbf{g}(\check{f}(a)) = \lambda \cdot \mathbf{g}(a) \quad \text{for } a \in p,$$

moreover

$$\mathbf{g} = \lim_{k \rightarrow \infty} \lambda^{-k} \circ t \circ \check{f}^{\circ k}$$

and the limit converges uniformly. Finally, the zero set of  $\mathbf{g}$  is the image of a section  $g: Z \rightarrow p$  and is exactly the set of points whose forward image under  $\check{f}$  remains bounded.

*Proof.* Define a function  $T: p \rightarrow \mathbb{R}$  by

$$T(a) \equiv t(\check{f}(a)) - \lambda \cdot t(a).$$

Note that  $T$  descends to a well defined continuous function  $T: Z \rightarrow \mathbb{R}$  since for an arbitrary  $r \in \mathbb{R}$  one has  $T(a + r) = t(\check{f}(a + r)) - \lambda \cdot t(a + r) = t(\check{f}(a) + \lambda r) - \lambda \cdot (t(a) + r) = T(a)$ .

One notes that since the function  $T$  is necessarily bounded if  $Z$  is compact then defining

$$\mathbf{g}(a) \equiv t(a) + \lambda^{-1} \cdot T(a) + \lambda^{-2} T(\check{f}(a)) + \lambda^{-3} T(\check{f}^{\circ 2}(a)) + \dots$$

gives a continuous function  $\mathbf{g}: p \rightarrow \mathbb{R}$  satisfying the above two properties.

Assume  $\mathbf{g}_1$  and  $\mathbf{g}_2$  are two such functions. Then  $\Delta \equiv \mathbf{g}_1 - \mathbf{g}_2: p \rightarrow \mathbb{R}$  is a function satisfying  $\Delta(a + r) = \Delta(a)$  for  $a \in p$  and  $r \in \mathbb{R}$  so  $\Delta$  descends to a continuous function  $\Delta: p \rightarrow \mathbb{R}$  satisfying  $\Delta(\check{f}(a)) = \lambda \cdot \Delta(a)$ . However since  $\lambda > 1$  one concludes that this is only possible if  $\Delta \equiv 0$  since  $M$  is compact so  $\Delta(M)$  has compact image in  $\mathbb{R}$ .

It is easy to check using the definition of  $T$  that  $\lambda^{-k} \circ t \circ \check{f}^{\circ k}(a)$  is exactly a partial sum of the first  $k$  terms of the above series and this gives the convergence result. The conclusion about the section  $g$  is trivial.  $\square$

The above construction can be carried through almost without modification for any subspace of  $H^1(Z, \mathcal{H})$  on which  $f^*$  is expanding. This gives an alternate way of understanding the convergence of preimages of sections. The point is that if  $s$  is any section of  $p$ , i.e. the potential of a current  $C$ , then  $\frac{1}{\lambda} f^*(C)$  is a current with potential which is the setwise preimage of  $s$  under  $\check{f}$  (this is easy to confirm from the construction of  $f$ ). The Green's trivialization  $\mathfrak{g}$  shows that  $\check{f}$  is uniformly repelling away from the image of the invariant section  $g$ . Thus as long as  $s$  is bounded in  $p$ , (not even necessarily continuous), then the successive preimages of  $s$  will converge uniformly to the section  $g$ . Since uniform convergence of potentials implies convergence of currents then the rescaled pullbacks of a current  $C$  converge to the current with potential  $g$ . We already have this as a theorem, so we have not restated it as such here. This is just an alternative approach. Note that in the case where  $Z = \mathbb{P}^2$  [FJ03] has given far more precise control of when the successive rescaled preimages of a current will converge to the eigencurrent.

## 7.1 Sections version with an Invariant Ample Bundle

It is also interesting to consider the special case where there is an invariant ample bundle with eigenvalue  $\lambda \geq 2$  an integer. Without loss of generality we assume  $\ell$  is very ample. The morphism of sheaves  $\log |\cdot| : \mathcal{O}^* \rightarrow \mathcal{H}$  induces a map from holomorphic line bundles to pluriharmonic bundles. We let  $p = \log |\ell|$  be the corresponding pluriharmonic bundle.

It is easy to see that there is a holomorphic map  $\ell \rightarrow \ell^\lambda$  which is of the form  $\sigma_\lambda : z \mapsto az^\lambda$ ,  $a \in \mathbb{C}^*$  on each fiber and is the identity on the base space. There is also a canonical holomorphic map  $\tilde{f} : f^*(\ell) \rightarrow \ell$  which is a line bundle map and is  $f$  on the base space.

One then defines the holomorphic map  $\check{f} : \ell \rightarrow \ell$  which is the composition of

$$\ell \xrightarrow{\sigma_k} \ell^k = f^*(\ell) \xrightarrow{\tilde{f}} \ell.$$

This map is of the form  $z \mapsto az^k$  on each fiber and is equal to the map  $f : Z \rightarrow Z$  on the base space. Let  $\ell^*$  denote  $\ell$  with its zero section removed, so that  $\log |\cdot| : \ell \rightarrow p$  is a well defined continuous map. Since the preimage

of the zero section of  $\ell$  under  $\check{f}$  is the zero section then  $\check{f}$  is a holomorphic self map of  $\ell^*$ . It is easy to confirm that  $\check{f}: \ell \rightarrow \ell$  can be rescaled so that the diagram

$$\begin{array}{ccc} \ell & \xrightarrow{\quad} & \ell \\ \downarrow \log |\cdot| & \check{f} & \downarrow \log |\cdot| \\ p & \xrightarrow{\quad} & p \\ & \check{f} & \end{array}$$

commutes.

Our Greens trivialization  $\mathbf{g}: p \rightarrow \mathbb{R}$  can be pulled back to give a Green's function  $\mathcal{G}: \ell^* \rightarrow \mathbb{R}$  on the punctured bundle  $\ell^*$ . It satisfies  $\mathcal{G}(\check{f}(w)) = \lambda \cdot \mathcal{G}(w)$  and  $\mathcal{G}(\beta w) = \mathcal{G}(w) + \log |\beta|$  for  $w \in \ell$  and  $\beta \in \mathbb{C}^*$ . Since  $\mathbf{g}$  is a trivialization of an  $\mathbb{R}$  bundle over a compact space,  $\mathbf{g}$  is proper. Since  $\log |\cdot|: \ell^* \rightarrow p$  is proper then  $\mathcal{G}$  is proper. Thus, in this setting one can construct a Greens function that is exactly analogous to the Green's function constructed on  $\mathbb{C}^{n+1}$  for a holomorphic endomorphism of  $\mathbb{P}^n$ . Potentially one could take advantage of the special geometry of very ample bundles to get information about the dynamics in this situation.

## 8 Bibliography

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