Linear perturbations of matched spacetimes: the gauge problem and background symmetries

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Abstract

We present a critical review about the study of linear perturbations of matched spacetimes including gauge problems. We analyse the freedom introduced in the perturbed matching by the presence of background symmetries and revisit the particular case of spherically symmetry in n-dimensions. This analysis includes settings with boundary layers such as brane world models and shell cosmologies.

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1 Introduction

An important aspect in any geometric gravitational theory is the analysis of how to match two spacetimes. This is true in particular for General Relativity and its perturbation theory. Despite the relevance and maturity of the matching theory one often finds papers where the matching conditions are not properly used. Most of the difficulties arise from the fact that the matching conditions are imposed in specific coordinate systems in a manner which is not completely coordinate independent. More specifically, matching two spacetimes requires identifying the boundaries pointwise, and sometimes this identification is done implicitly by fixing spacetime coordinates, without paying enough attention to the fact that solving the matching involves *finding* an identification of the boundary and that this should not be fixed a priori.

In perturbation theory this problem also arises, and it gets complicated by the fact that the fields to be matched (as the perturbed metric) are gauge dependent. So, in addition to a priori choices of identifications of the boundary, there is also the problem that particular gauges are often used. It may be argued that the matching theory must be gauge independent and therefore it can be performed in any gauge. This is true, but only when due care is taken to ensure that the choice of gauge does not restrict, a priori, the perturbed identification of the boundaries.

A complete description of the linearized matching conditions has been achieved only recently by Carter and Battye [5] and independently by Mukohyama [6]. To second order, the matching conditions have been recently found in [7]. Despite these papers, we believe that some confusion still lingers in the field, in particular with respect to the existing gauge invariant formulations. The aim of this paper is to try to clarify these issues. In order to do that, we will critically discuss some of the approaches proposed in the literature trying to make clear which are the implicit assumptions made and to what extent are they justified.

The first papers discussing the perturbed matching theory are, as far as we know, the classic papers by Gerlach and Sengupta [2, 3]. However, as explained below, their description of the perturbed matching theory contains imprecisions, and we will therefore start discussing their approach pointing out the difficulties they encounter. A first attempt to justify the claims in [2, 3] is due to Martín-García and Gundlach [4], who propose a different but nevertheless closely related set of linearized matching conditions. Pointing out the implicit assumption made by these authors will also help us to try to explain the subtleties inherent to the perturbed matching theory.

In [6] the linearized matching conditions are described for arbitrary backgrounds, perturbations and matching hypersurfaces, and then applied to the case of two background spacetimes with a high degree of symmetry, namely those which admit a maximal group of isometries acting on codimension two spacelike submanifolds (e.g. spherically symmetric spacetimes). In order to simplify the matching conditions, Mukohyama derives a set of matching conditions for so-called *doubly gauge invariants*. However, a gap arises in his final conclusions as the presented set of conditions for doubly gauge invariant quantities for the linearized matching of spacetimes are only shown to be *necessary conditions*. Analysing sufficiency touches directly on the issue we are trying to emphasize in this paper, so we devote one section to clarify this point, where we show how these conditions are, stricktly speaking, *not sufficient*. Since the matching conditions in terms of doubly gauge invariants are widely used in the literature, we consider important to close this gap. Moreover, the constructions of gauge invariant quantities using spherical harmonic decompositons leaves out the $l = 0$ and $l = 1$ sectors. We will discuss this issue and its consequences.

The paper is organized as follows. We start by summarising the perturbed matching conditions in Section 2, where we also describe the gauge freedom involved. Then, the procedures used in the classic papers [2, 3] together with the justifications and further developments in [4] are reviewed in Section 3. Section 4 focuses on the consequences of the existence of symmetries in the background configuration, which will have relevance in our final discussion. Section 5 has three subsections. The first one is devoted to present briefly the procedure and results discussed in [6] particularised to the case of spherically symmetric backgrounds. In the second subsection we analyse the sufficiency of the doubly gauge invariant matching conditions in [6]. The last subsection is devoted to the study of the freedom left in the perturbation of the matching hypersurface once the metric perturbations have been fixed at both sides. We finish with an appendix where explicit expressions for the discontinuities of the perturbed second fundamental forms in the spherical case are given. Some of these expressions are used in the main text.

2 Linearized matching

In this section we describe the gauge freedom involved in the linearised spacetime matching and summarise the perturbed matching conditions.

2.1 Gauge freedom

The purpose of the matching theory is to construct a new spacetime out of two spacetimes $M^{\pm}$ with boundary by *finding* a suitable diffeomorphism between the boundaries which allows for their pointwise identification. In particular, the matched spacetime cannot be thought to exist beforehand. Another aspect to bear in mind is that the matching conditions involve exclusively tensors on the identified boundary Σ and hence any coordinate system in $M^{\pm}$ is equally valid. This is well-known but it is still source of confusion sometimes.

In perturbed matching theory, not only the metrics are perturbed but also the matching hypersurfaces may be deformed. Furthermore, as for the metric, the "deviation" of the matching hypersurface is also a gauge dependent quantity. This can be best understood by viewing perturbations as ε-derivatives (at $\varepsilon = 0$) of a one-parameter family of spacetimes $(M_{\varepsilon}^{+}, g_{\varepsilon}^{+})$ with boundary Σ_{ε}^{+}. It is convenient to embed M_{ε}^{+} within a larger manifold (without boundary) V_{ε}^{+} to clarify the discussion. A priori, the manifolds $(M_{\varepsilon}^{+}, g_{\varepsilon})$ are completely distinct so it makes no direct sense to talk about ε-derivatives. It is necessary to identify first the different manifolds so that a single point p refers to one point on each of the manifolds. Obviously, there are infinite ways to identify the manifolds, all of them equally valid a priori. This freedom leads to the gauge dependence of the perturbed metric (and of any other geometrically defined tensor). The identification above may, or may not, map the boundaries Σ_{ε}^{+} among themselves. A priori, a point in Σ_{0}^{+} may be mapped, for $\varepsilon \neq 0$, to a point on Σ_{ε}^{+}, to a point interior to M_{ε}^{+} or to a point exterior to M_{ε}^{+} (within the extension V_{ε}^{+}) which is not part of the manifold. How can we then take derivatives with respect to ε at those later points? Since only derivatives at $\varepsilon = 0$ are needed, restricting to infinitesimal values of ε entails no loss of generality. Then, if for some small ε, a point $q \in \Sigma_{0}^{+}$ is mapped to the exterior of M_{ε}^{+}, it follows from differentiability with respect to ε that q is mapped, for the reverse value $-\varepsilon$, to a point interior to M_{ε}^{+}. Thus, perturbations can be defined at the boundary by taking one sided derivatives, i.e. to take limits $\varepsilon \longrightarrow 0$, with a sign restriction on ε (c.f. [7] for an alternative discussion).

However, an important issue remains: How do we describe the deformation of the boundary Σ_{0}^{+}? As a set of points each boundary Σ_{ε}^{+} maps, with the above identification, into a hypersurface of the background spacetime, which we call $\hat{\Sigma}_{\varepsilon}^{+}$. In general, this hypersurface will not coincide with Σ_{0}^{+} and may well touch it or cross it. This gives us an idea of how the boundary is deformed, but only as a subset, not pointwise. In order to know how the boundary actually moves within the background, we need to prescribe a priori a pointwise identification of Σ_{0}^{+} with Σ_{ε}^{+}. This identification is completely different and independent from the one described above involving spacetime points, and involves only the points on the boundaries. As before, there are infinitely many ways to identify the boundaries, and this defines a second gauge freedom, which involves objects intrinsically defined on the boundary. This gauge freedom will be referred as *hypersurface gauge*, as opposed to the usual *spacetime gauge* described above.

With both identifications chosen, the deformation of the boundary within the background can already be described: Fix a point q on the background boundary Σ_0^+. The identification of the boundaries defines a point q_ε on Σ_ε^+, for each ε. The spacetime identification takes this point q_ε and maps it into a point $\hat{q}_\varepsilon$ of the background M_0^+ (perhaps after a sign restriction on ε). Obviously $\hat{q}_\varepsilon$ belongs to the perturbed hypersurface $\hat{\Sigma}_\varepsilon^+$. We have therefore not only a deformation of the background hypersurface as a set of points, but also pointwise information. It only remains to take the tangent vector of the curve $\hat{q}_\varepsilon$ at $\varepsilon = 0$, i.e. $\vec{Z}^+ = \frac{d\hat{q}_\varepsilon}{d\varepsilon}|_{\varepsilon=0}$ which encodes completely the deformation of the matching hypersurface as seen from the background spacetime. Two final remarks are in order: (i) $\vec{Z}^+$ is defined exclusively on Σ_0^+, no extension thereof is defined or required and (ii) $\vec{Z}^+$ depends on both the spacetime and hypersurface gauges, since its defining curve is constructed using both identifications. However, decomposing $\vec{Z}^+ = Q^+ \vec{n}_+^0 + \vec{T}^+$, where $\vec{n}_+^0$ is the unit normal of Σ_0^+ (assumed non-null anywhere) and $\vec{T}^+$ is tangent to it, it turns out that Q^+ depends on the spacetime gauge but not on the hypersurface gauge. This is because changing the hypersurface gauge reorganizes the points within each $\hat{\Sigma}_\varepsilon^+$, but cannot modify any of them as a set of points.

Tensors defined intrinsically on the boundaries Σ_ε^+ are completely unaffected by the spacetime identification, and are therefore invariant under spacetime gauge transformations. Recall that the matching conditions involve only objects intrinsic to the matching hypersurfaces. Since the perturbed matching conditions are, formally, just their ε-derivatives, it follows by construction that the perturbed matching conditions must be gauge invariant under spacetime gauge transformations. This may seem surprising at first sight since the matching conditions must involve the perturbed metric, which is obviously gauge dependent. However, the conditions turn out to be gauge independent because they also involve the deformation vector $\vec{Z}^+$, which is spacetime gauge dependent. This vector is therefore of fundamental importance and must be taken into account in any sensible approach to the problem, as we shall see next.

2.2 Matching conditions

Let $(M_0^\pm, g_0^\pm)$ be n-dimensional spacetimes with non-null boundaries $\Sigma_0^\pm$. Matching them requires an identification of the boundaries, i.e. a pair of embeddings $\Phi_\pm : \Sigma_0 \longrightarrow M_0^\pm$ with $\Phi_\pm(\Sigma_0) = \Sigma_0^\pm$, where Σ_0 is an abstract copy of any of the boundaries. Let ξ^i $(i, j, \ldots = 1, \ldots, n-1)$ be a coordinate system on Σ_0. Tangent vectors to $\Sigma_0^\pm$ are obtained by $e_i^{\pm\alpha} = \frac{\partial \Phi_\pm^\alpha}{\partial \xi^i}$ $(\alpha, \beta, \ldots = 0, \ldots, n-1)$. There are also unique (up to orientation) unit normal vectors $n_\pm^{(0)\alpha}$ to the boundaries. We choose them so that if $n_+^{(0)\alpha}$ points towards M^+ then $n_-^{(0)\alpha}$ points outside of M^- or viceversa. The first and second fundamental are simply ${q^{(0)}}_{ij}{}^\pm \equiv e_i^{\pm\alpha} e_j^{\pm\beta} {g^{(0)}}_{\alpha\beta}|_{\Sigma^\pm}$, $\quad {K^{(0)}}_{ij}{}^\pm = -n_{\pm\alpha}^{(0)} e_i^{\pm\beta} \nabla_\beta^\pm e_j^{\pm\alpha}$. The matching conditions (in the absence of shells) require the equality of the first and second fundamental forms on $\Sigma_0^\pm$, i.e.

$$ {q^{(0)}}_{ij}{}^+ = {q^{(0)}}_{ij}{}^-, \quad {K^{(0)}}_{ij}{}^+ = {K^{(0)}}_{ij}{}^-. \tag{1} $$

Under a perturbation of the background metric $g_{pert}^\pm = g^{(0)\pm} + g^{(1)\pm}$ and of the matching hypersurfaces via $\vec{Z}^\pm = Q^\pm \vec{n^{(0)}}_\pm + \vec{T}^\pm$, the matching conditions will be perturbatively satisfied if and only if [6]

$$q^{(1)+}_{ij} = q^{(1)-}_{ij}, \qquad K^{(1)+}_{ij} = K^{(1)-}_{ij}, \tag{2}$$

with

$$\begin{aligned} q^{(1)}{}_{ij}{}^{\pm} &= \mathcal{L}_{\vec{T}^{\pm}} q^{(0)}{}_{ij}{}^{\pm} + 2Q^{\pm} K^{(0)}{}_{ij}{}^{\pm} + e_i^{\pm\alpha} e_j^{\pm\beta} g^{(1)}{}_{\alpha\beta}{}^{\pm}, \qquad (3)\\ K^{(1)}{}_{ij}{}^{\pm} &= \mathcal{L}_{\vec{T}^{\pm}} K^{(0)}{}_{ij}{}^{\pm} - \epsilon D_i D_j Q^{\pm} + Q^{\pm}(-n_{\pm}^{(0)\mu} n_{\pm}^{(0)\nu} R^{(0)\pm}_{\alpha\mu\beta\nu} e_i^{\pm\alpha} e_j^{\pm\beta} + K^{(0)}{}_{il}{}^{\pm} K^{(0)}{}^{l}_{j}{}^{\pm})\\ &+ \frac{\epsilon}{2} g^{(1)}{}_{\alpha\beta}{}^{\pm} n_{\pm}^{(0)\alpha} n_{\pm}^{(0)\beta} K^{(0)}{}_{ij}{}^{\pm} - n^{(0)}_{\pm\ \mu} S^{(1)\pm\mu}_{\alpha\beta} e_i^{\pm\alpha} e_j^{\pm\beta}, \qquad (4) \end{aligned}$$

where $\epsilon = n^{(0)}_{\alpha} n^{(0)\alpha}$, D is the covariant derivative of $(\Sigma, q^{(0)\pm})$ and $S^{(1)}{}^{\pm\alpha}_{\beta\gamma} \equiv \frac{1}{2}(\nabla^{\pm}_{\beta}\, g^{(1)}{}^{\pm\alpha}_{\gamma} + \nabla^{\pm}_{\gamma}\, g^{(1)}{}^{\pm\alpha}_{\beta} - \nabla^{\pm\,\alpha}\, g^{(1)}{}^{\pm}_{\beta\gamma})$.[1]

In these equations, $Q^{\pm}$ and $\vec{T}^{\pm}$ are a priori unknown quantities and fulfilling the matching conditions requires *showing* that two vectors $\vec{Z}^{\pm}$ exist such that (2) are satisfied. The spacetime gauge freedom can be exploited to fix either or both vectors $\vec{Z}^{\pm}$ a priori, but this should be avoided (or at least carefully analysed) if additional spacetime gauge choices are made, in order not to restrict a priori the possible matchings. Regarding the hypersurface gauge, this can be used to fix one of the vectors $\vec{T}^{+}$ or $\vec{T}^{-}$, but not both.

As already stressed the linearized matching conditions are by construction spacetime gauge invariant (in fact each of the tensors $q^{(1)}{}_{ij}{}^{\pm}$, $K^{(1)}{}_{ij}{}^{\pm}$ is). Moreover, the set of conditions (2) are hypersurface gauge invariant, provided the background is properly matched, since [6] under such a gauge transformation given by the vector $\vec{\zeta}$ on Σ_0, $q^{(1)}{}_{ij}$ transforms as $q^{(1)}{}_{ij} + \mathcal{L}_{\vec{\zeta}} q^{(0)}{}_{ij}$, and similarly for $K^{(1)}{}_{ij}$.

3 On previous spacetime gauge invariant formalisms

The first attempt to derive a general formalism for the matching conditions in linearized gravity is, to our knowledge, due to Gerlach and Sengupta [2]. Their approach is based on the description of the matching hypersurface Σ as a level set of a function f defined on the spacetime. Assuming the level sets $\{f = \text{const}\}$ to be timelike, a field of spacelike unit normals is defined as $n_\mu = (g^{\alpha\beta} f_{,\alpha} f_{,\beta})^{-1/2} f_{,\mu}$. The unperturbed matching conditions correspond to the continuity everywhere (in particular across Σ) of the tensors

$$q_{\alpha\beta} \equiv g_{\alpha\beta} - n_\alpha n_\beta, \qquad K_{\alpha\beta} \equiv q_\alpha{}^\mu q_\beta{}^\nu \nabla_\mu n_\nu, \tag{5}$$

which are the spacetime versions of the first and second fundamental forms introduced above. Being f defined everywhere, it makes sense to perturb it in order to describe the variation of the matching hypersurface. Obviously, by perturbing f one also perturbs n_μ. The perturbed matching conditions proposed in [2] read

$$q_\mu{}^\alpha q_\nu{}^\beta \triangle(q_{\alpha\beta})^+ = q_\mu{}^\alpha q_\nu{}^\beta \triangle(q_{\alpha\beta})^-, \qquad q_\mu{}^\alpha q_\nu{}^\beta \triangle(K_{\alpha\beta})^+ = q_\mu{}^\alpha q_\nu{}^\beta \triangle(K_{\alpha\beta})^-, \tag{6}$$

where $q_\beta{}^\alpha$ is the projector onto Σ, $\triangle$ stands for perturbation and $+$ and $-$ denote the quantities as computed from either side of the matching hypersurface Σ. These expressions involve the projections of the perturbations of $q_{\alpha\beta}$ and $K_{\alpha\beta}$ onto Σ. The need of

[1] We will abuse slightly the notation and refer to vectors on Σ_0 and their images on spacetime with the same symbol. The meaning should be clear from the context.

considering only the projected components is justified in [2] since the matching conditions need to be intrinsic to the matching hypersurfaces. However, Gerlach and Sengupta themselves note that conditions (6) are not gauge[2] invariant.

Since the main interest in [2, 3] refers to spherically symmetric backgrounds, this "ambiguity" is fixed in that case by finding suitable gauge invariant combinations of the linearized matching conditions, which turn out to give a correct set of necessary perturbed matching conditions in spherical symmetry. It should be stressed however, that the authors consider these gauge invariant subset to be sufficient also, with no further justification.

We know from the discussion in Sect. 2.1 above that (6) cannot be correct as it leads to a set of gauge dependent conditions. Since, on the other hand the proposal (6) may look plausible, it is of interest to point out where, and in which sense, it fails to be correct.

The first source of problems comes from assuming that the matched spacetime is given beforehand. Indeed, $q_{\alpha\beta}$ and $K_{\alpha\beta}$ are spacetime tensors and they can only exist (and be continuous) once the matched spacetime is constructed. But this is precisely the purpose of the matching conditions, so the conditions become circular. Another aspect of the same problem is that one can only talk about continuity once the pointwise identification of the boundaries is chosen. But a level set of a function defines only a set of points and not the way those points must be identified. A third instance of the same issue is that tensor components must be expressed in some basis, e.g. a common coordinate system covering both sides of Σ. But again this cannot be assumed a priori. It needs to be constructed.

Let us however mention that once the pointwise identification of the boundaries is chosen, the use of spacetime tensors is allowed provided they are finally projected onto the hypersurface. In that sense, and when properly used, using spacetime indices may simplify some calculations notably (see Carter and Battye, [5] where this notation is used to derive the perturbed matching conditions).

Besides this aspect (which already affects the background matching) the perturbed equations (6) suffer from one extra problem. The perturbations $\triangle(q_{\alpha\beta})(p)$ and $\triangle(K_{\alpha\beta})(p)$ at a point p in the background can be defined by taking ε-derivatives at fixed p and $\varepsilon = 0$ of the corresponding tensors (defined by $g_{\alpha\beta}(\varepsilon)$ and f_ε). For each value of ε, the matching conditions impose the continuity of $q_{\alpha\beta}(\varepsilon)$ and $K_{\alpha\beta}(\varepsilon)$ everywhere (with the caveat already mentioned regarding the identification of the boundaries). However, continuity of $\triangle(q_{\alpha\beta})$ and $\triangle(K_{\alpha\beta})$ at p would only follow if derivatives of continuous functions with respect to an external parameter were necessarily continuous (in our case, the derivative with respect to ε), which is not true in general. A trivial example is given by the function $u(\varepsilon, x) = |x + \varepsilon|$, whith $x \in \mathbb{R}$. For each ε this function is continuous. However, the derivative with respect to ε does not even exist at $x = 0, \varepsilon = 0$. This reflects the fact that subtracting continuous tensors at a fixed spacetime point p leads to objects that need not be continuous. This is in fact the main problem of (6) as linearized matching conditions.

An immediate question arises: Why is the gauge invariant subset of matching conditions found in [2, 3] for spherically symmetric backgrounds correct? In order to understand this, let us rewrite (6) using the formalism of section 2.2. First of all, since $\triangle(n_\alpha n_\beta)$ will contain, at least, one free $n^{(0)}_\alpha$, we have

$$q_\mu{}^\alpha q_\nu{}^\beta \triangle(q_{\alpha\beta})^{\pm} = q_\mu{}^\alpha q_\nu{}^\beta g^{(1)}{}^{\pm}_{\alpha\beta}. \tag{7}$$

[2]Throughout this section gauge will refer to spacetime gauge. Hypersurface gauges will only appear briefly towards the end of the section.

Moreover, a simple calculation gives $\triangle(\nabla_\alpha n_\beta) = \nabla_\alpha(\triangle n_\beta) - S^{(1)\mu}_{\alpha\beta} n^{(0)}_\mu$ and $\triangle(q^\alpha_\beta) = -g^{(1)\alpha\mu} n^{(0)}_\mu n^{(0)}_\beta + g^{(0)\alpha\mu}\triangle(n_\mu) n^{(0)}_\beta + n^{(0)\alpha}\triangle(n_\beta)$. These, together with standard properties of the projector, lead to

$$q_\mu{}^\alpha q_\nu{}^\beta \triangle(K_{\alpha\beta})^\pm = \left(a^{(0)}_\nu q_\mu{}^\alpha \triangle(n_\alpha) + q_\mu{}^\alpha q_\nu{}^\beta \nabla_\alpha(\triangle n_\beta) - q_\mu{}^\alpha q_\nu{}^\beta S^{(1)\rho}_{\alpha\beta} n^{(0)}_\rho \right)^\pm, \tag{8}$$

where $a^{(0)}_\nu \equiv n^{(0)\alpha}\nabla_\alpha n^{(0)}_\nu$. In general, these expressions do not agree with (3) and (4). However, when the gauges are chosen so that $\vec{Z}^\pm = 0$, then $\triangle f \equiv 0$ on Σ because the matching hypersurface is unperturbed as seen from the background. Consequently $\partial_\alpha(\triangle f) \propto n^{(0)}_\alpha$ on Σ, which implies $\triangle(n_\alpha) \stackrel{\Sigma}{=} h n^{(0)}_\alpha$ for some function h. Imposing $\vec{n}(\varepsilon)$ to be unit for all ε fixes $h = \frac{\epsilon}{2} g^{(1)\alpha\beta} n^{(0)}_\alpha n^{(0)}_\beta$. Inserting into (8) the matching conditions (6) become

$$\begin{aligned} \left(q_\mu{}^\alpha q_\nu{}^\beta g^{(1)}{}_{\alpha\beta}\right)^+ &= \left(q_\mu{}^\alpha q_\nu{}^\beta g^{(1)}{}_{\alpha\beta}\right)^-, \\ \left(\frac{\epsilon}{2} g^{(1)\alpha\beta} n^{(0)}_\alpha n^{(0)}_\beta K_{\mu\nu} - q_\mu{}^\alpha q_\nu{}^\beta S^{(1)\rho}_{\alpha\beta} n^{(0)}_\rho\right)^+ &= \left(\frac{\epsilon}{2} g^{(1)\alpha\beta} n^{(0)}_\alpha n^{(0)}_\beta K_{\mu\nu} - q_\mu{}^\alpha q_\nu{}^\beta S^{(1)\rho}_{\alpha\beta} n^{(0)}_\rho\right)^-, \end{aligned} \tag{9}$$

which agree with (2) (with the exception that (9) refers to spacetime tensors and (2) are defined on Σ). Since Gerlach and Sengupta derive a subset of gauge invariant matching conditions out of (6) in the spherically symmetric case and their conditions are correct in one gauge, it follows that the invariant subset is correct in any gauge. This is the reason why the results in [2, 3] involving spherically symmetric backgrounds turn out to be fine.

Substantial progress in the linearized matching problem was made by Martín-García and Gundlach [4]. These authors pointed out the lack of justification in [2, 3] for the choice of (6) as matching conditions. It was also argued that for spacetimes with boundary it only makes sense to define perturbations by using gauges where the perturbed matching hypersurface is mapped onto the background matching hypersurface. Perturbations in this gauge, called "surface gauge" (not to be confused with hypersurface gauge) are denoted by $\bar{\triangle}$, and its defining property is $\bar{\triangle} f = 0$. The idea was to write down the matching conditions in this gauge and then transform into any other gauge if necessary. As noticed by the authors, the surface gauge is not unique since there are still three degrees of freedom left, which correspond to the three directions tangent to Σ.

A relevant observation made in [4] was that the continuity of tensorial perturbations may depend on the index position in the tensors. The authors argue that the tensors truly intrinsic to the hypersurfaces are $q^{\alpha\beta}$, $K^{\alpha\beta}$ (with indices upstairs) and propose the following perturbed matching conditions

$$\bar{\triangle}(q^{\alpha\beta})^+ = \bar{\triangle}(q^{\alpha\beta})^-, \qquad \bar{\triangle}(K^{\alpha\beta})^+ = \bar{\triangle}(K^{\alpha\beta})^-, \tag{10}$$

which are demonstrated to become exactly (9). This shows the equivalence of both proposals in the surface gauge, as explicitly stated in [4]. This justifies partially the validity of both approaches in the surface gauge. However, the justification is not complete because of the issue we discuss next.

Indeed, conditions (10) still carry one implicit assumption that needs to be clarified. As already stressed the perturbed matching conditions have two inherent and independent degrees of gauge freedom. The approach by Martín-García and Gundlach involves only spacetime objects, and therefore can only notice the spacetime gauge freedom. This leads to an incorrect statement in [4], as it is not true that the linearized matching conditions read (10) in *any* surface gauge. Conditions (10) will only be valid when the spacetime gauge maps pairs of background points (identified, via the background matching) to pairs of points on the perturbed boundaries $\Sigma_\varepsilon^\pm$ which are also identified through the matching. Notice that not all surface gauges have this property. In explicit terms, this means that the vectors $\vec{Z}^\pm$ must (i) only have tangential components (so that we are in surface gauge) and (ii) have the same components when written in terms of an intrinsic basis of Σ_0. In less precise, but more intuitive terms, condition (ii) states that $\vec{Z}^+$ and $\vec{Z}^-$ are the same vector, i.e. that the gauges in both regions are chosen such that the displacement of one fixed point of the background hypersurface is identical in both regions (the displaced point, of course, stays on the unperturbed hypersurface, due to the choice of surface gauge). Observe finally that if $Q^\pm = 0$ and $\vec{T}^+ = \vec{T}^-$, then the linearized matching conditions (2) truly reduce to conditions (9), once the latter are projected on Σ. This shows the correctness of the approaches by Gerlach and Sengupta and Martín-García and Gundlach in special gauges.

4 Freedom in matching due to symmetries

We devote this section to the study of the consequences of the existence of background symmetries on perturbed spacetime matchings.

The existence of symmetries in the background configuration introduces two issues which are important to take into consideration: the first corresponds to the freedom introduced by the matching procedure, when preserving the symmetries, at the background level [9], c.f. [10] for an application. The second issue corresponds to the consequences that the symmetries in the background configuration may have on the perturbation of the matching.

It must be stressed here that the arbitrariness introduced by the presence of symmetries in the background configuration is completely independent from both the hypersurface and spacetime gauge freedoms. However, that arbitrariness is gauge dependent and therefore a gauge choice can be made to remove it. As we will show, an isometry in the background implies that there is a direction along which the difference $[\vec{T}] \equiv \vec{T}^+ - \vec{T}^-$ cannot be determined by the perturbed matching equations. But, as we have discussed at the end of section 2, one could eventually choose part of the spacetime gauges (if there is any freedom left) to fix $[\vec{T}]$. Note, finally, that a change of hypersurface gauge leaves $[\vec{T}]$ invariant.

4.1 Isometries

We shall now consider the presence of isometries in the background configuration. So, let us assume that one of the sides, say $(M_0^+, g^{(0)+})$, admits an isometry generated by the Killing vector field $\vec{\xi}^+$ tangent to the boundary Σ_0^+. The commutation of the Lie derivative and the pull-back implies [9]

$$\mathcal{L}_{\vec{\xi}^+} {q^{(0)}}_{ij}{}^+ = e_i^{+\alpha} e_j^{+\beta} \mathcal{L}_{\vec{\xi}^+} {g^{(0)}}_{\alpha\beta}{}^+|_{\Sigma_0} = 0,$$

which means that $\vec{\xi}^+$ is a Killing vector of $(\Sigma_0, {q^{(0)}}_{ij}{}^+)$. This implies from expression (3) that ${q^{(1)}}_{ij}{}^+$ is invariant under the transformation $\vec{T}^+ \to \vec{T}^+ + \vec{\xi}^+|_{\Sigma_0}$.

As for ${K^{(1)}}_{ij}{}^{+}$, from its expression (4), it is again clear that the previous transformations of $\vec{T}^{+}$ will leave ${K^{(1)}}_{ij}{}^{+}$ invariant provided $\mathcal{L}_{\vec{\xi}^{+}}{K^{(0)}}_{ij}{}^{+}=0$. But this is precisely the case since $\vec{\xi}^{+}$ is a Killing vector orthogonal to $\boldsymbol{n}^{(0)}_{+}$, which implies $\mathcal{L}_{\vec{\xi}^{+}}\boldsymbol{n}^{(0)}_{+}{}_{\beta}|_{\Sigma_0^+}=0$, and hence

$$\mathcal{L}_{\vec{\xi}^{+}}{K^{(0)}}_{ij}{}^{+}=e_i^{+\alpha}e_j^{+\beta}\mathcal{L}_{\vec{\xi}^{+}}(\nabla\boldsymbol{n}^{(0)}_{+})_{\alpha\beta}|_{\Sigma_0}=e_i^{+\alpha}e_j^{+\beta}\nabla_\alpha\mathcal{L}_{\vec{\xi}^{+}}\boldsymbol{n}^{(0)}_{+}{}_{\beta}|_{\Sigma_0}=0.$$

Of course, all this discussion also applies to the $(-)$ side.

The combination of the invariance of ${q^{(1)}}_{ij}{}^{\pm}$ and ${K^{(1)}}_{ij}{}^{\pm}$ leads to the fact that the first order perturbed matching conditions are invariant under a change of the vectors $\vec{T}^{\pm}$ along the direction of any isometry of the background configuration (preserved by the matching). Then, as expected, when symmetries are present the linearized matching conditions cannot determine the difference $[\vec{T}]$ completely: they leave undetermined the relative (between the two sides) deformation of the hypersurface along the direction of the symmetry. Note that, still, the *shape* of the perturbed hypersurface is completely determined, since that is driven by $Q^{\pm}$.

The overall picture is as follows: at the background level we have the arbitrariness of the identification of Σ_0^+ with Σ_0^- [9], which can be seen as a "sliding" between Σ_0^+ and Σ_0^-. The perturbation adds to this an arbitrary shift of the deformation of the matching hypersurface at each side along the orbits of the isometry group. As an example, in the description of stationary and axisymmetric compact bodies discussed in [10, 9], the background sliding corresponds to an arbitrary constant rotation of the interior with respect to the exterior. Note that, in that case, this rotation is only relevant because the exterior is taken to be asymptotically flat. As a result, two identical interiors can, in principle, give rise to two exteriors that differ by a constant rate rotation [10]. The shift of the surface deformation would, in principle, lead to an arbitrary constant rotation along the axial coordinate of the surface deformation of the body. Likewise, two identical perturbations in the interior of the body may produce two different perturbations in the exterior, which may differ by a relative constant rate rotation. A choice of spacetime gauge could be used to relate the deformations inside and outside. However, this may interfere with other gauge fixings that may have been made.

5 n-dimensional spherically symmetric backgrounds

In this section we shall revisit Mukohyama's theory for linearized matching in the special case of spherical symmetry. Similar results [6] hold for backgrounds admitting isometry groups of dimension $(n-1)(n-2)/2$ acting on non-null codimension-two orbits of arbitrary topology (strictly speaking the orbits need to be compact).

5.1 The approach of Mukohyama

Concentrating on one of the two spacetimes to be matched, either $+$ or $-$, we consider a spherically symmetric background metric of the form

$$g^{(0)}_{\alpha\beta}dx^\alpha dx^\beta=\gamma_{ab}dx^a dx^b+r^2\Omega_{AB}d\theta^A d\theta^B, \tag{11}$$

where γ_{ab} $(a,b,..=0,1)$ is a Lorentzian two-dimensional metric (depending only on $\{x^a\}$), $r>0$ is a function of $\{x^a\}$, and $\Omega_{AB}d\theta^A d\theta^B$ is the $n-2$ dimensional unit sphere metric with coordinates $\{\theta^A\}$ $(A,B,\ldots=2,3,\ldots,n-1)$.

A general spherically symmetric background hypersurface can be given in parametric form as

$$\Sigma_0 := \{x^0 = Z^{(0)0}(\lambda), x^1 = Z^{(0)1}(\lambda), \theta^A = \vartheta^A\}, \tag{12}$$

where $\{\xi^i\} = \{\lambda, \vartheta^A\}$ is a coordinate system in Σ_0 adapted to the spherical symmetry. The tangent vectors to Σ_0 read

$$\vec{e}_\lambda = \dot{Z}^{(0)0}\partial_{x^0} + \dot{Z}^{(0)1}\partial_{x^1}\Big|_{\Sigma_0}, \quad \vec{e}_{\vartheta^A} = \partial_{\theta^A}|_{\Sigma_0}, \tag{13}$$

where dot is derivative w.r.t. λ. With $N^2 \equiv -\epsilon e_\lambda{}^a e_\lambda{}^b \gamma_{ab}|_{\Sigma_0}$, so that $\epsilon = 1$ ($\epsilon = -1$) corresponds to a timelike (spacelike) hypersurface, the unit normal to Σ_0 reads

$$\boldsymbol{n}^{(0)} = \frac{\sqrt{-\det\gamma}}{N}\left(-\dot{Z}^{(0)1}dx^0 + \dot{Z}^{(0)0}dx^1\right)\Big|_{\Sigma_0}, \tag{14}$$

where the sign choice of N corresponds to the choice of orientation of the normal. The background induced metric and second fundamental form on Σ_0 read

$$q^{(0)}{}_{ij}d\xi^i d\xi^j = -\epsilon N^2 d\lambda^2 + r^2|_{\Sigma_0}\Omega_{AB}|_{\Sigma_0}d\vartheta^A d\vartheta^B, \tag{15}$$

$$K^{(0)}{}_{ij}d\xi^i d\xi^j = N^2\mathcal{K}d\lambda^2 + r^2\bar{\mathcal{K}}|_{\Sigma_0}\Omega_{AB}|_{\Sigma_0}d\vartheta^A d\vartheta^B, \tag{16}$$

where

$$\mathcal{K} \equiv N^{-2}e_\lambda{}^a e_\lambda{}^b\nabla_a n^{(0)}_b, \quad \bar{\mathcal{K}} = n^{(0)a}\partial_{x^a}\ln r.$$

It follows that the background matching conditions (1) are

$$N_+^2 = N_-^2, \quad r_+^2|_{\Sigma_0} = r_-^2|_{\Sigma_0}, \quad \mathcal{K}_+ = \mathcal{K}_-, \quad \bar{\mathcal{K}}_+ = \bar{\mathcal{K}}_-. \tag{17}$$

Using (3) and (4) we could now compute the first order perturbations $q^{(1)}{}_{ij}$ and $K^{(1)}{}_{ij}$ in terms of the above quantities and $\vec{Z}$ (or equivalently Q and $\vec{T}$), c.f. Eqs. (45) and (46) in [6]. Let us recall (see subsection 2.2) that while the individual tensors $q^{(1)}{}_{ij}$ and $K^{(1)}{}_{ij}$ are not hypersurface gauge invariant, their respective differences from the + and − sides (i.e. the linearized matching conditions) are. Those tensors depend of the hypersurface gauge through the tangent vectors $\vec{T}^+$ and $\vec{T}^-$, which under a gauge change transform simply by adding the gauge vector. It follows that only their difference $[\vec{T}]$ can appear in the linearized matching conditions. Consequently there are three degrees of freedom that cannot be fixed by the equations, but can be fixed by choosing the hypersurface gauge, for instance to set $\vec{T}^+$. Thus, the linearized matching conditions can be looked at as equations for the difference $[\vec{T}]$ as well as for Q^+ and Q^-, i.e. for five objects. If these equations admit solutions, then the linearized matching is possible and it is impossible otherwise.

Mukohyama emphasizes the convenience to look for doubly gauge invariant quantities to write down the linearized matching conditions, however the matching conditions are *already* gauge invariant (both for the spacetime and hypersurfaces gauges). Looking for gauge invariant combinations on each side amounts to writing equations where the difference vector $[\vec{T}]$ simply drops. Indeed, in many cases, knowing the value of such vector in a specific matching is not interesting. In that sense, using doubly gauge invariant quantities is useful as it lowers the number of equations to analyse. However, we want to stress that this is not related to obtaining gauge invariant linearized matching equations.

It is just related to not solving for superfluous information. In fact, a set of equations where also Q^+ and Q^- have disappeared would be even more convenient from this point of view, provided one is not interested in knowing how the hypersurfaces are deformed in the specific spacetime gauge being used.

Since the use of doubly gauge invariant matching conditions is used extensively, let us recall its main ingredients in order to discuss if they really are equivalent to the full set of linearized matching equations and in which sense.

To that aim Mukohyama [6], decomposes the perturbation tensors ${q^{(1)}}_{ij}$ and ${K^{(1)}}_{ij}$ in terms of scalar Y, vector V_A and tensor harmonics T_{AB} on the sphere, as[3]

$$\begin{aligned} {q^{(1)}}_{ij} d\xi^i d\xi^j &= \sum_{l=0}^{\infty} (\sigma_{00} Y d\lambda^2 + \sigma_{(Y)} T_{(Y)AB} d\vartheta^A d\vartheta^B) + \sum_{l=1}^{\infty} 2(\sigma_{(T)0} V_{(T)A} + \sigma_{(L)0} V_{(L)A}) d\lambda d\vartheta^A \\ &+ \sum_{l=2}^{\infty} (\sigma_{(T)} T_{(T)AB} + \sigma_{(LT)} T_{(LT)AB} + \sigma_{(LL)} T_{(LL)AB}) d\vartheta^A d\vartheta^B, \end{aligned} \tag{18}$$

$$\begin{aligned} {K^{(1)}}_{ij} d\xi^i d\xi^j &= \sum_{l=0}^{\infty} (\kappa_{00} Y d\lambda^2 + \kappa_{(Y)} T_{(Y)AB} d\vartheta^A d\vartheta^B) + \sum_{l=1}^{\infty} 2(\kappa_{(T)0} V_{(T)A} + \kappa_{(L)0} V_{(L)A}) d\lambda d\vartheta^A \\ &+ \sum_{l=2}^{\infty} (\kappa_{(T)} T_{(T)AB} + \kappa_{(LT)} T_{(LT)AB} + \kappa_{(LL)} T_{(LL)AB}) d\vartheta^A d\vartheta^B, \end{aligned} \tag{19}$$

where all the scalar coefficients depend only on λ. Each coefficient in the decomposition has indices l and m which have been dropped for notational simplicity. Notice that each coefficient σ and κ is defined in the range of l's appearing in the corresponding summatory. By construction, each of the σ and κ are spacetime-gauge invariant (but not hypersurface-gauge invariant). For $l \geq 2$ they can even be written down [6] explicity in terms of spacetime-gauge invariant quantities. In a similar way, the doubly gauge-invariant quantities presented in [6], are only defined for $l \geq 2$ (except $k_{(T)0}$, which is also defined for $l = 1$), and read

$$\begin{aligned}
l \geq 2: &\quad f_{00} \equiv \sigma_{00} - 2N\partial_\lambda \left(N^{-1}\chi\right), \\
l \geq 2: &\quad f \equiv \sigma_{(Y)} + \epsilon N^{-2}\chi\partial_\lambda \left(r^2|_{\Sigma_0}\right) + \frac{2}{n-2} k_l^2 \sigma_{(LL)}, \\
l \geq 2: &\quad f_0 \equiv \sigma_{(T)0} - r^2|_{\Sigma_0}\partial_\lambda \left(r^{-2}|_{\Sigma_0}\sigma_{(LT)}\right), \\
l \geq 2: &\quad f_{(T)} \equiv \sigma_{(T)}, \\
l \geq 2: &\quad k_{00} \equiv \kappa_{00} + \epsilon\mathcal{K}\sigma_{00} + \epsilon\chi\partial_\lambda\mathcal{K}, \\
l \geq 1: &\quad k_{(T)0} \equiv \kappa_{(T)0} - \bar{\mathcal{K}}\sigma_{(T)0}, \\
l \geq 2: &\quad k_{(L)0} \equiv \kappa_{(L)0} + \frac{1}{2}(\epsilon\mathcal{K} - \bar{\mathcal{K}})\sigma_{(L)0} + \frac{1}{2}(\epsilon\mathcal{K} + \bar{\mathcal{K}}) \left[\chi - r^2|_{\Sigma_0}\partial_\lambda(r^{-2}|_{\Sigma_0}\sigma_{(LL)})\right], \\
l \geq 2: &\quad k_{(LT)} \equiv \kappa_{(LT)} - \bar{\mathcal{K}}\sigma_{(LT)}, \\
l \geq 2: &\quad k_{(LL)} \equiv \kappa_{(LL)} - \bar{\mathcal{K}}\sigma_{(LL)}, \\
l \geq 2: &\quad k_{(Y)} \equiv \kappa_{(Y)} - \bar{\mathcal{K}}\sigma_{(Y)} + \epsilon N^{-2} r^2|_{\Sigma_0}\chi\partial_\lambda\bar{\mathcal{K}}, \\
l \geq 2: &\quad k_{(T)} \equiv \kappa_{(T)} - \bar{\mathcal{K}}\sigma_{(T)},
\end{aligned} \tag{20}$$

[3]The ranges of l's are not made explicit in [6] in order to include also non-compact homogeneous spaces, where the index l is continuous. However, to discuss sufficiency of the equations we need to be precise on the range of validity of each equation.

where $k_l^2 = l(l+n-3)$ and

$$l \geq 2: \qquad \chi \equiv \sigma_{(L)0} - r^2|_{\Sigma_0}\partial_\lambda(r^{-2}|_{\Sigma_0}\sigma_{(LL)}).$$

The orthogonality properties of the scalar, vector and tensor harmonics imply that the equalities of the coefficients σ and κ for each l and m is equivalent to the equality of the perturbation tensors (18) and (19) at both sides of Σ_0. Thus, recalling the notation $[f] \equiv f^+|_{\Sigma_0} - f^-|_{\Sigma_0}$, the equations

$$\begin{array}{ll} l \geq 0: & [\sigma_{00}] = [\sigma_{(Y)}] = 0 \\ l \geq 1: & [\sigma_{(L)0}] = [\sigma_{(T)0}] = 0 \\ l \geq 2: & [\sigma_{(T)}] = [\sigma_{(LT)}] = [\sigma_{(LL)}] = 0 \end{array} \tag{21}$$

$$\begin{array}{ll} l \geq 0: & [\kappa_{00}] = [\kappa_{(Y)}] = 0 \\ l \geq 1: & [\kappa_{(L)0}] = [\kappa_{(T)0}] = 0 \\ l \geq 2: & [\kappa_{(T)}] = [\kappa_{(LT)}] = [\kappa_{(LL)}] = 0 \end{array} \tag{22}$$

are equivalent to (2) and therefore correspond exactly to the linearized matching conditions in this setting. Notice that each of the equalities in (21) and (22) is in fact one equation for each l and m in the appropriate range. We will however refer to them simply as equations.

The full linearized matching conditions obviously *imply* the following equalities in terms of the doubly-gauge invariant quantities (20),

$$l \geq 2: \quad [f_{00}] = [f] = [f_0] = [f_{(T)}] = 0 \tag{23}$$

$$\begin{array}{ll} l \geq 1: & [k_{(T)0}] = 0 \\ l \geq 2: & [k_{00}] = [k_{(Y)}] = [k_{(L)0}] = [k_{(LL)}] = [k_{(LT)}] = [k_{(T)}] = 0. \end{array} \tag{24}$$

Whether these equations can be regarded as the full set of linearized matching conditions or not requires studying their sufficiency, i.e. whether they imply (21)-(22) or not. This point was not mentioned in [6] and in fact the answer turns out to be negative, although in a mild way, as we discuss in the next subsection.

5.2 On the sufficiency of the continuity of the doubly-gauge invariants

Let us recall that fulfilling the matching conditions requires finding two $\vec{Z}^{\pm}$ such that (21)-(22) are satisfied. The key issue for the matching is therefore to show existence of deformation vectors $\vec{Z}^{\pm}$ so that all the equations hold.

A plausibility argument in favour of the sufficiency of (23)-(24) comes from simple equation counting. Indeed, as already discussed, the linearized matching conditions are spacetime and hypersurface gauge invariant and therefore can only involve the difference vector $[\vec{T}]$, i.e. three quantities. Since constructing double gauge invariant quantities on each side eliminates this vector, the number of equations should be reduced exactly by three if they are to remain equivalent to the original set. This is precisely what happens as we go from the original forteen equations in (21)-(22) down to eleven equations in (23)-(24). This argument however is not conclusive, both because it is not rigorous and because each equation in those expressions is, in fact, a set of equations depending on l and m, and the range of l's changes with the equations. Let us therefore analyse this issue in detail. In particular we need to discuss what are the consequences of the non-existence of doubly gauge-invariant variables for $l=0$ and $l=1$ (except for $k_{(T)0}$ which exists for $l=1$), something not mentioned in [6].

Let us start by finding explicit expressions for σ's valid in the whole range of l's. As in [6], we decompose $g^{(1)}$ in harmonics as

$$\begin{aligned}
{g^{(1)}}_{\alpha\beta}dx^\alpha dx^\beta &= \sum_{l=0}^{\infty}(h_{ab}Y dx^a dx^b + h_{(Y)}T_{(Y)AB}d\theta^A d\theta^B) \\
&+\sum_{l=1}^{\infty}2(h_{(T)a}V_{(T)A} + h_{(L)a}V_{(L)A})dx^a d\theta^A \\
&+\sum_{l=2}^{\infty}(h_{(T)}T_{(T)AB} + h_{(LT)}T_{(LT)AB} + h_{(LL)}T_{(LL)AB})d\theta^A d\theta^B, \qquad (25)
\end{aligned}$$

and $\vec{Z}$ as

$$\begin{aligned}
Z_\alpha dx^\alpha &= \sum_{l=0}^{\infty} z_a Y dx^a + \sum_{l=1}^{\infty}(z_{(T)}V_{(T)A} + z_{(L)}V_{(L)A})d\theta^A \\
&= \sum_{l=0}^{\infty}\left(QY\boldsymbol{n}^{(0)} - \epsilon N^{-2}z_\lambda Y\boldsymbol{e}_\lambda\right) + \sum_{l=1}^{\infty}(z_{(T)}V_{(T)A} + z_{(L)}V_{(L)A})d\theta^A, \qquad (26)
\end{aligned}$$

which implies $T_\alpha dx^\alpha = \sum_{l=0}^{\infty}(-\epsilon N^{-2}z_\lambda Y\boldsymbol{e}_\lambda) + \sum_{l=1}^{\infty}(z_{(T)}V_{(T)A} + z_{(L)}V_{(L)A})d\theta^A$. Inserting these expressions into (2) and expanding in spherical harmonics it is straightforward to find

$$\begin{aligned}
l\geq 0: &\quad [\sigma_{00}] = 0 \Leftrightarrow [h_{\lambda\lambda}] + 2[Q]N^2\mathcal{K} + 2N\partial_\lambda\left(N^{-1}[z_\lambda]\right) = 0, \\
l\geq 1: &\quad [\sigma_{(L)0}] = 0 \Leftrightarrow [z_\lambda] + [h_{(L)\lambda}] + r^2|_{\Sigma_0}\partial_\lambda(r^{-2}|_{\Sigma_0}[z_{(L)}]) = 0, \qquad (27)\\
l\geq 2: &\quad [\sigma_{(LL)}] = 0 \Leftrightarrow [z_{(L)}] + [h_{(LL)}] = 0, \qquad (28)\\
l\geq 0: &\quad [\sigma_{(Y)}] = 0 \Leftrightarrow [h_{(Y)}] + 2[Q]r^2|_{\Sigma_0}\bar{\mathcal{K}} - \epsilon N^{-2}[z_\lambda]\partial_\lambda(r^2|_{\Sigma_0}) - \frac{2}{n-2}k_l^2[z_{(L)}] = 0, \\
l\geq 1: &\quad [\sigma_{(T)0}] = 0 \Leftrightarrow [h_{(T)\lambda}] + r^2|_{\Sigma_0}\partial_\lambda(r^{-2}|_{\Sigma_0}[z_{(T)}]) = 0, \\
l\geq 2: &\quad [\sigma_{(LT)}] = 0 \Leftrightarrow [z_{(T)}] + [h_{(LT)}] = 0, \qquad (29)\\
l\geq 2: &\quad [\sigma_{(T)}] = 0 \Leftrightarrow [h_{(T)}] = 0,
\end{aligned}$$

where $[h_{\lambda\lambda}], [h_{(L)\lambda}]$, etc. denote ${e_\lambda}^a{e_\lambda}^b[h_{ab}], {e_\lambda}^a[h_{(L)a}]$, etc. Later on we will also write down the explicit expressions for (22) but they are not needed in this subsection.

It is obvious by the form of f's and κ's (20) that the set of equations (21)-(22) are equivalent to (23)-(24) together with

$$l\geq 2: \quad [\sigma_{(L)0}] = [\sigma_{(LT)}] = [\sigma_{(LL)}] = 0 \qquad (30)$$

$$\begin{aligned}
l = 0,1: &\quad [\sigma_{00}] = [\sigma_{(Y)}] = 0 \\
l = 1: &\quad [\sigma_{(L)0}] = [\sigma_{(T)0}] = 0
\end{aligned} \qquad (31)$$

$$\begin{aligned}
l = 0,1: &\quad [\kappa_{00}] = [\kappa_{(Y)}] = 0 \\
l = 1: &\quad [\kappa_{(L)0}] = 0.
\end{aligned} \qquad (32)$$

Sufficiency of Mukohyama's doubly gauge invariant matching conditions would follow if these equations serve *exclusively* to determine the discontinuity $[\vec{T}]$, i.e. $[z_\lambda]$ for $l\geq 0$

and $[z_{(T)}], [z_{(L)}]$ for $l \geq 1$. Now, the explicit expressions (27), (29), (28) show that (30) determine uniquely $[z_\lambda]$, $[z_{(T)}]$ and $[z_{(L)}]$ for $l \geq 2$. So, restricted to the sector $l \geq 2$ Mukoyama's doubly gauge invariant matching conditions can be regarded as equivalent to the full set of matching conditions. Taking all l's into account, however, the equations turn out *not* to be sufficient. To show this, it is enough to display one equation involving the discontinuity of the background metric perturbations and $[Q]$ (but not $[\vec{T}]$) which holds as a consequence of the full set of matching conditions (21)-(22) *but not* as a consequence of (23)-(24). Using the fact that each $l = 1$ expression refers to $n - 1$ objects (one for each m), the number of equations in (31)-(32) is $7n - 3$, while the number of unkowns in $[\vec{T}]$ not yet determined by (30), i.e. $[z_\lambda]$ for $l = 0, 1$ and $[z_{(T)}], [z_{(L)}]$ for $l = 1$ is $3n - 2$, which is smaller. It is to be expected, therefore, that (31), (32) imply conditions where these variables do not appear. This can be made explicit, for instance, by combining $[\sigma_{00}]_{l=0} = 0$ with $[\sigma_{(Y)}]_{l=0} = 0$ which yields

$$l = 0: \qquad [h_{\lambda\lambda}] + 2[Q]N^2\mathcal{K} + 2N\partial_\lambda \left\{ \frac{\epsilon N}{\partial_\lambda(r^2|_{\Sigma_0})} \left([h_{(Y)}] + 2[Q]r^2|_{\Sigma_0}\bar{\mathcal{K}} \right) \right\} = 0,$$

whenever $\partial_\lambda(r^2|_{\Sigma_0}) \neq 0$. (If $\partial_\lambda(r^2|_{\Sigma_0}) = 0$ it is enough to consider $[\sigma_{(Y)}]_{l=0} = 0$.) This relation is enough to show that *the continuity of the doubly-gauge invariant variables of Mukohyama is not sufficient to ensure the existence of the perturbed matching.* Of course, this does not invalidate Mukohyama's approach in any way, which remains interesting and useful. It only means that, when using this approach to solve linearized matchings, one still needs to look more carefully into the $l = 0$ and $l = 1$ sector to make sure that the remaining equations (31) and (32) hold.

On the other hand, equations (31), (32) do not completely determine $[\vec{T}]$. The variable $[z_{(T)}]_{l=1}$ only appears in $[\sigma_{(T)0}]_{l=1} = 0$, in the term $\partial_\lambda(r^{-2}|_{\Sigma_0}[z_{(T)}])$. As a result, the matching conditions do not fix $[z_{(T)}]_{l=1}$ completely, but up to a constant factor times $r^2|_{\Sigma_0}$ (for each m). Recalling that $V_{(T)A}d\vartheta^A$ for $l = 1$ correspond to the three Killing vectors on the sphere, this arbitrary constant (for each m) accounts for the addition to $[\vec{T}]$ of an arbitrary Killing vector of the sphere. This is in accordance with the discussion in Section 4. We devote the following subsection to complete the study of the freedom left in the matching.

5.3 Freedom in the matching

As already emphasized, solving the linearized matching amounts to finding perturbation vectors $\vec{Z}^+$ and $\vec{Z}^-$. Assume now that a linearized matching between two given backgrounds and perturbations has been done. It is natural to ask what is the most general matching between those two spaces, i.e. what is the most general solution for $\vec{Z}^+$ and $\vec{Z}^-$ of the matching conditions. Geometrically, this means finding all the possible deformations of the matching hypersurface Σ_0 which allow the two spaces to be matched.

Since this problem is of interest not only when the full matching conditions are imposed but also in situations where layers of matter are present (e.g. in brane-world or shell cosmologies) so that jumps in the second fundamental forms are allowed, we will analyse this issue in two steps. First, we will study the equations involving the perturbed first fundamental forms and will determine the freedom they admit. On a second step we will write down the extra conditions coming from the equality of the second fundamental forms.

Thus, let us consider two perturbation configurations of the same background matching and denote their respective sets of difference variables on Σ_0 as $[f]$ and $[f]'$ for any given variable f. Now, we will define the difference between the two configurations as $< f >\equiv [f]' - [f]$ for any variable f. The assumption that the perturbation on each side is fixed once and for all implies $< g^{(1)} >= 0$. We are assuming that the linearized matching conditions are satisfied in each case, and so we can subtract them. Linearity implies that the differences of the linearised matching equations become equations for the difference vector $< \vec{Z} >$. The general solution of these equations clearly determines the freedom in the deformation of the hypesurface.

The difference of the equations in (21) for the two configurations using $< g^{(1)} >= 0$ give the following set of equations

$$
\begin{aligned}
l \geq 0: &\quad < \sigma_{00} >= 0 \;\rightarrow\; < Q > N^2 \mathcal{K} + N \partial_\lambda \left(N^{-1} < z_\lambda > \right) = 0, && (33)\\
l \geq 1: &\quad < \sigma_{(L)0} >= 0 \;\rightarrow\; < z_\lambda > + r^2|_{\Sigma_0} \partial_\lambda (r^{-2}|_{\Sigma_0} < z_{(L)} >) = 0, && (34)\\
l \geq 2: &\quad < \sigma_{(LL)} >= 0 \;\rightarrow\; < z_{(L)} >= 0, && (35)\\
l \geq 0: &\quad < \sigma_{(Y)} >= 0 \;\rightarrow\; 2 < Q > r^2|_{\Sigma_0} \bar{\mathcal{K}} - \epsilon N^{-2} < z_\lambda > \partial_\lambda (r^2|_{\Sigma_0}) \\
&\qquad\qquad - \frac{2}{n-2} k_l^2 < z_{(L)} >= 0, && (36)\\
l \geq 1: &\quad < \sigma_{(T)0} >= 0 \;\rightarrow\; \partial_\lambda (r^{-2}|_{\Sigma_0} < z_{(T)} >) = 0, && (37)\\
l \geq 2: &\quad < \sigma_{(LT)} >= 0 \;\rightarrow\; < z_{(T)} >= 0, && (38)\\
l \geq 2: &\quad < \sigma_{(T)} >= 0 \;\rightarrow\; 0 = 0.
\end{aligned}
$$

Expressions (35) and (38) readily determine $< z_{(L)} >_{l\geq 2} = < z_{(T)} >_{l \geq 2} = 0$, which substituted in (34) give $< z_\lambda >_{l\geq 2} = 0$. As a result, (36) for $l \geq 2$ lead to $< Q >_{l \geq 2} = 0$. Clearly all the equations for $l \geq 2$ are now satisfied. We now concentrate on the $l = 1$ equations. Equation (37) implies that $< z_{(T)} >_{l=1} = a r^2|_{\Sigma_0}$, where a is a constant for each m. Combining equations (33), (34) and (36) for $l = 1$ we obtain the following equation for $r^{-2}|_{\Sigma_0} < z_{(L)} >_{l=1}$,

$$
\begin{aligned}
&\bar{\mathcal{K}} \partial^2_\lambda (r^{-2}|_{\Sigma_0} < z_{(L)} >_{l=1}) + \left\{ (2\bar{\mathcal{K}} + \epsilon \mathcal{K}) \partial_\lambda (\ln r|_{\Sigma_0}) - \bar{\mathcal{K}} \partial_\lambda \ln N \right\} \partial_\lambda (r^{-2}|_{\Sigma_0} < z_{(L)} >_{l=1}) \\
&- \frac{N\mathcal{K}}{r^2} r^{-2}|_{\Sigma_0} < z_{(L)} >_{l=1} = 0, \qquad (39)
\end{aligned}
$$

while (34) and (33) determine $< z_\lambda >_{l=1}$ and $< Q >_{l=1}$ respectively (provided $\mathcal{K} \neq 0$, which occurs generically). The two equations for $l = 0$ can be rearranged onto

$$
\bar{\mathcal{K}} \partial_\lambda (N^{-1} < z_\lambda >_{l=0}) + N^{-1} < z_\lambda >_{l=0} \epsilon \mathcal{K} \partial_\lambda \ln (r|_{\Sigma_0}) = 0 \qquad (40)
$$

plus the equation (33) for $l = 0$, which determines $< Q >_{l=0}$.

Summarizing, we have found that the freedom in the deformation of the hypersurface compatible with the linearized matching conditions involving the first fundamental form is

$$
\begin{aligned}
[\vec{Z}]' - [\vec{Z}] = \sum_{l=0}^{1} &\left(< Q > Y \vec{n}^{(0)} - \epsilon N^{-2} < z_\lambda > Y \vec{e}_\lambda \right) \\
&+ a_m \vec{V}^m_{(T)} + r^{-2}|_{\Sigma_0} < z_{(L)} >_{l=1,m} \vec{V}^m_{(L)},
\end{aligned}
$$

where $r^{-2}|_{\Sigma_0} < z_{(L)} >_{l=1,m}$, satisfy (39), $< z_\lambda >_{l=0}$ satisfy (40) and the rest of the variables are completely determined as described above. The term in a_m corresponds to adding Killing vectors on the sphere, something already discussed in Section 4. The rest of terms involve combinations (with functions) of the conformal Killing vectors on the sphere and tangential vectors along λ. Notice that the coefficients of the conformal Killing (i.e. $< z_{(L)} >_{l=1,m}$) determine all the rest of the $l = 1$ coefficients. In particular when $< z_{(L)} >_{l=1,m}$ vanishes, then all the $l = 1$ terms vanish and the freedom becomes radially symmetric.

We now add to the analysis the difference of the equations in (22). Due to the fact that all coefficients in $< \vec{Z} >$ vanish for $l \geq 2$ we only need to consider the equations for $l = 0, 1$, i.e. (32). We refer the reader to Appendix A for the explicit expressions of (32) in terms of the metric perturbations and $\vec{Z}$. For the sake of completeness we also include all the explicit expressions of (22) in Appendix A. The difference of equations (32), see (44)-(46), whenever $< g^{(1)} >= 0$ read

$$l = 0, 1: \qquad < \kappa_{00} >= 0 \Leftrightarrow \tag{41}$$
$$- < QR^{(\gamma)}_{dbac} > n^{(0)d} n^{(0)a} e_\lambda{}^b e_\lambda{}^c - \epsilon \partial^2_\lambda < Q > + \frac{\epsilon}{2N^2} \partial_\lambda N^2 \partial_\lambda < Q > - \epsilon < Q > \mathcal{K}^2 N^2$$
$$- \epsilon \mathcal{K} N^2 \partial_\lambda (N^{-2} < z_\lambda >) - \epsilon \partial_\lambda (\mathcal{K} < z_\lambda >) = 0,$$
$$l = 1: \qquad < \kappa_{(L)0} >= 0 \Leftrightarrow \tag{42}$$
$$- \epsilon \partial_\lambda < Q > + \epsilon \mathcal{K} < z_\lambda > + \epsilon < Q > \partial_\lambda \ln(r|_{\Sigma_0}) + r^2|_{\Sigma_0} \bar{\mathcal{K}} \partial_\lambda (r^{-2}|_{\Sigma_0} < z_{(L)} >) = 0$$
$$l = 0, 1: \qquad < \kappa_{(Y)} >= 0 \Leftrightarrow \tag{43}$$
$$+ \frac{1}{2} N^{-2} \partial_\lambda (r^2|_{\Sigma_0}) \left(\partial_\lambda < Q > + \mathcal{K} < z_\lambda > \right) + \frac{1}{2} < Q n^{(0)a} n^{(0)b} \nabla_a \nabla_b r^2 >$$
$$- \frac{\epsilon}{2} N^{-2} e_\lambda{}^a < z_\lambda n^{(0)b} \nabla_b \nabla_a r^2 > + \frac{l(l+n-3)}{n-2} \left\{ \epsilon < Q > - 2\bar{\mathcal{K}} < z_{(L)} > \right\} = 0.$$

It can be checked that in general these equations overdetermine the previous equations, i.e. (39) and (40), although there may be particular cases for which they are compatible. Therefore, generically, they will imply that $< z_{(L)} >_{l=1,m} = 0$ and $< z_\lambda >_{l=0} = 0$, and thence all the rest of the variables vanish, $< z_\lambda >_{l=1,m} = < Q >_{l=1,m} = 0$, $< z_\lambda >_{l=0} = < Q >_{l=0} = 0$, so that the only freedom left is given by

$$[\vec{Z}]' - [\vec{Z}] = a_m \vec{V}^m_{(T)}.$$

Finding in which particular cases equations (39)-(43) are compatible is straightforward but tedious and will not be carried out explicitly here.

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A Appendix

For the sake of completeness we devote this appendix to present the explicit expressions of (22) in terms of the metric perturbations and $\vec{Z}$, which read

$$
\begin{aligned}
& l \geq 0: \qquad [\kappa_{00}] = 0 \Leftrightarrow && (44)\\
& \frac{\epsilon}{2} N^2 \mathcal{K}[h_{nn}] - \frac{1}{2} n^{(0)a} e_\lambda{}^b e_\lambda{}^c (2\nabla_c [h_{ab}] - \nabla_a [h_{bc}]) - [Q R^{(\gamma)}_{dbac}] n^{(0)d} n^{(0)a} e_\lambda{}^b e_\lambda{}^c \\
& -\epsilon \partial_\lambda^2 [Q] + \frac{\epsilon}{2N^2} \partial_\lambda N^2 \partial_\lambda [Q] - \epsilon [Q] \mathcal{K}^2 N^2 - \epsilon \mathcal{K} N^2 \partial_\lambda (N^{-2} [z_\lambda]) - \epsilon \partial_\lambda (\mathcal{K}[z_\lambda]) = 0, \\
& l \geq 1: \qquad [\kappa_{(L)0}] = 0 \Leftrightarrow && (45)\\
& -\frac{1}{2}[h_{n\lambda}] - \frac{1}{2} n^{(0)a} e_\lambda{}^b (\partial_b [h_{(L)a}] - \partial_a [h_{(L)b}]) - \epsilon \partial_\lambda [Q] - \epsilon \mathcal{K}[z_\lambda] \\
& + (\epsilon [Q] + [h_{(L)n}]) \partial_\lambda \ln(r|_{\Sigma_0}) + r^2|_{\Sigma_0} \bar{\mathcal{K}} \partial_\lambda (r^{-2}|_{\Sigma_0} [z_{(L)}]) = 0 \\
& l \geq 0: \qquad [\kappa_{(Y)}] = 0 \Leftrightarrow && (46)\\
& -\frac{\epsilon}{2} r^2|_{\Sigma_0} \bar{\mathcal{K}}[h_{nn}] + \frac{1}{2} N^{-2} \partial_\lambda (r^2|_{\Sigma_0}) \left(\epsilon [h_{n\lambda}] + \partial_\lambda [Q] + \mathcal{K}[z_\lambda] \right) + \frac{1}{2} [n^{(0)a} \partial_a h_{(Y)}] \\
& + \frac{1}{2} [Q n^{(0)a} n^{(0)b} \nabla_a \nabla_b r^2] - \frac{\epsilon}{2} N^{-2} e_\lambda{}^a [z_\lambda n^{(0)b} \nabla_b \nabla_a r^2] \\
& + \frac{l(l+n-3)}{n-2} \left\{ [h_{(L)n}] + \epsilon [Q] - 2\bar{\mathcal{K}}[z_{(L)}] \right\} = 0, \\
& l \geq 1: \qquad [\kappa_{(T)0}] = 0 \Leftrightarrow \\
& -\frac{1}{2} n^{(0)a} e_\lambda{}^b (\partial_b [h_{(T)a}] - \partial_a [h_{(T)b}]) + [h_{(T)n}] \partial_\lambda \ln(r|_{\Sigma_0}) + r^2|_{\Sigma_0} \bar{\mathcal{K}} \partial_\lambda (r^{-2}|_{\Sigma_0} [z_{(T)}]) = 0, \\
& l \geq 2: \qquad [\kappa_{(LT)}] = 0 \Leftrightarrow -\frac{1}{2}[h_{(T)n}] + \frac{1}{2} n^{(0)a} \partial_a [h_{(LT)}] + \bar{\mathcal{K}}[z_{(T)}] = 0, \\
& l \geq 2: \qquad [\kappa_{(LT)}] = 0 \Leftrightarrow -\frac{1}{2}[h_{(L)n}] + \frac{1}{2} n^{(0)a} \partial_a [h_{(LL)}] + \bar{\mathcal{K}}[z_{(L)}] - \frac{\epsilon}{2}[Q] = 0, \\
& l \geq 2: \qquad [\kappa_{(T)}] = 0 \Leftrightarrow \frac{1}{2} n^{(0)a} \partial_a [h_{(T)}] = 0.
\end{aligned}
$$

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