On (k, l, H)-kernels by walks and the H-class digraph

Hortensia Galeana-Sánchez and Miguel Tecpa-Galván

Abstract

Let H be a digraph possibly with loops and D a digraph without loops whose arcs are colored with the vertices of H (D is said to be an H-colored digraph). If $W = (x_0, \ldots, x_n)$ is an open walk in D and $i \in \{1, \ldots, n-1\}$, we say that there is an obstruction on x_i whenever $(color(x_{i-1}, x_i), color(x_i, x_{i+1})) \notin A(H)$.

A (k, l, H)-kernel by walks in an H-colored digraph D $(k \ge 2, l \ge 1)$, is a subset S of vertices of D, such that, for every pair of different vertices in S, every walk between them has at least k-1 obstructions, and for every $x \in V(D) \setminus S$ there exists an xS-walk with at most l-1 obstructions. This concept generalize the concepts of kernel, (k, l)-kernel, kernel by monochromatic paths, and kernel by H-walks. If D is an H-colored digraph, an H-class partition is a partition $\mathscr F$ of A(D) such that, for every $\{(u, v), (v, w)\} \subseteq A(D), (color(u, v), color(v, w)) \in A(H)$ iff there exists F in $\mathscr F$ such that $\{(u, v), (v, w)\} \subseteq F$. The H-class digraph relative to $\mathscr F$, denoted by $C_{\mathscr F}(D)$, is the digraph such that $V(C_{\mathscr F}(D)) = \mathscr F$, and $(F, G) \in A(C_{\mathscr F}(D))$ iff there exist $(u, v) \in F$ and $(v, w) \in G$ with $\{u, v, w\} \subseteq V(D)$.

We will show sufficient conditions on \mathscr{F} and $C_{\mathscr{F}}(D)$ to guarantee the existence of (k,l,H)-kernels by walks in H-colored digraphs, and we will show that some conditions are tight. For instance, we will show that if an H-colored digraph D has an H-class partition in which every class induces a strongly connected digraph, and has a obstruction-free vertex, then for every $k \geq 2$, D has a (k, k-1, H)-kernel by walks. Despite finding (k, l)-kernels is a NP-complete problem, some hypothesis presented in this paper can be verified in polynomial time.

(k, l)-kernel, H-colored digraph, H-kernel, (k, l, H)-kernel by walks. MSC class: 05C15, 05C20, 05C69.

1 Introduction.

For terminology and notation not defined here, we refer the reader to [3]. If D is a digraph and $x \in V(D)$, we denote by $A^+(x)$ the set $\{(x,v) \in A(D) : v \in V(D)\}$, $A^-(x)$ the set $\{(u,x) \in A(D) : u \in V(D)\}$, and $A(x) = A^-(x) \cup A^+(x)$. If D is a digraph without loops, a sink is a vertex x such that $A^+(x) = \emptyset$. If H is a digraph possibly with loops, a sink is a vertex x such that $A^+(x) \subseteq \{(x,x)\}$. If S_1 and S_2 is a subsets of V(D), we denote by $N^+(S_1)$ the proper out-neighbor of S_1 . In this paper we write walk, path and cycle, instead of directed walk, directed path, and directed cycle, respectively. If $W = (x_0, \ldots, x_n)$ is a walk (path), we say that W is an x_0x_n -walk $(x_0x_n$ -path). The length of W is the number n and it is denoted by I(W). If $I_1 = (z_0, \ldots, z_n)$ and $I_2 = (w_0, \ldots, w_m)$ are walks and $I_3 = w_0$, we denote by $I_3 \cup I_3$ the walk $I_3 \cup I_3$ the walk $I_3 \cup I_3$ the successor (respectively predecessor) of $I_3 \cup I_3$ belongs to a walk $I_3 \cup I_4$ the successor (respectively predecessor) of $I_3 \cup I_4$ the successor (respectively predecessor) of $I_3 \cup I_4$ the successor (respectively predecessor) of $I_4 \cup I_4$ the

If S_1 and S_2 are two disjoint subsets of V(D), a uv-walk in D is called an S_1S_2 -walk whenever $u \in S_1$ and $v \in S_2$. If $S_1 = \{x\}$ or $S_2 = \{x\}$, then we write xS_2 -walk or S_1x -walk, respectively. For a nonempty subset $F \subseteq A(D)$ the subdigraph arc-induced by F is denoted by $D\langle F \rangle$.

The concept of kernel was introduced by von Neumann and Morgenstern in [23] as a subset S of vertices of a digraph D, such that for every pair of different vertices in S, there is no arc between them, and every vertex not in S has at least one out-neighbor in S. This concept has been deeply and widely studied by several authors due to a large amount of theoretical and practical applications, by example [7], [9], [10] and [11]. In [6] Chvátal showed that deciding if a digraph has a kernel is an NP-complete problem, and a classical result proved by König [19] shows that every transitive digraph has a kernel.

The concept of kernel has been generalized over the years. A subset S of vertices of D is said to be a kernel by paths, if for every $x \in V(D) \setminus S$, there exists an xS-path (that is, S is absorbent by paths) and, for every pair of different vertices $\{u,v\} \subseteq S$, there is no uv-path in D (that is, S is independent by paths). This concept was introduced by Berge in [5], and it is a well known result that every digraph has a kernel by paths [5] (see Corollary 2 on p. 311). The concept of (k,l)-kernel was introduced by Borowiecki and Kwaśnik in [21] as follows: If $k \geq 2$, a subset S of vertices of a digraph D is a k-independent set, if for every pair of different vertices in S, every walk between them has length at least k. If $l \geq 1$, we say that S is an l-absorbent set if for every $x \in V(D) \setminus S$ there exists an xS-walk with length at most l. If $k \geq 2$ and $k \geq 1$, a k-kernel is a subset of k-kernel is a kernel. Notice that every 2-kernel is a kernel. Several sufficient conditions for the existence of k-kernels have been proved, as example see [14], [15] and [20]. In [14] the authors proved the following theorem:

Theorem 1.1. [14] If D is a symmetric digraph, then D has a k-kernel for every $k \geq 2$. Moreover, every maximal k-independent set in D is a k-kernel.

A digraph is m-colored if its arcs are colored with m colors. If D is an m-colored digraph, a path in D is called monochromatic (respectively, alternating) if all of its arcs are colored alike (respectively, consecutive arcs have different color). A subset S of vertices of D is a alternating by <math>alternating by alternating by <math>alternating by alternating by alternating by alternating <math>alternating by alternating by alternating by alternating <math>alternating by alternating by alternating by an alternating by an alternating by alternating by <math>alternating by alternating by alternating by an alternating by <math>alternating by alternating by alternating by <math>alternating by alternating by an alternating by <math>alternating by alternating by an alternating by <math>alternating by by an alternating by <math>alternation by by an alternating by <math>alternation by by an alternating by <math>alternation by an alternating by an alternating by <math>alternation by an alternating by <math>alternation by an alternating by an alternating by <math>alternation by an alternating by an alternating by <math>alternation by an alternating by an alternating by an alternating by <math>alternation by an alternating by an alterna

If D is an H-colored digraph, an H-class partition of A(D) is a partition of A(D), say \mathscr{F} , such that for every $\{(u,v),(v,w)\}\subseteq A(D), (\rho(u,v),\rho(v,w))\in A(H) \text{ if and only if there exists } F \text{ in } \mathscr{F} \text{ such that } \{(u,v),(v,w)\}\subseteq F.$ If $x\in V(D)$, we define $N_{\mathscr{F}}^-(x)=\{F\in\mathscr{F}:(u,x)\in F \text{ for some } u\in V(D)\},\ N_{\mathscr{F}}^+(x)=\{F\in\mathscr{F}:(x,v)\in F \text{ for some } v\in V(D)\},\ \text{ and } N_{\mathscr{F}}(x)=N_{\mathscr{F}}^+(x)\cup N_{\mathscr{F}}^-(x).$ If \mathscr{F} is an H-class partition of A(D), the H-class digraph relative to \mathscr{F} , denoted by $C_{\mathscr{F}}(D)$, is the digraph such that $V(C_{\mathscr{F}}(D))=\mathscr{F}$, and (F_i,F_j) is an arc in $C_{\mathscr{F}}(D)$, if and only if there exist $(u,v)\in F_i$ and $(v,w)\in F_j$ for some $\{u,v,w\}\subseteq V(D)$. Notice that $C_{\mathscr{F}}(D)$ can allow loops. Moreover, $\mathscr{C}(D)$ is a particular case of $C_{\mathscr{F}}(D)$ when H has only loops, every vertex in H has a loop and every class in \mathscr{F} consist in those arcs colored alike. An H-class partition \mathscr{F} is walk-preservative if for every $(F,G)\in A(C_{\mathscr{F}}(D))$ and $z\in V(D\langle F\rangle)$, there exists a zw-path in $D\langle F\rangle$ for some $w\in V(D\langle G\rangle)$. Notice that $w\in V(D\langle F\rangle)\cap V(D\langle G\rangle)$.

If $W = (x_0, \ldots, x_n)$ is a walk in an H-colored digraph D and $i \in \{0, \ldots, n-1\}$, we say that there is an obstruction on x_i iff $(\rho(x_{i-1}, x_i), \rho(x_i, x_{i+1})) \notin A(H)$ (indices are taken modulo n if $x_0 = x_n$). We denote by $O_H(W)$ the set $\{i \in \{0, \ldots, n-1\} :$ there is an obstruction on $x_i\}$. The H-length of W, denoted by $l_H(W)$, is defined as $l_H(W) = |O_H(W)| + 1$ if W is open, or $l_H(W) = |O_H(W)|$ otherwise. The H-length was primarily studied by Galeana-Sánchez and Sánchez-López in [16] for closed walks, and by Andenmatten, Galeana-Sáchez and Pach in [1] for open paths. Clearly, the usual length l(W) coincides with the H-length $l_H(W)$, in the very particular case when $l_H(W) = \emptyset$. An open walk in an $l_H(W)$ -colored digraph is an $l_H(W)$ -walk if and only if it has $l_H(W)$ -length 1.

It is worth mentioning that in [1], the authors defined the H-length for H-colored graphs and H-colored digraphs. A particular kind of H-coloring in graphs was studied by Szeider in [26] and, as a consequence, in [1] was proved that under the assumption $P \neq NP$, finding uv-paths of minimum H-length in H-colored graphs (H-colored digraphs) has no polynomial solution (although there is a polynomial algorithm to find uv-paths of minimum H-length for some H [1]).

Let D be an H-colored digraph and S a subset of vertices of D. If $l \ge 1$, we say that S is an (l, H)-absorbent set by paths (respectively, (l, H)-absorbent set by walks), if for every vertex $v \in V(D) \setminus S$ there exists a vS-path (respectively, vS-walk) whose H-length is at most l. If $k \ge 2$, we say that S is a (k, H)-independent set by paths (respectively, (k, H)-independent set by walks), if for every pair of different vertices in S, every path (respectively, walk) between them has H-length at least k. If $k \ge 2$ and $l \ge l$, we say that S is a (k, l, H)-kernel by paths (respectively, (k, l, H)-kernel by walks) if it is both (k, H)-independent by paths and (l, H)-absorbent by paths (respectively, (k, H)-independent by walks and (l, H)-absorbent by walks). If l = k - 1, a (k, l, H)-kernel by paths is called a (k, H)-kernel by paths (respectively, (k, H)-kernel by walks). It is straightforward to see that every kernel by H-paths (respectively, kernel by H-walks) is a (2, H)-kernel by paths (respectively, (2, H)-kernel by walks), and every (k, l)-kernel is a (k, l, H)-kernel by paths and a (k, l, H)-kernel by walks in D, if H has no arcs nor loops. Since finding (k, l)-kernels in digraphs is a NP-complete problem, finding

(k, l, H)-kernels in H-colored digraphs is a NP-complete problem.

On the other hand, the concepts of (k, l, H)-kernel by paths and (k, l, H)-kernel by walks are not equivalent. In [4], the authors showed an infinite family of digraphs with (2, H)-kernel by walks and no (2, H)-kernel by paths, and an infinite family of digraphs with (2, H)-kernel by paths and no (2, H)-kernel by walks.

In [18] the authors showed that, by applying conditions on $C_{\mathscr{F}}(D)$, it is possible to guarantee the existence of (k, H)-kernels by walks and (k, H)-kernels by paths in D. In the same spirit, in this paper we will show sufficient conditions on \mathscr{F} and $C_{\mathscr{F}}(D)$ in order to guarantee the existence of (k, l, H)-kernels by walks in H-colored digraphs, and we will show that some conditions are tight. Moreover, some hypothesis presented in this paper can be verified in polynomial time.

2 Preliminary results.

Lemma 2.1. Let D be a symmetric digraph and $\{k,l\} \subseteq \mathbb{N}$. If $2 \le k$ and $k-1 \le l$, then D has a (k,l)-kernel. Proof. It follows from Theorem 2.1 and the definition of (k,l)-kernel.

Lemma 2.2. Let D be an H-colored digraph, \mathscr{F} an H-class partition of A(D) and $x \in V(D)$. The following assertions holds:

- a) If $d^-(x) \neq 0$ and $d^+(x) \neq 0$, then for every $F_1 \in N^-_{\mathscr{F}}(x)$ and $F_2 \in N^+_{\mathscr{F}}(x)$, we have that $(F_1, F_2) \in A(C_{\mathscr{F}}(D))$.
- b) If x is obstruction-free in D and $d(x) \neq 0$, then there is a unique $F \in \mathscr{F}$ such that $x \in V(D(F))$.
- c) If T is a walk in D such that $O_H(T) = \emptyset$, then there is a unique $F \in \mathscr{F}$ such that $A(T) \subseteq F$.
- d) If u, v and w are three different vertices in D, T is a uv-walk, and T' is a vw-H-walk, then either $l_H(T \cup T') = l_H(T)$ or $l_H(T \cup T') = l_H(T) + 1$.
- *Proof.* a) If $F_1 \in N^-_{\mathscr{F}}(x)$ and $F_2 \in N^+_{\mathscr{F}}(x)$, then there exists $\{u,v\} \subseteq V(D)$ such that $(u,x) \in F_1$ and $(x,v) \in F_2$. It follows from definition of $C_{\mathscr{F}}(D)$ that $(F_1,F_2) \in A(C_{\mathscr{F}}(D))$.
- b) Since $d(x) \neq 0$, then there exists $F \in \mathscr{F}$ such that $x \in V(D(F))$. On the other hand, since x is obstruction-free, we have that $A(x) \subseteq F$, concluding that F is unique.
- c) If $T = (x_0, \ldots, x_n)$, then $(\rho(x_{i-1}, x_i), \rho(x_i, x_{i+1})) \in A(H)$ for every $i \in \{0, \ldots, n-1\}$ (indices modulo n if $x_0 = x_n$). Hence, it follows from definition of H-class partition that there is a unique $F \in \mathscr{F}$ such that $A(T) \subseteq F$.
- d) Suppose that $T = (z_0 = u, z_1, \dots, z_n = v)$ and $T' = (z_n = v, z_{n+1}, \dots, z_m = w)$. It is straightforward to see that $O_H(T \cup T') \subseteq O_H(T) \cup \{n\}$ and $O_H(T) \subseteq O_H(T \cup T')$, which implies that either $l_H(T \cup T') = l_H(T)$ or $l_H(T \cup T') = l_H(T) + 1$.

Lemma 2.3. If D is a digraph with no isolated vertices and K is a kernel by paths in D, then for every $x \in K$, $d_D^-(x) \neq 0$.

Proof. Proceeding by contradiction, suppose that there exists $x \in K$ such that $d_D^-(x) = 0$. Since D has no isolated vertices, then there exists $y \in V(D)$ such that $(x,y) \in A(D)$, which implies that $y \notin K$. Hence, there exists a yz-path in D, say P, such that $z \in K$. Since $d_D^-(x) = 0$, we have that $z \neq x$, which implies that $(x,y) \cup P$ is an xz-path in D with $\{x,z\} \subseteq K$, contradicting the independence by paths of K. Therefore, $d_D^-(x) \neq 0$. \square

Lemma 2.4. Let D be an H-colored digraph and \mathscr{F} an H-class partition of A(D). If \mathcal{S} is a nonempty subset of \mathscr{F} and K is a kernel by paths in $D\langle \cup_{F\in\mathcal{S}}F\rangle$, then for every $x\in K$, $N_{\mathscr{F}}^-(x)\cap \mathcal{S}\neq\emptyset$.

Proof. Let K be a kernel by paths in $D' = D\langle \cup_{F \in \mathcal{S}} F \rangle$ and $x \in K$. By Lemma 2.3 we have that $d_{D'}^-(x) \neq 0$. Hence, there exists $z \in V(D')$ such that $(z, x) \in A(D')$. It follows from definition of D' that $(z, x) \in F$ for some $F \in \mathcal{S}$, which implies that $F \in N_{\mathscr{F}}^-(x) \cap \mathcal{S}$. Hence, $N_{\mathscr{F}}^-(x) \cap \mathcal{S} \neq \emptyset$.

Lemma 2.5. Let D be an H-colored digraph and \mathscr{F} an H-class partition of A(D). If \mathcal{S} is an independent set in $C_{\mathscr{F}}(D)$, then $D\langle \cup_{F\in \mathcal{S}}F\rangle$ is an H-subdigraph of D.

Proof. Let $D' = D(\bigcup_{F \in \mathcal{S}} F)$ and $\{(u, v), (v, x)\} \subseteq A(D')$. We will prove that $(\rho(u, v), \rho(v, x)) \in A(H)$. By definition of D', there exists $\{F, G\} \subseteq \mathcal{S}$ such that $(u, v) \in F$ and $(v, x) \in G$. Hence, we have that $(F, G) \in A(C_{\mathscr{F}}(D))$. Since \mathcal{S} is an independent set in $C_{\mathscr{F}}(D)$, then F = G, which implies that $\{(u, v), (v, x)\} \subseteq F$. It follows from the fact that \mathscr{F} is an H-class partition of A(D) that $(\rho(u, v), \rho(v, x)) \in A(H)$. Therefore, D' is an H-subdigraph of D. □

Lemma 2.6. Let D be an H-colored digraph and \mathscr{F} an H-class partition of A(D). If D is strongly connected, then $C_{\mathscr{F}}(D)$ is strongly connected.

Proof. Let F and F' be different vertices in $C_{\mathscr{F}}(D)$, and $\{(x_0, x_1), (z_0, z_1)\} \subseteq A(D)$ such that $(x_0, x_1) \in F$ and $(z_0, z_1) \in F'$. It follows from the fact that D is strongly connected, that there exists an x_0z_1 -walk in D, say $C = (w_0, \ldots, w_n)$, whose initial arc is (x_0, x_1) and ending arc is (z_0, z_1) .

Since $(w_0, w_1) \in F$, $(w_{n-1}, w_n) \in F'$, and $F \neq F'$, we can conclude that $O_H(C) \neq \emptyset$ (Lemma 2.2) (c)). Let $O_H(C) = \{\alpha_1, \ldots, \alpha_r\}$ for some $r \geq 1$, and we can assume that $\alpha_i \leq \alpha_{i+1}$ whenever $i \in \{1, \ldots, r-1\}$. For every $i \in \{1, \ldots, r\}$, let $G_i \in \mathscr{F}$ such that $(w_{\alpha_i}, w_{\alpha_i}^+) \in G_i$. Notice that $P = (G_1, \ldots, G_r)$ is a walk in $C_{\mathscr{F}}(D)$, and $G_r = F'$.

If $\alpha_1 = 0$, then $F = G_1$, which implies that P is an FF'-walk in $C_{\mathscr{F}}(D)$. If $\alpha_1 \neq 0$, then $(F, G_1) \in A(C_{\mathscr{F}}(D))$, which implies that $(F, G_1) \cup P$ is an FF'-walk in $C_{\mathscr{F}}(D)$. Hence, we conclude that $C_{\mathscr{F}}(D)$ is strongly connected.

Corollary 2.7. Let D be a strongly connected H-colored digraph and \mathscr{F} a non-trivial H-class partition of A(D). If \mathcal{S} is an independent set in $C_{\mathscr{F}}(D)$, then $N_{C_{\mathscr{F}}(D)}^+(\mathcal{S}) \neq \emptyset$.

Proof. Since $C_{\mathscr{F}}(D)$ is non-trivial and strongly connected (Lemma 2.6), then $\mathcal{S} \neq V(C_{\mathscr{F}}(D))$. As $C_{\mathscr{F}}(D)$ is strongly connected, there exists an arc from \mathcal{S} to $V(C_{\mathscr{F}}(D)) \setminus \mathcal{S}$, which implies that $N^+_{C_{\mathscr{F}}(D)}(\mathcal{S}) \neq \emptyset$.

Lemma 2.8. Let D be an H-colored digraph, \mathscr{F} an H-class partition of A(D), \mathcal{S} a non-empty subset of $V(C_{\mathscr{F}}(D))$ such that $D\langle F \rangle$ is unilateral for every $F \in \mathcal{S}$, K a kernel by walks in $D\langle \bigcup_{F \in \mathcal{S}} F \rangle$, and $\{x, z\} \subseteq K$. If $x \in V(D\langle F_1 \rangle)$ and $z \in V(D\langle F_2 \rangle)$ for some $\{F_1, F_2\} \subseteq \mathcal{S}$, then $F_1 \neq F_2$.

Proof. Proceeding by contradiction, suppose that $F_1 = F_2$. Since $D\langle F_1 \rangle$ is unilateral, then either there exists an xz-path in $D\langle F_1 \rangle$ or there exists a zx-path in $D\langle F_1 \rangle$, say P. It follows that P is a path in $D\langle \bigcup_{F \in \mathcal{S}} F \rangle$ such that $\{x, z\} \subseteq K$, which contradicts the independence by paths of K. Therefore, $F_1 \neq F_2$.

Lemma 2.9. Let D be an H-colored digraph and \mathscr{F} an H-class partition of A(D) such that for every $F \in \mathscr{F}$, $D\langle F \rangle$ is strongly connected. The following assertions holds:

- a) F is walk-preservative.
- b) $C_{\mathscr{F}}(D)$ is a symmetric digraph.
- c) If $S \subseteq V(C_{\mathscr{F}}(D))$ is an independent set in $C_{\mathscr{F}}(D)$ and $\{F_1, F_2\} \subseteq S$, then $V(D\langle F_1 \rangle) \cap V(D\langle F_2 \rangle) = \emptyset$.

Proof. a) Let $(F,G) \in A(C_{\mathscr{F}}(D))$ and $x \in V(D\langle F \rangle)$. It follows from definition of $C_{\mathscr{F}}(D)$ that there exists $\{u,v,z\} \subseteq V(D)$ such that $(u,v) \in F$ and $(v,z) \in G$. Notice that $v \in V(D\langle F \rangle) \cap V(D\langle G \rangle)$. Since $D\langle F \rangle$ is strongly connected, then there exists an xv-walk in $D\langle F \rangle$, which implies that \mathscr{F} is walk-preservative.

- b) Let $(F_1, F_2) \in A(C_{\mathscr{F}}(D))$. It follows from definition of $C_{\mathscr{F}}(D)$ that there exists $\{u, v, z\} \subseteq V(D)$ such that $(u, v) \in F_1$ and $(v, z) \in F_2$. Since $(u, v) \in A(D\langle F_1 \rangle)$, then $D\langle F_1 \rangle$ is a nontrivial strongly connected digraph, which implies that there exists $u' \in V(D\langle F_1 \rangle)$ such that $(v, u') \in F_1$. In the same way, there exists $z' \in V(D\langle F_2 \rangle)$ such that $(z', v) \in F_2$. It follows from definition of $C_{\mathscr{F}}(D)$ that $(F_2, F_1) \in A(C_{\mathscr{F}}(D))$, concluding that $C_{\mathscr{F}}(D)$ is a symmetric digraph.
- c) Proceeding by contradiction, suppose that $V(D\langle F_1 \rangle) \cap V(D\langle F_2 \rangle) \neq \emptyset$, and consider $x \in V(D\langle F_1 \rangle) \cap V(D\langle F_2 \rangle)$. Since $D\langle F_1 \rangle$ and $D\langle F_2 \rangle$ are non-trivial strongly connected digraphs, then there exist $u \in V(D\langle F_1 \rangle)$ and $z \in V(D\langle F_2 \rangle)$ such that $(u, x) \in A(D\langle F_1 \rangle)$ and $(x, z) \in A(D\langle F_2 \rangle)$. It follows from definition of $C_{\mathscr{F}}(D)$ that $(F_1, F_2) \in A(C_{\mathscr{F}}(D))$, which is no possible since \mathcal{S} is an independent set in $C_{\mathscr{F}}(D)$. Therefore, $V(D\langle F_1 \rangle) \cap V(D\langle F_2 \rangle) = \emptyset$.

Proposition 2.10. Let D be an H-colored digraph and \mathscr{F} a non-trivial walk-preservative H-class partition of A(D) such that $C_{\mathscr{F}}(D)$ has no sinks. If S is an independent set in $C_{\mathscr{F}}(D)$, then there exists a kernel by paths in $D(\bigcup_{F \in S} F)$, say N, such that $N \subseteq V(D(\bigcup_{G \in N^+(S)} G))$.

Proof. We will denote by D_1 the digraph $D\langle \bigcup_{F\in\mathcal{S}}F\rangle$. Since $C_{\mathscr{F}}(D)$ has no sinks, then $N^+(\mathcal{S})\neq\emptyset$. Hence, denote by D_2 the digraph $D\langle \bigcup_{G\in N^+(\mathcal{S})}G\rangle$. Let N be a kernel by paths in D_1 intersecting $V(D_2)$ the most possible, that is, for every kernel by paths in D_1 , say N', we have that

$$|N \setminus V(D_2)| \le |N' \setminus V(D_2)| \tag{1}$$

Notice that possibly $N \cap V(D_2) = \emptyset$. We claim that $N \setminus V(D_2) = \emptyset$. Proceeding by contradiction, suppose that there exists $x_0 \in N \setminus V(D_2)$. Since $x_0 \in V(D_1)$, it follows from definition of D_1 that there exists $F \in \mathcal{S}$ such that $x_0 \in V(D\langle F \rangle)$. By hypothesis, $C_{\mathscr{F}}(D)$ has no sinks, which implies that there exists $G \in V(C_{\mathscr{F}}(D))$ such that $F \neq G$ and $(F,G) \in A(C_{\mathscr{F}}(D))$. It follows from the fact that \mathscr{F} is a walk-preservative H-class

partition that there exists an x_0z -path in $D\langle F \rangle$, say C, for some $z \in V(D\langle G \rangle)$. Notice that C is a path in D_1 , $z \in V(D_1) \cap V(D_2)$, and $z \neq x_0$. Moreover, since N is independent by paths in D_1 , then $z \notin N$. We will prove the following claims in order to get a contradiction.

Claim 1. There exists a zx_0 -path in D_1 .

Since $z \notin N$, there exists a zy-path in D_1 for some $y \in N$, say P'. It follows that $C \cup P'$ is an x_0y -walk in D_1 such that $x_0 \in N$ and $y \in N$, which implies that $x_0 = y$, concluding that P' is a zx_0 -path in D_1 .

Claim 2. $(N \setminus \{x_0\}) \cup \{z\}$ is a kernel by paths in D_1 .

In order to show that $N' = (N \setminus \{x_0\}) \cup \{z\}$ is an absorbent set by paths in D_1 , consider $u \in V(D_1) \setminus N'$. If $u = x_0$, then C is an $x_0 N'$ -path in D_1 . If $u \neq x_0$, then $u \in V(D_1) \setminus N$. It follows from the fact that N is an absorbent set by paths in D_1 that there exists a uv-path in D_1 , say P, for some $v \in N$. If $v \neq x_0$, then P is a uN'-path in D_1 . If $v=x_0$, then $P\cup C$ is a uN'-walk in D_1 , concluding that N' is an absorbent set by paths in D_1 .

Now we will show that N' is an independent set by paths in D_1 . Proceeding by contradiction, suppose that there exists a uv-path in D_1 , say T, where $\{u,v\}\subseteq N'$. Since N is an independent set by paths in D_1 , then $z \in \{u, v\}$. If z = u, it follows that $C \cup T$ is an x_0v -walk in D_1 , which contradicts the independence by paths of N. If v=z, by Claim 1 there exists a zx_0 -path in D_1 , say P, which implies that $T \cup P$ is a ux_0 -walk in D_1 , contradicting the independence by paths of N. Therefore, N' is an independent set by paths in D_1 , and the claim holds.

Notice that $|N' \setminus V(D_2)| = |N \setminus V(D_2)| - 1$, which is no possible by (1). Therefore, $N \setminus V(D_2) = \emptyset$, concluding that $N \subseteq V(D_2)$.

Proposition 2.11. Let D be an H-colored digraph with no isolated vertices, \mathscr{F} a walk-preservative H-class partition of A(D), and S an independent and l-absorbent set in $C_{\mathscr{F}}(D)$ for some $l \geq 1$. If K is a kernel by paths in $D(\bigcup_{F \in S} F)$, then K is an (l+1, H)-absorbent set by walks in D.

Proof. Let $D' = D(\bigcup_{F \in S} F)$ and $x_0 \in V(D) \setminus K$. If $x_0 \in V(D')$, since K is a kernel by paths in D', there exists an x_0K -path in D', say T. It follows from Lemma 2.5 that T is an H-path. Hence, $l_H(T)=1$, which implies that $l_H(T) \leq l+1$.

Now we will assume that $x_0 \notin V(D')$. By hypothesis, x_0 is no isolated, which implies that $x_0 \in V(D(F_0))$ for some $F_0 \in \mathscr{F}$. Notice that $F_0 \notin \mathcal{S}$ because $x_0 \notin V(D')$. Since \mathcal{S} is an l-absorbent set in $C_{\mathscr{F}}(D)$, we can consider an F_0F_t -path with minimum length in $C_{\mathscr{F}}(D)$, say $T=(F_0,\ldots,F_r)$, where $F_r\in \mathcal{S}$. Notice that $r\leq l$.

Since \mathscr{F} is walk-preservative, there exists $x_{\alpha_1} \in V(D\langle F_0 \rangle) \cap V(D\langle F_1 \rangle)$ such that there exists an $x_0 x_{\alpha_1}$ path in $D\langle F_0 \rangle$, say T_0 , and, for every $i \in \{2, \ldots, r-1\}$, there exist $x_{\alpha_i} \in V(D\langle F_{i-1} \rangle) \cap V(D\langle F_i \rangle)$ and $x_{\alpha_{i+1}} \in V(D\langle F_i \rangle) \cap V(D\langle F_{i+1} \rangle)$, such that there exists an $x_{\alpha_i} x_{\alpha_{i+1}}$ -path in $D\langle F_i \rangle$, say T_i . Notice that T_i is an H-path for every $i \in \{0, \ldots, r-1\}$, and $x_{\alpha_r} \in V(D')$. Moreover, since C is a path in $C_{\mathscr{F}}(D)$, then $A(T_i) \cap A(T_j) = \emptyset$ whenever $i \neq j$.

Now we consider $C = \bigcup_{i=0}^{r-1} T_i$, and suppose that $C = (z_1, \ldots, z_n)$. Notice that $z_n \in V(D')$ (because $z_n = x_{\alpha_r}$).

Claim 1. $l_H(C) \le l + 1$.

For every $i \in \{0, \ldots, r-2\}$, let $U_i = \{l \in \{1, \ldots, n-1\} : (z_{l-1}, z_l) \in F_i\}$ be, and $L = \{i \in \{0, \ldots, r-2\} : (z_{l-1}, z_l) \in F_i\}$ be, and $L = \{i \in \{0, \ldots, r-2\} : (z_{l-1}, z_l) \in F_i\}$ $U_i \neq \emptyset$. For every $i \in L$, define $\beta_i = maxU_i$.

We will show that $O_H(C) \subseteq \{\beta_i : i \in L\}$. If $m \in O_H(C)$, then it follows that $(\rho(z_{m-1}, z_m), \rho(z_m, z_{m+1})) \notin$ A(H). On the other hand, we have that $(z_{m-1}, z_m) \in A(T_j)$ for some $j \in \{0, \ldots, r-1\}$. Since T_j is an H-path, then $(z_m, z_{m+1}) \notin A(T_j)$, which implies that $m = \max U_j$ and $j \leq r - 2$. Hence, $m \in \{\beta_i : i \in L\}$. It follows that $O_H(C) \subseteq L$. Hence, $|O_H(C)| \leq |L|$, that is $|O_H(C)| \leq r - 1$, and we can conclude that $l_H(C) \leq l$.

If $z_n \in K$, then by Claim 1 we have that C is a x_0K -path with $l_H(C) \le l+1$. If $z_t \notin K$, since $z_n \in V(D')$, and K is an absorbent set by paths in D', then there exists a z_nK -path in D', say T_r . Hence, $C' = C \cup T_r$ is an x_0K -walk in D, and by Lemma 2.2, $l_H(C') \leq l_H(C) + 1$, which implies that $l_H(C') \leq l + 1$.

Therefore, K is an (l+1, H)-absorbent set by walks in D.

Notice that the conclusions in Proposition 2.11 are tight. We will show an example where K is not necessarily an (r, H)-absorbent set by walks in D for some $r \leq l + 1$. Consider the H-colored digraph showed in Figure 1, and for every $i \in \{1, ..., 6\}$ let $F_i = \{e \in A(D) : \rho(e) = c_i\}$. Clearly, $\mathscr{F} = \{F_i : i \in \{1, ..., 6\}\}$ is a walkpreservative H-class partition of A(D). Notice that $S = \{F_6\}$ is a 3-absorbent set in $C_{\mathscr{F}}(D)$. On the other hand, it is straightforward to see that $K = \{x_4\}$ is a kernel by paths in $D(F_6)$ which is not an (r, H)-absorbent by walks in D for every $r \in \{1, 2, 3\}$.

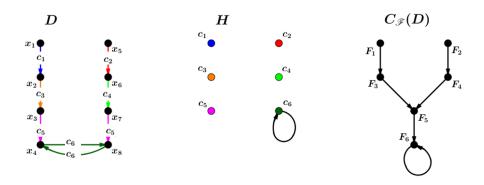


Figure 1:

On the other hand, if D is an H-colored digraph and \mathscr{F} is an H-class partition which is no walk-preservative, then Proposition 2.11 is not necessarily true. Consider the H-colored digraph showed in Figure 2. For every $i \in \{1, \ldots, 5\}$ let $F_i = \{e \in A(D) : \rho(e) = c_i\}$. It is straightforward to see that $\mathscr{F} = \{F_i : i \in \{1, \ldots, 5\}\}$ is an H-class partition of A(D). Notice that (F_2, F_3) is an arc of $C_{\mathscr{F}}(D)$ such that $x_3 \in V(D\langle F_2\rangle)$, and there is no x_3w -walk in $D\langle F_2\rangle$ with $w \in V(D\langle F_3\rangle)$, that is, \mathscr{F} is not a walk-preservative H-class partition of A(D). On the other hand, $S = \{F_5\}$ is a 4-absorbent set in $C_{\mathscr{F}}(D)$ but no kernel by paths in $D\langle F_5\rangle$ is a (5, H)-absorbent set by walks in D.

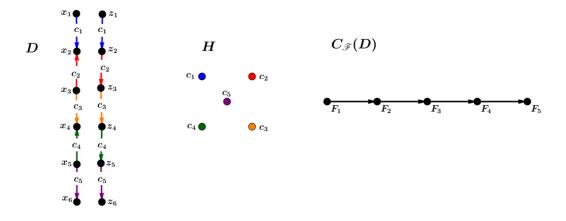


Figure 2:

3 Main results.

The following lemmas will be useful.

Lemma 3.1. Let D be a digraph with at least one arc, $W = \{x \in V(D) : d(x) = 0\}$, and $\{k,l\} \subseteq \mathbb{N}$ such that $2 \le k$ and $1 \le l$. If K is a (k,l,H)-kernel by walks in D - W, then $K \cup W$ is a (k,l,H)-kernel by walks in D.

Lemma 3.2. If D is an H-digraph, then every kernel by paths in D is a (k, l, H)-kernel by walks in D for every $k \ge 2$ and $l \ge 1$.

Proposition 3.3. Let D be an H-colored digraph with no isolated vertices, \mathscr{F} a walk-preservative H-class partition of A(D), and $S \subseteq V(C_{\mathscr{F}}(D))$ an independent and l-absorbent set in $C_{\mathscr{F}}(D)$ for some $l \geq 1$. If $N^+(S) = \emptyset$ and $k \geq 2$, then every kernel by paths in $D \setminus \bigcup_{F \in S} F \setminus S$ is a (k, l+1, H)-kernel by walks in D.

Proof. Let K be a kernel by paths in $D' = D\langle \cup_{F \in \mathcal{S}} F \rangle$. Since \mathscr{F} is walk preservative, it follows from Proposition 2.11 that K is an (l+1,H)-absorbent set by walks in D. Now, in order to prove that K is a (k,H)-independent set by walks in D for every $k \geq 2$, we will show that K is a path-independent set in D. Proceeding by contradiction, suppose that there exists an x_1x_n -path in D, say $T' = (x_1, \ldots, x_n)$, such that $\{x_1, x_n\} \subseteq K$. Consider $F_0 \in N_{\mathscr{F}}(x_1) \cap \mathcal{S}$ (Lemma 2.4). Hence, there exists $x_0 \in V(D)$ such that $(x_0, x_1) \in F_0 \cap A^-(x_1)$, and let $T = (x_0, x_1) \cup T'$ be.

If $A(T) \subseteq F_0$, then $A(T') \subseteq F_0$, which implies that T' is an x_1x_n -path in D', contradicting the fact that K is an independent set by paths in D'. Hence, $A(T) \not\subseteq F_0$. Let $t = min\{i \in \{1, ..., n-1\} : (x_i, x_{i+1}) \notin F_0\}$. It

follows that $(x_{t-1}, x_t) \in F_0$, and $(x_t, x_{t+1}) \in G$ for some $G \in \mathscr{F}$ with $F \neq G$. Notice that $(F, G) \in A(C_{\mathscr{F}}(D))$ and, since S is and independent set in $C_{\mathscr{F}}(D)$, then $G \notin S$. Therefore, $G \in N^+(S)$ which contradicts the assumption that $N^+(S) = \emptyset$. Hence, K is path-independent in D, which implies that K is a (k, H)-independent set by walks in D for every $k \geq 2$.

Therefore K is a (k, l+1, H)-kernel by walks in D for every $k \geq 2$.

Corollary 3.4. Let D be an H-colored digraph, \mathscr{F} a walk-preservative H-class partition of A(D), and $\{k,l\} \subseteq \mathbb{N}$ such that $k \geq 2$ and $l \geq 1$. If \mathcal{S} is a (k,l)-kernel in $C_{\mathscr{F}}(D)$ such that $N^+(\mathcal{S}) = \emptyset$, then D has a (k,l+1,H)-kernel by walks.

Proof. Let $W = \{x \in V(D) : d(x) = 0\}$ and K' a (k, l)-kernel in $C_{\mathscr{F}}(D)$ such that $N^+(\mathcal{S}) = \emptyset$. Since K' is an independent and l-absorbent set in $C_{\mathscr{F}}(D)$, it follows from Proposition 3.3 that D - W has a (k, l, H)-kernel by walks, say K. By Lemma 3.1, we can conclude that $K \cup W$ is a (k, l, H)-kernel by walks in D.

Proposition 3.5. Let D be an H-colored digraph, and \mathscr{F} a walk-preservative H-class partition of A(D) such that $C_{\mathscr{F}}(D)$ has a (k,l)-kernel, say \mathscr{S} . If the following conditions holds:

- a) $C_{\mathscr{F}}(D)$ has no sinks, and every cycle in $C_{\mathscr{F}}(D)$ is either a loop or has length at least k.
- b) For every $x \in V(D)$ such that $N_{\mathscr{F}}(x) \cap S \neq \emptyset$ and $N_{\mathscr{F}}(x) \cap N^+(\mathcal{S}) \neq \emptyset$, we have that $N_{\mathscr{F}}^-(x) \subseteq \mathcal{S}$.

Then D has a (k, l+1, H)-kernel by walks.

Proof. First, suppose that D has no isolated vertices. Let $D_1 = D\langle \cup_{F \in \mathcal{S}} F \rangle$. Since $C_{\mathscr{F}}(D)$ has no sinks and \mathcal{S} is an independent set in $C_{\mathscr{F}}(D)$, then $N^+(\mathcal{S}) \neq \emptyset$. Now, let $D_2 = D\langle \cup_{G \in N^+(\mathcal{S})} G \rangle$. By Proposition 2.10, we can consider a kernel by paths in D_1 , say K, such that $K \subseteq V(D_2)$. We will show that K is a (k, l+1, H)-kernel by walks in D.

Since \mathscr{F} is walk-preservative, it follows from Proposition 2.11 that K is an (l+1,H)-absorbent set by walks in D. It only remains to show that K is a (k,H)-independent set by walks in D. First, we will prove the following useful claim.

Claim 1. For every $x \in K$, $N_{\mathscr{F}}^-(x) \subseteq A(D_1)$.

If $x \in K$, then $x \in V(D_1) \cap V(D_2)$. Since $x \in V(D_1)$, it follows from definition of D_1 that there exists $F \in \mathcal{S}$ such that $A(x) \cap F \neq \emptyset$. Hence, $N_{\mathscr{F}}(x) \cap \mathcal{S} \neq \emptyset$. On the other hand, since $x \in V(D_2)$, an analogous proof will show that $N_{\mathscr{F}}(x) \cap N^+(\mathcal{S}) \neq \emptyset$. By hypothesis (b), we can conclude that $N_{\mathscr{F}}(x) \subseteq \mathcal{S}$.

In order to show that K is a (k, H)-independent set by walks in D, consider an x_0x_n -walk in D, say $T = (x_0, \ldots, x_n)$, such that $\{x_0, x_n\} \subseteq K$.

Claim 2. $O_H(T) \neq \emptyset$.

Proceeding by contradiction, suppose that $O_H(T) = \emptyset$. Hence, there exists $F' \in \mathscr{F}$ such that $A(T) \subseteq F'$, which implies that $(x_{n-1}, x_n) \in F'$. It follows from Claim 1 that $F' \in \mathcal{S}$. Hence, T is an x_0x_n -walk in D_1 , which contradicts the fact that K is an independent set by paths in D_1 . Therefore, $O_H(T) \neq \emptyset$ and the claim holds.

By Claim 2, suppose that $O_H(T) = \{\alpha_i : i \in \{1, ..., t\}\}$ where $t \geq 1$, and $\alpha_i \leq \alpha_{i+1}$ for every $i \in \{1, ..., t-1\}$. On the other hand, for every $i \in \{1, ..., t\}$ consider $F_i \in \mathscr{F}$ such that $(x_{\alpha_i}, x_{\alpha_i}^+) \in F_i$, and $F_0 \in \mathscr{F}$ such that $(x_0, x_1) \in F_0$. It follows from definition of $C_{\mathscr{F}}(D)$ that $T' = (F_0, F_1, ..., F_t)$ is a walk in $C_{\mathscr{F}}(D)$. Notice that $I_H(T) = I(T') + 1$ and, by Claim 1, $F_t \in \mathscr{S}$. Consider the following cases:

Case 1. $F_0 \in \mathcal{S}$.

If $F_0 \neq F_t$, as S is a k-independent set in $C_{\mathscr{F}}(D)$, then $l(T') \geq k$, which implies that $l_H(T) \geq k$. If $F_0 = F_t$, then T' is a closed walk in $C_{\mathscr{F}}(D)$ which is not a loop and, by hypothesis, $l(T') \geq k$. Hence, $l_H(T) \geq k$.

Case 2. $F_0 \in V(C_{\mathscr{F}}(D)) \setminus \mathcal{S}$.

By Lemma 2.4, consider $F \in N_{\mathscr{F}}(x_0) \cap \mathcal{S}$. Since $F_0 \notin \mathcal{S}$, then $F \neq F_0$. It follows from definition of $C_{\mathscr{F}}(D)$ that $T'' = (F, F_0) \cup T'$ is a walk in $C_{\mathscr{F}}(D)$. Notice that $l_H(T) = l(T'')$. If $F \neq F_t$, as \mathcal{S} is a k-independent set in $C_{\mathscr{F}}(D)$, we have that $l(T'') \geq k$, which implies that $l_H(T) \geq k$. If $F = F_t$, then T'' is a closed walk in $C_{\mathscr{F}}(D)$ which is not a loop and, by hypothesis, $l(T'') \geq k$. Hence, $l_H(T) \geq k$.

It follows from Case 1 and Case 2 that K is a (k, H)-independent set by walks in D. Therefore, K is a (k, l+1)-kernel by walks in D.

Now, suppose that $W = \{x \in V(D) : d(x) = 0\}$ is nonempty. By the the previous proof, we have that D - W has a (k, l+1, H)-kernel by walks, say K. By Lemma 3.1, we can conclude that $K \cup W$ is a (k, l+1, H)-kernel by walks in D.

As a consequence of Proposition 3.5, we have the following result for strongly connected digraphs.

Corollary 3.6. Let D be a strongly connected H-colored digraph and \mathscr{F} a walk-preservative H-class partition of A(D) such that $C_{\mathscr{F}}(D)$ has a (k,l)-kernel, say \mathscr{S} . If the following conditions holds:

- a) Every cycle in $C_{\mathscr{F}}(D)$ is either a loop or has length at least k.
- b) For every $x \in V(D)$ such that $N_{\mathscr{F}}(x) \cap S \neq \emptyset$ and $N_{\mathscr{F}}(x) \cap N^{+}(S) \neq \emptyset$, we have that $N_{\mathscr{F}}^{-}(x) \subseteq S$.

Then D has a (k, l+1, H)-kernel by walks.

Proof. If D is an H-digraph, it follows from Lemma 3.2 that for every $k \geq 2$ and $l \geq 1$, D has a (k, l, H)-kernel by walks. Hence, we may assume that D is not an H-digraph. It follows that $C_{\mathscr{F}}(D)$ is a non-trivial strongly connected digraph (Lemma 2.6), which implies that $C_{\mathscr{F}}(D)$ has no sinks. By Proposition 3.5 we have that D has a (k, l+1, H)-kernel by walks.

Proposition 3.7. Let D be an H-colored digraph with no isolated vertices, \mathscr{F} a walk-preservative H-class partition of A(D), and \mathcal{S} a (k,l)-kernel in $C_{\mathscr{F}}(D)$ such that $k \geq 3$ and $l \geq 1$. If for every $F \in \mathcal{S}$, $D\langle F \rangle$ is unilateral and has no sinks, then every kernel by paths in $D\langle \bigcup_{F \in \mathcal{S}} F \rangle$ is a (k-1,l+1,H)-kernel by walks in D.

Proof. Let K be a kernel by paths in $D_1 = D(\bigcup_{F \in \mathcal{S}} F)$. Since \mathscr{F} is a walk-preservative H-class partition, it follows from Proposition 2.11 that K is an (l+1,H)-absorbent set by walks in D. It only remains to show that K is a (k-1,H)-independent set by walks in D. Consider a walk in D, say $C = (x_0, \ldots, x_n)$, such that $\{x_0, x_n\} \subseteq K$.

Claim 1. $N_{\mathscr{F}}^-(x_0) \cap \mathcal{S} \neq \emptyset$ and $N_{\mathscr{F}}^+(x_n) \cap \mathcal{S} \neq \emptyset$.

Since $x_0 \in K$, it follows from Lemma 2.4 that $N_{\mathscr{F}}(x_0) \cap \mathcal{S} \neq \emptyset$. On the other hand, since $x_n \in V(D_1)$, then $x_n \in V(D\langle F \rangle)$ for some $F \in \mathcal{S}$. By hypothesis, $D\langle F \rangle$ has no sinks, which implies that $A^+(x_n) \cap F \neq \emptyset$. Hence $F \in N_{\mathscr{F}}^+(x_n)$, concluding that $F \in N_{\mathscr{F}}^+(x_n) \cap \mathcal{S}$, and the claim holds.

By Claim 1, consider $F' \in N_{\mathscr{F}}^-(x_0) \cap \mathcal{S}$ and $F'' \in N_{\mathscr{F}}^+(x_n) \cap \mathcal{S}$. It follows from Lemma 2.8 that $F' \neq F''$ (*).

Claim 2. $O_H(C) \neq \emptyset$.

Proceeding by contradiction, suppose that $O_H(C) = \emptyset$. Hence, there exists $F \in \mathscr{F}$ such that $A(C) \subseteq F$. By definition of $C_{\mathscr{F}}(D)$, we have that $\{(F',F),(F,F'')\}\subseteq A(C_{\mathscr{F}}(D))$. Since $\{F',F''\}\subseteq \mathcal{S}$ and \mathcal{S} is an independent set in $C_{\mathscr{F}}(D)$, then $F \neq F'$ and $F \neq F''$. Hence, (F',F,F'') is an F'F''-path in $C_{\mathscr{F}}(D)$, which is no possible since \mathcal{S} is a k-independent set with $k \geq 3$. Therefore, $O_H(C) \neq \emptyset$, and the claim holds.

By Claim 2, suppose that $O_H(C) = \{\alpha_i : i \in \{1, \dots, t\}\}$ where $t \geq 1$, and $\alpha_i \leq \alpha_{i+1}$ for every $i \in \{1, \dots, t-1\}$. For every $i \in \{1, \dots, t\}$, let $F_i \in \mathscr{F}$ such that $(x_{\alpha_i}, x_{\alpha_i}^+) \in F_i$, and $F_0 \in \mathscr{F}$ such that $(x_0, x_1) \in F_0$. Notice that $x_0 \in V(D\langle F_0 \rangle)$ and $x_n \in V(D\langle F_t \rangle)$. On the other hand, it follows from definition of $C_{\mathscr{F}}(D)$ that $C_0 = (F_0, F_1, \dots, F_t)$ is a walk in $C_{\mathscr{F}}(D)$. Consider the following cases:

Case 1. $F_0 \in \mathcal{S}$.

First, suppose that $F_t \in \mathcal{S}$. Notice that C_0 is a walk in $C_{\mathscr{F}}(D)$ such that $l_H(C) - 1 = l(C_0)$. On the other hand, we have that $\{F_0, F_t\} \subseteq \mathcal{S}$, $x_0 \in V(D\langle F_0 \rangle)$ and $x_n \in V(D\langle F_t \rangle)$, which implies that $F_0 \neq F_t$ (Lemma 2.8). Since \mathcal{S} is a k-independent set in $C_{\mathscr{F}}(D)$, we have that $l(C_0) \geq k$. We can conclude that $l_H(C) \geq k - 1$.

Now suppose that $F_t \notin \mathcal{S}$, and consider $C_1 = C_0 \cup (F_t, F'')$. Notice that C_1 is a walk in $C_{\mathscr{F}}(D)$ such that $l_H(C) = l(C_1)$. On the other hand, we have that $\{F_0, F''\} \subseteq \mathcal{S}$, $x_0 \in V(D\langle F_0 \rangle)$ and $x_n \in V(D\langle F'' \rangle)$, which implies that $F_0 \neq F''$ (Lemma 2.8). Since \mathcal{S} is a k-independent set in $C_{\mathscr{F}}(D)$, then $l(C_1) \geq k$. We can conclude that $l_H(C) \geq k - 1$.

Case 2. $F_0 \in V(C_{\mathscr{F}}(D)) \setminus \mathcal{S}$.

First, suppose that $F_t \in \mathcal{S}$, and consider $C_2 = (F', F_0) \cup C_0$. Notice that C_2 is a walk in $C_{\mathscr{F}}(D)$ such that $l_H(C) = l(C_2)$. On the other hand, we have that $\{F', F_t\} \subseteq \mathcal{S}$, $x_0 \in V(D\langle F' \rangle)$ and $x_n \in V(D\langle F_t \rangle)$, which implies that $F' \neq F_t$ (Lemma 2.8). Since \mathcal{S} is a k-independent set in $C_{\mathscr{F}}(D)$, then $l(C_2) \geq k$. We can conclude that $l_H(C) \geq k - 1$.

Now suppose that $F_t \notin \mathcal{S}$, and consider $C_3 = (F', F_0) \cup C_0 \cup (F_t, F'')$. Notice that C_3 is a walk in $C_{\mathscr{F}}(D)$ such that $l_H(C) + 1 = l(C_3)$. On the other hand, by (*) we have that $F' \neq F''$ and, since \mathcal{S} is a k-independent set in $C_{\mathscr{F}}(D)$, then $l(C_3) \geq k$. We can conclude that $l_H(C) \geq k - 1$.

It follows from the previous cases that K is a (k-1,H)-independent set by walks in D. Therefore, K is a (k-1,l+1)-kernel by walks in D.

Corollary 3.8. Let D be an H-colored digraph, \mathscr{F} a walk-preservative H-class partition of A(D) such that for every $F \in \mathscr{F}$, $D\langle F \rangle$ is unilateral and has no sinks. If $C_{\mathscr{F}}(D)$ has a (k,l)-kernel for some $k \geq 3$ and $l \geq 1$, then D has a (k-1,l+1,H)-kernel by walks.

Proof. First, suppose that D has no isolated vertices. If S is a (k,l)-kernel in $C_{\mathscr{F}}(D)$ with $k \geq 3$ and $l \geq 1$, then by Proposition 3.7, every kernel by paths in $D \langle \bigcup_{F \in S} F \rangle$ is a (k-1,l+1,H)-kernel by walks in D.

On the other hand, if $W = \{x \in V(D) : d(x) = 0\}$ is nonempty, then by the previous proof, we have that D - W has a (k - 1, l + 1, H)-kernel by walks, say K. By Lemma 3.1, we can conclude that $K \cup W$ is a (k - 1, l + 1, H)-kernel by walks in D.

Corollary 3.9. Let D be an H-colored digraph and \mathscr{F} an H-class partition of A(D). If for every $F \in \mathscr{F}$ we have that $D\langle F \rangle$ is strongly connected, then for every $k \geq 2$ and $l \geq k+1$, D has a (k,l,H)-kernel by walks.

Proof. First, suppose that D has no isolated vertices. Since $D\langle F \rangle$ is strongly connected for every $F \in \mathscr{F}$, we have that \mathscr{F} is walk-preservative (Lemma 2.9 (a)), and $D\langle F \rangle$ is unilateral and has no sinks for every $F \in \mathscr{F}$. On the other hand, by Lemma 2.9 (b), we have that $C_{\mathscr{F}}(D)$ is a symmetric digraph, which implies that $C_{\mathscr{F}}(D)$ has a (k+1,l-1)-kernel for every $k+1 \geq 3$ and $l-1 \geq k$ (Lemma 2.1). Hence, by Corollary 3.8, we have that D has a (k,l,H)-kernel by walks for every $k \geq 2$ and $l \geq k+1$.

On the other hand, if $W = \{x \in V(D) : d(x) = 0\}$ is nonempty, then by the previous proof, we have that D - W has a (k, l, H)-kernel by walks for every $k \ge 2$ and $l \ge k + 1$. By Lemma 3.1, we can conclude that D has a (k, l, H)-kernel by walks for every $k \ge 2$ and $l \ge k + 1$.

Proposition 3.10. Let D be an H-colored digraph, \mathscr{F} an H-class partition of A(D) and \mathscr{S} a (k,l)-kernel of $C_{\mathscr{F}}(D)$ for some $k \geq 3$ and $l \geq 1$. If for every $F \in \mathscr{S}$, $D\langle F \rangle$ is strongly connected and has a obstruction-free vertex in D, then D has a (k+1,l+1,H)-kernel by walks.

Proof. First, suppose that D has no isolated vertices. Let $S = \{F_1, \ldots, F_r\}$ for some $r \geq 1$, and for every $i \in \{1, \ldots, r\}$, let $z_i \in V(D\langle F_i \rangle)$ such that z_i is obstruction-free in D. By Lemma 2.9 (c) we have that $z_i \neq z_j$ whenever $\{i, j\} \subseteq \{1, \ldots, r\}$ and $i \neq j$.

Claim 1. $K = \{z_i : i \in \{1, ..., r\}\}$ is a kernel by paths in $D_1 = D \langle \bigcup_{F \in \mathcal{S}} F \rangle$.

In order to show that K is an absorbent set by paths in D_1 , consider $w \in V(D_1) \setminus K$. Since $w \in V(D_1)$, then there exists $j \in \{1, \ldots, r\}$ such that $w \in V(D\langle F_j \rangle)$. It follows from the fact that $D\langle F_j \rangle$ is strongly connected that there exists a wz_j -path in $D\langle F_j \rangle$, say P. Hence, P is a wK-path in D_1 , concluding that K is an absorbent set by paths in D_1 .

It only remains to show that K is a path-independent set in D_1 . It follows from Lemma 2.9 (c) that $V(D\langle F_i\rangle) \cap V(D\langle F_j\rangle) = \emptyset$ for every $\{i,j\} \subseteq \{1,\ldots r\}$ with $i \neq j$, which implies that there is no z_iz_j -path in D_1 for every $\{i,j\} \subseteq \{1,\ldots r\}$ with $i \neq j$. Hence, K is a path-independent set in D_1 , and the claim holds.

Now, we will show that K is a (k+1, l+1, H)-kernel by walks in D. In order to show that K is an (l+1, H)-absorbent set by walks in D, notice that \mathscr{F} is a walk-preservative partition of A(D) (Lemma 2.9 (a)), and, since K is a kernel by walks in D_1 , we can conclude from Proposition 2.11 that K is an (l+1, H)-absorbent set by walks in D.

It only remains to show that K is a (k+1, H)-independent set by walks in D. Consider $\{z_i, z_j\} \subseteq K$ with $i \neq j$, and a $z_i z_j$ -walk in D, say $C = (z_i = x_0, x_1, \ldots, x_n = z_j)$.

Claim 2. $O_H(C) \neq \emptyset$.

Proceeding by contradiction, suppose that $O_H(C) = \emptyset$, which implies that there exists $F \in \mathscr{F}$ such that $A(C) \subseteq F$. Since z_i and z_j are obstruction-free in D, then $F = F_i$ and $F = F_j$ (Lemma 2.2 (b)), concluding that $F_i = F_j$, which is no possible since $F_i \neq F_j$. Hence, $O_H(C) \neq \emptyset$ and the claim holds.

By Claim 2, suppose that $O_H(C) = \{\alpha_i : i \in \{1, \dots, t\}\}$ where $t \geq 1$, and $\alpha_i \leq \alpha_{i+1}$ for every $i \in \{1, \dots, t-1\}$. For every $i \in \{1, \dots, t\}$, let $G_i \in \mathscr{F}$ such that $(x_{\alpha_i}, x_{\alpha_i}^+) \in G_i$, and $G_0 \in \mathscr{F}$ such that $(x_0, x_1) \in G_0$. It follows from definition of $C_{\mathscr{F}}(D)$ that $C' = (G_0, G_1, \dots, G_t)$ is a walk in $C_{\mathscr{F}}(D)$. Notice that $z_i \in V(D\langle G_0 \rangle)$, $z_j \in V(D\langle G_t \rangle)$, and $l_H(C) = l(C') + 1$. Since z_i and z_j are obstruction-free in D, then $F_i = G_0$ and $F_j = G_t$ (Lemma 2.2 (b)), which implies that $\{G_0, G_t\} \subseteq \mathscr{S}$ and $G_0 \neq G_t$. Since \mathscr{S} is a k-independent set in $C_{\mathscr{F}}(D)$, then $l(C') \geq k$, which implies that $l_H(C) \geq k + 1$. Hence, K is a (k+1, H)-independent set by walks in D. Therefore, K is a (k+1, l+1, H)-kernel by walks in D.

On the other hand, if $W = \{x \in V(D) : d(x) = 0\}$ is nonempty, then by the previous proof, we have that D - W has a (k+1, l+1, H)-kernel by walks, say K. By Lemma 3.1, we can conclude that $K \cup W$ is a (k+1, l+1, H)-kernel by walks in D.

Theorem 3.11. If D is an H-colored digraph such that for every vertex $x \in V(D)$, there exists an xw-H-walk for some vertex obstruction-free w in D, then D has an kernel by H-walks.

Proof. First, we define the digraph D' whose vertex set consists in the obstruction-free vertices of D, and $(x,z) \in A(D')$ if and only if there exists an xz-H-walk in D. Now, we will show that D' has a kernel, say K, by proving that D' is a transitive digraph. Then, a simple proof will show that K is a kernel by H-walks in D.

In order to show that D' is a transitive digraph, consider $\{(u,v),(v,w)\}\subseteq A(D')$. It follows from definition of D' that there exists a uv-H-walk in D, say W_1 , and a vw-H-walk in D, say W_2 . Since v is a obstruction-free vertex in D, then we have that $W_1 \cup W_2$ is a uw-H-walk in D, which implies that $(u,w) \in A(D')$, concluding that D' is a transitive digraph.

Since D' is a transitive digraph, consider a kernel in D', say K. We will show that K is a kernel by H-walks in D. It follows from definition of D' and the fact that K is an independent set in D', that K is an independent set by H-walks in D. It only remains to show that K is an absorbent set by H-walks in D.

Consider $x \in V(D) \setminus K$. If x is a obstruction-free vertex in D, then $x \in V(D')$ and, since K is a kernel in D', there exists $w \in K$ such that $(x, w) \in A(D')$, which implies that there exists an xw-H-walk in D. If $x \notin V(D')$, then by hypothesis, there exists $z \in V(D')$ and an xz-H-walk in D, say W_1 . If $z \in K$, then W_1 is a xK-H-walk in D. If $z \notin K$, then there exists $w \in K$ such that $(z, w) \in A(D')$, which implies that there exists an zw-H-walk in D, say W_2 . Since z is a obstruction-free vertex in D, then $W_1 \cup W_2$ is an xK-H-walk in D, concluding that K is a kernel by H-walks in D.

Corollary 3.12. If D is an m-colored digraph such that for every vertex $x \in V(D)$, there exists a monochromatic xw-walk with color c, for some vertex w such that every arc in A(w) has color c, then D has a kernel by monochromatic paths.

Proof. Consider the digraph H whose vertices are the colors represented in A(D), and $A(H) = \{(c,c) : c \in V(H)\}$. Since D is an H-colored digraph satisfying the hypothesis on Lemma 3.11, it follows that D has a kernel by H-walks, which is a kernel by monochromatic paths in D.

Corollary 3.13. If D is an m-colored digraph such that for every vertex $x \in V(D)$, there exists a alternating xw-walk, for some vertex w such that $A^-(w)$ and $A^+(w)$ have no colors in common, then D has a kernel by properly colored walks.

Proof. Consider the digraph H whose vertices are the colors represented in A(D), and $A(H) = \{(c,d) : \{c,d\} \subseteq V(H), c \neq d\}$. Since D is an H-colored digraph satisfying the hypothesis on Theorem 3.11, it follows that D has a kernel by H-walks, which is a kernel by alternating walks in D.

Theorem 3.14. Let D be an H-colored digraph and \mathscr{F} an H-class partition of A(D) such that for every $F \in \mathscr{F}$, $D\langle F \rangle$ is strongly connected and has a obstruction-free vertex in D. For every $k \geq 2$, D has a (k, H)-kernel by walks.

Proof. Since $C_{\mathscr{F}}(D)$ is a symmetric digraph (Lemma 2.9 (b)), for every $k \geq 2$ we have that $C_{\mathscr{F}}(D)$ has a k-kernel (Theorem 1.1). By Theorem 3.10, D has a (k, H)-kernel by walks for every $k \geq 3$. On the other hand, it follows from Theorem 3.11 that D has a (2, H)-kernel by walks, concluding that D has a (k, H)-kernel by walks for every $k \geq 2$.

Acknowledgments

Hortensia Galeana-Sánchez is supported by CONACYT FORDECYT-PRONACES/39570/2020 and UNAM-DGAPA-PAPIIT IN102320. Miguel Tecpa-Galván is supported by CONACYT-604315.

References

- [1] C. Andenmatten: *H*-distance, *H*-*A*-kernel and in-state splitting in *H*-colored graphs and digraphs. (Master Thesis supervised by H. Galeana-Sánchez and J. Pach), École Polytechnique Fédérale de Lausanne, Switzerland, (2019).
- [2] P. Arpin, V. Linek: Reachability problems in edge-colored digraphs, Discrete. Math. 307, 2276-2289 (2007).
- [3] J. Bang-Jensen: G. Gutin Digraphs: Theory, Algorithms and Applications, Springer, Longdon, 2000.
- [4] G. Benítez-Bobadilla, H. Galeana-Sánchez, C. Hernández Cruz: Panchromatic patterns by paths, arXiv 1903.10031v1 https://arxiv.org/pdf/1903.10031.pdf, 24 Mar 2019

- [5] C. Berge: Graphs and Hypergraphs in: North-Holland Mathematical Library, North-Holland Publishing Co., Amsterdam (1985)
- [6] V. Chvátal: On the computational Complexity of Finding a kernel Report CRM300, Centre de Recherches Mathématiques, Université de Montréal (1973).
- [7] N. Creignou: The class of problems that are linearly equivalent to satisfiability or a uniform method for proving np-completeness, Theor. Comput. Sci., 145, 111-145 (1995).
- [8] P. Delgado-Escalante, H. Galeana-Sánchez: Restricted domination in arc-colored digraphs, AKCE Int. J. Comb. 1, 95-104 (2014).
- [9] Y. Dimopoulos, V. Magirou: A graph theoretic approach to default logic, Inform. Comput. 112, 239-256 (1994).
- [10] Y. Dimopoulos, A. Torres: Graph theoretical structures in logic programs and default theories, Theor. Comp. Sci. 170 (1–2), 209-244 (1996).
- [11] A.S. Fraenkel: Combinatorial game theory foundations applied to digraph kernels, Electron. J. Comb. 4 (2), (1997).
- [12] H. Galeana-Sánchez: Kernels by monochromatic paths and the color class digraph, Discuss. Math. Graph Theory 31, 273-281 (2011).
- [13] H. Galeana-Sánchez: Kernels in edge coloured digraphs, Discrete Math. 184, 87-99 (1998).
- [14] H. Galeana Sánchez, C. Hernández-Cruz: On the existence of (k, l)-kernels in infinite digraphs: a survey, Discuss. Math. 34, 431-466 (2014).
- [15] H. Galeana-Sánchez, C. Hernández-Cruz: k-kernels in generalizations of transitive digraphs, Discuss. Math. Graph Theory 31, 293-312 (2011).
- [16] H. Galeana-Sánchez, R. Sánchez-López: H-kernels and H-obstructions in H-colored digraphs, Discrete Math. 338, 2288-2294 (2015).
- [17] H. Galeana-Sánchez, R. Sánchez-López: Richardson's Theorem in H-coloured digraphs, Graphs and Combin. 32, 629-638 (2016).
- [18] H. Galeana-Sánchez, M. Tecpa-Galván: Some extensions for Richardson's theorem for (k, l, H)-kernels and (k, l, H)-kernels by walks in H-colored digraphs, submitted.
- [19] D. König: Theorie der endlichen undenlichen Graphen, Reprinted from Chelsea Publishing Company (1950).
- [20] M. Kwaśnik: On (k, l)-kernels, Graph Theory (Lagów, 1981) Lecture Notes in Math 1018, 114-121 (1983).
- [21] M. Kwaśnik: On (k, l)-kernels in graphs and their products (Ph.D. thesis supervised by M. Borowiecki), Technical University of Wroclaw, Wroclaw (1980).
- [22] V. Linek, B. Sands: A note on paths in edge-colored tournaments, Ars Combin. 44, 225-228 (1996).
- [23] J. von Neumann, O. Morgestern: Theory of Games and Economic Behavior, Princeton University Press, Princeton (1944).
- [24] K. B. Reid: Monotone reachability in arc-colored tournaments, Congr. Numer. 146, 131-141 (2000).
- [25] B. Sands, N. Sauer, R. Woodrow: On monochromatic paths in edge coloured digraphs, J. Combin. Theory Ser. B33, 271-275 (1982).
- [26] S. Szeider: Finding paths in graphs avoiding forbbiden transitions, Discrete Appl. Math. 126, 261-273 (2003).