INTERSECTION BODIES AND GENERALIZED COSINE TRANSFORMS

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Abstract. Intersection bodies represent a remarkable class of geometric objects associated with sections of star bodies and invoking Radon transforms, generalized cosine transforms, and the relevant Fourier analysis. The main focus of this article is interrelation between generalized cosine transforms of different kinds in the context of their application to investigation of a certain family of intersection bodies, which we call λ-intersection bodies. The latter include k-intersection bodies (in the sense of A. Koldobsky) and unit balls of finite-dimensional subspaces of L_p-spaces. In particular, we show that restrictions onto lower dimensional subspaces of the spherical Radon transforms and the generalized cosine transforms preserve their integral-geometric structure. We apply this result to the study of sections of λ-intersection bodies. New characterizations of this class of bodies are obtained and examples are given. We also review some known facts and give them new proofs.

Contents

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1. Introduction

This is an updated and extended version of our previous preprint [R5].

Intersection bodies interact with Radon transforms and encompass diverse classes of geometric objects associated to sections of star bodies. The concept of intersection body was introduced in the remarkable paper by Lutwak [Lu] and led to a breakthrough in the solution of the long-standing Busemann-Petty problem; see [G], [K4], [Lu], [Z2] for references and historical notes.

We remind some known facts that will be needed in the following. An origin-symmetric (o.s.) star body in $\mathbb{R}^n$, $n \ge 2$, is a compact set K with non-empty interior such that $tK \subset K$ $\forall t \in [0,1]$, $K = -K$, and the *radial function* $\rho_K(\theta) = \sup\{\lambda \ge 0 : \lambda\theta \in K\}$ is continuous on the unit sphere S^{n-1}. In the following, $\mathcal{K}^n$ denotes the set of all o.s. star bodies in $\mathbb{R}^n$, $G_{n,i}$ is the Grassmann manifold of i-dimensional linear subspaces of $\mathbb{R}^n$, and $\mathrm{vol}_i(\cdot)$ denotes the i-dimensional volume function. The *Minkowski functional* of a body $K \in \mathcal{K}^n$ is defined by $||x||_K = \min\{a \ge 0 : x \in aK\}$, so that $||\theta||_K = \rho_K^{-1}(\theta)$, $\theta \in S^{n-1}$.

Definition 1.1. [Lu] *A body $K \in \mathcal{K}^n$ is an intersection body of a body $L \in \mathcal{K}^n$ if $\rho_K(\theta) = \mathrm{vol}_{n-1}(L \cap \theta^\perp)$ for every $\theta \in S^{n-1}$, where $\theta^\perp$ is the central hyperplane orthogonal to θ.*

By taking into account that $\mathrm{vol}_{n-1}(L \cap \theta^\perp)$ in Definition 1.1 is a constant multiple of the Minkowski-Funk transform

$$(Mf)(\theta) = \int_{S^{n-1}\cap\theta^\perp} f(u)\, d_\theta u, \qquad f(u) = \rho_L^{n-1}(u),$$

Goodey, Lutwak and Weil [GLW] generalized Definition 1.1 as follows.

Definition 1.2. *A body $K \in \mathcal{K}^n$ is an intersection body if $\rho_K = M\mu$ for some even non-negative finite Borel measure μ on S^{n-1}.*

A sequence of bodies $K_j \in \mathcal{K}^n$ is said to be convergent to $K \in \mathcal{K}^n$ in the radial metric if $\lim\limits_{j\to\infty} ||\rho_{K_j} - \rho_K||_{C(S^{n-1})} = 0$.

Proposition 1.3. *The class of intersection bodies is the closure of the class of intersection bodies of star bodies in the radial metric.*

Proposition 1.4. *If K is an intersection body in $\mathbb{R}^n$, $n > 2$, then for every $i = 2, 3, \dots, n-1$ and every $\eta \in G_{n,i}$, $K \cap \eta$ is an intersection body in η.*

Regarding these two important propositions see [FGW], [GW] and a nice historical survey in [G].

Different generalizations of the concept of intersection body associated to lower dimensional sections were suggested in the literature; see, e.g., [K4], [RZ], [Z1]. The following one, which plays an important role in the study of the lower dimensional Busemann-Petty problem, is due to Zhang [Z1].

Definition 1.5. *We say, that a body $K \in \mathcal{K}^n$ belongs to Zhang's class $\mathcal{Z}_i^n$ if there is a non-negative finite Borel measure m on the Grassmann manifold $G_{n,i}$ such that $\rho_K^{n-i} = R_i^* m$, where R_i^* is the dual spherical Radon transform; see* (2.2), (2.5).

Another generalization was suggested by Koldobsky [K2] and described in detail in [K4]. This class of bodies will be our main concern.

Definition 1.6. [K4, p. 71] *A body $K \in \mathcal{K}^n$ is a k-intersection body of a body $L \in \mathcal{K}^n$ (we write $K = \mathcal{IB}_k(L)$) if*

$$\mathrm{vol}_k(K \cap \xi) = \mathrm{vol}_{n-k}(L \cap \xi^\perp) \qquad \forall \xi \in G_{n,k}. \tag{1.1}$$

We denote by $\mathcal{IB}_{k,n}$ the set of all bodies $K \in \mathcal{K}^n$ satisfying (1.1) for some $L \in \mathcal{K}^n$.

When $k = 1$, this definition coincides with Definition 1.1 up to a constant multiple. An analog of Definition 1.2 was given in the Fourier analytic terms as follows.

Definition 1.7. [K4, Definition 4.7] *A body $K \in \mathcal{K}^n$ is a k-intersection body if there is a non-negative finite Borel measure μ on S^{n-1}, so that for every Schwartz function ϕ,*

$$\int_{\mathbb{R}^n} ||x||_K^{-k} \phi(x)\, dx = \int_{S^{n-1}} \Big[\int_0^\infty t^{k-1} \hat{\phi}(t\theta)\, dt \Big]\, d\mu(\theta),$$

where $\hat{\phi}$ denotes the Fourier transform of ϕ.

The set of all k-intersection bodies in $\mathbb{R}^n$ will be denoted by $\mathcal{I}_k^n$.

Keeping in mind Proposition 1.3 for $k = 1$, one can alternatively define the class $\mathcal{I}_k^n$ as a closure of $\mathcal{IB}_{k,n}$ in the radial metric; cf. [Mi1, p. 532]. However, to apply results from [K4] to such class, equivalence of this definition to Definition 1.7 must be proved. We will do this in the more general situation in Section 5.2.

From Definitions 1.6 and 1.7 it is not clear, for which bodies $L \in \mathcal{K}^n$ the relevant k-intersection body $K = \mathcal{IB}_k(L)$ does exist. It is also not obvious which bodies actually constitute the class $\mathcal{I}_k^n$. The following important characterization is due to Koldobsky.

Theorem 1.8. [K4, Theorem 4.8] *A body $K \in \mathcal{K}^n$ is a k-intersection body if and only if $||\cdot||_K^{-k}$ represents a positive definite tempered distribution on $\mathbb{R}^n$, that is, the Fourier transform $(||\cdot||_K^{-k})^\wedge$ is a positive tempered distribution on $\mathbb{R}^n$.*

The concept of k-intersection body is related to another important development. For $K \in \mathcal{K}^n$, the quasi-normed space $(\mathbb{R}^n, ||\cdot||_K)$ is said to be isometrically embedded in $L_p,\ p > 0$, if there is a linear operator $T : \mathbb{R}^n \to L_p([0,1])$ so that $||x||_K = ||Tx||_{L_p([0,1])}$.

Theorem 1.9. [K4, Theorem 6.10] *The space $(\mathbb{R}^n, ||\cdot||_K)$ embeds isometrically in $L_p,\ p > 0,\ p \neq 2, 4, \ldots,$ if and only if $\Gamma(-p/2)(||\cdot||_K^p)^\wedge$ is a positive distribution on $\mathbb{R}^n \setminus \{0\}$.*

Following Theorems 1.9 and 1.8, one can formally say that $K \in \mathcal{I}_k^n$ if and only if $(\mathbb{R}^n, ||\cdot||_K)$ embeds isometrically in L_{-k}. This observation, combined with Definition 1.7, was used by A. Koldobsky to define the concept of "isometric embedding in L_p" for negative p.

Definition 1.10. [K4, Definition 6.14] *Let $0 < p < n,\ K \in \mathcal{K}^n$. The space $(\mathbb{R}^n, ||\cdot||_K)$ is said to be isometrically embedded in L_{-p} if there is a non-negative finite Borel measure μ on S^{n-1}, so that for every Schwartz function ϕ,*

$$\int_{\mathbb{R}^n} ||x||_K^{-p} \phi(x)\, dx = \int_{S^{n-1}} \Big[\int_0^\infty t^{p-1} \hat{\phi}(t\theta)\, dt \Big]\, d\mu(\theta),$$

where $\hat{\phi}$ denotes the Fourier transform of ϕ.

Origin-symmetric bodies K in this definition can be regarded as "unit balls of n-dimensional subspaces of L_{-p}". Comparing Definitions 1.10 and 1.7, one might call these bodies "p-intersection bodies". Since the meaning of the space L_{-p} itself is not specified in Definition 1.10 and since our paper is mostly focused on geometric properties of bodies (rather than embeddings in L_p), in the following we prefer to adopt another name *"λ-intersection body"*, where λ is a real number, that will be specified in due course. We denote the set of all λ-intersection bodies in $\mathbb{R}^n$ by $\mathcal{I}_\lambda^n$.

Contents of the paper. We will focus on intimate connection between intersection bodies, spherical Radon transforms, and generalized cosine transforms; see definitions in Section 2.2. This approach is motivated by the fact that the volume of a central cross section of a star body is expressed through the spherical Radon transform, and the latter is a member of the analytic family of the generalized cosine transforms. These transforms were introduced by Semyanistyi [Se] and arise (up to naming and normalization) in different contexts of analysis and geometry; see, e.g., [K4], [R1]-[RZ], [Sa2], [Sa3], [Str1], [Str2].

Sections 2-4 provide analytic background for geometric considerations in Sections 5-7. In Section 2 we establish our notation and define the generalized cosine transforms on the sphere and the relevant dual transforms on Grassmann manifolds. In Section 3 we present basic properties of these transforms, establish new relations between spherical Radon transforms and the generalized cosine transforms, and prove "restriction theorems", which are akin to trace theorems in Sobolev spaces. Section 4 deals with positive definite homogeneous distributions, that can be characterized in terms of the generalized cosine transforms. This section serves as a preparation for the forthcoming definition of the concept of λ-intersection body. We investigate which λ's are appropriate and why. In Section 5 we switch to geometry and define the class $\mathcal{I}^n_\lambda$ of λ-intersection bodies. The case $0<\lambda<n$ corresponds to the "unit balls of L_{-p}-spaces" in the spirit of Definition 1.10. The reader will find in this section new proofs of some known facts. We introduce the notion of λ-*intersection body of a star body* in $\mathbb{R}^n$, which extends Definition 1.6 to all $\lambda<n,\ \lambda\neq 0$. The class of all such bodies will be denoted by $\mathcal{IB}^n_\lambda$. We will prove that for all $\lambda<n,\ \lambda\neq 0,-2,-4,\dots,$ the class $\mathcal{I}^n_\lambda$ is the closure of $\mathcal{IB}^n_\lambda$ in the radial metric. The case $\lambda=1$ gives Proposition 1.3. It will be proved that all m-dimensional central sections of λ-intersection bodies are λ-intersection bodies in the corresponding m-planes provided $\lambda<m,\ \lambda\neq 0$.

The natural question arises: How to construct λ-intersection bodies? In Section 6 we give a series of examples; some of them are known and some are new. They can be obtained by utilizing auxiliary statements from Section 3. In particular, the famous embedding of Zhang's class $\mathcal{Z}^n_{n-k}$ into $\mathcal{I}^n_k$, which was first established in [K3] and studied in [Mi1], [Mi2], will be generalized to the case, when k is replaced by any $\lambda\in(0,n)$. Section 7 is devoted to the so called (q,ℓ)-balls, defined by

$$B^n_{q,\ell}=\{x=(x',x''):|x'|^q+|x''|^q\le 1;\ x'\in\mathbb{R}^{n-\ell},\ x''\in\mathbb{R}^\ell\},\quad q>0.$$

We show that if $0<q\le 2$, then $B^n_{q,\ell}\in\mathcal{I}^n_\lambda$ for all $\lambda\in(0,n)$. If $q>2$ and $n-3\le\lambda<n$, we still have $B^n_{q,\ell}\in\mathcal{I}^n_\lambda$. If $q>2$ and $0<\lambda<\lambda_0=\max(n-\ell,\ell)-2$, then $B^n_{q,\ell}\notin\mathcal{I}^n_\lambda$. The case, when $q>2,\ \ell>1$, and $\lambda_0\le\lambda<n-3$ represents an open problem.

In Section 8 we remind the generalized Busemann-Petty problem (GBP) for i-dimensional central sections of o.s. convex bodies in $\mathbb{R}^n$. This challenging problem is still open for $i=2$ and $i=3$ ($n\ge 5$). It actually inspires the whole investigation. Using properties of the generalized cosine transforms, we give a short direct proof of the fact that an affirmative answer to GBP implies that every smooth o.s. convex body in $\mathbb{R}^n$ with positive curvature is an $(n-i)$-intersection body. This fact was discovered by A. Koldobsky. The original proof in [K3] is based on the embedding $\mathcal{I}^n_{n-i} \subset \mathcal{Z}^n_i$ and Zhang's result [Z1, Theorem 6]. The latter heavily relies on the Hahn-Banach separation theorem. Our proof is more constructive and almost self-contained. We conclude the paper by Appendix, which is added for convenience of the reader.

The list of references at the end of the paper is far from being complete. Further references can be found in cited books and papers.

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2. Preliminaries

2.1. **Notation.** In the following, $\mathbb{N} = \{1, 2, \dots\}$ is the set of all natural numbers, S^{n-1} is the unit sphere in $\mathbb{R}^n$ with the area $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$; $C_e(S^{n-1})$ is the space of even continuous functions on S^{n-1}; $SO(n)$ is the special orthogonal group of $\mathbb{R}^n$; for $\theta \in S^{n-1}$ and $\gamma \in SO(n)$, $d\theta$ and $d\gamma$ denote the relevant invariant probability measures; $\mathcal{D}(S^{n-1})$ is the space of C^∞-functions on S^{n-1} equipped with the standard topology, and $\mathcal{D}'(S^{n-1})$ stands for the corresponding dual space of distributions. The subspaces of even test functions (distributions) are denoted by $\mathcal{D}_e(S^{n-1})$ ($\mathcal{D}'_e(S^{n-1})$); $G_{n,i}$ denotes the Grassmann manifold of i-dimensional subspaces ξ of $\mathbb{R}^n$ with the $SO(n)$-invariant probability measure $d\xi$; $\mathcal{D}(G_{n,i})$ is the space of infinitely differentiable functions on $G_{n,i}$.

We write $\mathcal{M}(S^{n-1})$ and $\mathcal{M}(G_{n,i})$ for the spaces of finite Borel measures on S^{n-1} and $G_{n,i}$; $\mathcal{M}_+(S^{n-1})$ and $\mathcal{M}_+(G_{n,i})$ are the relevant spaces of non-negative measures; $\mathcal{M}_{e+}(S^{n-1})$ denotes the space of even measures $\mu \in \mathcal{M}_+(S^{n-1})$. Given a function φ on $G_{n,i}$, we denote $\varphi^\perp(\eta) = \varphi(\eta^\perp), \quad \eta \in G_{n,n-i}$. Similarly, given a measure $\mu \in \mathcal{M}(G_{n,n-i})$, the corresponding "orthogonal measure" $\mu^\perp$ in $\mathcal{M}(G_{n,i})$ is defined by $(\mu^\perp, \varphi) = (\mu, \varphi^\perp)$, $\varphi \in C(G_{n,i})$.

Let $\{Y_{j,k}\}$ be an orthonormal basis of spherical harmonics on S^{n-1}. Here $j = 0, 1, 2, \dots,$ and $k = 1, 2, \dots, d_n(j)$, where $d_n(j)$ is the dimension of the subspace of spherical harmonics of degree j. Each function $\omega \in \mathcal{D}(S^{n-1})$ admits a decomposition $\omega = \sum_{j,k} \omega_{j,k} Y_{j,k}$ with the Fourier-Laplace coefficients $\omega_{j,k} = \int_{S^{n-1}} \omega(\theta) Y_{j,k}(\theta) d\theta$, which decay rapidly as $j \to \infty$. Each distribution $f \in \mathcal{D}'(S^{n-1})$ can be defined by

$(f, \omega) = \sum_{j,k} f_{j,k} \omega_{j,k}$ where $f_{j,k} = (f, Y_{j,k})$ grow not faster than j^m for some integer m. We will need the Poisson integral, which is defined for $f \in L^1(S^{n-1})$ by

$$(\Pi_t f)(\theta) = (1-t^2) \int_{S^{n-1}} f(u) |\theta - tu|^{-n} du, \quad 0 < t < 1, \tag{2.1}$$

and has the Fourier-Laplace decomposition $\Pi_t f = \sum_{j,k} t^j f_{j,k} Y_{j,k}$ [SW]. For $f \in \mathcal{D}'(S^{n-1})$, this decomposition serves as a definition of $\Pi_t f$. For harmonic analysis on the unit sphere, the reader is referred to [Le], [Mü], [Ne], [SW], and a survey article [Sa3].

2.2. **Basic integral transforms.** For integrable functions f on S^{n-1} and φ on $G_{n,i}$, $1 \le i \le n-1$, the spherical Radon transform $(R_i f)(\xi)$, $\xi \in G_{n,i}$, and its dual $(R_i^* \varphi)(\theta)$, $\theta \in S^{n-1}$, are defined by

$$(R_i f)(\xi) = \int_{\theta \in S^{n-1} \cap \xi} f(\theta)\, d_\xi \theta, \qquad (R_i^* \varphi)(\theta) = \int_{\xi \ni \theta} \varphi(\xi)\, d_\theta \xi, \tag{2.2}$$

where $d_\xi \theta$ and $d_\theta \xi$ denote the probability measures on the manifolds $S^{n-1} \cap \xi$ and $\{\xi \in G_{n,i} : \xi \ni \theta\}$, respectively. The precise meaning of the second integral is

$$(R_i^* \varphi)(\theta) = \int_{SO(n-1)} \varphi(r_\theta \gamma p_0)\, d\gamma, \qquad \theta \in S^{n-1}, \tag{2.3}$$

where p_0 is an arbitrarily fixed coordinate i-plane containing the north pole e_n and $r_\theta \in SO(n)$ is a rotation satisfying $r_\theta e_n = \theta$.

Operators R_i and R_i^* extend to finite Borel measures in a canonical way, using the duality

$$\int_{G_{n,i}} (R_i f)(\xi) \varphi(\xi) d\xi = \int_{S^{n-1}} f(\theta) (R_i^* \varphi)(\theta) d\theta. \tag{2.4}$$

Specifically, for $\mu \in \mathcal{M}(S^{n-1})$ and $m \in \mathcal{M}(G_{n,i})$, we define $R_i \mu \in \mathcal{M}(G_{n,i})$ and $R_i^* m \in \mathcal{M}(S^{n-1})$ by

$$(R_i \mu, \varphi) = \int_{S^{n-1}} (R_i^* \varphi)(\theta) d\mu(\theta), \quad (R_i^* m, f) = \int_{G_{n,i}} (R_i f)(\xi) dm(\xi), \tag{2.5}$$

where $\varphi \in C(G_{n,i})$, $f \in C(S^{n-1})$.

The generalized cosine transforms are defined by

$$(R_i^\alpha f)(\xi) = \gamma_{n,i}(\alpha) \int_{S^{n-1}} |\mathrm{Pr}_{\xi^\perp} \theta|^{\alpha+i-n} f(\theta)\, d\theta, \tag{2.6}$$

$$(\overset{*}{R}{}_i^\alpha \varphi)(\theta) = \gamma_{n,i}(\alpha) \int_{G_{n,i}} |\mathrm{Pr}_{\xi^\perp} \theta|^{\alpha+i-n} \varphi(\xi)\, d\xi, \tag{2.7}$$

$$\gamma_{n,i}(\alpha) = \frac{\sigma_{n-1}\,\Gamma((n-\alpha-i)/2)}{2\pi^{(n-1)/2}\,\Gamma(\alpha/2)}, \qquad Re\,\alpha > 0, \quad \alpha+i-n \neq 0, 2, 4, \ldots.$$

Here $\mathrm{Pr}_{\xi^\perp}\theta$ stands for the orthogonal projection of θ onto $\xi^\perp$, the orthogonal complement of $\xi \in G_{n,i}$. If f and φ are smooth enough, then integrals (2.2) can be regarded (up to a constant multiple) as members of the relevant analytic families (2.6) and (2.7); cf. Lemma 3.1. The particular case $i = n-1$ in (2.2) corresponds to the Minkowski-Funk transform

$$(2.8) \quad (Mf)(u) = \int_{\{\theta\,:\,\theta\cdot u=0\}} f(\theta)\, d_u\theta = (R_{n-1}f)(u^\perp), \qquad u \in S^{n-1},$$

which integrates a function f over great circles of codimension 1. This transform is a member of the analytic family

$$(2.9) \qquad (M^\alpha f)(u) = (R^\alpha_{n-1}f)(u^\perp) = \gamma_n(\alpha) \int_{S^{n-1}} f(\theta)|\theta\cdot u|^{\alpha-1}\, d\theta,$$

$$(2.10) \qquad \gamma_n(\alpha) = \frac{\sigma_{n-1}\,\Gamma\big((1-\alpha)/2\big)}{2\pi^{(n-1)/2}\Gamma(\alpha/2)}, \qquad Re\,\alpha > 0, \quad \alpha \neq 1, 3, 5, \ldots.$$

The values $\alpha = 1, 3, 5, \ldots$ are poles of the Gamma function $\Gamma((1-\alpha)/2)$. In some occasions we include these values into consideration and set

$$(2.11) \qquad (\tilde{M}^\alpha f)(u) = \int_{S^{n-1}} f(\theta)|\theta\cdot u|^{\alpha-1}\, d\theta.$$

Historical notes. Regarding spherical Radon transforms (2.2) and the Minkowski-Funk transform (2.8), see [GGG], [He], [R2], [R3]. The first detailed investigation of the analytic family $\{M^\alpha\}$ is due to Semyanistyi [Se], who showed that these operators naturally arise in the Fourier analysis of homogeneous functions. The case $\alpha = 2$ in (2.11) was known before, thanks to W. Blaschke, A.D. Alexandrov, and P. Lévy. Integrals (2.9) (sometimes with different normalization) arise in diverse areas of analysis and geometry; see [K4], [R1] - [R3], [Sa3], [Str1], and references therein. In convex geometry and Banach space theory, operators (2.11) with $\alpha - 1$ replaced by p are known as the p-cosine transforms. More general analytic families (2.6) and (2.7) were introduced in [R2].

3. Analytic Families of the Generalized Cosine Transforms

3.1. **Basic properties.** Below we review basic properties of integrals (2.6), (2.7), (2.9); see [R2], [R3] for more details. For integrable functions f and φ and $Re\,\alpha > 0$, integrals (2.6), (2.7) and (2.9) are absolutely convergent. When f and φ are infinitely differentiable, these integrals extend meromorphically to all $\alpha \in \mathbb{C}$.

Lemma 3.1. *If f and φ are continuous functions, then*

$$\lim_{\alpha\to 0} R_i^\alpha f = R_i^0 f = c_i\, R_i f, \qquad c_i = \frac{\sigma_{i-1}}{2\pi^{(i-1)/2}}; \tag{3.1}$$

$$\lim_{\alpha\to 0} \overset{*}{R}_i{}^\alpha \varphi = \overset{*}{R}_i{}^0 \varphi = c_i\, R_i^* \varphi, \tag{3.2}$$

$$\lim_{\alpha\to 0} M^\alpha f = M^0 f = c_{n-1}\, M f, \qquad c_{n-1} = \frac{\sigma_{n-2}}{2\pi^{(n-2)/2}}. \tag{3.3}$$

Hence, the Radon transform, its dual, and the Minkowski-Funk transform can be regarded (up to a constant multiple) as members of the corresponding analytic families $\{R_i^\alpha\}$, $\{\overset{*}{R}_i{}^\alpha\}$, $\{M^\alpha\}$.

Proof. Formulas (3.2) and (3.3) follow from (3.1). To prove (3.1), we write (2.6) in bi-spherical coordinates $\theta = u \sin\psi + v\cos\psi$, where

$$u \in S^{n-1}\cap\xi \sim S^{i-1}, \quad v \in S^{n-1}\cap\xi^\perp \sim S^{n-i-1}, \quad 0 \le \psi \le \pi/2.$$

$$d\theta = c\, \sin^{i-1}\psi \cos^{n-i-1}\psi\, d\psi du dv, \quad c = \sigma_{i-1}\sigma_{n-i-1}/\sigma_{n-1}.$$

This gives

$$\begin{aligned} (R_i^\alpha f)(\xi) &= c\,\gamma_{n,i}(\alpha) \int_0^{\pi/2} \sin^{i-1}\psi \cos^{\alpha-1}\psi\, d\psi \\ &\times \int_{S^{n-1}\cap\xi^\perp} dv \int_{S^{n-1}\cap\xi} f(u\sin\psi + v\cos\psi)\, du \\ &= \frac{c_i(\alpha)}{\Gamma(\alpha/2)} \int_0^1 t^{\alpha/2-1} F(t)\, dt, \end{aligned}$$

where

$$c_i(\alpha) = \frac{c\,\gamma_{n,i}(\alpha)\,\Gamma(\alpha/2)}{2} = \frac{\sigma_{i-1}\sigma_{n-i-1}}{2} \frac{\Gamma((n-\alpha-i)/2)}{2\pi^{(n-1)/2}} \;\to\; \frac{\sigma_{i-1}}{2\pi^{(i-1)/2}}$$

as $\alpha \to 0$, and

$$F(t) = (1-t^2)^{i/2-1} \int_{S^{n-1}\cap\xi^\perp} dv \int_{S^{n-1}\cap\xi} f(u\sqrt{1-t^2} + vt)\, du.$$

Since

$$\lim_{\alpha\to 0}\frac{1}{\Gamma(\alpha/2)}\int_0^1 t^{\alpha/2-1}F(t)\,dt = F(0) = \int_{S^{n-1}\cap\xi} f(u)du = (R_i f)(\xi),$$

we are done. □

Analytic continuation of integrals (2.9) can be realized in spherical harmonics as $M^\alpha f = \sum\limits_{j,k} m_{j,\alpha} f_{j,k} Y_{j,k}$, where

$$m_{j,\alpha} = \begin{cases} (-1)^{j/2}\,\dfrac{\Gamma(j/2+(1-\alpha)/2)}{\Gamma(j/2+(n-1+\alpha)/2)} & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd;} \end{cases} \tag{3.4}$$

see [R1], [R3]. If $f \in \mathcal{D}'(S^{n-1})$, then $M^\alpha f$ is a distribution defined by

$$(M^\alpha f, \omega) = (f, M^\alpha \omega) = \sum_{j,k} m_{j,\alpha}\, f_{j,k}\, \omega_{j,k}, \quad \omega \in \mathcal{D}(S^{n-1}); \quad \alpha \neq 1, 3, 5, \dots .$$

Lemma 3.2. *Let $\alpha, \beta \in \mathbb{C}$; $\alpha, \beta \neq 1, 3, 5, \dots .$ If $\alpha + \beta = 2 - n$ and $f \in \mathcal{D}_e(S^{n-1})$ (or $f \in \mathcal{D}'_e(S^{n-1})$), then*

$$M^\alpha M^\beta f = f. \tag{3.5}$$

If $\alpha, 2-n-\alpha \neq 1, 3, 5, \dots$, then M^α is an automorphism of the spaces $\mathcal{D}_e(S^{n-1})$ and $\mathcal{D}'_e(S^{n-1})$.

Proof. The equality (3.5) is equivalent to $m_{j,\alpha} m_{j,\beta} = 1$, $\alpha+\beta = 2-n$. The latter follows from (3.4). The second statement is a consequence of the standard theory of spherical harmonics [Ne], because the Fourier-Laplace multiplier $m_{j,\alpha}$ has a power behavior as $j \to \infty$. □

Corollary 3.3. *The Minkowski-Funk transform on the spaces $\mathcal{D}_e(S^{n-1})$ and $\mathcal{D}'_e(S^{n-1})$ can be inverted by the formula*

$$(M)^{-1} = c_{n-1}\, M^{2-n}, \qquad c_{n-1} = \frac{\sigma_{n-2}}{2\pi^{(n-2)/2}}. \tag{3.6}$$

Note that there is a wide variety of diverse inversion formulas for the Minkowski-Funk transform (see [GGG], [He], [R3] and references therein), but all of them are, in fact, different realizations of (3.6), depending on classes of functions.

3.2. **Auxiliary statements.** We establish some connections between operator families defined above.

Lemma 3.4. *Let $\alpha, \beta \in \mathbb{C}$; $\alpha, \beta \neq 1, 3, 5, \dots$. If $Re\,\alpha > Re\,\beta$, then $M^\alpha = M^\beta A_{\alpha,\beta}$, where $A_{\alpha,\beta}$ is a spherical convolution operator with the Fourier-Laplace multiplier*

$$a_{\alpha,\beta}(j) = \frac{\Gamma(j/2+(1-\alpha)/2)}{\Gamma(j/2+(n-1+\alpha)/2)} \frac{\Gamma(j/2+(n-1+\beta)/2)}{\Gamma(j/2+(1-\beta)/2)}, \tag{3.7}$$

so that $a_{\alpha,\beta}(j) \sim (j/2)^{\beta-\alpha}$ as $j \to \infty$. If α and β are real numbers satisfying $\alpha > \beta > 1-n$, $\alpha+\beta < 2$, then $A_{\alpha,\beta}$ is an integral operator such that $A_{\alpha,\beta} f \geq 0$ for every non-negative $f \in L^1(S^{n-1})$.

Proof. The first statement follows from (3.4). To prove the second one, we consider integral operators

$$(Q_+^{\mu,\nu} f)(x) = \frac{2}{\Gamma(\mu/2)} \int_0^1 (1-t^2)^{\mu/2-1} (\Pi_t f)(x)\, t^{n-\nu} dt, \tag{3.8}$$

$$(Q_-^{\mu,\nu} f)(x) = \frac{2}{\Gamma(\mu/2)} \int_1^\infty (t^2-1)^{\mu/2-1} (\Pi_{1/t} f)(x)\, t^{1-\nu} dt, \tag{3.9}$$

expressed through the Poisson integral (2.1). The Fourier-Laplace multipliers of $Q_+^{\mu,\lambda}$ and $Q_-^{\mu,\nu}$ are

$$\hat{q}_+^{\mu,\nu}(j) = \frac{\Gamma((j+n-\nu+1)/2)}{\Gamma((j+n-\nu+1+\mu)/2)}, \quad \hat{q}_-^{\mu,\nu}(j) = \frac{\Gamma((j+\nu-\mu)/2)}{\Gamma((j+\nu)/2)}. \tag{3.10}$$

They can be easily computed by taking into account that $\Pi_t \sim t^j$ in the Fourier-Laplace terms. If $f \in L^1(S^{n-1})$ and $0 < \mu < \nu < n$, then integrals (3.8) and (3.9) are absolutely convergent and obey $Q_\pm^{\mu,\nu} f \geq 0$ when $f \geq 0$. Comparing (3.10) and (3.7), we obtain a factorization $A_{\alpha,\beta} = Q_+^{\alpha-\beta,1-\beta} Q_-^{\alpha-\beta,1-\beta}$ (set $\mu = \alpha-\beta$, $\nu = 1-\beta$), which implies the second statement of the lemma. □

It is convenient to introduce a special notation for the spherical Radon transform and the generalized cosine transform with orthogonal argument. Assuming $\xi \in G_{n,i}$, we denote

$$(R_{n-i,\perp} f)(\xi) = (R_{n-i} f)(\xi^\perp), \qquad (R^\alpha_{n-i,\perp} f)(\xi) = (R^\alpha_{n-i} f)(\xi^\perp). \tag{3.11}$$

Lemma 3.5. *Let $f \in L^1(S^{n-1})$, $Re\,\alpha > 0$; $\alpha \neq 1, 3, 5, \dots$. Then*

$$(R_i M^\alpha f)(\xi) = c\,(R^{\alpha+i-1}_{n-i,\perp} f)(\xi), \qquad \xi \in G_{n,i}, \quad c = \frac{2\pi^{(i-1)/2}}{\sigma_{i-1}}, \tag{3.12}$$

or (replace i by $n-i$)

$$(R_{n-i,\perp} M^\alpha f)(\xi) = \frac{2\pi^{(n-i-1)/2}}{\sigma_{n-i-1}} (R_i^{\alpha+n-i-1} f)(\xi). \tag{3.13}$$

If $f \in \mathcal{D}_e(S^{n-1})$, then (3.12) and (3.13) extend to $Re\,\alpha \leq 0$ by analytic continuation.

Proof. For $Re\,\alpha > 0$,

$$(R_i M^\alpha f)(\xi) = \gamma_n(\alpha) \int_{S^{n-1}\cap\xi} d_\xi u \int_{S^{n-1}} f(\theta)|\theta\cdot u|^{\alpha-1}\, d\theta.$$

Since $|\theta\cdot u| = |\mathrm{Pr}_\xi\theta||v_\theta\cdot u|$ for some $v_\theta \in S^{n-1}\cap\xi$, by changing the order of integration, we obtain

$$(R_i M^\alpha f)(\xi) = \gamma_n(\alpha) \int_{S^{n-1}} f(\theta)|\mathrm{Pr}_\xi\theta|^{\alpha-1}\, d\theta \int_{S^{n-1}\cap\xi} |v_\theta\cdot u|^{\alpha-1} d_\xi u.$$

The inner integral is independent on v_θ and can be easily evaluated:

$$\begin{aligned}\int_{S^{n-1}\cap\xi} |v_\theta\cdot u|^{\alpha-1} d_\xi u &= \frac{\sigma_{i-2}}{\sigma_{i-1}} \int_{-1}^{1} |t|^{\alpha-1}(1-t^2)^{(i-3)/2}\, dt\\ &= \frac{2\pi^{(i-1)/2}\,\Gamma(\alpha/2)}{\sigma_{i-1}\,\Gamma((i+\alpha-1)/2)}.\end{aligned}$$

This implies (3.12). □

The following statement is dual to Lemma 3.5.

Lemma 3.6. *Let $\mu\in\mathcal{M}(G_{n,i})$, $\alpha\neq 1,3,5,\dots$. Then*

$$M^\alpha R_i^* \mu = c\, \overset{*}{R}{}^{\alpha+i-1}_{n-i} \mu^\perp, \qquad c = 2\pi^{(i-1)/2}/\sigma_{i-1}, \tag{3.14}$$

in the $\mathcal{D}'(S^{n-1})$-sense. If $Re\,\alpha > 0$ and μ is absolutely continuous with density $\varphi\in L^1(G_{n,i})$, then

$$M^\alpha R_i^* \varphi = c\, \overset{*}{R}{}^{\alpha+i-1}_{n-i} \varphi^\perp \tag{3.15}$$

almost everywhere on S^{n-1}. If $\varphi\in\mathcal{D}(G_{n,i})$, then (3.15) extends to all complex $\alpha\neq 1,3,5,\dots$ by analytic continuation.

Proof. Let $\omega\in\mathcal{D}_e(S^{n-1})$ (it suffices to consider only even test functions). By (2.4) and (3.12),

$$(M^\alpha R_i^*\mu, \omega) = (\mu, R_i M^\alpha \omega) = c\,(\mu, R^{\alpha+i-1}_{n-i,\perp}\omega) = c\,(\mu^\perp, R^{\alpha+i-1}_{n-i}\omega).$$

This gives the result. □

The next statement contains explicit representations of the right inverse of the dual Radon transform R_i^* (note that R_i^* is non-injective on $\mathcal{D}(G_{n,i})$ when $1 < i < n-1$).

Lemma 3.7. *Every function $f\in\mathcal{D}_e(S^{n-1})$ is represented as $f = R_i^* A f$, where $A : \mathcal{D}_e(S^{n-1}) \to \mathcal{D}(G_{n,i})$,*

$$Af = c_1\, R_i^{1-i} f = c_2\, R_{n-i,\perp} M^{2-n} f, \tag{3.16}$$

$$c_1 = \frac{\pi^{(1-i)/2}\sigma_{n-2}}{\sigma_{n-i-1}} = \frac{\Gamma((n-i)/2)}{\Gamma((n-1)/2)}, \qquad c_2 = \frac{\sigma_{n-2}}{2\pi^{n/2-1}}.$$

Proof. The coincidence of expressions in (3.16) follows from (3.13). To prove the first equality, we invoke spherical convolutions defined by analytic continuation of the integral

$$(3.17)\quad (Q^\alpha f)(\theta) = \frac{\sigma_{n-1}\Gamma((n-1-\alpha)/2)}{2\pi^{(n-1)/2}\Gamma(\alpha/2)} \int_{S^{n-1}} (1-|u\cdot\theta|^2)^{(\alpha-n+1)/2} f(u)du,$$

$Re\,\alpha > 0$, $\alpha - n \neq 0, 2, 4, \dots$, so that $Q^0 f = f$ [R2]. By Theorem 1.1 from [R2], $R_i^* R_i^\alpha f = c_1^{-1} Q^{\alpha+i-1} f$, and therefore (set $\alpha = 1-i$), $R_i^* R_i^{1-i} f = c_1^{-1} f$, as desired. □

The next statement provides an intriguing factorization of the Minkowski-Funk transform in terms of Radon transforms associated to mutually orthogonal subspaces. This factorization can be useful in different occurrences.

Theorem 3.8. *For $f \in L^1(S^{n-1})$ and $0 < i < n$,*

$$(3.18)\qquad Mf = R_i^* R_{n-i,\perp} f.$$

Proof. By (2.3),

$$\begin{aligned}
(R_i^* R_{n-i,\perp} f)(\theta) &= \int\limits_{SO(n-1)} (R_{n-i,\perp} f)(r_\theta \gamma \mathbb{R}^i)\, d\gamma \\
&= \int\limits_{SO(n-1)} (R_{n-i} f)(r_\theta \gamma \mathbb{R}^{n-i})\, d\gamma \\
&= \int\limits_{SO(n-1)} d\gamma \int\limits_{S^{n-1}\cap r_\theta\gamma\mathbb{R}^{n-i}} f(v)\, dv \\
&= \int\limits_{S^{n-1}\cap\mathbb{R}^{n-i}} dw \int\limits_{SO(n-1)} f(r_\theta \gamma w)\, d\gamma.
\end{aligned}$$

The inner integral is independent on $w \in S^{n-1} \cap \mathbb{R}^{n-i}$ and equals $(Mf)(\theta)$. This gives (3.18). □

3.3. **Restriction theorems.** Theorems of such type deal with traces of functions on lower dimensional subspaces and are well known, for instance, in the theory of function spaces. To the best of our knowledge, traces of functions represented by Radon transforms or, more generally, by the generalized cosine transforms , were not studied systematically and deserve particular attention, because they provide analytic background to a series of results related to sections of star bodies; cf. [R3, Sec. 3.5], [FGW]. Given a subspace $\eta \in G_{n,m}$ and $k < m$, we denote by $G_k(\eta)$ the manifold of all k-dimensional subspaces of η.

Theorem 3.9. *Let $f \in C_e(S^{n-1})$, $1 \le k < m < n$, $\lambda \neq 0, -2, -4, \dots$. If $Re\,\lambda < k$, then for every $\eta \in G_{n,m}$ and every $\xi \in G_k(\eta)$,*

$$(R^{k-\lambda}_{n-k} f)(\xi^\perp) = (R^{k-\lambda}_{m-k} T^\lambda_\eta f)(\xi^\perp \cap \eta), \tag{3.19}$$

where

$$(T^\lambda_\eta f)(u) = \tilde{c} \int\limits_{S^{n-1}\cap(\eta^\perp \oplus \mathbb{R}u)} f(w)|u \cdot w|^{m-\lambda-1}\, dw, \tag{3.20}$$

$$u \in S^{n-1} \cap \eta, \qquad \tilde{c} = \pi^{(m-n)/2}\, \sigma_{n-m}/2.$$

In particular (let $\lambda \to k$),

$$(R_{n-k} f)(\xi^\perp) = c\,(R_{m-k} T^k_\eta f)(\xi^\perp \cap \eta), \quad c = \frac{\pi^{(n-m)/2}\, \sigma_{m-k-1}}{\sigma_{n-k-1}}. \tag{3.21}$$

Proof. By (2.6),

$$(R^{k-\lambda}_{n-k} f)(\xi^\perp) = \gamma_{n,n-k}(k-\lambda) \int_{S^{n-1}} |\mathrm{Pr}_\xi \theta|^{-\lambda}\, f(\theta)\, d\theta.$$

We represent θ in bi-spherical coordinates as

$$\theta = u \cos\psi + v \sin\psi, \tag{3.22}$$

where

$$u \in S^{n-1} \cap \eta \sim S^{m-1}, \quad v \in S^{n-1} \cap \eta^\perp \sim S^{n-m-1}, \quad 0 \le \psi \le \pi/2,$$

$$d\theta = c''\, \sin^{n-m-1}\psi \cos^{m-1}\psi\, d\psi du dv, \quad c'' = \sigma_{m-1}\sigma_{n-m-1}/\sigma_{n-1}.$$

If $\xi \subset \eta$, then $|\mathrm{Pr}_\xi \theta| = |\mathrm{Pr}_\xi[\mathrm{Pr}_\eta \theta]| = |\mathrm{Pr}_\xi u| \cos\psi$, and therefore,

$$(R^{k-\lambda}_{n-k} f)(\xi^\perp) = \gamma_{m,m-k}(k-\lambda) \int_{S^{n-1}\cap\eta} |\mathrm{Pr}_\xi u|^{-\lambda} (T^\lambda_\eta f)(u)\, du,$$

where

$$\begin{aligned}
(T^\lambda_\eta f)(u) &= \frac{c''\, \gamma_{n,n-k}(k-\lambda)}{\gamma_{m,m-k}(k-\lambda)} \int_0^{\pi/2} \sin^{n-m-1}\psi \cos^{m-\lambda-1}\psi\, d\psi \\
&\times \int_{S^{n-1}\cap\eta^\perp} f(u\cos\psi + v \sin\psi)\, dv \\
&= \frac{\pi^{(m-n)/2}\, \sigma_{n-m}}{2} \int_{S^{n-1}\cap(\eta^\perp\oplus\mathbb{R}u)} f(w)|u\cdot w|^{m-\lambda-1}\, dw.
\end{aligned}$$

Formula (3.21) follows from (3.19) by (3.1). $\square$

Theorem 3.10. *Let $f \in D_e(S^{n-1})$, $\eta \in G_{n,m}$, $1<m<n$. Suppose that $f = M^{1-\lambda} g$, where $Re\,\lambda < m$, $\lambda \neq 0, -2, -4, \ldots$. Then the restriction of f onto η is represented as $f = M^{1-\lambda}_{S^{n-1}\cap\eta} T^\lambda_\eta g$, where T^λ_η has the form (3.20) and $M^{1-\lambda}_{S^{n-1}\cap\eta}$ denotes the same operator $M^{1-\lambda}$, but on the sphere $S^{n-1}\cap\eta$.*

Proof. For $Re\,\lambda < 1$, the statement is a particular case of Theorem 3.9 (set $k=1$). For other values of λ, the result follows by analytic continuation. $\square$

Remark 3.11. The restriction $\lambda \neq 0, -2, -4, \ldots$ in Theorems 3.9 and 3.10 is caused by the Gamma function $\Gamma(\lambda/2)$ in the numerator of the corresponding normalizing factor. It is evident from the proof, that both theorems remain true also for $\lambda = -2\ell, \ell \in \mathbb{N}$, if we remove the normalizing factor. Then $M^{1-\lambda}$ in Theorem 3.10 will be substituted for $\tilde{M}^{1+2\ell}$; see (2.11).

We will need the following generalization of Theorem 3.10.

Theorem 3.12. *Let $f \in C_e(S^{n-1})$, $\mu \in \mathcal{M}_{e+}(S^{n-1})$, and let $\eta \in G_{n,m}$, $1<m<n$. Suppose that $f = M^{1-\lambda}\mu$, if $\lambda < m$, $\lambda \neq -2\ell, \ell \in \mathbb{N}$, and $f = \tilde{M}^{1+2\ell}\mu$, if $\lambda = -2\ell$.*
(i) *There is a measure $\nu \in \mathcal{M}_{e+}(S^{n-1}\cap\eta)$ such that the restriction of f onto $S^{n-1}\cap\eta$ is represented as $f = M^{1-\lambda}_{S^{n-1}\cap\eta}\nu$.*
(ii) *If $d\mu(\theta) = g(\theta)d\theta$, $g \in C_e(S^{n-1})$, then* (i) *holds with $d\nu(\theta) = (T^\lambda_\eta g)(\theta)d\theta$, where $T^\lambda_\eta g$ has the form (3.20).*
(iii) *If $\lambda = -2\ell, \ell \in \mathbb{N}$, then* (i) *and* (ii) *hold with $M^{1-\lambda}_{S^{n-1}\cap\eta}$ substituted for $\tilde{M}^{1+2\ell}_{S^{n-1}\cap\eta}$.*

Proof. STEP 1. Let first $\lambda < m$, $\lambda \neq 0, -2, -4, \ldots$. We invoke the Poisson integral (2.1) so that

$$\Pi_t f = \Pi_t M^{1-\lambda}\mu = M^{1-\lambda} g_t, \qquad g_t = \Pi_t \mu \in \mathcal{D}_e(S^{n-1}), \quad t \in (0,1).$$

Since f is continuous, then $\Pi_t f$ converges to f as $t \to 0$ uniformly on S^{n-1}, and therefore, uniformly on $S^{n-1}\cap\eta$. Hence, for any test function $\omega \in \mathcal{D}(S^{n-1}\cap\eta)$, owing to Theorem 3.10, we have

$$\begin{aligned}
(f,\omega) &= \lim_{t\to 0}(\Pi_t f,\omega) = \lim_{t\to 0}(M^{1-\lambda} g_t,\omega) \\
&= \lim_{t\to 0}(M^{1-\lambda}_{S^{n-1}\cap\eta} T^\lambda_\eta g_t,\omega) = \lim_{t\to 0}(T^\lambda_\eta g_t, M^{1-\lambda}_{S^{n-1}\cap\eta}\omega) \\
&= \lim_{t\to 0}(\nu_t, M^{1-\lambda}_{S^{n-1}\cap\eta}\omega), \qquad \nu_t = T^\lambda_\eta g_t.
\end{aligned} \tag{3.23}$$

Thus, $\lim_{t\to 0}(\nu_t, M^{1-\lambda}_{S^{n-1}\cap\eta}\omega)$ exists for every $\omega \in \mathcal{D}(S^{n-1}\cap\eta)$. If ω is even, i.e., $\omega \in \mathcal{D}_e(S^{n-1}\cap\eta)$, then, by Lemma 3.2, we can replace ω by

$M^{1-m+\lambda}_{S^{n-1}\cap\eta}\omega$ and conclude that the limit $\lim\limits_{t\to 0}(\nu_t,\omega)$ is well-defined for every $\omega\in\mathcal{D}_e(S^{n-1}\cap\eta)$. Since $\nu_t=T^\lambda_\eta\Pi_t\mu$ is an even function and the generic test function $\omega\in\mathcal{D}(S^{n-1}\cap\eta)$ can be represented as $\omega_+ +\omega_-$, where $\omega_\pm$ are even and odd, respectively, it follows that the limit $\lim\limits_{t\to 0}(\nu_t,\omega)=\lim\limits_{t\to 0}(\nu_t,\omega_+)$ is well-defined for every $\omega\in\mathcal{D}(S^{n-1}\cap\eta)$ (not only for even ω, as stated above). Since $\mathcal{D}'(S^{n-1}\cap\eta)$ is weakly complete, there is an even distribution ν in $\mathcal{D}'(S^{n-1}\cap\eta)$ so that

$$(\nu,\omega)=\lim_{t\to 0}(\nu_t,\omega),\qquad \omega\in\mathcal{D}(S^{n-1}\cap\eta).$$

Furthermore, since $(\nu_t,\omega)=(T^\lambda_\eta\Pi_t\mu,\omega)$ is non-negative for every non-negative $\omega\in\mathcal{D}(S^{n-1}\cap\eta)$ and every $t\in(0,1)$, then ν is a positive distribution and, by Theorem 9.1, ν is a measure in $\mathcal{M}_{e+}(S^{n-1}\cap\eta)$. Thus, by (3.23), $(f,\omega)=\lim\limits_{t\to 0}(\nu_t,M^{1-\lambda}_{S^{n-1}\cap\eta}\omega)=(\nu,M^{1-\lambda}_{S^{n-1}\cap\eta}\omega)$, which means that $f=M^{1-\lambda}_{S^{n-1}\cap\eta}\nu$, as desired.

If $d\mu(\theta)=g(\theta)d\theta$, $g\in C_e(S^{n-1})$, then $\nu_t=T^\lambda_\eta\Pi_t g$ tends to $T^\lambda_\eta g$ uniformly on $S^{n-1}\cap\eta$ as $t\to 0$. Hence, by (3.23), $(f,\omega)=(T^\lambda_\eta g,M^{1-\lambda}_{S^{n-1}\cap\eta}\omega)$, which means $f=M^{1-\lambda}_{S^{n-1}\cap\eta}T^\lambda_\eta g$.

STEP 2. Consider the case $\lambda=-2\ell$, $\ell\in\mathbb{N}$, when $f=\tilde{M}^{1+2\ell}\mu$, $\mu\in\mathcal{M}_{e+}(S^{n-1})$, and the operator $T^\lambda_\eta=T^{-2\ell}_\eta$ has the form

$$(T^{-2\ell}_\eta h)(u)=\tilde{c}\int\limits_{S^{n-1}\cap(\eta^\perp\oplus\mathbb{R}u)}|u\cdot w|^{m+2\ell-1}\,h(w)\,dw,$$

cf. (3.20). For any functions $h\in C(S^{n-1})$ and $\omega\in C(S^{n-1}\cap\eta)$,

$$(T^{-2\ell}_\eta h,\omega)=(h,\overset{*}{T}{}^{-2\ell}_\eta\omega),\tag{3.24}$$

where

$$(\overset{*}{T}{}^{-2\ell}_\eta\omega)(\theta)=\frac{\Gamma(m/2)}{2\Gamma(n/2)}\,\omega\Big(\frac{\mathrm{Pr}_\eta\theta}{|\mathrm{Pr}_\eta\theta|}\Big)\,|\mathrm{Pr}_\eta\theta|^{2\ell}\in C(S^{n-1}).$$

Indeed, using bi-spherical coordinates (see (3.22)), we have

$$\begin{aligned}
(T_\eta^{-2\ell}h,\omega) &= \tilde{c}\int\limits_{S^{n-1}\cap\eta}\omega(u)du\int\limits_{S^{n-1}\cap(\eta^\perp\oplus\mathbb{R}u)} h(w)|u\cdot w|^{m+2\ell-1}\,dw\\
&= \frac{\tilde{c}\,\sigma_{n-m-1}}{\sigma_{n-m}}\int\limits_{S^{n-1}\cap\eta}\omega(u)du\int\limits_0^{\pi/2}\sin^{n-m-1}\psi\cos^{m+2\ell-1}\psi\,d\psi\\
&\times\int\limits_{S^{n-1}\cap\eta^\perp} h(u\cos\psi+v\sin\,\psi)\,dv\\
&= \frac{\tilde{c}\,\sigma_{n-m-1}}{c''\,\sigma_{n-m}}\int\limits_{S^{n-1}} h(\theta)\,\omega\Big(\frac{\mathrm{Pr}_\eta\theta}{|\mathrm{Pr}_\eta\theta|}\Big)\,|\mathrm{Pr}_\eta\theta|^{2\ell}\,d\theta = (h,\overset{*}{T}{}_\eta^{-2\ell}\omega).
\end{aligned}$$

Let $h=\Pi_t\mu$ and observe that the limit $\lim\limits_{t\to0}(T_\eta^{-2\ell}\Pi_t\mu,\omega)$ exists, because, by (3.24), $(T_\eta^{-2\ell}\Pi_t\mu,\omega)=(\Pi_t\mu,\overset{*}{T}{}_\eta^{-2\ell}\omega)\to(\mu,\overset{*}{T}{}_\eta^{-2\ell}\omega)$. Note that $(T_\eta^{-2\ell}\Pi_t\mu,\omega)\ge0$ for any non-negative $\omega\in C(S^{n-1}\cap\eta)$. Applying the standard completeness argument (as in Step 1), we conclude, that there is a measure $\nu\in\mathcal{M}_+(S^{n-1}\cap\eta)$ such that

$$\lim_{t\to0}(T_\eta^{-2\ell}\Pi_t\mu,\omega)=(\nu,\omega)\quad\forall\omega\in C(S^{n-1}\cap\eta).$$

Using this equality, for $f=\tilde{M}^{1+2\ell}\mu$ we obtain

$$\begin{aligned}
(f,\omega) &= \lim_{t\to0}(\Pi_t f,\omega)=\lim_{t\to0}(\Pi_t\tilde{M}^{1+2\ell}\mu,\omega)=\lim_{t\to0}(\tilde{M}^{1+2\ell}\Pi_t\mu,\omega)\\
&\quad\text{(use Theorem 3.10 and Remark 3.11)}\\
&= \lim_{t\to0}(\tilde{M}^{1+2\ell}_{S^{n-1}\cap\eta}T_\eta^{-2\ell}\Pi_t\mu,\omega)=\lim_{t\to0}(T_\eta^{-2\ell}\Pi_t\mu,\tilde{M}^{1+2\ell}_{S^{n-1}\cap\eta}\omega)\\
&= (\nu,\tilde{M}^{1+2\ell}_{S^{n-1}\cap\eta}\omega).
\end{aligned}$$

This gives the result.

If $d\mu(\theta)=g(\theta)d\theta$, $g\in C_e(S^{n-1})$, then, by Theorem 3.10 and Remark 3.11, for $\theta\in S^{n-1}\cap\eta$ we have

$$(\Pi_t f)(\theta)=(\Pi_t\tilde{M}^{1+2\ell}g)(\theta)=(\tilde{M}^{1+2\ell}\Pi_t g)(\theta)=(\tilde{M}^{1+2\ell}_{S^{n-1}\cap\eta}T_\eta^{-2\ell}\Pi_t g)(\theta).$$

Owing to continuity of the operators $\tilde{M}^{1+2\ell}_{S^{n-1}\cap\eta}$, $T_\eta^{-2\ell}$, and Π_t in the relevant spaces of continuous functions, by passing to the limit as $t\to0$, we obtain $f(\theta)=(\tilde{M}^{1+2\ell}_{S^{n-1}\cap\eta}T_\eta^{-2\ell}g)(\theta)$, $\theta\in S^{n-1}\cap\eta$, as desired. □

4. Positive Definite Homogeneous Distributions

We remind some known facts; see, e.g., [GS], [Le]. Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of rapidly decreasing C^∞-functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ its dual. The Fourier transform of $F \in \mathcal{S}'(\mathbb{R}^n)$ is defined by

$$\langle \hat{F}, \hat{\phi} \rangle = (2\pi)^n \langle F, \phi \rangle, \quad \hat{\phi}(y) = \int_{\mathbb{R}^n} \phi(x)\, e^{ix\cdot y}\, dx, \quad \phi \in \mathcal{S}(\mathbb{R}^n).$$

A distribution $F \in \mathcal{S}'(\mathbb{R}^n)$ is homogeneous of degree $\lambda \in \mathbb{C}$ if for any $\phi \in \mathcal{S}(\mathbb{R}^n)$ and any $a>0$, $\langle F, \phi(x/a) \rangle = a^{\lambda+n} \langle F, \phi \rangle$. Homogeneous distributions on $\mathbb{R}^n$ are intimately connected with distributions on S^{n-1}. Let first $f \in L^1(S^{n-1})$, $(E_\lambda f)(x) = |x|^\lambda f(x/|x|)$, $x \in \mathbb{R}^n \setminus \{0\}$. The operator E_λ generates a meromorphic $\mathcal{S}'$-distribution

$$\langle E_\lambda f, \phi \rangle = a.c. \int_0^\infty r^{\lambda+n-1} u(r) dr, \quad u(r) = \int_{S^{n-1}} f(\theta) \overline{\phi(r\theta)} d\theta,$$

where "$a.c.$" denotes analytic continuation in the λ-variable. The distribution $E_\lambda f$ is regular if $Re\, \lambda > -n$ and admits simple poles at $\lambda = -n, -n-1, \ldots$. The above definition extends to all distributions $f \in \mathcal{D}'(S^{n-1})$ by the formula

$$\langle E_\lambda f, \phi \rangle = a.c. \int_0^\infty r^{\lambda+n-1} u(r) dr, \quad u(r) = (f, \phi(r\theta)),^{1}$$

and the map $E_\lambda : \mathcal{D}'(S^{n-1}) \to \mathcal{S}'(\mathbb{R}^n)$ is weakly continuous. If f is orthogonal to all spherical harmonics of degree j, then the derivative $u^{(j)}(r)$ equals zero at $r=0$ and the pole at $\lambda = -n-j$ is removable. In particular, if f is an even distribution, i.e., $(f, \varphi) = (f, \varphi^-)$, $\varphi^-(\theta) = \varphi(-\theta) \quad \forall \varphi \in \mathcal{D}(S^{n-1})$, then the only possible poles of $E_\lambda f$ are $-n, -n-2, -n-4, \ldots$.

The Fourier transform of homogeneous distributions was extensively studied by many authors; see [Sa3] and references therein. We restrict our consideration to even distributions, when the operator family $\{M^\alpha\}$ defined by (2.9) naturally arises thanks to the formula

$$[E_{1-n-\alpha} f]^\wedge = 2^{1-\alpha} \pi^{n/2}\, E_{\alpha-1} M^\alpha f. \tag{4.1}$$

This formula amounts to Semyanistyi [Se]. If $f \in \mathcal{D}_e(S^{n-1})$, then (4.1) holds pointwise for $0 < Re\, \alpha < 1$ (see, e.g., Lemma 3.3 in [R1]) and extends in the S'-sense to all $\alpha \in \mathbb{C}$ satisfying

$$\alpha \notin \{1, 3, 5, \ldots\} \cup \{1-n, -n-1, -n-3, \ldots\}. \tag{4.2}$$

[1]Here and on, different notations $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$ are used for distributions on $\mathbb{R}^n$ and S^{n-1}, respectively.

Since $\mathcal{D}_e(S^{n-1})$ is dense in $\mathcal{D}'_e(S^{n-1})$ and the maps $E_{1-n-\alpha}$ and $E_{\alpha-1}$ are weakly continuous from $\mathcal{D}'_e(S^{n-1})$ to $\mathcal{S}'(\mathbb{R}^n)$, then (4.1) extends to all $f \in \mathcal{D}'_e(S^{n-1})$.

Regarding the cases excluded in (4.2), we note that if $\alpha = 1+2\ell$ for some $\ell = 0, 1, \ldots$, then (4.1) is meaningful if and only if f is orthogonal to all spherical harmonics of degree 2ℓ. If $\alpha = 1-n-2\ell$ for some $\ell = 0, 1, \ldots$, then, according to the spherical harmonic decomposition $f = \sum_{j,k} f_{j,k} Y_{j,k}$, j even, formula (4.1) is substituted for the following:

$$\begin{aligned}(4.3)\qquad [E_{2\ell} f]^\wedge(\xi) &= (2\pi)^n \sum_{j\le 2\ell} \sum_k f_{j,k} (-\Delta)^{\ell-j/2} Y_{j,k}(i\partial)\, \delta(\xi) \\ &+ 2^{n+2\ell}\pi^{n/2}\, E_{-n-2\ell} M^{1-n-2\ell}\Big[f - \sum_{j\le 2\ell}\sum_k f_{j,k} Y_{j,k}\Big](\xi),\end{aligned}$$

where $-\Delta$ is the Laplace operator, $\partial = (\partial/\partial\xi_1, \ldots, \partial/\partial\xi_n)$, and $\delta(\xi)$ is the delta function. It is worth noting that for $\alpha = 1, 3, 5, \ldots$, the distribution $[E_{1-n-\alpha} f]^\wedge$ can also be understood in the regularized sense without any orthogonality assumptions. However, such regularization does not preserve homogeneity; see [Sa1], [Sa3].

Our main concern is positivity and positive definiteness of even homogeneous distributions. The reader is referred to [GV] for the general theory. A distribution $F \in \mathcal{S}'(\mathbb{R}^n)$ is *positive* if $\langle F, \phi\rangle \ge 0$ for all non-negative $\phi \in \mathcal{S}(\mathbb{R}^n)$. A similar definition holds for distributions on the sphere and on $\mathbb{R}^n \setminus \{0\}$. A distribution $F \in \mathcal{S}'(\mathbb{R}^n)$ is *positive definite* if $\hat{F}$ is positive. For our purposes, it is important to know, which even homogeneous distributions are positive definite. Let us rewrite (4.1) and (4.2) with $1-n-\alpha$ replaced by $-\lambda$. We have

$$(4.4)\qquad [E_{-\lambda} f]^\wedge = 2^{n-\lambda}\pi^{n/2}\, E_{\lambda-n} M^{1+\lambda-n} f,$$

$$(4.5)\qquad \lambda \notin \Lambda_0, \quad \Lambda_0 = \{n, n+2, n+4\ldots\} \cup \{0, -2, -4, \ldots\}.$$

Theorem 4.1. *Let $\lambda \in \mathbb{R} \setminus \Lambda_0$, $f \in \mathcal{D}'_e(S^{n-1})$.*

(i) *If $\lambda < 0$ and $E_{-\lambda} f$ is a positive definite distribution, then $f = 0$.*

(ii) *For all $\lambda \in \mathbb{R} \setminus \Lambda_0$, the following statements are equivalent:*

(a) *$[E_{-\lambda} f]^\wedge$ is a positive distribution on $\mathbb{R}^n \setminus \{0\}$ (for $\lambda > 0$, this can be replaced by "$E_{-\lambda} f$ is a positive definite distribution on $\mathbb{R}^n$");*

(b) *$M^{1+\lambda-n} f \in \mathcal{M}_{e+}(S^{n-1})$;*

(c) *$f = M^{1-\lambda}\mu$ for some measure $\mu \in \mathcal{M}_{e+}(S^{n-1})$.*

Furthermore, for any real $\lambda \ne 0, -2, -4, \ldots$, and any $i = 1, 2, \ldots, n-1$, (c) *is equivalent to*

(d) *$R_i f = R^{i-\lambda}_{n-i,\perp}\mu$ for some measure $\mu \in \mathcal{M}_{e+}(S^{n-1})$.*

Proof. (i) Choose $\phi(x) = \exp(-|x|^m)\, p_{t,\theta}(x/|x|)$, where $m \in 2\mathbb{N}$ and $p_{t,\theta}(\cdot)$ is the Poisson kernel

$$p_{t,\theta}(u) = \frac{1-t^2}{(1-2tu\cdot\theta+t^2)^{n/2}}, \qquad 0<t<1; \quad u,\theta \in S^{n-1}. \tag{4.6}$$

Then $\langle E_{\lambda-n} M^{1+\lambda-n} f, \phi\rangle = c_\lambda (\Pi_t M^{1+\lambda-n} f)(\theta)$, where

$$c_\lambda = a.c. \int_0^\infty r^{\lambda-1}\exp(-r^m)\,dr = m^{-1}\Gamma(\lambda/m)$$

and $(\Pi_t M^{1+\lambda-n} f)(\theta)$ is the Poisson integral of $M^{1+\lambda-n} f$. If $E_{-\lambda} f$ is a positive definite distribution, then, by (4.4), $E_{\lambda-n} M^{1+\lambda-n} f$ is a positive distribution. On the other hand, if $\lambda<0$ and $m>-\lambda$, then $c_\lambda<0$. Hence $\langle E_{\lambda-n} M^{1+\lambda-n} f, \phi\rangle$ can be non-negative for *every* non-negative $\phi \in \mathcal{S}(\mathbb{R}^n)$ only if $(\Pi_t M^{1+\lambda-n} f)(\theta) = 0$ for every $0<t<1$ and $\theta \in S^{n-1}$. The latter implies $M^{1+\lambda-n} f = 0$, which is equivalent to $f=0$ because $M^{1+\lambda-n}$ is injective; see Lemma 3.2.

(ii) Let $[E_{-\lambda} f]^\wedge$ be a positive distribution on $\mathbb{R}^n \setminus \{0\}$. It means that for every $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\phi \ge 0$ and $0 \notin \text{supp}\phi$, $\langle [E_{-\lambda} f]^\wedge, \phi\rangle \ge 0$ or, by (4.4), $\langle E_{\lambda-n} M^{1+\lambda-n} f, \phi\rangle \ge 0$. Choose $\phi(x) = \psi(|x|)\omega(x/|x|)$, where $\omega \in \mathcal{D}(S^{n-1})$, $\omega \ge 0$, and ψ is a smooth non-negative function such that $\int_0^\infty r^{\alpha+n-2}\psi(r)dr = 1$ and $0 \notin \text{supp}\psi$. Then

$$\langle E_{\lambda-n} M^{1+\lambda-n} f, \phi\rangle = (M^{1+\lambda-n} f, \omega) \ge 0,$$

and therefore, $M^{1+\lambda-n} f \in \mathcal{M}_{e+}(S^{n-1})$; see Theorem 9.1.

Conversely, let $\mu = M^{1+\lambda-n} f \in \mathcal{M}_{e+}(S^{n-1})$ and let $\phi \in \mathcal{S}(\mathbb{R}^n)$; $\phi \ge 0$. In the case $\lambda<0$ we additionally assume $0 \notin \text{supp}\phi$. By (4.4),

$$\begin{aligned} \langle [E_{-\lambda} f]^\wedge, \phi\rangle &= 2^{n-\lambda}\pi^{n/2}\, \langle E_{\lambda-n}\mu, \phi\rangle \\ &= 2^{n-\lambda}\pi^{n/2} \int_0^\infty r^{\lambda-1} dr \int_{S^{n-1}} \phi(r\theta)d\mu(\theta) \ge 0. \end{aligned}$$

This proves equivalence of (a) and (b). Equivalence of (b) and (c) follows from Lemma 3.2.

Let us prove the equivalence of (c) and (d). If $R_i f = R^{i-\lambda}_{n-i,\perp}\mu$, $\mu \in \mathcal{M}_{e+}(S^{n-1})$, then, by (3.15),

$$\begin{aligned} (f, R_i^*\varphi) &= (R_i f, \varphi) = (R^{i-\lambda}_{n-i,\perp}\mu, \varphi) = (\mu, \overset{*}{R}{}^{i-\lambda}_{n-i}\varphi^\perp) \\ &= c^{-1}(\mu, M^{1-\lambda} R_i^*\varphi), \qquad \varphi \in \mathcal{D}(G_{n,i}). \end{aligned}$$

Since any function $\omega \in \mathcal{D}_e(S^{n-1})$ can be expressed as $\omega = R_i^*\varphi$ for some $\varphi \in \mathcal{D}(G_{n,i})$ (see Lemma 3.7), this gives $(f,\omega) = c^{-1}(\mu, M^{1-\lambda}, \omega)$ which is (c). Conversely, let $f = M^{1-\lambda}\mu$, $\mu \in \mathcal{M}_{e+}(S^{n-1})$, that is, $(f,\omega) = (\mu, M^{1-\lambda}, \omega)$ for every $\omega \in \mathcal{D}_e(S^{n-1})$. Choose $\omega = R_i^*\varphi$, $\varphi \in$

$\mathcal{D}(G_{n,i})$. Then, as above, $(f, R_i^* \varphi) = (\mu, M^{1-\lambda} R_i^* \varphi) = c\,(\mu, \overset{*}{R}{}^{i-\lambda}_{n-i} \varphi^\perp)$, which gives (d). $\square$

5. λ-INTERSECTION BODIES

5.1. **Definitions and comments.** We remind that $\mathcal{K}^n$ is the set of all origin-symmetric star bodies K in $\mathbb{R}^n$, $n \ge 2$; ρ_K and $||\cdot||_K$ are the radial function and the Minkowski functional of K. The following definitions and statements are motivated by Theorem 4.1 and the previous consideration. Let λ be a real number,

$$s_\lambda = \begin{cases} 1 & \text{if } \lambda > 0, \quad \lambda \neq n, n+2, n+4, \dots, \\ \Gamma(\lambda/2) & \text{if } \lambda < 0, \quad \lambda \neq -2, -4, \dots. \end{cases} \tag{5.1}$$

The values $\lambda = 0, n, n+2, n+4, \dots$ will not be considered in the following, but values $\lambda = -2, -4, \dots$ will be included. They become meaningful if we change normalization. For $\lambda \neq 0, n, n+2, n+4 \dots$, let $\mathcal{I}^n_\lambda$ be the set of bodies $K \in \mathcal{K}^n$, for which there is a measure $\mu \in \mathcal{M}_{e+}(S^{n-1})$ such that $s_\lambda \rho_K = M^{1-\lambda} \mu$ if $\lambda \neq -2\ell$, $\ell \in \mathbb{N}$, and $\rho_K = \tilde{M}^{1-\lambda} \mu \equiv \tilde{M}^{1+2\ell} \mu$, otherwise. The equality $s_\lambda \rho_K = M^{1-\lambda}\mu$ means that for any $\varphi \in \mathcal{D}(S^{n-1})$,

$$s_\lambda \int_{S^{n-1}} \rho_K^k(\theta) \varphi(\theta)\, d\theta = \int_{S^{n-1}} (M^{1-\lambda}\varphi)(\theta)\, d\mu(\theta),$$

where for $\lambda \ge 1$, $(M^{1-\lambda}\varphi)(\theta)$ is understood in the sense of analytic continuation. We remind the notation

$$\Lambda_0 = \{n, n+2, n+4 \dots\} \cup \{0, -2, -4, \dots\}.$$

Theorem 5.1. *For $\lambda \in \mathbb{R} \setminus \Lambda_0$, the following statements are equivalent:*

(a) $K \in \mathcal{I}^n_\lambda$*;*

(b) *The Fourier transform $[s_\lambda\, ||\cdot||_K^{-\lambda}]^\wedge$ is a positive distribution on $\mathbb{R}^n \setminus \{0\}$ (for $\lambda > 0$, this can be replaced by "$||\cdot||_K^{-\lambda}$ is a positive definite distribution on $\mathbb{R}^n$");*

(c) $s_\lambda\, M^{1+\lambda-n} \rho_K^\lambda \in \mathcal{M}_{e+}(S^{n-1})$*;*

The theorem is an immediate consequence of Theorem 4.1 if the latter is applied to $f = s_\lambda \rho_K^\lambda$. Another useful characterization is provided by Theorem 4.1 (d).

Theorem 5.2. *Let $\lambda \in \mathbb{R} \setminus \Lambda_0$. If $K \in \mathcal{I}^n_\lambda$, then for every $i \in \{1, 2, \dots, n-1\}$ there is a measure $\mu \in \mathcal{M}_{e+}(S^{n-1})$ such that $s_\lambda R_i \rho_K^\lambda = R^{i-\lambda}_{n-i,\perp} \mu$. Conversely, if*

$$s_\lambda R_i \rho_K^\lambda = R^{i-\lambda}_{n-i,\perp} \mu$$

for some $i \in \{1, 2, \dots, n-1\}$ and some $\mu \in \mathcal{M}_{e+}(S^{n-1})$, then $K \in \mathcal{I}^n_\lambda$.

Although $\mathcal{I}^n_\lambda$ was called "the set of bodies", the definition of this set is purely analytic and extra work is needed to understand what *bodies* (if any) actually constitute the class $\mathcal{I}^n_\lambda$.

The following comments will be helpful.

1. The case $\lambda > n$ is not so interesting, because by Theorem 5.1(c), $\mathcal{I}^n_\lambda$ is either empty (if $\Gamma((n-\lambda)/2) < 0$) or coincides with the whole class $\mathcal{K}^n$ (if $\Gamma((n-\lambda)/2) > 0$).

2. The case $\lambda \in (0, n)$ agrees with the concept of isometric embedding of the space $(\mathbb{R}^n, ||\cdot||_K)$ into $L_{-p},\ p = \lambda$; see Introduction. In the framework of this concept, all bodies $K \in \mathcal{I}^n_\lambda$ can be regarded as "unit balls of n-dimensional subspaces of $L_{-\lambda}$".

3. If $K \in \mathcal{I}^n_\lambda$, where $\lambda < 0$ (one can replace λ by $= -p,\ p > 0$), then

$$||u||^p_K = \int_{S^{n-1}} |\theta \cdot u|^p \, d\mu(\theta)$$

for some $\mu \in \mathcal{M}_{e+}(S^{n-1})$. This is the well known Lévy representation, characterizing isometric embedding of the space $(\mathbb{R}^n, ||\cdot||_K)$ into L_p; see Lemma 6.4 in [K4]. Statement (b) in Theorem 5.1 agrees with Theorem 1.9. Keeping this terminology, we can state the following

Proposition 5.3. *Let $p > -n,\ p \neq 0$. Then $(\mathbb{R}^n, ||\cdot||_K)$ embeds isometrically in L_p if and only if $K \in \mathcal{I}^n_{-p}$.*

4. If $\lambda = k \in \{1, 2, \ldots, n-1\}$, then $\mathcal{I}^n_\lambda = \mathcal{I}^n_k$ coincides with the class of k-intersection bodies; see Definition 1.7 and Theorem 1.8. Theorems 5.1 and 5.2 provide new characterizations of this class.

These comments inspire the following

Definition 5.4. *Let $\lambda < n,\ \lambda \neq 0$. A body $K \in \mathcal{K}^n$ is said to be a λ-intersection body if $K \in \mathcal{I}^n_\lambda$, or, in other words, if there is a measure $\mu \in \mathcal{M}_{e+}(S^{n-1})$ such that $s_\lambda \rho^\lambda_K = M^{1-\lambda}\mu$ if $\lambda \neq -2\ell,\ \ell \in \mathbb{N}$, and $\rho^{-2\ell}_K = \tilde{M}^{1+2\ell}\mu$, otherwise.*

The result of Theorem 5.2 for $\lambda = i = k$ can serve as an alternative definition of k-intersection bodies in terms of Radon transforms. This definition agrees with Definition 1.6 and mimics Definition 1.2.

Definition 5.5. *Let $k \in \{1, 2, \ldots, n-1\}$. A body $K \in \mathcal{K}^n$ is a k-intersection body if there is a non-negative measure μ on S^{n-1} such that*

$$(R_k \rho^k_K)(\xi) = (R_{n-k}\mu)(\xi^\perp), \qquad \xi \in G_{n,k}. \tag{5.2}$$

Equality (5.2) is understood in the weak sense according (2.5). Namely, for $\varphi \in C(G_{n,k})$ and $\varphi^\perp(\eta) = \varphi(\eta^\perp)$, $\eta \in G_{n,n-k}$, (5.2) means

$$\int_{G_{n,k}} (R_k \rho_K^k)(\xi)\varphi(\xi)\, d\xi = \int_{S^{n-1}} (R^*_{n-k}\varphi^\perp)(\theta)\, d\mu(\theta). \tag{5.3}$$

5.2. **λ-intersection bodies of star bodies and closure in the radial metric.** As we mentioned in Introduction, the class of intersection bodies, which coincides with $\mathcal{I}^n_\lambda$ when $\lambda = 1$, is the closure in the radial metric of the class of intersection bodies of star bodies. Below we extend this result to all $\lambda < n$, $\lambda \neq 0$, in the framework of the unique approach. We remind (see Definition 1.6) that $K \in \mathcal{K}^n$ is a k-intersection body of a body $L \in \mathcal{K}^n$ and write $K = \mathcal{IB}_k(L)$ if

$$\mathrm{vol}_k(K \cap \xi) = \mathrm{vol}_{n-k}(L \cap \xi^\perp) \qquad \forall \xi \in G_{n,k}. \tag{5.4}$$

Let $\mathcal{IB}_{k,n}$ be the set of all bodies $K \in \mathcal{K}^n$ satisfying (5.4) for some $L \in \mathcal{K}^n$.

How can we extend the purely geometric property (5.4) to non-integer values of k? To this end, we first express (5.4) in terms of the generalized cosine transforms (2.9).

Lemma 5.6. *If $K = \mathcal{IB}_k(L)$ is infinitely smooth, then*

$$\rho_L^{n-k} = c\, M^{1-n+k} \rho_K^k, \qquad \rho_K^k = c^{-1} M^{1-k} \rho_L^{n-k}, \tag{5.5}$$

$$c = \pi^{k-n/2}(n-k)/k.$$

Proof. We make use of (3.13), where we set $i = k$, $\alpha = 1 - n + k$ and $f = \rho_K^k$. By (3.1), this gives

$$R_k \rho_K^k = \tilde{c} R_{n-k,\perp} M^{1-n+k} \rho_K^k, \qquad \tilde{c} = \frac{\pi^{k-n/2}\, \sigma_{n-k-1}}{\sigma_{k-1}}. \tag{5.6}$$

On the other hand, if $K = \mathcal{IB}_k(L)$ is infinitely smooth, then, according to (5.4) and the equality

$$\mathrm{vol}_k(K \cap \xi) = \frac{\sigma_{k-1}}{k}\, (R_k \rho_K^k)(\xi), \tag{5.7}$$

we have

$$R_k \rho_K^k = \frac{k\, \sigma_{n-k-1}}{(n-k)\, \sigma_{k-1}}\, R_{n-k,\perp} \rho_L^{n-k}. \tag{5.8}$$

Comparing (5.6) and (5.8), owing to injectivity of the Radon transform, we obtain the first equality in (5.5). The second equality follows from the first one by (3.5). $\square$

Equalities (5.5) are extendable to non-integer values of k. We denote

$$c_{\lambda,n} = \pi^{\lambda-n/2}(n-\lambda)/\lambda,$$

and let s_λ be defined by (5.1).

Definition 5.7. *Let $\lambda < n$, $\lambda \neq 0$; $K, L \in \mathcal{K}^n$. We say that K is a λ-intersection body of L and write $K = \mathcal{IB}_\lambda(L)$ if $s_\lambda \rho_K^\lambda = c_{\lambda,n}^{-1} M^{1-\lambda} \rho_L^{n-\lambda}$ in the case $\lambda \neq -2\ell$, $\ell \in \mathbb{N}$, and $\rho_K^{-2\ell} = \tilde{M}^{1+2\ell}\rho_L^{n+2\ell}$, otherwise. The set of all λ-intersection bodies of star bodies will be denoted by $\mathcal{IB}_{\lambda,n}$. We also denote*

$$\mathcal{IB}^\infty_{\lambda,n} = \{K \in \mathcal{IB}_{\lambda,n} : \rho_K \in \mathcal{D}_e(S^{n-1})\}. \tag{5.9}$$

By (3.5), equality $s_\lambda \rho_K^\lambda = c_{\lambda,n}^{-1} M^{1-\lambda}\rho_L^{n-\lambda}$ is equivalent to $\rho_L^{n-\lambda} = s_\lambda\, c_{\lambda,n}\, M^{1-n+\lambda}\rho_K^\lambda$. Both equalities are generally understood in the sense of distributions, for instance,

$$s_\lambda(\rho_K^\lambda, \varphi) = c_{\lambda,n}^{-1}(\rho_L^{n-\lambda}, M^{1-\lambda}\varphi), \qquad \varphi \in \mathcal{D}(S^{n-1}).$$

If K (or L) is smooth, then $s_\lambda \rho_K^\lambda(\theta) = c_{\lambda,n}^{-1}(M^{1-\lambda}\rho_L^{n-\lambda})(\theta)$ pointwise for every $\theta \in S^{n-1}$.

Theorem 5.8. *Let $\lambda < n$, $\lambda \neq 0$. If $\lambda \neq -2\ell$, $\ell \in \mathbb{N}$, then the class $\mathcal{I}^n_\lambda$ of λ-intersection bodies is the closure of the classes $\mathcal{IB}_{\lambda,n}$ and $\mathcal{IB}^\infty_{\lambda,n}$ of λ-intersection bodies of star bodies in the radial metric:*

$$\mathcal{I}^n_\lambda = \mathrm{cl}\,\mathcal{IB}_{\lambda,n} = \mathrm{cl}\,\mathcal{IB}^\infty_{\lambda,n}. \tag{5.10}$$

If $\lambda = -2\ell$, $\ell \in \mathbb{N}$, then $\mathcal{I}^n_\lambda \subset \mathrm{cl}\,\mathcal{IB}_{\lambda,n} = \mathrm{cl}\,\mathcal{IB}^\infty_{\lambda,n}$.

Proof. STEP 1. We first prove that $\mathcal{I}^n_\lambda \subset \mathrm{cl}\,\mathcal{IB}^\infty_{\lambda,n}$. Let $K \in \mathcal{I}^n_\lambda$, i.e.,

(a) $s_\lambda \rho_K^\lambda = M^{1-\lambda}\mu$, $\mu \in \mathcal{M}_{e+}(S^{n-1})$, if $\lambda \neq -2\ell$, $\ell \in \mathbb{N}$, and

(b) $\rho_K^{-2\ell} = \tilde{M}^{1+2\ell}\mu$, otherwise.

Our aim is to define a sequence $K_j \in \mathcal{IB}^\infty_{\lambda,n}$ such that $\rho_{K_j} \to \rho_K$ in the C-norm. Consider the Poisson integral $\Pi_t \rho_K^\lambda$ (see (2.1)), that converges to ρ_K^λ in the C-norm when $t \to 1$. In the case (a), for any test function $\omega \in \mathcal{D}(S^{n-1})$ we have

$$(\Pi_t \rho_K^\lambda, \omega) = (\rho_K^\lambda, \Pi_t \omega) = s_\lambda^{-1}(\mu, M^{1-\lambda}\Pi_t\omega) = s_\lambda^{-1}(M^{1-\lambda}\Pi_t\mu, \omega).$$

Similarly, in the case (b), we have a pointwise equality $(\Pi_t\rho_K^{-2\ell})(\theta) = (\tilde{M}^{1+2\ell}\Pi_t\mu)(\theta)$, $\theta \in S^{n-1}$. Choose K_j so that $\rho_{K_j}^\lambda = \Pi_{t_j}\rho_K^\lambda$, where t_j is a sequence in $(0,1)$ approaching 1. Clearly, K_j converges to K in the radial metric. Moreover, $K_j \in \mathcal{IB}^\infty_{\lambda,n}$, because $\rho_{K_j}^\lambda = s_\lambda^{-1} c_{\lambda,n}^{-1} M^{1-\lambda}\rho_{L_j}^{n-\lambda}$ and $\rho_{K_j}^{-2\ell} = \tilde{M}^{1+2\ell}\rho_{L_j}^{n+2\ell}$, where the bodies L_j are defined by $\rho_{L_j}^{n-\lambda} = c_{\lambda,n}\Pi_{t_j}\mu$ in the case (a), and $\rho_{L_j}^{n+2\ell} = \Pi_{t_j}\mu$ in the case (b), respectively.

Conversely, let $K \in \mathrm{cl}\,\mathcal{IB}^\infty_{\lambda,n}$, $\lambda \neq -2,-4,\dots$. It means that there is a sequence of $K_j \in \mathcal{IB}^\infty_{\lambda,n}$ such that $\lim\limits_{j\to\infty} ||\rho_K - \rho_{K_j}||_{C(S^{n-1})} = 0$ and $s_\lambda \rho^\lambda_{K_j} = c^{-1}_{\lambda,n} M^{1-\lambda}\rho^{n-\lambda}_{L_j}$, $\rho_{L_j} \in \mathcal{D}_{e+}(S^{n-1})$. If $j \to \infty$, then for every $\omega \in \mathcal{D}(S^{n-1})$,

$$s_\lambda(\rho^\lambda_{K_j}, M^{1-n+\lambda}\omega) \to s_\lambda(\rho^\lambda_K, M^{1-n+\lambda}\omega) = s_\lambda(M^{1-n+\lambda}\rho^\lambda_K, \omega). \tag{5.11}$$

The right-hand side of (5.11) is non-negative, because by (3.5), for every j and every $\omega \in \mathcal{D}_{e+}(S^{n-1})$,

$$s_\lambda(\rho^\lambda_{K_j}, M^{1-n+\lambda}\omega) = c^{-1}_{\lambda,n}(M^{1-\lambda}\rho^{n-\lambda}_{L_j}, M^{1-n+\lambda}\omega) = c^{-1}_{\lambda,n}(\rho^{n-\lambda}_{L_j}, \omega) \geq 0.$$

By Theorem 9.1, it follows that $s_\lambda\, M^{1-n+\lambda}\rho^\lambda_K$ is a non-negative measure. We denote it by μ. By (3.5), for any $\omega \in \mathcal{D}(S^{n-1})$,

$$s_\lambda(\rho^\lambda_K, \omega) = s_\lambda(M^{1-n+\lambda}\rho^\lambda_K, M^{1-\lambda}\omega) = (\mu, M^{1-\lambda}\omega) = (M^{1-\lambda}\mu, \omega),$$

i.e., $K \in \mathcal{I}^n_\lambda$. This gives $\mathcal{IB}^\infty_{\lambda,n} \subset \mathcal{I}^n_\lambda$ and, by above, $\mathcal{I}^n_\lambda = \mathrm{cl}\,\mathcal{IB}^\infty_{\lambda,n}$.

STEP 2. It remains to prove that $\mathrm{cl}\,\mathcal{IB}^\infty_{\lambda,n} = \mathrm{cl}\,\mathcal{IB}_{\lambda,n}$. Since $\mathcal{IB}^\infty_{\lambda,n} \subset \mathcal{IB}_{\lambda,n}$, then $\mathrm{cl}\,\mathcal{IB}^\infty_{\lambda,n} \subset \mathrm{cl}\,\mathcal{IB}_{\lambda,n}$. To prove the opposite inclusion, let $K \in \mathrm{cl}\,\mathcal{IB}_{\lambda,n}$ and consider the case $\lambda \neq -2,-4,\dots$. We have to show that there is a sequence of smooth bodies K_j, which converges to K in the radial metric and such that $s_\lambda\rho^\lambda_{K_j} = c^{-1}_{\lambda,n}M^{1-\lambda}\rho^{n-\lambda}_{L_j}$ for some bodies $L_j \in \mathcal{K}^n$. Since $K \in \mathrm{cl}\,\mathcal{IB}_{\lambda,n}$, there is a sequence $\tilde{K}_j \in \mathcal{K}^n$ such that $\lim\limits_{j\to\infty} ||\rho_{\tilde{K}_j} - \rho_K||_{C(S^{n-1})} = 0$ and $s_\lambda\rho^\lambda_{\tilde{K}_j} = c^{-1}_{\lambda,n}M^{1-\lambda}\rho^{n-\lambda}_{\tilde{L}_j}$ for some bodies $\tilde{L}_j \in \mathcal{K}^n$. We define smooth bodies K_j and L_j by

$$\rho^\lambda_{K_j} = \Pi_{1-1/j}\rho^\lambda_{\tilde{K}_j}, \qquad \rho^{n-\lambda}_{L_j} = \Pi_{1-1/j}\rho^{n-\lambda}_{\tilde{L}_j},$$

where $\Pi_{1-1/j}$ stands for the Poisson integral with parameter $1-1/j$. Since operators $\Pi_{1-1/j}$ and $M^{1-\lambda}$ commute, then $s_\lambda\rho^\lambda_{K_j} = c^{-1}_{\lambda,n}M^{1-\lambda}\rho^{n-\lambda}_{L_j}$, and therefore, $K_j \in \mathcal{IB}^\infty_{\lambda,n}$. On the other hand, by the properties of the Poisson integral [SW],

$$|\rho^\lambda_{K_j} - \rho^\lambda_K| \leq |\Pi_{1-1/j}\rho^\lambda_{\tilde{K}_j} - \Pi_{1-1/j}\rho^\lambda_K| + |\Pi_{1-1/j}\rho^\lambda_K - \rho^\lambda_K| \to 0$$

as $j \to \infty$. It means, that $K \in \mathrm{cl}\,\mathcal{IB}^\infty_{\lambda,n}$ or $\mathrm{cl}\,\mathcal{IB}_{\lambda,n} \subset \mathrm{cl}\,\mathcal{IB}^\infty_{\lambda,n}$. Hence, by above, $\mathrm{cl}\,\mathcal{IB}_{\lambda,n} = \mathrm{cl}\,\mathcal{IB}^\infty_{\lambda,n}$. For $\lambda = -2,-4,\dots$, the argument is similar. $\square$

Remark 5.9. If $\lambda = -2,-4,\dots$, we cannot prove the coincidence of $\mathcal{I}^n_\lambda$ and $\mathrm{cl}\,\mathcal{IB}^\infty_{\lambda,n}$, because the proof of the embedding $\mathrm{cl}\,\mathcal{IB}^\infty_{\lambda,n} \subset \mathcal{I}^n_\lambda$ relies heavily on the fact that $M^{1-\lambda}$ is an isomorphism of $\mathcal{D}_e(S^{n-1})$. If $\lambda = -2,-4,\dots$, this is not so, and the operator $\tilde{M}^{1-\lambda}$ has a nontrivial kernel, which consists of spherical harmonics of degree $> 2\ell$; see [R1] for details.

It is interesting to translate Theorem 5.8 for $\lambda = -p,\ p > 0$, into the language of isometric embeddings. Ignoring a non-important positive constant factor and using polar coordinates, one can replace the equalities $s_\lambda \rho_K^\lambda = c_{\lambda,n}^{-1} M^{1-\lambda} \rho_L^{n-\lambda}$ and $\rho_K^{-2\ell} = \tilde{M}^{1+2\ell} \rho_L^{n+2\ell}$ in Definition 5.7 by

$$||u||_K^p = \int_L |x \cdot u|^p \, dx, \qquad u \in S^{n-1}. \tag{5.12}$$

Corollary 5.10.
(i) *A unit ball of every n-dimensional subspace of L_p, can be approximated in the radial metric by bodies K, defined by (5.12), where $L \in \mathcal{K}^n$ has a C^∞ boundary.*
(ii) *If, moreover, $p \neq 2, 4, \dots,$ then the set of unit balls of all n-dimensional subspaces of L_p, can be identified with the closure in the radial metric of the set of bodies K satisfying (5.12) for some smooth body $L \in \mathcal{K}^n$ (one can also consider arbitrary bodies $L \in \mathcal{K}^n$).*

5.3. **Central sections of λ-intersection bodies.** It is known, that a cross-section $K \cap \eta$ of a body $K \in I_k^n$ by any m-dimensional central plane η is a k-intersection body in η provided $1 \le k < m < n$. This fact was established in [Mi1, Proposition 3.17] by using Theorem 1.8 and a certain approximation procedure. Below we present more general results, including sections of k-intersection bodies of star bodies and the case of non-integer $k = \lambda$. These results are consequences of the restriction theorems from Section 3.3.

Theorem 5.11. *Let $1 \le k < m < n$, $\eta \in G_{n,m}$. If $K = \mathcal{IB}_k(L)$ in $\mathbb{R}^n$, then $K \cap \eta = \mathcal{IB}_k(\tilde{L})$ in η, where the body $\tilde{L}$ is defined by*

$$\rho_{\tilde{L}}^{m-k}(u) = c_{k,m,n} \int\limits_{S^{n-1} \cap (\eta^\perp \oplus \mathbb{R}u)} \rho_L^{n-k}(w) |u \cdot w|^{m-k-1} \, dw, \tag{5.13}$$

$$u \in S^{n-1} \cap \eta, \qquad c_{k,m,n} = \frac{(m-k)\,\sigma_{n-m}}{2(n-k)}.$$

Proof. By (5.7) and (3.21) (with $f = \rho_L^{n-k}$),

$$\begin{aligned} \mathrm{vol}_k(K \cap \xi) &= \mathrm{vol}_{n-k}(L \cap \xi^\perp) = \frac{\sigma_{n-k-1}}{n-k} (R_{n-k} \rho_L^{n-k})(\xi^\perp) \\ &= \frac{c\,\sigma_{n-k-1}}{n-k} (R_{m-k} T_\eta^k \rho_L^{n-k})(\xi^\perp \cap \eta) \qquad (5.14) \\ &= \frac{\sigma_{m-k-1}}{m-k} \, (R_{m-k} \rho_{\tilde{L}}^{m-k})(\xi^\perp \cap \eta) = \mathrm{vol}_{m-k}(\tilde{L} \cap \xi^\perp), \end{aligned}$$

as desired. □

Theorem 5.11 has the following generalization.

Theorem 5.12. *Let $1 < m < n$, $\eta \in G_{n,m}$ and suppose that $\lambda < m$, $\lambda \neq 0$. If $K = \mathcal{IB}_\lambda(L)$ in $\mathbb{R}^n$, then $K \cap \eta = \mathcal{IB}_\lambda(\tilde{L})$ in η, where the body $\tilde{L}$ is defined by*

$$\rho_{\tilde{L}}^{m-\lambda}(u) = \tilde{c} \int\limits_{S^{n-1}\cap(\eta^\perp \oplus \mathbb{R}u)} \rho_L^{n-\lambda}(w)|u \cdot w|^{m-\lambda-1}\, dw, \tag{5.15}$$

$$u \in S^{n-1} \cap \eta, \qquad \tilde{c} = \begin{cases} \dfrac{(m-\lambda)\,\sigma_{n-m}}{2(n-\lambda)} & \text{if } \lambda \neq -2\ell,\ \ell \in \mathbb{N}, \\[2ex] \pi^{(m-n)/2}\,\sigma_{n-m}/2 & \text{otherwise.} \end{cases}$$

Moreover, if $K \in \mathcal{I}_\lambda^n$ in $\mathbb{R}^n$, then $K \cap \eta \in \mathcal{I}_\lambda^m$ in η.

Proof. Let $\lambda \neq -2\ell$, $\ell \in \mathbb{N}$, and let $\theta \in S^{n-1} \cap \eta$. By Definition 5.7, $s_\lambda \rho_K^\lambda = c_{\lambda,n}^{-1} M^{1-\lambda} \rho_L^{n-\lambda}$, and Theorem 3.12 (with $f = s_\lambda \rho_K^\lambda$ and $g = c_{\lambda,n}^{-1}\rho_L^{n-\lambda}$) yields

$$s_\lambda \rho_K^\lambda(\theta) = (M^{1-\lambda}_{S^{n-1}\cap\eta} T_\eta^\lambda [c_{\lambda,n}^{-1}\rho_L^{n-\lambda}])(\theta) = c_{\lambda,m}^{-1}(M^{1-\lambda}_{S^{n-1}\cap\eta}\rho_{\tilde{L}}^{m-\lambda})(\theta),$$

where $\rho_{\tilde{L}}^{m-\lambda} = c\, T_\eta^\lambda \rho_L^{n-\lambda}$, $c = \pi^{(n-m)/2}(m-\lambda)/(n-\lambda)$. By Definition 5.7 and (3.20), we are done. If $\lambda = -2\ell$, $\ell \in \mathbb{N}$, then, as above,

$$\rho_K^{-2\ell}(\theta) = (\tilde{M}^{1+2\ell}_{S^{n-1}\cap\eta} T_\eta^{-2\ell} \rho_L^{n+2\ell})(\theta) = (M^{1-\lambda}_{S^{n-1}\cap\eta}\rho_{\tilde{L}}^{m+2\ell})(\theta)$$

where $\rho_{\tilde{L}}^{m+2\ell} = T_\eta^{-2\ell}\rho_L^{n+2\ell}$. This gives (5.15).

Furthermore, if $K \in \mathcal{I}_\lambda^n$, $\lambda \neq -2\ell$, $\ell \in \mathbb{N}$, then, by Definition 5.4, $s_\lambda \rho_K^\lambda = M^{1-\lambda}\mu$, $\mu \in \mathcal{M}_{e+}(S^{n-1})$. Hence, by Theorem 3.12, there is a measure $\nu \in \mathcal{M}_{e+}(S^{n-1}\cap\eta)$ such that the restriction of $s_\lambda\rho_K^\lambda$ onto $S^{n-1}\cap\eta$ is represented as $s_\lambda\rho_K^\lambda = M^{1-\lambda}_{S^{n-1}\cap\eta}\nu$. It means that $K \cap \eta \in \mathcal{I}_\lambda^m$ in η. In the case $\lambda = -2\ell$, $\ell \in \mathbb{N}$, the argument is similar. □

6. Examples of λ-intersection bodies

The definition of the classes $\mathcal{I}_\lambda^n$ and $\mathcal{IB}_{\lambda,n}$ and all known characterizations are purely analytic. Unlike the case $\lambda = 1$, when an intersection body of a star body is explicitly defined by a simple geometric procedure, it is not clear how can we construct $\lambda-$intersection bodies in the general case. Below we give some examples, when the radial function of a $\lambda-$intersection body can be explicitly determined. These examples utilize the generalized cosine transforms.

Example 6.1. Let $\lambda < 1$, $\lambda \neq 0$. This case is the simplest. Indeed, given a non-negative measure μ on S^{n-1}, the relevant $\lambda-$intersection body can be directly constructed by the formula $\rho_K^\lambda = M^{1-\lambda}\mu$, if $\lambda \neq -2\ell, \ell \in \mathbb{N}$, and $\rho_K^{-2\ell} = \tilde{M}^{1+2\ell}\mu$, otherwise. In other words (cf. (2.11)),

$$\rho_K^\lambda(u) = \int_{S^{n-1}} |\theta \cdot u|^{-\lambda}\, d\mu(\theta). \tag{6.1}$$

This fact (with λ replaced by $-p$) is a reformulation of Theorem 6.17 from [K4], which was stated in the language of isometric embeddings and relies on the P. Lévy characterization; see also Lemma 6.4 and Theorem 4.11 in [K4].

Example 6.2. If $n-3 \le \lambda < n,\ \lambda > 0$, then $\mathcal{I}_\lambda^n$ includes all origin-symmetric convex bodies in $\mathbb{R}^n$.

This fact is due to Koldobsky [K4, Corollary 4.9]. It can be proved using a slight modification of the argument from [R3, Sec. 7] as follows. By Theorem 5.1 (c), it suffices to check that for any o.s. convex body K, $M^{1+\lambda-n}\rho_K^\lambda \in \mathcal{M}_{e+}(S^{n-1})$. For $\lambda \ge n-1$, this is obvious. To handle the case $n-3 \le \lambda < n-1$, suppose first that K is infinitely smooth. Using polar coordinates, for $Re\,\alpha > 0$, we can write

$$(M^\alpha \rho_K^{\alpha+n-1})(u) = (\alpha+n-1)\,\gamma_n(\alpha) \int_K |x\cdot u|^{\alpha-1}\, dx. \tag{6.2}$$

Then $M^{1+\lambda-n}\rho_K^\lambda$ can be realized as analytic continuation (*a.c.*) at $\alpha = 1+\lambda-n$ of the right-hand side of (6.2). The latter can be written as

$$I(\alpha) = 2(\alpha+n-1)\gamma_n(\alpha) \int_0^\infty t^{\alpha-1} A_{K,u}(t)\, dt,$$

$A_{K,u}(t) = \mathrm{vol}_{n-1}(K \cap \{tu + u^\perp\})$. Taking analytic continuation (see [GS, Chapter 1]), for $-2 < \alpha < 0$ (which is equivalent to $n-3 \le \lambda < n-1$) we get

$$a.c. I(\alpha) = c_1 \int_0^\infty t^{\alpha-1}[A_{K,u}(t) - A_{K,u}(0)]\, dt.$$

Similarly, $a.c.I(\alpha)$ at $\alpha = -2$ (which corresponds to $\lambda = n-3$) is $c_2 A''_{K,u}(0)$. Following [GS], one can easily check that constants c_1 and c_2 are negative. Since K is convex, both analytic continuations are positive, and therefore $M^{1+\lambda-n}\rho_K^\lambda > 0$. If K is an arbitrary o.s. convex body, we approximate it in the radial metric by smooth o.s. convex bodies K_j. Then for any test function $\omega \in \mathcal{D}_+(S^{n-1})$, by the previous step we have

$$\begin{aligned}(M^{1+\lambda-n}\rho_K^\lambda, \omega) &= (\rho_K^\lambda, M^{1+\lambda-n}\omega) = \lim_{j\to\infty} (\rho_{K_j}^\lambda, M^{1+\lambda-n}\omega) \\ &= \lim_{j\to\infty} (M^{1+\lambda-n}\rho_{K_j}^\lambda, \omega) \ge 0.\end{aligned}$$

Hence, by Theorem 9.1, $M^{1+\lambda-n}\rho_K^\lambda$ is a non-negative measure and the proof is complete.

Example 6.3. If $\rho_K^\lambda = \overset{*}{R}{}^{i-\lambda}_{n-i}\nu$ for some $\nu \in \mathcal{M}_+(G_{n,n-i})$ and $\lambda \le i < n$, then $K \in \mathcal{I}_\lambda^n$.

Indeed, for any test function $\omega \in \mathcal{D}(S^{n-1})$, by (3.12) (with $\alpha = 1-\lambda$) we have

$$\begin{aligned}(\rho_K^\lambda, \omega) &= (\overset{*}{R}{}^{i-\lambda}_{n-i}\nu, \omega) = (\nu, R^{i-\lambda}_{n-i}\omega) = (\nu^\perp, R^{i-\lambda}_{n-i,\perp}\omega)\\ &= c^{-1}(\nu^\perp, R_i M^{1-\lambda}\omega) = c^{-1}(R_i^*\nu^\perp, M^{1-\lambda}\omega), \quad c = \frac{2\pi^{(i-1)/2}}{\sigma_{i-1}}.\end{aligned}$$

It means that for $0 < \lambda \le i < n$ and $\nu \in \mathcal{M}_+(G_{n,n-i})$,

$$\rho_K^\lambda = \overset{*}{R}{}^{i-\lambda}_{n-i}\nu \quad \Longleftrightarrow \quad \{\rho_K^\lambda = M^{1-\lambda}\mu, \quad \mu = c^{-1}R_i^*\nu^\perp\}. \tag{6.3}$$

By Definition 5.4, this gives the result. The particular case $\lambda = i$ implies the embedding into $\mathcal{I}_i^n$ of the Zhang's class $\mathcal{Z}_{n-i}^n$; see Definition 1.5. This embedding was proved in [K3] and [Mi1] in a different way; see also [Mi2], where it is proved that $\mathcal{Z}_{n-i}^n$ is a proper subset of $\mathcal{I}_i^n$ if $2 \le i \le n-2$.

Example 6.4. If $0 < (i-1)/2 < \lambda \le i < n$ and $\rho_K^\lambda = M^{i-\lambda}\mu$ for some $\mu \in \mathcal{M}_+(S^{n-1})$, then $K \in \mathcal{I}_\lambda^n$.

Indeed, by Lemma 3.4 (with $\alpha = i-\lambda,\ \beta = 1-\lambda$), $\rho_K^\lambda = M^{i-\lambda}\mu = M^{1-\lambda}A_{i-\lambda,1-\lambda}$, where $A_{i-\lambda,1-\lambda}$ is an integral operator which preserves positivity provided $i-\lambda > 1-\lambda > 1-n,\ (i-\lambda)+(1-\lambda) < 2$. This is just our case.

Example 6.5. One can construct bodies $K \in \mathcal{I}_\lambda^n$ from bodies $L \in \mathcal{I}_\delta^n$ by the formula $\rho_K = \rho_L^{\lambda/\delta}$ provided $0 < \delta < \lambda < n$.

Indeed, by Definition 5.4, there is a measure $\mu \in \mathcal{M}_+(S^{n-1})$ so that $\rho_L^\delta = M^{1-\delta}\mu$. Then, by Lemma 3.4 (with $\alpha = 1-\delta,\ \beta = 1-\lambda$), $\rho_K^\lambda = \rho_L^\delta = M^{1-\delta}\mu = M^{1-\lambda}A_{1-\delta,1-\lambda}\mu$, and we are done. This example generalizes the corresponding result from [Mi1, p. 533, Statement (c)], which was obtained in a different way for the case, when λ and δ are integers.

Example 6.6. Let

$$B_q^n = \{x \in \mathbb{R}^n : ||x||_q = \Big(\sum_{k=1}^n |x_k|^q\Big)^{1/q} \le 1\}. \tag{6.4}$$

If $0 < q \le 2$, then $B_q^n \in \mathcal{I}_\lambda^n$ for all $\lambda \in (0,n)$. If $2 < q < \infty$, $\lambda \in (0,n)$, then $B_q^n \in \mathcal{I}_\lambda^n$ if and only if $\lambda \ge n-3$.

Both statements are due to Koldobsky. The first one follows from the fact that for $0 < q \le 2$ the Fourier transform of $||x||_q^{-\lambda}$ is a positive $\mathcal{S}'$-distribution (see Lemmas 3.6 and 2.27 in [K4]). The second statement is a reformulation of Theorem 4.13 from [K4]. The "if" part is a consequence of Example 6.2.

7. (q, ℓ)-BALLS

In this section we consider one more example, which resembles Example 6.6, but does not fall into its scope and requires a separate consideration. Let

$$x = (x', x'') \in \mathbb{R}^n, \quad x' \in \mathbb{R}^{n-\ell} = \bigoplus_{j=1}^{n-\ell} \mathbb{R}e_j, \quad x'' \in \mathbb{R}^\ell = \bigoplus_{j=n-\ell+1}^{n} \mathbb{R}e_j,$$

$e_1, \ldots, \varepsilon_n$ being coordinate unit vectors. Consider the (q, ℓ)-ball

$$B^n_{q,\ell} = \{x : ||x||_{q,\ell} = (|x'|^q + |x''|^q)^{1/q} \le 1\}, \qquad q > 0. \tag{7.1}$$

We wonder, for which triples (q, ℓ, n), $B^n_{q,\ell}$ is a λ-intersection body. To study this problem, we need some preparation. Consider the Fourier integral

$$\gamma_{q,\ell}(\eta) = \int_{\mathbb{R}^\ell} e^{-|y|^q} e^{iy\cdot\eta}\, dy, \qquad \eta \in \mathbb{R}^\ell, \qquad q > 0. \tag{7.2}$$

The function $\gamma_{q,\ell}(\eta)$ is uniformly continuous on $\mathbb{R}^\ell$ and vanishes at infinity.

Lemma 7.1. *If $0 < q \le 2$, then $\gamma_{q,\ell}(\eta) > 0$ for all $\eta \in \mathbb{R}^\ell$.*

Proof. (Cf. [K4, p. 44, for $\ell = 1$]). For $\eta = 0$, the statement is obvious. It is known (see, e.g., [SW]), that

$$[e^{-t|\cdot|^2}]^\wedge(\eta) = \pi^{\ell/2} t^{-\ell/2} e^{-|\eta|^2/4t}, \qquad t > 0. \tag{7.3}$$

This gives the result for $q = 2$. Let $0 < q < 2$. By Bernstein's theorem [F, Chapter 18, Sec. 4], there is a non-negative finite measure μ_q on $[0, \infty)$ so that $e^{-z^{q/2}} = \int_0^\infty e^{-tz}\, d\mu_q(t)$, $z \in [0, \infty)$. Replace z by $|y|^2$ to get

$$e^{-|y|^q} = \int_0^\infty e^{-t|y|^2}\, d\mu_q(t). \tag{7.4}$$

Then (7.3) yields

$$\begin{aligned} \gamma_{q,\ell}(\eta) &= \int_{\mathbb{R}^\ell} e^{iy\cdot\eta} dy \int_0^\infty e^{-t|y|^2}\, d\mu_q(t) = \int_0^\infty d\mu_q(t) \int_{\mathbb{R}^\ell} e^{iy\cdot\eta} e^{-t|y|^2}\, dy \\ &= \pi^{\ell/2} \int_0^\infty t^{-\ell/2} e^{-|\eta|^2/4t}\, d\mu_q(t) > 0. \end{aligned}$$

The Fubini theorem is applicable here, because, by (7.4),

$$\int_{\mathbb{R}^\ell} |e^{iy\cdot\eta}|dy \int_0^\infty e^{-t|y|^2}\, d\mu_q = \int_{\mathbb{R}^\ell} e^{-|y|^q} dy < \infty.$$

□

Our next concern is the behavior of $\gamma_{q,\ell}(\eta)$ when $|\eta| \to \infty$. If q is even, then $e^{-|\cdot|^q}$ is a Schwartz function and therefore, $\gamma_{q,\ell}$ is infinitely smooth and rapidly decreasing. In the general case, we have the following.

Lemma 7.2. *For any $q > 0$,*

$$\lim_{|\eta|\to\infty} |\eta|^{\ell+q}\gamma_{q,\ell}(\eta) = 2^{\ell+q}\pi^{\ell/2-1}\Gamma(1+q/2)\Gamma((\ell+q)/2)\sin(\pi q/2). \tag{7.5}$$

Proof. For $\ell = 1$, this statement can be found in [PS, Chapter 3, Problem 154] and in [K4, p. 45]. In the general case, the proof is more sophisticated and relies on the properties of Bessel functions. By the well-known formula for the Fourier transform of a radial function (see, e.g., [SW]), we write $\gamma_{q,\ell}(\eta) = I(|\eta|)$, where

$$\begin{aligned} I(s) &= (2\pi)^{\ell/2} s^{1-\ell/2} \int_0^\infty e^{-r^q} r^{\ell/2} J_{\ell/2-1}(rs)\, dr \\ &= (2\pi)^{\ell/2} s^{-\ell} \int_0^\infty e^{-r^q} \frac{d}{dr}\left[(rs)^{\ell/2} J_{\ell/2}(rs)\right] dr. \end{aligned}$$

Integration by parts yields

$$I(s) = q(2\pi)^{\ell/2} s^{-\ell/2} \int_0^\infty e^{-r^q} r^{\ell/2+q-1} J_{\ell/2}(rs)\, dr.$$

Changing variable $z = s^q r^q$, we obtain

$$s^{\ell+q} I(s) = (2\pi)^{\ell/2} A(s^{1/q}), \qquad A(\delta) = \int_0^\infty e^{-z\delta} z^{\ell/2q} J_{\ell/2}(z^{1/q})\, dz.$$

We actually have to compute the limit $A_0 = \lim\limits_{\delta\to 0} A(\delta)$. To this end, we invoke Hankel functions $H_\nu^{(1)}(z)$, so that $J_\nu(z) = Re\, H_\nu^{(1)}(z)$ if z is real [Er]. Let $h_\nu(z) = z^\nu H_\nu^{(1)}(z)$. This is a single-valued analytic function in the z-plane with cut $(-\infty, 0]$. Using the properties of the Bessel functions [Er], we get

$$\lim_{z\to 0} h_\nu(z) = 2^\nu \Gamma(\nu)/\pi i, \tag{7.6}$$

$$h_\nu(z) \sim \sqrt{2/\pi}\, z^{\nu-1/2} e^{iz - \frac{\pi i}{2}(\nu+\frac{1}{2})}, \qquad z \to \infty. \tag{7.7}$$

Then we write $A(\delta)$ as $A(\delta) = Re \int_0^\infty e^{-z\delta} h_{\ell/2}(z^{1/q})\, dz$ and change the line of integration from $[0, \infty)$ to $\ell_\theta = \{z : z = re^{i\theta},\ r > 0\}$ for small $\theta < \pi q/2$. By Cauchy's theorem, owing to (7.6) and (7.7), we obtain $A(\delta) = Re \int_{\ell_\theta} e^{-z\delta} h_{\ell/2}(z^{1/q})\, dz$. Since for $z = re^{i\theta}$, $h_{\ell/2}(z^{1/q}) = O(1)$ when $r = |z| \to 0$ and $h_{\ell/2}(z^{1/q}) = O(r^{(\ell-1)/2q} e^{-r^{1/q}\sin(\theta/q)})$ as $r \to \infty$, by the Lebesgue theorem on dominated convergence, we get $A_0 = Re \int_{\ell_\theta} h_{\ell/2}(z^{1/q})\, dz$. To evaluate the last integral, we again use analyticity and replace ℓ_θ by $\ell_{\pi q/2} = \{z : z = re^{i\pi q/2},\ r > 0\}$ to get

$$A_0 = Re \Big[e^{i\pi q/2} \int_0^\infty h_{\ell/2}(r^{1/q} e^{i\pi/2})\, dr \Big].$$

To finalize calculations, we invoke McDonald's function $K_\nu(z)$ so that

$$h_\nu(z) = z^\nu H_\nu^{(1)}(z) = -\frac{2i}{\pi} (ze^{-i\pi/2})^\nu K_\nu(ze^{-i\pi/2}).$$

This gives

$$A_0 = \frac{2q}{\pi} \sin(\pi q/2) \int_0^\infty s^{\ell/2+q-1} K_{\ell/2}(s)\, ds.$$

The last integral can be explicitly evaluated by the formula 2.16.2 (2) from [PBM], and we obtain the result. □

Now we can proceed to studying (q,ℓ)-balls $B^n_{q,\ell}$; see (7.1). There is an intimate connection between geometric properties of the balls $B^n_{q,\ell}$ and the Fourier transform of the power function $||\cdot||^p_{q,\ell}$. The case $q = 2$ is well-known and associated with Riesz potentials; see, e.g., [St]. The relevant case of ℓ^n_q-balls, which agrees with $\ell = 1$ was considered in Example 6.6.

Lemma 7.3. *Let $q > 0$, $\xi = (\xi', \xi'') \in \mathbb{R}^n$, $\gamma_{q,\ell}(\xi'')$ and $\gamma_{q,n-\ell}(\xi')$ be the functions of the form (7.2). We define*

$$h_{p,q,\ell}(\xi) = \frac{q}{\Gamma(-p/q)} \int_0^\infty t^{n+p-1}\, \gamma_{q,n-\ell}(\xi' t)\, \gamma_{q,\ell}(\xi'' t)\, dt. \tag{7.8}$$

(i) *Let $\xi' \neq 0$ and $\xi'' \neq 0$. If q is even, then the integral (7.8) is absolutely convergent for all $p > -n$. Otherwise, it is absolutely convergent when $-n < p < 2q$. In these cases, $h_{p,q,\ell}(\xi)$ is a locally integrable function away from the coordinate subspaces $\mathbb{R}^\ell$ and $\mathbb{R}^{n-\ell}$.*

(ii) *If $-n < p < 0$, then $h_{p,q,\ell}(\xi) \in L^1_{loc}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$ and $(||\cdot||^p_{q,\ell})^\wedge(\xi) = h_{p,q,\ell}(\xi)$ in the sense of $\mathcal{S}'$-distributions. Specifically, for $\varphi \in \mathcal{S}(\mathbb{R}^n)$,*

$$\langle h_{p,q,\ell}\, , \hat{\varphi} \rangle = (2\pi)^n \langle ||\cdot||^p_{q,\ell}\, , \varphi \rangle. \tag{7.9}$$

Proof. (i) For any $0 < \varepsilon < a < \infty$,

$$\int_{\varepsilon<|\xi'|<a} d\xi' \int\limits_{\varepsilon<|\xi''|<a} |h_{p,q,\ell}(\xi;,\xi'')|\, d\xi''$$

$$\leq \frac{q}{|\Gamma(-p/q)|} \int\limits_0^\infty t^{n+p-1}\, dt \int\limits_{\varepsilon<|\xi'|<a} |\gamma_{q,n-\ell}(\xi' t)|\, d\xi' \int\limits_{\varepsilon<|\xi''|<a} |\gamma_{q,\ell}(\xi'' t)|\, d\xi''$$

$$= \frac{q}{|\Gamma(-p/q)|} \int\limits_0^\infty t^{p-1}\, dt \int\limits_{t\varepsilon<|z'|<ta} |\gamma_{q,n-\ell}(z')|\, dz' \int\limits_{t\varepsilon<|z''|<ta} |\gamma_{q,\ell}(z'')|\, dz''$$

$$= \frac{q}{|\Gamma(-p/q)|} \Big(\int\limits_0^1 + \int\limits_1^\infty \Big)(...) = \frac{q}{|\Gamma(-p/q)|}(I_1 + I_2).$$

The first integral is dominated by

$$c\, a^n \int\limits_0^1 t^{n+p-1}\, dt, \quad c = \sigma_{n-\ell-1}\sigma_{\ell-1} \max_{z'} |\gamma_{q,n-\ell}(z')| \max_{z''} |\gamma_{q,\ell}(z'')|$$

and is finite for $p > -n$. The second integral can be estimated by making use of Lemma 7.2. Specifically, if q is not an even integer, then

$$I_2 \leq c_\varepsilon \int\limits_1^\infty t^{p-1}\, dt \int\limits_{|z'|>t\varepsilon} \frac{dz'}{|z'|^{n-\ell+q}} \int\limits_{|z''|>t\varepsilon} \frac{dz''}{|z''|^{\ell+q}} \leq c_\varepsilon \int\limits_1^\infty t^{p-2q-1}\, dt.$$

If q is even, then $\gamma_{q,\ell}$ and $\gamma_{q,n-\ell}$ are rapidly decreasing and $I_2 \leq c_{\varepsilon,a} \int_1^\infty t^{p-2m-1}\, dt$ for any $m > 0$. This gives what we need.

(ii) If $-n < p < 0$, the same argument is applicable with $\varepsilon = 0$. In this case, I_2 does not exceed $||\gamma_{q,n-\ell}||_1 ||\gamma_{q,\ell}||_1 \int_1^\infty t^{p-1}\, dt$. The latter is finite when $p < 0$, because, by Lemma 7.2, $\gamma_{q,n-\ell}$ and $\gamma_{q,\ell}$ are integrable functions on respective spaces. When $\xi \to \infty$, one can readily check that $h_{p,q,\ell}(\xi) = O(|\xi|^m)$ for some $m > 0$, and therefore, $h_{p,q,\ell} \in \mathcal{S}'(\mathbb{R}^n)$.

To compute the Fourier transform $(|| \cdot ||_{q,\ell}^p)^\wedge(\xi)$, we replace $||x||_{q,\ell}^p$ by the formula

$$||x||_{q,\ell}^p = \frac{q}{\Gamma(-p/q)} \int\limits_0^\infty t^{p-1}\, e^{-|x'/t|^q - |x''/t|^q}\, dt, \qquad p < 0,$$

and note that the Fourier transform of the function $x \to e^{-|x'/t|^q - |x''/t|^q}$ is just $\gamma_{q,n-\ell}(\xi' t)\, \gamma_{q,\ell}(\xi'' t)$. Then

$$\begin{aligned}
\langle (||\cdot||^p_{q,\ell})^\wedge, \hat{\varphi}\rangle &= (2\pi)^n \langle ||\cdot||^p_{q,\ell}, \varphi\rangle \\
&= \frac{(2\pi)^n q}{\Gamma(-p/q)} \int\limits_0^\infty t^{p-1}\, dt \int\limits_{\mathbb{R}^n} e^{-|x'/t|^q - |x''/t|^q} \overline{\varphi(x)}\, dx \\
&= \frac{q}{\Gamma(-p/q)} \int\limits_0^\infty t^{n+p-1}\, dt \int\limits_{\mathbb{R}^n} \gamma_{q,n-\ell}(\xi' t)\, \gamma_{q,\ell}(\xi'' t)\, \overline{\hat{\varphi}(\xi)}\, d\xi.
\end{aligned}$$

Interchange of the order of integration in this argument can be easily justified using absolute convergence of integrals under consideration. □

Theorem 7.4. *If $0 < q \le 2$, $0 < \ell < n$, then $B^n_{q,\ell}$ is a λ-intersection body for any $0 < \lambda < n$.*

Proof. Owing to Lemma 7.1, the function (7.8) (with p replaced by $-\lambda$) is positive, and therefore, by Lemma 7.3, $||\cdot||^{-\lambda}_{q,\ell}$ represents a positive definite distribution. Now the result follows by Theorem 5.1. □

Consider the case $q > 2$. In this case $B^n_{q,\ell}$ is convex, and, owing to Example 6.2, $B^n_{q,\ell} \in \mathcal{I}^n_\lambda$ for all $n-3 \le \lambda < n$. What about $\lambda < n-3$? This case is especially intriguing.

Proposition 7.5. *If $q > 2$ and $0 < \lambda < \max(n-\ell, \ell) - 2$, then $||\cdot||^{-\lambda}_{q,\ell}$ is not a positive definite distribution and therefore, $B^n_{q,\ell} \notin \mathcal{I}^n_\lambda$.*

Proof. Let $0 < \lambda < n - \ell - 2$ and suppose the contrary, that $B^n_{q,\ell} \in \mathcal{I}^n_\lambda$. Consider the section of $B^n_{q,\ell}$ by the $(n-\ell+1)$-dimensional plane $\eta = \mathbb{R}e_n \oplus \mathbb{R}^{n-\ell}$. By Theorem 5.12, $B^n_{q,\ell} \cap \eta \in \mathcal{I}^{n-\ell+1}_\lambda$ in η, and therefore

$$||x_n e_n + x''||^\lambda_{q,\ell} = (|x_n|^q + |x''|^q)^{-\lambda/q}$$

is a positive definite distribution in η. By the second derivative text (see [K4, Theorem 4.19]) this is impossible if $0 < \lambda < n - \ell - 2$. A similar contradiction can be obtained if we assume $0 < \lambda < \ell - 2$ and consider the section of $B^n_{q,\ell}$ by the $(\ell+1)$-dimensional plane $\mathbb{R}e_1 \oplus \mathbb{R}^\ell$. □

Proposition 7.5 can be proved without using the second derivative text and Theorem 5.12 on sections of λ-intersection bodies; see [R4]. The bounds for λ appear to be the same.

Open problem. *Let $q > 2$, $\ell > 1$. Is $B^n_{q,\ell}$ a λ-intersection body if* $\max(n-\ell, \ell) - 2 < \lambda < n - 3$?

This problem does not occur in the case $\ell = 1$ as in Example 6.6.

8. The generalized cosine transforms and comparison of volumes

For $1 < i < n$, let $\mathrm{vol}_i(\cdot)$ denote the i-dimensional volume function. Suppose that i is fixed, and let A and B be o.s. convex bodies in $\mathbb{R}^n$ satisfying

$$\mathrm{vol}_i(A \cap \xi) \le \mathrm{vol}_i(B \cap \xi) \quad \forall \xi \in G_{n,i}. \tag{8.1}$$

Does it follow that

$$\mathrm{vol}_n(A) \le \mathrm{vol}_n(B) \quad ? \tag{8.2}$$

This question is known as the Generalized Busemann-Petty Problem (GBP); see [G], [RZ], [Z1].

Theorem 8.1. *If GBP (8.1)-(8.2) has an affirmative answer, then every smooth origin-symmetric convex body with positive curvature in* $\mathbb{R}^n$ *is an* $(n-i)$*-intersection body.*

Proof. Suppose that B is an o.s. convex body in $\mathbb{R}^n$ so that the radial function ρ_B is infinitely smooth, the boundary of B has a positive curvature and $B \notin \mathcal{I}^n_{n-i}$. By Definition 5.4, there is a function $\varphi \in \mathcal{D}_e(S^{n-1})$, which is negative on some open origin-symmetric set $\Omega \subset S^{n-1}$ and such that $\rho_B^{n-i} = M^{1+i-n}\varphi$. We choose a function $h \in \mathcal{D}_e(S^{n-1})$ so that $h \not\equiv 0$, $h(\theta) \ge 0$ if $\theta \in \Omega$ and $h(\theta) \equiv 0$ otherwise. Define an o.s. smooth body A by $\rho_A^i = \rho_B^i - \varepsilon M^{1-i}h$, $\varepsilon > 0$. If ε is small enough, then A is convex. Since by (3.12), $R_i M^{1-i} h = c\, R^0_{n-i,\perp} h \ge 0$, then $R_i \rho_A^i \le R_i \rho_B^i$, which gives (8.1). On the other hand, by (3.5),

$$(\rho_B^{n-i}, \rho_B^i - \rho_A^i) = \varepsilon (M^{1+i-n}\varphi, M^{1-i}h) = \varepsilon(\varphi, h) < 0,$$

or $(\rho_B^{n-i}, \rho_B^i) < (\rho_B^{n-i}, \rho_A^i)$. By Hölder's inequality, this implies $\mathrm{vol}_n(B) < \mathrm{vol}_n(A)$, which contradicts (8.2). □

Remark 8.2. As we noted in Introduction, Theorem 8.1 is not new, and its proof given in [K3] relies on a sequence of deep facts from functional analysis. The proof presented above is much more elementary and constructive. For instance, it allows us to keep invariance properties of the bodies under control. This advantage was essentially used in our paper [R4].

Theorem 8.1 and Proposition 7.5 imply the following

Corollary 8.3. *Let* $1 \le \ell \le n/2$; $i > \ell + 2$, $B = B^n_{4,\ell}$ (see (7.1)). *Then there is a smooth o.s. convex body* A *in* $\mathbb{R}^n$ *so that (8.1) holds but (8.2) fails.*

Setting $\ell = 1$ in this statement, we obtain the well-known Bourgain-Zhang theorem, which states that GBP has a negative answer when $3 < i < n$; see [BZ], [K4], [RZ] on this subject. For $i = 2$ and $i = 3$ ($n \geq 5$) the GBP is still open. An affirmative answer in these cases was obtained in [R4] for bodies having a certain additional symmetry.

9. Appendix

Every positive distribution $F \in \mathcal{S}'(\mathbb{R}^n)$ is given by a tempered non-negative measure μ, i.e., $\langle F, \phi \rangle = \int \phi(x) d\mu(x)$; see, e.g., [GV, p.147]). For convenience of the reader, we present a similar fact for the sphere.

Theorem 9.1. *A distribution $f \in \mathcal{D}'(S^{n-1})$ is positive if and only if there is a measure $\mu \in \mathcal{M}_+(S^{n-1})$ such that*

$$(f, \varphi) = \int_{S^{n-1}} \varphi(\theta) d\mu(\theta) \qquad \forall \varphi \in \mathcal{D}(S^{n-1}).$$

Proof. This statement is known, however, we could not find precise reference and decided to give a proof for convenience of the reader. The "if" part is obvious. To prove the " only if" part, we write a test function $\varphi \in \mathcal{D}(S^{n-1})$ as a sum $\varphi = \varphi_1 + i\varphi_2$, where $\varphi_1 = Re\, \varphi,\ \varphi_2 = Im\, \varphi$. Since $-||\varphi||_{C(S^{n-1})} \leq \varphi_j \leq ||\varphi||_{C(S^{n-1})}$, $j = 1, 2$, and f is positive, then

$$-(f, 1)\, ||\varphi||_{C(S^{n-1})} \leq (f, \varphi_j) \leq (f, 1)\, ||\varphi||_{C(S^{n-1})},$$

and therefore, $|(f, \varphi)| \leq |(f, \varphi_1)| + |(f, \varphi_2)| \leq 2(f, 1)\, ||\varphi||_{C(S^{n-1})}$. Since $\mathcal{D}(S^{n-1})$ is dense in $C(S^{n-1})$, then f extends as a linear continuous functional $\tilde{f}$ on $C(S^{n-1})$ and, by the Riesz theorem, there is a measure μ on S^{n-1} such that $(\tilde{f}, \omega) = \int_{S^{n-1}} \omega(\theta) d\mu(\theta)$ for every $\omega \in C(S^{n-1})$. In particular, $(f, \varphi) = (\tilde{f}, \varphi) = \int_{S^{n-1}} \varphi(\theta) d\mu(\theta)$ for every $\varphi \in \mathcal{D}(S^{n-1})$. By taking into account that every non-negative function $\omega \in C(S^{n-1})$ can be uniformly approximated by non-negative functions $\varphi_k \in \mathcal{D}(S^{n-1})$ (for instance, by Poisson integrals of ω), we get

$$\int_{S^{n-1}} \omega(\theta) d\mu(\theta) = \lim_{k \to \infty} \int_{S^{n-1}} \varphi_k(\theta) d\mu(\theta) = \lim_{k \to \infty} (f, \varphi_k) \geq 0.$$

The latter means that μ is non-negative. $\square$

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