Entropy, pressure, ground states and calibrated sub-actions for linear dynamics

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Abstract: Denote by X a Banach space and by $T : X \to X$ a bounded linear operator with non-trivial kernel satisfying suitable conditions. We consider the concepts of entropy - for T-invariant probability measures - and pressure for Hölder continuous potentials. We also prove the existence of ground states (the limit when temperature goes to zero) associated with such class of potentials when the Banach space X is equipped with a Schauder basis. We produce an example concerning weighted shift operators defined on the Banach spaces $c_0(\mathbb{R})$ and $l^p(\mathbb{R})$, $1 \leq p < +\infty$, where our results do apply. In addition, we prove the existence of calibrated sub-actions when the potential satisfies certain regularity conditions using properties of the so-called Mañé potential. We also exhibit examples of selection at zero temperature and explicit sub-actions in the class of Hölder continuous potentials.

Keywords: entropy, equilibrium states, ergodic optimization, ground states, invariant probabilities, linear dynamics, pressure, Ruelle operator, sub-actions, weighted shifts.

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1 Introduction

Ergodic Optimization is the branch of mathematics dedicated to the study of the properties of the set of invariant probability measures that maximize the value of the integral with respect to a fixed (at least continuous) potential A (see for instance [BLL13], [Gar17], [Jen06] and [Jen19]). The underlying dynamics of the above-mentioned papers is given by the shift map. This is the classical Ergodic Optimization framework.

Here we are interested in discrete-time Linear Dynamical Systems acting on a separable Banach space X. We point out that the dynamics of a bounded linear operator has some features which are quite different from the dynamics

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of the shift. For instance, if $v \in X$ is a periodic point for $T : X \to X$, then, the collection of all vectors in the one-dimensional subspace generated by v are also periodic points. Nowadays there exists substantial work on the dynamics of bounded linear operators acting on Banach spaces (see for instance [BM09, BCD$^+$18, BM20, GS91, GM14, GP11]).

In Statistical Mechanics the influence of the temperature Temp > 0 is described by considering the potential $\frac{1}{\text{Temp}} A$. It is common to introduce the parameter $t = \frac{1}{\text{Temp}}$. The Gibbs state associated with the potential $t\,A$ will be denoted by $\mu_{t\,A}$. When the potential is of Hölder class several nice properties can be derived for its corresponding Gibbs states (see for instance [PP90]). In [LMSV19] the authors show the existence of T-invariant probabilities with full support using the Gibbs state point of view.

Given a Hölder potential A we call equilibrium state for A an invariant one that maximizes a variational principle of pressure (to be defined later). For the case when the dynamics is given by the shift several papers addressed the question of showing that a Gibbs state for A is an equilibrium state for A as well (see for instance [PP90]). In the first part of our paper we will be interested in equilibrium states (they are invariant by the dynamics governed by the bounded linear operator T). The second part of the paper is dedicated to the topic of Ergodic Optimization.

Several results in Ergodic Optimization were developed using tools of Thermodynamical Formalism for compact and Polish metric spaces (see for instance [BLL13, BGT18, Gar17, GT12, Jen06, JMU06, JMU07]). This is so because maximizing probabilities for a potential A appear in a natural way as the limit of equilibrium states when the temperature Temp $= \frac{1}{t}$ goes to zero (which is the same to say that $t \to +\infty$). An invariant probability obtained as an accumulation point (as the limit of a subsequence $t_n \to +\infty$) is called a *ground state*. They are special in some sense because in addition, they maximize the entropy among all the maximizing probability measures of the potential A (see [BLL13, CLT01, FV18, JMU05, Kem11]).

In the case of the uniqueness of such accumulation point, we say that there exists *selection of probability at zero temperature*. This is the case for example when the maximizing probability is unique. One can show that for a generic Hölder potential, the maximizing probability is unique (see for instance [CLT01]). A very important result in the area shows that generically in the class of Hölder continuous potentials, the maximizing probability has support in a unique periodic orbit (see [Con16]).

In [Bre03], [CGU11] and [Lep05], it was proved (without assuming uniqueness of the maximizing probability) the existence of a unique accumulation point at zero temperature in the context of subshifts of finite type under the assumption that the potential depends only on finite coordinates. Se also [BGT18] and [GT12] for another kind of examples where there exists selection at zero temperature.

In [BCL$^+$11] and [vER07], the authors present examples of a Hölder continuous potential where selection at zero temperature does not occur.

For the shift map in the non-compact setting, similar results were proved showing the existence of ground states (the existence of accumulation points at zero temperature). Results in this direction were obtained for countable Markov shifts satisfying the so-called BIP property, topologically transitive countable Markov shifts, and full shifts defined on the lattice $\mathbb{R}^{\mathbb{N}}$ (see for instance [FV18, JMU05, Kem11, LMMS15, LV20a, SV20]).

The study of sub-actions and calibrated sub-actions provides important tools in the study of maximizing measures associated with Hölder continuous (or even continuous) potentials and its corresponding supports, This is so because a sub-action provides tools to identify the support of maximizing probabilities (see [BLL13, CLT01]). Several works addressed this issue in the context of XY models, expanding maps of the circle, subshifts of finite type, and even in non-compact settings such as shifts defined on Polish spaces. A helpful tool used to find sub-actions associated with a potential satisfying certain regularity is the so-called Mañé potential (see for details [BCL$^+$11, CF19, CLT01, Gar17, GL08, JMU07, LMMS15, LMST09])

The dynamics of bounded linear operators on Banach spaces (see [BM09, BCD$^+$18, Gil20, GP11]) present some special properties which are significantly different from the ones for the shift and continuous maps acting on Polish spaces. Questions of topological nature as expansivity, shadowing, transitivity, and structural stability in this linear setting were addressed in [BCD$^+$18, BM20, Gil20]. We refer the reader to the first part of [LMSV19] for a short account of some basic definitions and results concerning the theory of Linear Dynamics on vector spaces of infinite dimension.

For the discrete-time dynamical action of a bounded linear operator $T : X \to X$ on a Banach space X, the paper [GM14] present results about the existence of T-invariant probability measures with full support in the case that X is reflexive and separable. An extension of this result for a more general setting, including non-reflexive separable Banach spaces, was presented in [LMSV19]. This was obtained via the classical tool in Thermodynamic Formalism known as Ruelle-Perron-Frobenius Theorem. It was assumed that the kernel of the bounded linear operator L is nontrivial and of finite dimension. In order to define this operator, it was necessary in [LMSV19] to fix an *a priori* probability on the kernel of T with some properties.

For most of the reasoning of the present paper. we consider similar assumptions as in [LMSV19]. We introduce the concepts of entropy and pressure in the setting of Linear Dynamics on vector spaces of infinite dimension. Our first goal is to show that for a given potential $A : X \to \mathbb{R}$, satisfying some mild regularity assumptions, the Gibbs state μ_A obtained in Theorem 1 in [LMSV19] is in fact an equilibrium state associated with A i.e. it is satisfied a variational principle of pressure and the supremum is attained at μ_A.

Our second goal is to present results on the topic of Ergodic Optimization. We prove that the accumulation points of the family of probability measures $(\mu_{tA})_{t>1}$, when t goes to ∞, are maximizing probability measures for A. We point out the existence of accumulation points is not a trivial matter. This will require assuming some properties for the potential A in order to be able to show that a certain sequence of probabilities is tight. For some of our results, we will need some more strong assumptions on the *a priori* probability measure than the ones considered in [LMSV19]. Moreover, we show the existence of calibrated sub-actions when the potential is at least Hölder continuous. Our most important new assumption is to suppose that the Banach space X has a Schauder basis which is a property satisfied by a wide class of Banach spaces, such as, $c_0(\mathbb{R})$, $l^p(\mathbb{R})$, $1 \leq p < +\infty$ and any separable Hilbert space H. Moreover, we present examples concerning frequently hypercyclic and Devaney chaotic operators $L : X \to X$, which are defined on the Banach spaces $X = c_0(\mathbb{R})$ or $X = l^p(\mathbb{R})$, $1 \leq p < +\infty$. This class of operators, which were also considered in [LMSV19], are given by the equation

$$L((x_n)_{n\geq 1}) = (\alpha_n x_{n+1})_{n\geq 1} ,$$

where $(\alpha_n)_{n\geq 1}$ is a sequence of real numbers satisfying suitable conditions (see for instance [BM09, BCD$^+$18, BM20, GP11]). This type of operator is known in the classical mathematical literature as weighted shift.

At the end of the paper, we introduce the concept of Mañé potential in the setting of Linear Dynamics. Taking advantage of the Mañé potential we prove the existence of calibrated sub-actions associated with Hölder continuous potentials, under the assumption that the set of maximizing measures is nonempty. Furthermore, we show that it is not necessary to assume the hypothesis of summable variations on the potential in order to guarantee the result. In the last section we also present some examples where is possible to get an explicit expression for the calibrated sub-action and is guaranteed the selection at zero temperature.

The paper is organized as follows:

In section 2 we introduce basic definitions and state the main results to be obtained in the paper.

In section 3 we prove the existence of ground states in the matter of Linear Dynamics on vector spaces of infinite dimension. More precisely, in section 3.1 appears the proof of Theorem 1 which guarantees that any Gibbs state is, in fact, an equilibrium state. In section 3.2 appears the proof of Theorem 2, which guarantees the existence of ground states. Finally, in section 3.4 we present a particular example in the setting of weighted shifts.

In section 4 we show existence of calibrated sub-actions using the Mañé potential. More precisely, in section 4.1 appears the proof of Theorem 3 and in section 4.2 we present some examples to illustrate the theory.

The proofs of some of our results (in the setting Linear Dynamics) are similar to the analogous ones for the shift, in this case, we will just briefly mention to the reader references for the proof. For some other results, the proofs are quite different and we provide full details in this case.

2 Main results

In this section we present some basic definitions that are required for our reasoning and we also state the main results of the paper. Consider a Banach space X equipped with the norm $\|\cdot\|_X$. Given a bounded linear operator $T : X \to X$, we denote by $\|T\|_{co}$ the *co-norm* of T, which is given by

$$\|T\|_{co} := \inf\{\|T(x)\|_X : \|x\|_X = 1\} .$$

It is known that for any injective operator T is true that $\|T\|_{co} > 0$. Moreover, for each $x \in X$, we have that $\|T\|_{co}\|x\|_X \leq \|T(x)\|_X \leq \|T\|_{op}\|x\|_X$, where $\|T\|_{op}$ is the *operator norm* of T. From this follows the existence of the limit $\lim_{n\to+\infty}(\|T^n\|_{co})^{\frac{1}{n}}$. Indeed, the limit exists as a consequence of the sub-additivity of the sequence of real numbers $(\|T^n\|_{co})_{n\geq 1}$.

Hereafter, we assume that the Banach space X can be decomposed as the direct sum

$$X = \mathrm{Ker}(T) \oplus E ,$$

where $\mathrm{Ker}(T)$ denotes the *kernel* of T, which is assumed of finite dimension.

Then, defining

$$p(T) := \inf\{\|T(x)\|_X : \|x\|_X = 1,\, x \in E\} , \tag{1}$$

it follows that $p(T) > 0$ and $p(T)\|x\|_X \leq \|T(x)\|_X$ for each $x \in E$. Furthermore, under the assumption that for each $n \in \mathbb{N}$ the Banach space X admits a decomposition of the form

$$X = \mathrm{Ker}(T^n) \oplus E_n ,$$

where $E_{n+1} \subset E_n$, it follows that $p(T^n) > 0$. Besides that, as a consequence of the sub-additivity of the sequence $(p(T^n))_{n\geq 1}$, it follows the existence of the limit $\lim_{n\to+\infty}(p(T^n))^{\frac{1}{n}}$.

It is known that the convergence of the series $\sum_{n=1}^{+\infty} p(T^n)^{-\alpha}$ to a real number implies that the dynamical system $T : X \to X$ is Devaney chaotic and frequently hypercyclic (see for details [BM09, GP11]). In particular, the above implies that the dynamical system T is topologically transitive under the assumption that X is a separable Banach space.

For each $x \in X \setminus \{0\}$ and any $n \in \mathbb{N}$, we use the following notation

$$T^{-n}(x) := T^{-n}(\{x\}) = \{v \in X : T^n(v) = x\} .$$

Assuming that T is surjective but not injective, we obtain that the set $T^{-1}(x)$ is a non-empty and non-singleton set. Moreover, for each $v \in T^{-1}(x)$ it is satisfied the expression

$$T^{-1}(x) = \mathrm{Ker}(T) + \{v\} = \{z + v : z \in \mathrm{Ker}(T)\} .$$

In other words, $T^{-1}(x)$ is isometrically isomorphic to $\mathrm{Ker}(T)$; this property will be quite useful in the definition of the Ruelle operator (in the same way as in [LMSV19]).

Denote by $\mathcal{C}(X)$ the set of *continuous functions* from X into $\mathbb{R}$ and by $\mathcal{C}_b(X)$ the set of *bounded continuous functions* from X into $\mathbb{R}$. The set $\mathcal{C}_b(X)$ equipped with the uniform norm $\|\cdot\|_\infty$, which is given by $\|\varphi\|_\infty := \sup\{|\varphi(x)| : x \in X\}$, is a Banach space.

Definition 1. *We say that a potential $A \in \mathcal{C}(X)$ has* summable variations *with respect to the bounded linear operator $T : X \to X$, if*

$$V_T(A) := \sum_{n=1}^{+\infty} V_{T,n}(A) < +\infty \ , \tag{2}$$

where

$$V_{T,n}(A) := \sup\{|A(x) - A(y)| : \ T^i(x) = T^i(y), \ 1 \leq i \leq n\} \ .$$

We denote by $\mathcal{SV}_T(X)$ the set of potentials with summable variations. That is, the ones satisfying the equation in (2).

Given a function $\varphi \in \mathcal{C}(X)$ and $\alpha \in (0,1]$, define

$$\mathrm{Hol}_\varphi^\alpha := \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{\|x - y\|_X^\alpha} \ .$$

A function φ is called *α-Hölder continuous*, if $\mathrm{Hol}_\varphi^\alpha < +\infty$. The set of all the α-Hölder continuous functions from X into $\mathbb{R}$ is denoted by $\mathcal{H}_\alpha(X)$ and the set of all the *bounded α-Hölder continuous functions* is denoted by $\mathcal{H}_{b,\alpha}(X)$.

Given T as above, we consider an *a priori* probability measure ν on the kernel of T whose support is all the kernel. For instance, when the kernel has dimension one, we could take ν as the Gaussian distribution on $\mathbb{R}$ (which is isomorphic to the kernel of T) with mean zero and variance 1. More precisely, taking $\nu := f\, dr$, with $f(r) := \frac{1}{\sqrt{2\pi}} e^{-r^2/2}$.

Definition 2. *We say that a probability ν has* strong adapted tails *if, for any $\epsilon > 0$ there exists a sequence of positive numbers $(\kappa_n)_{n\geq 1}$ such that*

1. $\sum_{n=1}^{+\infty} \nu([-p(T^n)\kappa_n, \ p(T^n)\kappa_n]^c) < \epsilon$*;*
2. *the sequence $(\kappa_n)_{n\geq 1}$ belongs to $l^1(\mathbb{R})$.*

The above conditions are stronger than the ones presented in the definition of *adapted measure* that appears in [LMSV19]. Any probability measure satisfying the conditions of Definition 2 is also an adapted measure as defined in [LMSV19].

Now, we define a transfer operator acting on the set $\mathcal{C}_b(X)$, which will help us to find the so-called equilibrium states and ground states, via the existence of Gibbs states and the variational principle. Given a bounded above potential $A \in \mathcal{H}_\alpha(X)$ and an *a priori* Borelian probability measure ν, supported on $\mathrm{Ker}(T)$ and satisfying the strong adapted tails property, the *Ruelle operator* $\mathcal{L}_A$

of the potential $A : X \to \mathbb{R}$ is defined as the map assigning to each $\varphi \in \mathcal{C}_b(X)$ the function $\mathcal{L}_A(\varphi) \in \mathcal{C}_b(X)$ given by

$$\mathcal{L}_A(\varphi)(x) := \int_{z\in \mathrm{Ker}(T)} e^{A(z+v)}\varphi(z+v)d\nu(z), \quad v \in T^{-1}(x) \ . \tag{3}$$

We say that the potential A is *normalized* (for the *a priori* probabilty ν) if $\mathcal{L}_A(1) = 1$.

Denote by $\mathcal{B}(X)$ the set of *finite Borelian measures* on X. By well-known properties of the dual operator (see for instance [LMSV19]), we can define the dual Ruelle operator $\mathcal{L}_A^*$ as the map that assigns to each $\mu \in \mathcal{B}(X)$ the probability $\mathcal{L}_A^*(\mu)$, which for each $\varphi \in \mathcal{C}_b(X)$ satisfies

$$\int_X \varphi d\Big(\mathcal{L}_A^*(\mu)\Big) := \int_X \mathcal{L}_A(\varphi)d\mu \ .$$

Following the proof of the main Theorem in [LMSV19], with a modification in the part that guarantees the existence of the main eigenfunction (we adapt the reasoning of Section 3 in [CSS19] to the linear dynamics setting), it is not difficult to check that for any bounded above potential $A \in \mathcal{H}_\alpha(X) \cap \mathcal{SV}_T(X)$, there is an eigenvalue $\lambda_A > 0$ and a strictly positive eigenfunction $\psi_A \in \mathcal{H}_{b,\alpha}(X)$ associated with λ_A. Moreover, under the assumption that $\mathcal{L}_A(1) = 1$, it is possible to guarantees the existence of a *Gibbs state* μ_A (i.e. a fixed point of the operator $\mathcal{L}_A^*$).

The hypothesis on [LMSV19] was that the potential was bounded (above and below) and we will need here for some of our results that A is just bounded above.

Note that even in the case when A is not a normalized potential, the above claim implies the existence of an eigenprobability ρ_A (for the dual Ruelle operator $\mathcal{L}_A^*$) associated with the same eigenvalue λ_A. The probability ρ_A has the same support of the Gibbs state associated with the associated normalized potential (to be defined below). Indeed, since the potential

$$\overline{A} := A + \log(\psi_A) - \log(\psi_A \circ T) - \log(\lambda_A) \ .$$

is a normalized potential one gets that $\mathcal{L}_{\overline{A}}^*\mu_{\overline{A}} = \mu_{\overline{A}}$. Thus, denoting $\mu_A = \mu_{\overline{A}}$, it follows that $\rho_A = \frac{1}{\psi_A}d\mu_A$ satisfies the desired property. We call $\overline{A}$ the associated normalized potential for A.

We say that μ_A is the *Gibbs state* for the (non-normalized) potential A, ρ_A is the *conformal measure* associated with A and $\lambda_A > 0$ is the *main eigenvalue* of $\mathcal{L}_A$.

Hereafter, we use the notation $\mathcal{P}(X)$ for the set of *Borelian probability measures* on X and we denote by $\mathcal{P}_T(X)$ the ones that are *invariant* by the action of T.

Definition 3. *Given an* a priori *probability measure ν, supported on the Kernel of T, and $\mu \in \mathcal{P}_T(X)$, the* entropy *of μ is defined as*

$$h_\nu(\mu) := \inf\Big\{\int_X \log\Big(\frac{\mathcal{L}_0(u)}{u}\Big)d\mu : u \in \mathcal{C}_b(X),\ u > 0\Big\}. \tag{4}$$

Remark 1. *The values of $h_\nu(\mu)$ are non-positive and for the measure μ of maximal entropy the value $h_\nu(\mu) = 0$. In this last case, we can guarantee the existence of such μ, because the potential $B \equiv 0$ is a normalized potential.*

When considering the symbolic space $\{1, 2.., d\}^{\mathbb{N}}$, the shift $\sigma : \{1, 2.., d\}^{\mathbb{N}} \to \{1, 2.., d\}^{\mathbb{N}}$, and taking ν as the *counting measure* on $\{1, 2, .., d\}$, the real positive value one obtains from the analogous expression to (4) (using Ruelle operator, etc...) is exactly the Kolmogorov-Shannon entropy (see [Lo1, LMMS15, SV20]). If we take ν as the *normalized counting probability* on $\{1, 2, .., d\}$, then we get that the value obtained from (4) is the Kolmogorov-Shannon entropy minus the value $\log d$ (therefore, a non positive number) as explained in [LMMS15].

When the set of preimages of any point with respect to the dynamics is not a countable set, it is not appropriate to define entropy via dynamical partitions. This happens for instance when considering the shift acting on the symbolic space $M^{\mathbb{N}}$, where M is a compact metric space (like the case where M is the unitary circle). Then, alternatively, one can define entropy via the information provided by the Ruelle operator (which depends on an *a priori* probability measure ν as above in (4), and taking the Ruelle operator associated with the potential which is constant equal to zero). We point out that (in the general case) it is required to take ν as a probability (and not a measure) in order for the expression (4) to be well-defined . By taking ν as a probability (not a infinite measure) the value we obtain in (4) is non-positive, and the maximal possible value of the entropy of an invariant probability is the value zero. We refer the reader to [LMMS15, SV20] for more details.

In this way, it is natural to adapt this point of view in our setting; defining entropy of an T-invariant probability via (4). It is important to notice that (4) depends on the so-called *a priori* probability measure. Hence, fixing a probability (for instance the natural standard Gaussian measure on $\mathrm{Ker}(T)$), such definition results in a topological invariant with respect to the map T.

Definition 4. *Given a bounded above potential $A \in \mathcal{H}_\alpha(X) \cap \mathcal{SV}_T(X)$, we call the* pressure *of A the value*

$$P_\nu(A) := \sup\Big\{ h_\nu(\mu) + \int_X A d\mu : \mu \in \mathcal{P}_T(X) \Big\} .$$

In addition, we say that $\widehat{\mu} \in \mathcal{P}_T(X)$ is an equilibrium state *for A, if*

$$P_\nu(A) = h_\nu(\widehat{\mu}) + \int_X A d\widehat{\mu} .$$

The first result of our paper claims that given a potential A satisfying suitable conditions (see [LMSV19] for the assumptions), the set of Gibbs states associated with A is contained into the set of equilibrium states for A and $P_\nu(A) = \log(\lambda_A)$, where λ_A is the main eigenvalue of the transfer operator $\mathcal{L}_A$. Actually, in [LMSV19] the authors assume that the potential A defining the transfer operator is bounded and Hölder continuous. However, as we already mentioned, the proof in [LMSV19] also guarantees the existence of such Gibbs states when the potential A is Hölder continuous and bounded above.

The precise statement of our first result is the following:

Theorem 1. *Consider a separable Banach space X and a bounded linear operator $T : X \to X$, such that, T is surjective but not bijective. Assume also that $X = \mathrm{Ker}(T^n) \oplus E_n$, with $E_{n+1} \subset E_n$, for each $n \in \mathbb{N}$, and moreover suppose that $\sum_{n=1}^{+\infty} p(T^n)^{-\alpha} < +\infty$. For each bounded above potential $A \in \mathcal{H}_\alpha(X) \cap \mathcal{SV}_T(X)$ denote the Gibbs state associated with A by μ_A. Then, the following variational principle is satisfied*

$$P_\nu(A) = \log(\lambda_A) = h_\nu(\mu_A) + \int_X A d\mu_A \, .$$

That is, the Gibbs state μ_A associated with A is also an equilibrium state for A.

Theorem 1 is widely known in the mathematical literature as the variational principle of pressure, or the Ruelle Theorem. This result will be a quite useful instrument throughout the paper. We will use it as a tool to find the so-called *maximizing measures* and *ground states* of the potential A in the setting of Linear Dynamics. Below we establish conditions in order to guarantees the existence of such measures. We would like to point out to the reader that the proof of this variational principle depends exclusively on properties of the transfer operator defined in (3). However, we include it in order to facilitate the understanding of the theory in the setting that we are interested.

Definition 5. *A* Schauder basis *for the Banach space X is a sequence $(e_k)_{k\geq 1}$ of vectors in X, such that, for each $x \in X$ there is a unique sequence of real numbers $(\alpha_k)_{k\geq 1}$ satisfying*

$$\lim_{n\to+\infty} \Big\| x - \sum_{k=1}^{n} \alpha_k e_k \Big\|_X = 0 \, . \tag{5}$$

First note that any Banach space equipped with a Schauder basis results in a separable space. Furthermore, it is widely known that any Schauder basis for X induces a corresponding basis of *coordinate functions* for the dual space X', which is given by a sequence $(\pi_k)_{k\geq 1}$ on X', such that, for each $i, j \in \mathbb{N}$ it is satisfied $\pi_j(e_i) = \delta_{ij}$. Hereafter, we use the notation $x := \sum_{k=1}^{+\infty} \alpha_k e_k$ when the vector $x \in X$ satisfies the limit in (5) for the values $\alpha_k = \pi_k(x)$ for each $k \in \mathbb{N}$. Actually, note that the sequence $(\pi_k)_{k\geq 1}$ is a total subset of X', i.e., if $\pi_k(x) = 0$ for each $k \in \mathbb{N}$, it follows that $x = 0$.

There are several examples of Banach spaces equipped with a Schauder basis, for instance $X \in \{c_0(\mathbb{R}),\ l^p(\mathbb{R}),\ 1 \leq p < +\infty\}$ and $X = H$, where H is an arbitrary separable Hilbert space; these are typical examples of spaces satisfying that property. In particular, the set of coordinate functions agrees with the Schauder basis itself when $X = H$ is an arbitrary separable Hilbert space. Moreover, in the last case, the Schauder basis is called as *Hilbert basis* of H.

Definition 6. *Consider a Banach space X equipped with a Schauder basis $(e_k)_{k\geq 1}$ and coordinate functions $(\pi_k)_{k\geq 1}$. Define the set*

$$X_{i,j} := \{x \in X : \; j \leq |\alpha_i| \leq j+1\}$$

We say that a potential $A \in \mathcal{C}(X)$ satisfies the summability condition*, if for each $i \in \mathbb{N}$ is satisfied*

$$\sum_{j=1}^{+\infty} e^{\sup\{A(x) :\, x \in X_{i,j}\}} < +\infty \, . \tag{6}$$

In particular, the so-called *summability condition* implies that for each $i \in \mathbb{N}$ it is satisfied the property $\lim_{j\to+\infty} \sup\{A(x) : \; x \in X_{i,j}\} = -\infty$. Hence, in that case we get $\sup(A) < +\infty$, and therefore, the potential A cannot be a bounded below potential.

Definition 7. *Given a potential $A \in \mathcal{C}(X)$, a measure $\mu_{\max} \in \mathcal{P}_T(X)$ is called a* maximizing measure *for A, if $\int_X A d\mu_{\max} = m(A)$, where*

$$m(A) := \sup\Big\{\int_X A d\mu : \mu \in \mathcal{P}_T(X)\Big\} \, . \tag{7}$$

We denote by $\mathcal{P}_{\max}(A)$ the set of invariant probability measures attaining the supremum in (7)*, which is usually called as the set of* maximizing measures *of A.*

It is important to point out that for any potential A in $\mathcal{C}(X)$ satisfying the so-called summability condition, we have $m(A) < +\infty$. However, even assuming that condition we can have that $\mathcal{P}_{\max}(A) = \emptyset$. In the following example, we present a case where the set of maximizing measures is a non-empty subset of $\mathcal{P}_T(X)$. The above, under strong assumptions in the behavior of the potential.

Example 1. *When v is such that $T^k(v) = v$ the probability measure $\widetilde{\mu}$ with support in the periodic orbit $\{v, T(v), T^2(v), ..., T^{k-1}(v)\}$ of the form*

$$\widetilde{\mu} := \frac{1}{k} \sum_{j=0}^{k-1} \delta_{T^j(v)}$$

is T-invariant. Then, in this case $\frac{1}{k}\sum_{j=0}^{k-1} A(T^j(v)) \leq m(A)$. If for all $j = 0, 1, 2, ..., k-1$, we have that $A(T^j(v)) = m(A)$, it follows that $\widetilde{\mu}$ is maximizing for A

In order to prove the existence of ground states and maximizing measures in the case of bounded above potentials belonging to the set $\mathcal{H}_\alpha(X) \cap \mathcal{SV}(X)$, we will take advantage of certain properties of Gibbs states, which are consequences of the main Theorem in [LMSV19]. The next theorem claims that under suitable assumptions for the potential A, the set of ground states is a non-empty set.

Note that this is a non-trivial claim. This is so by the lack of compactness of the Banach space X and also by the fact that the closed balls are not compact in such vector spaces (by the Theorem of Riesz).

The statement of the result is as follows.

Theorem 2. *Let X be a Banach space equipped with a Schauder basis $(e_k)_{k\geq 1}$ and consider $T : X \to X$ a bounded linear operator surjective but not bijective. Assume also that $X = \mathrm{Ker}(T^n) \oplus E_n$, with $E_{n+1} \subset E_n$, for each $n \in \mathbb{N}$, and $\sum_{n=1}^{+\infty} p(T^n)^{-\alpha} < +\infty$. Then, for any potential $A \in \mathcal{H}_\alpha(X) \cap \mathcal{SV}_T(X)$ satisfying the summability condition, the family of equilibrium states $(\mu_{tA})_{t>1}$ has accumulation points at infinity - denoted generically by μ_∞. The probability μ_∞ will be maximizing for A.*

Our second interest in this paper is to be able to locate the union of the supports of maximizing measures associated with a fixed potential $A \in \mathcal{C}(X)$. A tool widely used for this purpose, in the classic literature on Ergodic Optimization, is based on calibrated sub-actions. In order to use this tool, some hypotheses are needed. All of this, of course, for the case where $\mathcal{P}_{\max}(A)$ is a non-empty set. The last Theorem provides conditions for the existence of maximizing probabilities.

We would like to mention that some results about the existence of maximizing measures (where the underlying space is not compact) for a certain class of potentials, defined on some Polish spaces with bounded expansive metrics, were presented in [JMU07].

Another class of potentials where $\mathcal{P}_{\max}(A)$ is a non-empty set is the following:

Definition 8. *Assume that the potential A belongs to $\mathcal{C}(X)$. We say that A satisfies the* maximizing property*, if there are vector subspaces $Y, Z \subset X$, such that $X = Y \oplus Z$, $\dim(Y) < +\infty$, and for each $x = x_y + x_z$ belonging to X, with $x_y \in Y$ and $x_z \in Z$, we have*

$$A(x) \leq A(x_y)\ , \tag{8}$$

and

$$\lim_{\|x_y\|_X \to +\infty} A(x_y) = -\infty\ . \tag{9}$$

Note that the maximizing property for A implies that $\sup(A) < +\infty$. In Lemma 4 we prove that under suitable conditions of regularity for the potential A, the maximizing property implies that $\mathcal{P}_{\max}(A)$ is a non-empty set.

In Lemma 3 we prove that $\mathcal{P}_{\max}(A) \neq \emptyset$ when there exist ground states (the last claim in Theorem 2)

Given A, the property $\mathcal{P}_{\max}(A) \neq \emptyset$ will guarantee that the sub-action obtained via the Mañé potential is well-defined .

The so-called sub-actions are also known in the mathematical literature as revelations (for details see [GL08, Jen06, Jen19])

Definition 9. *Given a potential $A \in \mathcal{C}(X)$, we say that a continuous function $V \in \mathcal{C}(X)$ is a* sub-action *associated with A, if satisfies the following inequality*

$$V \circ T \geq V + A - m(A)\ . \tag{10}$$

There are continuous potentials A without a sub-action (see for instance [Jen06]). In general, some regularity of the potential A is required for the existence of sub-actions (see [CLT01]).

It is easy to see that for any maximizing probability μ for A and each sub-action V associated with A, the support of μ is contained in the set

$$\Omega(A) := \{x \in X : \ V(T(x)) - V(x) - A(x) + m(A) = 0\} . \tag{11}$$

It is also true that in the case the support of a probability μ is contained in the set $\Omega(A)$, then, μ is a maximizing probability for A (see [CLT01]).

In this way, a sub-action can help to find explicitly the support of a maximizing measure (see for instance [CLT01, BLL13, Gar17]).

Definition 10. *Let V be a sub-action for A, if for any $x \in X$ there exists a point $y \in T^{-1}(x)$ which attains the equality in* (10) *i.e.*

$$V(x) = V(y) + A(y) - m(A) ,$$

we say that V is a calibrated sub-action*.*

There are cases where one can show the existence of maximizing probabilities for continuous potentials even if we cannot apply the Ruelle-Perron-Frobenius Theorem formalism. In this case, we can consider maximizing probabilities without talking about ground states (also known in the mathematical literature as limits at temperature zero). Sub-actions will be helpful anyway (see the expression in (11)). In example 2 we will consider a potential which is not bounded but exhibits an explicit sub-action.

In order to prove the existence of sub-actions we introduce an additional tool known in the classical mathematical literature as Mañé potential (for details see [CLT01, GL08] and section 3 in [CLO14] for the case of the finite shift).

Definition 11. *Given a potential $A \in \mathcal{H}_\alpha(X)$ we define the* Mañé potential ϕ_A *associated with A as the map given by*

$$\phi_A(x, y) := \lim_{\epsilon \to 0} \sup_{n \in \mathbb{N}} \sup_{\substack{x' \in T^{-n}(y) \\ \|x - x'\|_X < \epsilon}} S_n(A - m(A))(x') , \tag{12}$$

where $S_n(A)(x) := \sum_{j=0}^{n-1} A(T^j(x))$ is the n-th ergodic sum of the potential A.

We claim that under the assumptions in Theorem 3, we have that $\phi_A(x, y)$ belongs to $[-\infty, 0]$, for any pair of points $x, y \in X$ (for details of the proof see Section 4.1).

Now, we will present conditions to guarantee the existence of sub-actions associated with Hölder continuous potentials A via the use of the Mañé potential.

The statement of our third main result is the following:

Theorem 3. *Consider X a Banach space and $T : X \to X$ a bounded linear operator surjective but not injective. Assume that $X = \mathrm{Ker}(T^n) \oplus E_n$, with $E_{n+1} \subset E_n$, for each $n \in \mathbb{N}$, and suppose that $\sum_{n=1}^{+\infty} p(T^n)^{-\alpha} < +\infty$. Suppose also that $A \in \mathcal{H}_\alpha(X)$ and $\mathcal{P}_{\max}(A)$ is a non-empty set. Then, for each $x \in X$ the map $y \to \phi_A(x, y)$ belongs to $\mathcal{H}_\alpha(X)$ and it is a calibrated sub-action for A.*

An example of a potential A where the hypotheses of the above Theorem are satisfied is $A(x) = -\|x - v\|_X$, where v is a fixed point for T. In this case, the probability with support on v is maximizing for A (see Example 2 for a more complete discussion).

3 Existence of equilibrium states and maximizing measures

In this section, we present the proof of Theorem 1, which asserts the existence of equilibrium states for the Gibbs states obtained in the Ruelle-Perron-Frobenius Theorem formalism. We also present an example in the context of weighted shifts (see for instance [BM09, BCD$^+$18, BM20, GP11]). This will be proved in Subsection 3.1.

Subsection 3.2 is dedicated to another topic. We propose two different roads to guarantees the existence of maximizing measures. The first one, through the existence of the so-called ground states for potentials satisfying a condition of decay called in this paper the summability condition. The second one, using the so-called maximizing property, in which we assume, *in some way*, that the potential attains its maximum on a finite dimensional subspace of X. Our purpose is to be able to set conditions such that the Mañé potential is well-defined .

We assume throughout the section that X is a separable Banach space, $T : X \to X$ is a bounded linear operator surjective, but not bijective. We also assume that for each natural number n, we get $X = \mathrm{Ker}(T^n) \oplus E_n$, with $E_{n+1} \subset E_n$, and $\sum_{n=1}^{+\infty} p(T^n)^{-\alpha} < +\infty$.

3.1 A variational principle

Theorem 1 claims existence of an equilibrium state for each α-Hölder continuous potential A with summable variations. First note that, by [LMSV19], it is guaranteed existence of the eigenvalue $\lambda_A > 0$, a strictly positive eigenfunction $\psi_A \in \mathcal{H}_{b,\alpha}(X)$ and a Gibbs state $\mu_A \in \mathcal{P}_T(X)$. The Gibbs state is the natural candidate to be the equilibrium state associated with the potential A. The next result is the first step in order to prove that claim.

Lemma 1. *Consider $A \in \mathcal{H}_{b,\alpha}(X)$ and $T : X \to X$ satisfying the conditions of Theorem 1. Then, for any $\mu \in \mathcal{P}_T(X)$ it is satisfied the inequality*

$$h_\nu(\mu) + \int_X A d\mu \le \log(\lambda_A) \ . \tag{13}$$

Proof. Assume that $\mu \in \mathcal{P}_T(X)$. Then,

$$h_\nu(\mu) + \int_X A d\mu = \inf\Big\{\int_X \log\Big(\frac{\mathcal{L}_0(u)}{u}\Big) d\mu : u \in \mathcal{C}_b^+(X)\Big\} + \int_X A d\mu$$
$$\leq \int_X \log\Big(\frac{\mathcal{L}_0(e^A \psi_A)}{\psi_A}\Big) d\mu = \log(\lambda_A) \; .$$

□

In order to prove our first main result, we will show that the supremum in the expression of the left-hand of (13) is attained by the so-called Gibbs state obtained in Theorem 1 of [LMSV19].

Proof of Theorem 1. By Lemma 1, it follows that

$$P_\nu(A) = \sup\Big\{h_\nu(\mu) + \int_X A d\mu : \mu \in \mathcal{P}_T(X)\Big\} \leq \log(\lambda_A) \; . \tag{14}$$

On other hand, choosing $u_A = e^{\overline{A}}$, it follows that

$$\int_X \log\Big(\frac{\mathcal{L}_0(u_A)}{u_A}\Big) d\mu_A = -\int_X \overline{A} d\mu_A \; .$$

Furthermore, any $u \in \mathcal{C}_b^+(X)$ can be expressed as $u = ve^{\overline{A}}$, for some $v \in \mathcal{C}_b^+(X)$, which implies that

$$\int_X \log\Big(\frac{\mathcal{L}_0(u)}{u}\Big) d\mu = \int_X \log\Big(\frac{\mathcal{L}_{\overline{A}}(v)}{v}\Big) d\mu_A - \int_X \overline{A} d\mu_A \geq -\int_X \overline{A} d\mu_A \; .$$

That is,

$$h_\nu(\mu_A) = -\int_X \overline{A} d\mu_A = -\int_X A d\mu_A + \log(\lambda_A) \; . \tag{15}$$

Therefore, since $\mu_A \in \mathcal{P}(X)$, by (14) and (15), it follows that

$$h_\nu(\mu_A) + \int_X A d\mu_A = \log(\lambda_A) = \sup\Big\{h_\nu(\mu) + \int_X A d\mu : \mu \in \mathcal{P}_T(X)\Big\} := P(A) \, .$$

□

Given a potential A, the above variational principle only requires the existence of well-defined transfer operator $\mathcal{L}_A$ acting on the space of bounded continuous function and the existence of ψ_A satisfying $\mathcal{L}_A(\psi_A) = \lambda_A \psi_A$.

3.2 Accumulation points at zero temperature

In this Subsection, we present the proof of Theorem 2 which is the second main result of our paper. Our strategy is to adapt a classical result used for finding accumulation points of families of probability measures defined on non-compact metric spaces satisfying suitable hypothesis, widely known in the mathematical literature as Prohorov's conditions (see for instance [Pro56, Pro61]). In order to apply the result to our setting, it is necessary to define an adequate family of compact sets with large enough mass. In other words, we need to find sets with measure close to one for all the members of the family $(\mu_{tA})_{t>1}$; we have to show that the family of equilibrium states $(\mu_{tA})_{t>1}$ is tight.

Throughout this section we assume that the space X is equipped with a fixed Schauder basis $(e_k)_{k\geq 1}$. That is, for any $x \in X$ there is a unique sequence of real numbers $(\alpha_k)_{k\geq 1}$ satisfying equation (5). Furthermore, we assume that for each $k \in \mathbb{N}$ it is satisfied the equality $\alpha_k = \pi_k(x)$, where $(\pi_k)_{k\geq 1}$ is the corresponding basis of coordinate functions for the dual space X'.

Consider a strictly increasing sequence of natural numbers $(m_i)_{i\geq 1}$. Given $k \in \mathbb{N}$, we define

$$\Lambda_k := \{\pi_{m_1}, ..., \pi_{m_k}\} \subset X' \ . \tag{16}$$

Now, we define the function $\Pi_{\Lambda_k} : X \to \mathbb{R}^k$ assigning to each point $x \in X$ the value $\Pi_{\Lambda_k}(x) := (\pi_{m_1}(x), ..., \pi_{m_k}(x))$. Given any set of the form $E_k = \prod_{i=1}^{k}[a_i, b_i] \subset \mathbb{R}^k$, we say that $C_{\Lambda_k}(E_k)$ is a cylindrical set generated by E_k and Λ_k, if it is of the form

$$C_{\Lambda_k}(E_k) := \Pi_{\Lambda_k}^{-1}(E_k) \subset X \ . \tag{17}$$

We point out that the definition of cylindrical sets given above is a particular case of the one that appears on page pp. 406 in [Pro61]. It is easy to check that for any $k \in \mathbb{N}$ and each finite family of bounded closed intervals $\{[a_i, b_i]\}_{i=1}^{k}$, the set defined by

$$\bigcap_{i=1}^{k}\{x \in X : \ a_i \leq \pi_{m_i}(x) \leq b_i\} \ ,$$

is a cylindrical set. Indeed, we have

$$\bigcap_{i=1}^{k}\{x \in X : \ a_i \leq \pi_{m_i}(x) \leq b_i\} = \Pi_{\Lambda_k}^{-1}\Big(\prod_{i=1}^{k}[a_i, b_i]\Big).$$

By Lemma 3 in [Pro61], we can guarantee that a family of Borel probability measures $(\mu_s)_{s\in S}$ on X results in a tight family if, and only if, for any $\epsilon > 0$ there exists a compact set $\mathbb{K}_\epsilon \subset X$ and a finite family of linear functionals Λ_k (which we choose as defined by (16)), such that, for each $s \in S$ it is satisfied the inequality

$$\mu_s(\Pi_{\Lambda_k}^{-1}(\Pi_{\Lambda_k}(\mathbb{K}_\epsilon))) > 1 - \epsilon \ . \tag{18}$$

In the following Lemma, we show that any cylindrical set of the form (17) satisfies a kind of upper Gibbs inequality. This will be one of the main tools to prove the existence of accumulation points for the family of equilibrium states.

Lemma 2. *Consider a bounded above potential $A \in \mathcal{H}_\alpha(X) \cap \mathcal{SV}_T(X)$ and a collection of Borel sets $\{E_k\}_{k\geq 1}$ of the form $E_k = \prod_{i=1}^{k}[a_i, b_i]$, where $(a_i)_{i\geq 1}$ is a strictly decreasing sequence of real numbers and $(b_i)_{i\geq 1}$ is a strictly increasing sequence of real numbers with $a_1 < b_1$. Then, there is a constant $C > 1$, such that, for each $k \in \mathbb{N}$ and any $\widetilde{x}$ belonging to the cylindrical set $C_{\Lambda_k}(E_k)$ it is satisfied the following inequality*

$$\frac{\mu_A(C_{\Lambda_k}(E_k))}{e^{S_k A(\widetilde{x}) - k\log(\lambda_A)}} \leq C \ . \tag{19}$$

Furthermore, we have $C = e^{4V_T(A)}$.

Proof. In order to simplify our reasoning, we assume first that A is a normalized potential and, after that, we prove the result for an arbitrary potential. The above assumption guarantees that $\mathcal{L}_A(1) = 1$ and $\mathcal{L}_A^*(\mu_A) = \mu_A$. Since $C_{\Lambda_k}(E_k)$ is a closed subset of X, there exists a function $\phi \in \mathcal{C}(X)$, such that,

$$\mathbf{1}_{C_{\Lambda_k}(E_k)} \leq \phi \leq 1 \ .$$

Therefore, it follows that for any $\widetilde{x} \in C_{\Lambda_k}(E_k)$ it is satisfied

$$\begin{aligned}
&\mu_A(C_{\Lambda_k}(E_k)) \\
&= \int_X \mathbf{1}_{C_{\Lambda_k}(E_k)} d\mu_A \\
&\leq \int_X \phi d\mu_A \\
&= \int_X \mathcal{L}_A^k(\phi) d\mu_A \\
&= \int_X \int_{z_k \in \mathrm{Ker}(T^k)} \ldots \int_{z_1 \in \mathrm{Ker}(T)} e^{S_k A(z_k + v_k)} \phi(z_k + v_k) d\nu(z_1) \ldots d\nu(z_k) d\mu_A(x) \\
&\leq e^{S_k A(\widetilde{x}) + \sum_{i=1}^{k} V_{T,i}(A)} \int_X \int_{z_k \in \mathrm{Ker}(T^k)} \ldots \int_{z_1 \in \mathrm{Ker}(T)} \phi(z_k + v_k) d\nu(z_1) \ldots d\nu(z_k) d\mu_A(x) \\
&\leq e^{S_k A(\widetilde{x}) + V_T(A)} \int_X \mathcal{L}_A(1) d\mu_A \\
&= e^{S_k A(\widetilde{x}) + V_T(A)} \ ,
\end{aligned}$$

where $V_T(A)$ is given by (2). Therefore, it follows that

$$\frac{\mu_A(C_{\Lambda_k}(E_k))}{e^{S_k A(\widetilde{x})}} \leq e^{V_T(A)} \ . \tag{20}$$

Now, assume that $A \in \mathcal{H}_\alpha(X) \cap \mathcal{SV}_T(X)$ is not a normalized potential, we consider its corresponding normalization

$$\overline{A} = A + \log(\psi_A) - \log(\psi_A \circ T) - \log(\lambda_A) \ ,$$

where the potential $\psi_A \in \mathcal{H}_{b,\alpha}(X)$ and the value $\lambda_A > 0$ satisfy the equation $\mathcal{L}_A(\psi_A) = \lambda_A \psi_A$. It is easy to check that $\overline{A} \in \mathcal{H}_\alpha(X) \cap \mathcal{SV}_T(X)$, $\mathcal{L}_{\overline{A}}(1) = 1$ and $\mu_A = \mu_{\overline{A}}$ (see for instance [LMSV19]).

Note that for any $k \in \mathbb{N}$ and each $\widetilde{x} \in C_{\Lambda_k}(E_k)$

$$S_k \overline{A}(\widetilde{x}) = S_k A(\widetilde{x}) - k \log(\lambda_A) + \log(\psi_A)(\widetilde{x}) - \log(\psi_A)(T^k(\widetilde{x})) \ .$$

Besides that, since $\mathcal{L}_A(\psi_A) = \lambda_A \psi_A$, we have

$$e^{\log(\psi_A)} = \int_{z \in \mathrm{Ker}(T)} e^{(A + \log(\psi_A))(z+v)} d\nu(z), \ v \in T^{-1}(x) \ .$$

Then, is not difficult to check that for any $x, y \in X$ is satisfied $|\log(\psi_A)(x) - \log(\psi_A)(y)| < V_T(A)$ (a similar reasoning to the one used in the case of the function u in the proof of item a) of Theorem 1 in [LMSV19]). The above implies that $\sup(\log(\psi_A)) - \inf(\log(\psi_A)) \leq V_T(A)$ and $V_T(\overline{A}) \leq 3V_T(A)$. Thus, from (20), it follows immediately that

$$\begin{aligned} \frac{\mu_A(C_{\Lambda_k}(E_k))}{e^{S_k A(\widetilde{x}) - k\log(\lambda_A)}} &\leq e^{V_T(\overline{A}) + \log(\psi_A)(\widetilde{x}) - \log(\psi_A)(T^k(\widetilde{x}))} \\ &\leq e^{V_T(\overline{A}) + V_T(A)} \\ &\leq e^{4V_T(A)} \ . \end{aligned}$$

Hence, choosing the constant C as the value $e^{4V_T(A)}$, we get (19), and the proof is finished. ☐

Remark 2. *Note that Lemma 2 guarantees that for any set $X_{i,j}$ of the form described by Definition 6, we have*

$$\frac{\mu_A(X_{i,j})}{e^{A(\widetilde{x}) - \log(\lambda_A)}} \leq e^{4V_T(A)} \ .$$

The above is true because $X_{i,j} = \pi^{-1}([-(j+1), -j] \cup [j, (j+1)]$. This inequality will be of great importance in the proof of Theorem 2.

Take Λ_k as in (16). Now, we are able to present a collection of compact sets $\{\mathbb{K}_\epsilon : \epsilon \in (0,1)\}$ satisfying (18) for the family of equilibrium states $(\mu_{tA})_{t>1}$. Given $k \in \mathbb{N}$ and a strictly increasing sequence of natural numbers $(n_i)_{i \geq 1}$, we define the sets

$$X_{\Lambda_k} := \big\{x \in X : \ \pi_l(x) = 0, \ l \notin \{m_1, ..., m_k\}\big\}$$

and

$$\mathbb{K}_k := \Big\{x \in X : \ |\pi_{m_i}(x)| \leq n_{m_i}, \ 1 \leq i \leq k\Big\} \cap X_{\Lambda_k} \subset X \ . \tag{21}$$

Therefore, the set $\mathbb{K}_k$ is a non-empty compact subset of the Banach space X. Furthermore, it is not difficult to check that

$$\mathbb{K}_k = \Pi_{\Lambda_k}^{-1}\Big(\prod_{i=1}^{k} [-n_{m_i}, n_{m_i}]\Big) \cap X_{\Lambda_k} \ .$$

Now, we are able to present the proof of the second main result of this paper.

Proof of Theorem 2. First note that given $k \in \mathbb{N}$, $\Lambda_k = \{\pi_{m_1}, ..., \pi_{m_k}\}$, and $Y \subset X$, we have

$$\begin{aligned}\Pi_{\Lambda_k}(Y) &= (\pi_{m_1}(Y), ..., \pi_{m_k}(Y)) \\ &= (\pi_{m_1}(Y \cap X_{\Lambda_k}), ..., \pi_{m_k}(Y \cap X_{\Lambda_k})) \\ &= \Pi_{\Lambda_k}(Y \cap X_{\Lambda_k}) \ .\end{aligned}$$

Then, for each $k \in \mathbb{N}$, the set $\mathbb{K}_k$ defined in (21) satisfies

$$\begin{aligned}\Pi_{\Lambda_k}^{-1}\Big(\Pi_{\Lambda_k}(\mathbb{K}_k)\Big) &= \Pi_{\Lambda_k}^{-1}\Big(\Pi_{\Lambda_k}\Big(\Pi_{\Lambda_k}^{-1}\Big(\prod_{i=1}^{k}[-n_{m_i}, n_{m_i}]\Big) \cap X_{\Lambda_k}\Big)\Big) \\ &= \Pi_{\Lambda_k}^{-1}\Big(\Pi_{\Lambda_k}\Big(\Pi_{\Lambda_k}^{-1}\Big(\prod_{i=1}^{k}[-n_{m_i}, n_{m_i}]\Big)\Big)\Big) \\ &= \Pi_{\Lambda_k}^{-1}\Big(\prod_{i=1}^{k}[-n_{m_i}, n_{m_i}]\Big) \\ &\supset \bigcap_{i=1}^{+\infty} \pi_i^{-1}([-n_i, n_i]) \ .\end{aligned}$$

Therefore, for each $t > 1$, we have

$$\begin{aligned}\mu_{tA}\Big(\Pi_{\Lambda_k}^{-1}\Big(\Pi_{\Lambda_k}(\mathbb{K}_k)\Big)\Big) &\geq \mu_{tA}\Big(\bigcap_{i=1}^{+\infty} \pi_i^{-1}([-n_i, n_i])\Big) \\ &\geq 1 - \sum_{i=1}^{+\infty} \mu_{tA}(\{x \in X : \ |\alpha_i| \geq n_i\}) \\ &\geq 1 - \sum_{i=1}^{+\infty} \mu_{tA}\Big(\bigcup_{j=n_i}^{+\infty} \{x \in X : \ j \leq |\alpha_i| \leq j+1\}\Big) \\ &\geq 1 - \sum_{i=1}^{+\infty} \sum_{j=n_i}^{+\infty} \mu_{tA}(X_{i,j}) \ .\end{aligned}$$

Thus, in order to define each set $\mathbb{K}_\epsilon = \mathbb{K}_{k,\epsilon}$, with $\epsilon \in (0,1)$, it is enough to guarantee the existence of a strictly increasing sequence natural numbers $(n_i^\epsilon)_{i \geq 1}$, such that,

$$\sum_{j=n_i^\epsilon}^{+\infty} \mu_{tA}(X_{i,j}) < \frac{\epsilon}{2^i} \ .$$

This is so because in this case

$$\mu_{tA}\Big(\Pi_{\Lambda_k}^{-1}\Big(\Pi_{\Lambda_k}(\mathbb{K}_{k,\epsilon})\Big)\Big) \geq 1 - \sum_{i=1}^{+\infty} \sum_{j=n_i^\epsilon}^{+\infty} \mu_{tA}(X_{i,j}) > 1 - \sum_{i=1}^{+\infty} \frac{\epsilon}{2^i} = 1 - \epsilon \ . \quad (22)$$

Consider the potential $B \equiv 0$ which obviously belongs to the set $\mathcal{H}_\alpha(X) \cap \mathcal{SV}(X)$. It is easy to check that $\mathcal{L}_B(1) = 1$, and this implies that $\log(\lambda_B) = 0$. Then, by Theorem 1, it follows that

$$0 = h_\nu(\mu_B) \; .$$

Now, define

$$I := \int_X A d\mu_B \leq \sup(A) < +\infty \; .$$

By the above and Theorem 1, it follows that

$$\begin{aligned} \log(\lambda_{tA}) - tI &= h_\nu(\mu_{tA}) + t\int_X (A - I) d\mu_{tA} \\ &= h_\nu(\mu_{t(A-I)}) + t\int_X (A - I) d\mu_{tA} \\ &\geq h_\nu(\mu_B) + t\int_X (A - I) d\mu_B \\ &= h_\nu(\mu_B) = 0 \; . \end{aligned}$$

Therefore, by (19), for any pair $i, j \in \mathbb{N}$ and all $x \in X_{i,j}$

$$\begin{aligned} \mu_{tA}(X_{i,j}) &\leq e^{tA(x) - \log(\lambda_{tA}) + 4V_T(tA)} \\ &= e^{t(A(x)-I) - (\log(\lambda_{tA}) - tI) + 4tV_T(A)} \\ &\leq e^{t(A(x) - I + 4V_T(A))} \\ &\leq e^{t(\sup\{A(x) :\, x \in X_{i,j}\} - I + 4V_T(A))} \; . \end{aligned}$$

Now, by (6), it follows that for each $i \in \mathbb{N}$

$$\lim_{j \to +\infty} \sup\{A(x) :\; x \in X_{i,j}\} = -\infty \; .$$

The above implies the existence of $j_i \in \mathbb{N}$, such that, for any $j \geq j_i$ we have

$$\sup\{A(x) :\; x \in X_{i,j}\} - I + 4V_T(A) < 0 \; ,$$

which implies that

$$\mu_{tA}(X_{i,j}) \leq e^{\sup\{A(x) :\, x \in X_{i,j}\} - I + 4V_T(A)} \; . \tag{23}$$

Besides that, also by (6), given $\epsilon > 0$, we can find $n_i^\epsilon \geq j_i$, such that

$$\sum_{j=n_i^\epsilon}^{+\infty} e^{\sup\{A(x) :\, x \in X_{i,j}\}} < \frac{\epsilon}{2^i} e^{I - 4V_T(A)} \; . \tag{24}$$

Observe that (23) and (24) implies (22), and form this follows that for any $t > 1$,

$$\mu_{tA}\Big(\Pi_{\Lambda_k}^{-1}\Big(\Pi_{\Lambda_k}(\mathbb{K}_{k,\epsilon})\Big)\Big) > 1 - \epsilon \; .$$

From the above reasoning it follows that the family of equilibrium states $(\mu_{tA})_{t>1}$ is tight (for more details see Lemma 3 in [Pro61]). Therefore, it follows from the Prohorov's Theorem the existence of a sequence $(t_n)_{n\geq 1}$, such that, the sequence of equilibrium states $(\mu_{t_n A})_{n\geq 1}$ is convergent, with a limit denoted by μ_∞ (see for instance [Bil99, Pro56]).

It is important to point out that the limits of different subsequences do not have necessarily to be the same. $\square$

Remark 3. *In the case that* $\mathrm{Ker}(T^n) = \mathrm{span}\{e_1, ..., e_n\}$ *the condition* (6) *can be replaced by the condition*

$$\sum_{n=1}^{+\infty} e^{\sup\{A(x):\, x\in X_{1,n}\}} < +\infty\,.$$

Note that in this case, the summability only depends on the behavior of the first component of x with respect to the Schauder basis $(e_k)_{k\geq 1}$. For example, this property is satisfied in the case of weighted shifts.

3.3 Maximizing measures

In this section, we will show the existence of maximizing measures, associated with a certain class of potentials, in the framework of linear dynamical systems on Banach spaces of infinite dimension. This will be achieved under two different assumptions. First, we prove that the so-called ground states, whose existence was obtained in section 3.2, are maximizing measures. This will be proved for potentials A satisfying the hypothesis of Theorem 2. After that, we prove the existence of maximizing measures under the assumption that the potential A satisfies the maximizing property (see Definition 8). In these two cases, it will follow (see Lemma 5) that the Mañé potential only takes non-positive values, which is one of the main requirements which will be used in the proof of Theorem 3.

The following Lemma guarantees the existence of maximizing measures under the conditions of Theorem 2.

Lemma 3. *Assume that μ_∞ is a ground state obtained from the existence claimed by Theorem 2. Then, we have that $\mu_\infty \in \mathcal{P}_{\max}(A)$.*

Proof. The proof of this Lemma follows from Theorem 1 and the fact that $h_\nu(\mu) \leq 0$, for any $\mu \in \mathcal{P}_T(X)$. Indeed, it is not difficult to check that $m(A) = \lim_{t\to+\infty} \frac{\log(\lambda_{tA})}{t}$ (see for instance [BCL+11, LMMS15]). Therefore, for any

$\mu \in \mathcal{P}_\sigma(X)$ we have

$$\begin{aligned} m(A) &= \lim_{n\in\mathbb{N}} \frac{\log(\lambda_{t_n A})}{t_n} \\ &= \lim_{n\in\mathbb{N}} \frac{h_\nu(\mu_{t_n A})}{t_n} + \int_X A d\mu_{t_n A} \\ &\leq \lim_{n\in\mathbb{N}} \int_X A d\mu_{t_n A} \\ &= \int_X A d\mu_\infty \leq m(A) \; . \end{aligned}$$

The above implies that $\mu_\infty \in \mathcal{P}_{\max}(A)$. ☐

In the next Lemma, we also guarantee the existence of maximizing measures, but in this case, we assume the so-called maximizing property.

Lemma 4. *Consider a potential $A \in \mathcal{H}_\alpha(X)$ satisfying the maximizing property and let Y, Z be the subspaces of X satisfying* (8) *and* (9)*. If*

$$Y = \mathrm{span}\{v, ..., T^{k-1}(v)\}$$

for some periodic point $v \in X$ of period k, it follows that $\mathcal{P}_{\max}(A) \neq \emptyset$.

Proof. Consider subspaces $Y, Z \subset X$ satisfying the hypothesis of this Lemma. Note that under these assumptions all the points in the subspace Y are periodic points of period k.

Given a point $x \in X$, consider the values

$$m_n(A, x) := \frac{1}{n} S_n A(x)$$

and

$$m(A, x) := \limsup_{n\to+\infty} m_n(A, x) \; .$$

It is not difficult to verify that $m(A) \leq \sup_{x\in X} m(A, x)$ (see for instance [JMU07]). Moreover, $m(A, x_y) = m_k(A, x_y) \leq m(A)$, when x_y belongs to Y. Indeed, this is true because $m(A, x_y)$ is the value of the integral of A with respect to the periodic measure $\frac{1}{k}\sum_{j=0}^{k-1} \delta_{T^j(x_y)}$.

From the limit in (9), there is a constant $M > 0$, such that for each $x \in X$ with $\|x_y\| > M$,

$$m(A, x_y) < m(A) - 1 \; . \tag{25}$$

Define the set $X_M := \{x \in X : \|x_y\|_X \leq M\}$. Note that given a point $x \in X_M$, we can express such point as $x = x_y + x_z$ where $x_y \in Y$ has norm less than or equal to M and $x_z \in Z$, i.e. $X_M = Y_M + Z$, where $Y_M := \{y \in Y : \|y\|_X \leq M\}$. On other hand, since any point in Y is a periodic point of period k, it follows that

$$Y_0 := \bigcup_{j=0}^{+\infty} T^j(Y_M) = \bigcup_{j=0}^{k-1} T^j(Y_M) \; , \tag{26}$$

which implies that the set $Y_0 \subset Y$ is compact and invariant by the action of T.

Note that for each $x \in X$, such that, $x_y \in Y \setminus Y_0$, it is valid the inequality $\|x_y\| > M$, which implies (25). Thus, taking supremum on all the points $x_y \in Y \setminus Y_0$ and using (8), we obtain that

$$\sup_{\substack{x\in X\\ x_y\in Y\setminus Y_0}} m(A,x) \leq \sup_{x_y\in Y\setminus Y_0} m(A,x_y) \leq m(A) - 1\ . \tag{27}$$

On other hand, we have

$$\sup_{\substack{x\in X\\ x_y\in Y_0}} m(A,x) \leq \sup_{x_y\in Y_0} m(A,x_y) \leq m(A)\ , \tag{28}$$

where the first one of the inequalities is a consequence of (8) and the second one follows from the fact that all the points belonging to the T-invariant set Y_0 are in fact periodic points of period k.

By inequalities (27) and (28), we have $\sup_{x\in X} m(A,x) \leq m(A)$. Therefore, using that $m(A) \leq \sup_{x\in X} m(A,x)$ (see for instance [JMU07]), it follows that

$$\sup_{x\in X} m(A,x) = m(A)$$

Furthermore, by (8), we have

$$\sup_{x\in X} m(A,x) = \sup_{x_y\in Y_0} m(A,x_y) = m(A)$$

The above guarantees the existence of a maximizing measure supported on the compact set Y_0, which in fact is the limit in the weak* topology of periodic measures supported on periodic points belonging to Y_0. $\square$

Note that Lemmas 3 and 4 imply that $\mathcal{P}_{\max}(A)$ is a non-empty set. We claim that it will follow from all of the above that the Mañé potential is well-defined (there exist a finite supremum in expression (12)) and only takes non-positive values.

Lemma 5. *Assume that $\mathcal{P}_{\max}(A)$ is a non-empty set. Then,*

$$S_n(A - m(A)) \leq 0\ . \tag{29}$$

In particular, the Mañé potential ϕ_A associated with A is well-defined and takes only takes values in the set $[-\infty, 0]$.

Proof. Assume that the potential A satisfies the first condition of the Lemma. Since we are assuming that $\sum_{n=1}^{+\infty} p(T^n)^{-\alpha} < +\infty$, it follows that the dynamics T is Devaney chaotic, which implies that

$$\sup_{x\in X} m(A,x) = \sup_{x\in \mathrm{Per}(X)} m(A,x) \leq m(A)\ , \tag{30}$$

where $\mathrm{Per}(X)$ is the set of periodic points of X. Moreover, the last inequality in (30) follows from the fact that for any periodic orbit $\{v, ..., T^{k-1}(v)\}$ is satisfied the inequality

$$\int_X Ad\Big(\sum_{j=1}^{k-1} \delta_{T^j(v)}\Big) \leq m(A) \ .$$

Thus, by (30), we get (29). □

3.4 An example: weighted shifts

In this section, we present particular cases where the assumptions of Theorem 1 are true. We will consider the so-called weighted shifts in this section (see for details [BM09, BCD$^+$18, BM20, GP11, LMSV19]).

Consider the normed space $c_0(\mathbb{R})$ which is the set of sequences of real numbers $(x_n)_{n\geq 1}$ satisfying $\lim_{n\to+\infty} x_n = 0$ equipped with the norm

$$\|x\|_{c_0(\mathbb{R})} := \sup_{n\geq 1} |x_n| \ ,$$

and the normed space $l^p(\mathbb{R})$, $1 \leq p < +\infty$, which is the set of sequences satisfying $\sum_{n=1}^{+\infty} |x_n|^p < +\infty$ equipped with the norm

$$\|x\|_{l^p(\mathbb{R})} := \Big(\sum_{n=1}^{+\infty} |x_n|^p\Big)^{\frac{1}{p}} \ .$$

$c_0(\mathbb{R})$ and $l^p(\mathbb{R})$, $1 \leq p < +\infty$, are separable Banach spaces which are equipped with the Schauder basis $(e_k)_{k\geq 1}$, where $e_k = (\delta_{ik})_{i\geq 1}$. Moreover, in the case $1 < p < +\infty$, the space $l^p(\mathbb{R})$ is reflexive as well.

Consider $c, c' \in \mathbb{R}$ satisfying $0 < c < c'$ and a sequence $(\alpha_n)_{n\geq 1}$, such that, $\alpha_n \in (c, c')$ for each $n \in \mathbb{N}$. For each pair of natural numbers k, n define

$$\beta_k^n := \alpha_k \dots \alpha_{k+n-1} \ .$$

Now, fixing the number n, define

$$d_n := \inf\{\beta_k^n :\ k \in \mathbb{N}\} \ .$$

The weighted shift $L : X \to X$, where $X = c_0(\mathbb{R})$ or $X = l^p(\mathbb{R})$, $1 \leq p < +\infty$, is given by the linear map $L((x_n)_{n\geq 1}) = (\alpha_n x_{n+1})_{n\geq 1}$. It is well-known that the asymptotic behavior of the values β_k^n characterize the topological properties of the weighted shift L (see Remark 1 in [LMSV19] and [BM09, BCD$^+$18, BM20, GP11] for details). Indeed, taking $p(L^n) = d_n$, for each $n \in \mathbb{N}$, the map p satisfies definition (1). Thus, the convergence of the series $\sum_{n=1}^{+\infty} d_n^{-\alpha}$ to a real number implies that the weighted shift L is Devaney chaotic and topologically transitive.

Under the assumptions above, the Ruelle operator defined in (3) is given by

$$\mathcal{L}_A(\varphi)(x_1, x_2,\ ...) := \int_{\mathbb{R}} e^{A\left(r, \frac{x_1}{\alpha_1}, \frac{x_2}{\alpha_2},\ ...\right)} \varphi\Big(r, \frac{x_1}{\alpha_1}, \frac{x_2}{\alpha_2},\ ...\Big) d\nu(r) \ .$$

The next Propositions are a consequence of the main Theorems of this paper; under some strong assumptions in the values β_k^n and d_n.

Proposition 1. *Let X be $c_0(\mathbb{R})$ or $l^p(\mathbb{R})$, $1 \leq p < +\infty$, and $L : X \to X$ a weighted shift. Assume also that one of the following conditions is satisfied:*

i) $\lim_{n\to+\infty}(d_n)^{-\frac{1}{n}}$;

ii) $\sup\{\sum_{n=1}^{+\infty}(\beta_k^n)^{-1} : k \in \mathbb{N}\} < +\infty$.

Then, for each bounded above potential $A \in \mathcal{H}_\alpha(X) \cap \mathcal{SV}_L(X)$ the Gibbs state μ_A associated with A satisfies

$$\log(\lambda_A) = h_\nu(\mu_A) + \int_X A d\mu_A = P_\nu(A)\,.$$

Proof. By Lemma 2 in [LMSV19], items $i)$ and $ii)$ are equivalent to the property $\sum_{n=1}^{+\infty}(d_n)^{-\alpha} < +\infty$. Thus, our claim follows from Theorem 1. Indeed, this is true because $p(L^n) = d_n$, for each $n \in \mathbb{N}$. □

Proposition 2. *Let X be $c_0(\mathbb{R})$ or $l^p(\mathbb{R})$, $1 \leq p < +\infty$, and $L : X \to X$ a weighted shift. Assume also one of the following conditions:*

i) $\lim_{n\to+\infty}(d_n)^{-\frac{1}{n}}$;

ii) $\sup\{\sum_{n=1}^{+\infty}(\beta_k^n)^{-1} : k \in \mathbb{N}\} < +\infty$.

Then, for any bounded above potential $A \in \mathcal{H}_\alpha(X) \cap \mathcal{SV}_L(X)$, satisfying the summability condition, the family of equilibrium states $(\mu_{tA})_{t>1}$ has accumulation points at infinity which belongs to the set $\mathcal{P}_{\max}(A)$.

Proof. The proof follows the same reasoning already used in the proof of Proposition 1. □

Remark 4. *Note that Proposition 2 implies the existence of maximizing measures in the setting of weighted shifts which, in particular, will guarantee that the Mañé potential is well-defined by Lemma 5.*

4 Existence of sub-actions and some examples

In this section Theorem 3 will be proved. We will also present some properties of the Mañé potential and, moreover, we will show some explicit examples where calibrated sub-actions and maximizing measures do exist. Moreover, we show an explicit example in which there is selection at zero temperature. The first one of our examples concerns the existence of a calibrated sub-action in the framework of weighted shifts on the space $l^1(\mathbb{R})$. The other examples concern the uniqueness of the maximizing measure for potentials defined on weighted shifts in the spaces $c_0(\mathbb{R})$ and $l^p(\mathbb{R})$, $1 \leq p < +\infty$.

4.1 The Mañé potential

In this Subsection, we present the proof of Theorem 3 and also some results concerning the behavior of the Mañé potential. We will adapt for the Linear dynamics framework results from [GL08].

We assume throughout this section that X is a separable Banach space, $T : X \to X$ is a bounded linear operator surjective, but not bijective. We also assume that for each value of n, we get $X = \mathrm{Ker}(T^n) \oplus E_n$, with $E_{n+1} \subset E_n$, and $\sum_{n=1}^{+\infty} p(T^n)^{-\alpha} < +\infty$.

Proof of Theorem 3. Fix a certain point $x \in X$. We want to prove that the map $\phi_A(x, \cdot)$ is a sub-action. In order to do that, fixing another point $y \in X$, we consider $\epsilon > 0$, $n \in \mathbb{N}$, and $x' \in T^{-n}(y)$, such that, $\|x - x'\|_X < \epsilon$. Then, we get

$$S_{n+1}(A - m(A))(x') = S_n(A - m(A))(x') + A(y) - m(A) . \tag{31}$$

Since $T^{n+1}(x') = T(y)$, taking supremum in (31), first in the left side, and, after that in the right side, we obtain

$$\sup_{m\in\mathbb{N}} \sup_{\substack{x'\in T^{-m}(T(y)) \\ \|x-x'\|_X<\epsilon}} S_m(A-m(A))(x') \geq \sup_{n\in\mathbb{N}} \sup_{\substack{x'\in T^{-n}(y) \\ \|x-x'\|_X<\epsilon}} S_n(A-m(A))(x')+A(y)-m(A) .$$

Taking the limit, when ϵ goes to 0, in both sides of the above inequality, it follows that

$$\phi_A(x, T(y)) \geq \phi_A(x, y) + A(y) - m(A) .$$

That is, the map $\phi_A(x, \cdot)$ is a sub-action.

On other hand, taking supremum in (31), first on the right side, and after that on the left side, and subsequently, taking the limit when ϵ goes to 0, we obtain that

$$\phi_A(x, T(y)) \leq \phi_A(x, y) + A(y) - m(A) .$$

Therefore, $\phi_A(x, \cdot)$ is a calibrated sub-action. The above shows that the potential A is cohomologous to the constant $m(A)$, via the Mañé potential.

On other hand, we will get that $\phi_A(x, \cdot) \in \mathcal{H}_\alpha(X)$, as a consequence of the assumption $A \in \mathcal{H}_\alpha(X)$. Indeed, given $\epsilon > 0$ and $y^1, y^2 \in X$, it follows that for each $n \in \mathbb{N}$, $x^1 \in T^{-n}(y^1)$ and $x^2 \in T^{-n}(y^2)$, satisfying $\|x - x^1\|_X < \epsilon$ and $\|x - x^2\|_X < \epsilon$, we have

$$
\begin{aligned}
S_n(A-m(A))(x^1) &\le S_n(A-m(A))(x^2) + \sum_{j=0}^{n-1} |A(T^j(x^1)) - A(T^j(x^2))| \\
&\le S_n(A-m(A))(x^2) + \mathrm{Hol}_A^\alpha \sum_{j=0}^{n-1} \|T^j(x^1) - T^j(x^2)\|_X^\alpha \\
&\le S_n(A-m(A))(x^2) + \mathrm{Hol}_A^\alpha \Big(\sum_{j=1}^{n} p(T^j)^{-\alpha}\Big) \|y^1 - y^2\|_X^\alpha \\
&\le S_n(A-m(A))(x^2) + \mathrm{Hol}_A^\alpha \Big(\sum_{n=1}^{+\infty} p(T^n)^{-\alpha}\Big) \|y^1 - y^2\|_X^\alpha \ .
\end{aligned}
$$

Thus, taking the supremum on x^1 and x^2, after that, taking the supremum on all the natural numbers n, and, finally, taking the limit when ϵ goes to 0 - in the first and in the second one - on the above expressions, we get

$$
\phi_A(x, y^1) \le \phi_A(x, y^2) + \mathrm{Hol}_A^\alpha \Big(\sum_{n=1}^{+\infty} p(T^n)^{-\alpha}\Big) \|y^1 - y^2\|_X^\alpha \ .
$$

In an analogous way, we can also prove that

$$
\phi_A(x, y^2) \le \phi_A(x, y^1) + \mathrm{Hol}_A^\alpha \Big(\sum_{n=1}^{+\infty} p(T^n)^{-\alpha}\Big) \|y^1 - y^2\|_X^\alpha \ .
$$

The above implies that $\phi_A(x, \cdot) \in \mathcal{H}_\alpha(X)$, and, in particular, $\phi_A(x, y) \in \mathbb{R}$, for each $y \in X$, and this is the end of the proof. □

Now, we present an interesting property of the Mañé potential. The proof is valid on the general framework of Linear Dynamics on Banach spaces of infinite dimension. It shows a relation of the Mañé potential with a general subaction V. Analogous results on the framework of classical ergodic optimization appeared in [CLT01] and [GL08].

Proposition 3. *Suppose that the potential $A \in \mathcal{H}_\alpha(X)$ and $\mathcal{P}_{\max}(A)$ is a non-empty set. Let $V \in \mathcal{H}_\alpha(X)$ be a general sub-action for $A \in \mathcal{H}_\alpha(X)$, and ϕ_A the Mañé potential for A. Then, for any pair of points $x, y \in X$ we get*

$$
\phi_A(x, y) \le V(y) - V(x) \ . \tag{32}
$$

Proof. Fix the points $x, y \in X$. First note that for any point $x' \in X$, we have

$$
|V(x) - V(x')| \le \mathrm{Hol}_V^\alpha \|x - x'\|_X^\alpha \ .
$$

Moreover, since V is sub-action, it follows that for each $n \in \mathbb{N}$ it is also satisfied the inequality

$$
V(T^n(x')) - V(x') \ge S_n(A - m(A))(x') \ .
$$

Then, given $\epsilon > 0$, $n \in \mathbb{N}$ and $x' \in T^{-n}(y)$, such that, $\|x - x'\|_X < \epsilon$, we have

$$\begin{aligned} V(y) - V(x) &= V(T^n(x')) - V(x) \\ &\geq V(T^n(x')) - V(x') - \mathrm{Hol}_V^\alpha \|x - x'\|_X^\alpha \\ &> S_n(A - m(A))(x') - \mathrm{Hol}_V^\alpha \epsilon^\alpha \ . \end{aligned}$$

Therefore, taking the supremum among all the points $x' \in T^{-n}(y)$, satisfying that $\|x - x'\|_X < \epsilon$, and, after that, taking the supremum among all natural numbers n, and, finally, taking the limit when ϵ goes to 0, we get (32). ☐

4.2 Explicit examples of sub-actions and uniqueness of maximizing measure

In this Subsection, we present explicit examples of a calibrated sub-actions associated with bounded above Hölder continuous potentials. In some examples, we guaranteed selection at temperature zero temperature.

Example 2. *For fixed $c > 1$, consider the operator $L : l^1(\mathbb{R}) \to l^1(\mathbb{R})$ given by*

$$L((x_n)_{n\geq 1}) := (c\, x_{n+1})_{n\geq 1} \ .$$

Taking $v := (c^{-n+1})_{n\geq 1}$, it follows that for any $\alpha \in \mathbb{R}$ we get

$$L(\alpha v) = \alpha L(v) = \alpha v \ .$$

That is, αv is a fixed point for L for each $\alpha \in \mathbb{R}$. Consider the unbounded Hölder continuous potential

$$A(x) := -\ \|x - v\|_{l^1(\mathbb{R})} \ .$$

It follows that the delta Dirac measure δ_v is the unique maximizing probability measure for A and $m(A) = 0$.

We want to find a sub-action associated with the potential A i.e. a function V such that for all x is satisfied

$$V(L(x)) \geq A(x) + V(x). \tag{33}$$

Assuming that $c = 2$ and taking $V(x) := A(x)$, for each $x \in l^1(\mathbb{R})$, we want to show that (33) is true. Note that for $x = (x_n)_{n\geq 1} \in l^1(\mathbb{R})$ we have

$$\|x - v\|_{l^1(\mathbb{R})} = \sum_{n=1}^{+\infty} |x_n - 2^{-n+1}|$$

and

$$\|L(x) - v\|_{l^1(\mathbb{R})} = \sum_{n=1}^{+\infty} |2\, x_{n+1} - 2^{-n+1}| = 2 \sum_{n=1}^{+\infty} |x_{n+1} - 2^{-n}| \ .$$

Then, V is a sub-action for A, because

$$\begin{aligned} V(L(x)) = -\|L(x)-v\|_{l^1(\mathbb{R})} &= -2\sum_{n=1}^{+\infty} |x_{n+1} - 2^{-n}| \\ &\geq -2\sum_{n=1}^{+\infty} |x_n - 2^{-n+1}| \\ &= -2\|x-v\|_{l^1(\mathbb{R})} = A(x) + V(x) \ . \end{aligned}$$

Furthermore, this sub-action V is calibrated. Indeed, given a sequence $y = (y_n)_{n\geq 1}$ in $l^1(\mathbb{R})$, the L-preimages x of the point y are of the form

$$x = (x_n)_{n\geq 1} = \big(x_1, \frac{y_1}{2}, \frac{y_2}{2}, ...\big) \ .$$

Taking $\widetilde{x} := \big(1, \frac{y_1}{2}, \frac{y_2}{2}, ...\big)$, we get

$$\begin{aligned} V(y) = V(L(\widetilde{x})) &= -\|L(\widetilde{x}) - v\|_{l^1(\mathbb{R})} \\ &= -\sum_{n=1}^{+\infty} |y_n - 2^{-n+1}| \\ &= -2\sum_{n=1}^{+\infty} |\frac{y_n}{2} - 2^{-n}| \\ &= -2\,(\,|1-1| \,+\, |\frac{y_1}{2} - 2^{-1}| \,+\, |\frac{y_2}{2} - 2^{-2}| + ...) \\ &= -2\,\|\widetilde{x} - v\|_{l^1(\mathbb{R})} = A(\widetilde{x}) + V(\widetilde{x}) \ . \end{aligned}$$

The next examples present cases in which it can be guaranteed selection at zero temperature - for a Hölder continuous potential - defined on the spaces of sequences $c_0(\mathbb{R})$ and $l^p(\mathbb{R})$, $1 \leq p < +\infty$.

Example 3. *For fixed $c > 1$ and either $X = l^p(\mathbb{R})$ or $X = c_0(\mathbb{R})$, consider the operator $L : X \to X$ given by*

$$L((x_n)_{n\geq 1}) := (c\, x_{n+1})_{n\geq 1} \ .$$

Taking $v := (c^{-n+1})_{n\geq 1}$, it follows αv is a fixed point for each $\alpha \in \mathbb{R}$.

Consider a monotonous decreasing 1-Hölder continuous function r from $[0,1]$ into $\mathbb{R}$, such that, $r(0) = 1$ and suppose $\lim_{s\to+\infty} r(s) = 0$. Consider the potential

$$A(x) := -\, r(\|x-v\|_X)\, \|x-v\|_X \ .$$

It follows that $A \in \mathcal{H}_1(X)$, moreover, the delta Dirac measure δ_v is the unique maximizing probability measure for A and $m(A) = 0$. Therefore, there exists selection of probability at zero temperature.

Example 4. *For fixed $c_0, c_1 > 1$ and either $X = l^p(\mathbb{R})$ or $X = c_0(\mathbb{R})$, consider the operator $L : X \to X$ given by*

$$L((x_n)_{n\geq 1}) := (c_i\, x_{n+1})_{n\geq 1}\ ,$$

where $i \in \{0, 1\}$ is the unique number such that $n = 2k+i$ for some $k \in \mathbb{N}\cup\{0\}$. Then,

$$L^2((x_n)_{n\geq 1}) = (c_0\, c_1\, x_{n+2})_{n\geq 1}\ .$$

Taking

$$v := (1, 0, (c_0\, c_1)^{-1}, 0, (c_0\, c_1)^{-2}, 0, (c_0\, c_1)^{-3}, 0, ...)$$

and

$$w := (0, c_0\, (c_0\, c_1)^{-1}, 0, c_0\, (c_0\, c_1)^{-2}, 0, c_0\, (c_0\, c_1)^{-3}, 0, ...)\ ,$$

it follows that $L(v) = w$ and $L^2(v) = v$ i.e. v and w are periodic points of period two for the map L. Given α_1, α_2, the points of the form $\alpha_1 v + \alpha_2 w$ are points of period two, which generate a linear subspace of periodic points of period two.

Consider the potential

$$A(x) := -[r(\,\|x - v\|_X\,)\|x - v\|_X + r(\,\|x - w\|_X\,)\|x - w\|_X]\ ,$$

where r was defined above.

We have that $A \in \mathcal{H}_1(X)$, the periodic orbit $\{v, w\}$ of period two is such that $1/2\,\delta_v + 1/2\,\delta_w$, and it is the unique maximizing probability measure for the potential A and $m(A) = 0$.

Example 5. *For fixed $c > 1$, and either for $X = l^p(\mathbb{R})$ or $X = c_0(\mathbb{R})$, consider the operator $L : X \to X$ given by*

$$L((x_n)_{n\geq 1}) := (c\, x_{n+1})_{n\geq 1}\ .$$

Denote $W \subset X$ the infinite-dimensional vector space of the form

$$W := \{\ (x_1, 0, x_2, 0, ..., 0, x_n, 0, ...) :\ x \in X\}\ .$$

Define the potential

$$A(x) := -r(d_X(x, W))\, d_X(x, W)\ ,$$

where r was defined above and $d_X(x, W) := \inf\{\|x - w\|_X\ :\ w \in W\}$. Then, $A \in \mathcal{H}_1(X)$ and any $\mu \in \mathcal{P}_L(X)$, with support contained in W, is maximizing for A. For instance, the probability measure $t\delta_v + (1 - t)\delta_{L(v)}$, $t \in (0, 1)$, where

$$v := (1, 0, c^{-2}, 0, c^{-4}, 0, ..., 0, c^{-2n}, 0, ...)\ ,$$

is a maximizing measure for the potential A.

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