FROM DYADIC Λ_α TO Λ_α

WAEL ABU-SHAMMALA AND ALBERTO TORCHINSKY

Abstract. In this paper we show how to compute the Λ_α norm , $\alpha \geq 0$, using the dyadic grid. This result is a consequence of the description of the Hardy spaces $H^p(R^N)$ in terms of dyadic and special atoms.

Recently, several novel methods for computing the BMO norm of a function f in two dimensions were discussed in [9]. Given its importance, it is also of interest to explore the possibility of computing the norm of a BMO function, or more generally a function in the Lipschitz class Λ_α, using the dyadic grid in R^N. It turns out that the BMO question is closely related to that of approximating functions in the Hardy space $H^1(R^N)$ by the Haar system. The approximation in $H^1(R^N)$ by affine systems was proved in [2], but this result does not apply to the Haar system. Now, if $H^A(R)$ denotes the closure of the Haar system in $H^1(R)$, it is not hard to see that the distance $d(f, H^A)$ of $f \in H^1(R)$ to H^A is $\sim \big|\int_0^\infty f(x)\,dx\big|$, see [1]. Thus, neither dyadic atoms suffice to describe the Hardy spaces, nor the evaluation of the norm in BMO can be reduced to a straightforward computation using the dyadic intervals. In this paper we address both of these issues. First, we give a characterization of the Hardy spaces $H^p(R^N)$ in terms of dyadic and special atoms, and then, by a duality argument, we show how to compute the norm in $\Lambda_\alpha(R^N)$, $\alpha \geq 0$, using the dyadic grid.

We begin by introducing some notations. Let $\mathcal{J}$ denote a family of cubes Q in R^N, and $\mathcal{P}_d$ the collection of polynomials in R^N of degree less than or equal to d. Given $\alpha \geq 0$, $Q \in \mathcal{J}$, and a locally integrable function g, let $p_Q(g)$ denote the unique polynomial in $\mathcal{P}_{[\alpha]}$ such that $[g - p_Q(g)]\,\chi_Q$ has vanishing moments up to order $[\alpha]$.

For a locally square-integrable function g, we consider the maximal function $M^{\sharp,2}_{\alpha,\mathcal{J}}g(x)$ given by

$$M^{\sharp,2}_{\alpha,\mathcal{J}}g(x) = \sup_{x\in Q,\, Q\in\mathcal{J}} \frac{1}{|Q|^{\alpha/N}} \left(\frac{1}{|Q|}\int_Q |g(y) - p_Q(g)(y)|^2\,dy\right)^{1/2}.$$

1991 *Mathematics Subject Classification.* 42B30,42B35.

The Lipschitz space $\Lambda_{\alpha,\mathcal{J}}$ consists of those functions g such that $M^{\sharp,2}_{\alpha,\mathcal{J}}g$ is in L^∞, $\|g\|_{\Lambda_{\alpha,\mathcal{J}}} = \|M^{\sharp,2}_{\alpha,\mathcal{J}}g\|_\infty$; when the family in question contains all cubes in R^N, we simply omit the subscript $\mathcal{J}$. Of course, $\Lambda_0 = \mathrm{BMO}$.

Two other families, of dyadic nature, are of interest to us. Intervals in R of the form $I_{n,k} = [\,(k-1)2^n, k2^n]$, where k and n are arbitrary integers, positive, negative or 0, are said to be dyadic. In R^N, cubes which are the product of dyadic intervals of the same length, i.e., of the form $Q_{n,k} = I_{n,k_1}\times\cdots\times I_{n,k_N}$, are called dyadic, and the collection of all such cubes is denoted $\mathcal{D}$.

There is also the family $\mathcal{D}_0$. Let $I'_{n,k} = [(k-1)2^n, (k+1)2^n]$, where k and n are arbitrary integers. Clearly $I'_{n,k}$ is dyadic if k is odd, but not if k is even. Now, the collection $\{I'_{n,k} : n,k \text{ integers}\}$ contains all dyadic intervals as well as the shifts $[(k-1)2^n + 2^{n-1}, k\,2^n + 2^{n-1}]$ of the dyadic intervals by their half length. In R^N, put $\mathcal{D}_0 = \{Q'_{n,k} : Q'_{n,k} = I'_{n,k_1}\times\cdots\times I'_{n,k_N}\}$; $Q'_{n,k}$ is called a special cube. Note that $\mathcal{D}_0$ contains $\mathcal{D}$ properly.

Finally, given $I'_{n,k}$, let $I'^{L}_{n,k} = [(k-1)2^n, k2^n]$, and $I'^{R}_{n,k} = [k2^n, (k+1)2^n]$. The 2^N subcubes of $Q'_{n,k} = I'_{n,k_1}\times\cdots\times I'_{n,k_N}$ of the form $I'^{S_1}_{n,k_1}\times\cdots\times I'^{S_N}_{n,k_N}$, $S_j = L$ or R, $1\le j\le N$, are called the dyadic subcubes of $Q'_{n,k}$.

Let Q_0 denote the special cube $[-1,1]^N$. Given $\alpha\ge 0$, we construct a family $\mathcal{S}_\alpha$ of piecewise polynomial splines in $L^2(Q_0)$ that will be useful in characterizing Λ_α. Let A be the subspace of $L^2(Q_0)$ consisting of all functions with vanishing moments up to order $[\alpha]$ which coincide with a polynomial in $\mathcal{P}_{[\alpha]}$ on each of the 2^N dyadic subcubes of Q_0. A is a finite dimensional subspace of $L^2(Q_0)$, and, therefore, by the Graham-Schmidt orthogonalization process, say, A has an orthonormal basis in $L^2(Q_0)$ consisting of functions $p^1,\dots,p^M$ with vanishing moments up to order $[\alpha]$, which coincide with a polynomial in $\mathcal{P}_{[\alpha]}$ on each dyadic subinterval of Q_0. Together with each p^L we also consider all dyadic dilations and integer translations given by

$$p^L_{n,k,\alpha}(x) = 2^{n(N+\alpha)}p^L(2^nx_1+k_1,\dots,2^nx_N+k_N)\,,\quad 1\le L\le M\,,$$

and let

$$\mathcal{S}_\alpha = \{p^L_{n,k,\alpha} : n,k \text{ integers}, 1\le L\le M\}\,.$$

Our first result shows how the dyadic grid can be used to compute the norm in Λ_α.

Theorem A. *Let g be a locally square-integrable function and $\alpha\ge 0$. Then, $g\in\Lambda_\alpha$ if, and only if, $g\in\Lambda_{\alpha,\mathcal{D}}$ and $A_\alpha(g) = \sup_{p\in\mathcal{S}_\alpha}\big|\langle g,p\rangle\big| < \infty$. Moreover,*

$$\|g\|_{\Lambda_\alpha}\sim\|g\|_{\Lambda_{\alpha,\mathcal{D}}} + A_\alpha(g)\,.$$

Furthermore, it is also true, and the proof is given in Proposition 2.1 below, that $\|g\|_{\Lambda_\alpha}\sim\|g\|_{\Lambda_{\alpha,\mathcal{D}_0}}$. However, in this simpler formulation, the tree structure of the cubes in $\mathcal{D}$ has been lost.

The proof of Theorem A relies on a close investigation of the predual of Λ_α, namely, the Hardy space $H^p(R^N)$ with $0 < p = (\alpha + N)/N \le 1$. In the process we characterize H^p in terms of simpler subspaces: $H^p_{\mathcal{D}}$, or dyadic H^p, and $H^p_{\mathcal{S}_\alpha}$, the space generated by the special atoms in $\mathcal{S}_\alpha$. Specifically, we have

Theorem B. *Let $0 < p \le 1$, and $\alpha = N(1/p - 1)$. We then have*

$$H^p = H^p_{\mathcal{D}} + H^p_{\mathcal{S}_\alpha},$$

where the sum is understood in the sense of quasinormed Banach spaces.

The paper is organized as follows. In Section 1 we show that individual H^p atoms can be written as a superposition of dyadic and special atoms; this fact may be thought of as an extension of the one-dimensional result of Fridli concerning L^∞ 1-atoms, see [5] and [1]. Then, we prove Theorem B. In Section 2 we discuss how to pass from $\Lambda_{\alpha,\mathcal{D}}$, and $\Lambda_{\alpha,\mathcal{D}_0}$, to the Lipschitz space Λ_α.

1. Characterization of the Hardy spaces H^p

We adopt the atomic definition of the Hardy spaces H^p, $0 < p \le 1$, see [6] and [10]. Recall that a compactly supported function a with $[N(1/p-1)]$ vanishing moments is an L^2 p-atom with defining cube Q if $\operatorname{supp}(a) \subseteq Q$, and

$$|Q|^{1/p} \left(\frac{1}{|Q|} \int_Q |\, a(x)\, |^2 dx \right)^{1/2} \le 1\,.$$

The Hardy space $H^p(R^N) = H^p$ consists of those distributions f that can be written as $f = \sum \lambda_j a_j$, where the a_j's are H^p atoms, $\sum |\lambda_j|^p < \infty$, and the convergence is in the sense of distributions as well as in H^p. Furthermore,

$$\|f\|_{H^p} \sim \inf \Big(\sum |\lambda_j|^p \Big)^{1/p},$$

where the infimum is taken over all possible atomic decompositions of f. This last expression has traditionally been called the atomic H^p norm of f.

Collections of atoms with special properties can be used to gain a better understanding of the Hardy spaces. Formally, let $\mathcal{A}$ be a non-empty subset of L^2 p-atoms in the unit ball of H^p. The atomic space $H^p_{\mathcal{A}}$ spanned by $\mathcal{A}$ consists of those φ in H^p of the form

$$\varphi = \sum \lambda_j a_j\,, \quad a_j \in \mathcal{A}\,, \sum |\lambda_j|^p < \infty\,.$$

It is readily seen that, endowed with the atomic norm

$$\|\varphi\|_{H^p_{\mathcal{A}}} = \inf \Big\{ \Big(\sum |\lambda_j|^p \Big)^{1/p} : \varphi = \sum \lambda_j\, a_j\,, a_j \in \mathcal{A} \Big\}\,,$$

$H^p_{\mathcal{A}}$ becomes a complete quasinormed space. Clearly, $H^p_{\mathcal{A}} \subseteq H^p$, and, for $f \in H^p_{\mathcal{A}}$, $\|f\|_{H^p} \le \|f\|_{H^p_{\mathcal{A}}}$.

Two families are of particular interest to us. When $\mathcal{A}$ is the collection of all L^2 p-atoms whose defining cube is dyadic, the resulting space is $H^p_{\mathcal{D}}$, or dyadic H^p. Now, although $\|f\|_{H^p} \le \|f\|_{H^p_{\mathcal{D}}}$, the two quasinorms are not equivalent on $H^p_{\mathcal{D}}$. Indeed, for $p = 1$ and $N = 1$, the functions

$$f_n(x) = 2^n [\chi_{[1-2^{-n},1]}(x) - \chi_{[1,1+2^{-n}]}(x)]\,,$$

satisfy $\|f_n\|_{H^1} = 1$, but $\|f_n\|_{H^1_{\mathcal{D}}} \sim |n|$ tends to infinity with n.

Next, when $\mathcal{S}_\alpha$ is the family of piecewise polynomial splines constructed above with $\alpha = N(1/p - 1)$, in analogy with the one-dimensional results in [4] and [1], $H^p_{\mathcal{S}_\alpha}$ is referred to as the space generated by special atoms.

We are now ready to describe H^p atoms as a superposition of dyadic and special atoms.

Lemma 1.1. *Let a be an L^2 p-atom with defining cube Q, $0 < p \le 1$, and $\alpha = N(1/p - 1)$. Then a can be written as a linear combination of 2^N dyadic atoms a_i, each supported in one of the dyadic subcubes of the smallest special cube $Q_{n,k}$ containing Q, and a special atom b in $\mathcal{S}_\alpha$. More precisely, $a(x) = \sum_{i=1}^{2^N} d_i\, a_i(x) + \sum_{L=1}^{M} c_L\, p^L_{-n,-k,\alpha}(x)$, with $|d_i|\,, |c_L| \le c$.*

Proof. Suppose first that the defining cube of a is Q_0, and let $Q_1, \ldots, Q_{2^N}$ denote the dyadic subcubes of Q_0. Furthermore, let $\{e^1_i, \ldots, e^M_i\}$ denote an orthonormal basis of the subspace A_i of $L^2(Q_i)$ consisting of polynomials in $\mathcal{P}_{[\alpha]}$, $1 \le i \le 2^N$. Put

$$\alpha_i(x) = a(x)\chi_{Q_i}(x) - \sum_{j=1}^{M} \langle a\chi_{Q_i}, e^i_j\rangle\, e^i_j(x)\,, \quad 1 \le i \le 2^N\,,$$

and observe that $\langle \alpha_i, e^i_j\rangle = 0$ for $1 \le j \le M$. Therefore, α_i has $[\alpha]$ vanishing moments, is supported in Q_i, and

$$\|\alpha_i\|_2 \le \|a\chi_{Q_i}\|_2 + \sum_{j=1}^{M} \|a\chi_{Q_i}\|_2 \le (M+1)\, \|a\chi_{Q_i}\|_2\,.$$

So,

$$a_i(x) = \frac{2^{N(1/2\,-1/p)}}{M+1}\, \alpha_i(x)\,, \quad 1 \le i \le N\,,$$

is an L^2 p-dyadic atom. Finally, put

$$b(x) = a(x) - \frac{M+1}{2^{N(1/2\,-1/p)}} \sum_{i=1}^{2^N} a_i(x)\,.$$

Clearly b has $[\alpha]$ vanishing moments, is supported in Q_0, coincides with a polynomial in $\mathcal{P}_{[\alpha]}$ on each dyadic subcube of Q_0, and

$$\|b\|_2^2 \le \sum_{i=1}^{2^N} \sum_{j=1}^{M} |\langle a\chi_{Q_i}, e_j^i\rangle|^2 \le M\,\|a\|_2^2 \,.$$

So, $b \in A$, and, consequently, $b(x) = \sum_{L=1}^{M} c_L\, p^L(x)$, where

$$|c_L| = |\langle b, p^L\rangle| \le c\,, \quad 1 \le L \le M\,.$$

In the general case, let Q be the defining cube of a, side-length $Q = \ell$, and let n and $k = (k_1, \dots, k_N)$ be chosen so that $2^{n-1} \le \ell < 2^n$, and

$$Q \subset [(k_1 - 1)2^n, (k_1 + 1)2^n] \times \cdots \times [(k_N - 1)2^n, (k_N + 1)2^n]\,.$$

Then, $(1/2)^N \le |Q|/2^{nN} < 1$.

Now, given $x \in Q_0$, let a' be the translation and dilation of a given by

$$a'(x) = 2^{nN/p} a(2^n x_1 - k_1, \dots, 2^n x_N - k_N)\,.$$

Clearly, $[\alpha]$ moments of a' vanish, and

$$\|a'\|_2 = 2^{nN/p}\, 2^{-nN/2} \|a\|_2 \le c\,|Q|^{1/p} |Q|^{-1/2} \|a\|_2 \le c\,.$$

Thus, a' is a multiple of an atom with defining cube Q_0. By the first part of the proof,

$$a'(x) = \sum_{i=1}^{2^N} d_i\, a_i'(x) + \sum_{L=1}^{M} c_L\, p^L(x)\,, \quad x \in Q_0\,.$$

The support of each a_i' is contained in one of the dyadic subcubes of Q_0, and, consequently, there is a k such that

$$a_i(x) = 2^{-nN/p} a_i'(2^{-n} x_1 - k_1, \dots, 2^{-n} x_N - k_N)$$

a_i is an $L^2 p$ -atom supported in one of the dyadic subcubes of Q. Similarly for the p_L's. Thus,

$$a(x) = \sum_i d_i\, a_i(x) + \sum_{L=1}^{M} c_L p^L_{-n,-k,N(1/p-1)}(x)\,,$$

and we have finished. ∎

Theorem B follows readily from Lemma 1.1. Clearly, $H^p_{\mathcal{D}} + H^p_{\mathcal{S}_\alpha} \hookrightarrow H^p$. Conversely, let $f = \sum_j \lambda_j\, a_j$ be in H^p. By Lemma 1.1 each a_j can be written as a sum of dyadic and special atoms, and, by distributing the sum, we can write $f = f_d + f_s$, with f_d in $H^p_{\mathcal{D}}$, f_s in $H^p_{\mathcal{S}_\alpha}$, and

$$\|f_d\|_{H^p_{\mathcal{D}}}, \|f_s\|_{H^p_{\mathcal{S}_\alpha}} \le c \left(\sum |\lambda_j|^p \right)^{1/p}.$$

Taking the infimum over the decompositions of f we get $\|f\|_{H^p_{\mathcal{D}} + H^p_{\mathcal{S}_\alpha}} \le c\,\|f\|_{H^p}$, and $H^p \hookrightarrow H^p_{\mathcal{D}} + H^p_{\mathcal{S}_\alpha}$. This completes the proof.

The meaning of this decomposition is the following. Cubes in $\mathcal{D}$ are contained in one of the 2^N non-overlapping quadrants of R^N. To allow for the information carried by a dyadic cube to be transmitted to an adjacent dyadic cube, they must be connected. The $p^L_{n,k,\alpha}$'s channel information across adjacent dyadic cubes which would otherwise remain disconnected. The reader will have no difficulty in proving the quantitative version of this observation: Let T be a linear mapping defined on H^p, $0 < p \le 1$, that assumes values in a quasinormed Banach space X. Then, T is continuous if, and only if, the restrictions of T to $H^p_{\mathcal{D}}$ and $H^p_{\mathcal{S}_\alpha}$ are continuous.

2. Characterizations of Λ_α

Theorem A describes how to pass from $\Lambda_{\alpha,\mathcal{D}}$ to Λ_α, and we prove it next. Since $(H^p)^* = \Lambda_\alpha$ and $(H^p_{\mathcal{D}})^* = \Lambda_{\alpha,\mathcal{D}}$, from Theorem B it follows readily that $\Lambda_\alpha = \Lambda_{\alpha,\mathcal{D}} \cap (H^p_{\mathcal{S}_\alpha})^*$, so it only remains to show that $(H^p_{\mathcal{S}_\alpha})^*$ is characterized by the condition $A_\alpha(g) < \infty$.

First note that if g is a locally square-integrable function with $A_\alpha(g) < \infty$ and $f = \sum_{j,L} c_{j,L}\, p^L_{n_j,k_j,\alpha}$, since $0 < p \le 1$,

$$\begin{aligned} |\langle g, f\rangle| &\le \sum_{j,L} |c_{j,L}|\ |\langle g, p^L_{n_j,k_j,\alpha}\rangle| \\ &\le A_\alpha(g)\Big[\sum_{j,L} |c_{j,L}|^p\Big]^{1/p}, \end{aligned}$$

and, consequently, taking the infimum over all atomic decompositions of f in $H^p_{\mathcal{S}_\alpha}$, we get $g \in (H^p_{\mathcal{S}_\alpha})^*$ and $\|g\|_{(H^p_{\mathcal{S}_\alpha})^*} \le A_\alpha(g)$.

To prove the converse we proceed as in [3]. Let $Q_n = [-2^n, 2^n]^N$. We begin by observing that functions f in $L^2(Q_n)$ that have vanishing moments up to order $[\alpha]$ and coincide with polynomials of degree $[\alpha]$ on the dyadic subcubes of Q_n belong to $H^p_{\mathcal{S}_\alpha}$ and

$$\|f\|_{H^p_{\mathcal{S}_\alpha}} \le |Q_n|^{1/p-1/2}\|f\|_2\,.$$

Given $\ell \in (H^p_{\mathcal{S}_\alpha})^*$, for a fixed n let us consider the restriction of ℓ to the space of L^2 functions f with $[\alpha]$ vanishing moments that are supported in Q_n. Since

$$|\ell(f)| \le \|\ell\|\,\|f\|_{H^p_{\mathcal{S}_\alpha}} \le \|\ell\|\,|Q_n|^{1/p-1/2}\|f\|_2\,,$$

this restriction is continuous with respect to the norm in L^2 and, consequently, it can be extended to a continuous linear functional in L^2 and represented as

$$\ell(f) = \int_{Q_n} f(x)\, g_n(x)\, dx\,,$$

where $g_n \in L^2(Q_n)$ and satisfies $\|g_n\|_2 \le \|\ell\|\,|Q_n|^{1/p-1/2}$. Clearly, g_n is uniquely determined in Q_n up to a polynomial p_n in $\mathcal{P}_{[\alpha]}$. Therefore,

$$g_n(x) - p_n(x) = g_m(x) - p_m(x)\,, \quad \text{a.e. } x \in Q_{\min(n,m)}\,.$$

Consequently, if

$$g(x) = g_n(x) - p_n(x)\,, \quad x \in Q_n\,,$$

$g(x)$ is well defined a.e. and, if $f \in L^2$ has $[\alpha]$ vanishing moments and is supported in Q_n, we have

$$\begin{aligned} \ell(f) &= \int_{R^N} f(x)\, g_n(x)\, dx \\ &= \int_{R^N} f(x)\, [g_n(x) - p_n(x)]\, dx \\ &= \int_{R^N} f(x)\, g(x)\, dx\,. \end{aligned}$$

Moreover, since each $2^{nN/p} p^L(2^n \cdot + k)$ is an L^2 p-atom, $1 \le L \le M$, it readily follows that

$$\begin{aligned} A_\alpha(g) &= \sup_{1\le L\le M}\, \sup_{n,k\in Z} |\langle g, 2^{-n/p} p^L(2^n \cdot + k)\rangle| \\ &\le \|\ell\| \sup_L \|p^L\|_{H^p} \le \|\ell\|\,, \end{aligned}$$

and, consequently, $A_\alpha(g) \le \|\ell\|$, and $(H^p_{\mathcal{S}_\alpha})^*$ is the desired space. ■

The reader will have no difficulty in showing that this result implies the following: Let T be a bounded linear operator from a quasinormed space X into $\Lambda_{\alpha,\mathcal{D}}$. Then, T is bounded from X into Λ_α if, and only if, $A_\alpha(Tx) \le c\,\|x\|_X$ for every $x \in X$.

The process of averaging the translates of dyadic BMO functions leads to BMO, and is an important tool in obtaining results in BMO once they are known to be true in its dyadic counterpart, BMO_d, see [7]. It is also known that BMO can be obtained as the intersection of BMO_d and one of its shifted counterparts, see [8]. These results motivate our next proposition, which essentially says that $g \in \Lambda_\alpha$ if, and only if, $g \in \Lambda_{\alpha,\mathcal{D}}$ and g is in the Lipschitz class obtained from the shifted dyadic grid. Note that the shifts involved in this class are in all directions parallel to the coordinate axis and depend on the side-length of the cube.

Proposition 2.1. $\Lambda_\alpha = \Lambda_{\alpha,\mathcal{D}_0}$, *and* $\|g\|_{\Lambda_\alpha} \sim \|g\|_{\Lambda_{\alpha,\mathcal{D}_0}}$.

Proof. It is obvious that $\|g\|_{\Lambda_{\alpha,\mathcal{D}_0}} \le \|g\|_{\Lambda_\alpha}$. To show the other inequality we invoke Theorem A. Since $\mathcal{D} \subset \mathcal{D}_0$, it suffices to estimate $A_\alpha(g)$, or, equivalently, $|\langle g, p\rangle|$ for $p \in \mathcal{S}_\alpha$, $\alpha = N(1/p - 1)$. So, pick $p = p^L_{n,k,\alpha}$ in $\mathcal{S}_\alpha$. The defining cube Q of $p^L_{n,k,\alpha}$ is in $\mathcal{D}_0$, and, since $p^L_{n,k,\alpha}$ has $[\alpha]$ vanishing moments,

$\langle p^L_{n,k,\alpha}, p_Q(g)\rangle = 0$. Therefore,

$$\begin{aligned}|\langle g, p^L_{n,k,\alpha}\rangle| &= |\langle g - p_Q(g), p^L_{n,k,\alpha}\rangle| \\ &\le \|p^L_{n,k,\alpha}\|_2 \, \|g - p_Q(g)\|_{L^2(Q)} \\ &\le |Q|^{\alpha/N}|Q|^{1/2}\|p^L_{n,k,\alpha}\|_2 \, \|g\|_{\Lambda_{\alpha,\mathcal{D}_0}}.\end{aligned}$$

Now, a simple change of variables gives $|Q|^{\alpha/N}|Q|^{1/2}\|p^L_{n,k,\alpha}\|_2 \le 1$, and, consequently, also $A_\alpha(g) \le \|g\|_{\Lambda_{\alpha,\mathcal{D}_0}}$. ■

References

[1] W. Abu-Shammala, J.-L. Shiu, and A. Torchinsky, *Characterizations of the Hardy space H^1 and* BMO, preprint.

[2] H.-Q. Bui and R. S. Laugesen, *Approximation and spanning in the Hardy space, by affine systems*, Constr. Approx., to appear.

[3] A. P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distibution, II*, Advances in Math., **24** (1977), 101–171.

[4] G. S. de Souza, *Spaces formed by special atoms, I,* Rocky Mountain J. Math. **14** (1984), no. 2, 423–431.

[5] S. Fridli, *Transition from the dyadic to the real nonperiodic Hardy space*, Acta Math. Acad. Paedagog. Nyházi (N.S.) **16** (2000), 1–8, (electronic).

[6] J. García-Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, Notas de Matemática **116**, North Holland, Amsterdam, 1985.

[7] J. Garnett and P. Jones, BMO *from dyadic* BMO, Pacific J. Math. **99** (1982), no. 2, 351–371.

[8] T. Mei, BMO *is the intersection of two translates of dyadic* BMO, C. R. Math. Acad. Sci. Paris **336** (2003), no. 12, 1003–1006.

[9] T. M. Le and L. A. Vese, *Image decomposition using total variation and div*(BMO)*, Multiscale Model. Simul. **4**, (2005), no. 2, 390–423.

[10] A. Torchinsky, *Real-variable methods in harmonic analysis*, Dover Publications, Inc., Mineola, NY, 2004.

Department of Mathematics, Indiana University, Bloomington IN 47405
E-mail address: wabusham@indiana.edu

Department of Mathematics, Indiana University, Bloomington IN 47405
E-mail address: torchins@indiana.edu