Most Expected Winner: An Interpretation of Winners over Uncertain Voter Preferences

Haoyue Ping¹ and Julia Stoyanovich¹

¹New York University, USA {hp1326, stoyanovich}@nyu.edu

Abstract

It remains an open question how to determine the winner of an election given incomplete or uncertain voter preferences. One solution is to assume some probability space for the voting profile and declare the candidates having the best chance of winning to be the (co-)winners. We refer to this as the *Most* Probable Winner (MPW). In this paper, we propose an alternative winner interpretation for positional scoring rules — the Most Expected Winner (MEW), based on the expected performance of the candidates. This winner interpretation enjoys some desirable properties that the MPW does not. We establish the theoretical hardness of MEW over incomplete voter preferences, then identify a collection of tractable cases for a variety of voting profiles. An important contribution of this work is to separate the voter preferences into the generation step and the observation step, which gives rise to a unified voting profile combining both incomplete and probabilistic voting profiles.

1 Introduction

Voting is a mechanism to determine winners among the candidates in an election by aggregating the preferences of voters. In classical voting theory, each voter gives a complete preference (most frequently, a ranking) of all candidates. How voter preferences are aggregated is determined by a voting rule. A prominent class of voting rules, which assign scores to candidates based on their positions in the ranking of each voter and then sum up the scores for each candidate, are positional scoring rules, on which we focus in this paper.

In practice, voter preferences may well be incomplete and represented by partial orders. Since voting rules are defined over (complete) rankings, the solution is to replace each partial order with all of its linear extensions, each of which is a *completion* or a *possible world*¹ of the partial order. The preferences of all voters are referred to as a *voting profile*. A voting profile of complete rankings is a *complete voting profile*, while that of incomplete preferences is an *incomplete*

voting profile. Voters are assumed to cast their preferences independently. A *completion* of an incomplete voting profile is a complete voting profile obtained by replacing each voter's partial order with one of its completions.

There have been various interpretations of winners proposed for this setting. The most thoroughly studied are the necessary and possible winners [Konczak and Lang, 2005]. A candidate is an necessary winner (NW) if she wins in every possible world; she is a possible winner (PW) if she wins in at least one possible world. The NW and PW semantics are simple but they have shortcomings that make them impractical: The requirement for NW is so strict that there are often no winners available in a voting profile under this interpretation, while the requirement for PW does not differentiate between a candidate who only wins in one possible world and another candidate who only loses in one possible world.

[Bachrach *et al.*, 2010] assume that an incomplete voting profile of partial orders represents a uniform distribution over its completions, and prefer the candidates who enjoy victory in more possible worlds. We refer to this winner semantics as *Most Probable Winner* (MPW). While this semantics is well defined under any voting rule, and while it can be extended in a straight-forward way to incorporate the probability of a completion of a voting profile into the computation, computing a winner under MPW is known to be intractable already under plurality [Bachrach *et al.*, 2010].

[Hazon *et al.*, 2012] also investigates MPW but under a different setting, where voter preferences are specified explicitly by rankings and their associated probabilities. They proved that it is #P-hard to compute the candidate winning probabilities under plurality, k-approval, Borda, Copeland, and Bucklin, and provided an approximation algorithm.

In this paper, we propose the *Most Expected Winner* (MEW) as an alternative winner interpretation for incomplete voter preferences under positional scoring rules. Like MPW, it adopts the possible world semantics of incomplete voting profiles. However, in contrast to MPW that determines a winner by a (weighted) count of the possible worlds in which she wins, MEW follows the principle of score-based rules that high-scoring candidates should be favored. Specifically, an MEW is the candidate who has the highest expected score in a random possible world.

MEW and MPW are similar in that they both aggregate election results over all possible worlds and give a balanced

¹We use *completion* and *possible world* interchangeably.

Voter	$ \boldsymbol{\tau}_1 = \langle a, b, c \rangle$	$\tau_2 = \langle b, a, c \rangle$	$\tau_3 = \langle b, c, a \rangle$	$\tau_4 = \langle c, b, a \rangle$
\overline{x}	0.7	0.3	0	0
y	0	0	0.5	0.5

Table 1: A probabilistic voting profile for voters x and y. Each voter casts her vote independently of the other, leading to 4 possible worlds, listed with their probabilities: $\Pr(\tau_1, \tau_3) = 0.35$, $\Pr(\tau_1, \tau_4) = 0.35$, $\Pr(\tau_2, \tau_3) = 0.15$, and $\Pr(\tau_2, \tau_4) = 0.15$. Under the plurality rule, candidate a is the MPW with a winning probability of $\Pr(\tau_1, \tau_3) + \Pr(\tau_1, \tau_4) = 0.7$, while candidate b is the MEW with expected score $\Pr(\tau_2 \mid x) + \Pr(\tau_3 \mid y) = 0.3 + 0.5 = 0.8$.

evaluation of the candidates. Their difference lays in the aggregation methods, which will be discussed in detail in Section 3.2. In practice, MEW and MPW yield the same result often, but not always. Table 1 gives an example where MEW and MPW select different winners in an election with two voters and three candidates, under the plurality rule. In this election, each voter produces a full ranking drawn from a probability distribution over $\tau_1 = \langle a, b, c \rangle$, $\tau_2 = \langle b, a, c \rangle$, $\tau_3 = \langle b, c, a \rangle, \tau_4 = \langle c, b, a \rangle$, with probabilities of each ranking for each voter given in Table 1. Since the voter preferences are probabilistic distributions of rankings, the corresponding voting profile is named a probabilistic voting profile. Let $Pr(\tau_x, \tau_y)$ denote the probability that voter x and y cast ranking τ_x and τ_y . Assume there is no correlation between voters x and y, then $\Pr(\tau_x, \tau_y) = \Pr(\tau_x \mid x) \cdot \Pr(\tau_y \mid x)$ y), e.g., $\Pr(\tau_1, \tau_3) = 0.7 \cdot 0.5 = 0.35$. This voting profile effectively generates 4 possible worlds as in the caption of Table 1. The plurality rule rewards a candidate with 1 point every time she is ranked at the top of a ranking in the profile. So in the possible world of $Pr(\tau_1, \tau_3)$, candidates a obtains 1 point from τ_1 and b obtains 1 point from τ_3 , and both of them become (co-)winners in this possible world. After enumerating all 4 possible worlds, we find that candidate a is the MPW with probability 0.7 to be a (co-)winner, while candidate b is the MEW with expected score of 0.8 point.

By the way, if applying the Borda rule that rewards a candidate with the number of candidates ranked after her in a ranking every time, in the possible world of $\Pr(\tau_1, \tau_3)$, candidate a would obtain 2 points from τ_1 , while candidate b would obtain 1 point from τ_1 and 2 points from τ_3 . Candidate b would be the only NW of this profile and she would also be the MEW with an expected score of 2.8 points.

The main technical contribution of this paper is an investigation of computational complexity of MEW. (The problem statement is given formally in Section 3.3.) Another contribution is the modeling of uncertain voter preference for two distinct sources. The first is uncertainty in the preferences themselves: a voter may not feel strongly about the relative ordering of some pair of candidates, or, more generally, their preference may be drawn from some probability distribution [Marden, 1995]. We refer to this as uncertainty in preference generation. The second is uncertainty in the observation: a voting mechanism (e.g., approval ballots or a ranking of at most k < m candidates) may not allow the voters to fully reveal their preferences. With this understanding, in Section 2 we will classify voting profiles into prob-

Voting rule	$\mid m{r}_m$
plurality veto 2-approval Borda	$ \begin{array}{ c c c }\hline & (1,0,\ldots,0,0) \\ & (1,1,\ldots,1,0) \\ & (1,1,0,\ldots,0,0) \\ & (m-1,m-2,\ldots,1,0) \\ \end{array}$

Table 2: Examples of positional scoring rules.

abilistic profiles (uncertainty during generation), incomplete profiles (uncertainty during observation), and combined profiles. This classification gives us a framework within which to study the complexity of identifying MEW by computing expected scores of the candidates. Recall that $FP^{\#P}$ is a class of functions efficiently solvable with an oracle to some #P problem. A function f is $FP^{\#P}$ -hard if there is a polynomial-time Turing reduction from any $FP^{\#P}$ function to f. While it turns out that the MEW problem is $FP^{\#P}$ -complete under plurality, veto, and k-approval rules for the general case of uncertain profiles (Section 4), we will also identify interesting cases where it is tractable to compute the expected scores and determine the MEW (Section 5).

2 Uncertain Voting Profiles

We now propose a novel framework for representing uncertainty in voter preferences and introduce a *unified voting profile* that explicitly models different types of uncertainty that can arise in preference generation and elicitation.

Voting preliminaries. Let us denote by $C = \{c_1, \ldots, c_m\}$ a set of *candidates* or *items*², by $V = \{v_1, \ldots, v_n\}$ a set of voters, and by $P = (\tau_1, \ldots, \tau_n)$ a *complete voting profile* where τ_i is a ranking over C by voter v_i . Ranking τ is a bijection between candidates and ranks where $\tau(i)$ is the candidate at rank i and $\tau^{-1}(c)$ is the rank of candidate c in τ .

Positional scoring rules are arguably the most thoroughly studied voting rules. Let $r_m = (r_m(1),...,r_m(m))$ denote a positional scoring rule where $\forall 1 \leq i < j \leq m, r_m(i) \geq r_m(j)$ and $r_m(1) > r_m(m)$. It assigns a score $r_m(i)$ to the candidate at rank i. Table 2 lists several popular positional scoring rules.

The performance of a candidate c is the sum of her scores across the entire voting profile: $s(c, \mathbf{P}) = \sum_{\tau \in \mathbf{P}} s(c, \tau)$ where $s(c, \tau) = r_m(\tau^{-1}(c))$. Candidate w is a (co-)winner if her score is no less than the score of any other candidate.

Uncertainty in voter preferences. In classical voting theory, voters give complete rankings over candidates. However, in practice only partial preferences may be observed, due to the voting mechanism (e.g., when approval ballots or a ranking of at most k < m candidates are elicited), the uncertainty in preferences themselves [Marden, 1995], or both. Figure 1 represents uncertainty as two distinct steps: preference generation (Figure 1a) and preference observation (Figure 1b). Important special cases of voting profiles are discussed next.

²Candidates are used in the context of voting, while items are for general preference analysis. In this paper, they are interchangeable.

Uncertainty in profile generation. The most general form of voter preferences over rankings is a non-parametric probability distributions such as that given in Table 1. Let $P^{\mathsf{M}} = (\mathsf{M}_1, \dots, \mathsf{M}_n)$ denote a *probabilistic voting profile* where M_i is the ranking model of voter v_i . A *possible world* of P^{M} is a complete voting profile $P = (\tau_1, \dots, \tau_n)$ where each τ_i is sampled from M_i . It is assumed that so voters cast their ballots independently, i.e., $\Pr(P \mid P^{\mathsf{M}}) = \prod_{i=1}^n \Pr(\tau_i \mid \mathsf{M}_i)$. So P^{M} represents a probability distribution of its possible worlds $\Omega(P^{\mathsf{M}}) = \{P_1, \dots, P_z\}$.

Let s(c, M) and $s(c, P^M)$ denote the scores assigned to candidate c by model M and the probabilistic voting profile P^M , respectively. Note that they both are random variables.

Partial voting profiles are a special cases of probabilistic voting profiles, since they are based on the assumption that all completions are equally likely. Below are a few other cases.

A uniform voting profile, denoted by P^{U} , is a trivial case where no voter has any preference between any two candidates. Each voter's preferences are sampled from the uniform distribution over the rankings of all candidates. We list this special case for convenience of discussion.

There are specific important ranking models that we will describe in Section 5. The Mallows model [Mallows, 1957] is the best known, and it is further generalized by the Repeated Insertion Model (RIM) [Doignon *et al.*, 2004] and the ranking version of the Repeated Selection Model (RSM) [Chakraborty *et al.*, 2020]. Voting profiles consisting of Mallows, RIMs, and RSMs (ranking version) are denoted by P^{MAL} , P^{RIM} , and P^{rRSM} , respectively. We will see that when the generation step is based on these models, we have a computational advantage for computing the MEW.

Uncertainty in profile observation. A partial voting profile, denoted by $P^{\rm PO}$, consists of partial orders and represents a uniform distribution of its completions. What follows are its special cases.

A partitioned voting profile, denoted by P^{PP} , records preferences in the form of partitions: linear orders of item buckets, with no preference among the items in the same bucket.

A partial chain voting profile, denoted by P^{PC} , records preferences in the form of partial chains. A partial chain is a linear order of a subset of items, and there is no preference specified with regard to the remaining items.

A truncated voting profile, denoted by P^{TR} , records preferences in the form of truncated rankings. Let $\tau^{(t,b)}$ denote a truncated ranking with t items at the top and b items at the bottom. No preference information is specified for the middle part of the ranking. The $\tau^{(t,b)}$ is a special case of the partitioned preferences, with t+b+1 partitions.

Combined voting profiles. Most research works have assumed that a partial voting profile represents a uniform distribution over its completions. However, the assumption that all completions are equally likely may not hold in practice. We propose *combined voting profiles* P^{M+P} , where each voter is associated with both her original incomplete preferences P and a ranking model M. The ranking model M is her prior preference distribution, e.g., obtained from historical data. But after observing new evidence P, the probabilities of

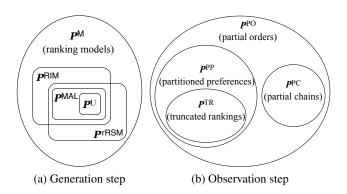


Figure 1: The Venn diagrams of voting profiles. The voter preferences in their mind are the ranking models in the generation step. Then we observe incomplete voting profiles that are essentially samples of the probabilistic voting profiles.

rankings that violate P collapse to zero, while the rest rankings that satisfy P have the same relative probabilities among each other. Formally speaking, the preferences of this voter become the posterior distribution of M conditioned on P.

In another sense, the combined voting profiles are also the most general form of voting profiles (same as $\boldsymbol{P}^{\text{M}}$) that unify all voting profiles so far. For example, a partitioned voting profile $\boldsymbol{P}^{\text{PP}}$ is essentially $\boldsymbol{P}^{\text{U+PP}}$, and a RIM voting profile $\boldsymbol{P}^{\text{RIM}}$ is essentially $\boldsymbol{P}^{\text{RIM}+\emptyset}$ where \emptyset means that the partial orders are the empty ones.

Among the tractable cases in Section 5, there are also combined voting profiles, which means that applying combined voting profiles not only gains theoretical benefits of more customized ranking distributions, but also becomes practical for certain combination of profiles.

3 Most Expected Winner

Given a general voting profile P^{M} and a positional scoring rule r_m , the performance of a candidate can be quantified by the expectation of her score in a random possible world.

Definition 1 (MEW). Given a voting profile P^{M} and a positional scoring rule r_m , candidate w is a Most Expected Winner, if and only if, $\mathbb{E}(s(w, P^{M})) = \max_{c \in C} \mathbb{E}(s(c, P^{M}))$.

Let $MEW(r_m, P^M)$ be the set of Most Expected Winners.

3.1 Alternative Interpretations

To gain an intuition for Most Expected Winner (MEW), we will now give two winner definitions that are equivalent to MEW. (Note that the proofs of all theorems in this section are presented in the Appendix.)

Least Expected Regret Winner

The MEW can also be regarded as the candidate who minimizes the expected regret in a random possible world. The concept of *regret* is borrowed from [Lu and Boutilier, 2011]. Let Regret(w, P) denote the regret value of choosing $w \in C$ as the winner given a complete voting profile P.

$$\operatorname{Regret}(w, \boldsymbol{P}) = \max_{c \in C} s(c, \boldsymbol{P}) - s(w, \boldsymbol{P})$$

Accordingly, the regret value Regret $(w, \mathbf{P}^{\mathsf{M}})$ over a probabilistic voting profile becomes a random variable.

$$\mathbb{E}(\operatorname{Regret}(w, \boldsymbol{P}^{\mathsf{M}})) = \sum_{\boldsymbol{P} \in \Omega(\boldsymbol{P}^{\mathsf{M}})} \operatorname{Regret}(c, \boldsymbol{P}) \cdot \Pr(\boldsymbol{P} \mid \boldsymbol{P}^{\mathsf{M}})$$

Candidate w is a Least Expected Regret Winner, if and only if, she minimizes the expected regret $\mathbb{E}(\text{Regret}(w, \mathbf{P}^{\mathsf{M}}))$.

Theorem 1. The MEW and the Least Expected Regret Winner are equivalent.

Meta-Election Winner

Recall that a voting profile P^{M} represents a probability distribution of possible worlds $\Omega(P^{M}) = \{P_1, \dots, P_z\}$. The Meta-Election Winner is defined as the candidate who wins a meta election with a large meta profile $P_M = (P_1, \dots, P_z)$ where rankings in P_i are weighted by $Pr(P_i \mid P^{M})$.

Theorem 2. The MEW and the Meta-Election Winner are equivalent.

3.2 **Comparing MEW and MPW**

The difference between MEW and MPW can be interpreted as different aggregation approaches across possible worlds.

Recall that $\Omega(P^{\mathsf{M}}) = \{P_1, \dots, P_z\}$ is the set of possible worlds of P^{M} , each possible world P_i is associated with a probability $p_i = \Pr(P_i \mid P^{\mathsf{M}})$ and $\sum_{i=1}^z p_i = 1$. Now let's see how the performance of a candidate c is aggregated across possible worlds. Let $\mathbb{1}()$ be the indicator function.

• MEW:
$$\mathbb{E}(s(c, \mathbf{P}^{\mathsf{M}})) = \sum_{i=1}^{z} s(c, \mathbf{P}_i) \cdot p_i$$

• MEW:
$$\mathbb{E}(s(c, \mathbf{P}^{\mathsf{M}})) = \sum_{i=1}^{z} s(c, \mathbf{P}_i) \cdot p_i$$

• MPW: $\Pr(c \text{ wins}) = \sum_{i=1}^{z} \mathbb{1}(c \text{ wins} \mid \mathbf{P}_i) \cdot p_i$

MEW estimates the average performance of a candidate, while MPW estimates the probability that she wins. As a result, MPW ignores the possible worlds if she cannot win.

Problem Statement

The MEW is determined based on the expected performance of the candidates. So the winner determination problem of MEW can be reduced to the problem of calculating the expected scores of candidates. That is, given a voting profile P^{M} , a positional scoring rule r_m , and a candidate $c \in C$, calculate $\mathbb{E}(s(c, \mathbf{P}^{\mathsf{M}}))$ the expected score of c. We name this problem Expected Score Computation (ESC).

Theorem 3. The problem of determining MEW can be reduced to the ESC problem.

Hardness of ESC

This section investigates the complexity of the ESC problem. We first prove the hardness of two closely related problems, the Fixed-rank Counting Problem and the Rank Estimation Problem. Then we demonstrate that the ESC problem is hard as well. The proofs to the theorems and lemmas in this section are presented in the Appendix.

4.1 Fixed-rank Counting Problem

Counting the number of linear extensions of a partial order is well-known to be #P-complete [Brightwell and Winkler, 1991]. The Fixed-rank Counting Problem (FCP) is interested in the number of linear extensions where an item is placed at a specific rank. Let $\Omega(\nu)$ denote the set of linear extensions of ν , and $N(c@j \mid \nu)$ denote the number of linear extensions in $\Omega(\nu)$ where item c is placed at rank j.

Definition 2 (FCP). Given a partial order ν over m items, an item c, and an integer $j \in [1, m]$, calculate $N(c@j \mid \nu)$, the number of linear extensions of ν where item c is placed at rank j.

[Loof, 2009] discussed this problem in his doctoral dissertation (Section 4.2.1). He proposed both exact and approximate algorithms, but did not provide a proof of complexity.

Theorem 4. The FCP is #P-complete.

The hardness of FCP facilitates the hardness proofs for the Rank Estimation Problems.

4.2 Rank Estimation Problem

Now we move on to the Rank Estimation Problem (REP). This problem can be regarded as the probabilistic version of the FCP. It calculates the probability that a given item is placed at a specific rank. But the REP is generalized from partial orders to arbitrary ranking distributions.

Definition 3 (REP). Given a ranking model M over m items, an item c, and an integer $j \in [1, m]$, calculate $Pr(c@j \mid M)$, the probability that item c is placed at rank j, by M.

For the convenience of discussions in the rest of this Section, we also define two special cases of the REP where items are placed at the top and bottom of the linear extensions.

Definition 4 (REP-t). Given a ranking model M of m items and an item c, calculate $Pr(c@1 \mid M)$, the probability that item c is placed at the top of a linear extension, by M.

Definition 5 (REP-b). Given a ranking model M of m items and an item c, calculate $Pr(c@m \mid M)$, the probability that item c is placed at the bottom of a linear extension, by M.

[Lerche and Sørensen, 2003] proposed an approximation for the REP over partial orders, but did not provide a formal complexity proof. [Bruggemann and Annoni, 2014] and [De Loof et al., 2011] also worked on a related problem, calculating the expected rank of an item in the linear extensions of a partial order. These works focused on approximation, lacking complexity proofs as well.

With the help of Theorem 4, it turns out that the REP and its two variants are all $FP^{\#P}$ -complete over partial orders.

Lemma 1. If the ranking model M is a partial order ν that represents a uniform distribution of $\Omega(\nu)$, the REP-t is $FP^{\#P}$ -complete.

Lemma 2. If the ranking model M is a partial order ν of m items that represents a uniform distribution of $\Omega(\nu)$, the REP-b is $FP^{\#P}$ -complete.

Theorem 5. If the ranking model M is a partial order ν that represents a uniform distribution of $\Omega(\nu)$, the Rank Estimation Problem is $FP^{\#P}$ -complete.

4.3 Complexity of ESC

The ESC problem is closely related to REP. First of all, ESC is no harder than REP over general voting profiles (Theorem 6), which lays the foundation of the identification of tractable cases in Section 5.

Theorem 6. Given a voting profile P^{M} and a positional scoring rule r_m , the ESC problem can be reduced to the REP.

Theorem 7. The REP is equivalent to the ESC problem over a collection of k-approval rules where $k = 0, \ldots, m$.

Theorem 7 provides an insight into the relation between REP and ESC in terms of computational complexity. There may be cases where the ESC problem is more tractable than the REP problem. But if a solver is available for the ESC problem over a collection of k-approval votes, this solver is computationally equivalent to the REP solver. Note that Theorem 7 is not limited to partial voting profiles.

Theorem 8. Given a partial voting profile P^{PO} , a distinguished candidate c, and plurality rule r_m , the ESC problem of calculating $\mathbb{E}(s(c \mid P^{PO}, r_m))$ is $FP^{\#P}$ -complete.

Theorem 9. Given a partial voting profile P^{PO} , a distinguished candidate c, and veto rule r_m , the ESC problem of calculating $\mathbb{E}(s(c \mid P^{PO}, r_m))$ is $FP^{\#P}$ -complete.

Theorem 10. Given a partial voting profile P^{PO} , a distinguished candidate c, and k-approval rule r_m , the ESC problem of calculating $\mathbb{E}(s(c \mid P^{PO}, r_m))$ is $FP^{\#P}$ -complete.

The three theorems above demonstrate the hardness of the ESC problem over partial voting profiles, under plurality, veto, and k-approval, respectively. In particular, ESC is $FP^{\#P}$ -complete even under plurality (Theorem 8).

5 Tractable Cases

The problem of determining MEW can be reduced to the ESC problem by Theorem 3, then further reduced to the REP by Theorem 6. This section will solve the MEW problem for general positional scoring rules for a variety of voting profiles by solving the REP.

5.1 Incomplete Voting Profiles

Section 4 has proved that calculating the expected scores of candidates given partial voting profiles is hard, even under plurality rule. But it turns out that the MEW problem has efficient algorithms for all special cases of partial voting profiles in Figure 1b, including partitioned voting profiles (covering truncated voting profiles) and partial chain voting profiles.

Theorem 11. Given positional scoring rule r_m , a partitioned voting profile $P^{PP} = (\nu_1^{PP}, \dots, \nu_n^{PP})$, and candidate w, determining $w \in MEW(r_m, P^{PP})$ is in O(mn).

Proof. Any $\nu^{PP} \in P^{PP}$ defines a set of consecutive ranks in the linear extensions of ν^{PP} for each of its partitions of candidates. Any candidate is equally likely to be positioned at these consecutive ranks. So the REP can be solved in O(1) for any candidate. Thus, the MEW problem can be solved in $O(nm^2)$ by calculating the expected scores of all candidates.

Algorithm 1 REP solver for RIM

```
Input: Item c, RIM(\sigma, \Pi) where |\sigma| = m
Output: \{k \to \Pr(c@k \mid \boldsymbol{\sigma}, \Pi) \mid k \in [1, m]\}
 1: \delta_0 := \emptyset, \mathcal{P}_0 := \{\delta_0\} and q_0(\delta_0) := 1
 2: for i = 1, ..., m do
         \mathcal{P}_i := \{\}
 3:
 4:
         for \delta \in \mathcal{P}_{i-1} do
 5:
             for j = 1, \ldots, i do
 6:
                 if \sigma(i) is c then
 7:
                     \delta' := \{c \to j\}
 8:
                 else if \delta = \{c \to k\} and j \le k then
                     \delta' := \{c \rightarrow k+1\}
 9:
                 else
10:
                    \delta' := \delta
11:
12:
                 end if
                 \mathcal{P}_i.add(\delta')
13:
14:
                 q_i(\delta') += q_{i-1}(\delta) \cdot \Pi(i,j)
15:
             end for
16:
          end for
17: end for
18: Obtain \{k \to \Pr(c@k \mid \boldsymbol{\sigma}, \Pi) \mid k \in [1, m]\} from q_m.
19: return \{k \to \Pr(c@k \mid \boldsymbol{\sigma}, \Pi) \mid k \in [1, m]\}
```

Theorem 12. Given positional scoring rule r_m , a partial chain voting profile $\mathbf{P}^{PC} = (\boldsymbol{\nu}_1^{PC}, \dots, \boldsymbol{\nu}_n^{PC})$, and candidate w, determining $w \in MEW(r_m, \mathbf{P}^{PC})$ is in $O(nm^2)$.

Proof. For any $\boldsymbol{\nu}^{\text{PC}} \in \boldsymbol{P}^{\text{PC}}$ and any candidate c, the $Pr(c@j \mid \boldsymbol{\nu}^{\text{PC}})$ is proportional to the degree of freedom to place the rest of the candidates, after fixing c at rank j.

- If $c \notin \boldsymbol{\nu}^{\text{PC}}$, $Pr(c@j \mid \boldsymbol{\nu}_i^{\text{PC}}) \propto \binom{m-1}{K} \cdot (m-1-K)!$ where $K = |\boldsymbol{\nu}_i^{\text{PC}}|$.
- If $c \in \boldsymbol{\nu}^{\operatorname{PC}}$, $Pr(c@j \mid \boldsymbol{\nu}^{\operatorname{PC}}) \propto \binom{j-1}{K_l} \cdot \binom{m-j}{K_r} \cdot (m-K)!$ where $K_l = |\{c' \mid c' \succ_{\boldsymbol{\nu}^{\operatorname{PC}}} c\}|, K_r = |\{c' \mid c \succ_{\boldsymbol{\nu}^{\operatorname{PC}}} c'\}|,$ and $K = |\boldsymbol{\nu}_i^{\operatorname{PC}}|.$

It takes $O(nm^2)$ to obtain the expected scores of all candidates and to determine whether w is a MEW.

5.2 Probabilistic Voting Profiles

While the problem of MEW determination is hard in general, it is tractable over specific ranking models such as the Mallows [Mallows, 1957] and its generalizations RIM [Doignon *et al.*, 2004] and RSM [Chakraborty *et al.*, 2020].

RIM, denoted by RIM(σ , Π), is a generative model that generalizes not only Mallows, but also the *Generalized Mallows* [Fligner and Verducci, 1986] and *multistage ranking models* [Fligner and Verducci, 1988]. It is parameterized by a reference ranking σ and a probability function Π where $\Pi(i,j)$ is the probability of inserting the i^{th} item $\sigma(i)$ at the j^{th} position $(1 \le j \le i)$ during ranking construction. The insert items in order of σ where $\sigma(i)$ is placed at the j^{th} position of τ with probability $\Pi(i,j)$.

Example 1. RIM($\langle a, b, c \rangle, \Pi$) generates $\tau = \langle c, a, b \rangle$ as follows. Initialize $\tau_0 = \langle \rangle$. When i = 1, $\tau_1 = \langle a \rangle$ by inserting a

into τ_0 with probability $\Pi(1,1)$. When i=2, $\tau_2=\langle a,b\rangle$ by inserting b into τ_1 at position 2 with probability $\Pi(2,2)$. When i=3, $\tau=\langle c,a,b\rangle$ by inserting c into τ_2 at position 1 with probability $\Pi(3,1)$. Overall, $\Pr(\tau \mid \langle a,b,c\rangle,\Pi)=\Pi(1,1) \cdot \Pi(2,2) \cdot \Pi(3,1)$.

Theorem 13. Given positional scoring rule r_m , a RIM voting profile $\mathbf{P}^{\mathsf{RIM}} = (\mathsf{RIM}_1, \dots, \mathsf{RIM}_n)$, and candidate w, determining $w \in \mathit{MEW}(r_m, \mathbf{P}^{\mathsf{RIM}})$ is in $O(nm^4)$.

Proof. Given any RIM ∈ P^{RIM} and any candidate c, the $Pr(c@j \mid \text{RIM})$ for $j=1,\ldots,m$ can be calculated by Algorithm 1 in $O(m^3)$. Algorithm 1 is a variant of RIMDP [Kenig *et al.*, 2018]. RIMDP calculates the marginal probability of a partial order over RIM via Dynamic Programming (DP). Algorithm 1 is simplified RIMDP in a sense that Algorithm 1 only tracks a particular item c, while RIMDP tracks multiple items to calculate the insertion ranges of items that satisfy the partial order. Note that Algorithm 1 calculates all m different values of j simultaneously. So it takes $O(nm \cdot m^3) = O(nm^4)$ to obtain the expected scores of m candidates over n RIMs to determine the MEW.

The complexity of MEW determination over RSM voting profiles is $O(nm^4)$ as well. Due to space restrictions, please refer to Section 8.2 in Appendix for more details.

5.3 Combined Voting Profiles

It is usually harder to compute the expected scores over combined voting profiles. Below are some cases where this problem is tractable. The first case is the RIMs combined with truncated rankings.

Theorem 14. Given positional scoring rule r_m , a voting profile $\mathbf{P}^{\mathsf{RIM}+TR} = \left((\mathsf{RIM}_1, \tau_1^{(t_1,b_1)}), \dots, (\mathsf{RIM}_n, \tau_n^{(t_n,b_n)}) \right)$, and candidate w, determining $w \in \mathit{MEW}(r_m, \mathbf{P}^{\mathsf{RIM}+TR})$ is in $O(nm^4)$.

Proof. Given any $(\mathsf{RIM}, \boldsymbol{\tau}^{(t,b)}) \in \boldsymbol{P}^{\mathsf{RIM}+\mathsf{TR}}$, candidate c, and rank j, if c is in the top or bottom part of $\boldsymbol{\tau}^{(t,b)}$, its rank has been fixed, which is a trivial case; If c is in the middle part of $\boldsymbol{\tau}^{(t,b)}$, we just need to slightly modify Algorithm 1 to calculate $\Pr(c@j \mid \mathsf{RIM}, \boldsymbol{\tau}^{(t,b)})$. Line 5 in Algorithm 1 enumerates values for j from 1 to i. The constraints made by $\boldsymbol{\tau}^{(t,b)}$ limits this insertion range of item $\boldsymbol{\sigma}(i)$. If $\boldsymbol{\sigma}(i)$ is in the top or bottom part of $\boldsymbol{\tau}^{(t,b)}$, its insertion position has been fixed by $\boldsymbol{\tau}^{(t,b)}$ and the inserted items of the top and bottom parts of $\boldsymbol{\tau}^{(t,b)}$ should be recorded as well by the state δ' ; If $\boldsymbol{\sigma}(i)$ is in the middle part of $\boldsymbol{\tau}^{(t,b)}$, $\boldsymbol{\sigma}(i)$ can be inserted into any position between the inserted top and bottom items.

Theoretically, the algorithm needs to track as many as (t+b+1) items. But the (t+b) items are fixed, which makes c the only item leading to multiple states during DP. The complexity of calculating $\Pr(c@j \mid \mathsf{RIM}, \tau^{(t,b)})$ for all j values is $O(m^3)$. It takes $O(nm^4)$ to calculate the expected scores of all candidates across all voters to determine the MEW. \square

Another tractable case is the Mallows combined with partitioned preferences. Let MAL (σ, ϕ) where $0 \le \phi \le 1$ denote a Mallows model [Mallows, 1957]. It defines a probability distribution of rankings: reference ranking σ at the

Profile (equiv.)	Complexity
$oldsymbol{P}^{ ext{U+PP}}$	$O(nm^2)$
$oldsymbol{P}^{ ext{U+PC}}$	$O(nm^2)$
$oldsymbol{P}^{RIM+\emptyset}$	$O(nm^4)$
$oldsymbol{P}^{rRSM+\emptyset}$	$O(nm^4)$
$oldsymbol{P}^{RIM+TR}$	$O(nm^4)$
$oldsymbol{P}^{MAL+PP}$	$O(nm^4)$
	$P^{ ext{U+PP}}$ $P^{ ext{U+PC}}$ $P^{ ext{RIM}+\emptyset}$ $P^{ ext{RSM}+\emptyset}$ $P^{ ext{RIM+TR}}$

Table 3: Tractability results of MEW under general positional scoring rule for various voting profiles, including partitioned preferences (PP), partial chains (PC), RIMs, rRSMs, and combined voting profiles. These profiles are also rewritten in the format of combined voting profiles.

center and other rankings closer to σ having higher probabilities. For a given ranking τ , $\Pr(\tau|\sigma,\phi) \propto \phi^{D(\sigma,\tau)}$ where $D(\sigma,\tau) = |(a,a')|a \succ_{\sigma} a',a' \succ_{\tau} a|$ is the Kendall-tau distance counting the number of disagreed preference pairs. If $\phi=1$, the Mallows becomes to a uniform distribution. The $\mathsf{MAL}(\sigma,\phi)$ is a special case for both $\mathsf{RIM}(\sigma,\Pi)$ by $\Pi(i,j) = \frac{\phi^{i-j}}{1+\phi+\ldots+\phi^{i-1}}$ and $\mathsf{rRSM}(\sigma,\Pi)$ by $\Pi(i,j) = \frac{\phi^{j-1}}{1+\phi+\ldots+\phi^{m-i}}$.

Theorem 15. Given positional scoring rule r_m , a voting profile $P^{\mathsf{MAL}+PP} = ((\mathsf{MAL}_1, \boldsymbol{\nu}_1^{PP}), \dots, (\mathsf{MAL}_n, \boldsymbol{\nu}_n^{PP}))$, and candidate w, determining $w \in \mathit{MEW}(r_m, P^{\mathsf{MAL}+PP})$ is in $O(nm^4)$.

Proof. Given any $(\mathsf{MAL}(\sigma,\phi), \boldsymbol{\nu}^{\mathsf{PP}}) \in \boldsymbol{P}^{\mathsf{MAL+PP}}$, candidate c, and rank j, consider calculating $Pr(c@j \mid \sigma,\phi,\boldsymbol{\nu}^{\mathsf{PP}})$. Let C_P denote the set of candidates in the same partition with c in $\boldsymbol{\nu}^{\mathsf{PP}}$. The relative orders between c and items out of C_P are already determined by $\boldsymbol{\nu}^{\mathsf{PP}}$. That is to say, for a nontrivial j value, $Pr(c@j \mid \sigma,\phi,\boldsymbol{\nu}^{\mathsf{PP}})$ is proportional to the exponential of the number of disagreed pairs within C_P . So we can construct a new Mallows model $\mathsf{MAL}'(\sigma',\phi)$ over C_P . It has the same ϕ as MAL and its reference ranking σ' is shorter than but consistent with σ . The $Pr(c@j \mid \mathsf{MAL}',\boldsymbol{\nu}^{\mathsf{PP}})$ for all non-trivial j values can be calculated in $O(|C_P|^3) < O(m^3)$ by Algorithm 1.

The MEW problem can be solved in $O(nm^4)$ by calculating the expected scores of all candidates across all voters to determine whether w is a MEW.

5.4 Summary

Table 3 summarizes the tractable cases. Combining this table with Figure 1, we can obtain more conclusions for a large number of specialized voting profiles. For example, the Mallows voting profile P^{MAL} is not listed in the Table, but its MEW complexity is bounded by $O(nm^4)$, since it is a special case of P^{RIM} . An interesting observation is that the MEW complexity over $P^{\text{RIM+PP}}$ is not tractable, but its special case $P^{\text{MAL+PP}}$ has a polynomial complexity. Although this table demonstrates that evaluating MEW over probabilistic voting profiles has higher complexities than that over incomplete voting profiles, e.g., $O(nm^4)$ for $P^{\text{MAL+PP}}$ but only $O(nm^2)$

for P^{PP} , the tractability results over a collection of probabilistic and combined voting profiles still give the MEW a computation advantage as a feasible option in practice.

6 Concluding Remarks

We embarked on two tasks: 1) modeling the uncertainty in voter preferences and 2) determining the winners accordingly. Distinguishing between uncertainties in preference generation and preference observation provides a powerful framework to describe and unify incomplete and probabilistic voting profiles. Then we proposed the Most Expected Winner for positional scoring rules, established its theoretical hardness, and identified a collection of tractable cases.

Our work can be used as a starting point for future studies. For example, the hardness of ESC is proved over only plurality, veto, and k-approval, which calls for investigation over other positional scoring rules such as Borda. When the MEW is intractable, it is often necessary to develop approximate solvers such as the AMP sampler for Mallows posteriors over partial orders [Lu and Boutilier, 2014]. The MEW can also be extended to other score-based rules, such as the Simpson rule and the Copeland rule.

Another direction is to consider voter preferences represented by additional ranking models [Marden, 1995], such as the Plackett-Luce (PL) model [Luce, 1959; Plackett, 1975]. [Noothigattu *et al.*, 2018; Zhao *et al.*, 2018] have studied preference aggregation over PL models, which is closely related to the MEW over voting profiles of PL models, and we plan to investigate this connection in the future.

7 Acknowledgements

This work was supported in part by NSF Grants No. 1916647 and 1916505.

References

- [Bachrach et al., 2010] Yoram Bachrach, Nadja Betzler, and Piotr Faliszewski. Probabilistic possible winner determination. In Proceedings of the Twenty-Fourth AAAI Conference on Artificial Intelligence, AAAI 2010, Atlanta, Georgia, USA, July 11-15, 2010, 2010.
- [Brightwell and Winkler, 1991] Graham R. Brightwell and Peter Winkler. Counting linear extensions is #p-complete. In *Proceedings of the 23rd Annual ACM Symposium on Theory of Computing, May 5-8, 1991, New Orleans, Louisiana, USA*, pages 175–181, 1991.
- [Bruggemann and Annoni, 2014] Rainer Bruggemann and Paola Annoni. Average heights in partially ordered sets. *MATCH Commun Math Comput Chem*, 71:117–142, 2014.
- [Chakraborty *et al.*, 2020] Vishal Chakraborty, Theo Delemazure, Benny Kimelfeld, Phokion G. Kolaitis, Kunal Relia, and Julia Stoyanovich. Algorithmic techniques for necessary and possible winners. *CoRR*, abs/2005.06779, 2020.

- [De Loof *et al.*, 2011] Karel De Loof, Bernard De Baets, and Hans De Meyer. Approximation of average ranks in posets. *Match Commun Math Comput Chem*, 66:219–229, 2011.
- [Doignon *et al.*, 2004] Jean-Paul Doignon, Aleksandar Pekeč, and Michel Regenwetter. The repeated insertion model for rankings: Missing link between two subset choice models. *Psychometrika*, 69(1):33–54, 2004.
- [Fligner and Verducci, 1986] Michael A Fligner and Joseph S Verducci. Distance based ranking models. *Journal of the Royal Statistical Society: Series B* (*Methodological*), 48(3):359–369, 1986.
- [Fligner and Verducci, 1988] Michael A Fligner and Joseph S Verducci. Multistage ranking models. *Journal of the American Statistical association*, 83(403):892–901, 1988.
- [Hazon *et al.*, 2012] Noam Hazon, Yonatan Aumann, Sarit Kraus, and Michael J. Wooldridge. On the evaluation of election outcomes under uncertainty. *Artif. Intell.*, 189:1–18, 2012.
- [Kenig et al., 2018] Batya Kenig, Lovro Ilijasic, Haoyue Ping, Benny Kimelfeld, and Julia Stoyanovich. Probabilistic inference over repeated insertion models. In Proceedings of the Thirty-Second AAAI Conference on Artificial Intelligence, (AAAI-18), New Orleans, Louisiana, USA, February 2-7, 2018, pages 1897–1904, 2018.
- [Konczak and Lang, 2005] Kathrin Konczak and Jérôme Lang. Voting procedures with incomplete preferences. In *Proc. IJCAI-05 Multidisciplinary Workshop on Advances in Preference Handling*, volume 20, 2005.
- [Lerche and Sørensen, 2003] Dorte Lerche and Peter B Sørensen. Evaluation of the ranking probabilities for partial orders based on random linear extensions. *Chemosphere*, 53(8):981–992, 2003.
- [Loof, 2009] Karel De Loof. *Efficient computation of rank probabilities in posets*. PhD thesis, Ghent University, 2009.
- [Lu and Boutilier, 2011] Tyler Lu and Craig Boutilier. Robust approximation and incremental elicitation in voting protocols. In *IJCAI 2011, Proceedings of the 22nd International Joint Conference on Artificial Intelligence, Barcelona, Catalonia, Spain, July 16-22, 2011*, pages 287–293, 2011.
- [Lu and Boutilier, 2014] Tyler Lu and Craig Boutilier. Effective sampling and learning for mallows models with pairwise-preference data. *J. Mach. Learn. Res.*, 15(1):3783–3829, 2014.
- [Luce, 1959] R.D. Luce. *Individual Choice Behavior: A Theoretical Analysis*. Wiley, 1959.
- [Mallows, 1957] C. L. Mallows. Non-null ranking models. *Biometrika*, 44:114–130, 1957.
- [Marden, 1995] John I Marden. Analyzing and modeling rank data. CRC Press, 1995.

[Noothigattu et al., 2018] Ritesh Noothigattu, Snehalkumar (Neil) S. Gaikwad, Edmond Awad, Sohan Dsouza, Iyad Rahwan, Pradeep Ravikumar, and Ariel D. Procaccia. A voting-based system for ethical decision making. In Proceedings of AAAI, New Orleans, Louisiana, USA, pages 1587–1594, 2018.

[Plackett, 1975] Robin L Plackett. The analysis of permutations. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, 24(2):193–202, 1975.

[Zhao et al., 2018] Zhibing Zhao, Haoming Li, Junming Wang, Jeffrey O. Kephart, Nicholas Mattei, Hui Su, and Lirong Xia. A cost-effective framework for preference elicitation and aggregation. In *Proceedings of UAI, Mon*terey, California, USA, pages 446–456, 2018.

8 Appendix

8.1 Additional proofs

THEOREM 1. The MEW and the Least Expected Regret Winner are equivalent.

 ${\it Proof.}$ The expected regret of a candidate w can be rewritten as follows.

$$\begin{split} & \mathbb{E}(\text{Regret}(w, \boldsymbol{P}^{\text{M}})) \\ &= \sum_{i=1}^{z} \text{Regret}(c, \boldsymbol{P}_{i}) \cdot \text{Pr}(\boldsymbol{P}_{i} \mid \boldsymbol{P}^{\text{M}}) \\ &= \sum_{i=1}^{z} \left(\max_{c \in C} s(c, \boldsymbol{P}_{i}) - s(w, \boldsymbol{P}_{i}) \right) \cdot \text{Pr}(\boldsymbol{P}_{i} \mid \boldsymbol{P}^{\text{M}}) \\ &= \sum_{i=1}^{z} \max_{c \in C} s(c, \boldsymbol{P}_{i}) \cdot \text{Pr}(\boldsymbol{P}_{i} \mid \boldsymbol{P}^{\text{M}}) - \mathbb{E}(s(w, \boldsymbol{P}^{\text{M}})) \end{split}$$

The first term $\sum_{i=1}^{z} \max_{c \in C} s(c, P_i) \cdot \Pr(P_i \mid P^{\mathsf{M}})$ is a constant value, when P^{M} and the voting rule are fixed. Thus, $\mathbb{E}(\operatorname{Regret}(w, P^{\mathsf{M}}))$ is minimized by maximizing $\mathbb{E}(s(w, P^{\mathsf{M}}))$, the expected score of the candidate w.

THEOREM 2. The MEW and the Meta-Election Winner are equivalent.

Proof. Let P^{M} denote a general voting profile with $\Omega(P^{\mathsf{M}}) = \{P_1, \dots, P_z\}$, and $P_M = (P_1, \dots, P_z)$ denote the large meta profile where rankings in P_i are weighted by $\Pr(P_i \mid P^{\mathsf{M}})$. According the definition of the Meta-Election Winner, $s(w, P_M) = \max_{c \in C} s(c, P_M)$. As a result, for any candidate c,

$$\mathbb{E}(s(c,\boldsymbol{P}^{\mathsf{M}})) = \sum_{\boldsymbol{P} \in \Omega(\boldsymbol{P}^{\mathsf{M}})} s(c,\boldsymbol{P}) \cdot \Pr(\boldsymbol{P} \mid \boldsymbol{P}^{\mathsf{M}}) = s(c,\boldsymbol{P}_{\!M})$$

Her expected score in P^{M} are precisely her score in P_{M} . The two winner definitions are optimizing the same metric.

THEOREM 3. The problem of determining MEW can be reduced to the ESC problem.

Proof. By solving the ESC problem m times for m candidates, the expected scores of all candidates are available to further determine the MEW.

THEOREM 4. The FCP is #P-complete.

Proof. First, we prove its membership in #P. The FCP is the counting version the following decision problem: given a partial order ν , an item c, and an integer j, determine whether ν has a linear extension $\tau \in \Omega(\nu)$ where c is ranked at j. This decision problem is obviously in NP, meaning that the FCP is in #P.

Then, we prove that the FCP is #P-hard by reduction. Recall that counting $|\Omega(\nu)|$, the number of linear extensions of a partial order ν , is #P-complete [Brightwell and Winkler, 1991]. This problem can be reduced to the FCP by $|\Omega(\nu)| = \sum_{j=1}^m N(c@j \mid \nu)$.

In conclusion, the FCP is #P-complete. \Box

LEMMA 1. If the ranking model M is a partial order ν that represents a uniform distribution of $\Omega(\nu)$, the REP-t is $FP^{\#P}$ -complete.

Proof. First, we prove that the REP-t is in FP^{#P}. Recall that $\Omega(\nu)$ is the linear extensions of a partial order ν , and $N(c@1 \mid \nu)$ is the number of linear extensions in $\Omega(\nu)$ where candidate c is at rank 1. Then $\Pr(c@1 \mid \nu) = N(c@1 \mid \nu)/|\Omega(\nu)|$. Consider that counting $N(c@1 \mid \nu)$ is in #P (Theorem 4) and counting $|\Omega(\nu)|$ is #P-complete [Brightwell and Winkler, 1991], so $\Pr(c@1 \mid \nu)$ is in FP^{#P}.

In the rest of this proof, we prove that the REP-t is #P-hard by reduction from the #P-complete problem of counting $|\Omega(\nu)|$.

Let c^* denote an item that has no parent in ν . Let ν_{-c^*} denote the partial order of ν with item c^* removed. If we are interested in the probability that c^* is placed at rank 1, we can write $\Pr(c^*@1 \mid \nu) = N(c^*@1 \mid \nu)/|\Omega(\nu)|$. The item c^* has been fixed at rank 1, so any placement of the rest items will definitely satisfy any relative order involving c^* . That is to say, the placement of the rest items just need to satisfy ν_{-c^*} , which leads to $N(c^*@1 \mid \nu) = |\Omega(\nu_{-c^*})|$.

For example, let $\nu' = \{c_1 \succ c_4, c_2 \succ c_4, c_3 \succ c_4\}$. Then $N(c_1@1 \mid \nu') = |\Omega(\nu'_{-c_1})| = |\Omega(\{c_2 \succ c_4, c_3 \succ c_4\})|$. Then we re-write $\Pr(c^*@1 \mid \nu) = N(c^*@1 \mid \nu)/|\Omega(\nu)| = N(c^*@1 \mid \nu)$

Then we re-write $\Pr(c^*@1 \mid \boldsymbol{\nu}) = N(c^*@1 \mid \boldsymbol{\nu})/|\Omega(\boldsymbol{\nu})| = |\Omega(\boldsymbol{\nu}_{-c^*})|/|\Omega(\boldsymbol{\nu})|$. The oracle for $\Pr(c^*@1 \mid \boldsymbol{\nu})$ manages to reduce the size of the counting problem from $|\Omega(\boldsymbol{\nu})|$ to $|\Omega(\boldsymbol{\nu}_{-c^*})|$. This oracle should be as hard as counting $|\Omega(\boldsymbol{\nu})|$. Thus calculating $\Pr(c^*@1 \mid \boldsymbol{\nu})$ is $\Pr^{\#P}$ -hard.

In conclusion, the REP-t is $P^{\#P}$ -complete.

LEMMA 2. If the ranking model M is a partial order ν of m items that represents a uniform distribution of $\Omega(\nu)$, the REP-b is $FP^{\#P}$ -complete.

Proof. This proof adopts the same approach as the proof of Lemma 1.

For the membership proof that the REP-b is in $FP^{\#P}$. let $N(c@m \mid \nu)$ denote the number of linear extensions in $\Omega(\nu)$ where candidate c is at the bottom rank m. Then $\Pr(c@m \mid \boldsymbol{\nu}) = N(c@m \mid \boldsymbol{\nu})/|\Omega(\boldsymbol{\nu})|$. Consider that counting $N(c@m \mid \nu)$ is in #P (Theorem 4) and counting $|\Omega(\nu)|$ is #P-complete [Brightwell and Winkler, 1991], so $\Pr(c@m \mid \boldsymbol{\nu})$ is in $FP^{\#P}$.

In the proof of Lemma 1, item c^* is an item with no parent in the partial order ν . In the current proof, item c^* is set to be an item with no child in ν . The ν_{-c^*} still denotes the partial order of ν but with item c^* removed. Then the probability that item c^* at the bottom rank m is $\Pr(c^*@m \mid \nu) = N(c^*@m)$ $|\nu\rangle/|\Omega(\nu)| = |\Omega(\nu_{-c^*})|/|\Omega(\nu)|$. The oracle for $\Pr(c^*@m)$ ν) manages to reduce the size of the counting problem again from $|\Omega(\nu)|$ to $|\Omega(\nu_{-c^*})|$. Thus, this oracle is #P-hard, and calculating $\Pr(c^*@m\mid \nu)$ is $\operatorname{FP}^{\#P}$ -hard.

In conclusion, the REP-b is $FP^{\#P}$ -complete.

THEOREM 5. If the ranking model M is a partial order ν that represents a uniform distribution of $\Omega(\nu)$, the Rank Estimation Problem is $\tilde{F}P^{\#P}$ -complete.

Proof. First, we prove that the REP is in $FP^{\#P}$. Recall that $\Omega(\nu)$ is the linear extensions of a partial order ν , and $N(c@j \mid \boldsymbol{\nu})$ is the number of linear extensions in $\Omega(\boldsymbol{\nu})$ where candidate c is at rank j. Then $Pr(c@j \mid \nu) =$ $N(c@j \mid \nu)/|\Omega(\nu)|$. Consider that counting $N(c@j \mid$ ν) is #P-complete (Theorem 4) and counting $|\Omega(\nu)|$ is #P-complete [Brightwell and Winkler, 1991] as well. So $\Pr(c@j \mid \boldsymbol{\nu})$ is in $FP^{\#P}$.

Lemma 1 demonstrates that REP-t, a special case of REP, is ${\rm FP}^{\#P}$ -hard. Thus REP is ${\rm \#P}$ -hard as well. In conclusion, REP is ${\rm FP}^{\#P}$ -complete.

THEOREM 6. Given a voting profile P^{M} and a positional scoring rule r_m , the ESC problem can be reduced to the REP.

Proof. Recall that the MEW w maximizes the expected score, i.e.,

$$s(w, \mathbf{P}^{\mathsf{M}}) = \max_{c \in C} \mathbb{E}(s(c, \mathbf{P}^{\mathsf{M}}))$$

The voting profile P^{M} contains n ranking distributions $\{M_1,\ldots,M_n\}$, so

$$\mathbb{E}(s(c, \mathbf{P}^{\mathsf{M}})) = \sum_{i=1}^{n} \mathbb{E}(s(c, \mathsf{M}_{i}))$$

where $\mathbb{E}(s(c, \mathsf{M}_i))$ is the expected score of c from voter v_i .

$$\mathbb{E}(s(c, \mathsf{M}_i)) = \sum_{j=1}^m Pr(c@j \mid \mathsf{M}_i) \cdot \boldsymbol{r}_m(j)$$

where c@j denotes candidate c at rank j, and $r_m(j)$ is the score of rank j.

Let \mathbb{T} denote the complexity of calculating $Pr(c@i \mid M_i)$. The original MEW problem can be solved by calculating $Pr(c@j \mid M_i)$ for all m candidates and n voters, which leads to the complexity of $O(n \cdot m \cdot \mathbb{T})$.

THEOREM 7. The REP is equivalent to the ESC problem over a collection of k-approval rules where k = 0, ..., m.

Proof. The ESC problem has be reduced to the REP (Theorem 6). This proof will focus on the other direction, i.e., reducing the REP to the ESC problem.

Let $Pr(c@j \mid M)$ denote the probability of placing candidate c at rank j over a ranking distribution M. Let P^{M} denote a single-voter profile consisting of only this ranking distribution M. Then the REP can be reduced to solving the ESC problem twice under j-approval and (j-1)-approval rules.

$$\begin{split} \Pr(c@j \mid \mathsf{M}) &= \mathbb{E}(s(c \mid \boldsymbol{P}^{\mathsf{M}}, j\text{-approval})) \\ &- \mathbb{E}(s(c \mid \boldsymbol{P}^{\mathsf{M}}, (j-1)\text{-approval})) \end{split}$$

THEOREM 8. Given a partial voting profile P^{PO} , a distinguished candidate c, and plurality rule r_m , the ESC problem of calculating $\mathbb{E}(s(c \mid P^{PO}, r_m))$ is $FP^{\#P}$ -complete.

Proof. Firstly, we prove the membership of the ESC problem as a $FP^{\#P}$ problem. Consider that the REP is $FP^{\#P}$ -complete over partial orders (Theorem 5), and the ESC problem can be reduced to the REP (Theorem 6) So the ESC problem is in $FP^{\#P}$ for partial voting profiles.

Secondly, we prove that the ESC problem is $FP^{\#P}$ -hard, even for plurality rule, by reduction from the REP-t that is $FP^{\#P}$ -hard (Lemma 1).

Let ν denote the partial order of the REP-t problem. Recall that the REP-t problem aims to calculate $Pr(c@1 \mid \nu)$ for a given item c. Let P^{ν} denote a voting profile consisting of just this partial order ν . The answer to the REP-t problem is the same as the answer to the corresponding ESC problem, i.e., $\Pr(c@1 \mid \boldsymbol{\nu}) = \mathbb{E}(s(c \mid \boldsymbol{P}^{\boldsymbol{\nu}}, \text{plurality}))$. So the ESC problem is FP#P-hard, even for plurality voting rule.

In conclusion, the ESC problem is $FP^{\#P}$ -complete, under plurality rule.

THEOREM 9. Given a partial voting profile P^{PO} , a distinguished candidate c, and veto rule r_m , the ESC problem of calculating $\mathbb{E}(s(c \mid P^{PO}, r_m))$ is $FP^{\#P}$ -complete.

Proof. This proof adopts the same approach as the proof of Theorem 8.

Firstly, the membership proof that the ESC is in $FP^{\#P}$ is based on the conclusions that the REP is $FP^{\#P}$ -complete over partial orders (Theorem 5), and that the ESC can be reduced to the REP (Theorem 6) So the ESC is in $FP^{\#P}$ for partial voting profiles.

Secondly, we prove that the ESC is $FP^{\#P}$ -hard, under veto voting rule, by reduction from the REP-b that is $FP^{\#P}$ -hard (Lemma 2).

Let ν denote the partial order of the REP-b problem. Recall that the REP-b problem aims to calculate $Pr(c@m \mid \nu)$

Algorithm 2 REP solver for rRSM

```
Input: Item c, rank k, rRSM(\sigma, \Pi)
Output: Pr(c@k \mid \boldsymbol{\sigma}, \Pi)
   1: \alpha_0 := |\{\sigma_i | \sigma_i \succ_{\sigma} c\}|, \beta_0 := |\{\sigma_i | c \succ_{\sigma} \sigma_i\}|
  2: \mathcal{P}_0 := \{ \langle \alpha_0, \beta_0 \rangle \} and q_0(\langle \alpha_0, \beta_0 \rangle) := 1
3: for i = 1, \dots, (k-1) do
  4:
              \mathcal{P}_i := \{\}
              for \langle \alpha, \beta \rangle \in \mathcal{P}_{i-1} do
  5:
  6:
                   if \alpha > 0 then
                        Generate a new state \langle \alpha', \beta' \rangle = \langle \alpha - 1, \beta \rangle.
  7:
                        \mathcal{P}_i.add(\langle \alpha', \beta' \rangle)
  8:
                         q_i(\langle \alpha', \beta' \rangle) += q_{i-1}(\langle \alpha, \beta \rangle) \cdot \sum_{i=1}^{\alpha} \Pi(i, j)
  9:
10:
                   end if
11:
                   if \beta > 0 then
                        Generate a new state \langle \alpha', \beta' \rangle = \langle \alpha, \beta - 1 \rangle.
12:
                        \mathcal{P}_i.add(\langle \alpha', \beta' \rangle)
13:
                        q_i(\langle \alpha', \beta' \rangle) += q_{i-1}(\langle \alpha, \beta \rangle) \cdot \sum_{j=\alpha+2}^{\alpha+1+\beta} \Pi(i,j)
14:
15:
              end for
16:
17: end for
18: return \sum_{\langle \alpha, \beta \rangle \in \mathcal{P}_{k-1}} q_{k-1}(\langle \alpha, \beta \rangle) \cdot \Pi(k, \alpha+1)
```

for a given item c. Let P^{ν} denote a voting profile consisting of just this partial order ν . The answer to the ESC indirectly solves the REP-b, i.e., $\Pr(c@m \mid \nu) = 1 - \mathbb{E}(s(c \mid P^{\nu}, \text{veto}))$. So the ESC problem is $\operatorname{FP}^{\#P}$ -hard under veto rule.

In conclusion, the ESC is $FP^{\#P}$ -complete, under veto rule.

THEOREM 10. Given a partial voting profile P^{PO} , a distinguished candidate c, and k-approval rule r_m , the ESC problem of calculating $\mathbb{E}(s(c \mid P^{PO}, r_m))$ is $FP^{\#P}$ -complete.

Proof. First the proof that the Expected Score Computation (ESC) is in $FP^{\#P}$ is the same as the proof of Theorem 8. Now we prove that the ESC problem is $FP^{\#P}$ -hard, under k-approval rule r_m , by reduction from the REP-t problem that is $FP^{\#P}$ -hard (Lemma 1).

Let ν denote the partial order of the REP-t problem. Recall that the REP-t problem aims to calculate $\Pr(c@1 \mid \nu)$ for a given item c. Let ν_+ denote a new partial order by inserting (k-1) ordered items $d_1 \succ \ldots \succ d_{k-1}$ into ν such that item d_{k-1} is preferred to every item in ν . Such placement of items $\{d_1,\ldots,d_{k-1}\}$ is to guarantee that all linear extensions of ν_+ start with $d_1 \succ \ldots \succ d_{k-1}$ and these linear extensions will be precisely the linear extensions of ν after removing $\{d_1,\ldots,d_{k-1}\}$.

Let P^{ν_+} denote a voting profile consisting of just this partial order ν_+ . The answer to the ESC problem for item c is $\mathbb{E}(s(c \mid P^{\nu_+}, k\text{-approval}))$. Since there is only one partial order ν_+ in the voting profile, $\mathbb{E}(s(c \mid P^{\nu_+}, k\text{-approval})) = \sum_{j=1}^k \Pr(c@j \mid \nu_+)$. Recall that the first (k-1) items in any linear extension of ν_+ starts with $d_1 \succ \ldots \succ d_{k-1}$, so $\forall 1 \leq j \leq (k-1), \Pr(c@j \mid \nu_+) = 0$, which leads to $\mathbb{E}(s(c \mid P^{\nu_+}, k\text{-approval}))$

 P^{ν_+} , k-approval)) = $\Pr(c@k \mid \nu_+)$. Since ν_+ is constructed by inserting (k-1) items before items in ν , $\Pr(c@k \mid \nu_+) = \Pr(c@1 \mid \nu)$. So $\mathbb{E}(s(c \mid P^{\nu_+}, k\text{-approval})) = \Pr(c@1 \mid \nu)$. The answer to the REP-t problem has been reduced to the ESC problem. So the ESC problem is $\operatorname{FP}^{\#P}$ -hard, under k-approval rule.

In conclusion, the ESC problem is $FP^{\#P}$ -complete, under k-approval rule.

8.2 MEW Tractability over RSM voting profiles

RSM [Chakraborty et~al., 2020] denoted by RSM(σ, p, Π) is another generalization of the Mallows. It is parameterized by a reference ranking σ , a probability function Π where $\Pi(i,j)$ is the probability of the j^{th} item selected at step i, and a probability function $p:\{1,...,m-1\} \rightarrow [0,1]$ where p(i) is the probability that the i^{th} selected item preferred to the remaining items. In contrast to the RIM that randomizes the item insertion position, the RSM randomized the item insertion order. In this paper, we will use RSM as a ranking model, i.e., $p\equiv 1$ such that it only outputs rankings. This ranking version is named rRSM and denoted by rRSM(σ,Π).

Example 2. rRSM (σ,Π) with $\sigma = \langle a,b,c \rangle$ generates $\tau = \langle c,a,b \rangle$ as follows. Initialize $\tau_0 = \langle \rangle$. When $i=1, \tau_1 = \langle c \rangle$ by selecting c with probability $\Pi(1,3)$, which making the remaining $\sigma = \langle a,b \rangle$. When $i=2, \tau_2 = \langle c,a \rangle$ by selecting a with probability $\Pi(2,1)$, which making the remaining $\sigma = \langle b \rangle$. When $i=3, \tau = \langle c,a,b \rangle$ by selecting b with probability $\Pi(3,1)$. Overall, $\Pr(\tau \mid \sigma,\Pi) = \Pi(1,3) \cdot \Pi(2,1) \cdot \Pi(3,1)$.

Theorem 16. Given a positional scoring rule r_m , a RSM voting profile $P^{\mathsf{rRSM}} = (\mathsf{rRSM}_1, \dots, \mathsf{rRSM}_n)$, and candidate w, determining $w \in MEW(r_m, P^{\mathsf{rRSM}})$ is in $O(nm^4)$.

Proof. Given any rRSM $\in \mathbf{P}^{\mathsf{RSM}}$, candidate c, and rank j, the $Pr(c@j \mid \mathsf{rRSM})$ is computed by Algorithm 2 in a fashion that is similar to Algorithm 1. This is also a Dynamic Programming (DP) approach. The states are in the form of $\langle \alpha, \beta \rangle$, where α is the number of items before c, and β is that after c in the remaining σ . For state $\langle \alpha, \beta \rangle$, there are $(\alpha+1+\beta)$ items in the remaining σ . Algorithm 2 only runs up to i=(k-1) (in line 3), since item c must be selected at step k and the rest steps do not change the rank of c any more. Each step i generates at most (i+1) states, corresponding to $[0,\ldots,i]$ items are selected from items before c in the original σ . The complexity of Algorithm 2 is bounded by $O(m^2)$. It takes $O(nm^4)$ to obtain the expected scores of all candidates and to determine the MEW.

Example 3. Let $\operatorname{rRSM}(\sigma,\Pi)$ denote a RSM where $\sigma = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4 \rangle$, and $\Pi = [[0.1, 0.3, 0.4, 0.2], [0.2, 0.5, 0.3], [0.3, 0.7], [1]]$. Assume we are interested in $\Pr(\sigma_2@3 \mid \sigma, \Pi)$, the probability of $\operatorname{rRSM}(\sigma,\Pi)$ placing σ_2 at rank 3.

- Before running RSM, there is $\alpha_0 = 1$ item before σ_2 and $\beta_0 = 2$ items after σ_2 in σ . So the initial state is $\langle \alpha_0, \beta_0 \rangle = \langle 1, 2 \rangle$, and $q_0(\langle 1, 2 \rangle) = 1$.
- At step i=1, the selected item can be either from $\{\sigma_1\}$ or $\{\sigma_3, \sigma_4\}$. So two new states are generated here.

- The σ_1 is selected with probability $\Pi(1,1) = 0.1$, which generates a new state $\langle 0,2 \rangle$, and $q_1(\langle 0,2 \rangle) = q_0(\langle 1,2 \rangle) \cdot \Pi(1,1) = 0.1$.
- An item $\sigma \in \{\sigma_3, \sigma_4\}$ is selected with probability $\Pi(1,3) + \Pi(1,4) = 0.6$, which generates a new state $\langle 1, 1 \rangle$, and $q_1(\langle 1, 1 \rangle) = q_0(\langle 1, 2 \rangle) \cdot 0.6 = 0.6$.

So $\mathcal{P}_1 = \{\langle 0, 2 \rangle, \langle 1, 1 \rangle\}$ and $q_1 = \{\langle 0, 2 \rangle \mapsto 0.1, \langle 1, 1 \rangle \mapsto 0.6\}$.

- At step i = 2, iterate states in \mathcal{P}_1 .
 - For state $\langle 0,2\rangle$, the selected item must be from the last two items in remaining reference ranking. A new state $\langle 0,1\rangle$ is generated with probability $\Pi(2,2)+\Pi(2,3)=0.8$.
 - For state $\langle 1, 1 \rangle$, the selected item is either the first

or last item in remaining reference ranking. A new state $\langle 0,1 \rangle$ is generated with probability $\Pi(2,1)=0.1$, and another state $\langle 1,0 \rangle$ is generated with probability $\Pi(2,3)=0.3$.

So
$$\mathcal{P}_2 = \{\langle 0, 1 \rangle, \langle 1, 0 \rangle\}$$
 and

$$\Box \ q_2(\langle 0, 1 \rangle) = q_1(\langle 0, 2 \rangle) \cdot 0.8 + q_1(\langle 1, 1 \rangle) \cdot 0.1 = 0.1 \cdot 0.8 + 0.6 \cdot 0.1 = 0.14$$

$$\Box q_2(\langle 1,0\rangle) = q_1(\langle 1,1\rangle) \cdot 0.3 = 0.6 \cdot 0.3 = 0.18$$

• At step i=3, item σ_2 must be selected to meet the requirement. For each state $\langle \alpha, \beta \rangle \in \mathcal{P}_2$, the rank of σ_2 is $(\alpha+1)$ in the corresponding remaining ranking. So $\Pr(\sigma_2@3 \mid \boldsymbol{\sigma}, \Pi) = q_2(\langle 0, 1 \rangle) \cdot \Pi(3, 1) + q_2(\langle 1, 0 \rangle) \cdot \Pi(3, 2) = 0.14 \cdot 0.3 + 0.18 \cdot 0.7 = 0.168$.