# Long Runs Imply Big Separators in Vector Addition Systems

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#### - Abstract -

Despite a very recent progress which settled the complexity of the reachability problem for Vector Addition Systems with States (VASSes) to be Ackermann-complete we still lack of lot of understanding for that problem. A striking example is the reachability problem for three-dimensional VASSes (3-VASSes): it is only known to be PSpace-hard and not known to be elementary. One possible approach which turned out to be successful for many VASS subclasses is to prove that to check reachability it suffices to inspect only runs of some bounded length. This approach however has its limitations, it is usually hard to design an algorithm substantially faster than the possible size of finite reachability sets in that VASS subclass. In 2010 Leroux has proven that non-reachability between two configurations implies separability of the source from target by some semilinear set, which is an inductive invariant. There can be a reasonable hope that it suffices to look for separators of bounded size, which would deliver an efficient algorithm for VASS reachability. In the paper we show that in VASSes fulfilling certain conditions existence of only long runs between some configurations implies existence of only big separators between some other configurations (and in a slightly modified VASS). Additionally we prove that a few known examples of hard VASSes fulfil the mentioned conditions. Therefore improving the complexity of the reachability problem (for any subclass) using the separators approach can meet serious obstacles.

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# 1 Introduction

The complexity of the reachability problem for Vector Addition Systems (VASes) was a challenging and natural problem for a few decades of research. First result about its complexity was ExpSpace-hardness by Lipton in 1976 [15]. Decidability was shown by Mayr in 1981 [16]. Later the construction of Mayr was further simplified and presented in a bit different light by Kosaraju [9] and Lambert [10] and currently is known under the name KLM decomposition after the three main inventors. First complexity upper bound was obtained by Leroux and Schmitz as recent as in 2016 [13], where the reachability problem was shown to be solvable in cubic-Ackermann time. A few years later the same authors have shown that the problem can be solved in Ackermann time and actually in primitive recursive time, when the dimension is fixed [14]. In the same time a lower bound of Tower-hardness was established for the reachability problem [4]. This year, very recently, two independent papers have shown Ackermann-hardness of the reachability problem [12, 6] thus settling the complexity of the problem to be Ackermann-complete.

However, despite this recent huge progress we still lack of lot of understanding about the reachability problem in VASSes. The most striking example is the problem for dimension three. The best known lower bound for the reachability problem for 3-VASSes is PSpace-

hardness inherited from the hardness result for 2-VASSes [1], while the best known upper bound is much bigger then Tower. Concretely speaking the problem can be solved in time  $\mathcal{F}_7$  [14], where  $\mathcal{F}_\alpha$  is the hierarchy of fast-growing complexity classes, see [18] (recall that Tower =  $\mathcal{F}_3$ ). Similarly for other low dimensional VASSes the situation is unclear. Currently the smallest dimension d for which an ExpSpace-hardness was published for d-VASSes is d=14 [4]. Therefore for any  $d\in[3,13]$  the problem can be in PSpace and is not known to be elementary. For some of those dimensions one can expect to get ExpSpace-hardness, but we conjecture that for 3-VASSes and 4-VASSes the reachability problem actually is elementary and the challenge is to find this algorithm. This complexity gap is just one witness of our lack of understanding of the structure of VASSes. One can hope that in the future we will be able to design efficient algorithms for the reachability problem even for VASSes in high dimensions under the condition that they belong to some subclass, for example they avoid some hard patterns. This can be quite important from a practical point of view. Understanding for which classes efficient (say PSpace) algorithms are possible may be thus another future goal. Therefore we think that the quest for better understanding the reachability problem is still valid, even though the complexity of the reachability problem for general VASSes is settled.

A common and very often successful approach to the reachability problems is proving a short run property, namely showing that in order to decide reachability it suffices to inspect only runs of some bounded length. This technique was exploited a lot in the area of VASSes. Rackoff proved his ExpSpace upper bound on the complexity of coverability problem [17] using this approach: he has shown that if there exist a covering run then there exist also a covering run of at most doubly-exponential length. Recently there was a lot of research about low dimensional VASSes and the technique of bounded run length was also used there. In [1] authors established complexity of the reachability problem for 2-VASSes to PSpace-completeness by proving that in order to decide reachability it suffices to inspect runs of at most exponential length. Similarly in the next paper in this line of research [7] it was shown that in 2-VASSes with unary transition representation it suffices to consider runs of polynomial length, which proves NL-completeness of the reachability problem.

This approach however meets a subtle obstacle when one tries to prove some upper bound on the complexity of the reachability problem in say d-VASSes. In order to show a bounded length property it is natural to try to unpump long runs in some way. Unpumping however can be very tricky when the run is close to some of the axes, as only a small modification of the run may cause some counters to become negative. A common approach to that problem is to modify a run when all its counters are high and therefore local run modifications cannot cause any problem. Such an approach is used for example in 2-VASSes [1] and in the KLM decomposition [9, 10, 16]. However not all the runs may have a configuration with all counter values high. Therefore it is very convenient to have a cycle, which increases all the counters simultaneously. Observe that existence of such a cycle implies that the set of reachable configurations is infinite. Thus using this approach is hard in the case when the reachability set is finite. For finite set in turn it is hard to design algorithm substantially faster than the size of the reachability set. It is well known that in d-VASSes finite reachability sets can be as large as  $F_{d-1}(n)$ , where n is the VASS size and  $F_{\alpha}$  is the hierarchy of very fast growing functions, see [18]. Therefore designing an algorithm breaking  $F_{d-1}(n)$  time for the reachability problem in d-VASSes (if such exist) may need some other approach. Breaking the barrier of the size of finite reachability set is possible in general, but probably usually very challenging. To the best of our knowledge the only nontrivial algorithm breaking it is the one in [7] doing a sophisticated analysis of possible behaviours of runs in 2-VASSes. This motivates a search for other techniques, which may be more suitable for designing fast

algorithms.

Leroux in his work [11] provided an algorithm for the reachability problem, which follows a completely different direction. He has shown that if there is no run from a configuration  $s \in \mathbb{N}^d$  to a configuration  $t \in \mathbb{N}^d$  in a VAS V then there exist a semilinear set S, called a separator, which (1) contains s, (2) does not contain t and (3) is an inductive invariant, namely if  $v \in S$  then also  $v + t \in S$  for any transition t of V such that  $v + t \in \mathbb{N}^d$ . Notice that existence of a separator clearly implies non-reachability between s and t, so by Leroux's work non-reachability and existence of separator are equivalent. Then the following simple algorithm decides whether t is reachable from s: run two semi-procedures, one looks for possible runs between s and t, longer and longer, another one looks for possible separators between s and t, bigger and bigger. Clearly either there exist a run or there exist a separator, so at some point algorithm will find it and terminate. From this perspective one can view runs and separators as a dual objects. If bounding the length of a run is nontrivial, maybe bounding the size of a separator can be another promising approach. Notice that providing for example an  $F_d(n)$  upper bound on the size of separators would provide an algorithm solving reachability problem and working in time around  $F_d(n)$ .

**Our contribution** Our main contribution are two theorems stating that in VASSes fulfilling certain conditions if there are only long runs between some its two configurations then in a small modification of this VASS there are only big separators for some other two configurations. We designed Theorem 4 to have relatively simple statement, but also to be sufficiently strong for our applications. Theorem 9 needs more sophisticated notions and more advanced tools to be proven, but it has potentially a broader spectrum of applications.

Additionally we have shown that two nontrivial constructions of hard VASSes, namely the 4-VASS from [5] and VASS used in the Tower-hardness construction from [4] fulfil the conditions proposed by us in Theorem 4. This indicates that for each VASS subclass  $\mathcal{F}$  (for example 3-VASSes) either (1) in order to prove better upper complexity bound for the reachability problem in  $\mathcal{F}$  one should focus more on proving short run property than small separator property or (2) there is a possibility of finding a VASS in  $\mathcal{F}$  with very long shortest runs and much smaller separators. However, in the latter case the mentioned VASS has to be constructed by the use of rather a different techniques than currently known, as it needs to violate conditions of Theorems 4 and 9.

We have not considered VASSes occurring in the most recent papers proving the Ackermann-hardness [12, 6], but it seems to us that these techniques are promising at least with respect to VASSes occurring in [6].

**Organisation of the paper** In Section 2 we introduce necessary notions and recall standard facts. Then in Section 3 we state and prove our two main results, Theorems 4 and 9. In Section 4 we provide two applications of Theorem 4, in Section 4.1 to the 4-VASS described in [5] and in Section 4.2 to VASSes occurring in the paper [4] proving the Tower-hardness of the reachability problem.

### 2 Preliminaries

**Basic notions** We denote by  $\mathbb{N}$  the set of nonnegative integers and by  $\mathbb{N}_+$  the set of positive integers. For  $a,b\in\mathbb{N}$  by [a,b] we denote the set  $\{a,a+1,\ldots,b-1,b\}$ . For a set S we write |S| to denote its size, i.e. the number of its elements. For two sets A,B we define  $A+B=\{a+b\mid a\in A,b\in A\}$  and  $AB=\{a\cdot b\mid a\in A,b\in B\}$ . In that context we often

simplify the notation and write x instead of the singleton set  $\{x\}$ , for example  $x + \mathbb{N}y$  denotes the set  $\{x\} + \mathbb{N}\{y\}$ . The description size of an irreducible fraction  $\frac{p}{q}$  is  $\max(|p|, |q|)$ , for  $r \in \mathbb{Q}$  its description size is the description size of its irreducible form.

For a d-dimensional vector  $x=(x_1,\ldots,x_d)\in\mathbb{R}^d$  and index  $i\in[1,d]$  we write x[i] for  $x_i$ . For  $S\subseteq[1,d]$  we write  $\operatorname{proj}_S(x)$  to denote the |S|-dimensional vector obtained from x by removing all the coordinates outside S. The norm of a vector  $x\in\mathbb{N}^d$  is  $\operatorname{norm}(x)=\max_{i\in[1,d]}|x[i]|$ . For  $i\in[1,d]$  the elementary vector  $e_i$  is the unique vector such that  $e_i[j]=0$  for  $j\neq i$  and  $e_i[i]=1$ . By  $0^d\in\mathbb{N}^d$  we denote the d-dimensional vector with all the coordinates being zero.

**Vector Addition Systems** A d-dimensional Vector Addition System with States (shortly d-VASS or just a VASS) consists of finite set of states Q and finite set of transitions  $T \subseteq Q \times \mathbb{Z}^d \times Q$ . Configuration of a d-VASS V = (Q,T) is a pair  $(q,v) \in Q \times \mathbb{N}^d$ , we often write q(v) instead of (q, v). For a configuration c = q(v) we write state (c) = q. For a set of vectors  $S \subseteq \mathbb{N}^d$  and state  $q \in Q$  we write  $q(S) = \{q(v) \mid v \in S\}$ . Transition t = (p, u, q) can be fired in configuration  $(r, v) \in Q \times \mathbb{N}^d$  if p = r and  $u + v \in \mathbb{N}^d$ . Then we write  $p(v) \xrightarrow{t} q(u + v)$ . The triple (p(u), t, q(u+v)) is called an anchored transition. A run is a sequence of anchored transitions  $\rho = (c_1, t_1, c_2), \dots, (c_n, t_n, c_{n+1})$ . Such a run  $\rho$  is then a run from configuration  $c_1$  to configuration  $c_{n+1}$  and traverses through configurations  $c_i$  for  $i \in [2, n]$ . We also say that  $\rho$  is from state $(c_1)$  to state $(c_{n+1})$ . If there is a run from configuration c to configuration c' we also say that c' is reachable from c or c reaches c' and write  $c \longrightarrow c'$ . Otherwise we write  $c \longrightarrow c'$ . The configuration c is the source of  $\rho$  while configuration c' is the target of  $\rho$ . For a configuration  $c \in Q \times \mathbb{N}^d$  we denote  $POST_V(c) = \{c' \mid c \longrightarrow c'\}$  the set of all the configurations reachable from c and  $PRE_V(c) = \{c' \mid c' \longrightarrow c\}$  the set of all the the configurations which reach c. The reachability problem for VASSes given a VASS V and two its configurations s and t asks whether s reaches t in V. For a VASS V by its size we denote the total number of bits needed to represent its states and transitions. A VASS is said to be binary if numbers in its transitions are encoded in binary. Effect of a transition  $(c,t,c') \in Q \times \mathbb{Z}^d \times Q$  is the vector  $t \in \mathbb{N}^d$ . We extend this notion naturally to anchored transitions and runs, effect of the run  $\rho = (c_1, t_1, c_2), \dots, (c_n, t_n, c_{n+1})$  is equal to  $t_1 + \dots + t_n$ . Vector Addition Systems (VASes) are VASSes with just one state, which therefore can be ignored. It is well known and simple to show that the reachability problems for VASes and for VASSes are polynomially interreducible. In this work we focus wlog. on the reachability problem for VASSes.

**Semilinear sets** For any vectors  $b, v_1, \ldots, v_k \in \mathbb{N}^d$  the set  $L = b + \mathbb{N}v_1 + \ldots + \mathbb{N}v_k$  is called a *linear* set. Then vector b is the *base* of L and vectors  $p_1, \ldots, p_k$  are *periods* of L. Set of vectors is *semilinear* if it is a finite union of linear sets. Set of VASS configurations  $S \subseteq Q \times \mathbb{N}^d$  is *semilinear* if it is a finite union of sets of the form  $q_i(S_i) \subseteq Q \times \mathbb{N}^d$ , where  $S_i$  are semilinear as sets of vectors. The *size* of a representation of a linear set is the sum of norms of its base and periods. The *size* of a representation of a semilinear set  $\bigcup_i L_i$  is the sum of sizes of representations of the sets  $L_i$ . The size of a semilinear set is the size of its smallest representation.

For a d-VASS V = (Q, T) and two its configurations  $s, t \in Q \times \mathbb{N}^d$  a set  $S \subseteq Q \times \mathbb{N}^d$  of configurations is a separator for (V, s, t) if it fulfils the following conditions: 1)  $s \in S$ , 2)  $t \notin S$ , 3) S is invariant under transitions of V, namely for any  $c \in S$  such that  $c \xrightarrow{t} c'$  for some  $t \in T$  we also have  $c' \in S$ . In our work we usually do not exploit by condition 3) by itself, but use the facts which are implied by all the conditions 1), 2) and 3) together:

 $POST(s) \subseteq S$  and  $PRE(t) \cap S = \emptyset$ .

**Well quasi-order on runs** We say that an order  $(X, \preceq)$  is a well-quasi order (wqo) if in every infinite sequence  $x_1, x_2, \ldots$  of elements of X there is a *domination*, i.e. there exist i < j such that  $x_i \preceq x_j$ .

Fix a d-VASS V = (Q, T). We define here a very useful order on runs, which turns out to be a wqo (a weaker version was originally introduced in [8]). For two configurations  $p(u), q(v) \in Q \times \mathbb{N}^d$  we write  $p(u) \leq q(v)$  if p = q and  $u[i] \leq v[i]$  for each  $i \in [1, d]$ . For two anchored transitions in  $(c_1, t, c_2), (c'_1, t', c'_2)$  we write  $(c_1, t, c_2) \leq (c'_1, t', c'_2)$  if  $t = t', c_1 \leq c'_1$  and  $c_2 \leq c'_2$  (notice that the last condition is actually implied by the previous two). For two runs  $\rho = m_1 \dots m_k$  and  $\rho' = m'_1 \dots m'_\ell$ , where all  $m_i$  for  $i \in [1, k]$  and  $m'_i$  for  $i \in [1, \ell]$  are anchored transitions we write  $\rho \leq \rho'$  if there exists a sequence of indices  $i_1 < i_2 < \dots < i_{k-1} < i_k = \ell$  such that for each  $j \in [1, k]$  we have  $m_j \leq m'_{i_j}$ . Notice there that setting  $i_k = \ell$  implies that  $\operatorname{trg}(\rho) \leq \operatorname{trg}(\rho')$ . All the subsequent considerations can be analogously applied in the case when  $i_1 = 1$ , but  $i_k$  does not necessarily equal k, which enforces that  $\operatorname{src}(\rho) \leq \operatorname{src}(\rho')$ . The following claim is a folklore, for a proof see Proposition 19 in [3].

 $\triangleright$  Claim 1. Order  $\triangleleft$  is a wgo on runs.

The order  $\leq$  has a nice property that runs bigger than a fixed one are additive in a certain sense. The following claim is also a folklore, for a proof see Lemma 23 in [2] (arXiv version of [3]).

ightharpoonup Claim 2. Let ho,  $ho_1$  and  $ho_2$  be runs of VASS V such that  $ho leq 
ho_1$ ,  $ho_2$ ,  $\operatorname{trg}(
ho_1) = \operatorname{trg}(
ho) + \delta_1$  and  $\operatorname{trg}(
ho_2) = \operatorname{trg}(
ho) + \delta_2$  for some  $\delta_1, \delta_2 \in \mathbb{N}^d$ . Then there exist a run ho' such that ho leq 
ho' and  $\operatorname{trg}(
ho') = \operatorname{trg}(
ho) + \delta_1 + \delta_2$ .

The following corollary can be easily shown by induction on n.

▶ Corollary 3. Let  $\rho \leq \rho'$  be runs of VASS V such that  $\operatorname{trg}(\rho') = \operatorname{trg}(\rho) + \delta$  for some  $\delta \in \mathbb{N}^d$ . Then for any  $n \in \mathbb{N}$  there exists a run  $\rho_n$  of V such that  $\operatorname{trg}(\rho_n) = \operatorname{trg}(\rho) + n\delta$ .

The above notions will be useful in the proof of Theorem 9.

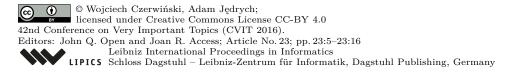
## 3 Separators

In this section we state and prove two main results. The first one is simpler both in the formulation and in the proof and sufficient to show applications in Section 4. The aim of the second one is to show a potentially more general scope of the applications and to formulate assumptions in a bit more canonical way.

## 3.1 Applied theorem

- ▶ Theorem 4. Let V be a d-VASS,  $s, t \in Q \times \mathbb{N}^d$  be two its configurations,  $q \in Q$  be a state and a line  $\alpha = a + \mathbb{N}\Delta$  for vectors  $a, \Delta \in \mathbb{N}^d$  be such that
- (1)  $\Delta = (\Delta[1], \Delta[2], 0^{d-2}) \in \mathbb{N}^d$ ,
- (2) there is no run from s to t,
- (3)  $q(\alpha) \subseteq POST_V(s)$ ,
- (4)  $q(\alpha + \mathbb{N}_+ e_2) \subseteq PRE_V(t)$ .

Then each separator for (V, s, t) contains a period  $r \cdot \Delta$  for some  $r \in \mathbb{Q}$ .



**Proof.** Consider an arbitrary separator S for (V, s, t). Let  $S_q = \{v \mid q(v) \in S, v \in \mathbb{N}^d\}$  be its part devoted to the state q and let  $S_q = \bigcup_{i \in I} L_i$ , where  $L_i$  are linear sets. As the line  $\alpha$  contains infinitely many points and  $\alpha \subseteq S_q$  then some of the linear sets  $L_i$  has to contain infinitely many points of  $\alpha$ . Denote this  $L_i$  by L, let  $L = b + \mathbb{N}p_1 + \ldots + \mathbb{N}p_k$ . We know that for arbitrarily big n we have  $a + n\Delta = b + n_1p_1 + \ldots + n_kp_k$  for some  $n_i \in \mathbb{N}$ . Whog. we can assume that  $n_i > 0$ , we just do not write the periods  $p_i$  with coefficient  $n_i = 0$ . Notice first that for each coordinate  $j \in [3,d]$  we have  $(a+n\Delta)[j] \leq \operatorname{norm}(a)$ . Therefore for  $p_i$  such that  $p_i[j] > 0$  for some  $j \in [3,d]$  we have  $n_i \leq \operatorname{norm}(a)$ . Let  $P_0$  be the set of periods  $p_i$  such that  $p_i[j] = 0$  for all  $j \in [3,d]$  and  $P_{\neq 0}$  be the set of the other periods  $p_i$ . We thus have

$$(a-b-\sum_{p_i\in P_{\neq 0}}n_ip_i)+n\Delta=\sum_{p_i\in P_0}n_ip_i,$$

where the sum  $v = (a - b - \sum_{p_i \in P_{\neq 0}} n_i p_i)$  has a norm bounded by  $B = \text{norm}(a) + \text{norm}(b) + k \cdot \text{norm}(a) \cdot \max_{p_i \in P_{\neq 0}} \text{norm}(p_i)$ , which is independent of n. If we restrict the equation to the first two coordinates of the considered vectors we have

$$(v[1], v[2]) + n(\Delta[1], \Delta[2]) = \sum_{p_i \in P_0} n_i(p_i[1], p_i[2]).$$
(1)

We aim at showing that one of  $p_i$  is equal to  $r \cdot \Delta$  for some  $r \in \mathbb{Q}$ . Recall that for each  $j \in [3,d]$  we have  $\Delta[j] = p_i[j] = 0$ , so it is enough to show that  $(p_i[1], p_i[2]) = r \cdot (\Delta[1], \Delta[2])$ . We first show the following claim.

 $\triangleright$  Claim 5. For each period  $p \in P_0$  we have  $\Delta[1] \cdot p[2] \leq \Delta[2] \cdot p[1]$ .

**Proof.** The intuitive meaning of  $\Delta[1] \cdot p[2] \leq \Delta[2] \cdot p[1]$  is that  $\frac{\Delta[1]}{\Delta[2]} \leq \frac{p[1]}{p[2]}$ , however we cannot write the fraction  $\frac{p[1]}{p[2]}$  as it might happen that p[2] = 0. Assume towards a contradiction that for some  $p \in P_0$  we have  $\Delta[1] \cdot p[2] > \Delta[2] \cdot p[1]$ . Let  $\delta = \Delta[1] \cdot p[2] - \Delta[2] \cdot p[1] > 0$ . Recall that  $b + n\Delta \in L$  for some  $n \in \mathbb{N}$ , therefore also  $b + n\Delta + \Delta[1] \cdot p \in L$  as  $p \in P_0$  is a period of L. Then however

$$\Delta[1] \cdot p = \Delta[1] \cdot (p[1], p[2], 0^{d-2}) = (\Delta[1] \cdot p[1], \Delta[1] \cdot p[2], 0^{d-2})$$
$$= (\Delta[1] \cdot p[1], \Delta[2] \cdot p[1] + \delta, 0^{d-2}) = p[1] \cdot \Delta + \delta \cdot e_2 \in \mathbb{N}\Delta + \mathbb{N}_+ e_2.$$

Therefore  $b+n\Delta+\Delta[1]\cdot p\in b+\mathbb{N}\Delta+\mathbb{N}_+e_2$  and it means that  $q(b+n\Delta+\Delta[1]\cdot p)\in \text{PRE}_V(t)$ . However we know that  $q(b+n\Delta+\Delta[1]\cdot p)\in q(S_q)\subseteq S$  and therefore separator S nonempty intersects the set  $\text{PRE}_V(t)$ . This is a contradiction with the definition of separator.

In order to show that  $p = r \cdot \Delta$  for some  $p \in P_0$  it is sufficient to show that  $\Delta[1] \cdot p[2] = \Delta[2] \cdot p[1]$ . Assume towards a contradiction that for all  $p \in P_0$  we have  $\Delta[1] \cdot p[2] \neq \Delta[2] \cdot p[1]$ . By Claim 5 we know that actually for all  $p \in P_0$  we have that  $\Delta[1] \cdot p[2] < \Delta[2] \cdot p[1]$ . Thus p[1] > 0 and we can equivalently write that for all  $p \in P_0$  we have that  $\frac{\Delta[2]}{\Delta[1]} > \frac{p[2]}{p[1]}$ . Let F be the maximal value of  $\frac{p[2]}{p[1]}$  for  $p \in P_0$ , clearly  $\frac{\Delta[2]}{\Delta[1]} > F$ , so

$$\Delta[2] > F \cdot \Delta[1]. \tag{2}$$

By (1) we know that

$$\frac{(v+n\Delta)[2]}{(v+n\Delta)[1]} \le F,$$

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as  $v + n\Delta$  is a positive linear combination of vectors  $p \in P_0$  and for each  $p \in P_0$  we have  $\frac{p[2]}{p[1]} \leq F$ . Therefore it holds

$$v[2] + n\Delta[2] \le F(v[1] + n\Delta[1])$$

and equivalently

$$n(\Delta[2] - F\Delta[1]) \le Fv[1] - v[2].$$

By (2) we have that  $\Delta[2] - F\Delta[1] > 0$  therefore

$$n \le \frac{Fv[1] - v[2]}{\Delta[2] - F\Delta[1]}.$$

This is in contradiction with the assumption that n can be arbitrarily big and finishes the proof.

▶ Remark 6. In Theorem 4 instead of separator for (V, s, t) one can consider a separator for (V', s, t), where V' is obtained from V by adding a loop in state q with the effect of decreasing the second counter. Indeed, it is easy to observe that all the points 1-4 in the theorem statement remain true after substitution of V by V'. Such a version of Theorem 4 is a bit more convenient for some of the applications.

### 3.2 Generalised theorem

Before we start introducing the notions needed for Theorem 9 we motivate the need of a generalisation of Theorem 4. Formulation of Theorem 4 is rather simple and it is sufficient for our applications, but it may seem a bit arbitrary. Sometimes in a VASS we have a ratio between some two counters x and y being a fraction with big description size. Then Theorem 4 is applicable and intuitively speaking it aims at proving that separators are big because of big description size of the fraction  $\frac{x}{y}$ . We can however very easily imagine that in some other, but related VASS a value of x is kept on some two counters as  $2x_1 + 3x_2$  and value of y is kept as a sum of three counters  $y_1 + y_2 + y_3$ . In that case we would like to reason about the description size of the fraction  $\frac{2x_1+3x_2}{y_1+y_2+y_3}$  and from it derive lower bounds on the size of separators. This motivates introduction of the linear functions defined below. When we deal with more than two counters we cannot easily speak about lines, which were natural in Theorem 4. This is the reason why we are forced to use a more abstract language in order to be prepared for the above mentioned simple applications.

**Greatest common divisors** We state here a fact about greatest common divisors, which is helpful in the sequel. By  $gcd(a_1, \ldots, a_k)$  we denote the greatest common divisor of all the numbers  $a_1, \ldots, a_k$ .

 $\triangleright$  Claim 7. For all natural numbers  $a_1, \ldots, a_k \leq M$  and for each  $S \geq M^2 - M$  which is divisible by  $\gcd(a_1, \ldots, a_k)$  there exist nonnegative coefficients  $b_1, \ldots, b_k \in \mathbb{N}$  such that  $S = a_1b_1 + \ldots + a_kb_k$ .

**Proof.** It is a folklore and can be easily proved by induction on k that there exist some coefficients  $b_1,\ldots,b_k\in\mathbb{Z}$  such that  $\gcd(a_1,\ldots,a_k)=\sum_{i=1}^k a_ib_i$ . Let us take such a solution which minimises the sum  $\sum_{i:b_i<0}|b_i|$ . We show that in this solution actually all the  $b_i$  are nonnegative. Assume otherwise and let  $b_i<0$  for some  $i\in[1,k]$ . As  $S\geq M^2-M$  then for some  $b_j\in[1,k]$  we have  $b_j\geq M$ . Then substituting  $b_j$  by  $b_j-a_i$  and  $b_i$  by  $b_i+a_j$  we obtain a solution with smaller  $\sum_{i:b_i<0}|b_i|$ , contradiction.

**Linear functions** Let a linear function Lin:  $\mathbb{N}^d \to \mathbb{N}$  be of a form  $\operatorname{Lin}(x_1,\ldots,x_d) = \sum_{i=1}^d n_i x_i$ , with all the coefficients  $n_i \in \mathbb{N}_+$ . We call a linear function  $\operatorname{reduced}$  if  $\gcd(n_1,\ldots,n_d) = 1$ , let us assume that Lin is reduced. Notice that each linear function is a reduced linear function multiplied by some natural number. Let  $M = \max_{i \in [1,d]} n_i$ . The  $\operatorname{support}$  of a linear function  $\operatorname{supp}(\operatorname{Lin}) \subseteq [1,d]$  is the set of coordinates for which coefficient  $n_i$  is nonzero. Let the set of vectors  $\operatorname{ZERO}(\operatorname{Lin}) \subseteq \mathbb{Z}^d$  contain all the vectors  $n_j e_i - n_i e_j$  for  $i,j \in \operatorname{supp}(\operatorname{Lin})$ . Clearly for any  $v \in \operatorname{ZERO}(\operatorname{Lin})$  and any  $u \in \mathbb{N}^d$  such that  $u + v \in \mathbb{N}^d$  we have  $\operatorname{Lin}(u + v) = \operatorname{Lin}(u)$ . The following claim tells that set  $\operatorname{ZERO}(\operatorname{Lin})$  in some way spans the set of vectors with the same value of Lin in case it is big.

 $\triangleright$  Claim 8. For any  $u, v \in \mathbb{N}^d$  such that  $\operatorname{Lin}(u) = \operatorname{Lin}(v) \ge d \cdot M^3$  there is a sequence of vectors  $u = x_0, x_1, \dots, x_k = v \in \mathbb{N}^d$  such that for all  $i \in [1, k]$  we have  $x_i - x_{i-1} \in \operatorname{ZERO}(\operatorname{Lin})$ .

**Proof.** We prove the claim by induction on d. For d=1 clearly u=v and there is nothing to show. Let us assume that claim holds for d-1, our aim is to prove it for d. Adding a vector  $y \in \text{ZERO}(\text{Lin})$  to  $x_{i-1}$  in order to obtain  $x_i = x_{i-1} + y \in \mathbb{N}^d$  we call a *step*. Clearly it is enough find a sequence of steps from u to v. The plan is to apply first such a sequence of steps from u to some u' such that u'[i] = v[i] for some  $i \in [1, d]$  and then show by induction assumption that a sequence of steps from u' to v exists as well.

For a subset of coordinates  $I \subseteq [1, d]$  and  $x \in \mathbb{N}^d$  by  $\operatorname{Lin}_I(x)$  we denote  $\sum_{i \in I} n_i x[i]$ . As  $\operatorname{Lin}(v) \geq d \cdot M^3$  there exists some  $j \in [1, d]$  such that  $\operatorname{Lin}_{[1, d] \setminus \{j\}}(v) \geq (d - 1)M^3$ . Assume wlog. that j = d, so

$$\operatorname{Lin}_{[1,d-1]}(v) \ge (d-1)M^3. \tag{3}$$

We first aim to reach u'' such that  $u''[d] - v[d] \ge M^2$ . Clearly until for some  $i \ne d$  we have  $u[i] \ge M$  we can apply the step  $n_i e_d - n_d e_i$  to u and increase value of u[d]. We continue this until we reach some u'' with  $\sum_{i=1}^{d-1} u''[i] < (d-1)M$ . This is indeed possible as  $\sum_{i=1}^{d-1} u''[i] \ge (d-1)M$  implies that for some  $i \ne d$  we have  $u''[i] \ge M$ . Then we have that  $\operatorname{Lin}_{[1,d-1]}(u'') < (d-1)M^2$  as  $M = \max_{i \in [1,d]} n_i$ , so  $n_d u''[d] > dM^3 - (d-1)M^2$ . By (3) we have  $n_d v[d] \le dM^3 - (d-1)M^3 = M^3$ . Therefore  $n_d (u''[d] - v[d]) > (d-1)(M^3 - M^2)$  and thus  $u''[d] - v[d] \ge (d-1)(M^2 - M) \ge M^2 - M$ . By Claim 7 we have that  $u''[d] - v[d] = \sum_{i=1}^{d-1} n_i x_i$  for some  $x_i \in \mathbb{N}$ . Therefore in order to obtain u' such that u'[d] = v[d] for each  $i \in [1, d-1]$  we apply for each  $i \in [1, d-1]$  exactly  $x_i$  number of times the step  $n_d e_i - n_i e_d$  to u''. Notice that all the other coordinates beside the d-th one increase, so these steps indeed lead to vectors with nonnegative coordinates. As  $\operatorname{Lin}_{i \in [1, d-1]}(u') = \operatorname{Lin}_{i \in [1, d-1]}(v) \ge (d-1)M^3$  we apply the induction assumption to show that indeed starting from u' one can reach v by a sequence of steps. This finishes the proof.

**Modification of a VASS** For two linear functions  $\operatorname{Lin}_1, \operatorname{Lin}_2 \in \mathbb{N}^d \to \mathbb{N}$  and a VASS V with state q we define VASS  $V_{\operatorname{Lin}_1,\operatorname{Lin}_2}^q$  as V with additional transitions those aim is to be able to increase the ratio  $\frac{\operatorname{Lin}_1(\cdot)}{\operatorname{Lin}_2(\cdot)}$ , but never decrease it. Let  $\operatorname{Lin}_1(x_1,\ldots,x_d) = \sum_{i=1}^d n_{i,1}x_i$  and let  $\operatorname{Lin}_2(x_1,\ldots,x_d) = \sum_{i=1}^d n_{i,2}x_i$ . The set of states of  $V_{\operatorname{Lin}_1,\operatorname{Lin}_2}^q$  is inherited from V similarly as the set of transitions of V. We additionally add to V transitions of the form (q,v,q), which are loops in the state  $q \in Q$ , of the following form:

- 1. for each coordinate  $i \in \text{supp}(\text{Lin}_2)$  add a loop which decreases coordinate i by one
- 2. for each coordinate  $i \notin \operatorname{supp}(\operatorname{Lin}_1) \cup \operatorname{supp}(\operatorname{Lin}_2)$  add two loops: one, which increases and one which decreases coordinate i by one
- 3. for each  $v \in \text{ZERO}(\text{Lin}_1)$  and each  $v \in \text{ZERO}(\text{Lin}_2)$  add a loop with effect v.

Notice that condition 2 means that we can freely modify in  $V_{\text{Lin}_1,\text{Lin}_2}^q$  the coordinates outside supp(Lin<sub>1</sub>)  $\cup$  supp(Lin<sub>2</sub>) We are ready to state the theorem.

- ▶ **Theorem 9.** Let Lin<sub>1</sub>, Lin<sub>2</sub>:  $\mathbb{N}^d \to \mathbb{N}$  be two reduced linear functions with disjoint supports. Let V = (Q, T) be a d-VASS,  $q \in Q$  be its state,  $s, t \in Q \times \mathbb{N}^d$  be two its configurations and  $R \in \mathbb{Q}$  be a rational number. Assume that
- (1) for each  $v \in \mathbb{N}^d$  if  $s \longrightarrow q(v)$  then  $\operatorname{Lin}_1(v) \geq R \cdot \operatorname{Lin}_2(v)$ ,
- (2) for each  $u, v \in \mathbb{N}^d$  if  $q(v) \longrightarrow t + u$  then  $\operatorname{Lin}_1(v u) \leq R \cdot \operatorname{Lin}_2(v u)$ ,
- (3) each run from state(s) to state(t) traverses through a configuration c with state(c) = q,
- (4) the set  $\{\operatorname{proj}_I(v) \mid s \longrightarrow q(v) \longrightarrow t\}$  is infinite, where  $I = \operatorname{supp}(\operatorname{Lin}_1) \cup \operatorname{supp}(\operatorname{Lin}_2)$ .

Then for any  $i \in \text{supp}(\text{Lin}_2)$  there is no run from s to  $t + e_i$  in  $V' = V^q_{\text{Lin}_1,\text{Lin}_2}$  and each separator for  $(V', s, t + e_i)$  contains a period p such that  $\text{proj}_I(p) \neq 0$  and  $\text{Lin}_1(p) = R \cdot \text{Lin}_2(p)$ .

**Proof.** Due to condition (4) in the theorem statement there exists an infinite sequence of vectors  $v_i \in \mathbb{N}^d$  with  $\operatorname{proj}_I(v_i)$  pairwise different such that  $s \longrightarrow q(v_i) \longrightarrow t$ . Let  $\rho_i^1$  be the corresponding runs from s to  $q(v_i)$  and  $\rho_i^2$  be the corresponding runs from  $q(v_i)$  to t. Recall that  $\leq$  is a well-quasi order and its modified version (denoted here  $\leq'$ ) with comparison on sources instead of targets is also a well-quasi order. Therefore there exist i < j such that  $\rho_i^1 \leq \rho_j^1$  and  $\rho_i^2 \leq' \rho_j^2$ . Let  $\Delta = v_j - v_i \in \mathbb{N}^d$ . Clearly  $\operatorname{proj}_I(\Delta) \neq 0$ , as  $\operatorname{proj}_I(v_i) \neq \operatorname{proj}_I(v_j)$ . Let  $a = v_i$ . By Corollary 3 we get that for any  $n \in \mathbb{N}$  there is a run of V from s to s there is a run in s from s to s. Therefore for any s is a have  $s \longrightarrow q(s+n\Delta) \longrightarrow s$ . By conditions (1) and (2) in the theorem statement we get that  $\operatorname{Lin}_1(s+n\Delta) = R \cdot \operatorname{Lin}_2(s+n\Delta)$  for any s in consequence  $\operatorname{Lin}_1(s) = R \cdot \operatorname{Lin}_2(s)$ . Notice that both  $\operatorname{Lin}_1(s) = R \cdot \operatorname{Lin}_2(s) > 0$ , because  $\operatorname{proj}_I(s) \neq 0$ .

We first show that  $s \to t + e_i$  in V'. Assume towards a contradiction that  $s \to t + e_i$ . By condition (3) we know that  $s \to q(v) \to t + e_i$  for some  $v \in \mathbb{N}^d$ . Observe first that conditions (1) and (2) still hold for reachability relation defined in V', as the added loops in state q do not invalidate them. Notice that by condition (2) setting  $u = e_i$  we have

$$\operatorname{Lin}_1(v) = \operatorname{Lin}_1(v - e_i) \le R \cdot \operatorname{Lin}_2(v - e_i) < R \cdot \operatorname{Lin}_2(v).$$

where the first equation follows from the fact that  $\operatorname{Lin}_1(e_i) = 0$ . On the other hand  $s \longrightarrow q(v)$  so by condition (1) we have  $\operatorname{Lin}_1(v) \ge R \cdot \operatorname{Lin}_2(v)$ , which is in contradiction with the above inequality. Thus indeed  $s \longrightarrow t + e_i$ .

Now we show that every separator for  $(V',s,t+e_i)$  contains an appropriate period p. This proof is quite similar to the proof of Theorem 4, but we need to deal with some more technicalities. Consider a separator  $S = \bigcup_{q \in Q} q(S_q)$  for  $(V',s,t+e_i)$ . Clearly  $S_q = \bigcup_{j \in J} L_j$ , where  $L_j$  are linear sets, needs to contain all the vectors  $a+n\Delta$  for  $n \in \mathbb{N}$ . Let L be one of the finitely many linear sets  $L_j$ , which contains infinitely many vectors among  $\{a+n\Delta \mid n \in \mathbb{N}\}$ . Let  $L=b+\mathbb{N}p_1+\ldots+\mathbb{N}p_k$  and P be the set of periods  $\{p_1,\ldots,p_k\}$ . We aim at showing that L contains a period p such that  $\operatorname{proj}_I(p) \neq 0$  and  $\operatorname{Lin}_1(p) = R \cdot \operatorname{Lin}_2(p)$ , namely  $\operatorname{Lin}_1(p) \cdot \operatorname{Lin}_2(\Delta) = \operatorname{Lin}_2(p) \cdot \operatorname{Lin}_1(\Delta)$ , as  $R = \frac{\operatorname{Lin}_1(\Delta)}{\operatorname{Lin}_2(\Delta)}$ . We first prove a claim analogous to Claim 5. This one is however much more challenging

We first prove a claim analogous to Claim 5. This one is however much more challenging to prove. For its purpose we have defined V' so intricately with the additional loops and in the proof we use a nontrivial Claim 8.

 $\triangleright$  Claim 10. For each period  $p \in P$  we have  $\operatorname{Lin}_1(\Delta) \cdot \operatorname{Lin}_2(p) \leq \operatorname{Lin}_2(\Delta) \cdot \operatorname{Lin}_1(p)$ .

**Proof.** Similarly as in Claim 5 we intuitively mean to show that for all  $p \in P$  we have  $\frac{\operatorname{Lin}_1(\Delta)}{\operatorname{Lin}_2(\Delta)} \le \frac{\operatorname{Lin}_1(p)}{\operatorname{Lin}_2(p)}$ , but this is not a formally correct statement as it may be that  $\operatorname{Lin}_2(p) = 0$ .

© Wojciech Czerwiński, Adam Jędrych; licensed under Creative Commons License CC-BY 4.0 42nd Conference on Very Important Topics (CVIT 2016). Editors: John Q. Open and Joan R. Access; Article No. 23; pp. 23:9–23:16 Leibniz International Proceedings in Informatics LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany Assume towards a contradiction that there is a period p such that  $\operatorname{Lin}_1(\Delta) \cdot \operatorname{Lin}_2(p) > \operatorname{Lin}_2(\Delta) \cdot \operatorname{Lin}_1(p)$ . Clearly  $a \in L$ , therefore also  $a + mp \in L$  for any  $m \in \mathbb{N}$ . We aim at showing that  $a + mp \longrightarrow a + n\Delta + e_i$  in V' for some  $m, n \in \mathbb{N}$ . This would lead to a contradiction as we know that  $a + n\Delta \longrightarrow t$ , so also  $a + n\Delta + e_i \longrightarrow t + e_i$ . Therefore we would have that  $a + mp \in L$  and also  $a + mp \longrightarrow a + n\Delta + e_i \longrightarrow t + e_i$ , which is a contradiction with the definition of the separator.

Let  $\operatorname{Lin}_2(x_1,\ldots,x_d)=\sum_{i=1}^d n_{i,2}x_i$  and let M be the maximal coefficient in  $\operatorname{Lin}_2$ , namely  $M=\max_{i\in[1,d]}n_{i,2}$ . We set  $m=dM^3\cdot\operatorname{Lin}_1(\Delta)$  and  $n=dM^3\cdot\operatorname{Lin}_1(p)$ . Observe now that  $\operatorname{Lin}_1(a+mp)=\operatorname{Lin}_1(a+n\Delta+e_i)$ . Indeed

$$\operatorname{Lin}_{1}(a+mp) = \operatorname{Lin}_{1}(v) + m \cdot \operatorname{Lin}_{1}(p) = \operatorname{Lin}_{1}(v) + dM^{3} \cdot \operatorname{Lin}_{1}(\Delta) \cdot \operatorname{Lin}_{1}(p)$$
$$= \operatorname{Lin}_{1}(v) + n \cdot \operatorname{Lin}_{1}(\Delta) = \operatorname{Lin}_{1}(a+n\Delta+e_{i}),$$

as  $i \notin \operatorname{supp}(\operatorname{Lin}_1)$ , so  $\operatorname{Lin}_1(e_i) = 0$ . Let  $p = p_1 + p_2 + p_{\operatorname{trash}}$  and  $\Delta = \Delta_1 + \Delta_2 + \Delta_{\operatorname{trash}}$ , where  $p_1, \Delta_1$  are positive only on  $\operatorname{supp}(\operatorname{Lin}_1)$ ,  $p_2, \Delta_2$  are positive only on  $\operatorname{supp}(\operatorname{Lin}_2)$  and  $p_{\operatorname{trash}}, \Delta_{\operatorname{trash}}$  are positive only outside  $\operatorname{supp}(\operatorname{Lin}_1) \cup \operatorname{supp}(\operatorname{Lin}_2)$ . First notice that thanks to transitions in V', which can freely modify coordinates outside  $\operatorname{supp}(\operatorname{Lin}_1) \cup \operatorname{supp}(\operatorname{Lin}_2)$  we can assume wlog. that  $p_{\operatorname{trash}} = \Delta_{\operatorname{trash}} = 0$ . Therefore by Claim 8 and because  $\operatorname{Lin}_1$  is reduced there is a run in V' from  $a + mp = a + mp_1 + mp_2$  to  $a + n\Delta_1 + mp_2$ , as  $\operatorname{Lin}_1(a + mp_1) = \operatorname{Lin}_1(a + n\Delta_1) \geq dM^3 \geq |\operatorname{supp}(\operatorname{Lin}_1)| \cdot M^3$ . We claim now that

$$\operatorname{Lin}_{2}(mp_{2}) = \operatorname{Lin}_{2}(n\Delta_{2} + e_{i} + v_{\operatorname{trash}}) \ge dM^{3} \ge |\operatorname{supp}(\operatorname{Lin}_{2})| \cdot M^{3}$$
(4)

for some  $v_{\text{trash}}$  positive only on supp(Lin<sub>2</sub>). This would finalise the argument as then

$$a + n\Delta_1 + mp_2 \longrightarrow a + n\Delta_1 + n\Delta_2 + e_i + v_{\text{trash}} \longrightarrow a + n\Delta_1 + n\Delta_2 + e_i = a + n\Delta + e_i$$

where the first relation  $\longrightarrow$  follows from Claim 8 and fact that  $\operatorname{Lin}_2$  is reduced and the second one follows from existence of transitions in V' which arbitrarily decrease any coordinate in  $\sup(\operatorname{Lin}_2)$ .

Recall first that  $\operatorname{Lin}_1(\Delta) \cdot \operatorname{Lin}_2(p) > \operatorname{Lin}_2(\Delta) \cdot \operatorname{Lin}_1(p)$ , so

$$\operatorname{Lin}_1(\Delta) \cdot \operatorname{Lin}_2(p) - \operatorname{Lin}_2(\Delta) \cdot \operatorname{Lin}_1(p) \ge 1$$

and that we set  $m = dM^3 \cdot \text{Lin}_1(\Delta)$  and  $n = dM^3 \cdot \text{Lin}_1(p)$ . Therefore

$$\operatorname{Lin}_2(mp_2) - \operatorname{Lin}_2(n\Delta_2) = dM^3 \cdot \operatorname{Lin}_1(\Delta) \cdot \operatorname{Lin}_2(p) - dM^3 \cdot \operatorname{Lin}_2(\Delta) \cdot \operatorname{Lin}_1(p) \ge dM^3.$$

So  $\operatorname{Lin}_2(mp_2) - \operatorname{Lin}_2(n\Delta_2 + e_i) \geq dM^3 - M \geq M^2 - M$ . Thus by Claim 7 there exist some  $b_1, \ldots, b_d \in \mathbb{N}$  such that  $\sum_{j=1}^d n_{j,2}b_j = \operatorname{Lin}_2(mp_2) - \operatorname{Lin}_2(n\Delta_2 + e_i)$ . Hence  $\operatorname{Lin}_2(mp_2) = \operatorname{Lin}_2(n\Delta_2 + e_i + \sum_{j=1}^n b_j e_j) \geq dM^3$  and we can set  $v_{\text{trash}} = \sum_{j=1}^n b_j e_j$  thus satisfying (4) and finishing the proof of the claim.

Now we exactly follow the lines of the proof of Claim 5. Let  $P_{\emptyset}$  be the set of all periods in P for which  $\operatorname{proj}_I(p) = \emptyset$  and  $P_{\neq \emptyset} = P \setminus P_{\emptyset}$ . Recall that we want to show existence of a period  $p \in P_{\neq \emptyset}$  fulfilling  $\operatorname{Lin}_1(p) \cdot \operatorname{Lin}_2(\Delta) = \operatorname{Lin}_2(p) \cdot \operatorname{Lin}_1(\Delta)$ . Assume towards a contradiction that there is no such period. Therefore by Claim 10 each period  $p \in P_{\neq \emptyset}$  satisfies  $\operatorname{Lin}_1(\Delta) \cdot \operatorname{Lin}_2(p) < \operatorname{Lin}_2(\Delta) \cdot \operatorname{Lin}_1(p)$ . In particular for each period  $p \in P_{\neq \emptyset}$  we have  $\operatorname{Lin}_1(p) > 0$ , thus we can equivalently write that for all  $p \in P_{\neq \emptyset}$  it holds

$$\frac{\operatorname{Lin}_2(p)}{\operatorname{Lin}_1(p)} < \frac{\operatorname{Lin}_2(\Delta)}{\operatorname{Lin}_1(\Delta)}.$$

© Wojciech Czerwiński, Adam Jędrych; licensed under Creative Commons License CC-BY 4.0 42nd Conference on Very Important Topics (CVIT 2016). Editors: John Q. Open and Joan R. Access; Article No. 23; pp. 23:10–23:16 Leibniz International Proceedings in Informatics LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany Let F be the maximal value of  $\frac{\text{Lin}_2(p)}{\text{Lin}_1(p)}$  for  $p \in P_{\neq \emptyset}$ , clearly  $\frac{\text{Lin}_2(\Delta)}{\text{Lin}_1(\Delta)} > F$ , so

$$\operatorname{Lin}_2(\Delta) > F \cdot \operatorname{Lin}_1(\Delta).$$
 (5)

Recall now that for arbitrary big n we have that  $a + n\Delta \in b + \mathbb{N}p_1 + \ldots + \mathbb{N}p_k$ , thus  $(a-b) + n\Delta \in \mathbb{N}p_1 + \ldots + \mathbb{N}p_k$ . Let v = a - b. We have then that

$$(\operatorname{Lin}_1(v+n\Delta),\operatorname{Lin}_2(v+n\Delta)) = \sum_{p_i \in P_{\neq \emptyset}} n_i(\operatorname{Lin}_1(p),\operatorname{Lin}_2(p)).$$

By the above we know that

$$\frac{\operatorname{Lin}_2(v + n\Delta)}{\operatorname{Lin}_1(v + n\Delta)} \le F,$$

as  $v+n\Delta$  is a positive linear combination of periods from P, for each  $p\in P_\emptyset$  we have  $\operatorname{Lin}_1(p)=\operatorname{Lin}_2(p)=0$  and for each  $p\in P_{\neq\emptyset}$  we have  $\frac{\operatorname{Lin}_2(p)}{\operatorname{Lin}_1(p)}\leq F$ . Therefore

$$\operatorname{Lin}_2(v) + n\operatorname{Lin}_2(\Delta) \le F(\operatorname{Lin}_1(v) + n\operatorname{Lin}_1(\Delta))$$

and equivalently

$$n(\operatorname{Lin}_2(\Delta) - F \cdot \operatorname{Lin}_1(\Delta)) \le F \cdot \operatorname{Lin}_1(v) - \operatorname{Lin}_2(v).$$

By (5) we have that  $\operatorname{Lin}_2(\Delta) - F \cdot \operatorname{Lin}_1(\Delta) > 0$  therefore

$$n \le \frac{F \cdot \operatorname{Lin}_1(v) - \operatorname{Lin}_2(v)}{\operatorname{Lin}_2(\Delta) - F \cdot \operatorname{Lin}_1(\Delta)}$$

This is in contradiction with the fact that n can be arbitrarily big and finishes the proof.

We remark one more time that Theorem 9 is not needed for our applications. The motivation behind stating and proving this theorem is to prepare for possible future applications and explore limits of our approach. In the proof of Theorem 9 one needs to use two additional techniques, which were not needed to prove Theorem 4: well quasi-order on runs and construction of modified VASS with auxiliary transitions. Moreover we believe that the statement of Theorem 9 is more robust and looks more natural than Theorem 4.

## 4 Applications

In this section we show how Theorem 4 can be used to obtain lower bounds on the separator size. In Section 4.1 we prove that using a construction from [5] one can obtain a 4-VASS with separators of at least doubly-exponential size. In Section 4.2 we show that there exist VASSes with separators of arbitrary high elementary size of a special shape. Existence of separators of arbitrary high elementary size is an easy consequence of Tower-hardness of VASS reachability problem, we provide in Theorem 13 a concrete instance of such a separator. It is not a big contribution, but the aim of proving Theorem 13 is rather to illustrate that our techniques can be quite easily applied to many existing VASS examples.

### 4.1 Doubly-exponential separator in a 4-VASS

The aim of this section is to show the following theorem.

▶ **Theorem 11.** There exists a family of binary 4-VASSes  $(V_n)_{n\in\mathbb{N}}$  of size  $\mathcal{O}(n^3)$  such that for some configurations  $s_n, t_n$  of  $V_n$  with  $\operatorname{norm}(s_n), \operatorname{norm}(t_n) \leq 1$  such that  $s_n$  does not reach  $t_n$  the smallest separator for  $(V_n, s_n, t_n)$  is of doubly-exponential size wrt. n.

The rest of this section is devoted to the proof of Theorem 11. Our construction is based on the construction of a family of binary 4-VASSes  $(U_n)_{n\in\mathbb{N}}$  with shortest run of doubly-exponential length, which is described in Section 5 in [5]. The proof follows a very natural line and is not a big challenge. It boils down to performing a small modification to 4-VASSes from [5] in order to assure conditions of Theorem 4 and then checking that indeed these conditions are satisfied. We sketch here the idea behind the construction of the mentioned family of 4-VASSes only into such an extend that we can explain the proof of Theorem 11, for details we refer to [5]. The whole construction is based on the following lemma (Lemma 12 in [5]).

▶ Lemma 12. For each  $n \ge 1$  there are n rational numbers

$$1 < f_1 < \ldots < f_n = 1 + \frac{1}{4^n}$$

of description size bounded by  $4^{n^2+n}$ , such that the description size of f defined as

$$f = (f_n)^{2^n} \cdot \dots \cdot (f_2)^{2^2} \cdot (f_1)^{2^1}$$
(6)

is bounded by  $4^{2(n^2+n)}$ .

The VASSes  $U_n$  are constructed as follows. Valuation of the four counters  $(x_1, x_2, x_3, x_4)$  in the distinguished *initial* state is initially  $0^4$ , the run is accepting if it finishes in the distinguished *final* state also with valuation  $0^4$ . Each run of  $U_n$  consists of n+2 phases: the *initial phase*, n phases corresponding to fractions  $f_n, f_{n-1}, \ldots, f_2, f_1$ , respectively and the *final phase*. In each run after the initial phase the counter valuation is equal to (N, N, 0, 0) for some nondeterministically guessed value  $N \in \mathbb{N}$ . In every accepting run the phase corresponding to the fraction  $f_i$  results only in multiplying the second counter  $x_2$  by a value  $f_i^{2^i}$ . Therefore in such an accepting run counter valuation after phases corresponding to fractions  $f_n, \ldots, f_i$  is the following

$$(x_1, x_2, x_3, x_4) = (N, N \cdot f_n^{2^n} \cdot \ldots \cdot f_i^{2^i}, 0, 0).$$

In particular after all the n phases corresponding to fractions the second counter is equal to

$$N \cdot f_n^{2^n} \cdot \ldots \cdot f_1^{2^1} = N \cdot f,$$

where the equality follows from Lemma 12. Let  $q_i$  be the state of  $U_n$  after the initial phase, n-i phases corresponding to fractions  $f_n, \ldots, f_{i+1}$  and just before the phase corresponding to fraction  $f_i$  (in the case when i>0). It is important that due to Claim 15 in [5] any reachable configuration of the form  $q_i(\mathsf{x}_1,\mathsf{x}_2,\mathsf{x}_3,\mathsf{x}_4)$  satisfies  $\mathsf{x}_2 \leq f_n^{2^n} \cdot \ldots \cdot f_i^{2^i} \cdot \mathsf{x}_1$ . In particular for any reachable configuration  $q_0(\mathsf{x}_1,\mathsf{x}_2,\mathsf{x}_3,\mathsf{x}_4)$  we have that  $\mathsf{x}_2 \leq f \cdot \mathsf{x}_1$ , as  $f = f_n^{2^n} \cdot \ldots \cdot f_1^{2^1}$ . Let  $f = \frac{a}{b}$ . In the final phase the transition with the effect (-b, -a, 0, 0) is applied in a loop. One can easily see that in order to reach counter values  $0^4$  after the final phase we need to have in the state  $q_0$  values satisfying the equality  $\mathsf{x}_2 = f \cdot \mathsf{x}_1$ . Similarly any configuration  $q_{n-1}(\mathsf{x}_1,\mathsf{x}_2,\mathsf{x}_3,\mathsf{x}_4)$  on an accepting run needs to satisfy  $\mathsf{x}_2 = f_n^{2^n} \cdot \mathsf{x}_1$ . Let  $f_i = \frac{a_i}{b_i}$  for all  $i \in [1,n]$ . Then in any accepting run  $\mathsf{x}_1$  needs to be divisible by  $b_n^{2^n}$ , which is a doubly-exponential number wrt. n. This forces any accepting run of  $U_n$  to be doubly-exponential.

We slightly modify 4-VASSes  $U_n$  in order to obtain 4-VASSes  $V_n$ , which fulfil conditions of Theorem 4. VASS  $V_n$  is obtained from  $U_n$  by adding at the end of  $U_n$  two instructions: 1) decrease of  $\mathsf{x}_2$  by 1; and then 2) a loop, which decreases  $\mathsf{x}_2$  by an arbitrary nonnegative number. We first show that  $V_n$  indeed satisfies conditions of Theorem 4 and then we argue how this finishes the proof of Theorem 11. Let  $q_{\rm in}$  be the initial state of VASS  $U_n$ ,  $q_{\rm out}$  be its final state and  $q_{\rm last}$  be the state after the final phase, but before applying the above mentioned decreases of  $\mathsf{x}_2$ . We set  $s_n = q_{\rm in}(0^4)$  and  $t_n = q_{\rm out}(0^4)$ . Let  $N = \prod_{i=1}^n b_i^{2^i}$ . We set  $\Delta = (N, N \cdot f_n^{2^n}, 0, 0) \in \mathbb{N}^4$  and fix state  $q = q_{n-1}$ . We claim that VASS  $V_n$  together with two configurations  $s_n$  and  $t_n$ , state q and vector  $\Delta$  satisfies conditions of Theorem 4.

Condition (1) is satisfied trivially as  $\Delta = (N, N \cdot f_n^{2^n}, 0, 0)$ . In order to see that (2) is satisfied recall that if  $V_n$  reaches  $q_0(\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3, \mathsf{x}_4)$  then we have that  $\mathsf{x}_2 \leq \frac{a}{b} \mathsf{x}_1$ . In the final phase we subtract in the loop vector (b, a, 0, 0), but it does not change this inequality. Finally subtracting at least 1 at  $\mathsf{x}_2$  implies that any reachable configuration  $q_{\text{out}}(\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3, \mathsf{x}_4)$  satisfies  $\mathsf{x}_2 + 1 \leq \frac{a}{b} \cdot \mathsf{x}_1$ . This is not true for  $\mathsf{x}_1 = \mathsf{x}_2 = 0$ , which shows that condition (2) is indeed satisfied. Observe now that for each  $k \in \mathbb{N}$  configuration  $q_n(kN, kN, 0, 0)$  after the initial phase is reachable from  $s_n$ . Thus also configuration  $q_{n-1}(kN, kN \cdot f_n^{2^n}, 0, 0)$  is reachable from  $s_n$  if we multiply the second counter by  $f_n^{2^n}$ , which proves condition (3). Observe also that starting in a configuration  $q_{n-1}(kN, kN \cdot f_n^{2^n}, 0, 0)$  one can reach valuation  $q_{\text{last}}(0^4)$  before the decreases of  $\mathsf{x}_2$ . Therefore after adding  $e_2 = (0, 1, 0, 0)$  to both sides we get that  $q_{n-1}(kN, kN \cdot f_n^{2^n} + 1, 0, 0) \longrightarrow q_{\text{last}}(0, 1, 0, 0) \longrightarrow t_n$  for any  $k \in \mathbb{N}$ . Because of an option of decreasing  $\mathsf{x}_2$  many times in  $q_{\text{last}}$  we get that  $q_{n-1}(kN, kN \cdot f_n^{2^n} + \ell, 0, 0) \longrightarrow t_n$  for any  $k \in \mathbb{N}$ , which equivalent to the condition (4).

Therefore by Theorem 4 we have that each separator for  $(V_n, s_n, t_n)$  contains a period  $\Delta \cdot r \in \mathbb{N}^4$  for some  $r \in \mathbb{Q}$ . Let  $\Delta \cdot r = (\mathsf{x}_1, \mathsf{x}_2, 0, 0) \in \mathbb{N}^4$ . We know that  $\mathsf{x}_2 = \mathsf{x}_1 \cdot \left(\frac{a_n}{b_n}\right)^{2^n}$ , so in order for  $\mathsf{x}_2 \in \mathbb{N}$  we need to have  $b_n^{2^n} \mid \mathsf{x}_1$ . This implies that  $\mathsf{x}_1$  is doubly-exponential with respect to n. Therefore size of the period  $\Delta \cdot r$  and thus size of the separator is doubly-exponential with respect to n.

## 4.2 Tower size separators

In this section we show the following result.

- ▶ Theorem 13. There exists a family of VASSes  $(V_n)_{n\in\mathbb{N}}$  of size polynomial wrt. n such that for some configurations  $s_n, t_n$  of  $V_n$  with  $\operatorname{norm}(s_n) = \operatorname{norm}(t_n) = 0$  the following is true:
- $\bullet$   $s_n \longrightarrow t_n$ , but the shortest run is n-fold exponential,
- $s_n \not\longrightarrow t_n + e$  for some elementary vector e and each separator for  $(V_n, s_n, t_n + e)$  is of at least n-fold exponential size.

The rest of this section is dedicated to prove Theorem 13. Construction of VASSes  $V_n$  is based on constructions in [4], but we do not follow exactly [4] in order to avoid some technicalities and simplify the construction. As the construction [4] is pretty involved we decided only to sketch the intuition behind the constructed VASSes and use many of its properties without a detailed explanation. The main goal of this section is to provide an intuition why Theorem 4 is indeed applicable to that case, for details of the construction we refer to the original paper [4]. Essentially speaking Theorem 4 is applicable to VASSes in which the configurations on the accepting run are distinguished from the others once by keeping some specific ratio of counter values. The bigger the description size of the ratio the bigger is the separator. In VASSes from [4] the description size is n-fold exponential, which implies the lower bound on the size of separators.

We say that a counter is B-bounded if its value is upper bounded by B along the whole run. In [4] counters are assured to be B-bounded (for various values of B) in the following way. In order to guarantee that counter x initially equal to 0 is B-bounded we introduce another counter  $\bar{x}$  which is initialised to B and an invariant  $a + \bar{a} = B$  is kept throughout the whole run. The construction of [4] strongly relies on the fact that having a triple (x,y,z)=(B,C,BC) one can simulate C/2 zero-tests for a B-bounded counter. One zero-test for counter a is realised as follows. We decrease y by y. Then we enter two loops, the first one with the effect (-1,1,-1) on counters  $(a,\bar{a},z)$  and the second one with the effect (1,-1,-1) on the same counters. It is easy to see that for a B-bounded counter a the maximal possible decrease on a after these two loops is equal to a and it can only be realised if before and after the loops we have a and a and a are along the a and a are the loops we have a and a and a are the loops we have a and a are the loops where a and a are the loops where a are the loops where

Let  $3!^n$  be the number 3 followed by n applications of the factorial function. For example  $3!^1 = 3$ ,  $3!^2 = 6$  and  $3!^3 = 720$ . Roughly speaking VASS in [4] consists of a sequence of n gadgets, such that in every accepting run they compute triples of the form  $(3!^i, k_i, 3!^i \cdot k_i)$  for some nondeterministically guessed values  $k_i \in \mathbb{N}$ . The first gadget  $\mathcal{B}$  computes triple  $(3, k_1, 3 \cdot k_1)$  in a very easy way: after increasing its counters by (3, 0, 0) it fires a nondeterministically guessed number  $k_1$  of times a loop with the effect equal to (0, 1, 3). Next we have a sequence of n-1 gadgets  $\mathcal{F}$ , the i-th one inputing a triple  $(3!^i, k_i, 3!^i \cdot k_i)$  and outputting a triple  $(3!^{i+1}, k_{i+1}, 3!^{i+1} \cdot k_{i+1})$  on some other set of three counters. It is important to mention that correct value of the output triple requires that after the run values of the input triple are all zero. The last triple  $(3!^n, k_n, 3!^i \cdot k_n)$  is used in [4] to simulate  $3!^n$ -bounded counters of a counter automaton. Here however we modify this construction in order to obtain VASSes  $V_n$ , which fulfil the conditions of Theorem 13. As all the triples use different counters the presented VASS has at least dimension 3n. In the original construction of [4] some of the counters were actually reused in order to decrease the dimension. Here we allow for a wasteful use of counters, this however do not change the idea of the construction.

The family of VASSes  $V_n$  is defined as follows. We distinguish an initial state  $q_{\rm in}$  of  $V_n$ and a final state  $q_{\text{out}}$  of  $V_n$ . For  $i \in [1, n]$  let  $q_i$  be the state after the i-th gadget. From the above description we get that in state  $q_i$  valuation of some three counters is equal to  $(3!^i, k_i, 3!^i \cdot k_i)$ . Let us denote these counters  $(\mathsf{x}_i, \mathsf{y}_i, \mathsf{z}_i)$ . Thus in  $q_n$  we reach counter values  $(x_n, y_n, z_n) = (3!^n, k_n, 3!^n \cdot k_n)$  for some  $k_n \ge 0$ . Then after state  $q_n$  we define a state  $q_{\text{dec}}$  in which we decrease values of  $y_n$  and  $z_n$  by applying some nonzero number of zero-tests for  $3!^n$ -bounded counters. This operation can be seen as a loop decreasing counters  $(x_n, y_n, z_n)$ by  $(0,1,3!^n)$ , but of course subtracting  $(0,1,3!^n)$  is not realised by a single transition, but by some smaller sub-gadget of our VASS. Then we go to a state  $q_{\text{out}}$  in which we can decrease in a loop the counter  $x_n$  and as well the counter  $y_n$ . We define  $s_n = q_{in}(0^d)$  and  $t_n = q_{out}(0^d)$ for an appropriate dimension d. We set a vector e to be zero on all the coordinates beside  $y_n$ on which it is set to be one. We claim that  $V_n$  with configurations  $s_n$ ,  $t_n$  and an elementary vector e satisfy conditions Theorem 13. It is easy to see (assuming all the above remarks about the construction of [4]) that all the accepting runs need to traverse through a configuration  $q_n(3!^n, k, 3!^n \cdot k)$  for some  $k \geq 1$ , which implies that all the accepting runs have at least n-fold exponential length. It therefore remains to show the second point of Theorem 13, we apply Theorem 4 for that purpose.

As counters  $x_n$ ,  $y_n$  and  $z_n$  are important let us assume wlog. that they correspond to the first three coordinates in our notation, respectively. Let  $\Delta = (0, 1, 3!^n, 0^{d-3}) \in \mathbb{N}^d$ , namely  $\Delta[i] = 0$  for all  $i \notin \{y_n, z_n\}$ ,  $\Delta[y_n] = 1$  and  $\Delta[z_n] = 3!^n$ . Properties of  $V_n$  can be summarised in the following claim, which can be derived from [4].

 $\triangleright$  Claim 14. If  $s_n \longrightarrow q_{\text{dec}}(x,y,z,0^{d-3})$  then  $x=3!^n$  and  $y\leq 3!^nz$ . Moreover for any  $y\in\mathbb{N}$ 

we have 
$$s_n \longrightarrow q_{\text{dec}}(3!^n, y, 3!^n \cdot y, 0^{d-3})$$
.

Now we aim at showing that VASS  $V_n$  together with configurations  $s_n$ ,  $t_n + e$ , vector  $\Delta$  and state  $q_{\text{dec}}$  fulfils conditions of Theorem 4. It is immediate to see that condition (1) is satisfied as only second and third coordinates in  $\Delta$  are nonzero (we allow for reordering the coordinates in Theorem 4 without loss of generality). In order to show (2) we rely on Claim 14. We have  $t_n + e = q_{\text{out}}(0, 1, 0, 0^{d-3})$ , thus if  $s_n \longrightarrow t_n + e$  we need to have  $s_n \longrightarrow q_{\text{dec}}(x, y, 0, 0^{d-3}) \longrightarrow t_n + e$  for some y > 0. This is however a contradiction with Claim 14, as then  $y \cdot 3!^n > 0$ . By Claim 14 we also immediately derive condition (3). To show condition (4) notice that because of the loop in  $q_{\text{dec}}$  of effect  $(0, -1, -3!^n)$  on counters  $(x_n, y_n, z_n)$  we have

$$q_{\mathrm{dec}}(0,k+\ell,k\cdot 3!^n,0^{d-3}) \longrightarrow q_{\mathrm{dec}}(0,\ell,0^{d-2}) \longrightarrow q_{\mathrm{out}}(0,\ell-1,0^{d-2}) \longrightarrow q_{\mathrm{out}}(0,1,0^{d-2}) = t_n + e^{-t_n}$$

for any  $k, \ell \geq 1$ . This shows that indeed Theorem 4 can be applied to  $V_n$ . Thus each separator for  $(V_n, s_n, t_n + e)$  contains a period of a form  $(r, r \cdot 3!^n, 0, 0^{d-3}) \in \mathbb{N}$ . As  $r \in \mathbb{N}$  the number  $r \cdot 3!^n$  is n-fold exponential and thus the size of any separator for  $(V_n, s_n, t_n + e)$  is n-fold exponential, which finishes the proof of Theorem 13.

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