A note on a PDE approach to option pricing under xVA

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Abstract

In this paper we study partial differential equations (PDEs) that can be used to model value adjustments. Different value adjustments denoted generally as xVA are nowadays added to the risk-free financial derivative values and the PDE approach allows their easy incorporation. The aim of this paper is to show how to solve the PDE analytically in the Black-Scholes setting to get new semi-closed formulas that we compare to the widely used Monte-Carlo simulations and to the numerical solutions of the PDE. Particular example of collateral taken as the values from the past will be of interest.

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Contents

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1 Introduction

The famous Black-Scholes model for option pricing in an arbitrage-free environment was introduced in 1973 by Black and Scholes (1973) and Merton (1973). The pioneering model was very popular in the beginning and widely used by practitioners. One of the basic assumptions of the Black-Scholes model is that the volatility of the underlying asset is constant. This hypothesis was later discarded after Rubinstein (1985) observed that the real-market prices of out-of-the-money call options with short maturity were significantly higher than the predictions of the model. Hull and White (1987) introduced a model with stochastic volatility to fix this issue. The stock market crash of October 1987 with the infamous Black Monday permanently changed the shape of the volatility surface, see (Hull, 2009, Chapter 18.3). After the crash the implied volatility developed the marked volatility smile as observed today that Rubinstein (1994) explained with the fear of traders regarding the possibility of a similar event. As a consequence, stochastic volatility models by Hull and White (1987), Stein and Stein (1991), Heston (1993), and others became the standard in option pricing. The financial crisis that started in 2007 substantially changed the rules for derivative pricing again. The possibility to borrow and lend any amount of money at a risk-free interest rate is a crucial assumption of all the above mentioned models since its standardly used in the hedging arguments for the price valuation of the options. As the crisis culminated in the bankruptcy of Lehman Brothers and other big financial entities this conjecture became obsolete. Since the default of a trading partner is not in the scope of the classic option pricing models the traders had to react to this shortage in the models and included a collateral against default in many trades. Both parties of a contract have to consider the potential default of the counterparty. The traders respond to this bilateral counterparty risk with credit value adjustments (CVA), Sorenson and Bollier (1994) and Jarrow and Turnbull (1995), to have a collateral against default of the counterparty. The corresponding collateral of the counterparty, e.g., a bank, is covered by a debit value adjustment (DVA), Duffie and Huang (1996). In addition to this adjustments, derivative contracts can also contain a collateral, also called a collateral value adjustment (COLVA), regulated by the credit support annex (CSA). For more details on risk management see Hull (2018). Convexity adjustments that describe the differences between the prices of collateralized and non-collateralized derivatives were studied by Piterbarg (2010). Funding cost adjustments (FCA) are other common price adjustments that traders use to include funding cost in the price of the contract. According to Burgard and Kjaer (2011a) and Hull and White (2012) FCA have no theoretical basis. Since traders use them anyway Hull and White (2014) studied the adjustments with focus on their practical realization. There exist also other types of adjustments like capital value adjustments (KVA), margin value adjustments (MVA), and liquidity value adjustments (LVA) but we will restrict ourselves just to CVA, DVA, and FCA. The various adjustments to the value of the derivative are subsumed as xVA. A comprehensive guide to xVA is the book by Gregory (2020).

In analogy to the Black-Scholes approach, Burgard and Kjaer (2011b) derived a partial differential equation (PDE) for the adjusted derivative price under bilateral counterparty risk and funding costs. The partial differential equation is linear if the mark-to-market value at default is considered to be the non-default value of the derivative and nonlinear if the mark-to-market value is given by the adjusted value of the derivative with all value adjustments. The latter case is less meaningful from the practitioner's point of view since the close out is based in the risk-free mark-to-market for contracts that follow the 2002 ISDA Master Agreement[1]. Hence, we will study the linear partial differential equation introduced in Burgard and Kjaer (2013) which takes collaterals into account.

Value adjustments are normally applied to portfolios with various derivatives. Nevertheless, we consider a single option contract to analyze the influence of the different adjustments. The macro view, i.e., the homogeneous cost view for the whole portfolio, essentially follows from the micro view of single contracts. We derive the analytic solution of the PDE of Burgard and Kjaer (2013), obtain a semi-closed pricing formula for the value adjustments, and compare the formula to widely

[1]available online at `https://www.isda.org/book/2002-isda-master-agreement-english/` [cit. 2019-09-12].

used Monte-Carlo simulations and numerical solutions of the partial differential equations as in Arregui, Salvador, and Vázquez (2017) and Arregui, Salvador, Ševčovič, and Vázquez (2018). The problem has recently been studied for the Heston model in Salvador and Oosterlee (2021).

The paper is structured in the following way. We introduce the PDE for the correction between the adjusted price with xVA and the risk-free price of a derivative contract and we discuss various collaterals in Section 2. In Section 3, we introduce the pricing formula and give details on the numerical schemes, before numerical results are presented in Section 4. We conclude in Section 5.

2 Preliminaries

We consider a generic derivative contract (e.g. option), possibly collateralized, between an issuer B and a counterparty C, with an economic value $\check{V}$ that incorporates the risk of default of counterparty and issuer and any net funding costs the issuer may encounter prior to own default. We are mainly interested in the correction $U = \check{V} - V$ to the risk-free Black-Scholes price V. Using the Black-Scholes PDE for V one can derive (Burgard and Kjaer, 2013) the corresponding PDE for U.

Let $\mathcal{A}$ be the standard Black-Scholes operator

$$\mathcal{A} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (q_S - \gamma_S) S \frac{\partial}{\partial S},$$

where $\sigma > 0$ is the constant volatility and $q_S - \gamma_S$ is the effective financing rate of the market factor hedge S. We will study the following final value problem for function $U(t, S)$

$$\begin{cases} \dfrac{\partial U}{\partial t}(t,S) + \mathcal{A}U(t,S) - (r + \lambda_\mathrm{B} + \lambda_\mathrm{C})U(t,S) = f(V(t,S)), & \text{for } t \in [0,T), S \geq 0, \\ U(T,S) = 0, & \text{for } S \geq 0. \end{cases} \tag{1}$$

where the inhomogeneity f has the form

$$\begin{aligned} f &= f_\mathrm{CVA} + f_\mathrm{DVA} + f_\mathrm{FCA} + f_\mathrm{COLVA}, \\ f_\mathrm{CVA} &= -\lambda_\mathrm{C}(g_\mathrm{C} - V), \\ f_\mathrm{DVA} &= -\lambda_\mathrm{B}(g_\mathrm{B} - V), \\ f_\mathrm{FCA} &= \lambda_\mathrm{B} \cdot \varepsilon_h, \\ f_\mathrm{COLVA} &= s_X \cdot X, \end{aligned}$$

with constants $\lambda_\mathrm{B}, \lambda_\mathrm{C}, s_X$ and the hedge error ε_h described below. Here, $\lambda_\mathrm{B}, \lambda_\mathrm{C}$ are the default intensities of the counterparties, ε_h describes the size of the hedge error, and s_X is the difference between the collateral rate and the risk-free rate. X denotes the collateral and we have

$$\begin{aligned} g_\mathrm{C} &= R_\mathrm{C}(V - X)^+ - (V - X)^- + X, \\ g_\mathrm{B} &= (V - X)^+ - R_\mathrm{B}(V - X)^- + X, \end{aligned}$$

where $R_\mathrm{B}, R_\mathrm{C} \in [0,1]$ are the recovery rates of the derivative position. For this regular bilateral close-out the hedge error ε_h becomes

$$\varepsilon_h = (1 - R_B)(V - X)^+. \tag{2}$$

The positive and negative part of a function φ are defined as $\varphi^+ = \max(\varphi, 0)$ and $\varphi^- = \max(-\varphi, 0)$, respectively. We remark that Burgard and Kjaer (2013) use the definition $\min(\varphi, 0)$ for the negative part resulting in slightly different equations.

We are interested in different standard types of collaterals. For un-collateralized derivative contracts, we have $X = 0$ and the terms f_{CVA}, f_{DVA} and f_{FCA} simplify to

$$f_{\mathrm{CVA}} = \lambda_{\mathrm{C}}(1 - R_{\mathrm{C}})V^{+}, \quad f_{\mathrm{DVA}} = -\lambda_{\mathrm{B}}(1 - R_{\mathrm{B}})V^{-}, \quad f_{\mathrm{FCA}} = \lambda_{\mathrm{B}}(1 - R_{\mathrm{B}})V^{+}.$$

In one-way CSA, the issuer posts collateral provided that the risk-free value is out-of-the-money. That corresponds to $X = -V^{-}$ and we get

$$f_{\mathrm{CVA}} = \lambda_{\mathrm{C}}(1 - R_{\mathrm{C}})V^{+}, \quad f_{\mathrm{DVA}} = 0, \quad f_{\mathrm{FCA}} = \lambda_{\mathrm{B}}(1 - R_{\mathrm{B}})V^{+}.$$

For gold-plated two-way CSA with collateral $X = V$ there are no CVA, DVA, and FCA. If the collateral rate equals the risk-free rate then the term f_{COLVA} vanishes and the adjusted price $\check{V}$ of a two-way CSA becomes the risk-less price V.

In real world situations, there is most likely a collateral that depends on V non-smoothly, non-linearly or even discontinuously. For further reading about these dependencies we refer the reader for example to ISDA: Legal Guidelines for Smart Derivatives Contracts: Collateral [2].

A popular choice for a collateral among practitioners is the *value from the previous day*, i.e. in more general case $X = V(t - t_0, S)$. With the notation $\Delta_t V = V(t, S) - V(t - t_0, S)$ we obtain

$$\begin{aligned}
f_{\mathrm{CVA}} &= -\lambda_{\mathrm{C}}(R_{\mathrm{C}}(\Delta_t V)^{+} - (\Delta_t V)^{-} - V(t - t_0, S)), \\
f_{\mathrm{DVA}} &= -\lambda_{\mathrm{B}}((\Delta_t V)^{+} - R_{\mathrm{B}}(\Delta_t V)^{-} - V(t - t_0, S)), \\
f_{\mathrm{FCA}} &= \lambda_{\mathrm{B}}(1 - R_{\mathrm{B}})(\Delta_t V)^{+},
\end{aligned}$$

for this collateral. Typically t denotes the current time, $t_0 = 1/252$ represents one business day and hence $S(t - t_0)$ is the value of the underlying asset price from the previous day. If V is the Black-Scholes call option price, we have that

$$V(t - t_0, S(t - t_0)) = N(d_1(t - t_0, S(t - t_0)))S(t - t_0) + K\mathrm{e}^{-r(T-t-t_0)}N(d_2(t - t_0, S(t - t_0)), \quad (3)$$

where $N(\cdot)$ is the standard normal CDF and

$$d_{1,2}(s, x) = \frac{1}{\sigma\sqrt{T - s}}\left[\ln\frac{x}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)(T - s)\right].$$

Applying the Feynman-Kac Theorem to PDE (1) gives us (Burgard and Kjaer, 2013) the semi-closed solution in the form

$$U = U_{\mathrm{CVA}} + U_{\mathrm{DVA}} + U_{\mathrm{FCA}} + U_{\mathrm{COLVA}}, \quad (4)$$

with

$$\begin{aligned}
U_{\mathrm{CVA}} &= -\int_t^T \lambda_{\mathrm{C}}(u)D_{r+\lambda_{\mathrm{B}}+\lambda_{\mathrm{C}}}(t, u)\mathbb{E}_t[V(u) - g_{\mathrm{C}}(V(u), X(u))]\,\mathrm{d}u, \\
U_{\mathrm{DVA}} &= -\int_t^T \lambda_{\mathrm{B}}(u)D_{r+\lambda_{\mathrm{B}}+\lambda_{\mathrm{C}}}(t, u)\mathbb{E}_t[V(u) - g_{\mathrm{B}}(V(u), X(u))]\,\mathrm{d}u, \\
U_{\mathrm{FCA}} &= -\int_t^T \lambda_{\mathrm{B}}(u)D_{r+\lambda_{\mathrm{B}}+\lambda_{\mathrm{C}}}(t, u)\mathbb{E}_t[\varepsilon_h(u)]\,\mathrm{d}u, \\
U_{\mathrm{COLVA}} &= -\int_t^T s_X D_{r+\lambda_{\mathrm{B}}+\lambda_{\mathrm{C}}}(t, u)\mathbb{E}_t[X(u)]\,\mathrm{d}u,
\end{aligned}$$

where $D_{\hat{r}}(t, u) = \exp\left\{-\int_t^u \hat{r}(v)\,\mathrm{d}v\right\}$ is the discount factor between t and u for a rate $\hat{r}$. The measure of the expectations in these equations is such that S drifts at rate $(q_S - \gamma_S)$. The sum of DVA and FCA is sometimes referred to as funding valuation adjustments (FVA).

[2] available online at `https://www.isda.org/2019/09/12/legal-guidelines-for-smart-derivatives-contracts-collateral/` [cit. 2019-09-12].

3 Methodology

In this section we present our main results, in particular we show how to analytically solve the corresponding PDE by a transformation to the heat equation and present a direct relation between the correction and the risk-less price for certain contracts with non-negative payoff. We discuss the boundary conditions for the numerical schemes and provide details on the numerics of all three used approaches: finite differences, the semi-closed solution formula, and the Monte-Carlo method.

We perform the standardly used change of variables to the time till maturity and the logarithm of the stock price

$$\tau = T - t \quad \text{and} \quad x = \ln S, \tag{5}$$

and introduce new functions $u(\tau, x) \cong U(t, S)$ and $v(\tau, x) \cong V(t, S)$. After applying (5) to the PDE (1) we get

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} + \left(q_S - \gamma_S - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial x} - (r + \lambda_\mathrm{B} + \lambda_\mathrm{C})u - f.$$

If we set

$$\rho = q_S - \gamma_S - \frac{1}{2}\sigma^2$$

to simplify the notation then we can write the initial value problem for the correction $u(\tau, x)$ as

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 u}{\partial x^2} + \rho\dfrac{\partial u}{\partial x} - (r + \lambda_\mathrm{B} + \lambda_\mathrm{C})u - f, & \text{for } \tau \in (0, T], x \in \mathbb{R}, \\ u(0, x) = 0, & \text{for } x \in \mathbb{R}. \end{cases} \tag{6}$$

The PDE (6) is parabolic and has a unique weak solution (Evans, 2010).

3.1 Solution by transformation to heat equation

Black and Scholes obtained their famous formula by transforming the PDE to the homogeneous heat equation from physics. If we want to get a heat equation from (6), we have to perform successive transformations.

We start by introducing $\tilde{u}(\tau, x) = \mathrm{e}^{(r+\lambda_\mathrm{B}+\lambda_\mathrm{C})\tau} u(\tau, x)$ to simplify the PDE (6) to

$$\frac{\partial \tilde{u}}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 \tilde{u}}{\partial x^2} + \rho\frac{\partial \tilde{u}}{\partial x} - f \cdot \mathrm{e}^{(r+\lambda_\mathrm{B}+\lambda_\mathrm{C})\tau}.$$

Next, we use a suitable shift $\tilde{\tilde{u}}(\tau, x) = \tilde{u}(\tau, x - \rho\tau)$ to obtain the *non-homogeneous heat equation*

$$\frac{\partial \tilde{\tilde{u}}}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 \tilde{\tilde{u}}}{\partial x^2} - f \cdot \mathrm{e}^{(r+\lambda_\mathrm{B}+\lambda_\mathrm{C})\tau}. \tag{7}$$

We observe that $\tilde{\tilde{u}}(0, x) = u(0, x) = 0$ for all $x \in \mathbb{R}$ and we can write the solution of (7) as

$$\tilde{\tilde{u}}(\tau, x) = -\int_0^\tau \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\frac{1}{2}\sigma^2(\tau - s)}} \mathrm{e}^{-\frac{(z-x)^2}{2\sigma^2(\tau-s)}} \cdot f(v(s, z))\,\mathrm{d}z\right) \mathrm{e}^{(r+\lambda_\mathrm{B}+\lambda_\mathrm{C})s}\,\mathrm{d}s.$$

We perform one more substitution

$$y = \frac{z - x}{\sqrt{2\sigma^2(\tau - s)}}$$

to have the solution in the form

$$\tilde{\tilde{u}}(\tau, x) = -\frac{1}{\sqrt{\pi}}\int_0^\tau \left(\int_{-\infty}^{\infty} \mathrm{e}^{-y^2} \cdot f(v(s, x + \sqrt{2\sigma^2(\tau - s)}y))\,\mathrm{d}y\right) \mathrm{e}^{(r+\lambda_\mathrm{B}+\lambda_\mathrm{C})s}\,\mathrm{d}s.$$

Finally, we apply the inverse substitutions to obtain the formula for the solution of (6) in terms of the original variables

$$u(\tau, x) = -\frac{1}{\sqrt{\pi}} \int_0^\tau \left(\int_{-\infty}^{\infty} \mathrm{e}^{-y^2} \cdot f(v(s, x + \rho\tau + \sqrt{2(\tau - s)}\sigma y)) \,\mathrm{d}y \right) \mathrm{e}^{-(r+\lambda_{\mathrm{B}}+\lambda_{\mathrm{C}})(\tau-s)} \,\mathrm{d}s. \tag{8}$$

Remark 3.1. *Another representation of the same solution can be obtained by means of the Fourier transform*

$$\hat{u}(\tau, k) := \mathcal{F}\left[u(\tau, x)\right] = \int_{-\infty}^{\infty} u(\tau, x)\mathrm{e}^{-ikx} \,\mathrm{d}x,$$
$$\hat{f}(\tau, k) := \mathcal{F}\left[f(v(\tau, x))\right] = \int_{-\infty}^{\infty} f(v(\tau, x))\mathrm{e}^{-ikx} \,\mathrm{d}x,$$

where i denotes the complex unit. If we apply the Fourier transform to the PDE (6)*, we get*

$$\frac{\partial \hat{u}}{\partial \tau} = -\frac{1}{2}\sigma^2 k^2 \hat{u} + ik\rho\hat{u} - (r + \lambda_B + \lambda_C)\hat{u} - \hat{f} = g(k)\hat{u} - \hat{f}, \tag{9}$$

where we denoted

$$g(k) = -\frac{1}{2}\sigma^2 k^2 + ik\rho - (r + \lambda_B + \lambda_C).$$

We introduce $\hat{h}(\tau, k) = \mathrm{e}^{-g(k)\tau}\hat{u}(\tau, k)$ to transform (9) *to the ODE*

$$\frac{\partial \hat{h}}{\partial \tau} = -\mathrm{e}^{-g(k)\tau}\hat{f}$$

that can be solved directly

$$\hat{h}(\tau, k) = -\int_0^\tau \mathrm{e}^{-g(k)s}\hat{f}(s, k) \,\mathrm{d}s.$$

Hence, the solution of (9) *has the form*

$$\hat{u}(\tau, k) = -\int_0^\tau \mathrm{e}^{g(k)(\tau-s)}\hat{f}(s, k) \,\mathrm{d}s$$

and we can apply the inverse Fourier transform to obtain the solution of the original equation (6)

$$\begin{aligned} u(\tau, x) = \mathcal{F}^{-1}\left[\hat{u}(\tau, k)\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\tau, k)e^{ikx} \,\mathrm{d}k \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{ikx} \left(\int_0^\tau \mathrm{e}^{(-\frac{1}{2}\sigma^2 k^2 + ik\rho - (r+\lambda_B+\lambda_C))(\tau-s)}\hat{f}(s, k) \,\mathrm{d}s \right) \mathrm{d}k. \end{aligned} \tag{10}$$

To show that two different forms (8) *and* (10) *of the solution are equivalent one may use for example similar techniques as Baustian, Mrázek, Pospíšil, and Sobotka (2017).*

3.2 Non-negative payoffs

Many derivative contracts like call options have a non-negative payoff resulting in a non-negative risk-less price v of the derivative, i.e., we have $v = v^+$ and $v^- = 0$. If we consider un-collateralized contracts or one-way CSA for such derivatives, as described in Section 2, then the term f for the value adjustments simplifies to $f(v) = (\lambda_{\mathrm{C}}(1 - R_{\mathrm{C}}) + \lambda_{\mathrm{B}}(1 - R_{\mathrm{B}}))v$ and we can represent the correction u in dependence of the risk-less price v. We get

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 u}{\partial x^2} + \rho\dfrac{\partial u}{\partial x} - (r + \lambda_{\mathrm{B}} + \lambda_{\mathrm{C}})u - c_f v, & \text{for } \tau \in (0, T], x \in \mathbb{R}, \\ u(0, x) = 0, & \text{for } x \in \mathbb{R}. \end{cases} \tag{11}$$

with $c_f = (\lambda_\mathrm{C}(1-R_\mathrm{C}) + \lambda_\mathrm{B}(1-R_\mathrm{B}))$ and recall that v satisfies the Black-Scholes equation

$$\begin{cases} \dfrac{\partial v}{\partial \tau} = \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 u}{\partial x^2} + \rho \dfrac{\partial u}{\partial x} - rv, & \text{for } \tau \in (0,T], x \in \mathbb{R}, \\ v(0,x) = h(x), & \text{for } x \in \mathbb{R}, \end{cases}$$

where $h(x) \geq 0$ is the payoff function of the derivative. We observe that the function $\frac{-c_f v}{\lambda_\mathrm{B}+\lambda_\mathrm{C}}$ is a solution of (11) but has the non-trivial initial value

$$\left(\frac{-c_f v}{\lambda_\mathrm{B}+\lambda_\mathrm{C}}\right)(0,x) = \frac{-c_f}{\lambda_\mathrm{B}+\lambda_\mathrm{C}} h(x).$$

Fortunately, we can make use of the function $w = \mathrm{e}^{-(\lambda_\mathrm{B}+\lambda_\mathrm{C})\tau} v$ that satisfies $w(0,x) = h(x)$ and

$$\frac{\partial w}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 w}{\partial x^2} + \rho \frac{\partial w}{\partial x} - (r + \lambda_\mathrm{B} + \lambda_\mathrm{C}) w$$

to obtain the solution

$$u(\tau,x) = \frac{c_f}{\lambda_\mathrm{B}+\lambda_\mathrm{C}}(w(\tau,x) - v(\tau,x)) = \frac{\lambda_\mathrm{C}(1-R_\mathrm{C}) + \lambda_\mathrm{B}(1-R_\mathrm{B})}{\lambda_\mathrm{B}+\lambda_\mathrm{C}} \left(\mathrm{e}^{-(\lambda_\mathrm{B}+\lambda_\mathrm{C})\tau} - 1\right) v(\tau,x)$$

of the initial value problem (11) with trivial initial data. We can use this representation of the solution with any solution formula for v like the Black-Scholes formula or a Fourier transform based formula to directly derive the correction term u.

3.3 Boundary conditions

Both problems (1) and (6) are defined either on a half-line in S or a line in x, respectively. However, if we want to compare the numerical solution of the problem (6) with the solution provided by MC or formula (8), we must reduce ourselves to a finite interval in x. In such case, we need to define boundary conditions.

The PDE in (1) is a good starting point here, rather than the PDE in (6). Let us assume that we work on the finite interval $[0, S_\mathrm{max}]$ with $S_\mathrm{max} \in \mathbb{R}^+$ sufficiently large. If we set $S = 0$, the PDE in (1) reduces to

$$\frac{\partial U}{\partial t} = (r + \lambda_\mathrm{B} + \lambda_\mathrm{C}) \cdot U + \lambda_\mathrm{B} \cdot \varepsilon_h,$$

which can be solved to get

$$U(t,0) = \frac{\lambda_\mathrm{B}\varepsilon_h}{r + \lambda_\mathrm{B} + \lambda_\mathrm{C}} \left(e^{(r+\lambda_\mathrm{B}+\lambda_\mathrm{C})(t-T)} - 1\right). \tag{12}$$

Hence, we have this Dirichlet boundary condition on the left for the PDE in (1).

Now, if we divide the PDE in (1) by S^2 and approach the limit $S \to +\infty$, we get

$$\lim_{S \to +\infty} \frac{\partial^2 U}{\partial S^2}(t,S) = 0.$$

We are actually assuming here some behaviour of $U(t,S)$ as $S \to +\infty$. More precisely, the growth of $U(t,S)$ in S should be sub-quadratic for large S. In pricing equations for V, a common approach is to consider the so-called *zero gamma condition* (in't Hout, 2017, Sec. 2.3), i.e.,

$$\frac{\partial^2 V}{\partial S^2}(t, S_\mathrm{max}) = 0, \quad 0 \leq t \leq T,$$

and hence it is natural to assume the same type of condition for the correction U

$$\frac{\partial^2 U}{\partial S^2}(t, S_\mathrm{max}) = 0, \quad 0 \leq t \leq T, \tag{13}$$

for some sufficiently large $S_{\max}$. This approach was also already used, e.g., by Castillo, Ferreiro, García-Rodríguez, and Vázquez (2013) and Arregui, Salvador, and Vázquez (2017). It is possible to assume here $U(t, S_{\max}) = c_0(t) + c_1(t)S$ as it was done in these papers, substitute it into (1) and compute c_0, c_1 either numerically or in some cases also analytically. However, we will stick with (13) as our boundary condition. It is easier to implement, does not require any additional lengthy computations and most importantly it provides us with significantly better results, when comparing the numerical solution of the PDE to Monte-Carlo (MC) or formula (8).

Let us consider two numbers $x_{\min}, x_{\max} \in \mathbb{R}$ such that

$$-\infty < x_{\min} < 0 < x_{\max} < +\infty.$$

Due to the transformation (5) and derived boundary conditions (12),(13), it seems reasonable to consider boundary conditions

$$\begin{aligned} u(\tau, x_{\min}) &= \frac{\lambda_B \varepsilon_h}{r + \lambda_{\mathrm{B}} + \lambda_{\mathrm{C}}} \left(\mathrm{e}^{-(r+\lambda_{\mathrm{B}}+\lambda_{\mathrm{C}})\tau} - 1 \right), \\ \frac{\partial^2 u}{\partial x^2}(\tau, x_{\max}) &= 0, \end{aligned}$$

for all $\tau \in [0, T]$ completing the problem (6).

3.4 Numerical solution using finite differences

The goal of this subsection is to introduce the approximate scheme for the transformed problem (6) completed by boundary conditions (14),(15). We intend to use the idea of the method of lines, i.e., a separate discretization in time τ and space x (see, e.g., (LeVeque, 2007, Chapter 13)). The whole problem is implemented in the software MATLAB, which possesses several useful solvers for the evolution in time. Hence, we will approximate the spatial part of the problem (6),(14),(15), which results in the system of ODEs. Then we will use a suitable ODE solver in MATLAB to iterate it in time. The PDE of the problem (6) is typical advection–diffusion–reaction equation. Since the problem is situated in one spatial dimension, it is sufficient to use finite differences.

As we mentioned in the previous subsection, we work on the interval $[x_{\min}, x_{\max}]$ in x. Let us discretize this interval by the uniform grid with constant step size $\delta = x_{i+1} - x_i$ for $i = 0, 1, \ldots, N_x - 1$, hence, there is $x_0 = x_{\min}$ and $x_{N_x} = x_{\max}$. We denote the approximate value of $u(\tau, x_i)$ by U_i and the value of $f(V(\tau, x_i))$ by F_i.

We acknowledge that it is quite common to use a non-uniform grid in x as was presented in other works, e.g., (Rouah, 2013, Chap. 10) or (in't Hout, 2017, Sec. 4.2), where the mapping function can be for example hyperbolic sine. One can then use very rough grid while maintaining reasonable density of points around the strike, where the solution changes most significantly. This approach can be useful if one aims to make large amount of numerical experiments (for different parameters of the model for example) with low computation cost. However, if we look at our problem from the point of view of numerical mathematics, the solution of (6) (see Figure 1) does not include any dramatic oscillations or boundary layers. Hence, we can achieve a reasonable solution with relatively rough uniform grid. Since our goal in this paper is to compare different approaches of solving the problem (6), e.g., for different step sizes δ, using the non-uniform grid is not necessary or even desirable.

We approximate derivatives in (6) by

$$\frac{\partial^2 u}{\partial x^2}(\tau, x_i) \approx \frac{U_{i-1} - 2U_i + U_{i+1}}{\delta^2} \quad \text{and} \quad \frac{\partial u}{\partial x}(\tau, x_i) \approx \frac{U_{i+1} - U_{i-1}}{2\,\delta}$$

for every $i = 1, \ldots, N_x - 1$. Both of these differences are central with the error $O(\delta^2)$. Let us mention here that since our problem is diffusion–dominated, the first order central difference approximating the advection term does not bring in any oscillations and thus no stabilization is necessary.

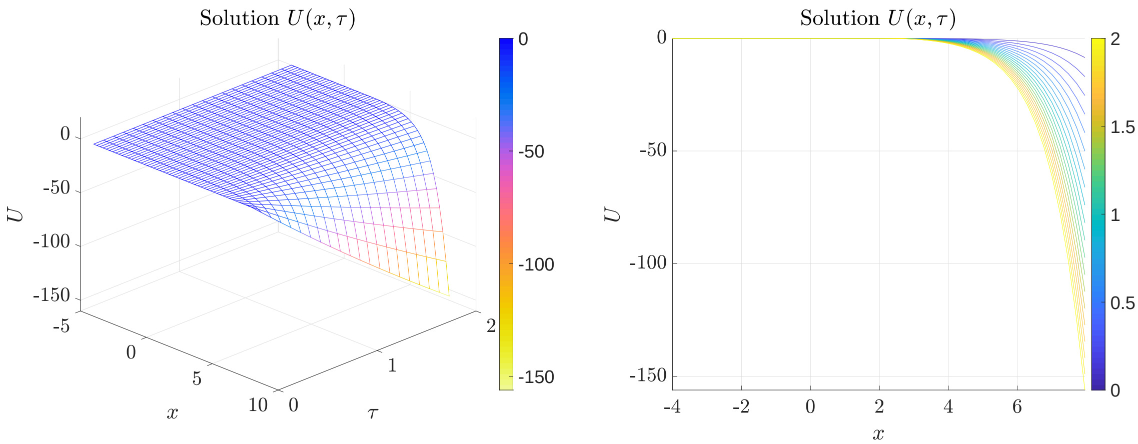

Figure 1: Example of the solution computed by finite differences. The first picture contains the 3D plot of the solution in (x, τ). Contours of the solution for different values of τ are depicted on the right. The plotted solution was computed for the case $x_{\text{min}} = -4$, $x_{\text{max}} = 8$, and the strike $K = 15$ (the other parameters are stated in the next section).

To deal with the boundary condition (15) at x_{max}, we employ the classical idea of a ghost point, i.e., we consider the point $x_{N_x+1} = x_{N_x} + \delta$ outside of the grid. With this point, we can approximate the second derivative in (15) again by the second order central difference, which yields

$$\frac{U_{N_x-1} - 2U_{N_x} + U_{N_x+1}}{\delta^2} = 0. \tag{16}$$

Hence, we have the system of ODEs

$$\frac{\partial U_i}{\partial \tau} = \frac{1}{2}\sigma^2 \cdot \frac{U_{i-1} - 2U_i + U_{i+1}}{\delta^2} + \rho \cdot \frac{U_{i+1} - U_{i-1}}{2\,\delta} - (r + \lambda_{\text{B}} + \lambda_{\text{C}}) \cdot U_i - F_i, \quad \text{for } i = 1, \dots, N_x - 1. \tag{17}$$

The equation (17) for $i = N_x$ contains the value U_{N_x+1} outside of the grid, i.e., the value at the ghost point. Expressing U_{N_x+1} from (16) and substituting it into (17) for $i = N_x$ give us the equation

$$\frac{\partial U_{N_x}}{\partial \tau} = \rho \cdot \frac{U_{N_x} - U_{N_x-1}}{\delta} - (r + \lambda_{\text{B}} + \lambda_{\text{C}}) \cdot U_{N_x} - F_{N_x}, \tag{18}$$

for the computation of the value U_{N_x} at the boundary x_{max}.

We are using MATLAB's `ode15s` solver for the evolution of (17), (18) in time. It is known that if the coefficient in front of the second derivative in (6) is small, the problem could be stiff. Hence, it is safer option to use `ode15s`, which is built to deal with stiff problems, rather than `ode45`. There are of course other options for the evolution in time, for example methods of ADI type like Douglas method - see, e.g., (Rouah, 2013, Chap. 10) or (in't Hout, 2017, Chap. 13). This type of methods is often used for problems in multiple spatial dimensions. Despite the fact, that our problem is in one spatial dimension, we could use ADI method to split the iterative process into three stages, the advection part, the diffusion part and the source part. This approach is very viable and would be better than for example Euler's method, lowering the restriction of the CFL condition. However, `ode15s` solver is adaptive in time and very capable of dealing with this kind of problems, as well.

3.5 Numerical solution using the transformation to heat equation

In order to evaluate the double integral in (8) numerically, a suitable quadrature has to be chosen. Although there exist many sophisticated quadratures, see for example monographs by Krylov (1962); Stoer and Bulirsch (2002); Davis and Rabinowitz (2007); Dahlquist and Åke (2008), for the clarity of our numerical comparison we use the simple trapezoid method. Let us consider an equidistant partition of the interval $[0, \tau]$

$$s_i = i \cdot \delta_s, \quad i = 0, 1, \ldots, N_s$$

for given $\delta_s \in \mathbb{R}$ and $N_s \in \mathbb{N}$ such that $N_s \cdot \delta_s = \tau$. The inner integral needs to be restricted to a bounded interval in order for the trapezoid method to be applicable. Let $y_{\max} > 0$ be sufficiently large number which shall be specified in the forthcoming sections and let $\delta_y \in \mathbb{R}$ and $N_y \in \mathbb{N}$ be such that $N_y \cdot \delta_y = 2y_{\max}$. We then define the partition of the interval $[-y_{\max}, y_{\max}]$ as

$$y_i = -y_{\max} + i \cdot \delta_y, \quad i = 0, 1, \ldots, N_y.$$

We can now express the trapezoid value as

$$U_{\mathrm{HE}}(\tau, x) = -\frac{1}{4\sqrt{\pi}} \sum_{i=0}^{N_s-1} \sum_{j=0}^{N_y-1} \left(g(s_{i+1}, y_{i+1}) - g(s_i, y_i) \right) \delta_y \, \delta_s, \tag{19}$$

in which

$$g(s, y) = g(s, y; \tau, x) = \mathrm{e}^{-y^2-(r+\lambda_B+\lambda_C)(\tau-s)} f(V(s, x + \rho\tau + \sqrt{2(\tau-s)}\rho y)).$$

3.6 Numerical solution using the Monte-Carlo method

Currently, the most widely used method to evaluate the formula (4) with the subsequent expressions is the Monte-Carlo simulation. The incorporation of a change of variables and the simplification of the term $D_{\hat{r}}(t, u)$ in (4) results in

$$\begin{aligned} U_*(t, S) &= -\int_t^T \mathrm{e}^{(r+\lambda_B+\lambda_C)(u-t)} \mathbb{E}[f_*(V(u, S(u))) \,|\, S(t) = S] \,\mathrm{d}u, \\ &= -\int_0^{T-t} \mathrm{e}^{(r+\lambda_B+\lambda_C)s} \mathbb{E}[f_*(V(t+y, S(t+y))) \,|\, S(t) = S] \,\mathrm{d}s, \end{aligned}$$

in which $* \in \{$ CVA, DVA, FCA, COLVA $\}$ and $S(t)$ is modelled as the geometric Brownian motion. After the change of variables (5) we get

$$U_*(\tau, x) = -\int_0^\tau e^{(r+\lambda_B+\lambda_C)s} \mathbb{E}[f_*(V(T-\tau+s, \ln(S(T-\tau+s)))) \,|\, \ln(S(T-\tau)) = x] \,\mathrm{d}s. \tag{20}$$

We shall compute the integral term in (20) again via the trapezoid method on the interval $[0, \tau]$ with partition

$$s_i = i \cdot \delta_s, \quad i = 0, 1, \ldots, N_s$$

with $\delta_s \in \mathbb{R}$, $N_s \in \mathbb{N}$ and $\delta_s \cdot N_s = \tau$. The integral (20) can be now computed as

$$\hat{U}_*(\tau, x) = -\frac{1}{2} \sum_{i=0}^{N_s-1} (g_*(s_{i+1}) - g_*(s_i))\delta_s$$

in which

$$g_*(s_i) := g_*(s_i; x) = \mathrm{e}^{(r+\lambda_B+\lambda_C)s} \varepsilon_*(x),$$

where $\varepsilon_*(x)$ is the Monte-Carlo estimate of the corresponding conditional expectation. Finally

$$U_{\mathrm{MC}}(\tau, x) = \hat{U}_{\mathrm{CVA}}(\tau, x) + \hat{U}_{\mathrm{DVA}}(\tau, x) + \hat{U}_{\mathrm{FCA}}(\tau, x) + \hat{U}_{\mathrm{COLVA}}(\tau, x) \tag{21}$$

yields the numerically computed value of (4). In practice, $\varepsilon_*(x)$ is usually efficiently estimated using the Quasi-Monte-Carlo method with low discrepancy Sobol sequences - see, e.g., Renzitti, Bastani, and Sivorot (2020). The classical Monte-Carlo estimates will be calculated in the forthcoming section which is sufficient for the accuracy comparison purposes.

4 Numerical results

In this section we compare the pricing formula (8) with the Monte-Carlo pricing approach using the formula (4) and a numerical solution of the PDE (6), namely with the solution obtained using the finite differences method (FDM). Also, we distinguish two collateral strategies: the un-collateralized contract, $X = 0$, which leads to more intricate right source term of (1) compared to other choices and the delayed collateral, $X = V(t - t_0, S(t - t_0))$. We do not consider one-way CSA since the corresponding adjustments are quite similar to contracts with no collateral. We recall that the different collaterals were discussed in Section 2.

4.1 Un-collateralized contract, $X = 0$

We fixed parameters $\sigma = 0.25$, $q_s = 0.03$, $\gamma_s = 0$, $r = 0.03$, $\lambda_B = 0.02$, $\lambda_C = 0.05$, $R_B = 0.4$, $R_C = 0.4$, $s_X = 0.012$, $X = 0$, $T = 2$, ε_h as in (2) and we valued the European call option with strike price $K = 15$.

For all the tests, we considered the pricing formula (8) to be the benchmark solution which we computed with parameters $\delta_y = 2^{-3}$, $\delta_s = 2^{-10}$, $y_{\max} = 100$. These parameters seem to be the threshold ones such that finer discretization and larger integration region would not improve the accuracy significantly.

The experiments pursued two main directions. First, we studied the behaviour of the finite difference scheme in Section 3.4 while varying the spatial discretization δ_x, the temporal discretization δ_τ and the integration region $[x_{\min}, x_{\max}]$. The second experiment is focused on the Monte-Carlo formula (21) and its accuracy with respect to the number of simulated paths, denoted by N. In order to mitigate the approximative behaviour of the boundary condition (18) and to eliminate the growing region size in the latter case we compare the values in the region $\Omega = [\ln(K) - 1, \ln(K) + 1] \times [0, \tau]$. The error was quantified via the estimate of the $L_{\max}$ norm, i.e.,

$$\|g\|_{L_{\max}} = \max\{|g(i \cdot \delta_x, j \cdot \delta_s)|\,;\, i = 0, 1, \ldots, N_\tau,\, j = 0, 1, \ldots, N_x\}$$

and the estimate of the L_2-norm

$$\|g\|_{L_2} = \left(\sum_{i=0}^{N_\tau - 1} \sum_{j=0}^{N_x - 1} \left(g(x_{i+1}, \tau_{j+1}) - g(x_i, \tau_j)\right)^2 \, \delta_x \, \delta_\tau \right)^{\frac{1}{2}},$$

While evaluating the performance of the finite difference scheme, we fixed $-x_{\min} = x_{\max} = 5$ and let $\delta_x = 2^{-2i}$, $i = 1, \ldots, 4$, and $\delta_\tau = 2^{-2j}$, $j = 1, \ldots, 5$, vary. The results are depicted in Figure 2 which show a good agreement of the two approaches. As can be seen, the error is almost independent of the temporal discretization fineness which can be caused by the fact, that the `ode15s` solver chooses the time step adaptively.

Next, we fixed $\delta_x = 2^{-6}$, $\delta_\tau = 2^{-3}$, $x_{\min} = -4$ and let $x_{\max} = 4, 5, \ldots, 8$, vary. We do not vary $x_{\min}$, because $\mathrm{e}^{x_{\min}} = \mathrm{e}^{-4} \approx 0.018$ is sufficiently close to zero and taking more negative $x_{\min}$ would not provide much difference. As can be seen in Figure 3 the influence of the region truncation is notable for extremely small regions exclusively. Further enlargement does not significantly improve the accuracy.

Finally, we set $\delta_x = 2^{-1}$, $\delta_\tau = 2^{-2}$, $-x_{\min} = x_{\max} = 5$ (note that these parameters influence only the number of points we use to compute the error since only the schemes (19), (21) were considered) and the Monte-Carlo time step $\delta_s = 2^{-10}$. The dependence of the error of the Monte-Carlo method on the number of simulated paths $N = 10^{3+0.5j}$, $j = 0, \ldots, 4$ is depicted in Figure 4.

4.2 Collateral as the delayed value, $X = V(t - t_0, S(t - t_0))$

Let us now consider the collateral to be the delayed value as in (3) where $S(t - t_0)$ is approximated by its backward-discounted value, i.e.,

$$S(t - t_0) \approx S(t) \exp\left\{-t_0(q_s - \gamma_s - \frac{1}{2}\sigma^2)\right\}.$$

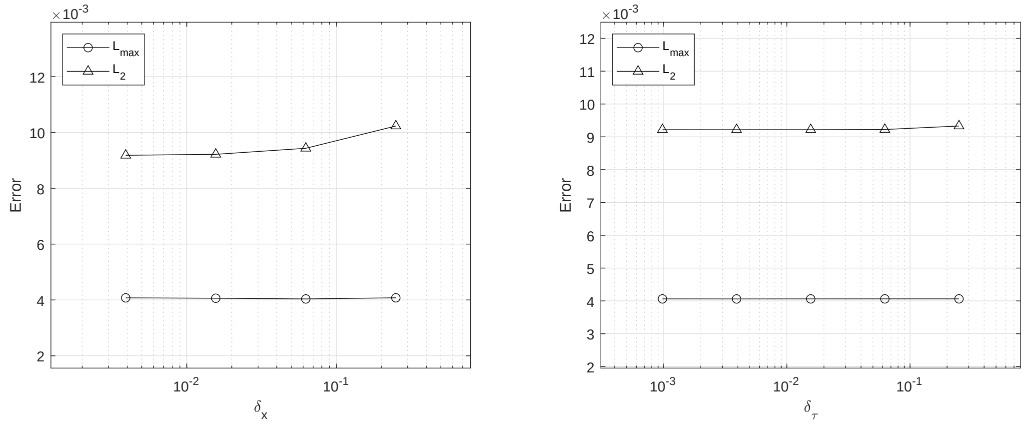

Figure 2: Comparison of finite difference scheme (FD) (Section 3.4) and the heat equation formula (HE) (Section 3.5). The left panel captures the situation with fixed $\delta_\tau = 2^{-6}$ and the right panel depicts the case with fixed $\delta_x = 2^{-6}$.

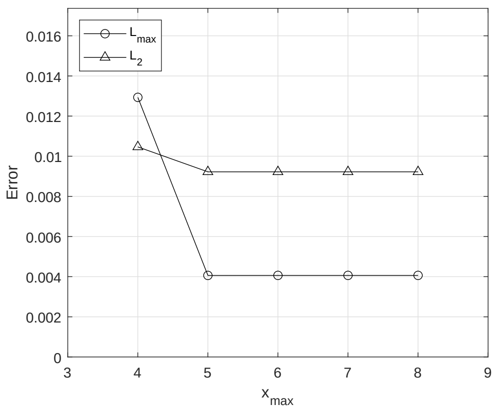

Figure 3: Comparison of finite difference scheme (FD) (Section 3.4) and the heat equation formula (HE) (Section 3.5) with $\delta_x = 2^{-6}$, $\delta_\tau = 2^{-3}$ and $x_{\max}$ variable.

When dealing with one underlying only, Monte-Carlo simulation can be performed using the Brownian bridge pinned at t and $t - t_0$ which leads to the same approximative underlying value at $t - t_0$. Also, one needs to adjust the time discretization in the computation of the integral appropriately for the partition points to hit exactly the past time $t - t_0$.

The model parameters are unchanged compared to the previous experiments except for $s_X = 0.02$. This is a necessary adjustment. Indeed, if we denote $X_1 = V(t - t_1, S(t - t_1))$, $X_2 = V(t-t_2, S(t-t_2))$ for any delay terms $t_1, t_2 \in \mathbb{R}_0^+$ then since $V - X_i \leq 0$ for $i = 1, 2$, the difference of the right hand side of (1) can be expressed as

$$f_1 - f_2 = (-\lambda_B(1 - R_B) + s_X)(X_1 - X_2).$$

If $\lambda_B(1 - R_B) = s_X$ the values of the same derivatives are equal regardless of the delay t_i.

The delay term was set to be $t_0 = 1/252$ (assuming 252 trading days per year). The difference between the values of the derivatives with no delay ($X = V$) and with delay ($X = V(t - t_0, S(t - t_0)))$ is depicted in the left panel of Figure 5. Similar test in the virtue of Figure 4 for the heat equation formulation and the Monte-Carlo simulation is show in the right panel of Figure 5.

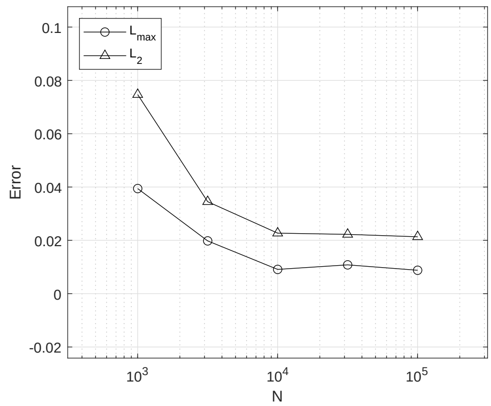

Figure 4: Comparison of the Monte-Carlo solution (MC) (Section 3.6) and the heat equation formula (HE) (Section 3.5) for variable number of simulated paths N.

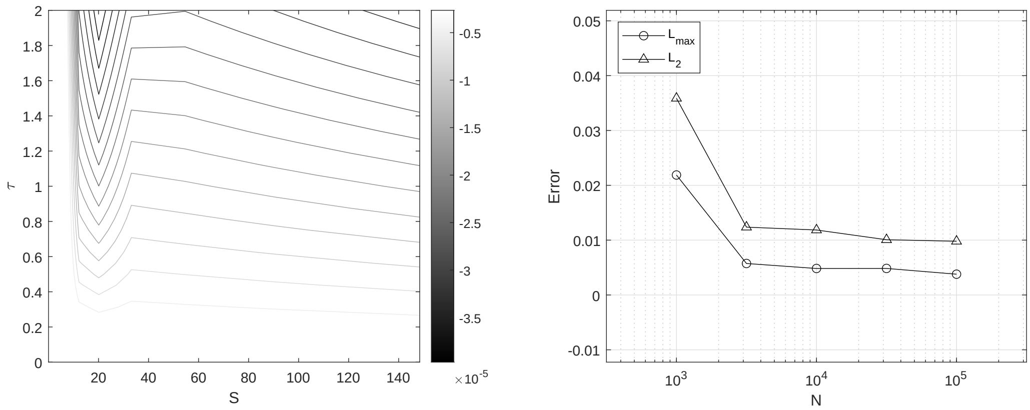

Figure 5: Left panel: Contour plot of the difference of the valued of the European call option with the undelayed collateral $X = V$ and the previous-day delayed collateral $X = V(t - t_0, S(t - t_0))$ with $t_0 = 1/252$. Right panel: The comparison of the heat equation formula and the Monte-Carlo simulation for the delayed collateral.

5 Conclusion

The aim of this paper was to bring attention to analytical solutions of the partial differential equation used in value adjustment modelling. Although the PDE approach is known for more than a decade (Burgard and Kjaer, 2011b), presented derivation of semi-closed formulas has not been published until now. When dealing with one underlying asset only, we can easily apply the transformation of the xVA PDE to the heat equation with known solution and hence obtain a fast and reliable semi-closed pricing formula that confirmed to be consistent with the Monte-Carlo simulations and with the numerical solution of the PDE obtained by the finite differences method.

A special attention was paid to the example with collateral taken as the value from the past, typically one day back is considered in practice. As a further research it might be interesting to study also what happens if the collateral is posted daily, weekly, monthly, etc. This however poses new analytical and numerical challenges, since the inhomogeneity f is then discontinuous.

It is worth mentioning that xVA is often considered in a multi-dimensional setting, i.e., with multiple risk factors whose driving Wiener processes are instantaneously correlated. Such a setting then leads to a PDE with multiple spatial dimensions that could also be suited for transformation to heat equation as presented in this paper. A detailed study of the multi-dimensional setting was not in the scope of this paper but will be a subject of future research. Nevertheless, our analytic approach still might be of practical interest in the sense that one may need to study the valuation adjustment for each of the underlyings independently in order to identify their individual contribution to the risk. In this case, an analytic formula is definitely much more convenient than any Monte-Carlo simulation. On the other hand, calculating xVA for highly complex portfolio is nowadays performed only by means of Monte-Carlo and Quasi-Monte-Carlo simulations.

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