DISTRIBUTION OF INTEGRAL FOURIER COEFFICIENTS OF A MODULAR FORM OF HALF INTEGRAL WEIGHT MODULO PRIMES

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ABSTRACT. Recently, Bruinier and Ono classified cusp forms $f(z) := \sum_{n=0}^{\infty} a_f(n)q^n \in S_{\lambda+\frac{1}{2}}(\Gamma_0(N), \chi) \cap \mathbb{Z}[[q]]$ that does not satisfy a certain distribution property for modulo odd primes p. In this paper, using Rankin-Cohen Bracket, we extend this result to modular forms of half integral weight for primes $p \geq 5$. As applications of our main theorem we derive distribution properties, for modulo primes $p \geq 5$, of traces of singular moduli and Hurwitz class number. We also study an analogue of Newman's conjecture for overpartitions.

1. Introduction and Results

Let $M_{\lambda+\frac{1}{2}}(\Gamma_0(N), \chi)$ and $S_{\lambda+\frac{1}{2}}(\Gamma_0(N), \chi)$ be the spaces, respectively, of modular forms and cusp forms of weight $\lambda + \frac{1}{2}$ on $\Gamma_0(N)$ with a Dirichlet character χ whose conductor divides N. If $f(z) \in M_{\lambda+\frac{1}{2}}(\Gamma_0(N), \chi)$, then $f(z)$ has the form

$$f(z) = \sum_{n=0}^{\infty} a(n)q^n,$$

where $q := e^{2\pi i z}$. It is well-known that the coefficients of f are related to interesting objects in number theory such as the special values of L-function, class number, traces of singular moduli and so on. In this paper, we study congruence properties of the Fourier coefficients of $f(z) \in M_{\lambda+\frac{1}{2}}(\Gamma_0(N), \chi) \cap \mathbb{Z}[[q]]$ and their applications.

Recently, Bruinier and Ono proved in [3] that $g(z) \in S_{\lambda+\frac{1}{2}}(\Gamma_0(N), \chi) \cap \mathbb{Z}[[q]]$ has a special form (see (2.1)) by modulo p when p is an odd prime and the coefficients of $f(z)$ do not satisfy the following property for p:

Property A. If M is a positive integer, we say that a sequence $\alpha(n) \in \mathbb{Z}$ satisfies Property A for M if for every integer r

$$\sharp\{\, 1 \leq n \leq X \mid \alpha(n) \equiv r \pmod{M} \text{ and } \gcd(M, n) = 1\} \\ \gg_{r,M} \begin{cases} \frac{\sqrt{X}}{logX} & \text{if } r \not\equiv 0 \pmod{M}, \\ X & \text{if } r \equiv 0 \pmod{M}. \end{cases}$$

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Let

$$\theta(f(z)) := \frac{1}{2\pi i} \cdot \frac{d}{dz} f(z) = \sum_{n=1}^{\infty} n \cdot a(n) q^n.$$

Using Rankin-Cohen Bracket (see (2.3)), we prove that there exists

$$\widetilde{f}(z) \in S_{\lambda+p+1+\frac{1}{2}}(\Gamma_0(4N), \chi) \cap \mathbb{Z}[[q]]$$

such that $\theta(f(z)) \equiv \widetilde{f}(z) \pmod{p}$. We extend the results in [3] to modular forms of half integral weight.

Theorem 1. *Let λ be a non-negative integer. We assume that $f(z) = \sum_{n=0}^{\infty} a(n) q^n \in M_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi) \cap \mathbb{Z}[[q]]$, where χ is a real Dirichlet character. If $p \geq 5$ is a prime and there exists a positive integer n for which $\gcd(a(n), p) = 1$ and $\gcd(n, p) = 1$, then at least one of the following is true:*

(1) The coefficients of $\theta^{p-1}(f(z))$ satisfies Property A for p.

(2) There are finitely many square-free integers n_1, $n_2, \cdots, n_t$ for which

$$\theta^{p-1}(f(z)) \equiv \sum_{i=1}^{t} \sum_{m=0}^{\infty} a(n_i m^2) q^{n_i m^2} \pmod{p}. \tag{1.1}$$

Moreover, if $\gcd(4N, p) = 1$ and an odd prime ℓ divides some n_i, then

$$p | (\ell-1)\ell(\ell+1)N \text{ or } \ell \mid N.$$

Remark 1.1. Note that for every odd prime $p \geq 5$,

$$\theta^{p-1}(f(z)) \equiv \sum_{\substack{n>0 \\ p \nmid n}} a(n) q^n \pmod{p}.$$

As an applications of Theorem 1, we study the distribution of traces of singular moduli modulo primes $p \geq 5$. Let $j(z)$ be the usual j-invariant function. We denote by F_d the set of positive definite binary quadratic forms

$$F(x,y) = ax^2 + bxy + cy^2 = [a,b,c]$$

with discriminant $-d = b^2 - 4ac$. For each $F(x,y)$, let α_F be the unique complex number in the complex upper half plane, which is a root of $F(x,1)$. We define $\omega_F \in \{1,2,3\}$ as

$$\omega_F := \begin{cases} 2 & \text{if } F \sim_\Gamma [a,0,a], \\ 3 & \text{if } F \sim_\Gamma [a,a,a], \\ 1 & \text{otherwise}, \end{cases}$$

where $\Gamma := SL_2(\mathbb{Z})$. Here, $F \sim_\Gamma [a,b,c]$ denotes that $F(x,y)$ is equivalent to $[a,b,c]$. From these notations, we define the Hecke trace of singular moduli.

Definition 1.2. If $m \geq 1$, then we define the *mth Hecke trace of the singular moduli of discriminant* $-d$ as

$$t_m(d) := \sum_{F \in F_d/\Gamma} \frac{j_m(\alpha_F)}{\omega_F},$$

where F_d/Γ denotes a set of $\Gamma-$equivalence classes of F_d and

$$j_m(z) := j(z)|T_0(m) = \sum_{\substack{d|m \\ ad=m}} \sum_{b=0}^{d-1} j\left(\frac{az+b}{d}\right).$$

Here, $T_0(m)$ denotes the normalized mth weight zero Hecke operator.

Note that $t_1(d) = t(d)$, where

$$t(d) := \sum_{F \in F_d/\Gamma} \frac{j(\alpha_F) - 744}{\omega_F}$$

is the usual trace of singular moduli. Let

$$h(z) := \frac{\eta(z)^2}{\eta(2z)} \cdot \frac{E_4(4z)}{\eta(4z)^6}$$

and $B_m(1, d)$ denote the coefficient of q^d in $h(z)|T(m^2, 1, \chi_0)$, where

$$E_4(z) := 1 + 240 \sum_{n=1}^{\infty} \sum_{d|n} d^3 q^n, \ \eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),$$

and χ_0 is a trivial character. Here, $T(m^2, \lambda, \chi)$ denotes the mth Hecke operator of weight $\lambda + \frac{1}{2}$ with a Dirichlet chracter χ (see VI. §3. in [5] or (2.5)). Zagier proved in [11] that for all m and d

$$t_m(d) = -B_m(1, d). \tag{1.2}$$

Using these generating functions, Ahlgren and Ono studied the divisibility properties of traces and Hecke traces of singular moduli in terms of the factorization of primes in imaginary quadratic fields (see [2]). For example, they proved that a positive proportion of the primes ℓ has the property that $t_m(\ell^3 n) \equiv 0 \pmod{p^s}$ for every positive integer n coprime to ℓ such that p is inert or ramified in $\mathbb{Q}\left(\sqrt{-n\ell}\right)$. Here, p is an odd prime, and s and m are integers with $p \nmid m$. In the following theorem, we give the distribution of traces and Hecke traces of singular moduli modulo primes p.

Theorem 2. *Suppose that $p \geq 5$ is a prime such that $p \equiv 2 \pmod 3$.*

(1) Then, for every integer r, $p \nmid r$,

$$\sharp\{\, 1 \leq n \leq X \mid t_1(n) \equiv r \pmod p\} \gg_{r,p} \begin{cases} \frac{\sqrt{X}}{logX} & \text{if } r \not\equiv 0 \pmod p \\ X & \text{if } r \equiv 0 \pmod p. \end{cases}$$

(2) Then, a positive proportion of the primes ℓ has the property that

$$\sharp\{\, 1 \leq n \leq X \mid t_\ell(n) \equiv r \pmod p\} \gg_{r,p} \begin{cases} \frac{\sqrt{X}}{logX} & \text{if } r \not\equiv 0 \pmod p \\ X & \text{if } r \equiv 0 \pmod p. \end{cases}$$

for every integer r, $p \nmid r$.

As another application we study the distribution of Hurwitz class number modulo primes $p \geq 5$. The Hurwitz class number $H(-N)$ is defined as follows: the class number of quadratic forms of the discriminant $-N$ where each class C is counted with multiplicity $\frac{1}{Aut(C)}$. The following theorem gives the distribution of Hurwitz class number modulo primes $p \geq 5$.

Theorem 3. *Suppose that $p \geq 5$ is a prime. Then, for every integer r*

$$\sharp\{\, 1 \leq n \leq X \mid H(n) \equiv r \pmod p\} \gg_{r,p} \begin{cases} \frac{\sqrt{X}}{logX} & \text{if } r \not\equiv 0 \pmod p, \\ X & \text{if } r \equiv 0 \pmod p. \end{cases}$$

We also use the main theorem to study an analogue of Newman's conjecture for overpartitions. Newman's conjecture concerns the distribution of the ordinary partition function modulo primes p.

Newman's Conjecture. *Let $P(n)$ be an ordinary partition function. If M is a positive integer, then for every integer r there are infinitely many nonnegative integer n for which $P(n) \equiv r \pmod M$.*

This conjecture was already studied by many mathematicians (see Chapter 5. in [8]). The overpartition of a natural number n is a partition of n in which the first occurrence of a number may be overlined. Let $\bar{P}(n)$ be the number of the overpartition of an integer n. As an analogue of Newman's conjecture, the following theorem gives a distribution property of $\bar{P}(n)$ modulo odd primes p.

Theorem 4. *Suppose that $p \geq 5$ is a prime such that $p \equiv 2 \pmod 3$. Then, for every integer r,*

$$\sharp\{\, 1 \leq n \leq X \mid \bar{P}(n) \equiv r \pmod p\} \gg_{r,p} \begin{cases} \frac{\sqrt{X}}{logX} & \text{if } r \not\equiv 0 \pmod p, \\ X & \text{if } r \equiv 0 \pmod p. \end{cases}$$

Remark 1.3. When $r \equiv 0 \pmod p$, Theorem 2, 3 and 4 were proved in [2] and [10].

Next sections are detailed proofs of theorems: Section 2 gives a proof of Theorem 1. In Section 3, we give the proofs of Theorem 2, 3, and 4.

2. Proof of Theorem 1

We begin by stating the following theorem proved in [3].

Theorem 2.1 ([3]). *Let λ be a non-negative integer. Suppose that $g(z) = \sum_{n=0}^{\infty} a_g(n)q^n \in S_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi) \cap \mathbb{Z}[[q]]$, where χ is a real Dirichlet character. If p is an odd prime and a positive integer n exists for which $\gcd(a_g(n), p) = 1$, then at least one of the following is true:*

(1) If $0 \leq r < p$, then

$$\sharp\{\ 1 \leq n \leq X \mid a_g(n) \equiv r \pmod p\} \gg_{r,M} \begin{cases} \frac{\sqrt{X}}{logX} & \text{if } r \not\equiv 0 \pmod p, \\ X & \text{if } r \equiv 0 \pmod p. \end{cases}$$

(2) There are finitely many square-free integers $n_1, n_2, \cdots, n_t$ for which

$$g(z) \equiv \sum_{i=1}^{t} \sum_{m=0}^{\infty} a_g(n_i m^2) q^{n_i m^2} \pmod p. \tag{2.1}$$

Moreover, if $\gcd(p, 4N) = 1$, $\epsilon \in \{\pm 1\}$, and $\ell \nmid 4Np$ is a prime with $\left(\frac{n_i}{\ell}\right) \in \{0, \epsilon\}$ for $1 \leq i \leq t$, then $(\ell - 1)g(z)$ is an eigenform modulo p of the half-integral weight Hecke operator $T(\ell^2, \lambda, \chi)$. In particular, we have

$$(\ell - 1)g(z)|T(\ell^2, \lambda, \chi) \equiv \epsilon\chi(p) \left(\frac{(-1)^{\lambda}}{\ell}\right) \left(\ell^{\lambda} + \ell^{\lambda-1}\right)(\ell - 1)g(z) \pmod p. \tag{2.2}$$

Recall that $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi) \cap \mathbb{Z}[[q]]$. Thus, to apply Theorem 2.1, we show that there exists a cusp form $\widetilde{f}(z)$ such that $\widetilde{f}(z) \equiv \theta^{p-1}(f(z)) \pmod p$ for a prime $p \geq 5$.

Lemma 2.2. *Suppose that $p \geq 5$ is a prime and*

$$f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_{\lambda+\frac{1}{2}}(\Gamma_0(N), \chi) \cap \mathbb{Z}[[q]].$$

Then, there exists a cusp form $\widetilde{f}(z) \in S_{\lambda+(p+1)(p-1)+\frac{1}{2}}(\Gamma_0(N), \chi) \cap \mathbb{Z}[[q]]$ such that

$$\widetilde{f}(z) \equiv \theta^{p-1}(f(z)) \pmod p.$$

Proof of Lemma 2.2. For $F(z) \in M_{\frac{k_1}{2}}(\Gamma_0(N), \chi_1)$ and $G(z) \in M_{\frac{k_2}{2}}(\Gamma_0(N), \chi_2)$, let

$$[F(z), G(z)]_1 := \frac{k_2}{2}\theta(F(z)) \cdot G(z) - \frac{k_1}{2}F(z) \cdot \theta(G(z)). \tag{2.3}$$

This operator is referred to as a Rankin-Cohen 1-bracket, and it was proved in [4] that

$$[F(z), G(z)]_1 \in S_{\frac{k_1+k_2}{2}+2}(\Gamma_0(N), \chi_1\chi_2\chi'),$$

where $\chi' = 1$ if $\frac{k_1}{2}$ and $\frac{k_2}{2} \in \mathbb{Z}$, $\chi'(d) = \left(\frac{-4}{d}\right)^{\frac{k_i}{2}}$ if $\frac{k_i}{2} \in \mathbb{Z}$ and $\frac{k_{3-i}}{2} \in \frac{1}{2} + \mathbb{Z}$, and $\chi'(d) = \left(\frac{-4}{d}\right)^{\frac{k_1+k_2}{2}}$ if $\frac{k_1}{2}$ and $\frac{k_2}{2} \in \frac{1}{2} + \mathbb{Z}$.

For even $k \geq 4$, let

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sum_{d|n} d^{k-1} q^n$$

be the usual normalized Eisenstein series of weight k. Here, the number B_k denotes the kth Bernoulli number. The function $E_k(z)$ is a modular form of weight k on $SL_2(\mathbb{Z})$, and

$$E_{p-1}(z) \equiv 1 \pmod{p} \tag{2.4}$$

(see [6]). From (2.3) and (2.4), we have

$$[E_{p-1}(z), f(z)]_1 \equiv \theta(f(z)) \pmod{p}$$

and $[E_{p-1}(z), f(z)]_1 \in S_{\lambda+p+1+\frac{1}{2}}(\Gamma_0(N), \chi)$. Repeating this method $p-1$ times, we complete the proof. □

Using the following lemma, we can deal with the divisibility of $a_g(n)$ for positive integers n, $p \nmid n$, where $g(z) = \sum_{n=1}^{\infty} a_g(n) q^n \in S_{\lambda+\frac{1}{2}}(\Gamma_0(N), \chi) \cap \mathbb{Z}[[q]]$.

Lemma 2.3 (see Chapter 3 in [8]). *Suppose that $g(z) = \sum_{n=1}^{\infty} a_g(n) q^n \in S_{\lambda+\frac{1}{2}}(\Gamma_0(N), \chi)$ has coefficients in $\mathcal{O}_K$, the algebraic integers of some number field K. Furthermore, suppose that $\lambda \geq 1$ and that $\boldsymbol{m} \subset \mathcal{O}_K$ is an ideal norm M.*

1. *Then, a positive proportion of the primes $Q \equiv -1 \pmod{4MN}$ has the property that*
$$g(z) | T(Q^2, \lambda, \chi) \equiv 0 \pmod{\boldsymbol{m}}.$$
2. *Then a positive proportion of the primes $Q \equiv 1 \pmod{4MN}$ has the property that*
$$g(z) | T(Q^2, \lambda, \chi) \equiv 2g(z) \pmod{\boldsymbol{m}}.$$

We can now prove Theorem 1.

Proof of Theorem 1. From Lemma 2.2, there exists a cusp form

$$\widetilde{f}(z) \in S_{\lambda+(p+1)(p-1)+\frac{1}{2}}(\Gamma_0(N), \chi) \cap \mathbb{Z}[[q]]$$

such that

$$\widetilde{f}(z) \equiv \theta^{p-1}(f(z)) \pmod{p}.$$

Note that, for $F(z) = \sum_{n=0}^{\infty} a_F(n) q^n \in M_{k+\frac{1}{2}}(\Gamma_0(N), \chi)$ and each prime $Q \nmid N$, the half-integral weight Hecke operator $T(Q^2, \lambda, \chi)$ is defined as

(2.5)
$$F(z) | T(Q^2, k, \chi) := \sum_{n=0}^{\infty} \left(a_F(Q^2 n) + \chi^*(Q) \left(\frac{n}{Q}\right) Q^{k-1} a_F(n) + \chi^*(Q^2) Q^{2k-1} a_F(n/Q^2) \right) q^n,$$

where $\chi^*(n) := \chi^*(n)\left(\frac{(-1)^k}{n}\right)$ and $a_F(n/Q^2) = 0$ if $Q^2 \nmid n$. If $F(z)|T(Q^2,k,\chi) \equiv 0 \pmod{p}$ for a prime $Q \nmid N$, then we have

$$\begin{aligned} a_F(Q^2 \cdot Qn) + \chi^*(Q)\left(\frac{Qn}{Q}\right)Q^{k-1}a_F(Qn) + \chi^*(Q^2)Q^{2k-1}a_F\left(Qn/Q^2\right) \\ \equiv a_F(Q^3n) \equiv 0 \pmod{p} \end{aligned}$$

for every positive integer n such that $\gcd(Q,n) = 1$. Thus, we have the following by Lemma 2.3-(1):

$$\sharp\{\, 1 \le n \le X \mid a(n) \equiv 0 \pmod{p} \text{ and } \gcd(p,n) = 1\} \gg X.$$

We apply Theorem 2.1 with $\widetilde{f}(z)$. Then the purpose of the remaining part of the proof is to show the following: if $\gcd(p,4N) = 1$, an odd prime ℓ divides some n_i, and

$$\theta^{p-1}(f(z)) \equiv \sum_{i=1}^{t}\sum_{m=0}^{\infty} a(n_im^2)q^{n_im^2} \pmod{p}, \tag{2.6}$$

then $p|(\ell-1)\ell(\ell+1)N$ or $\ell \mid N$. We assume that there exists a prime ℓ_1 such that $\ell_1|n_1$, $p \nmid (\ell_1-1)\ell_1(\ell_1+1)N$ and $\ell \mid N$. We also assume that $n_t = 1$ and that $n_i \nmid n_1$ for every i, $2 \le i \le t-1$. Then, we can take a prime ℓ_i for each i, $2 \le i \le t-1$, such that $\ell_i|n_i$ and $\ell_i \nmid n_1$. For convention, we define

$$\left(\frac{n}{2}\right) := \begin{cases} (-1)^{(n-1)^2/8} & \text{if } n \text{ is odd,} \\ 0 & \text{otherwise,} \end{cases}$$

and $\chi_Q(d) := \left(\frac{d}{Q}\right)$ for a prime Q. Let $\psi(d) := \prod_{i=2}^{t-1}\chi_{\ell_i}(d)$. We take a prime β such that $\psi(n_1)\chi_\beta(n_1) = -1$. If we denote the ψ-twist of $\widetilde{f}(z)$ by $\widetilde{f}_\psi(z)$ and the $\psi\chi_\beta$-twist of $\widetilde{f}(z)$ by $\widetilde{f}_{\psi\chi_\beta}(z)$, then

$$\widetilde{f}_{\psi\chi_\beta^2}(z) - \widetilde{f}_{\psi\chi_\beta}(z) \equiv 2 \sum_{\gcd(m,\beta\prod_{j\ge 2}\ell_j)=1} a(n_1m^2)q^{n_1m^2} \pmod{p}$$

and $\widetilde{f}_{\psi\chi_\beta}(z) \in S_{\lambda+(p+1)(p-1)+\frac{1}{2}}(\Gamma_0(N\alpha^2\beta^2),\chi) \cap \mathbb{Z}[[q]]$ (see Chapter 3 in [8]). Note that

$$\gcd(N\alpha^2\beta^2, p) = \gcd(N\alpha^2\beta^2, \ell_1) = 1.$$

Thus, $(\widetilde{f}_\psi(z) - \widetilde{f}_{\psi\chi_\beta}(z))|T(\ell_1^2, \lambda+(p+1)(p-1), \chi)$ satisfies the formula (2.2) of Theorem 2.1 for both of $\epsilon = 1$ and $\epsilon = -1$. This results in a contradiction since

$$(\widetilde{f}_\psi(z) - \widetilde{f}_{\psi\chi_\beta}(z))|T(\ell_1^2, \lambda+(p+1)(p-1), \chi) \not\equiv 0 \pmod{p}$$

and $p \ge 5$. Thus, we complete the proof. $\square$

3. Proofs of Theorem 2, 3, and 4

3.1. **Proof of Theorem 2.** Note that $h(z) = \frac{\eta(z)^2}{\eta(2z)} \cdot \frac{E_4(4z)}{\eta(4z)^6}$ is a meromorphic modular form. In [2] it was obtained a holomorphic modular form on $\Gamma_0(4p^2)$ whose Fourier coefficients generate traces of singular moduli modulo p (see the formula (3.1) and (3.2)). Since the level of this modular form is not relatively prime to p, we need the following proposition.

Proposition 3.1 ([1]). *Suppose that $p \geq 5$ is a prime. Also, suppose that $p \nmid N$, $j \geq 1$ is an integer, and*

$$g(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_{\lambda+\frac{1}{2}}(\Gamma_0(Np^j)) \cap \mathbb{Z}[[q]].$$

Then, there exists a cusp form $G(z) \in S_{\lambda'+\frac{1}{2}}(\Gamma_0(N)) \cap \mathbb{Z}[[q]]$ such that

$$G(z) \equiv g(z) \pmod{p},$$

where $\lambda' + \frac{1}{2} = (\lambda + \frac{1}{2})p^j + p^e(p-1)$ for a sufficiently large $e \in \mathbb{N}$.

Using Theorem 1 and Proposition 3.1, we give the proof of Theorem 2.

Proof of Theorem 2. Let

$$h_{1,p}(z) := h(z) - \left(\frac{-1}{p}\right) h_{\chi_p}(z), \tag{3.1}$$

where $h_{\chi_p}(z)$ is the χ_p-twist of $h(z)$. From (1.2), we have

$$h_{1,p}(z) := -2 - \sum_{\substack{0<d\equiv 0,3 \pmod 4 \\ p \mid d}} t_1(d)q^d - 2 \sum_{\substack{0<d\equiv 0,3 \pmod 4 \\ \left(\frac{-d}{p}\right)=-1}} t_1(d)q^d$$

and

$$\begin{aligned} h_{m,p}(z) &:= h_{1,p}(z) | T(m^2, 1, \chi_0) \\ &= -2 - \sum_{\substack{0<d\equiv 0,3 \pmod 4 \\ p \mid d}} t_m(d)q^d - 2 \sum_{\substack{0<d\equiv 0,3 \pmod 4 \\ \left(\frac{-d}{p}\right)=-1}} t_m(d)q^d \end{aligned}$$

for every positive integer m. Let

$$F_p(z) := \frac{\eta(4z)^{p^2}}{\eta(4pz)}.$$

It was proved in [2] that if α is a sufficiently large positive integer, then $h_{1,p}(z)F_p(z)^{\alpha} \in M_{\frac{3}{2}+k_0}(\Gamma_0(4p^2))$ and

$$h_{1,p}(z)F_p(z)^{\alpha} \equiv h_{1,p}(z) \pmod{p}, \tag{3.2}$$

where $k_0 = \alpha \cdot \frac{p^2-1}{2}$. Lemma 2.2 and Proposition 3.1 imply that there exists $f_{1,p}(z) \in S_{\lambda'+\frac{1}{2}}(\Gamma_0(4)) \cap \mathbb{Z}[[q]]$ such that

$$f_{1,p}(z) \equiv -2 \sum_{\substack{0<d\equiv 0,3 \pmod 4 \\ \left(\frac{-d}{p}\right)=-1}} t_m(d)q^d \pmod p,$$

where $\lambda' = (k_0 + 1 + (p-1)(p+1) + \frac{1}{2})p^2 + p^e(p-1)$ for a sufficiently large $e \in \mathbb{N}$.

We assume that the coefficients of $f_{1,p}(z)$ do not satisfy Property A for an odd prime $p \equiv 2 \pmod 3$. Note that $\left(\frac{-3}{p}\right) = -1$ and that $p \nmid (3-1)3(3+1)$. So, Theorem 1 implies that

$$2t_1(3) \equiv 0 \pmod p.$$

This results in a contradiction since $2t_1(3) = 2^4 \cdot 31$. Thus, we obtain a proof when $m = 1$.

For every odd prime ℓ, we have

$$\begin{aligned} f_{1,p}(z)|T(\ell^2, \lambda', \chi_0) &\equiv \theta^{p-1}(h_{1,p}(z))|T(\ell^2, \lambda', \chi_0) \\ &\equiv \theta^{p-1}(h_{1,p}(z)|T(\ell^2, 1, \chi_0)) \equiv \theta^{p-1}(h_{\ell,p}(z)) \pmod p. \end{aligned}$$

Moreover, Lemma 2.3 implies that a positive proportion of the primes ℓ satisfies the property

$$f_{1,p}(z)|T(\ell^2, \lambda', \chi_0) \equiv 2f_{1,p} \pmod p.$$

This completes the proof. □

3.2. **Proofs of Theorem 3.** The following theorem gives the formula for the Hurwitz class number in terms of the Fourier coefficients of a modular form of half integral weight.

Theorem 3.2. *Let $T(z) := 1 + 2\sum_{n=1}^{\infty} q^{n^2}$. If integers $r_3(n)$ are defined as*

$$\sum_{n=0}^{\infty} r_3(n)q^n := T(z)^3,$$

then

$$r(n) = \begin{cases} 12H(-4n) & \text{if } n \equiv 1,2 \pmod 4, \\ 24H(-n) & \text{if } n \equiv 3 \pmod 8, \\ r(n/4) & \text{if } n \equiv 0 \pmod 4, \\ 0 & \text{if } n \equiv 7 \pmod 8. \end{cases}$$

Note that $T(z)$ is a half integral weight modular form of weight $\frac{1}{2}$ on $\Gamma_0(4)$. Combining Theorem 1 and Theorem 3.2, we derive the proof of Theorem 3.

Proof of Theorem 3. Let $G(z)$ be the $\left(\frac{4}{\cdot}\right)$-twist of $T(z)^3$. Then, from Theorem 3.2, we have

$$G(z) = 1 + \sum_{n\equiv 1 \pmod 4} 12H(-4n)q^n + \sum_{n \equiv 3 \pmod 8} 24H(-n)q^n$$

and $G(z) \in M_{\frac{3}{2}}(\Gamma_0(16))$. Note that $24H(-3) = 8$. This gives the complete proof by Theorem 1. $\square$

3.3. **Proofs of Theorem 4.** In the following, we prove Theorem 4.

Proof of Theorem 4. Let

$$W(z) := \frac{\eta(2z)}{\eta(z)^2}.$$

It is known that

$$W(z) = \sum_{n=0}^{\infty} \bar{P}(n) q^n$$

and that $W(z)$ is a weakly holomorphic modular form on $\Gamma_0(16)$. Let

$$G(z) := \left(W(z) - \left(\frac{-1}{p}\right) W_{\chi_p}(z) \right) F_p(z)^{p^\beta},$$

where $F_p(z) = \frac{\eta(4z)^{p^2}}{\eta(4p^2z)}$ and β are positive integers. Then we have

$$G(z) \equiv 2 \sum_{\substack{0<n \\ \left(\frac{-n}{p}\right)=-1}} \bar{P}(n) q^n + \sum_{\substack{0<n \\ p \mid n}} \bar{P}(n) q^n \pmod{p}.$$

We claim that there exists a positive integer β such that $G(z)$ is a holomorphic modular form of half integral weight on $\Gamma_0(16p^2)$. To prove our claim, we follow the arguments of Ahlgren and Ono ([1], Lemma 4.2). Note that, by a well-known criterion, $F_p(z)$ is a holomorphic modular form on $\Gamma_0(4p^2)$ that vanishes at each cusp $\frac{a}{c} \in \mathbb{Q}$ for which $p^2 \nmid c$ (see [7]). This implies that $G(z)$ is a weakly holomorphic modular form on $\Gamma_0(16p^2)$. If β is sufficiently large, then $G(z)$ is holomorphic except at each cusp $\frac{a'}{c'}$ for which $p^2 | c'$.

Thus, we prove that $G(z)$ is holomorphic at $\frac{1}{2^m p^2}$ for $0 \leq m \leq 3$. Let, for odd d,

$$\epsilon_d := \begin{cases} 1 \text{ if } d \equiv 1 \pmod{4}, \\ i \text{ if } d \equiv 3 \pmod{4}. \end{cases}$$

If $f(z)$ is a function on the complex upper half plane, $\lambda \in \mathbb{Z}$, and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$, then we define the usual slash operator by

$$f(z) \mid_{\lambda+\frac{1}{2}} \gamma := \left(\frac{c}{d}\right)^{2\lambda+1} \epsilon_d^{-1-2\lambda} (cz+d)^{-\lambda-\frac{1}{2}} f\left(\frac{az+b}{cz+d}\right).$$

Let $g := \sum_{v=1}^{\infty} \left(\frac{v}{p}\right) e^{2\pi i v/p}$ be the usual Gauss sum. Note that

$$W_{\chi_p}(z) = \frac{g}{p} \sum_{v=1}^{p-1} \left(\frac{v}{p}\right) W(z)|_{-\frac{1}{2}} \left(\begin{smallmatrix} 1 & -v/p \\ 0 & 1 \end{smallmatrix}\right).$$

Choose an integer k_v satisfying

$$16k_v \equiv 15v \pmod{p}.$$

Then, we have

$$\begin{pmatrix} 1 & -\frac{v}{p} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2^m p^2 & 1 \end{pmatrix} = \gamma_{v,m} \begin{pmatrix} 1 & 0 \\ 2^m p^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{16v}{p} + \frac{16k_v}{p} \\ 0 & 1 \end{pmatrix}, \tag{3.3}$$

where

$$\gamma_{v,m} = \begin{pmatrix} 1 - 2^{m+4}p(v + k_v + 2^m v^2 p - 2^m v k_v p) & \frac{1}{p}(15v - 16k_v - 2^{m+4}(v^2 p + v k_v p)) \\ 2^{2m}p^2(-16vp + 16k_v p) & 2^{m+4}vp - 2^{m+4}k_v p + 1 \end{pmatrix}.$$

Note that $W(z)$ has its only pole at $z \sim 0$ up to $\Gamma_0(16)$. Since $\gamma_{v,m} \in \Gamma_0(16)$, the formula (3.3) implies that $W_{\chi_p}(z)$ is holomorphic at $2^m p^2$ for $1 \le m \le 3$. Thus, $G(z)$ is holomorphic at $2^m p^2$ for $1 \le m \le 3$.

If $m = 0$, then we have

$$W(z)|_{-\frac{1}{2}}\gamma_{v,0} = \left(\frac{-16vp^3 + 16k_v p^3}{16vp - 16k_v p + 1}\right) W(z) = \left(\frac{p^2(-vp + k_v p)}{16vp - 16k_v p + 1}\right) W(z) = W(z).$$

Note that

$$W(z)|_{-\frac{1}{2}} \left(\begin{smallmatrix} 1 & 0 \\ p^2 & 1 \end{smallmatrix}\right) = \alpha \cdot q^{-\frac{1}{16}} + O(1), \tag{3.4}$$

where α is a nonzero complex number. The q-expansion of $W_{\chi_p}(z)$ at $\frac{1}{p^2}$ is given by

$$W_{\chi_p}(z)|_{-\frac{1}{2}} \left(\begin{smallmatrix} 1 & 0 \\ p^2 & 1 \end{smallmatrix}\right). \tag{3.5}$$

Using (3.3) and (3.4), the only term in (3.5) with a negative exponent on q is the term

$$\frac{g}{p}\alpha q^{-\frac{1}{16}} \sum_{v=1}^{p-1} \left(\frac{v}{p}\right) e^{\frac{2\pi i}{p}(v - k_v)}.$$

If N is defined by $16N \equiv 1 \pmod{p}$, then we have

$$\frac{g}{p}\alpha q^{-\frac{1}{16}} \sum_{v=1}^{p-1} \left(\frac{v}{p}\right) e^{\frac{2\pi i}{p}(v-k_v)} = \frac{g}{p}\alpha q^{-\frac{1}{16}} \sum_{v=1}^{p-1} \left(\frac{v}{p}\right) e^{\frac{2\pi i}{p}Nv} = \frac{g^2}{p}\alpha q^{-\frac{1}{16}} = \left(\frac{-1}{p}\right) \alpha q^{-\frac{1}{16}}.$$

Thus, we have that

$$(W(z) - W_{\chi_p}(z))|_{-\frac{1}{2}} \left(\begin{smallmatrix} 1 & 0 \\ p^2 & 1 \end{smallmatrix}\right) = O(1).$$

This implies that $G(z)$ is a holomorphic modular form of half integral weight on $\Gamma_0(16p^2)$. Noting that

$$\bar{P}(3) = 8,$$

the remaining part of the proof is similar to that in Theorem 3. Thus, it is omitted. $\square$

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