APPROXIMATE SOLUTIONS TO THE DIRICHLET PROBLEM FOR HARMONIC MAPS BETWEEN HYPERBOLIC SPACES

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ABSTRACT. Our main result in this paper is the following: Given H^m, H^n hyperbolic spaces of dimensional $m \geq 2$ and n corresponding, and given a Holder function $f = (f^1, ..., f^{n-1}) : \partial H^m \to \partial H^n$ between geometric boundaries of H^m and H^n. Then for each $\epsilon > 0$ there exists a harmonic map $u : H^m \to H^n$ which is continuous up to the boundary (in the sense of Euclidean) and $u|_{\partial H^m} = (f^1, ..., f^{n-1}, \epsilon)$.

1. INTRODUCTION

Let H^m and H^n are hyperbolic spaces with dimensions $m \geq 2$ and n correspondingly. For convenience, we use the upper-half space models for H^m and H^n. So $H^m = \{(x^1, ..., x^m) \in \mathbf{R}^m : x^m > 0\}$, $H^n = \{(y^1, ..., y^n) \in \mathbf{R}^n : y^n > 0\}$ with metrics

$$d^2_{H^m} = \frac{1}{(x^m)^2}((dx^1)^2 + ... + (dx^m)^2),$$

$$d^2_{H^n} = \frac{1}{(y^n)^2}((dy^1)^2 + ... + (dy^n)^2).$$

So the tension fields of $u = (y^1, ..., y^n)$ is

$$\tau^\alpha = (x^m)^2(\Delta_0 y^\alpha - \frac{m-2}{x^m}\frac{\partial y^\alpha}{\partial x^m} - \frac{2}{y^n} < \nabla_0 y^\alpha, \nabla_0 y^n >),$$

for $1 \leq \alpha \leq n - 1$ and

$$\tau^n(u) = (x^m)^2(\Delta_0 y^n - \frac{m-2}{x^m}\frac{\partial y^\alpha}{\partial x^m} + \frac{1}{y^n}(\sum_{\alpha=1}^{n-1} |\nabla_0 y^\alpha|^2 - |\nabla_0 y^n|^2)),$$

where ∇_0 is the Euclidean gradient and Δ_0 is the Euclidean Laplacian.

A C^2 map $u : H^m \to H^n$ is called a harmonic map if $\tau(u)^s = 0$ for all $s = 1, 2, ..., n$. The literature about harmonic maps between Riemannian manifolds are abundant, we refer the readers to the classical work [4].

One of the interesting problems for harmonic maps is that of the Dirichlet problem at infinity: Given ∂H^m and ∂H^n geometric boundaries of H^m and H^n, and given a continuous map $f : \partial M \to \partial N$ (here continuity is understood in the sense

of Euclidean), is there a harmonic map $u : H^m \to H^n$ such that in Euclidean sense u is continuous up to the boundary ∂H^m and takes boundary value f?

For this problem with some more requirements for the smoothness of f, there are many results. In three papers [8], [9] and [7], Li and Tam established the existence and uniqueness of a harmonic function u which is C^1 up to the boundary and has boundary value f, provided f is C^1. But for more general types of f, according to our knowledge, there is no answer to the existence of a solution u.

In this paper we establish the existence of approximate solutions to the Dirichlet problem for harmonic maps between two hyperbolic spaces with prescribed boundary value. More explicitly, we prove the following result

Theorem 1. *Let $f : H^m \to H^n$ be a bounded uniformly continuous. Let functions g and φ be as in Section 2. Assume that $\int_0^1 t^{-1} g(t) dt < \infty$, in particular, this condition is satisfied if f is Holder continuous. For each $\epsilon > 0$, there exists a harmonic map $u_\epsilon : H^m \to H^n$ which is continuous up to the boundary ∂H^m and $u|_{\partial H^m} = (f^1, ..., f^{n-1}, \epsilon)$.*

Our strategy for proving this result is the follows: First, we construct an initial map, i.e., a C^2 map $v = (v^1, ..., v^{n-1}, v^n) : H^m \to H^n$ which has boundary value f for any continuous map $f : \partial H^m \to H^n$. For this step we follows the ideas in [9], with some changes: Since the function f needs not to differentiable, we can not take v^n as in [9], and the function v^n of ours is a function of one variable x^m. Then, we use this function to produce harmonic maps $u_\epsilon : H^m \to H^n$ which takes boundary value $(f^1, ..., f^{n-1}, v^n + \epsilon)$ for every $\epsilon > 0$.

2. Initial maps

In this part, we use the techniques in [9] to construct good initial maps v having the map $f : \partial H^m \to \partial H^n$ as the boundary value.

Let $f : \mathbf{R}^{m-1} \to \mathbf{R}^{n-1}$ be a uniformly continuous bounded function. Let $g : H^m \to (0.\infty)$ be C^2, bounded and

$$\lim_{x^m \to 0} g(x', x^m) = 0,$$

uniformly in x'.

We denote by $v = \{f, g\} : H^m \to H^n$ the extension of f defined as follows

$$v^\alpha(x', x^m) = \frac{2}{m \omega_m} \int_{R^{m-1}} \frac{x^m f^\alpha(y')}{(|x' - y'|^2 + (x^m)^2)^{m/2}} dy',$$

for $1 \le \alpha \le n - 1$ and

$$v^n(x', x^m) = g(x', x^m).$$

By results in [9] (pp. 628-630) we have
(i) v is C^2 and up to the boundary given by $x^m = 0$ it is continuous.
(ii) If $1 \le \alpha \le n - 1$ then

$$\lim_{x^m \to 0} x^m |\nabla_0 v^\alpha| = 0,$$

uniformly in x'.

Moreover, by estimates of elliptic PDEs (see Theorem 2.10 in [5]), noting that v^α is bounded, there exists constants $C > 0$ such that

(2.1) $$\max\{(x^m)^3 |D^3 v^\alpha|, (x^m)^2 |D^2 v^\alpha|, (x^m) |\nabla_0 v^\alpha|\} \le C.$$

We put

$$g(r) = \sup_{x',y' \in \mathbf{R}^{m-1}, \ |x'-y'| \leq r} |f(y') - f(x')|,$$

and

$$\varphi(r) = \int_0^\infty \frac{r}{s^2 + r^2} g(s) ds.$$

Since g is monotone it follows that g is Lebesgue measurable. Moreover, since g is bounded, we see that φ is well-defined.

Using polar coordinates with center at x' we see that there exists a constant $C > 0$ such that

$$\int_{\mathbf{R}^{m-1}} \frac{x^m |f(y') - f(x')|}{(|x' - y'|^2 + (x^m)^2)^{m/2}} \leq C\varphi(x^m),$$

for all $x' \in \mathbf{R}^{m-1}$.

Since f is uniformly continuous we see that

$$\lim_{r \to 0} g(r) = 0.$$

Now we show that

$$\lim_{x^m \to 0} \varphi(x^m) = 0.$$

Indeed, for any $\epsilon > 0$, we find $\delta > 0$ such that

$$g(s) \leq \epsilon,$$

if $0 < s \leq \delta$. So, if $K = \sup_{s \in \mathbf{R}} g(s)$ we have

$$
\begin{aligned}
\varphi(r) &= \int_0^\delta \frac{r}{s^2 + r^2} g(s) ds + \int_\delta^\infty \frac{r}{s^2 + r^2} g(s) ds \\
&\leq \int_0^\delta \epsilon \frac{r}{s^2 + r^2} ds + \int_\delta^\infty K \frac{r}{s^2 + r^2} ds \\
&= \epsilon \arctan(\delta/r) + K(\pi/2 - \arctan(\delta/r)).
\end{aligned}
$$

Letting $r \to 0$ we see that

$$\limsup_{r \to 0} \varphi(r) \leq \epsilon\pi/2.$$

Since $\epsilon > 0$ is arbitrary, we see that

$$\lim_{r \to 0} \varphi(r) = 0.$$

Thus, if we put $v = \{f, \varphi(x^m)\}$ we see that v is an extension of f. Moreover we have the following result

Lemma 1. *Let $f : \partial H^m \to \partial H^n$ be nonconstant, uniformly continous and bounded. Put $v = \{f, \varphi(x^m)\}$ as above. Then v is smooth, up to the boundary it is continuous, $v|_{\mathbf{R}^{m-1}} = f$ and there exists $C > 0$ such that for x^m near 0 we have*

$$\|\tau(v)\|^2 \leq C.$$

Proof. By Section 6 in [9] we have

$$|(x^m)\nabla_0 v^\alpha| \le C_3|\varphi(x^m)|,$$

where $1 \le \alpha \le n-1$ and C_3 is a positive constant.

Directly computation gives

$$\varphi'(r) = \int_0^\infty \frac{s^2 - r^2}{(s^2 + r^2)^2} g(s)ds,$$

$$\varphi''(r) = \int_0^\infty \frac{-2r}{(s^2 + r^2)^2} g(s)ds + \int_0^\infty \frac{-4r(s^2 - r^2)}{(s^2 + r^2)^3} g(s)ds.$$

So

$$\max\{|r\varphi'(r)|, |r^2\varphi''(r)|\} \le C_4\varphi(r),$$

where C_4 is a constant.

Since g is increasing, g' exists almost everywhere and $g' \ge 0$. Using integration by parts, noting that $\frac{d}{ds}(\frac{-s}{s^2+r^2}) = \frac{s^2-r^2}{(s^2+r^2)^2}$, we have

$$\begin{aligned}
\varphi'(r) &= \int_0^\infty \frac{s^2 - r^2}{(s^2 + r^2)^2} g(s)ds \\
&= \frac{-s}{s^2 + r^2} g(s)\Big|_0^\infty + \int_0^\infty \frac{s}{s^2 + r^2} g'(s)ds \\
&= \int_0^\infty \frac{s}{s^2 + r^2} g'(s)ds.
\end{aligned}$$

Differentiating the last term in above equality we get

$$\varphi''(r) = -\int_0^\infty \frac{2rs}{(s^2 + r^2)^2} g'(s)ds.$$

Since f is nonconstant we see easily that $g' \not\equiv 0$ (in fact, we don't need this restriction since we can add g with a non-constant positive function, for example $(x^m)^{1/2}$). So since $g' \ge 0$, it follows from above equalities that

$$\varphi'(r) > 0,$$

and

$$|r\varphi''(r)| \le C_5\varphi'(r),$$

where C_5 is a positive constant. Then use the formula for the tension field we are done. $\qquad\square$

3. Proof of Theorem 1

Proof. Fixed $\epsilon > 0$. We define $v_\epsilon : H^m \to H^n$ as follows:

$$v_\epsilon(x) = (v^1(x), v^2(x), ..., v^{n-1}(x), \varphi(x^m) + \epsilon).$$

For each $\delta > 0$ denote $u_{\epsilon,\delta} : H^m \supseteq \Omega_\delta = \{x^m > \delta\} \to H^n$ the harmonic map taking value v_ϵ on $\partial\Omega_\delta$.

By inequality (2.1) in [2] and properties of v and v_ϵ (see Lemma 1) we have

$$\Delta_{H^m} d_{H^n}(u_{\epsilon,\delta}, v_\epsilon) \ge -|\tau(v_\epsilon)| \ge -C\frac{\varphi(x^m)}{\varphi(x^m) + \epsilon} \ge -C\frac{1}{\epsilon}\varphi(x^m),$$

for all $x \in \Omega_\delta$, and here C is one constant from Lemma 1.

We claim that the function

$$\psi(r) = \int_0^r \int_s^\infty u^{-2}\varphi(u) \ du \ ds$$

is well-defined for $r \geq 0$. In fact, using the formula for φ we have

$$\psi(r) = \int_0^r \int_s^\infty u^{-2}\varphi \ du \ ds = \int_0^r \int_s^\infty \int_0^\infty u^{-1}(u^2+t^2)^{-1}g(t)dt \ du \ ds.$$

Since the integrand is non-negative, using Fubini's theorem we have

$$\int_0^r \int_s^\infty \int_0^\infty u^{-1}(u^2+t^2)^{-1}g(t)dt \ du \ ds \ = \ \int_0^r \int_0^\infty \int_s^\infty u^{-1}(u^2+t^2)^{-1}g(t)du \ dt \ ds$$

$$= \ \int_0^r \int_0^\infty \frac{1}{2t^2}\log(1+\frac{t^2}{s^2})g(t)dt \ ds$$

$$= \ \int_0^\infty \int_0^r \frac{1}{2t^2}\log(1+\frac{t^2}{s^2})g(t)ds \ dt$$

$$= \ \int_0^\infty \frac{\pi t^2 - 2\arctan(\frac{t}{r})t^2 + r\log(1+\frac{t^2}{r^2})}{t^3}g(t)dt.$$

Now since $g(t)$ is bounded we have

$$\int_0^\infty \frac{\pi t^2 - 2\arctan(\frac{t}{r})t^2}{t^3}g(t)dt$$

is convergent. Fixed $r \geq 0$, near $t = 0$ we have

$$\frac{r\log(1+\frac{t^2}{r^2})}{t^3}g(t) \approx t^{-1}g(t),$$

and when $t \to \infty$ we have

$$\frac{r\log(1+\frac{t^2}{r^2})}{t^3}g(t) \approx t^{-3}g(t),$$

hence since $g(t)$ is bounded and the assumption that $\int_0^1 t^{-1}g(t)$ converges, our claim is verified.

We use the same ψ to denote the function $\psi : H^m \to \mathbf{R}$ defined by $\psi(x) = \psi(x^m)$ for $x = (x^1, ..., x^{m-1}, x^m) \in H^m$. Now we have $\psi'(r) = \int_r^\infty u^{-2}\varphi \ du > 0$ and $\psi''(r) = -r^{-2}\varphi(r)$, since $m \geq 2$ we have

$$\Delta_{H^m}(-\psi(x)) = -(x^m)^2[\psi''(x^m) - \frac{(m-2)}{x^m}\psi'(x^m)] \geq -(x^m)^2\psi''(x^m) = \varphi(r).$$

Hence

$$\Delta_{H^m}(d_{H^n}(u_{\epsilon,\delta}, v_\epsilon) - C\frac{1}{\varphi(\epsilon)}\psi) \geq 0,$$

for $x \in \Omega_\delta$. Hence by maximum principle we have

$$\sup_{x \in \Omega} d_{H^n}(u_{\epsilon,\delta}, v_\epsilon) \leq C\frac{1}{\epsilon}\psi(x^m).$$

This bound for $d_{H^n}(u_{\epsilon,\delta}, v_\epsilon)$ is independent of δ, hence by standard arguments (see the proof of Theorem 6.4 in [9]) we have a harmonic map $u_\epsilon : H^m \to H^n$ which is the subsequent limit of $u_{\epsilon,\delta}$. Moreover for all $x \in H^m$ we have

$$d_{H^n}(u_\epsilon, v_\epsilon) \leq C\frac{1}{\epsilon}\psi(x^m).$$

Hence
$$\lim_{x^m \to 0} d_{H^n}(u_\epsilon, v_\epsilon) = 0,$$

which shows that u_ϵ is continuous up to the boundary and takes boundary value
$v_\epsilon(x^1, ..., x^{m-1}, 0) = (f^1, f^2, ..., f^{n-1}, \epsilon)$. $\square$

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