Much ado about 248

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Abstract

In this note we present three representations of a 248-dimensional Lie algebra, namely the algebra of Lie point symmetries admitted by a system of five trivial ordinary differential equations each of order forty-four, that admitted by a system of seven trivial ordinary differential equations each of order twenty-eight and that admitted by one trivial ordinary differential equation of order two hundred and forty-four.

1 Introduction

A system of n ordinary differential equations each of order $M > 1$,

$$u_k^{(M)} = f_k(u_j^{(s)}, t), \quad j,k = 1,n, \quad s = 0, M-1, \tag{1}$$

has a variable number of Lie point symmetries depending upon the structure of the functions f_k. The maximal dimension D of the algebra of admitted Lie point symmetries can be obtained by the formulæ [9]

$$M = 2 \implies D = n^2 + 4n + 3 \tag{2}$$
$$M > 2 \implies D = n^2 + Mn + 3. \tag{3}$$

Some explicit numbers are given in Table 1.

Recently the elaboration of the elements of the Lie algebra, $E8$, of order 248 has been variously announced [3, 7, 13, 17, 16] in the serious popular media. The authoritative source is the Atlas of Lie Groups and Representations [2] which is funded by the National Science Foundation through the American Institute of Mathematics [1]. The results of the E8 computation were announced in a talk at MIT by David Vogan on Monday, March 19, 2007, and the details may be found at [15]. The Atlas of Lie Groups and Representations is a project to make available information about representations of semisimple Lie groups over real and p-adic fields. Of particular importance is the problem of the unitary dual, *ie* the classification of all of the irreducible unitary representations of a given Lie group. The goal of the Atlas of Lie

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Table 1: : The maximal dimension of the algebra of admitted Lie point symmetries for systems of equations of varying order (horizontal) and number (vertical).

$n \backslash M$	2	3	4	5	6	7	8	9	10
1	8	7	8	9	10	11	12	13	14
2	15	13	15	17	19	21	23	25	27
3	24	21	24	27	30	33	36	39	42
4	35	31	35	39	43	47	51	55	59
5	48	43	48	53	58	63	68	73	78
6	63	57	63	69	75	81	87	93	99
7	80	73	80	87	94	101	108	115	122
8	99	91	99	107	115	123	131	139	147
9	120	111	120	129	138	147	156	165	174
10	143	133	143	153	163	173	183	193	203

Groups and Representations is to classify the unitary dual of a real Lie group, G, by computer. A step in this direction is to compute the admissible representations of G including their Kazhdan-Lusztig-Vogan polynomials. The computation for $E8$ was an important test of the technology. While the computation is an impressive achievement, it is only a small step towards the unitary dual and should not be ranked as important as the original work of Kazhdan, Lusztig, Vogan, Beilinson, Bernstein *et al.* (See for example [4, 5, 6, 11, 12, 14, 18, 8].) Nevertheless the result was regarded as being suitable for a concerted campaign of publicity to heighten awareness of Mathematics in the community at large:
“Symmetrie ist möglicherweise das erfolgreichste Prinzip der Physik überhaupt” [7].
“Un groupe de chercheurs américains et européens, parmi lesquels on trouve deux Français, est parvenu à décoder une des structures les plus vastes de l’histoire des mathématiques” [13].
“It may be that some day this calculation can help physicists to understand the universe” [17].
“Eighteen mathematicians spent four years and 77 hours of supercomputer computation to describe this structure” [16].

In this note we demonstrate three representations of a Lie algebra of dimension 248. The two of us spent four hours and 77 seconds of pocket-calculator computation to describe these three structures.

2 Three simple systems

For $D = 248$ formula (2) does not have integral solutions and so there is no system of second-order ordinary differential equations of maximal symmetry possessing a 248-dimensional algebra of its Lie point symmetries[1]. About formula (3) the factors of 248-3=245 are 1, 5 and 7 (49 is out of question because $49^2 > 245$). Consequently

[1] Is this another instance of the intrinsically uniqueness of Classical Mechanics?

possible values of n are 1, 5 and 7. The corresponding values of M are 244, 44 and 28, respectively. The systems of maximal symmetry are easily obtained as one simply puts $f_k = 0 \; \forall \; k$. Thus the systems we construct are the simplest representations of the equivalence class under point transformation of systems of equations of maximal symmetry.

Firstly we consider the following system:

$$u_k^{(44)} = 0, \qquad k = 1, 5. \tag{4}$$

It is easy to show that this simple system admits a 248-dimensional algebra of its Lie point symmetries since $5^2 + 5 \cdot 44 + 3 = 248$. The algebra is generated by the operators

$$\begin{aligned} \Gamma_1 &= t^2\partial_t + 43t\textstyle\sum_{i=1}^5 u_i\partial_{u_i}, \\ \Gamma_2 &= t\partial_t, \\ \Gamma_3 &= \partial_t, \\ \Gamma_{i,k} &= u_k\partial_{u_i}, \quad k = 1, 5, \; i = 1, 5 \\ \Gamma_{i+5,s} &= t^s\partial_{u_i}, \quad s = 0, 43, \; i = 1, 5. \end{aligned} \tag{5}$$

Secondly we consider the system

$$u_r^{(28)} = 0, \qquad r = 1, 7. \tag{6}$$

This equally simple system admits a 248-dimensional algebra ($7^2 + 7 \cdot 28 + 3 = 248$) of its Lie point symmetries generated by

$$\begin{aligned} \Gamma_1 &= t^2\partial_t + 27t\textstyle\sum_{j=1}^7 u_j\partial_{u_j}, \\ \Gamma_2 &= t\partial_t, \\ \Gamma_3 &= \partial_t, \\ \Gamma_{j,r} &= u_r\partial_{u_j}, \quad r = 1, 7, \; j = 1, 7 \\ \Gamma_{j+7,n} &= t^n\partial_{u_j}, \quad n = 0, 27, \; j = 1, 7. \end{aligned} \tag{7}$$

Thirdly and finally the scalar equation,

$$u^{(244)} = 0, \tag{8}$$

admits a 248-dimensional Lie algebra ($1^2 + 1 \cdot 244 + 3 = 248$) of its point symmetries generated by the operators

$$\begin{aligned} \Gamma_1 &= t^2\partial_t + 243tu\partial_u, \\ \Gamma_2 &= t\partial_t, \\ \Gamma_3 &= \partial_t, \\ \Gamma_4 &= u\partial_u, \\ \Gamma_{n+5} &= t^n\partial_u, \quad n = 0, 243. \end{aligned} \tag{9}$$

3 Conclusion

We have demonstrated three representations of Lie algebras of dimension 248 which is the dimension of $E8$. Although the algebras we present are not simple, their method of construction is. The reason for this simplicity is that we used representations for systems of equations of maximal symmetry. We do not deny that larger systems, be that in order or number, of less than maximal symmetry could possibly have an algebra of dimension 248, but even on the assumption that such systems be linear the complexity of the calculation becomes immense [10] and defeats the purpose of the present note.

Note that we have used the simplest forms for the generators of the algebras of the three systems, (4), (6) and (8), for our primary interest is the demonstration of the existence of the algebras. Normally one would use combinations which reflect subalgebraic structures. For example in the case of (8) for which the algebra is obviously $sl(250, I\!R)$ one would replace Γ_2 with $\tilde{\Gamma}_2 = 2t\partial_t + 243u\partial_u$ to underline the subalgebraic structure $\{sl(2, I\!R) \oplus A_1\} \oplus_s 244A_1$, where Γ_1, $\tilde{\Gamma}_2$ and Γ_3 constitute a representation of $sl(2, I\!R)$, Γ_4 reflects the homogeneity of the equation in the dependent variable and the 244-element abelian subalgebra is composed of the solution symmetries, so called because the coefficient functions are solutions of (8).

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