

Quantum Group of Isometries in Classical and Noncommutative Geometry

by

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Abstract

We formulate a quantum generalization of the notion of the group of Riemannian isometries for a compact Riemannian manifold, by introducing a natural notion of smooth and isometric action by a compact quantum group on a classical or noncommutative manifold described by spectral triples, and then proving the existence of a universal object (called the quantum isometry group) in the category of compact quantum groups acting smoothly and isometrically on a given (possibly noncommutative) manifold satisfying certain regularity assumptions. In fact, we identify the quantum isometry group with the universal object in a bigger category, namely the category of ‘quantum families of smooth isometries’, defined along the line of Woronowicz and Soltan. We also construct a spectral triple on the Hilbert space of forms on a noncommutative manifold which is equivariant with respect to a natural unitary representation of the quantum isometry group. We give explicit description of quantum isometry groups of commutative and noncommutative tori, and in this context, obtain the quantum double torus defined in [11] as the universal quantum group of holomorphic isometries of the noncommutative torus.

1 Introduction

Since the formulation of quantum automorphism groups by Wang ([15], [16]), following suggestions of Alain Connes, many interesting examples of such quantum groups, particularly the quantum permutation groups of finite sets and finite graphs, have been extensively studied by a number of mathematicians (see, e.g. [1], [2], [17] and references therein), who have also found applications to and interaction with areas like free probability and subfactor theory. The underlying basic principle of defining a quantum automorphism group corresponding to some given mathematical structure (for example, a

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finite set, a graph, a C^* or von Neumann algebra) consists of two steps : first, to identify (if possible) the group of automorphisms of the structure as a universal object in a suitable category, and then, try to look for the universal object in a similar but bigger category by replacing groups by quantum groups of appropriate type. However, most of the work done so far concern some kind of quantum automorphism groups of a ‘finite’ structure, for example, of finite sets or finite dimensional matrix algebras. It is thus quite natural to try to extend these ideas to the ‘infinite’ or ‘continuous’ mathematical structures, for example classical and noncommutative manifolds. In the present article, we have made an attempt to formulate and study the quantum analogues of the groups of Riemannian isometries, which play a very important role in the classical differential geometry. The group of Riemannian isometries of a compact Riemannian manifold M can be viewed as the universal object in the category of all compact metrizable groups acting on M , with smooth and isometric action. Therefore, to define the quantum isometry group, it is reasonable to consider a category of compact quantum groups which act on the manifold (or more generally, on a noncommutative manifold given by spectral triple) in a ‘nice’ way, preserving the Riemannian structure in some suitable sense, to be precisely formulated. In this article, we have given a definition of such ‘smooth and isometric’ action by a compact quantum group on a (possibly noncommutative) manifold, extending the notion of smooth and isometric action by a group on a classical manifold. Indeed, the meaning of isometric action is nothing but that the action should commute with the ‘Laplacian’ coming from the spectral triple, and we should mention that this idea was already present in [2], though only in the context of a finite metric space or a finite graph. The universal object in the category of such quantum groups, if it exists, should be thought of as the quantum analogue of the group of isometries, and we have been able to prove its existence under some regularity assumptions, all of which can be verified for a general compact connected Riemannian manifold as well as the standard examples of noncommutative manifolds. Motivated by the ideas of Woronowicz and Soltan, we actually consider a bigger category. The isometry group of a classical manifold, viewed as a compact metrizable space (forgetting the group structure), can be seen to be the universal object of a category whose object-class consists of subsets (not necessarily subgroups) of the set of smooth isometries of the manifold. Then it can be proved that this universal compact set has a canonical group structure. A natural quantum analogue of this has been formulated by us, called the category of ‘quantum families of smooth isometries’. The underlying C^* -algebra of the quantum isometry group has been identified with its universal object and

moreover, it is shown to be equipped with a canonical coproduct making it into a compact quantum group.

We believe that a detailed study of quantum isometry groups will not only give many new and interesting examples of compact quantum groups, it will also contribute to the understanding of quantum group covariant spectral triples. In fact, we have made some progress in this direction already by constructing a spectral triple (which is often closely related to the original spectral triple) on the Hilbert space of forms which is equivariant with respect to a canonical unitary representation of the quantum isometry group.

In a companion article [3] with J. Bhowmick, we provide explicit computations of quantum isometry groups of a few classical and noncommutative manifolds. However, we briefly quote some of main results of [3] in the present article. One interesting observation is that the quantum isometry group of the noncommutative two-torus \mathcal{A}_θ (with the canonical spectral triple) is (as a C^* algebra) a direct sum of two commutative and two noncommutative tori, and contains as a quantum subgroup (which is universal for certain class of isometric actions called holomorphic isometries) the ‘quantum double-torus’ discovered and studied by Hajac and Masuda ([11]).

2 Definition of the quantum isometry group

2.1 Isometry groups of classical manifolds

We begin with a well-known characterization of the isometry group of a (classical) compact Riemannian manifold. Let (M, g) be a compact Riemannian manifold and let $\Omega^1 = \Omega^1(M)$ be the space of smooth one-forms, which has a right Hilbert- $C^\infty(M)$ -module structure given by the $C^\infty(M)$ -valued inner product $\langle\langle \cdot, \cdot \rangle\rangle$ defined by

$$\langle\langle \omega, \eta \rangle\rangle (m) = \langle \omega(m), \eta(m) \rangle|_m,$$

where $\langle \cdot, \cdot \rangle|_m$ is the Riemannian metric on the cotangent space T_m^*M at the point $m \in M$. The Riemannian volume form allows us to make Ω^1 a pre-Hilbert space, and we denote its completion by \mathcal{H}_1 . Let $\mathcal{H}_0 = L^2(M, \text{dvol})$ and consider the de-Rham differential d as an unbounded linear map from \mathcal{H}_0 to \mathcal{H}_1 , with the natural domain $C^\infty(M) \subset \mathcal{H}_0$, and also denote its closure by d . Let $\mathcal{L} := -d^*d$. The following identity can be verified by direct and easy computation using the local coordinates :

$$(\partial \mathcal{L})(\phi, \psi) \equiv \mathcal{L}(\bar{\phi}\psi) - \mathcal{L}(\bar{\phi})\psi - \bar{\phi}\mathcal{L}(\psi) = 2 \langle\langle d\phi, d\psi \rangle\rangle \quad \text{for } \phi, \psi \in C^\infty(M) \quad (*).$$

Proposition 2.1 *A smooth map $\gamma : M \rightarrow M$ is a Riemannian isometry if and only if γ commutes with \mathcal{L} in the sense that $\mathcal{L}(f \circ \gamma) = (\mathcal{L}(f)) \circ \gamma$ for all $f \in C^\infty(M)$.*

Proof :

If γ commutes with \mathcal{L} then from the identity (*) we get for $m \in M$ and $\phi, \psi \in C^\infty(M)$:

$$\begin{aligned}
& \langle d\phi|_{\gamma(m)}, d\psi|_{\gamma(m)} \rangle_{\gamma(m)} \\
&= \langle \langle d\phi, d\psi \rangle \rangle (\gamma(m)) \\
&= \frac{1}{2}(\partial \mathcal{L}(\phi, \psi) \circ \gamma)(m) \\
&= \frac{1}{2}\partial \mathcal{L}(\phi \circ \gamma, \psi \circ \gamma)(m) \\
&= \langle \langle d(\phi \circ \gamma), d(\psi \circ \gamma) \rangle \rangle (m) \\
&= \langle d(\phi \circ \gamma)|_m, d(\psi \circ \gamma)|_m \rangle_m \\
&= \langle (d\gamma|_m)^*(d\phi|_{\gamma(m)}), (d\gamma|_m)^*(d\psi|_{\gamma(m)}) \rangle_m,
\end{aligned}$$

which proves that $(d\gamma|_m)^* : T_{\gamma(m)}^*M \rightarrow T_m^*M$ is an isometry. Thus, γ is a Riemannian isometry.

Conversely, if γ is an isometry, both the maps induced by γ on \mathcal{H}_0 and \mathcal{H}_1 , i.e. $U_\gamma^0 : \mathcal{H}_0 \rightarrow \mathcal{H}_0$ given by $U_\gamma^0(f) = f \circ \gamma$ and $U_\gamma^1 : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ given by $U_\gamma^1(f d\phi) = (f \circ \gamma)d(\phi \circ \gamma)$ are unitaries. Moreover, $d \circ U_\gamma^0 = U_\gamma^1 \circ d$ on $C^\infty(M) \subset \mathcal{H}_0$. From this, it follows that $\mathcal{L} = -d^*d$ commutes with U_γ^0 . \square

Now let us consider a compact metrizable (i.e. second countable) space Y with a continuous map $\theta : M \times Y \rightarrow M$. We abbreviate $\theta(m, y)$ as ym and denote by ξ_y the map $M \ni m \mapsto ym$. Let $\alpha : C(M) \rightarrow C(M) \otimes C(Y) \cong C(M \times Y)$ be the map given by $\alpha(f)(m, y) := f(ym)$ for $y \in Y, m \in M$ and $f \in C(M)$. For a state ϕ on $C(Y)$, denote by α_ϕ the map $(\text{id} \otimes \phi) \circ \alpha : C(M) \rightarrow C(M)$. We shall also denote by \mathcal{C} the subspace of $C(M) \otimes C(Y)$ generated by elements of the form $\alpha(f)(1 \otimes \psi)$, $f \in C(M), \psi \in C(Y)$. Since $C(M)$ and $C(Y)$ are commutative algebras, it is easy to see that \mathcal{C} is a $*$ -subalgebra of $C(M) \otimes C(Y)$. Then we have the following

Theorem 2.2 (i) \mathcal{C} is norm-dense in $C(M) \otimes C(Y)$ if and only if for every $y \in Y$, ξ_y is one-to-one.
(ii) The map ξ_y is C^∞ for every $y \in Y$ if and only if $\alpha_\phi(C^\infty(M)) \subseteq C^\infty(M)$ for all ϕ .
(iii) Under the hypothesis of (ii), each ξ_y is also an isometry if and only if

α_ϕ commutes with $(\mathcal{L} - \lambda)^{-1}$ for all state ϕ and all λ in the resolvent of \mathcal{L} (equivalently, α_ϕ commutes with the Laplacian \mathcal{L} on $C^\infty(M)$).

Proof :

(i) First, assume that ξ_y is one-to-one for all y . By Stone-Weirstrass Theorem, it is enough to show that \mathcal{C} separates points. Take $(m_1, y_1) \neq (m_2, y_2)$ in $M \times Y$. If $y_1 \neq y_2$, we can choose $\psi \in C(Y)$ which separates y_1 and y_2 , hence $(1 \otimes \psi) \in \mathcal{C}$ separates (m_1, y_1) and (m_2, y_2) . So, we can consider the case when $y_1 = y_2 = y$ (say), but $m_1 \neq m_2$. By injectivity of ξ_y , we have $ym_1 \neq ym_2$, so there exists $f \in C(M)$ such that $f(ym_1) \neq f(ym_2)$, i.e. $\alpha(f)(m_1, y) \neq \alpha(f)(m_2, y)$. This proves the density of \mathcal{C} .

For the converse, we argue as in the proof of Proposition 3.3 of [14]. Assume that \mathcal{C} is dense in $C(M) \otimes C(Y)$, and let $y \in Y$, $m_1, m_2 \in M$ such that $ym_1 = ym_2$. That is, $\alpha(f)(1 \otimes \psi)(m_1, y) = \alpha(f)(1 \otimes \psi)(m_2, y)$ for all $f \in C(M)$, $\psi \in C(Y)$. By the density of \mathcal{C} we get $\chi(m_1, y) = \chi(m_2, y)$ for all $\chi \in C(M \times Y)$, so $(m_1, y) = (m_2, y)$, i.e. $m_1 = m_2$.

(ii) The ‘if part’ of (ii) follows by considering the states corresponding to point evaluation, i.e. $C(Y) \ni \psi \mapsto \psi(y)$, $y \in Y$. For the converse, we note that an arbitrary state ϕ corresponds to a regular Borel measure μ on Y so that $\phi(h) = \int h d\mu$, and thus, $\alpha_\phi(f)(m) = \int f(ym) d\mu(y)$ for $f \in C(M)$. From this, by interchanging differentiation and integration (which is allowed by the Dominated Convergence Theorem, since μ is a finite measure) we can prove that $\alpha_\phi(f)$ is C^∞ whenever f is so.

The assertion (iii) follows from Proposition 2.1 in a straightforward way.

□

Let us recall a few well-known facts about the Laplacian \mathcal{L} , viewed as a negative self-adjoint operator on the Hilbert space $L^2(M, d\text{vol})$. It is known (see [12] and references therein) that \mathcal{L} has compact resolvents and all its eigenvectors belong to $C^\infty(M)$. Moreover, it follows from the Sobolev Embedding Theorem that

$$\bigcap_{n \geq 1} \text{Dom}(\mathcal{L}^n) = C^\infty(M).$$

Let $\{e_{ij}, j = 1, \dots, d_i; i = 1, 2, \dots\}$ be the set of (normalised) eigenvectors of \mathcal{L} , where $e_{ij} \in C^\infty(M)$ is an eigenvector corresponding to the eigenvalue λ_i , $|\lambda_1| < |\lambda_2| < \dots$. We have the following:

Lemma 2.3 *The complex linear span of $\{e_{ij}\}$ is norm-dense in $C(M)$.*

Proof :

This is a consequence of the asymptotic estimates of eigenvalues λ_i , as

well as the uniform bound of the eigenfunctions e_{ij} . For example, it is known ([9], Theorem 1.2) that there exist constants C, C' such that $\|e_{ij}\|_\infty \leq C|\lambda_i|^{\frac{n-1}{4}}$, $d_i \leq C'|\lambda_i|^{\frac{n-1}{2}}$, where n is the dimension of the manifold M . Now, for $f \in C^\infty(M) \subseteq \bigcap_{k \geq 1} \text{Dom}(\mathcal{L}^k)$, we write f as an a-priori L^2 -convergent series $\sum_{ij} f_{ij} e_{ij}$ ($f_{ij} \in \mathbb{C}$), and observe that $\sum |f_{ij}|^2 |\lambda_i|^{2k} < \infty$ for every $k \geq 1$. Choose and fix sufficiently large k such that $\sum_{i \geq 0} |\lambda_i|^{n-1-2k} < \infty$, which is possible due to the well-known Weyl asymptotics of eigenvalues of \mathcal{L} . Now, by the Cauchy-Schwarz inequality and the estimate for d_i , we have

$$\sum_{ij} |f_{ij}| \|e_{ij}\|_\infty \leq C(C')^{\frac{1}{2}} \left(\sum_{ij} |f_{ij}|^2 |\lambda_i|^{2k} \right)^{\frac{1}{2}} \left(\sum_{i \geq 0} |\lambda_i|^{n-1-2k} \right)^{\frac{1}{2}} < \infty.$$

Thus, the series $\sum_{ij} f_{ij} e_{ij}$ converges to f in sup-norm, so $\text{Sp}\{e_{ij}, j = 1, 2, \dots, d_i; i = 1, 2, \dots\}$ is dense in sup-norm in $C^\infty(M)$, hence in $C(M)$ as well. \square

Let us denote $\text{Sp}\{e_{ij}, j = 1, \dots, d_i; i \geq 1\}$ by \mathcal{A}_0^∞ from now on. We shall now show that $C^\infty(M)$ can be replaced by the smaller subspace \mathcal{A}_0^∞ in Theorem 2.2. We need a lemma for this, which will be useful later on too.

Lemma 2.4 *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and for $i = 1, 2$, let \mathcal{L}_i be (possibly unbounded) self-adjoint operator on \mathcal{H}_i with compact resolvents, and let \mathcal{V}_i be the linear span of eigenvectors of \mathcal{L}_i . Moreover, assume that there is an eigenvalue of \mathcal{L}_i for which the eigenspace is one-dimensional, say spanned by a unit vector ξ_i . Let Ψ be a linear map from \mathcal{V}_1 to \mathcal{V}_2 such that $\mathcal{L}_2 \Psi = \Psi \mathcal{L}_1$ and $\Psi(\xi_1) = \xi_2$. Then we have*

$$\langle \xi_2, \Psi(x) \rangle = \langle \xi_1, x \rangle \quad \forall x \in \mathcal{V}_1. \quad (1)$$

Proof:

By hypothesis on Ψ , it is clear that there is a common eigenvalue, say λ_0 , of \mathcal{L}_1 and \mathcal{L}_2 , with the eigenvectors ξ_1 and ξ_2 respectively. Let us write the set of eigenvalues of \mathcal{L}_i as a disjoint union $\{\lambda_0\} \cup \Lambda_i$ ($i = 1, 2$), and let the corresponding orthogonal decomposition of \mathcal{V}_i be given by $\mathcal{V}_i = \mathbb{C}\xi_i \oplus \bigoplus_{\lambda \in \Lambda_i} \mathcal{V}_i^\lambda \equiv \mathbb{C}\xi_i \oplus \mathcal{V}_i'$, say, where \mathcal{V}_i^λ denotes the eigenspace of \mathcal{L}_i corresponding to the eigenvalue λ . By assumption, Ψ maps \mathcal{V}_1^λ to \mathcal{V}_2^λ whenever λ is an eigenvalue of \mathcal{L}_2 , i.e. $\mathcal{V}_2^\lambda \neq \{0\}$, and otherwise it maps \mathcal{V}_1^λ into $\{0\}$. Thus, $\Psi(\mathcal{V}_1') \subseteq \mathcal{V}_2'$. Now, (1) is obviously satisfied for $x = \xi_1$, so it is enough to prove (1) for all $x \in \mathcal{V}_1'$. But we have $\langle \xi_1, x \rangle = 0$ for $x \in \mathcal{V}_1'$, and since $\Psi(x) \in \mathcal{V}_2' = \mathcal{V}_2 \cap \{\xi_2\}^\perp$, it follows that $\langle \xi_2, \Psi(x) \rangle = 0 = \langle \xi_1, x \rangle$. \square

Lemma 2.5 *Let Y and α be as in Theorem 2.2. Then the following are equivalent.*

- (a) *For every $y \in Y$, ξ_y is smooth isometric.*
- (b) *For every state ϕ on $C(Y)$, we have $\alpha_\phi(\mathcal{A}_0^\infty) \subseteq \mathcal{A}_0^\infty$, and $\alpha_\phi \mathcal{L} = \mathcal{L} \alpha_\phi$ on \mathcal{A}_0^∞ .*

Proof:

We prove only the nontrivial implication (b) \Rightarrow (a). Assume (b) that α_ϕ leaves \mathcal{A}_0^∞ invariant and commutes with \mathcal{L} on it, for every state ϕ . To prove that α is a smooth isometric action, it is enough (see the proof of Theorem 2.2) to prove that $\alpha_y(\mathcal{A}^\infty) \subseteq \mathcal{A}^\infty$ for all $y \in Y$, where $\alpha_y(f) := (\text{id} \otimes \text{ev}_y)(f) = f \circ \xi_y$, ev_y being the evaluation at the point y . Let M_1, \dots, M_k be the connected components of the compact manifold M . Thus, the Hilbert space $L^2(M, \text{dvol})$ admits an orthogonal decomposition $\oplus_{i=1}^k L^2(M_i, \text{dvol})$, and the Laplacian \mathcal{L} is of the form $\oplus_i \mathcal{L}_i$ where \mathcal{L}_i denotes the Laplacian on M_i . Since each M_i is connected, we have $\text{Ker}(\mathcal{L}_i) = \mathbb{C}\chi_i$, where χ_i is the constant function on M_i equal to 1. Now, we note that for fixed y and i , the image of M_i under the continuous function ξ_y must be mapped into a component, say M_j . Thus, by applying Lemma 2.4 with $\mathcal{H}_1 = L^2(M_i)$, $\mathcal{H}_2 = L^2(M_j)$, $\Psi = \xi_y$ and the L^2 -continuity of the map $f \mapsto \alpha_y(f) = f \circ \xi_y$, we have

$$\int_{M_j} \alpha_y(f)(x) \text{dvol}(x) = \int_{M_i} f(x) \text{dvol}(x)$$

for all f in the linear span of eigenvectors of \mathcal{L}_i , hence (by density) for all f in $L^2(M_i)$. It follows that $\int_M \alpha_y(f) \text{dvol} = \int_M f \text{dvol}$ for all $f \in L^2(M)$, in particular for all $f \in C(M)$. Since α_y is a $*$ -homomorphism on $C(M)$, we have

$$\langle \alpha_y(f), \alpha_y(g) \rangle = \int_M \alpha_y(\overline{f}g) \text{dvol} = \int_M \overline{f}g \text{dvol} = \langle f, g \rangle,$$

for all $f, g \in C(M)$. Thus, α_y extends to an isometry on $L^2(M)$, to be denoted by the same notation, which by our assumption commutes with the self-adjoint operator \mathcal{L} on the core \mathcal{A}_0^∞ , and hence α_y commutes with \mathcal{L}^n for all n . In particular it leaves invariant the domains of each \mathcal{L}^n , which implies $\alpha_y(\mathcal{A}^\infty) \subseteq \mathcal{A}^\infty$. \square

In view of the fact that the set of isometries of M , denoted by $ISO(M)$, is a compact second countable (i.e. compact metrizable) group, we see that $ISO(M)$ is the maximal compact second countable group acting on M such that the action is smooth and isometric. In other words, if we consider a category whose objects are compact metrizable groups acting smoothly and

isometrically on M , and morphisms are the group homomorphisms commuting with the actions on M , then $ISO(M)$ (with its canonical action on M) is the initial object of this category. However, one can take a more general viewpoint and consider the category of compact metrizable spaces Y equipped with a continuous map $\theta : M \times Y \rightarrow M$ satisfying (i)-(iii) of Theorem 2.2, or equivalently, the pair of commutative unital C^* -algebras $\mathcal{B} = C(Y)$ and a unital C^* -homomorphism $\alpha : C(M) \rightarrow C(M) \otimes \mathcal{B}$ satisfying the conditions (i)-(iii). The set of isometries $ISO(M)$ (as a topological space) can be identified with the universal object of this category, and then one can prove that it has a group structure.

It is quite natural to formulate a quantum analogue of the above, by considering, in the spirit of Woronowicz and Soltan (see [19] and [13]), ‘quantum families of isometries’, which can be defined to be a pair (\mathcal{B}, α) where \mathcal{B} is a (not necessarily commutative) C^* -algebra and $\alpha : C(M) \rightarrow C(M) \otimes \mathcal{B}$ is unital C^* -homomorphism satisfying (i)-(iii) of Theorem 2.2, i.e. the linear span of $\alpha(C(M))(1 \otimes \mathcal{B})$ (which is not necessarily a $*$ -subalgebra any more, \mathcal{B} being possibly noncommutative) is norm-dense in $C(M) \otimes \mathcal{B}$ and for every state ϕ on \mathcal{B} , the map α_ϕ keeps $C^\infty(M)$ invariant and commutes with the Laplacian \mathcal{L} . The morphisms of this category are obvious. We shall prove that this category has a universal object, and this universal object can be equipped with a canonical quantum group structure. This will define the quantum isometry group of a manifold. However, we shall go beyond classical manifolds and define quantum isometry group $QISO(\mathcal{A}^\infty, \mathcal{H}, D)$ for a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$, with \mathcal{A}^∞ being unital, and satisfying certain assumptions. To this end, we need to carefully formulate the notion of Laplacian in noncommutative geometry, which is the goal of the next subsection.

2.2 Laplacian in noncommutative geometry

Given a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$, we recall from [10] and [6] the construction of the space of one-forms. We have a derivation from \mathcal{A}^∞ to the \mathcal{A}^∞ - \mathcal{A}^∞ bimodule $\mathcal{B}(\mathcal{H})$ given by $a \mapsto [D, a]$. This induces a bimodule morphism π from $\Omega^1(\mathcal{A}^\infty)$ (the bimodule of universal one-forms on \mathcal{A}^∞) to $\mathcal{B}(\mathcal{H})$, such that $\pi(\delta(a)) = [D, a]$, where $\delta : \mathcal{A}^\infty \rightarrow \Omega^1(\mathcal{A}^\infty)$ denotes the universal derivation map. We set $\Omega_D^1 \equiv \Omega_D^1(\mathcal{A}^\infty) := \Omega^1(\mathcal{A}^\infty)/\text{Ker}(\pi) \cong \pi(\Omega^1(\mathcal{A}^\infty)) \subseteq \mathcal{B}(\mathcal{H})$. Assume that the spectral triple is of compact type and has a finite dimension in the sense of Connes ([6]), i.e. there is some $p > 0$ such that the operator $|D|^{-p}$ (interpreted as the inverse of the restriction of $|D|^p$ on the closure of its range, which has a finite co-dimension

since D has compact resolvents) has finite nonzero Dixmier trace, denoted by Tr_ω (where ω is some suitable Banach limit, see, e.g. [6], [10]). Consider the canonical ‘volume form’ τ coming from the Dixmier trace, i.e. $\tau : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{C}$ defined by $\tau(A) := \frac{1}{Tr_\omega(|D|^{-p})} Tr_\omega(A|D|^{-p})$. Let us at this point assume that the spectral triple is QC^∞ , i.e. \mathcal{A}^∞ and $\{[D, a], a \in \mathcal{A}^\infty\}$ are contained in the domains of all powers of the derivation $[|D|, \cdot]$. Under this assumption, τ is a positive faithful trace on the C^* -subalgebra generated by \mathcal{A}^∞ and $\{[D, a] \mid a \in \mathcal{A}^\infty\}$, and the GNS Hilbert space $L^2(\mathcal{A}^\infty, \tau)$ is denoted by \mathcal{H}_D^0 . Similarly, we equip Ω_D^1 with a semi-inner product given by $\langle \eta, \eta' \rangle := \tau(\eta^* \eta')$, and denote the Hilbert space obtained from it by \mathcal{H}_D^1 . The map $d_D : \mathcal{H}_D^0 \rightarrow \mathcal{H}_D^1$ given by $d_D(\cdot) = [D, \cdot]$ is an unbounded densely defined linear map. Let us assume the following:

Assumption(i) (a) d_D is closable (the closure is denoted again by d_D);
(b) $\mathcal{A}^\infty \subseteq \text{Dom}(\mathcal{L})$, where $\mathcal{L} := -d_D^* d_D$ and \mathcal{A}^∞ is viewed as a dense subspace of \mathcal{H}_D^0 ;

At this point, let us show that this assumption is valid under a very natural condition on the spectral triple.

Lemma 2.6 *Suppose that for every element $a \in \mathcal{A}^\infty$, the map $\mathbb{R} \ni t \mapsto \alpha_t(X) := \exp(itD)X\exp(-itD)$ is differentiable at $t = 0$ in the norm-topology of $\mathcal{B}(\mathcal{H})$, where $X = a$ or $[D, a]$. Then the assumption (i) is satisfied. Moreover, in this case, \mathcal{L} maps \mathcal{A}^∞ into the weak closure of \mathcal{A}^∞ in $\mathcal{B}(\mathcal{H}_D^0)$.*

Proof :

We first observe that $\tau(\alpha_t(A)) = \tau(A)$ for all t and for all $A \in \mathcal{B}(\mathcal{H})$, since $\exp(itD)$ commutes with $|D|^{-p}$. If moreover, A belongs to the domain of norm-differentiability (at $t = 0$) of α_t , i.e. $\frac{\alpha_t(A) - A}{t} \rightarrow i[D, A]$ in operator-norm, then it follows from the property of the Dixmier trace that $\tau([D, A]) = \frac{1}{i} \lim_{t \rightarrow 0} \frac{\tau(\alpha_t(A)) - \tau(A)}{t} = 0$. Now, since by assumption we have the norm-differentiability at $t = 0$ of $\alpha_t(A)$ for A belonging to the $*$ -subalgebra (say \mathcal{B}) generated by \mathcal{A}^∞ and $[D, \mathcal{A}^\infty]$, it follows that $\tau([D, A]) = 0 \forall A \in \mathcal{B}$. Let us now fix $a, b, c \in \mathcal{A}^\infty$ and observe that

$$\begin{aligned} & \langle a d_D(b), d_D(c) \rangle \\ &= \tau((a d_D(b))^* d_D(c)) \\ &= -\tau([D, [D, b^*]a^*c]) + \tau([D, [D, b^*]a^*]c) \end{aligned}$$

$$= \tau([D, [D, b^*]a^*]c),$$

using the fact that $\tau([D, [D, b^*]a^*]c) = 0$. This implies

$$| \langle a d_D(b), d_D(c) \rangle | \leq \| [D, [D, b^*]a^*] \| \tau(c^*c)^{\frac{1}{2}} = \| [D, [D, b^*]a^*] \| \| c \|_2,$$

where $\|c\|_2 = \tau(c^*c)^{\frac{1}{2}}$ denotes the L^2 -norm of $c \in \mathcal{H}_D^0$. This proves that $a d_D(b)$ belongs to the domain of d_D^* for all $a, b \in \mathcal{A}^\infty$, so in particular d_D^* is dense, i.e. d_D is closable. Moreover, taking $a = 1$, we see that $d_D(\mathcal{A}^\infty) \subseteq \text{Dom}(d_D^*)$, or in other words, $\mathcal{A}^\infty \subseteq \text{Dom}(d_D^* d_D)$. This proves (i)(a) and (i)(b). The last sentence in the statement of the lemma can be proven along the line of Theorem 2.9, page 129, [10]. \square

We need few more assumptions on the operator \mathcal{L} to define the quantum isometry group.

Assumption (ii): \mathcal{L} has compact resolvents,

Assumption (iii): $\mathcal{L}(\mathcal{A}^\infty) \subseteq \mathcal{A}^\infty$;

Assumption (iv): Each eigenvector of \mathcal{L} (which has a discrete spectrum, hence a complete set of eigenvectors) belongs to \mathcal{A}^∞ ;

Assumption (v) ('connectedness assumption'): the kernel of \mathcal{L} is one-dimensional, spanned by the identity 1 of \mathcal{A}^∞ , viewed as a unit vector in \mathcal{H}_D^0 .

We call \mathcal{L} the noncommutative Laplacian and T_t the noncommutative heat semigroup. We summarize some simple observations in form of the following

Lemma 2.7 (a) *If the assumptions (i)-(v) are valid, then for $x \in \mathcal{A}^\infty$, we have $\mathcal{L}(x^*) = (\mathcal{L}(x))^*$.*

(b) *If $T_t := \exp(t\mathcal{L})$ maps \mathcal{H}_D^0 into \mathcal{A}^∞ for all $t > 0$, then the assumption (iv) is satisfied.*

Proof :

It follows by simple calculation using the facts that τ is a trace and $d_D(x^*) = -(d_D(x))^*$ that

$$\begin{aligned} & \tau(\mathcal{L}(x^*)^* y) \\ &= -\tau(d_D(x) d_D(y)) = -\tau(d_D(y) d_D(x)) = \tau((d_D(y^*))^* d_D(x)) \\ &= \langle y^*, \mathcal{L}(x) \rangle = \tau(y \mathcal{L}(x)) = \tau(\mathcal{L}(x) y), \end{aligned}$$

for all $y \in \mathcal{A}^\infty$. By density of \mathcal{A}^∞ in \mathcal{H}_D^0 (a) follows. To prove (b), we note that if $x \in \mathcal{H}_D^0$ is an eigenvector of \mathcal{L} , say $\mathcal{L}(x) = \lambda x$ ($\lambda \in \mathbb{C}$), then we have $T_t(x) = e^{\lambda t} x$, hence $x = e^{-\lambda t} T_t(x) \in \mathcal{A}^\infty$. \square

Since by assumption, \mathcal{L} has a countable set of eigenvalues each with finite multiplicity, let us denote them by $\lambda_0 = 0, \lambda_1, \lambda_2, \dots$ with $V_0 = \mathbb{C} 1, V_1, V_2, \dots$ be corresponding eigenspaces (finite dimensional), and for each i , let $\{e_{ij}, j = 1, \dots, d_i\}$ be an orthonormal basis of V_i . By Assumption (iv), $V_i \subseteq \mathcal{A}^\infty$ for each i , V_i is closed under $*$, and moreover, $\{e_{ij}^*, j = 1, \dots, d_i\}$ is also an orthonormal basis for V_i , since $\tau(x^*y) = \tau(yx^*)$ for $x, y \in \mathcal{A}^\infty$. We also make the following

Assumption (vi) The complex linear span of $\{e_{ij}, i = 0, 1, \dots; j = 1, \dots, d_i\}$, say \mathcal{A}_0^∞ , is norm-dense in \mathcal{A}^∞ .

Definition 2.8 We say that a spectral triple satisfying the assumptions (i)-(vi) is *admissible*.

Remark 2.9 We have just seen that classical spectral triple $(\mathcal{A}^\infty = C^\infty(M), \mathcal{H}, D)$, where M is compact connected spin manifold, \mathcal{H} is the L^2 space of square integrable spinors and D is the Dirac operator, is indeed admissible in our sense. Later on we shall discuss how we can weaken the connectedness assumption as well, thus accommodating a general classical (commutative) spectral triple in our set-up. Moreover, the standard examples of noncommutative spectral triples, e.g. those on \mathcal{A}_θ , quantum Heisenberg manifold etc., do belong to the admissible class.

Lemma 2.10 Let us assume that the spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ is admissible. Let $\Psi : \mathcal{A}_0^\infty \rightarrow \mathcal{A}_0^\infty$ be a (norm-) bounded linear map, such that $\Psi(1) = 1$, and $\Psi \circ \mathcal{L} = \mathcal{L} \circ \Psi$ on the subspace \mathcal{A}_0^∞ spanned (algebraically) by $V_i, i = 1, 2, \dots$. Then $\tau(\Psi(x)) = \tau(x)$ for all $x \in \mathcal{A}^\infty$.

Proof :

By Lemma 2.4 with $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_D^0$, $\xi_1 = \xi_2 = 1$, we have $\tau(\Psi(x)) = \tau(x)$ for all $x \in \mathcal{A}_0^\infty$. By the norm-continuity of Ψ and τ it extends to the whole of \mathcal{A}^∞ . \square

2.3 Definition and existence of the quantum isometry group

We begin by recalling the definition of compact quantum groups and their actions from [18]. A compact quantum group is given by a pair (\mathcal{S}, Δ) , where \mathcal{S} is a unital separable C^* algebra equipped with a unital C^* -homomorphism $\Delta : \mathcal{S} \rightarrow \mathcal{S} \otimes \mathcal{S}$ (where \otimes denotes the injective tensor product) satisfying

- (ai) $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$ (co-associativity), and
- (aia) the linear span of $\Delta(\mathcal{S})(\mathcal{S} \otimes 1)$ and $\Delta(\mathcal{S})(1 \otimes \mathcal{S})$ are norm-dense in $\mathcal{S} \otimes \mathcal{S}$.

It is well-known (see [18]) that there is a canonical dense $*$ -subalgebra \mathcal{S}_0 of \mathcal{S} , consisting of the matrix coefficients of the finite dimensional unitary (co)-representations of \mathcal{S} , and maps $\epsilon : \mathcal{S}_0 \rightarrow \mathbb{C}$ (co-unit) and $\kappa : \mathcal{S}_0 \rightarrow \mathcal{S}_0$ (antipode) defined on \mathcal{S}_0 which make \mathcal{S}_0 a Hopf $*$ -algebra.

We say that the compact quantum group (\mathcal{S}, Δ) acts on a unital C^* algebra \mathcal{B} , if there is a unital C^* -homomorphism $\alpha : \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{S}$ satisfying the following :

- (bi) $(\alpha \otimes id) \circ \alpha = (id \otimes \Delta) \circ \alpha$, and
- (bia) the linear span of $\alpha(\mathcal{B})(1 \otimes \mathcal{S})$ is norm-dense in $\mathcal{B} \otimes \mathcal{S}$.

Let us now recall the concept of universal quantum groups as in [17], [15] and references therein. We shall use most of the terminologies of [15], e.g. Woronowicz C^* -subalgebra, Woronowicz C^* -ideal etc, however with the exception that we shall call the Woronowicz C^* algebras just compact quantum groups, and not use the term compact quantum groups for the dual objects as done in [15]. For $Q \in GL_n(\mathbb{C})$, let $A_u(Q)$ denote the universal compact quantum group generated by $u_{ij}, i, j = 1, \dots, n$ satisfying the relations

$$uu^* = I_n = u^*u, \quad u'Q\bar{u}Q^{-1} = I_n = Q\bar{u}Q^{-1}u',$$

where $u = ((u_{ij}))$, $u' = ((u_{ji}))$ and $\bar{u} = ((u_{ij}^*))$. The coproduct, say $\tilde{\Delta}$, is given by,

$$\tilde{\Delta}(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}.$$

We refer the reader to [17] for a detailed discussion on the structure and classification of such quantum groups. Let us denote by \mathcal{U}_i the quantum group $A_{d_i}(I)$, where d_i is dimension of the subspace V_i . We fix a representation $\beta_i : V_i \rightarrow V_i \otimes \mathcal{U}_i$ of \mathcal{U}_i on the Hilbert space V_i , given by $\beta_i(e_{ij}) = \sum_k e_{ik} \otimes u_{kj}^{(i)}$, for $j = 1, \dots, d_i$, where $u^{(i)} \equiv u_{kj}^{(i)}$ are the generators of \mathcal{U}_i as discussed before. Thus, both $u^{(i)}$ and $\bar{u}^{(i)}$ are unitaries. It follows from [15] that the representations β_i canonically induce a representation $\beta = *_i \beta_i$ of the free product $\mathcal{U} := *_i \mathcal{U}_i$ (which is a compact quantum group, see [15] for the details) on the Hilbert space \mathcal{H}_D^0 , such that the restriction of β on V_i coincides with β_i for all i .

In view of the characterization of smooth isometric action on a classical manifold, we make the following definitions.

Definition 2.11 *A quantum family of smooth isometries of a noncommutative manifold \mathcal{A}^∞ (or, more precisely on the corresponding spectral triple) is a pair (\mathcal{S}, α) where \mathcal{S} is a separable unital C^* -algebra, $\alpha : \overline{\mathcal{A}} \rightarrow \overline{\mathcal{A}} \otimes \mathcal{S}$ (where $\overline{\mathcal{A}}$ denotes the C^* algebra obtained by completing \mathcal{A}^∞ in the norm of $\mathcal{B}(\mathcal{H}_D^0)$) is a unital C^* -homomorphism, satisfying the following:*

- (a) $\overline{\text{Sp}}(\alpha(\overline{\mathcal{A}})(1 \otimes \mathcal{S})) = \overline{\mathcal{A}} \otimes \mathcal{S}$,
- (b) $\alpha_\phi := (\text{id} \otimes \phi) \circ \alpha$ maps \mathcal{A}_0^∞ into itself and commutes with \mathcal{L} on \mathcal{A}_0^∞ , for every state ϕ on \mathcal{S} .

In case the C^ -algebra \mathcal{S} has a coproduct Δ such that (\mathcal{S}, Δ) is a compact quantum group and α is an action of (\mathcal{S}, Δ) on $\overline{\mathcal{A}}$, we say that (\mathcal{S}, Δ) acts smoothly and isometrically on the noncommutative manifold.*

Fix a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$. Consider the category \mathbf{Q} with the object-class consisting of all quantum families of isometries (\mathcal{S}, α) of the given noncommutative manifold, and the set of morphisms $\text{Mor}((\mathcal{S}, \alpha), (\mathcal{S}', \alpha'))$ being the set of unital C^* -homomorphisms $\phi : \mathcal{S} \rightarrow \mathcal{S}'$ satisfying $(\text{id} \otimes \phi) \circ \alpha = \alpha'$. We also consider another category \mathbf{Q}' whose objects are triplets $(\mathcal{S}, \Delta, \alpha)$ where (\mathcal{S}, Δ) is a compact quantum group acting smoothly and isometrically on the given noncommutative manifold, with α being the corresponding action. The morphisms are the homomorphisms of compact quantum groups which are also morphisms of the underlying quantum families. The forgetful functor $F : \mathbf{Q}' \rightarrow \mathbf{Q}$ is clearly faithful, and we can view $F(\mathbf{Q}')$ as a subcategory of \mathbf{Q} .

Let us assume from now on that the spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ is admissible. Our aim is to prove the existence of a universal object in \mathbf{Q} . We shall also prove that the (unique upto isomorphism) universal object belongs to $F(\mathbf{Q}')$, and its pre-image in \mathbf{Q}' is a universal object in the category \mathbf{Q}' . To this end, we need some preparatory results.

Lemma 2.12 *Consider an admissible spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ and let (\mathcal{S}, α) be a quantum family of smooth isometries of the spectral triple. Moreover, assume that the action α is faithful in the sense that there is no proper C^* -subalgebra \mathcal{S}_1 of \mathcal{S} such that $\alpha(\mathcal{A}^\infty) \subseteq \mathcal{A}^\infty \otimes \mathcal{S}_1$. Then $\tilde{\alpha} : \mathcal{A}^\infty \otimes \mathcal{S} \rightarrow \mathcal{A}^\infty \otimes \mathcal{S}$ defined by $\tilde{\alpha}(a \otimes b) : \alpha(a)(1 \otimes b)$ extends to an \mathcal{S} -linear unitary on the Hilbert \mathcal{S} -module $\mathcal{H}_D^0 \otimes \mathcal{S}$, denoted again by $\tilde{\alpha}$. Moreover, we can find a C^* -isomorphism $\phi : \mathcal{U}/\mathcal{I} \rightarrow \mathcal{S}$ between \mathcal{S} and a quotient of \mathcal{U} by a C^* -ideal*

\mathcal{I} of \mathcal{U} , such that $\alpha = (\text{id} \otimes \phi) \circ (\text{id} \otimes \Pi_{\mathcal{I}}) \circ \beta$ on $\mathcal{A}^\infty \subseteq \mathcal{H}_D^0$, where $\Pi_{\mathcal{I}}$ denotes the quotient map from \mathcal{U} to \mathcal{U}/\mathcal{I} .

If, furthermore, there is a compact quantum group structure on \mathcal{S} given by a coproduct Δ such that $(\mathcal{S}, \Delta, \alpha)$ is an object in \mathbf{Q}' , the map $\alpha : \mathcal{A}^\infty \rightarrow \mathcal{A}^\infty \otimes \mathcal{S}$ extends to a unitary representation (denoted again by α) of the compact quantum group (\mathcal{S}, Δ) on \mathcal{H}_D^0 . In this case, the ideal \mathcal{I} is a Woronowicz C^* -ideal and the C^* -isomorphism $\phi : \mathcal{U}/\mathcal{I} \rightarrow \mathcal{S}$ is a morphism of compact quantum groups.

Proof :

Let ω be any state on \mathcal{S} . Since the action $\alpha : \mathcal{A}^\infty \rightarrow \mathcal{A}^\infty \otimes \mathcal{S}$ is smooth and isometric, we conclude by Lemma 2.10 that $\tau(\alpha_\omega(x)) = \tau(x)\omega(1)$ for all $x \in \overline{\mathcal{A}}$. Since ω is arbitrary, we have $(\tau \otimes \text{id})\alpha(x) = \tau(x)1_{\mathcal{S}}$ for all $x \in \overline{\mathcal{A}}$. So, $\langle \alpha(x), \alpha(y) \rangle_{\mathcal{S}} = \langle x, y \rangle 1_{\mathcal{S}}$, where $\langle \cdot, \cdot \rangle_{\mathcal{S}}$ denotes the \mathcal{S} -valued inner product of the Hilbert module $\mathcal{H}_D^0 \otimes \mathcal{S}$. This proves that $\tilde{\alpha}$ defined by $\tilde{\alpha}(x \otimes b) := \alpha(x)(1 \otimes b)$ ($x \in \mathcal{A}^\infty, b \in \mathcal{S}$) extends to an \mathcal{S} -linear isometry on the Hilbert \mathcal{S} -module $\mathcal{H}_D^0 \otimes \mathcal{S}$. Moreover, since $\alpha(\mathcal{A}^\infty)(1 \otimes \mathcal{S})$ is norm-dense in $\overline{\mathcal{A}} \otimes \mathcal{S}$, it is clear that the \mathcal{S} -linear span of the range of $\alpha(\mathcal{A}^\infty)$ is dense in the Hilbert module $\mathcal{H}_D^0 \otimes \mathcal{S}$, or in other words, the isometry $\tilde{\alpha}$ has a dense range, so it is a unitary.

Since α_ω leaves each V_i invariant, it is clear that α maps V_i into $V_i \otimes \mathcal{S}$ for each i . Let $v_{kj}^{(i)}$ ($j, k = 1, \dots, d_i$) be the elements of \mathcal{S} such that $\alpha(e_{ij}) = \sum_k e_{ik} \otimes v_{kj}^{(i)}$. Note that $v_i := ((v_{kj}^{(i)}))$ is a unitary in $M_{d_i}(\mathbb{C}) \otimes \mathcal{S}$. Moreover, the $*$ -subalgebra generated by all $\{v_{kj}^{(i)}, i, j, k \geq 1\}$ must be dense in \mathcal{S} by the assumption of faithfulness.

We have already remarked that $\{e_{ij}^*\}$ is also an orthonormal basis of V_i , and since α , being a C^* -action on $\overline{\mathcal{A}}$, is $*$ -preserving, we have $\alpha(e_{ij}^*) = (\alpha(e_{ij}))^* = \sum_k e_{ik}^* \otimes v_{kj}^{(i)*}$, and therefore $((v_{kj}^{(i)*}))$ is also unitary. By universality of \mathcal{U}_i , there is a C^* -homomorphism from \mathcal{U}_i to \mathcal{S} sending $u_{kj}^{(i)}$ to $v_{kj}^{(i)}$, and by definition of the free product, this induces a C^* -homomorphism, say Π , from \mathcal{U} onto \mathcal{S} , so that $\mathcal{U}/\mathcal{I} \cong \mathcal{S}$, where $\mathcal{I} := \text{Ker}(\Pi)$.

In case \mathcal{S} has a coproduct Δ making it into a compact quantum group and α is a quantum group action, it is easy to see that the subalgebra of \mathcal{S} generated by $v_{kj}^{(i)}$ is a Hopf algebra, with $\Delta(v_{kj}^{(i)}) = \sum_l v_{kl}^{(i)} \otimes v_{lj}^{(i)}$. From this, it follows that Π is Hopf-algebra morphism, hence \mathcal{I} is a Woronowicz C^* -ideal. \square

Before we state and prove the main theorem, let us note the following

elementary fact about C^* -algebras.

Lemma 2.13 *Let \mathcal{C} be a C^* algebra and \mathcal{F} be a nonempty collection of C^* -ideals (closed two-sided ideals) of \mathcal{C} . Then for any $x \in \mathcal{C}$, we have*

$$\sup_{I \in \mathcal{F}} \|x + I\| = \|x + I_0\|,$$

where I_0 denotes the intersection of all I in \mathcal{F} and $\|x + I\| = \inf\{\|x - y\| : y \in I\}$ denotes the norm in \mathcal{C}/I .

Proof :

It is clear that $\sup_{I \in \mathcal{F}} \|x + I\|$ defines a norm on \mathcal{C}/I_0 , which is in fact a C^* -norm since each of the quotient norms $\|\cdot + I\|$ is so. Thus the lemma follows from the uniqueness of C^* norm on the C^* algebra \mathcal{C}/I_0 . \square

Theorem 2.14 *For any admissible spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$, the category \mathbf{Q} of quantum families of smooth isometries has a universal (initial) object, say (\mathcal{G}, α_0) . Moreover, \mathcal{G} has a coproduct Δ_0 such that (\mathcal{G}, Δ_0) is a compact quantum group and $(\mathcal{G}, \Delta_0, \alpha_0)$ is a universal object in the category \mathbf{Q}' of compact quantum groups acting smoothly and isometrically on the given spectral triple. The action α_0 is faithful.*

Proof :

Recall the C^* -algebra \mathcal{U} considered before, and the map β from \mathcal{H}_D^0 to $\mathcal{H}_D^0 \otimes \mathcal{U}$. By our definition of β , it is clear that $\beta(\mathcal{A}_0^\infty) \subseteq \mathcal{A}_0^\infty \otimes_{\text{alg}} \mathcal{U}$. However, β is only a linear map (unitary) but not necessarily a $*$ -homomorphism. We shall construct the universal object as a suitable quotient of \mathcal{U} . Let \mathcal{F} be the collection of all those C^* -ideals \mathcal{I} of \mathcal{U} such that the composition $\Gamma_{\mathcal{I}} := (id \otimes \Pi_{\mathcal{I}}) \circ \beta : \mathcal{A}_0^\infty \rightarrow \mathcal{A}_0^\infty \otimes_{\text{alg}} (\mathcal{U}/\mathcal{I})$ extends to a C^* -homomorphism from $\bar{\mathcal{A}}$ to $\bar{\mathcal{A}} \otimes (\mathcal{U}/\mathcal{I})$, where $\Pi_{\mathcal{I}}$ denotes the quotient map from \mathcal{U} onto \mathcal{U}/\mathcal{I} . This collection is nonempty, since the trivial one-dimensional C^* -algebra \mathbb{C} gives an object in \mathbf{Q} and by Lemma 2.12 we do get a member of \mathcal{F} . Now, let \mathcal{I}_0 be the intersection of all ideals in \mathcal{F} . We claim that \mathcal{I}_0 is again a member of \mathcal{F} . Since any C^* -homomorphism is contractive, we have $\|\Gamma_{\mathcal{I}}(a)\| \equiv \|\beta(a) + \bar{\mathcal{A}} \otimes \mathcal{I}\| \leq \|a\|$ for all $a \in \mathcal{A}_0^\infty$ and $\mathcal{I} \in \mathcal{F}$. By Lemma 2.13, we see that $\|\Gamma_{\mathcal{I}_0}(a)\| \leq \|a\|$ for $a \in \mathcal{A}_0^\infty$, so $\Gamma_{\mathcal{I}_0}$ extends to a norm-contractive map on $\bar{\mathcal{A}}$ by the density of \mathcal{A}_0^∞ in $\bar{\mathcal{A}}$. Moreover, for $a, b \in \bar{\mathcal{A}}$ and for $\mathcal{I} \in \mathcal{F}$, we have $\Gamma_{\mathcal{I}}(ab) = \Gamma_{\mathcal{I}}(a)\Gamma_{\mathcal{I}}(b)$. Since $\Pi_{\mathcal{I}} = \Pi_{\mathcal{I}} \circ \Pi_{\mathcal{I}_0}$, we can rewrite the homomorphic property of $\Gamma_{\mathcal{I}}$ as

$$\Gamma_{\mathcal{I}_0}(ab) - \Gamma_{\mathcal{I}_0}(a)\Gamma_{\mathcal{I}_0}(b) \in \bar{\mathcal{A}} \otimes (\mathcal{I}/\mathcal{I}_0).$$

Since this holds for every $\mathcal{I} \in \mathcal{F}$, we conclude that $\Gamma_{\mathcal{I}_0}(ab) - \Gamma_{\mathcal{I}_0}(a)\Gamma_{\mathcal{I}_0}(b) \in \bigcap_{\mathcal{I} \in \mathcal{F}} \bar{\mathcal{A}} \otimes (\mathcal{I}/\mathcal{I}_0) = (0)$, i.e. $\Gamma_{\mathcal{I}_0}$ is a homomorphism. In a similar way, we can show that it is a $*$ -homomorphism. Since each β_i is a unitary representation of the compact quantum group \mathcal{U}_i on the finite dimensional space V_i , it follows that $\beta_i(V_i)(1 \otimes \mathcal{U}_i)$ is total in $V_i \otimes \mathcal{U}_i$. In particular, for any $v_i \in V_i$ (i arbitrary), the element $v_i \otimes 1_{\mathcal{U}_i} = v_i \otimes 1_{\mathcal{U}}$ belongs to the linear span of $\beta_i(V_i)(1 \otimes \mathcal{U}_i) \subset \beta(V_i)(1 \otimes \mathcal{U})$. Thus, $\mathcal{A}_0^\infty \otimes 1_{\mathcal{U}}$ is contained in the linear span of $\beta(\mathcal{A}_0^\infty)(1 \otimes \mathcal{U})$ and hence $\mathcal{A}_0^\infty \otimes 1_{\frac{\mathcal{U}}{\mathcal{I}_0}}$ is linearly spanned by $\Gamma_{\mathcal{I}_0}(\mathcal{A}_0^\infty)(1 \otimes \mathcal{U}/\mathcal{I}_0)$.

By the norm-density of \mathcal{A}_0^∞ in $\bar{\mathcal{A}}$ and the contractivity of the quotient map, it follows that $\bar{\mathcal{A}} \otimes \mathcal{U}/\mathcal{I}_0$ is the closed linear span of $\Gamma_{\mathcal{I}_0}(\mathcal{A}_0^\infty)(1 \otimes \mathcal{U}/\mathcal{I}_0)$. This completes the proof that $(\mathcal{U}/\mathcal{I}_0, \Gamma_{\mathcal{I}_0})$ is indeed an object of \mathbf{Q} .

We now show that $\mathcal{G} := \mathcal{U}/\mathcal{I}_0$ is a universal object in \mathbf{Q} . To see this, consider any object (\mathcal{S}, α) of \mathbf{Q} . Without loss of generality we can assume the action to be faithful, since otherwise we can replace \mathcal{S} by the C^* -subalgebra generated by the elements $\{v_{kj}^{(i)}\}$ appearing in the proof of Lemma 2.12. But by Lemma 2.12 we can further assume that \mathcal{S} is isomorphic with \mathcal{U}/\mathcal{I} for some $\mathcal{I} \in \mathcal{F}$. Since $\mathcal{I}_0 \subseteq \mathcal{I}$, we have a C^* -homomorphism from $\mathcal{U}/\mathcal{I}_0$ onto \mathcal{U}/\mathcal{I} , sending $x + \mathcal{I}_0$ to $x + \mathcal{I}$, which is clearly a morphism in the category \mathbf{Q} . This is indeed the unique such morphism, since it is uniquely determined on the dense subalgebra generated by $\{u_{kj}^{(i)} + \mathcal{I}_0, i, j, k \geq 1\}$ of \mathcal{G} .

To construct the coproduct on $\mathcal{G} = \mathcal{U}/\mathcal{I}_0$, we first consider $\alpha^{(2)} = (\Gamma_{\mathcal{I}_0} \otimes \text{id}) \circ \Gamma_{\mathcal{I}_0} : \bar{\mathcal{A}} \rightarrow \bar{\mathcal{A}} \otimes \mathcal{G} \otimes \mathcal{G}$. It is easy to verify that $(\mathcal{G} \otimes \mathcal{G}, \alpha^{(2)})$ is an object in the category \mathbf{Q} , so by the universality of $(\mathcal{G}, \Gamma_{\mathcal{I}_0})$, we have a unique unital C^* -homomorphism $\Delta_0 : \mathcal{G} \rightarrow \mathcal{G} \otimes \mathcal{G}$ satisfying

$$(\text{id} \otimes \Delta_0) \circ \Gamma_{\mathcal{I}_0}(x) = \alpha^{(2)}(x) \quad \forall x \in \bar{\mathcal{A}}.$$

Taking $x = e_{ij}$, we get

$$\sum_l e_{il} \otimes (\pi_{\mathcal{I}_0} \otimes \pi_{\mathcal{I}_0}) \left(\sum_k u_{lk}^{(i)} \otimes u_{kj}^{(i)} \right) = \sum_l e_{il} \otimes \Delta_0(\pi_{\mathcal{I}_0}(u_{lj}^{(i)})).$$

Comparing coefficients of e_{il} , and recalling that $\tilde{\Delta}(u_{lj}^{(i)}) = \sum_k u_{lk}^{(i)} \otimes u_{kj}^{(i)}$ (where $\tilde{\Delta}$ denotes the coproduct on \mathcal{U}), we have

$$(\pi_{\mathcal{I}_0} \otimes \pi_{\mathcal{I}_0}) \circ \tilde{\Delta} = \Delta_0 \circ \pi_{\mathcal{I}_0} \tag{2}$$

on the linear span of $\{u_{jk}^{(i)}, i, j, k \geq 1\}$, and hence on the whole of \mathcal{U} . This implies that Δ_0 maps $\mathcal{I}_0 = \text{Ker}(\pi_{\mathcal{I}_0})$ into $\text{Ker}(\pi_{\mathcal{I}_0} \otimes \pi_{\mathcal{I}_0}) = (\mathcal{I}_0 \otimes 1 + 1 \otimes \mathcal{I}_0) \subset$

$\mathcal{U} \otimes \mathcal{U}$. In other words, \mathcal{I}_0 is a Hopf C^* -ideal, and hence $\mathcal{G} = \mathcal{U}/\mathcal{I}_0$ has the canonical compact quantum group structure as a quantum subgroup of \mathcal{U} . It is clear from the relation (2) that Δ_0 coincides with the canonical coproduct of the quantum subgroup $\mathcal{U}/\mathcal{I}_0$ inherited from that of \mathcal{U} . It is also easy to see that the object $(\mathcal{G}, \Delta_0, \Gamma_{\mathcal{I}_0})$ is universal in the category \mathbf{Q}' , using the fact that (by Lemma 2.12) any compact quantum group (\mathcal{G}, Φ) acting smoothly and isometrically on the given spectral triple is isomorphic with a quantum subgroup \mathcal{U}/\mathcal{I} , for some Hopf C^* -ideal \mathcal{I} of \mathcal{U} .

Finally, the faithfulness of α_0 follows from the universality by standard arguments which we briefly sketch. If $\mathcal{G}_1 \subset \mathcal{G}$ is a $*$ -subalgebra of \mathcal{G} such that $\alpha_0(\overline{\mathcal{A}}) \subseteq \overline{\mathcal{A}} \otimes \mathcal{G}_1$, it is easy to see that $(\mathcal{G}_1, \Delta_0, \alpha_0)$ is also a universal object, and by definition of universality of \mathcal{G} it follows that there is a unique morphism, say j , from \mathcal{G} to \mathcal{G}_1 . But the map $j \circ i$ is a morphism from \mathcal{G} to itself, where $i : \mathcal{G}_1 \rightarrow \mathcal{G}$ is the inclusion. Again by universality, we have that $j \circ i = \text{id}_{\mathcal{G}}$, so in particular, i is onto, i.e. $\mathcal{G}_1 = \mathcal{G}$. \square

Definition 2.15 *We shall call the universal object (\mathcal{G}, Δ_0) obtained in the theorem above the quantum isometry group of $(\mathcal{A}^\infty, \mathcal{H}, D)$ and denote it by $QISO(\mathcal{A}^\infty, \mathcal{H}, D)$, or just $QISO(\mathcal{A}^\infty)$ (or sometimes $QISO(\overline{\mathcal{A}})$) if the spectral triple is understood from the context.*

Remark 2.16 *Assume that an admissible spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ also satisfies the condition (i) of Lemma 2.5, i.e. $\bigcap \text{Dom}(\mathcal{L}^n) = \mathcal{A}^\infty$. Let $\alpha : \overline{\mathcal{A}} \rightarrow \overline{\mathcal{A}} \otimes \mathcal{S}$ be a smooth isometric action on \mathcal{A}^∞ by a compact quantum group \mathcal{S} . We recall from the proof of Lemma 2.12 that the map $\tilde{\alpha}$ from $\overline{\mathcal{A}} \otimes \mathcal{S}$ to itself extends to an \mathcal{S} -linear unitary on the Hilbert \mathcal{S} -module $\mathcal{H}_D^0 \otimes \mathcal{S}$, i.e. $\tilde{\alpha}$ can be viewed as a unitary in $\mathcal{B}(\mathcal{H}_D^0) \otimes \mathcal{S}$. Clearly, for any state ϕ on \mathcal{S} , we have $\alpha_\phi = (\text{id} \otimes \phi)(\tilde{\alpha}) \in \mathcal{B}(\mathcal{H}_D^0)$. Now, by the definition of a smooth isometric action, the bounded operator α_ϕ commutes with the self-adjoint operator \mathcal{L} on \mathcal{A}_0^∞ , which is a core for \mathcal{L} . So, α_ϕ must commute with \mathcal{L}^n for all n , and in particular keeps $\mathcal{A}^\infty = \bigcap_n \text{Dom}(\mathcal{L}^n)$ invariant.*

Remark 2.17 *Let us now briefly indicate how one can weaken the hypothesis of connectedness. Such an extension of our results is desirable to accommodate the classical spaces, including the finite sets and graphs, in our framework. One possible approach could be to consider the category of compact quantum group actions α which are not only ‘smooth’ and ‘isometric’ in our sense, but also satisfy the τ -invariance condition, i.e. $(\tau \otimes \text{id})(\alpha(a)) = \tau(a)1$. It is easy to see that the connectedness assumption has been used by us only to prove that the τ -invariance is automatic for*

smooth isometric actions. Thus, if we work in the smaller category of such τ -invariant actions only, the proof of Theorem 2.14 does go through and thus we can prove the existence of a universal object, to be defined as the quantum isometry group. It is easy to see that for the algebra of functions on a finite set, with the spectral triple given by $D = 0$, this quantum isometry group coincides with the quantum permutation group defined by Wang.

Remark 2.18 *It is easy to see how to extend our formulation and results to spectral triples which are not necessarily of type II, i.e. when the trace τ is replaced by some non-tracial positive functional. Indeed, our construction will go through in such a situation more or less verbatim, by replacing the universal quantum groups $A_{d_i}(I)$ by $A_{d_i}(Q_i)$ for some suitable choice of matrices Q_i coming from the modularity property of τ .*

2.4 Construction of quantum group-equivariant spectral triples

In this subsection, we shall briefly discuss the relevance of quantum isometry group to the problem of constructing quantum group equivariant spectral triples, which is important to understand the role of quantum groups in the framework of noncommutative geometry. There has been a lot of activity in this direction recently, see, for example, the articles by Chakraborty and Pal ([5]), Connes ([7]), Landi et al ([8]) and the references therein. In the classical situation, there exists a natural unitary representation of the isometry group $G = ISO(M)$ of a manifold M on the Hilbert space of forms, so that the operator $d + d^*$ (where d is the de-Rham differential operator) commutes with the representation. Indeed, $d + d^*$ is also a Dirac operator for the spectral triple given by the natural representation of $C^\infty(M)$ on the Hilbert space of forms, so we have a canonical construction of G -equivariant spectral triple. Our aim in this subsection is to generalize this to the noncommutative framework, by proving that $d_D + d_D^*$ is equivariant with respect to a canonical unitary representation on the Hilbert space of ‘noncommutative forms’ (see, for example, [10] for a detailed discussion of such forms).

Consider an admissible spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ and moreover, make the assumption of Lemma 2.6, i.e. assume that $t \mapsto e^{itD} x e^{-itD}$ is norm-differentiable at $t = 0$ for all x in the $*$ -algebra \mathcal{B} generated by \mathcal{A}^∞ and $[D, \mathcal{A}^\infty]$.

Lemma 2.19 *In the notation of Lemma 2.6, we have the following (where $b, c \in \mathcal{A}^\infty$):*

$$d_D^*(d_D(b)c) = -\frac{1}{2} (b\mathcal{L}(c) - \mathcal{L}(b)c - \mathcal{L}(bc)). \quad (3)$$

Proof:

Denote by $\chi(b, c)$ the right hand side of euqation (3) and fix any $a \in \mathcal{A}^\infty$. Using the facts the the functional τ is a faithful trace on the $*$ -algebra \mathcal{B} , $\mathcal{L} = -d_D^* d_D$ and that $[D, X] = 0$ for any X in \mathcal{B} , we have,

$$\begin{aligned}
\tau(a^* \chi(b, c)) &= -\frac{1}{2} \{ \tau(a^* b \mathcal{L}(c)) + \tau(ca^* \mathcal{L}(b)) + \tau(a^* \mathcal{L}(bc)) \} \\
&= \frac{1}{2} \{ \tau([D, a^* b][D, c]) - \tau([D, ca^*][D, b]) - \tau([D, a^*][D, bc]) \} \\
&= \frac{1}{2} \{ \tau(a^* [D, b][D, c]) - \tau([D, c]a^* [D, b]) - \tau(c[D, a^*][D, b]) - \tau([D, a^*][D, b]c) \} \\
&= -\tau([D, a^*][D, b]c) \\
&= \tau([D, a]^* [D, b]c) \\
&= \langle d_D(a), d_D(b)c \rangle \\
&= \tau(a^* (d_D^* (d_D(b)c))).
\end{aligned}$$

□

From this, we get the following by a simple computation:

$$\langle ad_D(b), a' d_D(b') \rangle = -\frac{1}{2} \tau(b^* \Psi(a^* a, b')), \quad (4)$$

for $a, b, a', b' \in \mathcal{A}^\infty$, and where $\Psi(x, y) := \mathcal{L}(xy) - \mathcal{L}(x)y + x\mathcal{L}(y)$. Now, let us denote the quantum isometry group of the given spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ by $(\mathcal{G}, \Delta, \alpha)$. Let \mathcal{A}_0 denote the $*$ -algebra generated by \mathcal{A}_0^∞ , \mathcal{G}_0 denote $*$ -algebra of \mathcal{G} generated by matrix elements of irreducible representations. Clearly, $\alpha : \mathcal{A}_0 \rightarrow \mathcal{A}_0 \otimes_{\text{alg}} \mathcal{G}_0$ is a Hopf-algebraic action of \mathcal{G}_0 on \mathcal{A}_0 . Define $\tilde{\Psi} : (\mathcal{A}_0 \otimes_{\text{alg}} \mathcal{G}_0) \times (\mathcal{A}_0 \otimes_{\text{alg}} \mathcal{G}_0) \rightarrow \mathcal{A}_0 \otimes_{\text{alg}} \mathcal{G}_0$ by

$$\tilde{\Psi}((x \otimes q), (x' \otimes q')) := \Psi(x, x') \otimes (qq').$$

It follows from the relation $(\mathcal{L} \otimes \text{id}) \circ \alpha = \alpha \circ \mathcal{L}$ on \mathcal{A}_0 that

$$\tilde{\Psi}(\alpha(x), \alpha(y)) = \alpha(\Psi(x, y)). \quad (5)$$

We now define a linear map $\alpha^{(1)}$ from the linear span of $\{ad_D(b) : a, b \in \mathcal{A}_0\}$ to $\mathcal{H}_D^1 \otimes \mathcal{G}$ by setting

$$\alpha^{(1)}(ad_D(b)) := \sum_{i,j} a_i^{(1)} d_D(b_j^{(1)}) \otimes a_i^{(2)} b_j^{(2)},$$

where for any $x \in \mathcal{A}_0$ we write $\alpha(x) = \sum_i x_i^{(1)} \otimes x_i^{(2)} \in \mathcal{A}_0 \otimes_{\text{alg}} \mathcal{G}_0$ (summation over finitely many terms). We shall sometimes use the Sweedler convention of writing the above simply as $\alpha(x) = x^{(1)} \otimes x^{(2)}$. It then follows from the identities (4) and (5), and also the fact that $(\tau \otimes \text{id})(\alpha(a)) = \tau(a)1$ for all $a \in \mathcal{A}_0$ that

$$\begin{aligned}
& \langle ad_D(b), a' d_D(b') \rangle_{\mathcal{G}} \\
&= -\frac{1}{2}(\tau \otimes \text{id})(\alpha(b^*) \tilde{\Psi}(\alpha(a^* a'), \alpha(b'))) \\
&= -\frac{1}{2}(\tau \otimes \text{id})(\alpha(b^*) \alpha(\Psi(a^* a', b'))) \\
&= -\frac{1}{2}(\tau \otimes \text{id})(\alpha(b^* \Psi(a^* a', b'))) \\
&= -\frac{1}{2} \tau(b^* \Psi(a^* a', b')) 1_{\mathcal{G}} \\
&= \langle ad_D(b), a' d_D(b') \rangle_{1_{\mathcal{G}}}.
\end{aligned}$$

This proves that $\alpha^{(1)}$ is indeed well-defined and extends to a \mathcal{G} -linear isometry on $\mathcal{H}_D^1 \otimes \mathcal{G}$, to be denoted by $U^{(1)}$, which sends $(ad_D(b)) \otimes q$ to $\alpha^{(1)}(ad_D(b))(1 \otimes q)$, $a, b \in \mathcal{A}_0$, $q \in \mathcal{G}$. Moreover, since the linear span of $\alpha(\mathcal{A}_0^\infty)(1 \otimes \mathcal{G})$ is dense in $\mathcal{H}_D^0 \otimes \mathcal{G}$, it is easily seen that the range of the isometry $U^{(1)}$ is the whole of $\mathcal{H}_D^1 \otimes \mathcal{G}$, i.e. $U^{(1)}$ is a unitary. In fact, from its definition it can also be shown that $U^{(1)}$ is a unitary representation of the compact quantum group \mathcal{G} on \mathcal{H}_D^1 .

In a similar way, we can construct unitary representation $U^{(n)}$ of \mathcal{G} on the Hilbert space of n -forms for any $n \geq 1$, by defining

$$U^{(n)}((a_0 d_D(a_1) d_D(a_2) \dots d_D(a_n)) \otimes q) = a_0^{(1)} d_D(a_1^{(1)}) \dots d_D(a_n^{(n)}) \otimes (a_0^{(2)} a_1^{(2)} \dots a_n^{(2)} q), \quad a_i \in \mathcal{A}_0^\infty,$$

(using Sweedler convention) and verifying that it extends to a unitary. We also denote by $U^{(0)}$ the unitary representation $\tilde{\alpha}$ on \mathcal{H}_D^0 discussed before. Finally, we have a unitary representation $U = \bigoplus_{n \geq 0} U^{(n)}$ of \mathcal{G} on $\tilde{\mathcal{H}} := \bigoplus_n \mathcal{H}_D^n$, and also extend d_D as a closed densely defined operator on $\tilde{\mathcal{H}}$ in the obvious way, by defining $d_D(a_0 d_D(a_1) \dots d_D(a_n)) = d_D(a_0) \dots d_D(a_n)$. It is now straightforward to see the following:

Theorem 2.20 *The operator $D' := d_D + d_D^*$ is equivariant in the sense that $U(D' \otimes 1) = (D' \otimes 1)U$.*

We point out that there is a natural representation π of $\overline{\mathcal{A}}$ on $\tilde{\mathcal{H}}$ given by $\pi(a)(a_0 d_D(a_1) \dots d_D(a_n)) = a a_0 d_D(a_1) \dots d_D(a_n)$, and $(\pi(\mathcal{A}^\infty), \tilde{\mathcal{H}}, D')$ is indeed a spectral triple, which is \mathcal{G} -equivariant.

Although the relation between spectral properties of D and D' is not clear in general, in many cases of interest (e.g. when there is an underlying type $(1, 1)$ spectral data in the sense of [10]) these two Dirac operators are closely related. As an illustration, consider the canonical spectral on the noncommutative 2-torus \mathcal{A}_θ , which is discussed in some details in the next section. In this case, the Dirac operator D acts on $L^2(\mathcal{A}_\theta, \tau) \otimes \mathbb{C}^2$, and it can easily be shown (see [10]) that the Hilbert space of forms is isomorphic with $L^2(\mathcal{A}_\theta, \tau) \otimes \mathbb{C}^4 \cong L^2(\mathcal{A}_\theta) \otimes \mathbb{C}^2$; thus D' is essentially same as D in this case.

3 Examples and computations

We give some simple yet interesting explicit examples of quantum isometry groups here. However, we give only some computational details for the first example, and for the rest, the reader is referred to a companion article ([3]).

Example 1 : commutative tori

Consider $M = \mathbb{T}$, the one-torus, with the usual Riemannian structure. The $*$ -algebra $\mathcal{A}^\infty = C^\infty(M)$ is generated by one unitary U , which is the multiplication operator by z in $L^2(\mathbb{T})$. The Laplacian is given by $\mathcal{L}(U^n) = -n^2 U^n$. If a compact quantum group $(\mathcal{S}, \Delta_\mathcal{S})$ acts on \mathcal{A}^∞ smoothly, let $A_n, n \in \mathbb{Z}$ be elements of \mathcal{S} such that $\alpha_0(U) = \sum_n U^n \otimes A_n$ (here $\alpha_0 : \mathcal{A}^\infty \rightarrow \mathcal{A}^\infty \otimes_{\text{alg}} \mathcal{S}$ is the \mathcal{S} -action on \mathcal{A}^∞). Note that this infinite sum converges at least in the topology of the Hilbert space $L^2(\mathbb{T}) \otimes L^2(\mathcal{S})$, where $L^2(\mathcal{S})$ denotes the GNS space for the Haar state of \mathcal{S} . It is clear that the condition $(\mathcal{L} \otimes id) \circ \alpha_0 = \alpha_0 \circ \mathcal{L}$ forces to have $A_n = 0$ for all but $n = \pm 1$. The conditions $\alpha_0(U)\alpha_0(U)^* = \alpha_0(U)^*\alpha_0(U) = 1 \otimes 1$ further imply the following:

$$\begin{aligned} A_1^* A_1 + A_{-1}^* A_{-1} &= 1 = A_1 A_1^* + A_{-1} A_{-1}^*, \\ A_1^* A_{-1} &= A_{-1}^* A_1 = A_1 A_{-1}^* = A_{-1} A_1^* = 0. \end{aligned}$$

It follows that $A_{\pm 1}$ are partial isometries with orthogonal domains and ranges. Say, A_1 has domain P and range Q . Hence the domain and range of A_{-1} are respectively $1 - P$ and $1 - Q$. Consider the unitary $V = A + B$, so that $VP = A$, $V(1 - P) = B$. Now, from the fact that $(\mathcal{L} \otimes id)(\alpha_0(U^2)) = \alpha_0(\mathcal{L}(U^2))$ it is easy to see that the coefficient of $1 \otimes 1$ in the expression of $\alpha_0(U)^2$ must be 0, i.e. $AB + BA = 0$. From this, it follows that V and P commute and therefore $P = Q$. By straightforward calculation using the facts that V is unitary, P is a projection and V and P commute, we

can verify that α_0 given by $\alpha_0(U) = U \otimes VP + U^{-1} \otimes V(1 - P)$ extends to a $*$ -homomorphism from \mathcal{A}^∞ to $\mathcal{A}^\infty \otimes C^*(V, P)$ satisfying $(\mathcal{L} \otimes id) \circ \alpha_0 = \alpha_0 \circ \mathcal{L}$. It follows that the C^* algebra $QISO(\mathbb{T})$ is commutative and generated by a unitary V and a projection P , or equivalently by two partial isometries A, B such that $A^*A = AA^*, B^*B = BB^*, AB = BA = 0$. So, as a C^* algebra it is isomorphic with $C(\mathbb{T}) \oplus C(\mathbb{T}) \cong C(\mathbb{T} \times \mathbb{Z}_2)$. The coproduct (say Δ_0) can easily be calculated from the requirement of co-associativity, and the Hopf algebra structure of $QISO(\mathbb{T})$ can be seen to coincide with that of the semi-direct product of \mathbb{T} by \mathbb{Z}_2 , where the generator of \mathbb{Z}_2 acts on \mathbb{T} by sending $z \mapsto \bar{z}$.

We summarize this in form of the following.

Theorem 3.1 *The universal quantum group of isometries $QISO(\mathbb{T})$ of the one-torus \mathbb{T} is isomorphic (as a quantum group) with $C(\mathbb{T} \rtimes_{\mathbb{Z}_2}) = C(ISO(\mathbb{T}))$.*

We can easily extend this result to higher dimensional commutative tori, and can prove that the quantum isometry group coincides with the classical isometry group. This is some kind of rigidity result, and it will be interesting to investigate the nature of quantum isometry groups of more general classical manifolds.

Example 2 : Noncommutative torus; holomorphic isomrtries

Next we consider the simplest and well-known example of noncommutative manifold, namely the noncommutative two-torus \mathcal{A}_θ , where θ is a fixed irrational number (see [6]). It is the universal C^* algebra generated by two unitaries U and V satisfying the commutation relation $UV = \lambda VU$, where $\lambda = e^{2\pi i\theta}$. There is a canonical faithful trace τ on \mathcal{A}_θ given by $\tau(U^m V^n) = \delta_{mn}$. We consider the canonical spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$, where \mathcal{A}^∞ is the unital $*$ -algebra spanned by U, V , $\mathcal{H} = L^2(\tau) \oplus L^2(\tau)$ and D is given by

$$D = \begin{pmatrix} 0 & d_1 + id_2 \\ d_1 - id_2 & 0 \end{pmatrix},$$

where d_1 and d_2 are closed unbounded linear maps on $L^2(\tau)$ given by $d_1(U^m V^n) = mU^m V^n$, $d_2(U^m V^n) = nU^m V^n$. It is easy to compute the space of one-forms Ω_D^1 (see [4], [10], [6]) and the Laplacian $\mathcal{L} = -d^*d$ is given by $\mathcal{L}(U^m V^n) = -(m^2 + n^2)U^m V^n$. For simplicity of computation, instead of the full quantum isometry group we at first concentrate on an interesting quantum subgroup $\mathcal{G} = QISO^{\text{hol}}(\mathcal{A}^\infty, \mathcal{H}, D)$, which is the universal quantum group which leaves invariant the subalgebra of \mathcal{A}^∞ consisting of polynomials in U, V and 1, i.e. span of $U^m V^n$ with $m, n \geq 0$. The proof of existence and uniqueness of such a universal quantum group is more or

less identical to the proof of existence and uniqueness of QISO. We call \mathcal{G} the quantum group of “holomorphic” isometries, and observe in the theorem stated below without proof (see [3]) that this quantum group is nothing but the quantum double torus studied in [11].

Theorem 3.2 *Consider the following co-product $\Delta_{\mathcal{B}}$ on the C^* algebra $\mathcal{B} = C(\mathbb{T}^2) \oplus \mathcal{A}_{2\theta}$, given on the generators A_0, B_0, C_0, D_0 as follows (where A_0, D_0 correspond to $C(\mathbb{T}^2)$ and B_0, C_0 correspond to $\mathcal{A}_{2\theta}$)*

$$\Delta_{\mathcal{B}}(A_0) = A_0 \otimes A_0 + C_0 \otimes B_0, \quad \Delta_{\mathcal{B}}(B_0) = B_0 \otimes A_0 + D_0 \otimes B_0,$$

$$\Delta_{\mathcal{B}}(C_0) = A_0 \otimes C_0 + C_0 \otimes D_0, \quad \Delta_{\mathcal{B}}(D_0) = B_0 \otimes C_0 + D_0 \otimes D_0.$$

Then (\mathcal{B}, Δ_0) is a compact quantum group and it has an action α_0 on \mathcal{A}_θ given by

$$\alpha_0(U) = U \otimes A_0 + V \otimes B_0, \quad \alpha_0(V) = U \otimes C_0 + V \otimes D_0.$$

Moreover, $(\mathcal{B}, \Delta_{\mathcal{B}})$ is isomorphic (as quantum group) with $\mathcal{G} = QISO^{\text{hol}}(\mathcal{A}^\infty, \mathcal{H}, D)$.

We refer to [3] for a proof of the above result, and to [11] for the computation of the Haar stat and representation theory of the compact quantum group \mathcal{G} .

Example 3 : Noncommutative Torus; full quantum isometry group

By similar but somewhat tedious calculations (see [3]) one can also describe explicitly the full quantum isometry group $QISO(\mathcal{A}^\infty, \mathcal{H}, D)$. It is as a C^* algebra has eight direct summands, four of which are isomorphic with the commutative algebra $C(\mathbb{T}^2)$, and the other four are irrational rotation algebras.

Theorem 3.3 *$QISO(\mathcal{A}_\theta) = \oplus_{k=1}^8 C^*(U_{k1}, U_{k2})$ (as a C^* algebra), where for odd k , U_{k1}, U_{k2} are the two commuting unitary generators of $C(\mathbb{T}^2)$, and for even k , $U_{k1}U_{k2} = \exp(4\pi i\theta)U_{k2}U_{k1}$, i.e. they generate $\mathcal{A}_{2\theta}$. The (co)-action on the generators U, V (say) of \mathcal{A}_θ are given by the following :*

$$\alpha_0(U) = U \otimes (U_{11} + U_{31}) + V \otimes (U_{52} + U_{62}) + U^{-1} \otimes (U_{21} + U_{41}) + V^{-1} \otimes (U_{72} + U_{82}),$$

$$\alpha_0(V) = U \otimes (U_{51} + U_{71}) + V \otimes (U_{12} + U_{22}) + U^{-1} \otimes (U_{61} + U_{81}) + V^{-1} \otimes (U_{32} + U_{42}).$$

From the co-associativity condition, the co-product of $QISO(\mathcal{A}_\theta)$ can easily be calculated. For the detailed description of the coproduct, counit, antipode and study of the representation theory of $QISO(\mathcal{A}_\theta)$, the reader is referred to [3]. It is interesting to mention here that the quantum isometry

group of \mathcal{A}_θ is a Rieffel type deformation of the isometry group (which is same as the quantum isometry group) of the commutative two-torus. The commutative two-torus is a subgroup of its isometry group, but when the isometry group is deformed into $QISO(\mathcal{A}_\theta)$, the subgroup relation is not respected, and the deformation of the commutative torus, which is $\mathcal{A}_{2\theta}$, sits in $QISO(\mathcal{A}_\theta)$ just as a C^* subalgebra (in fact a direct summand) but not as a quantum subgroup any more. This perhaps provides some explanation of the non-existence of any Hopf algebra structure on the noncommutative torus.

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