The Hardy-Lorentz Spaces $H^{p,q}(R^n)$

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Abstract

In this paper we consider the Hardy-Lorentz spaces $H^{p,q}(R^n)$, with $0 < p \le 1$, $0 < q \le \infty$. We discuss the atomic decomposition of the elements in these spaces, their interpolation properties, and the behavior of singular integrals and other operators acting on them.

The real variable theory of the Hardy spaces represents a fruitful setting for the study of maximal functions and singular integral operators. In fact, it is because of the failure of these operators to preserve L^1 that the Hardy space H^1 assumes its prominent role in harmonic analysis. Now, for many of these operators, the role of L^1 can just as well be played by $H^{1,\infty}$, or Weak H^1. However, although these operators are amenable to $H^1 - L^1$ and $H^{1,\infty} - L^{1,\infty}$ estimates, interpolation between H^1 and $H^{1,\infty}$ has not been available. Similar considerations apply to H^p and Weak H^p for $0 < p < 1$.

The purpose of this paper is to provide an interpolation result for the Hardy-Lorentz spaces $H^{p,q}$, $0 < p \le 1$, $0 < q \le \infty$, including the case of Weak H^p as and end point for real interpolation. The atomic decomposition is the key ingredient in dealing with interpolation since in this context neither truncations are available, nor reiteration applies.

The paper is organized as follows. The Lorentz spaces, including criteria that assure membership in $L^{p,q}$, $0 < p < \infty$, $0 < q \le \infty$, are discussed in Section 1. In Section 2 we show that distributions in $H^{p,q}$ have an atomic decomposition in terms of H^p atoms with coefficients in an appropriate mixed norm space. An interesting application of this decomposition is to $H^{p,q} - L^{p,\infty}$ estimates for Calderón-Zygmund singular integral operators, $p < q \le \infty$. Also, by manipulating the different levels of the atomic decomposition, we show that, for $0 < q_1 < q < q_2 \le \infty$, $H^{p,q}$ is an intermediate space between H^{p,q_1} and H^{p,q_2}. This result applies to Calderón-Zygmund singular integral operators, including those with variable kernels, Marcinkiewicz integrals, and other operators.

1 The Lorentz spaces

The Lorentz space $L^{p,q}(R^n) = L^{p,q}$, $0 < p < \infty$, $0 < q \le \infty$, consists of those measurable functions f with finite quasinorm $\|f\|_{p,q}$ given by

$$\|f\|_{p,q} = \left(\frac{q}{p}\int_0^\infty [t^{1/p} f^*(t)]^q \,\frac{dt}{t}\right)^{1/q}, \quad 0 < q < \infty\,,$$

$$\|f\|_{p,\infty} = \sup_{t>0}\,[t^{1/p} f^*(t)]\,, \quad q = \infty\,.$$

The Lorentz quasinorm may also be given in terms of the distribution function $m(f,\lambda) = |\{x \in R^n : |f(x)| > \lambda\}|$, loosely speaking, the inverse of the non-increasing rearrangement f^* of f. Indeed, we have

$$\|f\|_{p,q} = \left(\frac{q}{p}\int_0^\infty \lambda^{q-1} m(f,\lambda)^{q/p}\, d\lambda\right)^{1/q} \sim \Big(\sum_k \big[2^k m(f,2^k)^{1/p}\big]^q\Big)^{1/q}\,,$$

when $0 < q < \infty$, and

$$\|f\|_{p,\infty} = \sup_k\; 2^k m(f,2^k)^{1/p}\,, \quad q = \infty\,.$$

Note that, in particular, $L^{p,p} = L^p$, and $L^{p,\infty}$ is weak L^p.

The following two results are useful in verifying that a function is in $L^{p,q}$.

Lemma 1.1. *Let $0 < p < \infty$, and $0 < q \le \infty$. Assume that the non-negative sequence $\{\mu_k\}$ satisfies $\{2^k\mu_k\} \in \ell^q$. Further suppose that the non-negative function φ verifies the following property: there exists $0 < \varepsilon < 1$ such that, given an arbitrary integer k_0, we have $\varphi \le \psi_{k_0} + \eta_{k_0}$, where ψ_{k_0} is essentially bounded and satisfies $\|\psi_{k_0}\|_\infty \le c\,2^{k_0}$, and*

$$2^{k_0\varepsilon p}\, m(\eta_{k_0}, 2^{k_0}) \le c\sum_{k_0}^{\infty}[2^{k\varepsilon}\mu_k]^p\,.$$

Then, $\varphi \in L^{p,q}$, and $\|\varphi\|_{p,q} \le c\,\|\{2^k\mu_k\}\|_{\ell^q}$.

Proof. It clearly suffices to verify that $\|\{2^k\,|\{\varphi > \gamma\,2^k\}|^{1/p}\}\|_{\ell^q} < \infty$, where γ is an arbitrary positive constant. Now, given k_0, let ψ_{k_0} and η_{k_0} be as above, and put $\gamma = c+1$, where c is the constant in the above inequalities; for this choice of γ, $\{\varphi > \gamma\,2^{k_0}\} \subseteq \{\eta_{k_0} > 2^{k_0}\}$.

When $q = \infty$, we have

$$2^{k_0\varepsilon}\, m(\eta_{k_0}, 2^{k_0})^{1/p} \le c\Big(\sum_{k_0}^{\infty} [2^{-k(1-\varepsilon)}\, 2^k\, \mu_k]^p \Big)^{1/p} \le c\, 2^{-k_0(1-\varepsilon)} \sup_{k \ge k_0} [\, 2^k\, \mu_k] \,.$$

Thus, $2^{k_0}\, m(\eta_{k_0}, 2^{k_0})^{1/p} \le \sup_{k\ge k_0} [\, 2^k\, \mu_k]$, and, consequently,

$$2^{k_0}\, m(\varphi, \gamma\, 2^{k_0})^{1/p} \le c\, \|\{2^k \mu_k\}\|_{\ell^\infty} \,, \quad \text{all } k_0.$$

When $0 < q < \infty$, let $1 - \varepsilon = 2\delta$ and rewrite the right-hand side above as

$$\sum_{k_0}^{\infty} \frac{1}{2^{k\delta p}} [2^{k(1-\delta)} \mu_k]^p \,.$$

When $p < q$, by Hölder's inequality with exponent $r = q/p$ and its conjugate r', this expression is dominated by

$$\Big(\sum_{k_0}^{\infty} \frac{1}{2^{k\,\delta p r'}} \Big)^{1/r'} \Big(\sum_{k_0}^{\infty} \Big[2^{k(1-\delta)} \mu_k \Big]^{rp} \Big)^{1/r}$$
$$\le c\, 2^{-k_0\,\delta p} \Big(\sum_{k_0}^{\infty} \Big[2^{k(1-\delta)} \mu_k \Big]^{q} \Big)^{p/q},$$

and, when $0 < q \le p$, $r < 1$, and we get a similar bound by simply observing that it does not exceed

$$2^{-k_0\delta p} \Big(\sum_{k_0}^{\infty} [2^{k(1-\delta)} \mu_k]^p \Big)^{r/r} \le 2^{-k_0\delta p} \Big(\sum_{k_0}^{\infty} \Big[2^{k(1-\delta)} \mu_k \Big]^{q} \Big)^{p/q}.$$

Whence, continuing with the estimate, we have

$$2^{k_0\varepsilon p}\, m(\eta_{k_0}, 2^{k_0}) \le c\, 2^{-k_0\delta p} \Big(\sum_{k_0}^{\infty} \Big[2^{k(1-\delta)} \mu_k \Big]^{q} \Big)^{p/q},$$

which yields, since $1 - \varepsilon = 2\,\delta$,

$$2^{k_0}\, m(\varphi, \gamma\, 2^{k_0})^{1/p} \le c\, 2^{k_0\,\delta} \Big(\sum_{k_0}^{\infty} \Big[2^{k(1-\delta)} \mu_k \Big]^{q} \Big)^{1/q}.$$

Thus, raising to the q and summing, we get

$$\sum_{k_0} \left[2^{k_0}\, m(\varphi, \gamma\, 2^{k_0})^{1/p}\right]^q \le c \sum_{k_0} 2^{k_0\,\delta\, q} \sum_{k=k_0}^{\infty} \left[2^{k(1-\delta)}\mu_k\right]^q ,$$

which, upon changing the order of summation in the right-hand side of the above inequality, is bounded by

$$\sum_{k} \left[2^{k(1-\delta)}\mu_k\right]^q \Big[\sum_{k_0=-\infty}^{k} 2^{k_0\,\delta\, q} \Big] \le c \sum_{k} \left[2^{k}\mu_k\right]^q . \quad \blacksquare$$

The reader will have no difficulty in verifying that, for Lemma 1.1 to hold, it suffices that ψ_{x_0} satisfies

$$m(\psi_{x_0}, 2^{k_0})^{1/p} \le c\, \mu_{k_0} , \quad \text{all } k_0 .$$

This holds, for instance, when $\|\psi_{x_0}\|_r^r \le c\, 2^{k_0 r}\mu_{k_0}^p$, $0 < r < \infty$. In fact, the assumptions of Lemma 1.1 correspond to the limiting case of this inequality as $r \to \infty$.

Another useful condition is given by our next result, the proof is left to the reader.

Lemma 1.2. *Let $0 < p < \infty$, and let the non-negative sequence $\{\mu_k\}$ be such that $\{2^k\mu_k\} \in \ell^q$, $0 < q \le \infty$. Further, suppose that the non-negative function φ satisfies the following property: there exists $0 < \varepsilon < 1$ such that, given an arbitrary integer k_0, we have $\varphi \le \psi_{k_0} + \eta_{k_0}$, where ψ_{k_0} and η_{k_0} satisfy*

$$2^{k_0 p} m(\psi_{k_0}, 2^{k_0})^{\varepsilon} \le c \sum_{-\infty}^{k_0} \left[2^k \mu_k^{\varepsilon}\right]^p , \quad 0 < \varepsilon < \min(1, q/p) ,$$

$$2^{k_0 \varepsilon} |\{\eta_{k_0} > 2^{k_0}\}| \le c \sum_{k_0}^{\infty} \left[2^{k\varepsilon}\mu_k\right]^p .$$

Then, $\varphi \in L^{p,q}$, and $\|\varphi\|_{p,q} \le c\, \|\{2^k\mu_k\}\|_{\ell^q}$.

We will also require some basic concepts from the theory of real interpolation. Let A_0, A_1, be a compatible couple of quasinormed Banach spaces, i.e., both A_0 and A_1 are continuously embedded in a larger topological vector space. The Peetre K functional of $f \in A_0 + A_1$ at $t > 0$ is defined by

$$K(t, f; A_0, A_1) = \inf_{f=f_0+f_1} \|f_0\|_0 + t\,\|f_1\|_1\,,$$

where $f = f_0 + f_1$, $f_0 \in A_0$ and $f_1 \in A_1$.

In the particular case of the L^q spaces, the K functional can be computed by Holmstedt's formula, see [12]. Specifically, for $0 < q_0 < q_1 \le \infty$, let α be given by $1/\alpha = 1/q_0 - 1/q_1$. Then,

$$K(t, f; L^{q_0}, L^{q_1}) \sim \left(\int_0^{t^\alpha} f^*(s)^{q_0} ds\right)^{1/q_0} + t\left(\int_{t^\alpha}^{\infty} f^*(s)^{q_1} ds\right)^{1/q_1}.$$

The intermediate space $(A_0, A_1)_{\eta,\,q}$, $0 < \eta < 1$, $0 < q < \infty$, consists of those f's in $A_0 + A_1$ with

$$\|f\|_{(A_0,A_1)_{\eta,\,q}} = \left(\int_0^\infty \left[t^{-\eta} K(t, f; A_0, A_1)\right]^q \frac{dt}{t}\right)^{1/q} < \infty\,,$$

$$\|f\|_{(A_0,A_1)_{\eta,\infty}} = \sup_{t>0} \left[t^{-\eta} K(t, f; A_0, A_1)\right] < \infty\,, \quad q = \infty\,.$$

Finally, for the L^q and $L^{p,q}$ spaces, we have the following result. Let $0 < q_1 < q < q_2 \le \infty$, and suppose that $1/q = (1-\eta)/q_1 + \eta/q_2$. Then, $L^q = (L^{q_1}, L^{q_2})_{\eta,q}$, and, $L^{1,q} = (L^{1,q_1}, L^{1,q_2})_{\eta,q}$, see [4].

2 The Hardy-Lorentz spaces $H^{p,q}$

In this paper we adopt the atomic characterization of the Hardy spaces H^p, $0 < p \le 1$. Recall that a compactly supported function a with $[n(1/p - 1)]$ vanishing moments is an H^p atom with defining interval I (of course, I is a cube in R^n), if $\mathrm{supp}(a) \subseteq I$, and $|I|^{1/p}\,|a(x)| \le 1$. The Hardy space $H^p(R^n) = H^p$ consists of those distributions f that can be written as $f = \sum \lambda_j a_j$, where the a_j's are H^p atoms, $\sum |\lambda_j|^p < \infty$, and the convergence is in the sense of distributions as well as in H^p. Furthermore,

$$\|f\|_{H^p} \sim \inf \left(\sum |\lambda_j|^p\right)^{1/p},$$

where the infimum is taken over all possible atomic decompositions of f. This last expression has traditionally been called the atomic H^p norm of f.

C. Fefferman, Rivière and Sagher identified the intermediate spaces between the Hardy space H^{p_0}, $0 < p_0 < 1$, and L^∞, as

$$(H^{p_0}, L^\infty)_{\eta,q} = H^{p,q}, \quad 1/p = (1-\eta)/p_0\,,\ 0 < q \le \infty\,,$$

where $H^{p,q}$ consists of those distributions f whose radial maximal function $Mf(x) = \sup_{t>0} |(f * \varphi_t)(x)|$ belongs to $L^{p,q}$. Here φ is a compactly supported, smooth function with nonvanishing integral, see [10]. R. Fefferman and Soria studied in detail the space $H^{1,\infty}$, which they called Weak H^1, see [11].

Just as in the case of H^p, $H^{p,q}$ can be characterized in a number of different ways, including in terms of non-tangential maximal functions and Lusin functions. In what follows we will calculate the quasinorm of f in $H^{p,q}$ by the means of the expression

$$\left\| \left\{ 2^k m(Mf, 2^k)^{1/p} \right\} \right\|_{\ell^q}, \quad 0 < p \le 1,\ 0 < q \le \infty\,,$$

where Mf is an appropriate maximal function of f.

Passing to the atomic decomposition of $H^{p,q}$, the proof is divided in two parts. First, we construct an essentially optimal atomic decomposition; Parilov has obtained independently this result for $H^{1,q}$ when $1 \le q$, see [14]. Also, R. Fefferman and Soria gave the atomic decomposition of Weak H^1, see [11], and Alvarez the atomic decomposition of Weak H^p, $0 < p < 1$, see [2].

Theorem 2.1. *Let $f \in H^{p,q}$, $0 < p \le 1$, $0 < q \le \infty$. Then f has an atomic decomposition $f = \sum_{j,k} \lambda_{j,k} a_{j,k}$, where the $a_{j,k}$'s are H^p atoms with defining intervals $I_{j,k}$ that have bounded overlap uniformly for each k, the sequence $\{\lambda_{j,k}\}$ satisfies $\big(\sum_k \big[\sum_j |\lambda_{j,k}|^p\big]^{q/p}]\big)^{1/q} < \infty$, and the convergence is in the sense of distributions. Furthermore, $\big(\sum_k \big[\sum_j |\lambda_{j,k}|^p\big]^{q/p}]\big)^{1/q} \sim \|f\|_{H^{p,q}}$.*

Proof. The idea of constructing an atomic decomposition using Calderón's reproducing formula is well understood, so we will only sketch it here, for further details, see [5] and [18]. Let $Nf(x) = \sup\{|(f * \psi_t)(y)| : |x - y| < t\}$ denote the non-tangential maximal function of f with respect to a suitable smooth function ψ with nonvanishing integral. One considers the open sets $\mathcal{O}_k = \{Nf > 2^k\}$, all integers k, and builds the atoms with defining interval associated to the intervals, actually cubes, of the Whitney decomposition of $\mathcal{O}_k$, and hence satisfying all the required properties. More precisely, one constructs a sequence of bounded functions f_k with norm not exceeding $c\,2^k$ for each k, and such that $f - \sum_{|k|\le n} f_k \to 0$ as $n \to \infty$ in the sense of distributions. These functions have the further property that $f_k(x) = \sum_j \alpha_{j,k}(x)\,$, where $|\alpha_{j,k}(x)| \le c\,2^k$, c is a constant, each $\alpha_{j,k}$ has vanishing moments up to order $[n(1/p-1)]$ and is supported in $I_{j,k}$ - roughly one of the Whitney cubes -, where the $I_{j,k}$'s have bounded overlaps for each k, uniformly in k. It only remains now to scale $\alpha_{j,k}$,

$$\alpha_{j,k}(x) = \lambda_{j,k}\, a_{j,k}(x)\,,$$

and balance the contribution of each term to the sum. Let $\lambda_{j,k} = 2^k |I_{j,k}|^{1/p}$. Then, $a_{j,k}(x)$ is essentially an H^p atom with defining interval $I_{j,k}$, and one has $\Big(\sum_j |\lambda_{j,k}|^p\Big)^{1/p} \sim 2^k\,|\mathcal{O}_k|^{1/p}$. Thus,

$$\Big\|\Big(\sum_j |\lambda_{j,k}|^p\Big)^{1/p}\Big\|_{\ell^q} \sim \Big\|\big\{2^k\,|\mathcal{O}_k|^{1/p}\big\}\Big\|_{\ell^q} \sim \|f\|_{H^{p,q}}\,, \quad 0 < q \le \infty\,. \quad \blacksquare$$

As an application of this atomic decomposition, the reader should have no difficulty in showing directly the C. Fefferman, Rivière, Sagher characterization of $H^{p,q}$, see [10].

Another interesting application of this decomposition is to $H^{p,q} - L^{p,\infty}$ estimates for Calderón-Zygmund singular integral operators T, $p < q \le \infty$. This approach combines the concept of p-quasi local operator of Weisz, see [17], with the idea of variable dilations of R. Fefferman and Soria, see [11]. Intuitively, since Hörmander's condition implies that T maps H^1 into L^1, say, for T to be defined in $H^{1,s}$, $1 < s \le \infty$, some strengthening of this condition is required. This is accomplished by the variable dilations. Moreover, since we will include $p < 1$ in our discussion, as p gets smaller, more regularity of the kernel of T will be required. This justifies the following definition.

Given $0 < p \le 1$, let $N = [n(1/p-1)]$, and, associated to the kernel $k(x,y)$ of a Calderón-Zygmund singular integral operator T, consider the modulus of continuity ω_p given by

$$\omega_p(\delta) = \sup_I \frac{1}{|I|}\int_{R^n\setminus(2/\delta)I} \Big[\int_I |\,k(x,y) - \sum_{|\alpha|\le N} (y-y_I)^\alpha k_\alpha(x,y_I)|\,dy\,\Big]^p dx\,,$$

where $0 < \delta \le 1$, and the sup is taken over the collection of arbitrary intervals I of R^n centered at y_I. Here, for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$,

$$k_\alpha(x, y_I) = \frac{1}{\alpha!} D^\alpha k(x,y)\big]_{y=y_I}\,.$$

$\omega_p(\delta)$ controls the behavior of T on atoms. More precisely, if a is an H^p atom with defining interval I, and $0 < \delta < 1$, observe that

$$T(a)(x) = \int_I [k(x,y) - \sum_{|\alpha| \le N} (y - y_I)^\alpha k_\alpha(x, y_I)]\, a(y)\, dy\,,$$

and, consequently,

$$\int_{R^n \setminus (2/\delta) I} |T(a)(x)|^p\, dx \le \omega_p(\delta)\,.$$

We are now ready to prove the $H^{p,q} - L^{p,\infty}$ estimate for a Calderón-Zygmund singular integral operator T with kernel $k(x,y)$.

Theorem 2.2. *Let $0 < p \le 1$, and $p < q \le \infty$. Assume that a Calderón-Zygmund singular integral operator T is of weak-type (r,r) for some $1 < r < \infty$, and that the modulus of continuity ω_p of the kernel k satisfies a Dini condition of order $q/(q-p)$, namely,*

$$A_{p,q} = \Big[\int_0^1 \omega_p(\delta)^{q/(q-p)} \frac{d\delta}{\delta} \Big]^{(q-p)/q} < \infty\,.$$

Then T maps $H^{p,q}$ continuously into $L^{p,\infty}$, and $\|Tf\|_{p,\infty} \le c\, A_{p,q}^{1/p}\, \|f\|_{H^{p,q}}$.

Proof. We need to show that

$$2^{k_0 p} m(Tf, 2^{k_0}) \le c\, \|f\|_{H^{p,q}}^p\,, \quad \text{all } k_0\,.$$

Let $f = \sum_k \sum_j \lambda_{j,k} a_{j,k}$, be the atomic decomposition of f given in Theorem 2.1, and set $f_1 = \sum_{k \le k_0} \sum_j \lambda_{j,k} a_{j,k}$, and $f_2 = f - f_1$. Further, let $\mu_k = \Big(\sum_j |\lambda_{j,k}|^p\Big)^{1/p}$, and recall that $\|\{\mu_k\}\|_{\ell^q} \sim \|f\|_{H^{p,q}}$.

Since $\|f_1\|_r^r \le c\, 2^{k_0(r-p)} \|f\|_{H^{p,\infty}}^p$, we have

$$2^{pk_0} m(Tf_1, 2^{k_0}) \le c\, \|f\|_{H^{p,\infty}}^p\,.$$

Next, put $I^*_{j,k} = 2^{1/n}(3/2)^{p(k-k_0)/n} I_{j,k}$, and let

$$\Omega = \bigcup_{k>k_0} \bigcup_j I^*_{j,k}\,.$$

Since $|I^*_{j,k}| = 2(3/2)^{p(k-k_0)}|I_{j,k}| \sim 2^{-k_0 p}(3/4)^{p(k-k_0)}|\lambda_{j,k}|^p$, we get

$$\begin{aligned}
|\,\Omega| &\le \sum_{k>k_0}\sum_j |I^*_{j,k}| \le c\,2^{-k_0 p} \sum_{k>k_0}(3/4)^{p(k-k_0)}\sum_j |\lambda_{j,k}|^p \\
&\le c\,2^{-k_0 p}\Big[\sup_{k>k_0}\mu_k\Big]^p \le c\,2^{-k_0 p}\|f\|^p_{H^{p,\infty}}\,.
\end{aligned}$$

Also, since $0 < p \le 1$, it readily follows that

$$|T(f_2)(x)|^p \le \sum_{k>k_0}\sum_j |\lambda_{j,k}|^p |T(a_{j,k})(x)|^p\,,$$

and, by Tonelli and the estimate for $T(a)$, we have

$$\begin{aligned}
\int_{R^n\setminus\Omega} |T(f_2)(x)|^p\,dx &\le \sum_{k>k_0}\sum_j |\lambda_{j,k}|^p \int_{R^n\setminus I^*_{j,k}} |T(a_{j,k})(x)|^p\,dx \\
&\le \sum_{k>k_0} \omega_p\Big(\Big(\frac{2}{3}\Big)^{p(k-k_0)/n}\Big)\,\mu_k^p \\
&\le \Big(\sum_{k>0}\omega_p\Big(\Big(\frac{2}{3}\Big)^{pk/n}\Big)^{q/(q-p)}\Big)^{(q-p)/q}\,\|\mu_k\|^p_{\ell^q} \\
&\le c\,\Big[\int_0^1 \omega_p(\delta)^{q/(q-p)}\frac{d\delta}{\delta}\Big]^{(q-p)/q}\,\|f\|^p_{H^{p,q}}\,.
\end{aligned}$$

This bound gives at once

$$2^{pk_0}\,|\{x\notin\Omega : |T(f_2)(x)| > 2^{k_0}\}| \le c\,A_{p,q}\,\|f\|^p_{H^{p,q}}\,,$$

which implies that

$$\begin{aligned}
2^{pk_0} m(Tf_2, 2^{k_0-1}) &\le 2^{pk_0}\left[|\Omega| + |\{x\notin\Omega : |T(f_2)(x)| > 2^{k_0-1}\}|\right] \\
&\le c\,\|f\|^p_{H^{p,\infty}} + c\,A_{p,q}\,\|f\|^p_{H^{p,q}}\,.
\end{aligned}$$

Finally,

$$\begin{aligned}
2^{k_0 p} m(Tf, 2^{k_0}) &\le 2^{k_0 p} m(Tf_1, 2^{k_0-1}) + 2^{k_0 p} m(Tf_2, 2^{k_0-1}) \\
&\le c\,\|f\|^p_{H^{p,\infty}} + c\,A_{p,q}\,\|f\|^p_{H^{p,q}}\,,
\end{aligned}$$

and, since $\|f\|_{H^{p,\infty}} \le c\,\|f\|_{H^{p,q}}$ for all q, we have finished. ■

We pass now to the converse of Theorem 2.1. It is apparent that a condition that relates the coefficients λ_j with the corresponding atoms a_j involved in an atomic decomposition of the form $\sum_j \lambda_j a_j(x)$ is relevant here. More precisely, if I_j denotes the supporting interval of a_j, let

$$\mathcal{I}_k = \{ j : 2^k \le |\lambda_j|/|I_j|^{1/p} < 2^{k+1} \} ,$$

and, for $\lambda = \{\lambda_j\}$, put

$$\|\lambda\|_{[p,q]} = \Big(\sum_k \Big[\sum_{j \in \mathcal{I}_k} |\lambda_j|^p \Big]^{q/p} \Big)^{1/q} .$$

We then have,

Theorem 2.3. *Let $0 < p \le 1$, $0 < q \le \infty$, and let f be a distribution given by $f = \sum_j \lambda_j \, a_j(x)$, where the a_j's are H^p atoms, and the convergence is in the sense of distributions. Further, assume that the family $\{I_j\}$ consisting of the supports of the a_j's has bounded overlap at each level $\mathcal{I}_k$ uniformly in k, and $\|\lambda\|_{[p,q]} < \infty$. Then, $f \in H^{p,q}$, and $\|f\|_{H^{p,q}} \le c \, \|\lambda\|_{[p,q]}$.*

Proof. Let $Mf(x) = \sup_{t>0} |(f * \psi_t)(x)|$ denote the radial maximal function of f with respect to a suitable smooth function ψ with support contained in $\{|x| \le 1\}$ and nonvanishing integral. We will verify that Mf satisfies the conditions of Lemma 1.1 and is thus in $L^{p,q}$.

Fix an integer k_0 and let

$$g(x) = \sum_{k<k_0} \sum_{j \in \mathcal{I}_k} \lambda_j a_j(x) .$$

Since $\|Mg\|_\infty \le \|g\|_\infty$ it suffices to estimate $|g(x)|$. Let C be the bounded overlap constant for the family of the supports of the a_j's. Then, for $j \in \mathcal{I}_k$,

$$|\lambda_j| \; |a_j(x)| = \frac{1}{|I_j|^{1/p}} \, |\lambda_j| \; |I_j|^{1/p} \, |a_j(x)| \le 2^k \chi_{I_j}(x) ,$$

and, consequently,

$$|g(x)| \le \sum_{k<k_0} 2^k \sum_j \chi_{I_j}(x) \le C \, 2^{k_0} .$$

Next, let

$$h(x) = \sum_{k \ge k_0} \sum_{j \in \mathcal{I}_k} \lambda_j a_j(x) .$$

Since a_j has $N = [n(1/p-1)]$ vanishing moments, it is not hard to see that, if I_j is the defining interval of a_j and I_j is centered at x_j, and $\gamma = (n+N+1)/n > 1/p$, then, with c independent of j, $\varphi_j(x) = Ma_j(x)$ satisfies

$$\varphi_j(x) \le c\,\frac{|I_j|^{\gamma-1/p}}{(|I_j| + |x-x_j|^n)^\gamma}\,.$$

Thus, if $1/\gamma < \varepsilon p < 1$,

$$Mh(x)^{\varepsilon p} \le c \sum_{j\in\mathcal{I}_k, k\ge k_0} \frac{(|\lambda_j|\;|I_j|^{\gamma-1/p})^{\varepsilon p}}{(|I_j| + |x-x_j|^n)^{\gamma\varepsilon p}}\,,$$

which, upon integration, yields

$$\int_{R^n} Mh(x)^{\varepsilon p}\,dx \le c \sum_{j\in\mathcal{I}_k, k\ge k_0} (|\lambda_j|\;|I_j|^{\gamma-1/p})^{\varepsilon p} \int_{R^n} \frac{1}{(|I_j| + |x-x_j|^n)^{\gamma\varepsilon p}}\,dx\,.$$

The integrals in the right-hand side above are of order $|I_j|^{1-\gamma\varepsilon p}$ and, consequently, by Chebychev's inequality,

$$2^{k_0\varepsilon p}|\{Mh > 2^{k_0}\}| \le c \sum_{j\in\mathcal{I}_k, k\ge k_0} |\lambda_j|^{\varepsilon p}\,|I_j|^{1-\varepsilon} \le c \sum_{k\ge k_0} 2^{k\varepsilon p} \sum_{j\in\mathcal{I}_k} |I_j|\,.$$

Thus, Lemma 1.1 applies with $\varphi = Mf$, $\psi_{k_0} = Mg$, $\eta_{k_0} = Mh$, and $\mu_k = \left(\sum_{j\in\mathcal{I}_k} |I_j|\right)^{1/p}$, and we get

$$\big\|\{2^k\, m(Mf,2^k)^{1/p}\}\big\|_{\ell^q} \le c \left\|\left\{2^k\Big(\sum_{j\in\mathcal{I}_k}|I_j|\Big)^{1/p}\right\}\right\|_{\ell^q},$$

which, since

$$|I_j| \sim \frac{|\lambda_j|^p}{2^{kp}}\,,\quad j\in\mathcal{I}_k\,,$$

is bounded by $c\,\|\lambda\|_{[p,q]}$, $0<q\le\infty$. ∎

The next result is of interest because it applies to arbitrary decompositions in $H^{p,q}$. The proof relies on Lemma 1.2, and is left to the reader.

Theorem 2.4. *Let $0<p\le 1$, $0<q\le\infty$, and let f be a distribution given by $f = \sum_j \lambda_j\, a_j(x)$, where the a_j's are H^p atoms, and the convergence is in the sense of distributions. Further, assume that $\|\lambda\|_{[\eta,q]} < \infty$ for some $0<\eta<\min(p,q)$. Then, $f\in H^{p,q}$, and $\|f\|_{H^{p,q}} \le c\,\|\lambda\|_{[\eta,q]}$.*

2.1 Interpolation between Hardy-Lorentz spaces

We are now ready to identify the intermediate spaces of a couple of Hardy-Lorentz spaces with the same first index $p \le 1$.

Theorem 2.5. *Let $0 < p \le 1$. Given $0 < q_1 < q < q_2 \le \infty$, define $0 < \eta < 1$ by the relation $1/q = (1-\eta)/q_1 + \eta/q_2$. Then, with equivalent quasinorms,*

$$H^{p,q} = (H^{p,q_1}, H^{p,q_2})_{\eta,q}\,.$$

Proof. Since the non-tangential maximal function Nf of a distribution f in H^{p,q_1} is in L^{p,q_1}, and that of f in H^{p,q_2} is in L^{p,q_2}, we have

$$K(t, Nf; L^{p,q_1}, L^{p,q_2}) \le c\, K(t, f; H^{p,q_1}, H^{p,q_2})\,.$$

Thus,

$$\|Nf\|_{p,q} \sim \|Nf\|_{(L^{p,q_1},L^{p,q_2})_{\eta,q}} \le c\,\|f\|_{(H^{p,q_1},H^{p,q_2})_{\eta,q}}\,,$$

and $(H^{p,q_1}, H^{p,q_2})_{\eta,q} \hookrightarrow H^{p,q}$.

To show the other embedding, with the notation in the proof of Theorem 2.1, write $f = \sum_k \sum_j \lambda_{j,k} a_{j,k}$, and recall that for every integer k, the level set $\mathcal{I}_k = \{j : |\lambda_{j,k}|/|I_{j,k}|^{1/p} \sim 2^k\}$ contains exclusively the sequence $\{\lambda_{j,k}\}$. Let $\mu_k^p = \sum_{j\in\mathcal{I}_k} |\lambda_{j,k}|^p$. By construction, $\sum_k \mu_k^q \sim \|f\|^q_{H^{p,q}}$. Now, rearrange $\{\mu_k\}$ into $\{\mu_l^*\}$, and, for each $l \ge 1$, let k_l be such that $\mu_{k_l} = \mu_l^*$. For $l_0 \ge 1$, let $\mathcal{K}_{l_0} = \{k_1, \dots, k_{l_0}\}$, and put $f_{1,l_0} = \sum_{k\in\mathcal{K}_{l_0}} \sum_j \lambda_{j,k} a_{j,k}$ and $f_{2,l_0} = f - f_{1,l_0}$. Then, by Theorem 2.2, $f_{1,l_0} \in H^{p,q_1}$, $f_{2,l_0} \in H^{p,q_2}$, and, with the usual interpretation for $q_2 = \infty$,

$$\|f_{1,l_0}\|_{H^{p,q_1}} \le c\Big(\sum_1^{l_0} \mu_l^{*q_1}\Big)^{1/q_1}, \quad \|f_{2,l_0}\|_{H^{p,q_2}} \le c\Big(\sum_{l_0+1}^{\infty} \mu_l^{*q_2}\Big)^{1/q_2}.$$

So, for $t > 0$ and every positive integer l_0, we have

$$K(t, f; H^{p,q_1}, H^{p,q_2}) \le c\Big[\Big(\sum_1^{l_0} \mu_l^{*q_1}\Big)^{1/q_1} + t\Big(\sum_{l_0+1}^{\infty} \mu_l^{*q_2}\Big)^{1/q_2}\Big].$$

Now, by Homstedt's formula, there is a choice of l_0 such that the right-hand side above $\sim K(t, \{\mu_k\}; \ell^{q_1}, \ell^{q_2})$, and, consequently,

$$K(t, f; H^{p,q_1}, H^{p,q_2}) \le c\, K(t, \{\mu_k\}; \ell^{q_1}, \ell^{q_2})\,.$$

Thus,

$$\begin{aligned}\|f\|_{(H^{p,q_1},H^{p,q_2})_{\eta,q}} &\le c\,\|\{\mu_k\}\|_{(\ell^{q_1},\ell^{q_2})_{\eta,q}}\\ &\le c\,\|\{\mu_k\}\|_{\ell_q} \le c\,\|f\|_{H^{p,q}}\,,\end{aligned}$$

and $H^{p,q} \hookrightarrow (H^{p,q_1}, H^{p,q_2})_{\eta,q}$. ■

The reader will have no difficulty in verifying that Theorem 2.5 gives that if T is a continuous, sublinear map from H^1 into L^1, and from $H^{1,\infty}$ into $L^{1,\infty}$, then $\|Tf\|_{1,q} \le c\,\|f\|_{H^{1,q}}$ for $1 < q < \infty$. This observation has numerous applications. For instance, consider the Calderón-Zygmund singular integral operators with variable kernel defined by

$$T_\Omega(f)(x) = \text{ p.v.} \int_{R^n} \frac{\Omega(x,\, x-y)}{|x-y|^n}\, f(y)\, dy\,.$$

Under appropriate growth and smoothness assumptions on Ω, T_Ω maps H^1 continuously into L^1, see [6], and $H^{1,\infty}$ continuously into $L^{1,\infty}$, see [8]. Thus, if Ω satisfies the assumptions of both of these results, T_Ω maps $H^{1,q}$ continuously into $L^{1,q}$ for $1 < q < \infty$. A similar result follows by invoking the characterization of $H^{1,q}$ given by C. Fefferman, Rivière and Sagher. However, in this case the $H^p - L^p$ estimate requires additional smoothness of Ω, as shown, for instance, in [6]. Similar considerations apply to the Marcinkiewicz integral, see [9], and [7].

Finally, when $p < 1$, our results cover, for instance, the δ-CZ operators satisfying $T^*(1) = 0$ discussed by Alvarez and Milman, see [3]. These operators, as well as a more general related class introduced in [15], preserve H^p and $H^{p,\infty}$ for $n/(n+\delta) < p \le 1$, and, consequently, by Theorem 2.5, they also preserve $H^{p,q}$ for p in that same range, and $q > p$.

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