# LYAPUNOV EXPONENT FOR PRODUCTS OF RANDOM ISING TRANSFER MATRICES: THE BALANCED DISORDER CASE

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Abstract. We analyze the top Lyapunov exponent of the transfer matrix for the nearest neighbor Ising chain with random external field. This Lyapunov exponent coincides with the free energy of the Ising chain with random external field, but it also plays a central role in the analysis of the two dimensional Ising model with columnar disorder and of the quantum chain with transverse random field. We obtain its sharp behavior in the large interaction limit when the external field is centered: this *balanced* case turns out to be *critical* in many respects. From a mathematical standpoint we precisely identify the behavior of the top Lyapunov exponent of a product of two dimensional random matrices close to a diagonal random matrix for which top and bottom Lyapunov exponents coincide. In particular, the Lyapunov exponent is only log-Hölder continuous.

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## 1. Introduction and results

### 1.1. Background.

The simple two dimensional matrix of the form

$$M = M(\varepsilon, Z) := \begin{pmatrix} 1 & \varepsilon \\ \varepsilon Z & Z \end{pmatrix}, \tag{1.1}$$

with $\varepsilon \geq 0$ and $Z \geq 0$ repeatedly appears in physics, notably in the Ising model context [13, 23]:

(I1) If $d = 1$ the transfer matrix of the Ising model with external magnetic field $h$ and nearest neighbor interaction potential $J$ is equal to $e^{h+J}$ times (1.1), with $\varepsilon = e^{-2J}$ and $Z = e^{-2h}$. Therefore the free energy density of the model is $h+J$ plus the largest among the two eigenvalues of $M$.

(I2) If $d = 2$ on the square lattice and for $h = 0$, one can still express the partition function of the model in terms of the largest eigenvalue of $M$ with an adequate choice of $\varepsilon(\theta, J_1)$ and $Z = Z(J_1, J_2)$ where $J_1$ and $J_2$ are the horizontal and vertical nearest neighbor interactions and $\theta$ is a *phase* arising from a Fourier transform in the nonrandom direction. In particular the free energy density can be expressed as the integral of such an eigenvalue, for fixed $J_1$ and $J_2$, over $\theta \in [-\pi, \pi]$, with the small $\theta$ (long wave-lengths) being relevant for the critical behavior. In this limit, i.e. $\theta$ small, $\varepsilon(\theta, J_1)$ becomes proportional to $|\theta|$ and we point out also that the $\theta$, or $\varepsilon$, small limit clearly corresponds to large $J$ in ((I1)).

(I3) The ground state analysis of the $d = 1$ quantum Ising field with traverse field leads once again to the same matrix problem and it is directly connected to the $d = 2$ classical model discussed in ((I2)), see e.g. [20].

For the $d = 1$ (non quantum) case there is no phase transition, i.e. the free energy is a real analytic function, and this is just the regularity of the leading eigenvector of a matrix with positive entries. But for $d = 2$, or in the $d = 1$ quantum case, there is a phase transition; this is possible due to the fact that $\varepsilon(\theta, J_1) = 0$ for $\theta = 0$ and $\pi$, so we are no longer in the context of matrices with positive entries and the two eigenvalues may coincide (when $J_1$ and $J_2$ are positive, this can happen only if $\theta = 0$). These solutions are to a certain extent *robust* with respect to introduction of (some types of) disorder. For example in $d = 1$ if $h$ varies from site to site and it is the realization of a sequence $(h_j)_{j\in\mathbb{Z}}$ of random variables (for example, IID), then the free energy expression still holds, provided we replace the leading eigenvalue of the single matrix with the (top) Lyapunov exponent of the sequence:

$$\mathcal{L}(\varepsilon) := \lim_{n\to\infty} \frac{1}{n}\mathbb{E} \log \| M(\varepsilon, Z_1)M(\varepsilon, Z_2)\cdots M(\varepsilon, Z_n)\| , \tag{1.2}$$

where $\|\cdot\|$ is an arbitrary norm of matrices and of course $Z_j = e^{-2h_j}$, $\varepsilon = e^{-2J}$. Analogously, in the $d = 2$ and with columnar disorder – $J_2$ is replaced by an IID sequence – the free energy is equal to the Lyapunov exponent averaged over the phase [22, 23, 27].

The large interaction limit for the disordered external field one dimensional Ising model, or zero phase limit for the $d = 2$ of quantum case, has been repeatedly analyzed. Notably in [12] B. Derrida and H. Hilhorst – origin of the acronym DH – claimed that when $\mathbb{E}[\log Z] \neq 0$ and $\mathbb{E}[Z] > 1$ , unless $Z$ is restricted to a discrete set of values of the form $\{0\} \cup \{\dots, z^{-1}, 1, z, z^2, \dots\}$,

$$\mathcal{L}(\varepsilon) - (\mathbb{E}[\log Z])_+ \overset{\varepsilon\searrow 0}{\sim} C_Z \varepsilon^{2|\alpha|} , \tag{1.3}$$

where $\alpha$ is the only nonzero real solution of $\mathbb{E}[Z^\alpha] = 1$ and $C_Z$ is a positive constant. The assumption on the values of $Z$ is important: in [12] an example without this property is given in which $C_Z$ must be replaced by a log-periodic function is given. In [14] (1.3) has been proven under the stronger assumption that the law of $Z$ has a compact support bounded away from zero and that $Z$ has a $C^1$ density. Beyond the application, this result identifies a singular behavior of the Lyapunov exponents and enters the field of inquiry into the regularity of Lyapunov exponents (e.g. [19, 2, 28]) as a case in which the singularity is sharply identified.

In this work we focus on $\mathbb{E}[\log Z] = 0$: since we assume also $Z$ non trivial, we have that $\mathbb{E}[Z^\alpha] = 1$ is solved only by $\alpha = 0$. We will discuss in § 1.4 what is known if $\mathbb{E}[\log Z] \neq 0$ and $\mathbb{E}[Z] \leq 1$: in this case $|\alpha| \geq 1$ as opposed to $|\alpha| \in (0, 1)$ of (1.3).

The $\mathbb{E}[\log Z] = 0$, or $\alpha = 0$, case has been considered in [24, (4.34) and pp. 1218-1220] where the authors claim, via formal expansions, that for $\varepsilon \searrow 0$

$$\mathcal{L}(\varepsilon) = \frac{C_{Z,1}}{|\log \varepsilon| + C_{Z,2}} + O\left(\varepsilon^2\right) , \tag{1.4}$$

for a specific class of laws of $Z$ (superposition of one (bi-)exponential and one Dirac delta). Moreover, the 2-scale approach of [12] can be generalized and leads to a prediction in the spirit of (1.4) for *arbitrary* distributions of $Z$ [11]. See also the discussion on *weak disorder limits* in Section 1.4.

Our aim is obtaining a mathematical control on $\mathcal{L}(\varepsilon)$ for $\varepsilon \searrow 0$ for a wide class of laws for $Z$. We anticipate that our results will hold asymptotically under the assumption that $Z$ has a suitably regular density and without requiring the support to be bounded (but both $Z$ and $1/Z$ are in $\mathbb{L}^p$ for a $p > 1$). As in [14], we will start from the 2-scale idea of [12]. What is done in [12] is to guess a probability measure – we call it the DH probability – that should be sufficiently close to the Furstenberg probability (on the projective space, i.e. simply the sector $[0, \pi/2]$ because we work with positive matrices in dimension two) with which one can explicitly express the Lyapunov exponent. The measure essentially concentrates near 0, but a much finer description is needed. The 2-scale idea is about gluing together an $\varepsilon \searrow 0$ limit problem near 0 and another limit problem near $\pi/2$. Both problems are non trivial and they require a quantitative understanding of the invariant measure of a chain that appears in the context of one dimensional Random Walks in Random Environment (RWRE) [7, 17, 16] and in random affine iterations [6]. Even if the problems at 0 and $\pi/2$ are in a sense *dual problems*, when $\mathbb{E}[\log Z] \neq 0$ they are actually different in nature: one of the two chains is positive recurrent, the other is transient. In [14] we first gave a rigorous construction of the DH probability – this is essentially an asymptotic matching problem – and then we showed that this probability is *sufficiently close* to the Furstenberg probability to yield (1.3). In the key second step we exploited a contraction property, under the random matrix action, of a suitable norm that depends on a parameter $\beta \in (0, \alpha)$: the contraction factor is precisely $\mathbb{E}[Z^\beta] < 1$.

When $\mathbb{E}[\log Z] = 0$ there are two major changes with respect to the $\mathbb{E}[\log Z] \neq 0$ case :

(1) The two limit problems change nature: they become two qualitatively identical problems – if $Z$ has the same law as $1/Z$ they are the same problem – and they are both null recurrent. These chains are associated to a balanced RWRE, also known as *Sinai walk*, and they are a particular critical random difference equation: these chains do not have an invariant probability, but they do have a *unique* $\sigma$-finite measure on which we need a sharp control in order to perform the gluing procedure. The gluing procedure builds the DH measure, which is a probability measure and is close to the invariant probability of the chain we are interested in (which is positive recurrent).

(2) $\mathbb{E}[\log Z] = 0$ means $\alpha = 0$, so there is no room to apply the contractive procedure of [14]. As a matter of fact, this problem is *critical* in the sense of [1, 4] and the contraction property exploited in [14] is simply not there. We take this occasion to stress that $\mathbb{E}[\log Z] = 0$ is (rather, is expected to be, since mathematically the problem is open) also the criticality condition for the two dimensional Ising model with columnar disorder ([22, 23] and [8, App. A]).

We will nevertheless take the DH path: we are going to explain in § 1.3 how we do it, notably how we deal with the lack of the contraction property. But first let us introduce more formally the model and state our main result.

1.2. **The main result.** For what follows it is more natural to work with $\mathsf{z}_j := \log Z_j$ and let $\mathsf{z}$ be a variable that is distributed like the $\mathsf{z}_j$'s. On $\mathsf{z}$ we assume:

(1) exponential integrability, namely that there exists $\delta > 0$ such that
$$\mathbb{P}\left(|\mathsf{z}| > x\right) \, = \, O\left(\exp\left(-\delta x\right)\right) . \tag{1.5}$$

(2) $\mathsf{z}$ has a density $\zeta$ which is uniformly $\theta$-Hölder continuous, for a $\theta \in (0, 1]$, i.e.
$$\sup_{x \neq y} \frac{|\zeta(x) - \zeta(y)|}{|x - y|^\theta} \, < \, \infty . \tag{1.6}$$

We remark that these hypotheses imply that $\zeta$ is locally bounded and that $\zeta(x) = O(\exp(-\delta'\theta|x|/(1 + \theta))$ for every $\delta' \in (0, \delta)$ (see Lemma B.1(1)). In particular, $\zeta \in \mathbb{L}^p$ for every $p \in [1, \infty]$.

Of course we assume $\mathbb{E}[\mathsf{z}] = 0$.

**Theorem 1.1.** *There exist three constants $\kappa_1 > 0$, $\kappa_2 \in \mathbb{R}$ and $\eta \in (0,1)$ such that, for $\varepsilon \to 0$,*

$$\mathcal{L}(\varepsilon) \,=\, \frac{\kappa_1}{\log(1/|\varepsilon|) + \kappa_2} + O\left(|\varepsilon|^{\eta}\right) . \tag{1.7}$$

$\kappa_1$, $\kappa_2$ and $\eta$ depend only on the law of $Z$. For $\kappa_j$ we have a semi-explicit expression, see (3.12), and for $\eta$ we give an explicit lower bound. Note that Theorem 1.1 is stated also for $\varepsilon < 0$: we can pass from $\varepsilon$ to $-\varepsilon$ by conjugation via the diagonal matrix with $(1,-1)$ on the diagonal.

Theorem 1.1 shows in particular that $\mathcal{L}(\cdot)$ is not Hölder continuous: this is not in contrast with [19, 2] because we are looking at the neighborhood of random matrices in which there is no separation between the two Lyapunov exponents (in the Osedelec sense [3, Ch. IV]). We take this occasion also to point out that with respect to the well known *Halperin example* in the context of Anderson localization [28, App. 3], the singularity identified by Theorem 1.1, as well as (1.3), is present also if $\zeta$ is $C^{\infty}$ or even analytic and it is not related to low regularity of the law of $\mathtt{z}$, like in the Halperin case.

1.3. **A walk through our approach: the DH strategy for $\alpha = 0$.** In order to explain our approach we make a change of perspective on the problem by setting

$$k \,:=\, -\log \varepsilon\,, \tag{1.8}$$

so that the original matrix $M$ of (1.1) becomes

$$\begin{pmatrix} 1 & \exp(-k) \\ \exp(-k+\mathtt{z}) & \exp(\mathtt{z}) \end{pmatrix}, \tag{1.9}$$

and if we parametrize $P(\mathbb{R}^2) \cong \mathbb{R}$ with the coordinates $(1, \exp(x))^{\mathtt{T}}$ we readily see that the action of the matrix (1.9) is

$$x \,\mapsto\, z + \log\left(\frac{e^{-k}+e^{x}}{1+e^{x-k}}\right) \,=\, z + h_k(x)\,. \tag{1.10}$$

We observe that $h_k(\cdot)$ is odd and $x \mapsto x - h_k(x)$, which is also odd, is small if $x \in (-k,k)$ and far from the boundary points $\pm k$:

$$x - h_k(x) \,=\, \log\left(\frac{1+e^{x-k}}{1+e^{-x-k}}\right) \,=\, O\left(e^{-(k-x)}\right) + O\left(e^{-(k+x)}\right), \tag{1.11}$$

so $h_k(x)$ is *very close* to $x$ on an interval that approaches $\mathbb{R}$ in the limit $k \to \infty$ (see Fig. 2).

Denote by $X = (X_n)_{n=0,1,\ldots}$ the Markov chain generated by the map (1.10), that is

$$X_{n+1} \,=\, \mathtt{z}_{n+1} + h_k\left(X_n\right) . \tag{1.12}$$

Since the image of $h_k$ is $(-k,k)$ the process $(X_n)$ *hardly leaves* $(-k,k)$ and when the process is in $(-k,k)$ and far from the boundaries it is *close to being* a random walk with centered increments.

$X$ is an irreducible positive recurrent Markov chain (see the beginning of Appendix A) and via its invariant probability $\nu_k$ one has that

$$\mathcal{L}(k) \overset{\text{a.o.n.}}{=} \mathcal{L}(\exp(-k))) \,=\, \int_{\mathbb{R}} \log(1+\exp(-k-x))\nu_k(\,\mathrm{d}x)\,, \tag{1.13}$$

where the first equality is an abuse of notation (a.o.n.) that we will systematically commit. The formula (1.13) will be further explained later on (see in particular (2.7)), but it is what follows by specializing the Furstenberg formula for the top Lyapunov exponent [3, Th. 3.6] to our context.

Since $X$ is close to being a symmetric random walk in the bulk and because of the strong containing effect at the boundary it is natural to expect that the invariant probability is going to be close to the Lebesgue measure times a suitable constant $1/C_k$ in the bulk, and this measure should decay quickly outside $(-k, k)$. If this is the case $C_k \sim 2k$ in the $k \to \infty$ limit. If we insert this guess into (1.13) we readily see that the leading contribution comes from $x$ close to $-k$ on the scale $k$: by this we mean an interval centered in $-k$ of diverging length $o(k)$. We are therefore interested in focusing on how the process looks from $-k$. So we consider $(X_n + k)$ and we readily see that the process that appears for $k \to \infty$ is

$$Y_{n+1} \,=\, \mathtt{z}_{n+1} + h\left(Y_n\right) \qquad \text{with } h(y) \,=\, y + \log(1+\exp(-y))\,. \tag{1.14}$$

This new Markov chain, of which we will give a detailed treatment, has a strong repulsion at zero, forcing $Y$ to live most of time in the positive semi-axis. But there is no mechanism that bounds $Y$ from above: in fact, $Y$ is null recurrent and the non normalizable invariant measure does approach a multiple of the Lebesgue measure far from the origin. The random iteration (1.14) is the critical random difference equation that emerges in the analysis of Sinai walks [1, 4, 6, 29]; consequently, the literature is extensive, notably on the behavior of the invariant measure of the $Y$ process at $+\infty$. However the focus for us is twofold:

(1) characterizing the local part of the invariant measure and the behavior at $-\infty$;
(2) obtaining a sharp estimate on the behavior at $+\infty$.

While point (1) is central because it determines the leading behavior of (1.13), point (2) is as important because the invariant measure of $Y$ just provides the DH guess on the negative and positive semi-axes, separately (they are essentially symmetric problems). But we need a sharp asymptotic analysis at $+\infty$ of the two measures, i.e. at the origin which is very far from both $+k$ and $-k$, to glue them together. The results available on this problem are obtained in too general a context and they are too weak for our purposes.

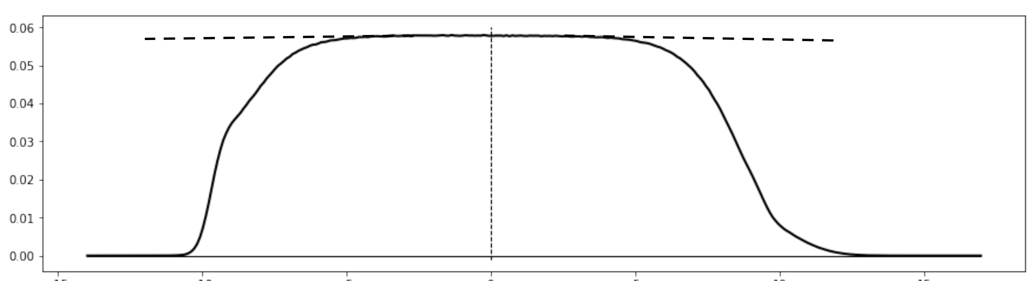


FIGURE 1. The solid line gives density of the invariant probability for $k = 10$ and $\mathtt{z} = \xi(-1/2 + 3\eta/10) + (1-\xi)(\eta+1)$, with $\xi$ a Bernoulli with success probability $2/3$ and $\eta$ a standard Gaussian, with $\xi$ and $\eta$ independent. The distribution of $\mathtt{z}$ is asymmetric and bimodal. The densities of the two non normalizable measures that build the DH measure are (essentially) given by the prolongation of the plateau to the right for one measure, and to the left for the other one (dashed lines).

Once the DH probability $\nu_k^{\mathrm{DH}}$ (explicit!) is built, its closeness to the invariant probability $\nu_k$ (not explicit!) has to be established. What is directly accessible is the action of the (one step) Markov Kernel $T = T_k$ on $\nu_k^{\mathrm{DH}}$ and we certainly want $T\nu_k^{\mathrm{DH}} - \nu_k^{\mathrm{DH}}$ *small.* In

[14] this closeness is estimated for $\alpha > 0$ in terms of a norm $|||\cdot|||_\beta$ indexed by $\beta \in (0, \alpha)$. And the key point is that $\left|\left|\left|\nu_k^{\mathrm{DH}} - \nu_k\right|\right|\right|_\beta$ is bounded above by $c_\beta \left|\left|\left|T\nu_k^{\mathrm{DH}} - \nu_k^{\mathrm{DH}}\right|\right|\right|_\beta$, with $c_\beta = 1/(1 - \mathbb{E}[Z^\beta])_+$ (if $\alpha < 0$, i.e. $\mathbb{E}[Z^\alpha] > 1$, we work with $\mathbb{E}[Z^{-\beta}]$). $|||\cdot|||_0$ is well defined too: it is simply the $\mathbb{L}^1$ norm of the primitive of $\cdot$ which is in $\mathbb{L}^1$. And in fact $|||\cdot|||_0$ is, or could be, a good norm for our purposes: the problem is that $c_0 = +\infty$.

To get around this problem we take an approach that avoids using contraction properties. In fact we show that the bound holds with $c_0 = c_0(k)$ which is $O(k^2)$ (up to logarithmic corrections). The divergence of $c_0(k)$ can be overcome if $\left|\left|\left|T\nu_k^{\mathrm{DH}} - \nu_k^{\mathrm{DH}}\right|\right|\right|_0$ decays faster that $k^{-2}$. With our hypotheses, this decay is exponential in $k$.

**Remark 1.2.** *At this stage the dominant role of $-k$ with respect to $+k$ may appear strange. But this is just an artifact of the choice of* (1.13) *which is the formula that one obtains when looking at the exponential growth of the* $(1, 1)$ *entry of the matrix. But $+k$ takes the leading stand if we consider the formula stemming out of the exponential growth of the* $(2, 2)$ *entry, see* (2.7)*. Of course, the Lyapunov exponent does not depend on the choice of entry. This is discussed further in Remark 3.2.*

### 1.4. **Generalizations, more on the literature and organization of the paper.**

*The case* $\mathbb{E}[\log Z] \neq 0$ *and* $\mathbb{E}[Z] \leq 1$. Under these hypotheses $\mathbb{E}[Z^\alpha] = 1$ has a solution for $|\alpha| \geq 1$ unless $\mathbb{P}(Z \in (0, 1]) = 1$ and this last case can be looked upon as $\alpha = \pm\infty$ case (we assume in any case that $Z$ is not a constant). One can find in [12] an argument in favor of the fact that $\mathcal{L}(\varepsilon) = \sum_{j=0}^{\lfloor|\alpha|\rfloor} C_j \varepsilon^{2j} + C_\alpha \varepsilon^{2|\alpha|} + o(\varepsilon^{2|\alpha|})$, at least for $\alpha \notin \mathbb{Z}$. Results in this direction can be found in [15] that we cite also for more complete results. Let us point out, however, that identifying the singular behavior, that is the presence of $C_\alpha \varepsilon^{2|\alpha|}$, is an open problem and the approach we employ does not seem to be appropriate when this is a subleading term (i.e., $|\alpha| \geq 1$).

*Weak disorder limits.* The weak disorder limit corresponds in dynamical terms to a very slow dynamics. This allows to rescale time and the arising dynamics is a two dimensional system of stochastic ODEs that can be exactly solved. This limit dynamics has the remarkable property that the Furstenberg probability has an explicit formula which leads to an expression for the Lyapunov exponent in terms of Bessel functions (to our knowledge this appeared first in the works of McCoy and Wu [22], but similar computations for related systems were already in the literature, see[8, 9, 10, 20, 26] for recent works on weak disorder limits and historical overviews). Deriving the full asymptotic expansion of the Lyapunov exponent is then a somewhat cumbersome, but straightforward, exercise [8, Prop. 1.3]. Particularly relevant for us is the $\varepsilon \searrow 0$ limit behavior of the Lyapunov exponent for $\alpha = 0$ [8, (1.11)] which can be refined to [8, end of Sec. 4]

$$\frac{1}{4\left(\log(1/\varepsilon) - \log 2 - \gamma\right)} + O\left(\varepsilon^2\right) , \tag{1.15}$$

where $\gamma$ is the Euler-Mascheroni constant and the $O(\varepsilon^2)$, which can be explicitly expressed, is non zero. Of course the last formula should be compared to our main result (1.7).

As already pointed out in [8], the extremely sharp matching of the *finite disorder* case and the *infinitesimal disorder* case is mathematically quite surprising: not only the leading order matches, but the structure of the subheading terms is the same (to the extent of what is known). However there does not appear to be any way to recover finite disorder results from infinitesimal disorder computations. This is particularly unfortunate for the

$d = 2$ columnar disorder case [22, 23, 27]: McCoy and Wu used the weak disorder limit to infer results about the model and setting forth a precise and highly non trivial prediction for the critical behavior that represents a challenge for mathematicians (see [8, App. A] and references therein).

*What about lower regularity of the distribution of the disorder?* Our approach does not capture only the leading behavior, but also the main subleading correction and gives a very small error term, much like (1.15) which stems instead from an exact computation. Clearly in estimating the error term there seems to be a lot of room: for example, it would be sufficient to show (*the one step estimate*) that $\|T\nu_k^{\mathrm{DH}} - \nu_k^{\mathrm{DH}}\|_1 = O(k^{-c})$, for a $c > 2$. But our approach naturally yields an exponential estimate, see (3.10). This approach (i.e., estimating the inverse Laplace transform) strongly relies on an appropriate regularity of $\zeta$, i.e. asymptotic decay in Fourier space (the imaginary direction for the Laplace transform), and yields leading term, subleading correction and exponential bounds. It is really not clear whether our one step bound procedure could go through with less regularity.

As a matter of fact, we are unable to treat the *exactly solvable case* of [24]: in this case the Laplace transform of the invariant $\sigma$-finite measures are more explicit, but they have a poor decay in the imaginary direction.

*More general matrices.* All we did can be greatly generalized (and this discussion applies to [14] too). Matrices of the form

$$\begin{pmatrix} \widetilde{Z} & \varepsilon \widetilde{Z} \\ \varepsilon Z & Z \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \varepsilon Z' \\ \varepsilon Z'' & Z \end{pmatrix}, \tag{1.16}$$

can be dealt with in a straightforward way under suitable hypotheses: for the first example, if $\widetilde{Z} > 0$ too, with no independence requirement with $Z$, simply because $\widetilde{Z}$ can be factored out and the key variable becomes $Z/\widetilde{Z}$. For the second example, the analysis does not contain much novelty if $Z$, $Z'$ and $Z''$ are independent and positive and if we require on the marginal laws of $Z'$ and $Z''$ the same integrability and regularity we have required for $Z$. A complete analysis of two dimensional matrices with positive entries approaching for $\varepsilon \searrow 0$ a diagonal matrix might be of interest (but it is probably rather cumbersome and the Ising model motivation is lost). Some higher dimension matrices generalizations are treated in [15, App. A].

*Organization of what follows.* In Section 2 we study the main Markov chain $X$. We state in Proposition 2.2 the crucial bound that we call *reduction to one step estimate*. We also study the auxiliary chain $Y$ that is central in the construction of the DH probability.

In Section 3 we construct the DH probability and perform the one step bound: the proof of our main result, Theorem 1.1, is in this section (right after Proposition 3.3).

In Section 4 we prove the *reduction to one step estimate*, i.e. Proposition 2.2.

## 2. The underlying Markov chains

In general, if $Z$ is a random variable, we use $G_Z(x) = \mathbb{P}(Z > x)$ and $F_Z(x) = 1 - G_Z(x)$. This notation is used also for finite measures $\mu$: $G_\mu(x) = \mu((x, \infty))$ and $F_\mu(x) = \mu((-\infty, x])$. We will work also with non normalizable ($\sigma$-finite) measures, notably with measures $\mu$ such that $\mu((-\infty, x]) < \infty$ for every $x$, and the notation $F_\mu(x)$ will be used also in this case.

2.1. **About the main Markov chain $X$.** We start the analysis of the $X$ chain defined by (1.12).

**Lemma 2.1.** *We have*

$$TG_{X_n}(x) \,:=\, G_{X_{n+1}}(x) \,=\, G_{\mathsf{z}}(k+x) + \int_{\mathbb{R}} G_{X_n}(y) h_k'(y) \zeta\left(x - h_k(y)\right) \mathrm{d}y\,, \tag{2.1}$$

*and*

$$TF_{X_n}(x) \,:=\, F_{X_{n+1}}(x) \,=\, F_{\mathsf{z}}(-k+x) + \int_{\mathbb{R}} F_{X_n}(y) h_k'(y) \zeta\left(x - h_k(y)\right) \mathrm{d}y\,. \tag{2.2}$$

*Proof.* By using that $h_k(\cdot)$ is an increasing bijection from $\mathbb{R}$ to $(-k,k)$ we see that

$$\begin{aligned} G_{X_{n+1}}(x) \,&=\, \mathbb{P}\left(\mathsf{z}_{n+1} + h_k\left(X_n\right) > x\right) \\ &=\, \mathbb{P}\left(x - \mathsf{z}_{n+1} \le -k\right) + \mathbb{P}\left(h_k\left(X_n\right) > x - \mathsf{z}_{n+1};\, |x - \mathsf{z}_{n+1}| < k\right) \\ &=\, \mathbb{P}\left(\mathsf{z} \ge x + k\right) + \mathbb{P}\left(X_n > h_k^{-1}\left(x - \mathsf{z}_{n+1}\right);\, |x - \mathsf{z}_{n+1}| < k\right) \\ &=\, G_{\mathsf{z}}(k+x) + \int_{-k+x}^{k+x} G_{X_n}\left(h_k^{-1}(x-z)\right) \zeta(z)\, \mathrm{d}z\,, \end{aligned} \tag{2.3}$$

and by the change of variable $z = x - h_k(y)$ we complete the proof of the first identity. The proof of the second identity is of course exactly analogous (or can be directly derived from the first). □

The map $T$ can then be extended by linearity to act on $G$ defined by $G(x) = G_{\nu_+}(x) - G_{\nu_-}(x)$, $\nu_\pm$ finite measures (so $|G(-\infty)| < \infty$):

$$TG(x) \,=\, G(-\infty) G_{\mathsf{z}}(k+x) + \int_{-k+x}^{k+x} G\left(h_k^{-1}(x-z)\right) \zeta(z)\, \mathrm{d}z\,. \tag{2.4}$$

In the applications we consider $\nu$ is the difference of two probability measures, so $G(-\infty) = 0$ and $T$ reduces to $T_0$:

$$T_0 G(x) \,:=\, \int_{\mathbb{R}} G(y) h_k'(y) \zeta\left(x - h_k(y)\right) \mathrm{d}y\,, \tag{2.5}$$

which is well defined also in the slightly different set-up of $G \in \mathbb{L}^1$ because (1.6) implies that $\|\zeta\|_\infty < \infty$.

The following bound will be crucial; Section 4 is devoted to its proof.

**Proposition 2.2.** *There exists $C > 0$ and $k_0$ such that for every $k \ge k_0$ and $G \in L^1(\mathbb{R};\mathbb{R})$*

$$\sum_{n=0}^{\infty} \left\|T_0^n G\right\|_1 \,\le\, k^2 \left(\log k\right)^C \|G\|_1\,. \tag{2.6}$$

It is well known that the Lyapunov exponent can be expressed in terms the invariant probability for the action of the random matrix on the projective circle [3, Ch. 2] and for two by two matrices with positive entries we can work on $(0, \pi/2)$ or, considering the tangent of this angle, on $(0, \infty)$ (see [14, (1.6)] or [12, Sec. 2]). Our parametrization choice $P(\mathbb{R}^2) \cong \mathbb{R}$, recall (1.9)-(1.10), just corresponds to applying the logarithm to the tangent of the angle and it suffices to apply this change of variables to the expressions in [12, 14] (which take advantage of the specific form of the matrix under consideration to obtain a simpler expression than the standard Furstenberg formula): we obtain that, by writing $\nu_k$ for the invariant measure on $\mathbb{R}$, the Lyapunov exponent $\mathcal{L}(k)$ is equal to $L_k[G_{\nu_k}]$ with

$$L_k[G] := \int_{\mathbb{R}} \frac{1}{1+e^{k-x}} G(x)\,\mathrm{d}x = \int_{\mathbb{R}} \frac{1}{1+e^{k+x}} (1-G(x))\,\mathrm{d}x\,, \tag{2.7}$$

which will be used for $G(x) = \nu((x,\infty))$ and $\nu$ a probability. We readily see that

$$|L_k[G_1] - L_k[G_2]| \le \|G_1 - G_2\|_1\,. \tag{2.8}$$

We thus have the following important corollary to Proposition 2.2:

**Corollary 2.3.** *With $C$ and $k_0$ as in Proposition 2.2, for $k \ge k_0$ and any probability $\gamma$ such that $TG_\gamma - G_\gamma \in \mathbb{L}^1$*

$$|\mathcal{L}(k) - L_k[G_\gamma]| \le \|G_{\nu_k} - G_\gamma\|_1 \le k^2 (\log k)^C \|TG_\gamma - G_\gamma\|_1\,. \tag{2.9}$$

*Proof.* Since the Markov chain $(X_n)$ is irreducible and positive recurrent (see the beginning of Section A), $\lim_n T^n G_\gamma(x) = G_{\nu_k}(x)$ for every $x \in \mathbb{R}$ which is a continuity point of $G_{\nu_k}(\cdot)$ (we will see that $G_{\nu_k}(\cdot)$ is continuous, see Remark 2.8, but at this stage this is not needed since the set of discontinuities of $G_{\nu_k}(\cdot)$ is countable, hence of Lebesgue measure zero). Therefore, by Fatou's Lemma, we have that

$$\begin{aligned}\|G_{\nu_k} - G_\gamma\|_1 &\le \liminf_N \left\|T^N G_\gamma - G_\gamma\right\|_1 \\ &\le \sum_{n=0}^{\infty} \|T_0^n (TG_\gamma - G_\gamma)\|_1 \le k^2 (\log k)^C \|TG_\gamma - G_\gamma\|_1\,, \end{aligned} \tag{2.10}$$

and the proof is complete by (2.8). $\square$

2.2. **Looking from the edge: the reduced chain $Y$.** If we sit on $-k$, that is if we make it our new origin, in the limit as $k \to \infty$ the Markov chain becomes

$$Y_{n+1} = \mathsf{z}_{n+1} + h(Y_n) \qquad \text{with } h(y) = y + \log(1+\exp(-y))\,. \tag{2.11}$$

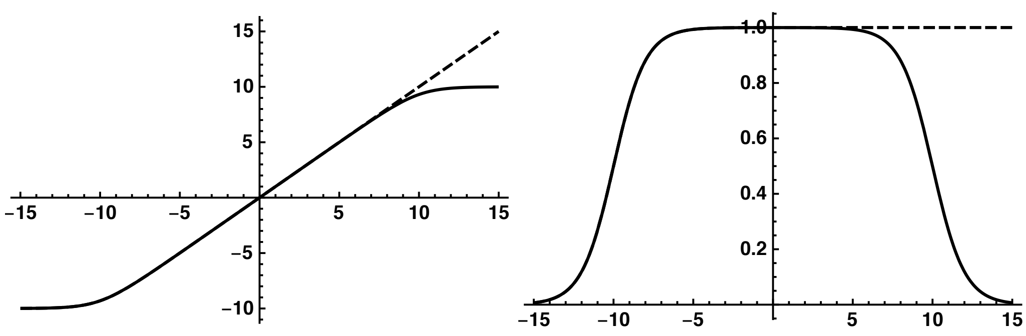


FIGURE 2. For $k = 10$: on the left the plot of $h_k(\cdot)$ (solid line) and of $x \mapsto h(x+k) - k$ (dashed line); on the right the derivative of the same functions.

**Proposition 2.4.** *The Markov chain $Y$ has a unique invariant measure $\nu$ with $\nu((-\infty, x]) < \infty$ for every $x \in \mathbb{R}$, but $\nu(\mathbb{R}) = \infty$.*

For non-normalizable measures uniqueness is of course meant *up to a multiplicative constant* and we note that $\nu$ is characterized by

$$\int g(z)\nu(\,\mathrm{d}z) \;=\; \int\int g(z+h(y))\zeta(z)\,\mathrm{d}z\,\nu(\,\mathrm{d}y) \quad \text{for every measurable } g \geq 0\,. \tag{2.12}$$

The proof of Proposition 2.4 is given for completeness in the Appendix: the Markov chain $Y$ is a (very) particular case of the critical random difference equation problem, see [1, 4, 6] and references therein. We sketch a concise proof of Proposition 2.4 because the context of [1, 4, 6] is really much wider and Proposition 2.4 can be proven with substantially easier arguments.

We conclude this section by studying the asymptotic behavior of $F_\nu$. From the results of [4, 6] one can extract that $F_\nu(x) = \nu((-\infty, x]) \sim cx$ for $x \to \infty$; however our techniques require a stronger result, see (2.13) below, which requires substantially more constraining requirements than the hypotheses in [4, 6].

Here is a preliminary result that allows the use of the Laplace transform. Besides being an immediate consequence of the analysis in [4, 6], it can also be extracted from [14, Lemma 3.3]: since context and notations here are slightly different, we give the proof in the appendix. We make a definite choice of $F_\nu$ by stipulating that $F_\nu\,(x_0) = 1$ for an $x_0$ which is (strictly) inside the support of $\nu$.

**Lemma 2.5.** *There exists $c > 0$ such that $F_\nu(x) \leq c\exp(cx)$ for every $x \geq x_0$.*

Here is the main result of this section:

**Proposition 2.6.** *There exist three constants $m_\zeta > 0$, $c_\zeta \in \mathbb{R}$ and $\varrho_\zeta \in (0,1)$ such that for $x \to \infty$*

$$F_\nu(x) \;=\; m_\zeta x + c_\zeta + O\left(\exp(-\varrho_\zeta x)\right)\,. \tag{2.13}$$

*Moreover, for every $\delta' \in (0,\delta)$ (recall (1.5) for $\delta$) we have for $x \to -\infty$*

$$F_\nu(x) \;=\; O\left(\exp(\delta' x)\right)\,. \tag{2.14}$$

The constants carry the subscript $\zeta$ because they are primarily determined by $\zeta$, but $m_\zeta$ and $c_\zeta$ depend also on the arbitrary choice of $x_0$: $c_\zeta/m_\zeta$ and $\varrho_\zeta$ instead depend only on $\zeta$.

*Proof.* We start with (2.13). We are going to use the Laplace transform for (non negative) functions and measures supported on $[0,\infty)$: $\widehat{f}(u) := \int_{\mathbb{R}} \exp(-ux) f(\,\mathrm{d}x) = \int_{[0,\infty)} \exp(-ux) f(\,\mathrm{d}x)$ and, if $f$ is absolutely continuous, $\widehat{f}(u) = \int_0^\infty \exp(-ux) f(x)\,\mathrm{d}x$. If $F_f(x) \leq \exp(cx)$ for $x$ large, then $\widehat{f}(u)$ is analytic in the complex half plane $\Re u > c$ and $|\widehat{f}(u)| \leq \widehat{f}(\Re u)$.

In order to use the Laplace transform to study the right-tail of $\nu$ we introduce $\widetilde{Y}_n := \log(1+\exp(Y_n))$; then $(\widetilde{Y}_n)$ is a Markov chain with

$$\widetilde{Y}_{n+1} \;=\; \log\left(1+\exp\left(\mathsf{z}_{n+1}+\widetilde{Y}_n\right)\right) \;=\; \mathsf{z}_{n+1}+\widetilde{Y}_n + \log\left(1+\exp\left(-\left(\mathsf{z}_{n+1}+\widetilde{Y}_n\right)\right)\right)\,, \tag{2.15}$$

analogous to (2.11). The chains $(\widetilde{Y}_n)$ and $(Y_n)$ are equivalent and the invariant measure $\mu$ of $(\widetilde{Y}_n)$ is directly related to $\nu$. In particular $\mu$ is $\sigma$-finite, $F_\mu(x) = 0$ for every $x \leq 0$ and, by Proposition 2.4, $F_\mu(x) < \infty$ for every $x$. In fact, we have $F_\nu(x) = F_\mu(\log(1+\exp(x)))$

so we can replace $F_\nu(x)$ with $F_\mu(x)$ in (2.13) obtaining an equivalent expression. Note the following characterization of $\mu$: for measurable $g:(0,\infty)\to[0,\infty)$

$$\int g\,\mathrm{d}\mu \;=\; \int\int g\left(\log\left(1+\exp(-z-x)\right)\right)\zeta(\,\mathrm{d}z)\mu(\,\mathrm{d}x)\,. \tag{2.16}$$

By Lemma 2.5 we have that $\widehat{F}_\mu(u)$ is analytic in the domain $\Re u > c > 0$. We recall also that, in the same domain, $\widehat{\mu}(u) = u\widehat{F}_\mu(u)$. We call $b_\mu$ the *optimal value of* $c$:

$$b_\mu := \inf\{u\in\mathbb{R}:\, \widehat{\mu}(u)<\infty\}\,, \tag{2.17}$$

so $b_\mu \le c$. On the other hand Proposition 2.4 tells us that $b_\mu \ge 0$, because $\nu(\mathbb{R}) = \mu([0,\infty)) = \infty$. What we are going to show now is that $b_\mu = 0$.

By (2.16) we readily see that

$$\widehat{\mu}(u) \;=\; \int\int (1+\exp(x+y))^{-u}\zeta(\,\mathrm{d}x)\mu(\,\mathrm{d}y)\,. \tag{2.18}$$

We now use the identity [25, (5.13.1)] (with $a=0$, $b=u$ and $s=w$, $a$, $b$ and $s$ are the notations in [25])

$$(1+z)^{-u} \;=\; \frac{1}{2\pi i\Gamma(u)}\int_{w_0-i\infty}^{w_0+i\infty}\Gamma(w)\Gamma(u-w)z^{-w}\,\mathrm{d}w\,, \tag{2.19}$$

which holds if $w_0\in(0,\Re u)$ and for every $z\in\mathbb{C}\setminus i(-\infty,0)$. We recall also the *generalized Stirling* formula [25, (5.11.9)]:

$$|\Gamma(x+iy)| \overset{|y|\to\infty}{\sim} \sqrt{2\pi}|y|^{x-1/2}\exp(-\pi|y|/2)\,, \tag{2.20}$$

for $|y|\to\infty$, uniformly for $x$ in bounded intervals. Therefore by inserting (2.19) into (2.18) and by using the Fubini-Tonelli Theorem we obtain that for $b_\mu < w_0 < \Re u$ we have

$$\widehat{\mu}(u) \;=\; \frac{1}{2\pi i\Gamma(u)}\int_{w_0-i\infty}^{w_0+i\infty}\Gamma(w)\Gamma(u-w)\widehat{\zeta}(w)\widehat{\mu}(w)\,\mathrm{d}w\,. \tag{2.21}$$

The next step is moving $w_0$ to the right of $\Re u$: more precisely to $(\Re u,\Re u+1)$. Given the decay for large imaginary part of the Gamma function, this can be done, but the stripe $\{z\in\mathbb{C}:\, w_0<\Re z<w_0+1\}$ contains the simple pole of $\Gamma(w-u)$ at $w=u$. Since the residue of the integrand (times the prefactor) is $-\widehat{\zeta}(u)\widehat{\mu}(u)$ and by remarking that we are performing the integral in the clockwise sense, we obtain that for $w_0\in(\Re u,\Re u+1)$

$$\widehat{\mu}(u) \;=\; \widehat{\zeta}(u)\widehat{\mu}(u) + \frac{1}{2\pi i\Gamma(u)}\int_{w_0-i\infty}^{w_0+i\infty}\Gamma(w)\Gamma(u-w)\widehat{\zeta}(w)\widehat{\mu}(w)\,\mathrm{d}w\,, \tag{2.22}$$

which we rewrite as

$$\widehat{\mu}(u) \;=\; \frac{1}{2\pi i\Gamma(u)(1-\widehat{\zeta}(u))}\int_{w_0-i\infty}^{w_0+i\infty}\Gamma(w)\Gamma(u-w)\widehat{\zeta}(w)\widehat{\mu}(w)\,\mathrm{d}w \;=:\; P(u)I(u)\,, \tag{2.23}$$

with $P(\cdot)$ the pre-factor and $I(\cdot)$ the integral. We remark that this allows to analytically continue $I(u)$ to smaller values of $\Re u$, even down to $\Re u < w_0-1$ and $w_0$ is chosen larger (but arbitrarily close to $b_\mu$). But $P(u)$ has a singularity at $u=0$ as we are going to explain:

- $\Gamma(u)\sim 1/u$ has a simple pole at $u=0$ (like for all the negative integers) and these are the only singularities; moreover the $\Gamma$ function has no zero.

- $u \mapsto 1-\widehat{\zeta}(u)$ is an entire function with $1-\widehat{\zeta}(u) \sim -\sigma^2 u^2/2$, $\sigma^2 := \int x^2 \zeta(\,\mathrm{d}x) > 0$: so, $1-\widehat{\zeta}(u)$ a double zero at the origin. We remark also that (with standard probabilistic notation) $\widehat{\zeta}(-it) = \varphi_{\mathtt{z}}(t)$, i.e. it is the characteristic function of a continuous random variable which yields $|\widehat{\zeta}(it)| < 1$ for $t \neq 0$ ($|\varphi_{\mathtt{z}}(t)| = 1$ for a $t \neq 0$ directly implies that $\mathtt{z}$ is discrete). By the Riemann-Lebesgue Lemma we have $|\widehat{\zeta}(it)| = o(1)$ for $|t|$ large. Therefore there exists $\varrho > 0$ such that $1-\widehat{\zeta}(u) = 0$ for $\Re u > -\varrho$ only for $u = 0$. We assume from now on that $\varrho \le 1$ (in any case, $I(\cdot)$ has been analytically continued only up to $\Re u > -1$).

So $P(\cdot)$ does contribute a simple pole at $z = 0$: a priori there is still the possibility that $I(0) = 0$ making this singularity removable, but in fact this is not possible because we have already remarked that $b_\mu \ge 0$.

We have therefore proven not only that $b_\mu = 0$, but also that $\widehat{\mu}$ can be meromorphically extended to $\{z \in \mathbb{C} : \Re z > -\varrho\}$ and the only singularity in this region is the simple pole at zero

$$\widehat{\mu}(u) \,=\, \frac{m_\zeta}{u} + c_\zeta + uH(u)\,, \tag{2.24}$$

with $m_\zeta > 0$ because $\widehat{\mu}(u) > 0$ for $u > 0$, $c_\zeta \in \mathbb{R}$ and $H(\cdot)$ is an analytic function on the domain $\{z \in \mathbb{C} : \Re z > -\varrho\}$. This of course tells us that also $\widehat{F}_\mu$ can be meromorphically extended to the same domain and

$$\widehat{F}_\mu(u) \,=\, \frac{m_\zeta}{u^2} + \frac{c_\zeta}{u} + H(u)\,. \tag{2.25}$$

We now use the classical Mellin-Bromwich formula for the inverse Laplace transform: for every $b > b_\mu = 0$

$$F_\mu(x) \,=\, \frac{1}{2\pi i} \lim_{R\to\infty} \int_{b-iR}^{b+iR} \widehat{F}_\mu(u) \exp(ux)\,\mathrm{d}u\,. \tag{2.26}$$

We introduce the rectangular contour $\mathtt{C}_R$ made of the segment of integration in (2.26), the segment $\{u : \Re u = -\varrho_\zeta$ and $\Im u \in [-R, R]\}$ and by the two segments $\{u : \Im u = \pm R$ and $\Re u \in [-\varrho_\zeta, a]\}$. The orientation is counter-clockwise and $\varrho_\zeta$ is chosen in $(0, \varrho)$. The integration along this contour is equal to the residue of the double pole at the origin:

$$\widehat{F}_\mu(u)\exp(ux) \,=\, \Big(\frac{m_\zeta}{u^2} + \frac{c_\zeta}{u} + O(1)\Big)\left(1 + xu + O(u^2)\right) \,=\, \frac{m_\zeta}{u^2} + \frac{c_\zeta + m_\zeta x}{u} + O(1)\,, \tag{2.27}$$

hence the residue is $c_\zeta + m_\zeta x$ and

$$\frac{1}{2\pi i} \int_{\mathtt{C}_R} \widehat{F}_\mu(u)\exp(ux)\,\mathrm{d}u \,=\, c_\zeta + m_\zeta x\,. \tag{2.28}$$

We are therefore left with showing that:

(1) the contribution of the integrals along the segments $\{u : \Im u = \pm R$ and $\Re u \in [-\varrho_\zeta, a]\}$ vanishes as $R \to \infty$;
(2) the contribution of the integrals along $\{u : \Re u = -\varrho_\zeta$ and $\Im u \in [-R, R]\}$ is $O(\exp(-\varrho_\zeta x))$ for $x \to \infty$.

Both estimates rely on

**Lemma 2.7.** *We consider $w_0 > 0$ and a closed interval $I \subset (w_0 - 1/2, w_0)$. For $|w_1| \to \infty$ and uniformly in $\Re u \in I$*

$$\frac{1}{|\Gamma(u)|} \int_{w_0 - i\infty}^{w_0 + i\infty} |\Gamma(w)|\, |\Gamma(u-w)|\, \mathrm{d}w \,=\, O\left(|\Im u|^{1/2}\right) \tag{2.29}$$

Lemma 2.7 is [14, Lemma 4.4], but the proof can also be obtained directly from (2.20). We remark that the dominant contribution comes from integrating for $w$ in a neighborhood of the real axis.

Moreover for both (1) and (2) the crucial formula is (2.23) because, in conjunction with (2.29) and recalling that $\widehat{F}_\mu(u) = \widehat{\mu}(u)/u$, it tells us that

$$|\widehat{F}_\mu(u)| \,=\, O\left(\left|\widehat{\zeta}(u)\right| |\Im u|^{-1/2}\right), \tag{2.30}$$

for $|\Im u|$ large and uniformly for $\Re u$ in the interval we consider.

Therefore the contour integration in point (1) is $O(1/R^{1/2})$ and for what concerns point (2) we have that for every $R$

$$\left| \int_{-\varrho_\zeta - iR}^{-\varrho_\zeta + iR} \widehat{F}_\mu(u) \exp(ux)\, \mathrm{d}u \right| \,\le\, \exp(-\varrho_\zeta x) \int_{-\infty}^{\infty} \left| \widehat{F}_\mu\left(-\varrho_\zeta + iy\right) \right| \mathrm{d}y\,, \tag{2.31}$$

with the integral in the right-hand side that converges because of (B.1). This completes the proof of (2.13).

For (2.14) we use (2.12) with $g = \mathbf{1}_{(-\infty,x]}$, that is

$$F_\nu(x) \,=\, \int\int \mathbf{1}_{(-\infty,x]}(z + h(y))\nu(\,\mathrm{d}y)\zeta(\,\mathrm{d}z)\,. \tag{2.32}$$

By splitting the integral in the $y$ variable into positive and negative values and by using that $h(y) \ge 0$ for $y$ negative and $h(y) \ge y$ for $y$ positive we see that

$$\begin{aligned} F_\nu(x) \,&\le\, F_\nu(0)F_\zeta(x) + \int\int \mathbf{1}_{(-\infty,x]}(z+y)\mathbf{1}_{(0,\infty)}(y)\nu(\,\mathrm{d}y)\zeta(\,\mathrm{d}z) \\ &\le\, F_\nu(0)F_\zeta(x) + \int\int \mathbf{1}_{(-\infty,x-z]}(y)\mathbf{1}_{(0,\infty)}(x-z)\nu(\,\mathrm{d}y)\zeta(\,\mathrm{d}z) \\ =\, F_\nu(0)F_\zeta(x) &+ \int F_\nu(x-z)\mathbf{1}_{(0,\infty)}(x-z)\zeta(\,\mathrm{d}z) \le F_\nu(0)F_\zeta(x) + C\int_0^\infty y\zeta(x-y)\,\mathrm{d}y\,, \end{aligned} \tag{2.33}$$

where in the last step $C$ is a positive constant and we have used (2.13). But $\int_0^\infty y\zeta(x - y)\,\mathrm{d}y = \int_{-\infty}^x F_\zeta(z)\,\mathrm{d}z$ and (2.14) is established.

The proof of Proposition 2.6 is therefore complete. □

**Remark 2.8.** (2.32) *also characterizes $\nu_k$ if we replace $h(\cdot)$ with $h_k(\cdot)$. From this we also have the characterization*

$$F_{\nu_k}(x) \,=\, \int_{\mathbb{R}} F_\zeta(x - h_k(y))\nu_k(\,\mathrm{d}y)\,, \tag{2.34}$$

*from which we see that $F_{\nu_k}$ is $C^1$, so $\nu_k$ has a density, because $0 \le \zeta(x - h_k(y)) \le \|\zeta\|_\infty$.*

*Similarly $\nu$ has a density, but the argument needs to be refined. Equation* (2.34) *still holds also without the subscripts $k$, but because $\nu(\mathbb{R}) = \infty$ we need an appropriate upper bound for $\zeta(x-h(y))$ in order to take the derivative under the integral. Since $\nu((-\infty, x]) <$*

*$\infty$ for every $x$, $\zeta \in \mathbb{L}^\infty$ reduces the problem to finding and upper bound for $y > 0$ and large (of course we can focus on $x$ in a compact set). But this follows because there exists $\varepsilon > 0$ and $z_0$ such that $\zeta(z) \le \exp(\varepsilon z)$ for $z \le z_0$. In order to show this and to make the argument more readable, let us replace $\zeta(z)$ with $\zeta(-z)$. If $\zeta(z) \le \exp(-\varepsilon z)$ for $z$ larger than some value $z_0$ is false, then there exists a sequence $(z_j)$ with $\lim_j z_j = \infty$ and $\zeta(z_j) > \exp(-\varepsilon z_j)$. Uniform $\theta$-Hölder continuity implies that $\zeta(z) \ge \exp(-\varepsilon z_j)/2$ for $|z - z_j| \le ((e^{-\varepsilon z_j})/2C)^{1/\theta}$. So, possibly for $j$ large, $\int_{z_j/2}^{\infty} \zeta(z)\,\mathrm{d}z \ge \exp(-\varepsilon z_j)((e^{-\varepsilon z_j})/2C)^{1/\theta}$. Since the left-hand side decays exponentially, by choosing $\varepsilon > 0$ small we reach a contradiction. So we have shown that $\zeta$ vanishes exponentially fast: using that $\nu((-\infty, x])$ has linear asymptotic growth, $F_\nu \in C^1$ follows.*

## 3. The DH probability, the one step bound and the proof the main result

We now define the DH probability, i.e. the measure that we expect (and will prove) to be close to the invariant probability $\nu_k$. It is built by gluing together the invariant measure for the edge process at $k$ and for the one at $-k$. Unless $\zeta$ is symmetric, the two edge limit problems are not the same, since the one on the left involves $x \mapsto \zeta(x)$ as jump density probability and the one one the right involves $x \mapsto \zeta(-x)$. So the cumulative function on the left (respectively, right) edge will be denoted by $F_\triangleleft(\cdot)$ (respectively, $F_\triangleright(\cdot)$). Moreover we choose to normalize the cumulative functions so that they are, for $x \to \infty$, equivalent to $x$. Proposition 2.6 than says that there exist $c_s \in \mathbb{R}$, $s$ is $\triangleleft$ or $\triangleright$, and $\varrho \in (0,1)$ such that for $x \to \infty$

$$F_s(x) = x + c_s + O\left(\exp(-\varrho x)\right) . \tag{3.1}$$

We define the DH probability by giving its integrated tail probability $G_k(x) = \nu_k^{\mathrm{DH}}((x,\infty))$ (cf. Sec. 1.3):

$$G_k(x) \,:=\, \begin{cases} F_\triangleright(k-x)/C_k & \text{if } x \ge 0\,, \\ 1 - (F_\triangleleft(x+k)/C_k) & \text{if } x \le 0\,, \end{cases} \tag{3.2}$$

where the fact that the definition must be consistent at $x = 0$ fixes the value of $C_k > 0$ which, therefore, by (3.1), satisfies

$$C_k \,=\, 2k + c_\triangleleft + c_\triangleright + O(\exp(-\varrho k))\,. \tag{3.3}$$

We register also that for the cumulative probability $F_k(\cdot) = 1 - G_k(\cdot)$ we have

$$F_k(x) \,:=\, \begin{cases} 1 - (F_\triangleright(k-x)/C_k) & \text{if } x \ge 0\,, \\ F_\triangleleft(x+k)/C_k & \text{if } x \le 0\,. \end{cases} \tag{3.4}$$

The next fact is straightforward computation, but it is of course central for us:

**Lemma 3.1.** *For $k \to \infty$*

$$L_k[G_k] \,=\, \frac{1}{C_k}\int_{\mathbb{R}} \frac{F_\triangleright(y)}{1+e^y}\,\mathrm{d}y + O(\exp(-k)) \,=\, \frac{1}{C_k}\int_{\mathbb{R}} \frac{F_\triangleleft(y)}{1+e^y}\,\mathrm{d}y + O(\exp(-k))\,. \tag{3.5}$$

*Proof.* By the first equality in (2.7) we have

$$L_k[G_k] \,=\, \frac{1}{C_k}\int_0^\infty \frac{F_\triangleright(k-x)}{1+e^{k-x}}\,\mathrm{d}x + \int_{-\infty}^0 \frac{G_k(x)}{1+e^{k-x}}\,\mathrm{d}x\,. \tag{3.6}$$

Using $G_k(x) \in [0,1]$ we readily see that the second addendum is smaller than $2\exp(-k)$. Moreover

$$\int_0^\infty \frac{F_\triangleright(k-x)}{1+e^{k-x}}\,\mathrm{d}x \;=\; \int_{\mathbb{R}} \frac{F_\triangleright(y)}{1+e^y}\,\mathrm{d}y - \int_k^\infty \frac{F_\triangleright(y)}{1+e^y}\,\mathrm{d}y \;=\; \int_{\mathbb{R}} \frac{F_\triangleright(y)}{1+e^y}\,\mathrm{d}y + O\left(k\exp(-k)\right)\,, \tag{3.7}$$

where in the last step we used (3.1).

Repeating the same argument starting with the second equality in (2.7) gives the same expression with $F_\triangleleft$ replacing $F_\triangleright$. □

**Remark 3.2.** (3.5) *of course implies* $\int (1+e^y)^{-1}F_s(y)\,\mathrm{d}y$ *does not depend on* $s$. *Equivalently, by integration by parts:*

$$\int_{\mathbb{R}} \log\left(1+e^{-y}\right)\nu_\triangleright(\,\mathrm{d}y) \;=\; \int_{\mathbb{R}} \log\left(1+e^{-y}\right)\nu_\triangleleft(\,\mathrm{d}y)\,. \tag{3.8}$$

*One way to see this directly is to observe that by* (1.12) *and* (1.10) *we have* $\int_{\mathbb{R}}(x - h_k(x))\nu_k(\,\mathrm{d}x) = 0$, *with* $\nu_k$ *the invariant probability of the Markov chain defined by* (1.12). *But this is equivalent to*

$$\int_{\mathbb{R}} \log\left(1+e^{k-x}\right)\nu_k(\,\mathrm{d}x) \;=\; \int_{\mathbb{R}} \log\left(1+e^{-k-x}\right)\nu_k(\,\mathrm{d}x)\,, \tag{3.9}$$

*and* (3.8) *follows by exploiting the strong form of convergence – see Remark 3.4 – we have for* $2k\nu_k\Theta_{-k}^{-1}$ *towards* $\nu_\triangleleft$ *and for* $2k\nu_k\Theta_k^{-1}$ *towards* $\nu_\triangleright$*: here* $\Theta_a(x) := x + a$.

**Proposition 3.3.** *There exists* $\eta > 0$ *such that*

$$\left\|TG_k - G_k\right\|_1 \;=\; O\left(\exp(-\eta k)\right)\,. \tag{3.10}$$

The proof is written so that the constant $\eta$ in (3.10) can be chosen arbitrarily in $(0, \min(\delta/2, \theta)/2, \varrho)$, with $\delta$ in (1.5), $\theta$ in (1.6) and $\varrho$ is the constant $\varrho_\zeta$ in Proposition 2.6: a slight modification of the proof yields that we can choose $\eta \in (0, \min(\delta, \theta, \varrho))$.

Before proving Proposition 3.3 *we remark in an official way* that it is the last brick needed for our main result.

*Proof of Theorem 1.1.* Recall that $\nu_k$ is the invariant probability of the main chain $X$. It suffices to choose $G_\gamma = G_k$ and apply Corollary 2.3. By using Proposition 3.3 we obtain that

$$\left|\mathcal{L}(k) - L_k[G_k]\right| \;\le\; \left\|G_{\nu_k} - G_k\right\|_1 \;=\; O\left(k^2\left(\log k\right)^C \exp(-\eta k)\right)\,. \tag{3.11}$$

We then conclude by Lemma 3.1 and by exploiting the expression (3.3) for $C_k$. We obtain

$$\kappa_1 \;=\; \frac{1}{2}\int_{\mathbb{R}} \frac{F_\triangleleft(y)}{1+e^y}\,\mathrm{d}y \;=\; \frac{1}{2}\int_{\mathbb{R}} \frac{F_\triangleright(y)}{1+e^y}\,\mathrm{d}y \quad \text{and} \quad \kappa_2 = \frac{1}{2}\left(c_\triangleleft + c_\triangleright\right)\,. \tag{3.12}$$

□

**Remark 3.4.** *We remark that a byproduct of the* $\mathbb{L}^1$ *control in* (3.11) *is that* $2k\nu_k\theta_{-k}^{-1}$ *converges vaguely for* $k \to \infty$ *towards* $\nu_\triangleleft$ *and* $2k\nu_k\theta_k^{-1}$ *converges vaguely towards* $\nu_\triangleright$. *Vague convergence, i.e. with test functions that are compactly supported and* $C^0$, *is an elegant statement, but* (3.11) *is much stronger and, in particular, it solves the issue raised in Remark 3.2.*

*Proof of Proposition 3.3.* Of course

$$\|TG_k - G_k\|_1 \,=\, \left\|(TG_k - G_k)\mathbf{1}_{(-\infty,0)}\right\|_1 + \left\|(TG_k - G_k)\mathbf{1}_{(0,\infty)}\right\|_1\,, \tag{3.13}$$

and, even if the two terms on the right may be (and typically are) different unless $\zeta$ is symmetric, they can be treated in the same way because both of them can be written as

$$A_{k,1} \,:=\, \int_{-\infty}^{0}\left|\int_{\mathbb{R}} F(y)h_k'(y)\zeta(x-h_k(y))\,\mathrm{d}y - F(x)\right|\,\mathrm{d}x\,, \tag{3.14}$$

with $F(\cdot)$ which is $F_k(\cdot)$, given in (3.4), for the first addendum in the right-hand side of (3.13), and $F(\cdot)$ that is instead replaced by the right-hand side in (3.4) with $\triangleleft$ and $\triangleright$ exchanged. Arbitrarily, we choose to work with the first addendum and the bound we are after is achieved in two steps.

The first step in controlling $A_{k,1}$ is to remark that we can avoid the nuisance of the fact that $F_k(y)$ has two different expressions according to the sign of $y$. Namely, we want to switch to:

$$A_{k,2} \,:=\, \frac{1}{C_k}\int_{-\infty}^{0}\left|\int_{\mathbb{R}} F_{\triangleleft}(y+k)h_k'(y)\zeta(x-h_k(y))\,\mathrm{d}y - F_{\triangleleft}(x+k)\right|\,\mathrm{d}x\,. \tag{3.15}$$

We can do this because

$$|A_{k,2}-A_{k,1}| \,\le\, \int_{-\infty}^{0}\int_{0}^{\infty}\left|\frac{F_{\triangleleft}(k+y)+F_{\triangleright}(k-y)}{C_k}-1\right| h_k'(y)\zeta(x-h_k(y))\,\mathrm{d}y\,\mathrm{d}x\,, \tag{3.16}$$

and, by (3.1) and (3.3), for $y \ge 0$ we have

$$\left|\frac{F_{\triangleleft}(k+y)+F_{\triangleright}(k-y)}{C_k}-1\right| \,\le\, \frac{C}{C_k}\times\begin{cases}\exp(-\varrho(k-y)) & \text{if } y\in[0,k]\,,\\ y-k+1 & \text{if } y>k\,,\end{cases} \tag{3.17}$$

for a suitably chosen $C>0$. So $|A_{k,2}-A_{k,1}|$ is bounded above by

$$\int_0^k \exp(-\varrho(k-y))F_\zeta(-h_k(y))h_k'(y)\,\mathrm{d}y + \int_k^\infty (y-k+1)F_\zeta(-h_k(y))h_k'(y)\,\mathrm{d}y\,, \tag{3.18}$$

times $C/C_k$. So, by keeping in mind that $F_\zeta(-x)=O(\exp(-\delta x)$, that $h_k'(y)=O(\exp(-(y-k)))$ for $y-k\to\infty$, that $h_k'(y)\in(0,1)$ and by remarking that $y-h_k(y)\le\log 2$ for $y\in[0,k]$, with adequate choice of $C$

$$|A_{k,2}-A_{k,1}| \,\le\, \frac{C}{C_k}\left(\int_0^k e^{-\varrho(k-y)}e^{-\delta y}\,\mathrm{d}y + e^{-\delta k}\int_k^\infty (y-k+1)e^{-(y-k)}\,\mathrm{d}y\right) \,=\, O\left(e^{-(\delta\wedge\varrho)k}\right), \tag{3.19}$$

with $\delta\wedge\varrho=\min(\delta,\varrho)$.

The second step is the control of $A_{k,2}$. For this we introduce

$$\mathtt{h}_k(x) \,:=\, h(x+k)-k \tag{3.20}$$

and remark that

$$0 \,\le\, \mathtt{h}_k(x)-h_k(x) \,=\, \log(1+\exp(x-k)) \,=\, \begin{cases}O(\exp(x-k)) & \text{for } x-k\to-\infty\,,\\ O(x-k) & \text{for } x-k\to+\infty\,,\end{cases} \tag{3.21}$$

$$0 \,\le\, \mathtt{h}_k'(x)-h_k'(x) \,=\, 1/(1+\exp(k-x)) \,=\, \begin{cases}O(\exp(x-k)) & \text{for } x-k\to-\infty\,,\\ \le 1 & \text{for every } x \text{ and } k\,.\end{cases} \tag{3.22}$$

Moreover we stress that $h_k'(x)$ and $\mathtt{h}_k'(x)$ are in $(0,1)$ for every $x$ and, with reference to Figure 2, the two functions almost coincide for $x \ll k$. They start to differ when $x$ approaches $k$ from the left because $\mathtt{h}_k'(x)$ keeps being very close to one and $\mathtt{h}_k(x) \sim x$.

Since by the characterizing property of $F_\triangleleft(\cdot)$

$$\int_{\mathbb{R}} F_\triangleleft(y+k)\mathtt{h}_k'(y)\zeta(x-\mathtt{h}_k(y))\,\mathrm{d}y - F_\triangleleft(x+k) = \int_{\mathbb{R}} F_\triangleleft(y)h'(y)\zeta(x-h(y))\,\mathrm{d}y - F_\triangleleft(x) \;=\; 0, \tag{3.23}$$

we have

$$A_{k,2} \;=\; \frac{1}{C_k}\int_{-\infty}^0 \left|\int_{\mathbb{R}} F_\triangleleft(y+k)\left(h_k'(y)\zeta(x-h_k(y)) - \mathtt{h}_k'(y)\zeta(x-\mathtt{h}_k(y))\right)\mathrm{d}y\right|\mathrm{d}x\,, \tag{3.24}$$

We split the integral over $\mathbb{R}$ according to $y < -3k/2$, $y \in [-3k/2, k/2]$ and $y > k/2$. We have

$$\begin{gathered}\int_{-\infty}^0 \left|\int_{k/2}^{\infty} F_\triangleleft(y+k)\left(h_k'(y)\zeta(x-h_k(y)) - \mathtt{h}_k'(y)\zeta(x-\mathtt{h}_k(y))\right)\mathrm{d}y\right|\mathrm{d}x \;\le \\ C\int_{-\infty}^0\int_{k/2}^{\infty}(y+k)\left(h_k'(y)\zeta(x-h_k(y)) + \zeta(x-\mathtt{h}_k(y))\right)\mathrm{d}y\,\mathrm{d}x \;= \\ C\int_{k/2}^{\infty}(y+k)\left(h_k'(y)F_\zeta(-h_k(y)) + F_\zeta(-\mathtt{h}_k(y))\right)\mathrm{d}y \;\le\; C'\int_{k/2}^{\infty} y\left(e^{(k-y)\wedge 0-\frac{\delta}{2}k} + e^{-\delta y}\right)\mathrm{d}y,\end{gathered} \tag{3.25}$$

which is $O(k^2\exp(-\delta k/2))$. Similarly with $\delta' \in (0,\delta)$ (recall (2.14))

$$\begin{gathered}\int_{-\infty}^0 \left|\int_{-\infty}^{-3k/2} F_\triangleleft(y+k)\left(h_k'(y)\zeta(x-h_k(y)) - \mathtt{h}_k'(y)\zeta(x-\mathtt{h}_k(y))\right)\mathrm{d}y\right|\mathrm{d}x \;\le \\ C\int_{-\infty}^{-3k/2} e^{\delta'(y+k)}e^{y+k}(F_\zeta(-h_k(y)) + F_\zeta(-\mathtt{h}_k(y))\,\mathrm{d}y \;\le\; 2C\int_{-\infty}^{-3k/2} e^{\delta'(y+k)}e^{y+k}\,\mathrm{d}y\;,\end{gathered} \tag{3.26}$$

which is $O(\exp(-k(1+\delta')/2))$. We have therefore obtained that $A_{k,2}$ is equal to

$$\frac{1}{C_k}\int_{-\infty}^0 \left|\int_{-3k/2}^{k/2} F_\triangleleft(y+k)\left(h_k'(y)\zeta(x-h_k(y)) - \mathtt{h}_k'(y)\zeta(x-\mathtt{h}_k(y))\right)\mathrm{d}y\right|\mathrm{d}x + O\left(ke^{-\frac{\delta'}{2}k}\right). \tag{3.27}$$

For the last estimate we use

$$\begin{gathered}\left|\int_{-3k/2}^{k/2} F_\triangleleft(y+k)\left(h_k'(y)\zeta(x-h_k(y)) - \mathtt{h}_k'(y)\zeta(x-\mathtt{h}_k(y))\right)\mathrm{d}y\right| \;\le \\ \int_{-3k/2}^{k/2} F_\triangleleft(y+k)\Big(\underbrace{\left|h_k'(y) - \mathtt{h}_k'(y)\right|\zeta(x-h_k(y))}_{1} + \underbrace{\mathtt{h}_k'(y)\left|\zeta(x-h_k(y)) - \zeta(x-\mathtt{h}_k(y))\right|}_{2}\Big)\,\mathrm{d}y\,,\end{gathered} \tag{3.28}$$

and we write the last line as $T_1(x) + T_2(x)$ in the obvious way. Since $F_\triangleleft(y+k) \le 2k$ for $y \le k/2$

$$\begin{aligned}\int_{-\infty}^0 T_1(x)\,\mathrm{d}x \,&=\, \int_{-3k/2}^{k/2} F_\triangleleft(y+k)\left|h_k'(y) - \mathtt{h}_k'(y)\right| F_\zeta(-h_k(y))\,\mathrm{d}y \\ &\le\, 2k\int_{-3k/2}^{k/2}\left|h_k'(y) - \mathtt{h}_k'(y)\right|\,\mathrm{d}y \,\le\, Ck\int_{-3k/2}^{k/2} e^{y-k}\,\mathrm{d}y \,\le\, O\left(k\exp(-k/2)\right).\end{aligned} \tag{3.29}$$

Moreover

$$\begin{aligned}\int_{-\infty}^0 T_2(x)\,\mathrm{d}x \,&=\, \int_{-\infty}^0\int_{-3k/2}^{k/2} F_\triangleleft(y+k)\mathtt{h}_k'(y)\left|\zeta(x-h_k(y)) - \zeta(x-\mathtt{h}_k(y))\right|\,\mathrm{d}y\,\mathrm{d}x \\ &\le\, \int_{-2k}^0\int_{-3k/2}^{k/2} F_\triangleleft(y+k)C_\zeta\left|h_k(y)) - \mathtt{h}_k(y)\right|^\theta\,\mathrm{d}y\,\mathrm{d}x \\ &\quad +C\int_{-3k/2}^{k/2} F_\triangleleft(y+k)\left(F_\zeta(-2k-h_k(y)) + F_\zeta(-2k-\mathtt{h}_k(y))\right)\mathrm{d}y \\ \le C'k\int_{-2k}^0\mathrm{d}x\int_{-3k/2}^{k/2}&\exp(\theta(y-k))\,\mathrm{d}y + 8Ck^2F_\zeta(-k) \,=\, O\left(k^2\exp\left(-\left(\frac{\theta}{2}\wedge\delta\right)k\right)\right),\end{aligned} \tag{3.30}$$

where in the second step we have exploited the regularity of $\zeta$ for $x \in (-2k, 0)$ – the constant $C_\zeta$ is the left-hand side in (1.6) – and, for smaller $x$'s we have just bounded $|\zeta(x-h_k(y)) - \zeta(x-\mathtt{h}_k(y))|$ with $\zeta(x-h_k(y)) + \zeta(x-\mathtt{h}_k(y))$ and we have performed the integral over $x$, using once again $F_\triangleleft(y+k) \le 2k$ in the range we consider.

The proof of Proposition 3.3 is therefore complete. □

## 4. Diffusion estimate: the proof of Proposition 2.2

In this section we essentially estimate the speed of convergence to stationarity of the chain $X$. The process is essentially a symmetric random walk far from the boundary, but the positive recurrent character crucially depends on the visits to the boundary. By a preliminary manipulation we reduce the estimate we need to estimating the expectation of the time of first exit from $(-k, k)$ and this is achieved by diffusion approximation on $(-k + 2\log\log k, k - 2\log\log k)$ and by a rough estimate on the remaining portions of length $2\log\log k$.

*Proof of Proposition 2.2.* In order to work with more explicit constants we give the proof under the assumption that both $\mathbb{P}(\mathtt{z} > \log 2) > 0$ and $\mathbb{P}(\mathtt{z} < -\log 2) > 0$: we explain in Remark 4.2 how the proof can be easily generalized.

Our proof involves two auxiliary Markov chains, closely related to $X$, which we now introduce.

The first one is $X^\sim = (X_n^\sim)_{n\in\mathbb{N}_0}$ with state space $\mathbb{R} \cup \{\mathcal{C}\}$. For $X^\sim$ the state $\mathcal{C}$ is absorbing (*cemetery*), that is $X_n^\sim = \mathcal{C}$ implies $X_m^\sim = \mathcal{C}$ for every $m$ larger than $n$. On the other hand if $X_n^\sim = y \in \mathbb{R}$ the probability that $X_{n+1}^\sim = \mathcal{C}$ is $1 - h_k'(y)$. Therefore if $X_n^\sim = y \in \mathbb{R}$ the chain *survives* with probability $h_k'(y)$ and, in this case, it jumps to $x$ with transition density $\zeta(x - h_k(y))$. Note that $h_k'(x)$ is even, decreasing for $x > 0$,

$h'_k(x) = O(\exp(-k+x)) + O(\exp(-k-x))$ and

$$h'_k(\pm k) \,=\, \frac{1}{2} - \frac{1}{e^{2k}+1} \,<\, \frac{1}{2}\,, \tag{4.1}$$

so $h'_k(x) < 1/2$ if $|x| \geq k$, and $h'_k(x) \leq h'_k(0) = 1$ for every $x$.

By the observations we just made it is easily seen that the chain $X^{\sim}$ can be coupled with a simpler chain $X^{\approx} = (X^{\approx}_n)_{n\in\mathbb{N}_0}$ that has a smaller *death probability*. The transition kernel for $X^{\approx}$ from $y \in (-k,k)$ to $x \in \mathbb{R}$ is given by $\zeta(x - h_k(y))$ and it is therefore a probability: transitions from $(-k,k)$ to $\mathcal{C}$ are forbidden. On the other hand the kernel from $y \in \mathbb{R}\setminus(-k,k)$ to $x \in \mathbb{R}$ is $\zeta(x-h_k(y))/2$ and the probability of going from $y \in \mathbb{R}\setminus(-k,k)$ to $\mathcal{C}$ is $1/2$. $\mathcal{C}$ is absorbing also for this chain.

Of course these two chains are *dominated* by the even simpler chain $X = (X_n)_{n\in\mathbb{N}_0}$ for which the death probability is zero: its natural state space is just $\mathbb{R}$, but of course we can keep $\mathbb{R}\cup\{\mathcal{C}\}$ as state space and the absorbing state does not communicate with $\mathbb{R}$, and the transition kernel from $y$ to $x$ is the probability density $\zeta(x - h_k(y))$. In fact $X$ is just the main Markov chain we consider in this walk, i.e. (1.12).

We denote by $\mathbb{P}_y$ the (joint) law of the chains $X$, $X^{\sim}$ and $X^{\approx}$ and the index $y$ denotes the (common) initial condition.

Let us assume that $G(\cdot)$ is non negative (the general result is recovered by linearity). Of course

$$\int_{\mathbb{R}} T_0 G(x)\,\mathrm{d}x \,=\, \int_{\mathbb{R}} h'_k(y)\zeta(x-h_k(y))G(y)\,\mathrm{d}y\,, \tag{4.2}$$

and a direct computation shows that for $n = 2, 3, \ldots$

$$\int_{\mathbb{R}} T_0^n G(x)\,\mathrm{d}x \,=\, \int_{\mathbb{R}^n} h'_k(y_1)\left(\prod_{j=1}^{n-1}\zeta(y_j - h_k(y_{j+1}))h'_k(y_{j+1})\right)\,\mathrm{d}y_1\ldots\mathrm{d}y_n\,. \tag{4.3}$$

Therefore for $n \in \mathbb{N}$ we have the probabilistic representation and bound:

$$\begin{aligned}\int_{\mathbb{R}} T_0^n G(x)\,\mathrm{d}x \,=\, \int_{\mathbb{R}} G(y)\mathbb{P}_y\left(X^{\sim}_n \neq \mathcal{C}\right)\,\mathrm{d}y \,&\leq\, \int_{\mathbb{R}} G(y)\mathbb{P}_y\left(X^{\approx}_n \neq \mathcal{C}\right)\,\mathrm{d}y\\ &=\, \int_{\mathbb{R}} G(y)\mathbb{E}_y\left[2^{-H(n)}\right]\,\mathrm{d}y\,,\end{aligned} \tag{4.4}$$

with $H(n) := |\{j = 0, 1, \ldots, n : |X_j| \geq k\}|$. Let us set $\tau_0^H = 0$ and, for $j \in \mathbb{N}$, $\tau_j^H := \inf\{n \in \mathbb{N} : H(n) = j\}$, so $\tau_j^H$ is the time of the $j^{\text{th}}$ entry of $X$ into $\mathbb{R}\setminus(-k,k)$. With these notations, by the Fubini-Tonelli Theorem, we have

$$\sum_{n=0}^{\infty}\int_{\mathbb{R}} G(y)\mathbb{E}_y\left[2^{-H(n)}\right]\,\mathrm{d}y \,=\, \int_{\mathbb{R}} G(y)\mathbb{E}_y\left[\sum_{n=0}^{\infty}2^{-H(n)}\right]\,\mathrm{d}y\,, \tag{4.5}$$

and, again by the Fubini-Tonelli Theorem and using the definitions:

$$\mathbb{E}_y\left[\sum_{n=0}^{\infty}2^{-H(n)}\right] \,=\, \sum_{j=0}^{\infty}2^{-j}\mathbb{E}_y\left[\sum_{n=0}^{\infty}\mathbf{1}_{H(n)=j}\right] \,=\, \sum_{j=0}^{\infty}2^{-j}\mathbb{E}_y\left[\tau_{j+1}^H - \tau_j^H\right] \,\leq\, 2\sup_y \mathbb{E}_y\left[\tau_1^H\right]\,. \tag{4.6}$$

For every $r > 0$ we set $\tau_r := \inf\{n \geq 0 : X_n \in \mathbb{R} \setminus (-r, r)\}$. So $\tau_1^H = \tau_k$ and the steps we have just developed yield that for $G(\cdot) \geq 0$

$$\sum_{n=0}^{\infty} \int_{\mathbb{R}} T_0^n G(x)\,\mathrm{d}x \,\leq\, 2 \sup_y \mathbb{E}_y\left[\tau_k\right] \int_{\mathbb{R}} G(x)\,\mathrm{d}x\,, \tag{4.7}$$

which readily implies for $G \in L^1$

$$\sum_{n=0}^{\infty} \|T_0^n G\|_1 \,\leq\, \sum_{n=0}^{\infty} \int_{\mathbb{R}} T_0^n |G(x)|\,\mathrm{d}x \,\leq\, 2 \sup_y \mathbb{E}_y\left[\tau_k\right] \|G\|_1\,. \tag{4.8}$$

We are going to show:

**Lemma 4.1.** *There exists $C > 0$ and $k_0 > 0$ such that for every $k \geq k_0$*

$$\sup_x \mathbb{E}_x\left[\tau_k\right] \,\leq\, \frac{1}{2} k^2 \left(\log k\right)^C\,. \tag{4.9}$$

Lemma 4.1 and (4.8) imply (2.6) and Proposition 2.2 is proven. □

**Remark 4.2.** *The purpose of the assumption that $\mathbb{P}(\mathsf{z} > \log 2)$ and $\mathbb{P}(\mathsf{z} < -\log 2)$ are non zero is to guarantee that the chain $X$ can exit $(-k, k)$. Under this assumption in fact $\mathbb{P}(X_{n+1} \geq k | X_n = x) > 0$ for $x$ sufficiently close to $k$ (and analogous statement at $-k$): note in fact that $h_k(\pm k) = \pm(k - \log 2 + \log(1 + e^{-2k})$. If either $\mathbb{P}(\mathsf{z} > \log 2) = 0$ or $\mathbb{P}(\mathsf{z} < -\log 2) = 0$, the definitions of the chain $X^{\approx}$ should be modified by stipulating that the chain may step to the absorbing state $\mathcal{C}$ only when $|x| \geq k - c$, with $c > 0$ such that $(k - c) - h_k(k - c)(> 0)$ and $(k - c) + h_k(-(k - c))(< 0)$ are in the interior of the support of $\zeta$. Of course in this case the probability of jumping to $\mathcal{C}$ is no longer $1/2$: since*

$$h_k'(\pm(k - c)) \,=\, 1 - \frac{1}{e^c + 1} - \frac{1}{e^{2k-c} + 1} \,<\, 1 - \frac{1}{e^c + 1}\,, \tag{4.10}$$

*we can choose this probability equal to $1 - 1/(e^c + 1)$. These changes do not affect Proposition 4.9 because they only affect the constants $C$ and $k_0$, whose precise values are unimportant.*

*Proof of Lemma 4.1.* We recall that $\zeta$ is centered and $\sigma^2 = \int x^2 \zeta(x)\,\mathrm{d}x > 0$. We claim that it suffices to show that for $k \geq k_0$

$$\inf_{x \in \mathbb{R}} \mathbb{P}_x\left(\tau_k \leq k^2\right) \,\geq\, 2(\log k)^{-C} =: p_k\,. \tag{4.11}$$

In fact (4.11) implies that for $j \in \mathbb{N}$ and every $x$

$$\mathbb{P}_x\left(\tau_k > (j+1)k^2\right) \,\leq\, \mathbb{P}_x\left(\tau_k > jk^2\right)(1 - p_k)\,, \tag{4.12}$$

and therefore (also for $j = 0$)

$$\mathbb{P}_x\left(\left\lceil \frac{\tau_k}{k^2} \right\rceil > j\right) \,=\, \mathbb{P}_x\left(\tau_k > jk^2\right) \,\leq\, (1 - p_k)^j\,, \tag{4.13}$$

which implies

$$\mathbb{E}_x[\tau_k] \le k^2 \mathbb{E}_x\left[\left\lceil \frac{\tau_k}{k^2} \right\rceil\right] = k^2 \sum_{j=0}^{\infty} \mathbb{P}_x\left(\tau_k > jk^2\right)$$
$$\le k^2 \sum_{j=0}^{\infty} (1-p_k)^j = \frac{k^2}{p_k} = \frac{1}{2} k^2 (\log k)^C . \quad (4.14)$$

Let us prove (4.11): it suffices to prove that both

$$\inf_{x \ge 0} \mathbb{P}_x\left(\tau_k \le k^2, X_{\tau_k} > 0\right) \quad \text{and} \quad \inf_{x \le 0} \mathbb{P}_x\left(\tau_k \le k^2, X_{\tau_k} < 0\right) \quad (4.15)$$

are bounded below by $2(\log k)^{-C}$. These two estimates are *equivalent* up to the fact that $\zeta$ is not symmetric (but this affects only the choice of the constants and in an obvious way). So we just treat the first expression, i.e. when the chain exits on the right.

We take this occasion also to remark that if we take two chains $X$ and $X'$, with $X_0 = x$ and $X'_0 = x'$, both satisfying (1.12), coupled by the common randomness $(\mathbf{z}_n)$, we have $X_{n+1} - X'_{n+1} = h_k(X_n) - h_k(X'_n)$, with $h_k(\cdot)$ increasing, so the sign of $X_n - X'_n$ does not depend on $n$. Therefore $\mathbb{P}_0\left(\tau_k \le k^2, X_{\tau_k} > 0\right) \ge \mathbb{P}_{x'}\left(\tau_k \le k^2, X_{\tau_k} > 0\right)$ for $x \ge x'$. So it suffices to check that

$$\mathbb{P}_0\left(\tau_k \le k^2, X_{\tau_k} > 0\right) \ge 2(\log k)^{-C} . \quad (4.16)$$

Moreover, what the process does outside $(-k, k)$ is irrelevant, so, when $|X_n| \ge k$, we replace $h_k(X_n)$ in (1.12) with $X_n$. This means that $h_k(x)$ is redefined as $h_k(x)\mathbf{1}_{(-k,k)}(x) + x\mathbf{1}_{\mathbb{R}\setminus(-k,k)}(x)$. Note that this corresponds to choosing a different increasing function $h_k(\cdot)$.

To show that (4.16) holds we choose $\delta > 0$ so that $\inf_{x \in [0,k)} \mathbb{P}_x(X_1 - X_0 \ge \delta | X_0 = x) = p_\delta > 0$. Note that this infimum is achieved at $x = k$ and, since we have trivialized the drift outside $(-k, k)$, the infimum over $x \in [0, k)$ can be taken over $x \in [0, \infty)$. We introduce also the stopping time

$$\tau'_k := \inf\left\{n \ge \tau_{k-2\log k} : |X_n - (k - 2\log k)| \ge 2\log k - 2\log\log k\right\} , \quad (4.17)$$

and we remark that

$$\mathbb{P}_0\left(\tau_k \le k^2, X_{\tau_k} > 0\right) \ge \mathbb{P}_0\Big(\tau_{k-2\log k} \le k^2/2 \text{ and } X_{\tau_{k-2\log k}} \ge k - 2\log k ,$$
$$\tau'_k - \tau_{k-2\log k} \le (2\log k)^2 \text{ and } X_{\tau'_k} \ge k - 2\log\log k ,$$
$$X_{\tau_{k-2\log\log k}+j} - X_{\tau_{k-2\log\log k}+j-1} \ge \delta \text{ for } j = 1, 2, \ldots, \lceil (2/\delta)\log\log k \rceil\Big) . \quad (4.18)$$

Let us read this complicated-looking expression: each item in the list that follows corresponds to the three lines of the right-hand side of (4.18).

(1) We ask that in less then $k^2/2$ steps the chain exits the interval $(-k + 2\log k, k - 2\log k)$ from the right: we will show that, by Brownian motion approximation, this probability is positive uniformly in $k$. For this to work we need the drift to be negligible on the relevant (diffusive) time scale, which given the form of $h_k$ requires us to stop sufficiently far (i.e. $2\log k$) from the boundary.

(2) We ask that the chain exits on the right the interval of length $2(2\log k - 2\log\log k)$ centered in $k - 2\log k$ in $4(\log k)^2$ steps or less. Note that in the unlikely (favorable) event that $\tau'_k = \tau_{k-2\log k}$ then $X_{\tau_{k-2\log k}} \ge k - 2\log\log k$ is verified, so we can assume $X_{\tau_{k-2\log}} \in [k - 2\log k, k - 2\log\log k)$ and again we can treat this step by Brownian motion approximation. The drift is larger, but still negligible because

the time span is much shorter than what we considered before. This will cost a probability factor bounded away from zero, like for (1). We also remark that, on this event, $\tau'_k = \tau_{k-2\log\log k}$.

(3) Once the chain is at a distance at most $2\log\log k$ from the right boundary point $k$, we ask that it goes straight out of the domain by making steps of length at least $\delta > 0$: this will cost a factor smaller than one (but bounded below by $p_\delta > 0$) at each step. Since the number of steps is proportional to $\log\log k$ this costs a probability factor that vanishes like a power of $1/\log k$. Note that we have modified the chain outside of $(-k, k)$, by removing the boundary repulsion, so that this ballistic strategy can be performed also once the chain is beyond $k$.

Before starting the lower bound estimate for the right-hand side of (4.18) we anticipate that, by the Strong Markov Property, we will be able to perform three separate estimates: each estimate corresponding to one of the three items in the list.

Corresponding to the first item we aim at showing that

$$\mathbb{P}_0\left(\tau_{k-2\log k} \le k^2/2 \text{ and } X_{\tau_{k-2\log k}} \ge k - 2\log k\right) \tag{4.19}$$

is bounded away from zero, uniformly in $k \ge k_0$. The event does not change if we replace $h_k(x)$ with $x$ for $|x| \ge k - 2\log k$ in defining the Markov chain $X$: so we will do so. Moreover we define the process $X_{t,k} := X_{tk^2}/k$ if $tk^2$ is an integer and otherwise we define $X_{t,k}$ by affine interpolation so the trajectory is continuous. We are going to show, via a diffusion approximation, that the sequence of processes $(X_{\cdot,k})_{k\in\mathbb{N}}$, with $X_{\cdot,k}$ a random element of $C^0([0,T];\mathbb{R})$ (with $T > 0$ arbitrary), converges in law to a Brownian motion with variance $\sigma^2 = \int x^2\zeta(x)\,\mathrm{d}x$. This is formally stated in the following Lemma.

**Lemma 4.3.** *For every continuous and bounded function $F$ from $C^0([0,T);\mathbb{R})$ to $\mathbb{R}$ we have that*

$$\lim_{k\to\infty} \left|\mathbb{E}_0\left[F\left(X_{\cdot,k}\right)\right] - E_0[F(B^\sigma_\cdot)]\right| \,=\, 0\,, \tag{4.20}$$

*where, under $P_0$, $B^\sigma_\cdot = \sigma B_\cdot$ with $B_\cdot$ a standard Brownian motion.*

*Proof of Lemma 4.3.* We apply the diffusion approximation procedure detailed in [31, pp. 266–272]. By [31, Assumptions (2.4)-(2.5)-(2.6), Theorem 11.2.3] it sufficed to check three conditions:

(1) The (vanishing) drift condition:

$$\lim_{k\to\infty} \sup_{y:\,|y|\le k-2\log k} k\left|\int (x-y)\zeta(x-h_k(y))\,\mathrm{d}x\right| \,=\, 0\,. \tag{4.21}$$

Note that we can restrict to $|x| \le k - 2\log k$ because the process has been modified so to be centered outside of this interval. (4.21) holds because, using first $\int x\zeta(x)\,\mathrm{d}x = 0$ and then (1.11), we obtain

$$\left|\int (x-y)\zeta(x-h_k(y))\,\mathrm{d}x\right| \,=\, |h_k(y) - y| \,\le\, \exp(-2\log k) \,=\, \frac{1}{k^2}\,, \tag{4.22}$$

and (4.21) is verified.

(2) The control of the variance:

$$\lim_{k\to\infty} \sup_{y:\,|y|\le k-2\log k} \left|\int (x-y)^2\zeta(x-h_k(y))\,\mathrm{d}x - \sigma^2\right| \,=\, 0\,, \tag{4.23}$$

and this is a direct consequence of the fact that, in the range of $y$ that we consider, $|y - h_k(y)|$ is bounded by the boundary case $y = k - 2\log k$ and the resulting expression vanishes as $k \to \infty$.

(3) The control on large jumps: for every $\varepsilon > 0$

$$\lim_{k\to\infty} k \sup_{x\in\mathbb{R}} \mathbb{P}\left(\left|\mathtt{z}_1 + (h_k(x) - x)\,\mathbf{1}_{[-(k-2\log k),+(k-2\log k)]}(x)\right| > \varepsilon k\right) \,=\, 0\,, \tag{4.24}$$

which is (largely) verified because $h_k(x) - x = O(1/k^2)$ for $|x| \le k - 2\log k$ and because $\mathtt{z}_1$ has finite exponential moments.

This completes the proof of Lemma 4.3. □

Lemma 4.3 implies

$$\mathbb{P}_0\left(\tau_{k-2\log k} \le k^2/2 \text{ and } X_{\tau_{k-2\log k}} \ge k - 2\log k\right) \stackrel{k\to\infty}{\longrightarrow} P_0\left(\mathtt{t}_\sigma \le 1/2,\, B^\sigma_{\mathtt{t}_\sigma} = 1\right) \,=:\, p_\sigma\,, \tag{4.25}$$

and $\mathtt{t}_\sigma$ is the hitting time of $(-1,1)^\complement$ by $B^\sigma_\cdot$. Hence for $k$ sufficiently large

$$\mathbb{P}_0\left(\tau_{k-2\log k} \le k^2/2 \text{ and } X_{\tau_{k-2\log k}} \ge k - 2\log k\right) \,\ge\, \frac{p_\sigma}{2}\,. \tag{4.26}$$

We now restart from time $\tau_{k-2\log k}$ and use the Strong Markov Property. $X_{\tau_{k-2\log k}} \in [k - 2\log k, k - 2\log\log k)$ and we apply a Brownian motion approximation to the chain $(X_{\tau_{k-2\log k+j}})_{j=0,1,\ldots}$ on the time scale $(\log k)^2$: the steps are analogous to the ones of the previous step. Also in this case it is more concise to resort to the comparison argument explained right before (4.16), so that the starting point of our chain can and will be chosen equal to $k - 2\log k$. Therefore, by recentering, i.e. by translating the system so that $k - 2\log k$ becomes the origin (of course we have to shift accordingly the repulsion), it suffices to show that

$$\lim_{k\to\infty} \mathbb{P}_0\left(\tau_{2\log k - 2\log\log k} \le (2\log k)^2,\, X_{\tau_{2\log k-2\log\log k}} > 0\right) \,=\, P_0\left(\tau \le \mathtt{t}_\sigma, B^\sigma_{\mathtt{t}_\sigma} = 1\right) \,>0\,, \tag{4.27}$$

But (4.27) is just a close analog of Lemma 4.3 and the key point is that the proof is essentially the same up to replacing $k$ with $\log k$: note notably that now the repulsion is much stronger, $O(\exp(-2\log\log k)) = O(1/(\log k)^2)$ uniformly in the interval, but this yields a negligible drift because time is $O(\log k)$.

Therefore we arrive at: there exists $p'_\sigma > 0$ and $k_0$ such that

$$\begin{aligned}\inf_{k\ge k_0} \mathbb{P}_0\Big(&\tau_{k-2\log k} \le k^2/2 \text{ and } X_{\tau_{k-2\log k}} \ge k - 2\log k\,,\\ &\tau'_k - \tau_{k-2\log k} \le (2\log k)^2 \text{ and } X_{\tau'_k} \ge k - 2\log\log k\Big) \,\ge\, p'_\sigma\end{aligned} \tag{4.28}$$

Now we apply again the Strong Markov Property using the fact that $X_{\tau_{k-2\log\log k}} \ge k - 2\log\log k$ and the last estimate is just a product estimate that leads to (recall (4.18))

$$\mathbb{P}_0\left(\tau_k \le k^2\,, X_{\tau_k} \ge k\right) \,\ge\, p'_\sigma\, p_\delta^{\lceil (2/\delta)\log\log k\rceil} \,\ge\, \frac{1}{(\log k)^C}\,, \tag{4.29}$$

with $C = (3/\delta)\log(1/p_\delta)$ and $k$ sufficiently large. This completes the proof of Lemma 4.1. □

## Appendix A. Complementary results on the auxiliary chains

The analysis of the basic properties of the $X$ chain can of course be found for example in the first two chapters of [3]: by basic properties we mean existence and uniqueness of an invariant probability, that follow from irreducibility and positive recurrence. We choose not to discuss these issues in detail because we do give below more details about the $Y$ chain that is a bit more delicate to deal with – in particular, the invariant measure is not normalizable – and because for the $Y$ chain we need a few results that depend on our restricted framework. For $Y$ we are going to exploit [21] and we will in particularly give a Lyapunov function to show recurrence: for $X$ we have positive recurrence because of the rather evident *confinement properties* that can be made explicit using for example the Lyapunov function $x \mapsto (|x|-k)_+^2)$ and [21, Th. 11.3.4].

*Proof of Proposition 2.4.* We distinguish here among the case in which the support of $\zeta$ is bounded away from $-\infty$ and when it is not.

In the first case it is straightforward to see that if $-c = \inf\{x : \zeta(x) > 0\} < 0$ then the process eventually enters $(-\log(\exp(c)-1), \infty)$ and does not leave this set: in fact $-\log(\exp(c)-1)$ is the fixed point of $y \mapsto h(y) - c$. Moreover $Y$ is irreducible, more precisely $\psi$-irreducible in the terminology for example of [21], with $\psi$ any probability with a density that is positive on $(-\log(\exp(c)-1), \infty)$ and zero on the complement: this is a direct consequence of the fact that $\mathbf{z}$ has a density, of (2.11) and of the fact that $-c$ is the left edge of the support of the transition probability. Recurrence of $Y$ follows for example by applying the criterion in [18, Th. 3.1], or we can apply [21, Th. 8.0.2] with a Lyapunov function equal to $\log((x-M)_+ + 1)$, with $M = M(\sigma) > 0$ suitably chosen.

If the support of $\zeta$ is not bounded below, then the process is still $\psi$-irreducible and this time $\psi$ is any probability with positive density. Recurrence can be established in the same way: the repulsion from the left is very strong. An explicit Lyapunov function is $x^2 \mathbf{1}_{(-\infty,0)(x)} + \mathbf{1}_{[0,\infty)}(x) \log((x-M)_+ + 1)$ [21, Th. 8.0.2] .

So in both cases $\nu$ exists, it is $\sigma$-finite and it is unique [21, Th. 10.4.9]). Of course $\nu$ is characterized by (2.12). In particular for every bounded Borel set $B$

$$\nu(B) \,=\, \int_B \left( \int_{\mathbb{R}} \zeta(z - h(y)) \nu(\,\mathrm{d}y) \right) \mathrm{d}z \,, \tag{A.1}$$

and we use this formula to show that $\nu(B) < \infty$ for every bounded Borel set. To show this we use the fact that $\zeta \geq \varepsilon \mathbf{1}_{(a,b)}$ for suitably chosen positive constants $\varepsilon, a$ and $b$. Assume that there exists a bounded Borel set $B_1$ with $\nu(B_1) = \infty$. Then there exists $x_0 \in B_1$ such that $\nu((x_0 - 1/n, x_0 + 1/n)) = \infty$ for every $n$ and this directly yields $\int_{\mathbb{R}} \zeta(z - h(y)) \nu(\,\mathrm{d}y) = \infty$ for every $z$ such that $z + h(x_0) \in (a,b)$. But in this case, by (A.1), $\nu(B) = \infty$ for every $B \subset (a - h(x_0), b - h(x_0))$ of positive Lebesgue measure, which is impossible because $\nu$ is $\sigma$-additive.

To establish $\nu(\mathbb{R}) = \infty$ we suppose that $\nu(\mathbb{R}) < \infty$, and we assume that $\nu$ is a probability. In this case we remark that the process defined recursively by $S_{n+1} := \mathbf{z}_{n+1} + S_n$ and by $S_0 := Y_0$ satisfies $S_n \leq Y_n$ for every $n$. Hence $\mathbb{P}(S_n \geq \sqrt{n}) \leq \mathbb{P}(Y_n \geq \sqrt{n})$. But the Central Limit Theorem implies that the left-hand side converges as $n \to \infty$ to a positive number, while the right-hand side vanishes in the same limit because $(Y_n)$ is tight by the Ergodic Theorem applied to our (supposedly) positive recurrent process. Hence $\nu(\mathbb{R}) = \infty$.

For the left tail property we start by remarking that using (2.12) with $g = \mathbf{1}_{(a,b+1)}$, $a < b$, and restricting the integral in the right-hand side to $y \in (-\infty, \log(e-1))$, i.e.

$h(y) \in (0,1)$, and to $x \in (a,b)$

$$\nu((a,b+1)) \,=\, \nu \times \zeta\left(\{(y,z) : \, a < z + h(y) < b+1\}\right) \,\geq\, \nu\left(\{y : \, h(y) \in (0,1)\}\right) \zeta((a,b))\,, \tag{A.2}$$

that is $\nu((a,b+1)) \geq \nu((-\infty, \log(e-1)))\zeta((a,b))$. By choosing $a$ and $b$ such that $\zeta((a,b)) > 0$ we see that $\nu((-\infty, \log(e-1))) < \infty$. Therefore $\nu((-\infty,x]) < \infty$ for every $x$ and the proof is complete. □

*Proof of Lemma 2.5.* the measure. By setting $g = \mathbf{1}_{(-\infty,x]}$ in (2.12) we see that for every $x \in \mathbb{R}$

$$F_\nu(x) \,=\, \int F_\zeta(x - h(y))\nu(\,\mathrm{d}y)\,. \tag{A.3}$$

Since both $h(\cdot)$ and $F_\zeta(\cdot)$ are non decreasing we have that $y \mapsto F_\zeta(x - h(y))$ is non increasing. So, by (A.3), we have that for every $x$ and every $z$

$$F_\nu(x) \,\geq\, \int_{-\infty}^{z} F_\zeta(x-h(y))\nu(\,\mathrm{d}y) \,\geq\, F_\zeta(x - h(z))F_\nu(z)\,. \tag{A.4}$$

Set $-\widetilde{c} := \inf\{y : \, F_\zeta(y) > 0\} \in [-\infty, 0)$: one extracts form (1.14) that $\inf\{y : \, F_\nu(y) > 0\} = -\log(\exp(\widetilde{c}) - 1)$. Now consider the sequence $(x_j)$, $x_0$ the value we have chosen for $F_\nu(x_0) = 1$, and, for $j \in \mathbb{N}$, $x_j = x_{j-1} + \rho$ with $\rho > 0$ such that $x_0 - h(x_1) = x_0 - h(x_0 + \rho) > -\widetilde{c}$, so that $q := F_\zeta(x_0 - h(x_1)) > 0$. Such a choice of $\rho > 0$ is possible because $x_0 > -\log(\exp(\widetilde{c}) - 1)$ so

$$x_0 - h(x_0) \,=\, -\log(1 + \exp(-x_0)) \,>\, -\widetilde{c}\,. \tag{A.5}$$

Note moreover that for $j \in \mathbb{N}$ we have $x_j - h(x_{j+1}) = -\rho - \log(1 + \exp(-x_{j+1})) \geq -\rho - \log(1+\exp(-x_1)) = x_0 - h(x_1)$. So $F_\zeta(x_j - h(x_{j+1})) \geq q$ for every $j$. Therefore from (A.4) we infer that

$$F_\nu(x_{j+1}) \,\leq\, \frac{1}{q^j}\,. \tag{A.6}$$

Since $F_\nu$ is non decreasing this yields the claim. □

## Appendix B. About the Fourier transform of $\zeta$

With our Laplace transform notation $\widehat{\zeta}(-it) = \varphi_{\mathtt{z}}(t)$ is the characteristic function of $\mathtt{z}$, that is the Fourier transform of the law of $\mathtt{z}$. In this Appendix we show how our hypotheses (1.5) and (1.6) on the distribution $\zeta$ imply that the characteristic function satisfies the bound

$$\int_{\mathbb{R}} \frac{|\varphi_{\mathtt{z}}(t)|}{(1+|t|)^{1/2}}\,\mathrm{d}t \,<\infty\,, \tag{B.1}$$

and Lemma B.1 directly yields (B.1), thanks to our hypotheses (1.5) and (1.6). We use $\hat{f}(t) := \widehat{f}(-it) = \int_{\mathbb{R}} f(x)\exp(itx)\,\mathrm{d}x$.

**Lemma B.1.** *Assume that there exists $\theta > 0$ such that $|f(x) - f(y)| \leq |x-y|^\theta$ for every $x \neq y \in \mathbb{R}$ and that there exists $c > 0$ such that $\int_{\mathbb{R}} |f(x)|\exp(c|x|)\,\mathrm{d}x < \infty$. Then*

1. *$f(x) = O(\exp(-b|x|))$ with $b = c\theta/(1-\theta)$;*
2. *$\int_{\mathbb{R}}(|\hat{f}(t)|/(1+|t|)^{1/2})\,\mathrm{d}t < \infty$.*

*Proof.* It suffices to consider the case $x \to \infty$. We proceed by contradiction: assume that there exists a sequence of positive numbers $(x_n)$ with $\lim_n x_n = +\infty$ and $|f(x_n)| \geq$

$2\exp(-bx_n)$ for every $n$. Then for every $x \in [x_n, x_n + \exp(-bx_n/\theta)]$ we have $|f(x)| \geq \exp(-bx_n)$. Therefore

$$\int_{x_n}^{x_n+\exp(-bx_n/\theta)} |f(x)| \exp(cx)\, \mathrm{d}x \;\geq\; \exp(-bx_n/\theta)\exp(-bx_n)\exp(cx_n) \;=\; 1\,, \tag{B.2}$$

which is impossible because $\int_{\mathbb{R}} |f(x)| \exp(c|x|)\, \mathrm{d}x < \infty$. Therefore (1) is established.

For (2) we start by remarking that (1) implies that the continuous function $f$ is in $\mathbb{L}^p$ for every $p \geq 1$, in particular for $p = 2$. Letting $(\omega_2(h))^2 := \int_{\mathbb{R}} (f(x+h) - f(x))^2\, \mathrm{d}x$, for $h \in [-1/2, 1/2]$ and $K > 2$ we have

$$\begin{aligned}(\omega_2(h))^2 \;&\leq\; \int_{-K}^{K} (f(x+h) - f(x))^2\, \mathrm{d}x + C \int_{K-h}^{\infty} \exp(-2bx)\, \mathrm{d}x \\ &\leq\; 2K|h|^{2\theta} + C' \exp(-2bK)\,,\end{aligned} \tag{B.3}$$

so if we choose $K = (\theta/b)\log(1/|h|)$ – if $(\theta/b)\log(2) \leq 2$ we just choose a smaller value for $b$ – we have that there exists $C > 0$ such that for $|h| \leq 1/2$.

$$\omega_2(h) \;\leq\; C|h|^\theta \sqrt{\log(1/|h|)}\,; \tag{B.4}$$

also evidently $\sup_h \omega_2(h) \leq \sqrt{2}\|f\|_2$. These estimates imply that for every $\alpha \in (0, \theta)$

$$\int_{\mathbb{R}} \frac{(\omega_2(h))^2}{|h|^{1+2\alpha}}\, \mathrm{d}h \;<\; \infty\,, \tag{B.5}$$

which implies [30, Ch. 5: (46) and paragraph after (47)] that

$$\int_{\mathbb{R}} \left|\hat{f}(t)\right|^2 (1+|t|)^{2\alpha}\, \mathrm{d}t \;<\; \infty\,. \tag{B.6}$$

Since

$$\left(\int_{\mathbb{R}} \frac{\left|\hat{f}(t)\right|}{(1+|t|)^{1/2}}\, \mathrm{d}t\right)^2 \;\leq\; \int_{\mathbb{R}} \left|\hat{f}(t)\right|^2 (1+|t|)^{2\alpha}\, \mathrm{d}t \int_{\mathbb{R}} \frac{1}{(1+|t|)^{1+2\alpha}}\, \mathrm{d}t\,, \tag{B.7}$$

the proof of part (2) is complete. □

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