

An algorithm for the classification of smooth Fano polytopes

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Abstract

We present an algorithm that produces the classification list of smooth Fano d -polytopes for any given $d \geq 1$. The input of the algorithm is a single number, namely the positive integer d . The algorithm has been used to classify smooth Fano d -polytopes for $d \leq 7$. There are 7622 isomorphism classes of smooth Fano 6-polytopes and 72256 isomorphism classes of smooth Fano 7-polytopes.

1 Introduction

Isomorphism classes of smooth toric Fano varieties of dimension d correspond to isomorphism classes of so-called smooth Fano d -polytopes, which are fully dimensional convex lattice polytopes in \mathbb{R}^d , such that the origin is in the interior of the polytopes and the vertices of every facet is a basis of the integral lattice $\mathbb{Z}^d \subset \mathbb{R}^d$. Smooth Fano d -polytopes have been intensively studied for the last decades. They have been completely classified up to isomorphism for $d \leq 4$ ([1], [18], [3], [15]). Under additional assumptions there are classification results valid in every dimension.

To our knowledge smooth Fano d -polytopes have been classified in the following cases:

- When the number of vertices is $d + 1$, $d + 2$ or $d + 3$ ([9],[2]).
- When the number of vertices is $3d$, which turns out to be the upper bound on the number of vertices ([6]).
- When the number of vertices is $3d - 1$ ([19]).
- When the polytopes are centrally symmetric ([17]).
- When the polytopes are pseudo-symmetric, i.e. there is a facet F , such that $-F$ is also a facet ([8]).
- When there are many pairs of centrally symmetric vertices ([5]).

- When the corresponding toric d -folds are equipped with an extremal contraction, which contracts a toric divisor to a point ([4]) or a curve ([16]).

Recently a complete classification of smooth Fano 5-polytopes has been announced ([12]). The approach is to recover smooth Fano d -polytopes from their image under the projection along a vertex. This image is a *reflexive* $(d - 1)$ -polytope (see [3]), which is a fully-dimensional lattice polytope containing the origin in the interior, such that the dual polytope is also a lattice polytope. Reflexive polytopes have been classified up to dimension 4 using the computer program PALP ([10],[11]). Using this classification and PALP the authors of [12] succeed in classifying smooth Fano 5-polytopes.

In this paper we present an algorithm that classifies smooth Fano d -polytopes for any given $d \geq 1$. We call this algorithm SFP (for Smooth Fano Polytopes). The input is the positive integer d , nothing else is needed. The algorithm has been implemented in C++, and used to classify smooth Fano d -polytopes for $d \leq 7$. For $d = 6$ and $d = 7$ our results are new:

Theorem 1.1. *There are 7622 isomorphism classes of smooth Fano 6-polytopes and 72256 isomorphism classes of smooth Fano 7-polytopes.*

The classification lists of smooth Fano d -polytopes, $d \leq 7$, are available on the authors homepage: <http://home.imf.au.dk/oebro>

A key idea in the algorithm is the notion of a special facet of a smooth Fano d -polytope (defined in section 3.1): A facet F of a smooth Fano d -polytope is called *special*, if the sum of the vertices of the polytope is a non-negative linear combination of vertices of F . This allows us to identify a finite subset \mathcal{W}_d of the lattice \mathbb{Z}^d , such that any smooth Fano d -polytope is isomorphic to one whose vertices are contained in \mathcal{W}_d (theorem 3.6). Thus the problem of classifying smooth Fano d -polytopes is reduced to the problem of considering certain subsets of \mathcal{W}_d .

We then define a total order on finite subsets of \mathbb{Z}^d and use this to define a total order on the set of smooth Fano d -polytopes, which respects isomorphism (section 4). The SFP-algorithm (described in section 5) goes through certain finite subsets of \mathcal{W}_d in increasing order, and outputs smooth Fano d -polytopes in increasing order, such that any smooth Fano d -polytope is isomorphic to exactly one in the output list.

As a consequence of the total order on smooth Fano d -polytopes, the algorithm needs not consult the previous output to check for isomorphism to decide whether or not to output a constructed polytope.

2 Smooth Fano polytopes

We fix a notation and prove some simple facts about smooth Fano polytopes.

The convex hull of a set $K \in \mathbb{R}^d$ is denoted by $\text{conv}K$. A *polytope* is the convex hull of finitely many points. The dimension of a polytope P is the dimension of the affine hull, $\text{aff}P$, of the polytope P . A *k-polytope* is a polytope of dimension k . A *face* of a polytope is the intersection of a supporting hyperplane with the polytope. Faces of polytopes are polytopes. Faces of dimension 0 are called *vertices*, while faces of codimension 1 and 2 are called *facets* and *ridges*, respectively. The set of vertices of a polytope P is denoted by $\mathcal{V}(P)$.

Definition 2.1. A convex lattice polytope P in \mathbb{R}^d is called a smooth Fano d -polytope, if the origin is contained in the interior of P and the vertices of every facet of P is a \mathbb{Z} -basis of the lattice $\mathbb{Z}^d \subset \mathbb{R}^d$.

We consider two smooth Fano d -polytopes P_1, P_2 to be *isomorphic*, if there exists a bijective linear map $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d$, such that $\varphi(\mathbb{Z}^d) = \mathbb{Z}^d$ and $\varphi(P_1) = P_2$.

Whenever F is a $(d-1)$ -simplex in \mathbb{R}^d , such that $0 \notin \text{aff}F$, we let $u_F \in (\mathbb{R}^d)^*$ be the unique element determined by $\langle u_F, F \rangle = \{1\}$. For every $w \in \mathcal{V}(F)$ we define $u_F^w \in (\mathbb{R}^d)^*$ to be the element where $\langle u_F^w, w \rangle = 1$ and $\langle u_F^w, w' \rangle = 0$ for every $w' \in \mathcal{V}(F)$, $w' \neq w$. Then $\{u_F^w | w \in \mathcal{V}(F)\}$ is the basis of $(\mathbb{R}^d)^*$ dual to the basis $\mathcal{V}(F)$ of \mathbb{R}^d .

When F is a facet of a smooth Fano polytope and $v \in \mathcal{V}(P)$, we certainly have $\langle u_F, v \rangle \in \mathbb{Z}$ and

$$\langle u_F, v \rangle = 1 \iff v \in \mathcal{V}(F) \quad \text{and} \quad \langle u_F, v \rangle \leq 0 \iff v \notin \mathcal{V}(F).$$

The lemma below concerns the relation between the elements u_F and $u_{F'}$, when F and F' are adjacent facets.

Lemma 2.2. Let F be a facet of a smooth Fano polytope P and $v \in \mathcal{V}(F)$. Let F' be the unique facet which intersects F in a ridge R of P , $v \notin \mathcal{V}(R)$. Let $v' = \mathcal{V}(F') \setminus \mathcal{V}(R)$.

Then

1. $\langle u_F^v, v' \rangle = -1$.
2. $\langle u_F, v' \rangle = \langle u_{F'}, v \rangle$.
3. $\langle u_{F'}, x \rangle = \langle u_F, x \rangle + \langle u_F^v, x \rangle (\langle u_F, v' \rangle - 1)$ for any $x \in \mathbb{R}^d$.
4. In particular,

- $\langle u_F^v, x \rangle < 0$ iff $\langle u_{F'}, x \rangle > \langle u_F, x \rangle$.
- $\langle u_F^v, x \rangle > 0$ iff $\langle u_{F'}, x \rangle < \langle u_F, x \rangle$.
- $\langle u_F^v, x \rangle = 0$ iff $\langle u_{F'}, x \rangle = \langle u_F, x \rangle$.

for any $x \in \mathbb{R}^d$.

5. Suppose $x \neq v'$ is a vertex of P where $\langle u_F^v, x \rangle < 0$. Then $\langle u_F, v' \rangle > \langle u_F, x \rangle$.

Proof. The sets $\mathcal{V}(F)$ and $\mathcal{V}(F')$ are both bases of the lattice \mathbb{Z}^d and the first statement follows.

We have $v + v' \in \text{span}(F \cap F')$, and then the second statement follows.

Use the previous statements to calculate $\langle u_{F'}, x \rangle$.

$$\begin{aligned} \langle u_{F'}, x \rangle &= \langle u_{F'}, \sum_{w \in \mathcal{V}(F)} \langle u_F^w, x \rangle w \rangle \\ &= \sum_{w \in \mathcal{V}(F) \setminus \{v\}} \langle u_F^w, x \rangle + \langle u_F^v, x \rangle \langle u_{F'}, v \rangle \\ &= \langle u_F, x \rangle + \langle u_F^v, x \rangle (\langle u_{F'}, v \rangle - 1) \\ &= \langle u_F, x \rangle + \langle u_F^v, x \rangle (\langle u_F, v' \rangle - 1). \end{aligned}$$

As $\langle u_F, v' \rangle - 1 < 0$ the three equivalences follow directly.

Suppose there is a vertex $x \in \mathcal{V}(P)$, such that $\langle u_F^v, x \rangle < 0$ and $\langle u_F, v' \rangle \leq \langle u_F, x \rangle$. Then

$$\langle u_{F'}, x \rangle = \langle u_F, x \rangle + \langle u_F^v, x \rangle (\langle u_F, v' \rangle - 1) \geq \langle u_F, x \rangle - (\langle u_F, v' \rangle - 1) \geq 1.$$

Hence x is on the facet F' . But this cannot be the case as $\mathcal{V}(F') = \{v'\} \cup \mathcal{V}(F) \setminus \{v\}$. Thus no such x exists.

And we're done. \square

In the next lemma we show a lower bound on the numbers $\langle u_F^w, v \rangle$, $w \in \mathcal{V}(F)$, for any facet F and any vertex v of a smooth Fano d -polytope.

Lemma 2.3. *Let F be a facet and v a vertex of a smooth Fano polytope P . Then*

$$\langle u_F^w, v \rangle \geq \begin{cases} 0 & \langle u_F, v \rangle = 1 \\ -1 & \langle u_F, v \rangle = 0 \\ \langle u_F, v \rangle & \langle u_F, v \rangle < 0 \end{cases}$$

for every $w \in \mathcal{V}(F)$.

Proof. When $\langle u_F, v \rangle = 1$ the statement is obvious.

Suppose $\langle u_F, v \rangle = 0$ and $\langle u_F^w, v \rangle < 0$ for some $w \in \mathcal{V}(F)$. Let F' be the unique facet intersecting F in the ridge $\text{conv}\{\mathcal{V}(F) \setminus \{w\}\}$. By lemma 2.2 $\langle u_{F'}, v \rangle > 0$. As $\langle u_{F'}, v \rangle \in \mathbb{Z}$ we must have $\langle u_{F'}, v \rangle = 1$. This implies $\langle u_F, v \rangle = -1$.

Suppose $\langle u_F, v \rangle < 0$ and $\langle u_F^w, v \rangle < \langle u_F, v \rangle \leq -1$ for some $w \in \mathcal{V}(F)$. Let $F' \neq F$ be the facet containing the ridge $\text{conv}\{\mathcal{V}(F) \setminus \{w\}\}$, and let w' be the unique vertex in $\mathcal{V}(F') \setminus \mathcal{V}(F)$. Then by lemma 2.2

$$\langle u_{F'}, v \rangle = \langle u_F, v \rangle + \langle u_F^w, v \rangle (\langle u_F, w' \rangle - 1) \geq \langle u_F, v \rangle - \langle u_F^w, v \rangle.$$

If $\langle u_F, v \rangle - \langle u_F^w, v \rangle > 0$, then v is on the facet F' . But this is not the case as $\langle u_F^w, v \rangle < -1$. We conclude that $\langle u_F^w, v \rangle \geq \langle u_F, v \rangle$. \square

When F is a facet and v a vertex of a smooth Fano d -polytope P , such that $\langle u_F, v \rangle = 0$, we can say something about the face lattice of P .

Lemma 2.4 ([7] section 2.3 remark 5(2), [13] lemma 5.5). *Let F be a facet and v be vertex of a smooth Fano polytope P . Suppose $\langle u_F, v \rangle = 0$. Then $\text{conv}\{\{v\} \cup \mathcal{V}(F) \setminus \{w\}\}$ is a facet of P for every $w \in \mathcal{V}(F)$ with $\langle u_F^w, v \rangle = -1$.*

Proof. Follows from the proof of lemma 2.3. \square

3 Special embeddings of smooth Fano polytopes

In this section we find a concrete finite subset \mathcal{W}_d of \mathbb{Z}^d with the nice property that any smooth Fano d -polytope is isomorphic to one whose vertices are contained in \mathcal{W}_d . The problem of classifying smooth Fano d -polytopes is then reduced to considering subsets of \mathcal{W}_d .

3.1 Special facets

The following definition is a key concept.

Definition 3.1. *A facet F of a smooth Fano d -polytope P is called special, if the sum of the vertices of P is a non-negative linear combination of $\mathcal{V}(F)$, that is*

$$\sum_{v \in \mathcal{V}(P)} v = \sum_{w \in \mathcal{V}(F)} a_w w, \quad a_w \geq 0.$$

Clearly, any smooth Fano d -polytope has at least one special facet.

Let F be a special facet of a smooth Fano d -polytope P . Then

$$0 \leq \langle u_F, \sum_{v \in \mathcal{V}(P)} v \rangle = d + \sum_{v \in \mathcal{V}(P), \langle u_F, v \rangle < 0} \langle u_F, v \rangle,$$

which implies $-d \leq \langle u_F, v \rangle \leq 1$ for any vertex v of P . By using the lower bound on the numbers $\langle u_F^w, v \rangle$, $w \in \mathcal{V}(F)$ (see lemma 2.3), we can find an explicit finite subset of the lattice \mathbb{Z}^d , such that every $v \in \mathcal{V}(P)$ is contained in this subset. In the following lemma we generalize this observation to subsets of $\mathcal{V}(P)$ containing $\mathcal{V}(F)$.

Lemma 3.2. *Let P be a smooth Fano polytope. Let F be a special facet of P and let V be a subset of $\mathcal{V}(P)$ containing $\mathcal{V}(F)$, whose sum is ν .*

$$\nu = \sum_{v \in V} v.$$

Then

$$\langle u_F, \nu \rangle \geq 0$$

and

$$\langle u_F^w, \nu \rangle \leq \langle u_F, \nu \rangle + 1$$

for every $w \in \mathcal{V}(F)$.

Proof. For convenience we set $U = \mathcal{V}(P) \setminus V$ and $\mu = \sum_{v \in U} v$. Since F is a special facet we know that

$$0 \leq \langle u_F, \sum_{v \in \mathcal{V}(P)} v \rangle = \langle u_F, \nu \rangle + \langle u_F, \mu \rangle.$$

The set $\mathcal{V}(F)$ is contained in V so $\langle u_F, v \rangle \leq 0$ for every v in U , hence $\langle u_F, \nu \rangle \geq 0$.

Suppose that for some $w \in \mathcal{V}(F)$ we have $\langle u_F^w, \nu \rangle > \langle u_F, \nu \rangle + 1$. By lemma 2.3 we know that

$$\langle u_F^w, v \rangle \geq \begin{cases} -1 & \langle u_F, v \rangle = 0 \\ \langle u_F, v \rangle & \langle u_F, v \rangle < 0 \end{cases}$$

for every vertex $v \in \mathcal{V}(P) \setminus \mathcal{V}(F)$. There is at most one vertex v of P , $\langle u_F, v \rangle = 0$, with negative coefficient $\langle u_F^w, v \rangle$ (lemma 2.4). So

$$\langle u_F^w, \mu \rangle \geq \langle u_F, \mu \rangle - 1.$$

Now, consider $\langle u_F^w, \sum_{v \in \mathcal{V}(P)} v \rangle$.

$$\langle u_F^w, \sum_{v \in \mathcal{V}(P)} v \rangle = \langle u_F^w, \nu \rangle + \langle u_F^w, \mu \rangle > \langle u_F, \nu \rangle + \langle u_F, \mu \rangle = \langle u_F, \sum_{v \in \mathcal{V}(P)} v \rangle.$$

But this implies that $\langle u_F^x, \sum_{v \in \mathcal{V}(P)} v \rangle$ is negative for some $x \in \mathcal{V}(F)$. A contradiction. \square

Corollary 3.3. *Let F be a special facet and v any vertex of a smooth Fano d -polytope. Then $-d \leq \langle u_F, v \rangle \leq 1$ and*

$$\left. \begin{array}{c} 0 \\ -1 \\ \langle u_F, v \rangle \end{array} \right\} \leq \langle u_F^w, v \rangle \leq \left\{ \begin{array}{ll} 1 & , \langle u_F, v \rangle = 1 \\ d-1 & , \langle u_F, v \rangle = 0 \\ d + \langle u_F, v \rangle & , \langle u_F, v \rangle < 0 \end{array} \right.$$

for every $w \in \mathcal{V}(F)$.

Proof. For $\langle u_F, v \rangle = 1$ the statement is obvious. When $\langle u_F, v \rangle = 0$ the coefficients of v with respect to the basis $\mathcal{V}(F)$ is bounded below by -1 (lemma 2.3), so no coefficient exceeds $d-1$.

So the case $\langle u_F, v \rangle < 0$ remains. The lower bound is by lemma 2.3. Use lemma 3.2 on the subset $V = \mathcal{V}(F) \cup \{v\}$ to prove the upper bound. \square

3.2 Special embeddings

Let (e_1, \dots, e_d) be a fixed basis of the lattice $\mathbb{Z}^d \subset \mathbb{R}^d$.

Definition 3.4. *Let P be a smooth Fano d -polytope. Any smooth Fano d -polytope Q , with $\text{conv}\{e_1, \dots, e_d\}$ as a special facet, is called a special embedding of P , if P and Q are isomorphic.*

Obviously, for any smooth Fano polytope P , there exists at least one special embedding of P . As any polytope has finitely many facets, there exists only finitely many special embeddings of P .

Now we define a subset of \mathbb{Z}^d which will play an important part in what follows.

Definition 3.5. *By \mathcal{W}_d we denote the maximal set (with respect to inclusion) of lattice points in \mathbb{Z}^d such that*

1. *The origin is not contained in \mathcal{W}_d .*
2. *The points in \mathcal{W}_d are primitive lattice points.*
3. *If $a_1 e_1 + \dots + a_d e_d \in \mathcal{W}_d$, then $-d \leq a \leq 1$ for $a = a_1 + \dots + a_d$ and*

$$\left. \begin{array}{c} 0 \\ -1 \\ a \end{array} \right\} \leq a_i \leq \left\{ \begin{array}{ll} 1 & , a = 1 \\ d-1 & , a = 0 \\ d+a & , a < 0 \end{array} \right.$$

for every $i = 1, \dots, d$.

The next theorem is one of the key results in this paper. It allows us to classify smooth Fano d -polytopes by considering subsets of the explicitly given set \mathcal{W}_d .

Theorem 3.6. *Let P be an arbitrary smooth Fano d -polytope, and Q any special embedding of P . Then $\mathcal{V}(Q)$ is contained in the set \mathcal{W}_d .*

Proof. Follows directly from corollary 3.3 and the definition of \mathcal{W}_d . □

4 Total ordering of smooth Fano polytopes

In this section we define a total order on the set of smooth Fano d -polytopes for any fixed $d \geq 1$.

Throughout the section (e_1, \dots, e_d) is a fixed basis of the lattice \mathbb{Z}^d .

4.1 The order of a lattice point

We begin by defining a total order \preceq on \mathbb{Z}^d .

Definition 4.1. Let $x = x_1e_1 + \dots + x_de_d$, $y = y_1e_1 + \dots + y_de_d$ be two lattice points in \mathbb{Z}^d . We define $x \preceq y$ if and only if

$$(-x_1 - \dots - x_d, x_1, \dots, x_d) \leq_{lex} (-y_1 - \dots - y_d, y_1, \dots, y_d),$$

where \leq_{lex} is the lexicographical ordering on the product of $d+1$ copies of the ordered set (\mathbb{Z}, \leq) .

The ordering \preceq is a total order on \mathbb{Z}^d .

Example. $(0, 1) \prec (-1, 1) \prec (1, -1) \prec (-1, 0)$.

Let V be any nonempty finite subset of lattice points in \mathbb{Z}^d . We define $\max V$ to be the maximal element in V with respect to the ordering \preceq . Similarly, $\min V$ is defined to be the minimal element in V .

A important property of the ordering is shown in the following lemma.

Lemma 4.2. Let P be a smooth Fano d -polytope, such that $\text{conv}\{e_1, \dots, e_d\}$ is a facet of P . For every $1 \leq i \leq d$, let $v_i \neq e_i$ denote the vertex of P , such that $\text{conv}\{e_1, \dots, e_{i-1}, v_i, e_{i+1}, \dots, e_d\}$ is a facet of P . Then $v_i = \min\{v \in \mathcal{V}(P) \mid \langle u_F^{e_i}, v \rangle < 0\}$.

Proof. By lemma 2.2.(1) the vertex v_i is in the set $\{v \in \mathcal{V}(P) \mid \langle u_F^{e_i}, v \rangle < 0\}$, and by lemma 2.2.(5) and the definition of the ordering \preceq , v_i is the minimal element in this set. \square

In fact, we have chosen the ordering \preceq to obtain the property of lemma 4.2, and any other total order on \mathbb{Z}^d having this property can be used in what follows.

4.2 The order of a smooth Fano d -polytope

We can now define an ordering on finite subsets of \mathbb{Z}^d . The ordering is defined recursively.

Definition 4.3. Let X and Y be finite subsets of \mathbb{Z}^d . We define $X \preceq Y$ if and only if $X = \emptyset$ or

$$Y \neq \emptyset \wedge (\min X \prec \min Y \vee (\min X = \min Y \wedge X \setminus \{\min X\} \preceq Y \setminus \{\min Y\})).$$

Example. $\emptyset \prec \{(0, 1)\} \prec \{(0, 1), (-1, 1)\} \prec \{(0, 1), (1, -1)\} \prec \{(-1, 1)\}$.

When W is a nonempty finite set of subsets of \mathbb{Z}^d , we define $\max W$ to be the maximal element in W with respect to the ordering of subsets \preceq . Similarly, $\min W$ is the minimal element in W .

Now, we are ready to define the order of a smooth Fano d -polytope.

Definition 4.4. Let P be a smooth Fano d -polytope. The order of P , $\text{ord}(P)$, is defined as

$$\text{ord}(P) := \min\{\mathcal{V}(Q) \mid Q \text{ a special embedding of } P\}.$$

The set is non-empty and finite, so $\text{ord}(P)$ is well-defined.

Let P_1 and P_2 be two smooth Fano d -polytopes. We say that $P_1 \leq P_2$ if and only if $\text{ord}(P_1) \preceq \text{ord}(P_2)$. This is indeed a total order on the set of isomorphism classes of smooth Fano d -polytopes.

4.3 Permutation of basisvectors and presubsets

The group S_d of permutations of d elements acts on \mathbb{Z}^d in the obvious way by permuting the basisvectors:

$$\sigma.(a_1 e_1 + \dots + a_d e_d) := a_1 e_{\sigma(1)} + \dots + a_d e_{\sigma(d)} \quad , \quad \sigma \in S_d.$$

Similarly, S_d acts on subsets of \mathbb{Z}^d :

$$\sigma.X := \{\sigma.x \mid x \in X\}.$$

In this notation we clearly have for any special embedding P of a smooth Fano d -polytope

$$\text{ord}(P) \preceq \min\{\sigma.\mathcal{V}(P) \mid \sigma \in S_d\}.$$

Let V and W be finite subsets of \mathbb{Z}^d . We say that V is a *presubset* of W , if $V \subseteq W$ and $v \prec w$ whenever $v \in V$ and $w \in W \setminus V$.

Example. $\{(0, 1), (-1, 1)\}$ is a presubset of $\{(0, 1), (-1, 1), (1, -1)\}$, while $\{(0, 1), (1, -1)\}$ is not.

Lemma 4.5. Let P be a smooth Fano polytope. Then every presubset V of $\text{ord}(P)$ is the minimal element in $\{\sigma.V \mid \sigma \in S_d\}$.

Proof. Let $\text{ord}(P) = \{v_1, \dots, v_n\}$, $v_1 \prec \dots \prec v_n$. Suppose there exists a permutation σ and a k , $1 \leq k \leq n$, such that

$$\sigma.\{v_1, \dots, v_k\} = \{w_1, \dots, w_k\} \prec \{v_1, \dots, v_k\},$$

where $w_1 \prec \dots \prec w_k$. Then there is a number j , $1 \leq j \leq k$, such that $w_i = v_i$ for every $1 \leq i < j$ and $w_j \prec v_j$.

Let σ act on $\{v_1, \dots, v_n\}$.

$$\sigma.\{v_1, \dots, v_n\} = \{x_1, \dots, x_n\} \quad , \quad x_1 \prec \dots \prec x_n.$$

Then $x_i \preceq v_i$ for every $1 \leq i < j$ and $x_j \prec v_j$. So $\sigma.\text{ord}(P) \prec \text{ord}(P)$, but this contradicts the definition of $\text{ord}(P)$. \square

5 The SFP-algorithm

In this section we describe an algorithm that produces the classification list of smooth Fano d -polytopes for any given $d \geq 1$. The algorithm works by going through certain finite subsets of \mathcal{W}_d in increasing order (with respect to the ordering defined in the previous section). It will output a subset V iff $\text{conv}V$ is a smooth Fano d -polytope P and $\text{ord}(P) = V$.

Throughout the whole section (e_1, \dots, e_d) is a fixed basis of \mathbb{Z}^d and I denotes the $(d-1)$ -simplex $\text{conv}\{e_1, \dots, e_d\}$.

5.1 The SFP-algorithm

The SFP-algorithm consists of three functions,

SFP, **AddPoint** and **CheckSubset**.

The finite subsets of \mathcal{W}_d are constructed by the function **AddPoint**, which takes a subset V , $\{e_1, \dots, e_d\} \subseteq V \subseteq \mathcal{W}_d$, together with a finite set \mathcal{F} , $I \in \mathcal{F}$, of $(d-1)$ -simplices in \mathbb{R}^d as input. It then goes through every v in the set

$$\{v \in \mathcal{W}_d \mid \max V \prec v\}$$

in increasing order, and recursively calls itself with input $V \cup \{v\}$ and some set \mathcal{F}' of $(d-1)$ -simplices of \mathbb{R}^d , $\mathcal{F} \subseteq \mathcal{F}'$. In this way subsets of \mathcal{W}_d are considered in increasing order.

Whenever **AddPoint** is called, it checks if the input set V is the vertex set of a special embedding of a smooth Fano d -polytope P such that $\text{ord}(P) = V$, in which case the polytope $P = \text{conv}V$ is outputted.

For any given integer $d \geq 1$ the function **SFP** calls the function **AddPoint** with input $\{e_1, \dots, e_d\}$ and $\{I\}$. In this way a call **SFP**(d) will make the algorithm go through every finite subset of \mathcal{W}_d containing $\{e_1, \dots, e_d\}$, and smooth Fano d -polytopes are outputted in strictly increasing order.

It is vital for the effectiveness of the SFP-algorithm, that there is some efficient way to check if a subset $V \subseteq \mathcal{W}_d$ is a presubset of $\text{ord}(P)$ for some smooth Fano d -polytope P . The function **AddPoint** should perform this check before the recursive call **AddPoint**(V, \mathcal{F}').

If P is any smooth Fano d -polytope, then any presubset V of $\text{ord}(P)$ is the minimal element in the set $\{\sigma.V \mid \sigma \in S_d\}$ (by lemma 4.5). In other words, if there exists a permutation σ such that $\sigma.V \prec V$, then the algorithm should not make the recursive call **AddPoint**(V).

But this is not the only test we wish to perform on a subset V before the recursive call. The function **CheckSubset** performs another test: It takes a subset V , $\{e_1, \dots, e_d\} \subseteq V \subseteq \mathcal{W}_d$ as input together with a finite set of $(d-1)$ -simplices \mathcal{F} , $I \in \mathcal{F}$, and returns a set \mathcal{F}' of $(d-1)$ -simplices containing \mathcal{F} , if there exists a special embedding P of a smooth Fano d -polytope, such that

1. V is a presubset of $\mathcal{V}(P)$
2. \mathcal{F} is a subset of the facets of P

This is proved in theorem 5.1. If no such special embedding exists, then **CheckSubset** returns false in many cases, but not always! Only when **CheckSubset**(V, \mathcal{F}) returns a set \mathcal{F}' of simplices, we allow the recursive call **AddPoint**(V, \mathcal{F}').

Given input $V \subseteq \mathcal{W}_d$ and a set \mathcal{F} of $(d-1)$ -simplices of \mathbb{R}^d , the function **CheckSubset** works in the following way: Suppose V is a presubset of $\mathcal{V}(P)$ for some special embedding P of a smooth Fano d -polytope and \mathcal{F} is a subset of the facets of P . Deduce as much as possible of the face lattice of P and look for contradictions to the lemmas stated in section 2. The more facets we know of P , the more restrictions we can put on the vertex set $\mathcal{V}(P)$, and then on V . If a contradiction arises, return false. Otherwise, return the deduced set of facets of P .

The following example illustrates how the function **CheckSubset** works.

5.2 An example of the reasoning in CheckSubset

Let $d = 5$ and $V = \{v_1, \dots, v_8\}$, where

$$\begin{aligned} v_1 &= e_1, v_2 = e_2, v_3 = e_3, v_4 = e_4, v_5 = e_5 \\ v_6 &= -e_1 - e_2 + e_4 + e_5, v_7 = e_2 - e_3 - e_4, v_8 = -e_4 - e_5. \end{aligned}$$

Suppose P is a special embedding of a smooth Fano 5-polytope, such that V is a presubset of $\mathcal{V}(P)$. Certainly, the simplex I is a facet of P .

Notice, that V does not violate lemma 3.2.

$$v_1 + \dots + v_8 = e_2 + e_5.$$

If V did contradict lemma 3.2, then the polytope P could not exist, and **CheckSubset**($V, \{I\}$) should return false.

For simplicity we denote any k -simplex $\text{conv}\{v_{i_1}, \dots, v_{i_k}\}$ by $\{i_1, \dots, i_k\}$.

Since $\langle u_I, v_6 \rangle = 0$, the simplices $F_1 = \{2, 3, 4, 5, 6\}$ and $F_2 = \{1, 3, 4, 5, 6\}$ are facets of P (lemma 2.4).

There are exactly two facets of P containing the ridge $\{1, 2, 4, 5\}$. One of them is I . Suppose the other one is $\{1, 2, 4, 5, 9\}$, where v_9 is some lattice point not in V , $v_9 \in \mathcal{V}(P)$. Then $\langle u_I, v_9 \rangle > \langle u_I, v_7 \rangle$ by lemma 2.2.(5) and then $v_9 \prec v_7$ by the definition of the ordering of lattice points \mathbb{Z}^d . But then V is not a presubset of $\mathcal{V}(P)$. This is the nice property of the ordering of \mathbb{Z}^d , and the reason why we chose it as we did. We conclude that $F_3 = \{1, 2, 4, 5, 7\}$ is a facet of P , and by similar reasoning $F_4 = \{1, 2, 3, 5, 8\}$ and $F_5 = \{1, 2, 3, 4, 8\}$ are facets of P .

Now, for each of the facets F_i and every point $v_j \in V$, we check if $\langle u_{F_i}, v_j \rangle = 0$. If this is the case, then by lemma 2.4 $\text{conv}(\{v_j\} \cup \mathcal{V}(F_i) \setminus \{w\})$ is a facet of P for every $w \in \mathcal{V}(F_i)$ where $\langle u_{F_i}^w, v_j \rangle < 0$. In this way we get that

$$\{2, 4, 5, 6, 7\} , \{1, 4, 5, 6, 7\} , \{1, 2, 3, 7, 8\} , \{1, 3, 5, 7, 8\}$$

are facets of P .

We continue in this way, until we cannot deduce any new facet of P . Every time we find a new facet F we check that v is beneath F (that is $\langle u_F, v \rangle \leq 1$) and that lemma 2.3 holds for any $v \in V$. If not, then **CheckSubset**($V, \{I\}$) should return false.

If no contradiction arises, **CheckSubset**($V, \{I\}$) returns the set of deduced facets.

5.3 The SFP-algorithm in pseudo-code

Input: A positive integer d .

Output: A list of special embeddings of smooth Fano d -polytopes, such that

1. Any smooth Fano d -polytope is isomorphic to one and only one polytope in the output list.
2. If P is a smooth Fano d -polytope in the output list, then $\mathcal{V}(P) = \text{ord}(P)$.
3. If P_1 and P_2 are two non-isomorphic smooth Fano d -polytopes in the output list and P_1 preceeds P_2 in the output list, then $\text{ord}(P_1) \prec \text{ord}(P_2)$.

SFP (an integer $d \geq 1$)

1. Construct the set $V = \{e_1, \dots, e_d\}$ and the simplex $I = \text{conv}V$.
2. Call the function **AddPoint**($V, \{I\}$).
3. End program.

AddPoint (a subset V where $\{e_1, \dots, e_d\} \subseteq V \subseteq \mathcal{W}_d$, a set of $(d-1)$ -simplices \mathcal{F} in \mathbb{R}^d where $I \in \mathcal{F}$)

1. If $P = \text{conv}(\mathcal{V}(V))$ is a smooth Fano d -polytope and $\mathcal{V}(V) = \text{ord}(P)$, then output P .
2. Go through every $v \in \mathcal{W}_d$, $\max \mathcal{V}(V) \prec v$, in increasing order with respect to the ordering \prec :
 - (a) If **CheckSubset**($V \cup \{v\}, \mathcal{F}$) returns false, then goto (d). Otherwise let \mathcal{F}' be the returned set of $(d-1)$ -simplices.

- (b) If $V \cup \{v\} \neq \min\{\sigma.(V \cup \{v\}) \mid \sigma \in S_d\}$, then goto (d).
- (c) Call the function **AddPoint**($V \cup \{v\}, \mathcal{F}'$).
- (d) Let v be the next element in \mathcal{W}_d and go back to (a).

3. Return

CheckSubset (a subset V where $\{e_1, \dots, e_d\} \subseteq V \subseteq \mathcal{W}_d$, a set of $(d-1)$ -simplices \mathcal{F} in \mathbb{R}^d where $I \in \mathcal{F}$)

1. Let $\nu = \sum_{v \in V} v$.
2. If $\langle u_I, \nu \rangle < 0$, then return false.
3. If $\langle u_I^{e_i}, \nu \rangle > 1 + \langle u_I, \nu \rangle$ for some i , then return false.
4. Let $\mathcal{F}' = \mathcal{F}$.
5. For every $i \in \{1, \dots, d\}$: If the set $\{v \in V \mid \langle u_I^{e_i}, v \rangle < 0\}$ is equal to $\{\max V\}$, then add the simplex $\text{conv}(\{\max V\} \cup \mathcal{V}(I) \setminus \{e_i\})$ to \mathcal{F}' .
6. If there exists $F \in \mathcal{F}'$ such that $\mathcal{V}(F)$ is not a \mathbb{Z} -basis of \mathbb{Z}^d , then return false.
7. If there exists $F \in \mathcal{F}'$ and $v \in V$ such that $\langle u_F, v \rangle > 1$, then return false.
8. If there exists $F \in \mathcal{F}'$, $v \in V$ and $w \in \mathcal{V}(F)$, such that

$$\langle u_F^w, v \rangle < \begin{cases} 0 & \langle u_F, v \rangle = 1 \\ -1 & \langle u_F, v \rangle = 0 \\ \langle u_F, v \rangle & \langle u_F, v \rangle < 0 \end{cases}$$

then return false.

9. If there exists $F \in \mathcal{F}'$, $v \in V$ and $w \in \mathcal{V}(F)$, such that $\langle u_F, v \rangle = 0$ and $\langle u_F^w, v \rangle = -1$, then consider the simplex $F' = \text{conv}(\{v\} \cup \mathcal{V}(F) \setminus \{w\})$. If $F' \notin \mathcal{F}'$, then add F' to \mathcal{F}' and go back to step 6.
10. Return \mathcal{F}' .

5.4 Justification of the SFP-algorithm

The following theorems justify the SFP-algorithm.

Theorem 5.1. *Let P be a special embedding of a smooth Fano d -polytope and V a presubset of $\mathcal{V}(P)$, such that $\{e_1, \dots, e_d\} \subseteq V$. Let \mathcal{F} be a set of facets of P .*

*Then **CheckSubset**(V, \mathcal{F}) returns a subset \mathcal{F}' of the facets of P and $\mathcal{F} \subseteq \mathcal{F}'$.*

Proof. By lemma 3.2 the subset V will pass the tests in step 2 and 3 in **CheckSubset**.

The function **CheckSubset** constructs a set \mathcal{F}' of $(d-1)$ -simplices containing the input set \mathcal{F} . We now wish to prove that every simplex F in \mathcal{F}' is a facet of P : By the assumptions the subset $\mathcal{F} \subseteq \mathcal{F}'$ consists of facets of P . Consider the addition of a simplex F_i , $1 \leq i \leq d$, in step 5:

$$F_i = \text{conv}(\{\max V\} \cup \mathcal{V}(I) \setminus \{e_i\}).$$

As $\max V$ is the only element in the set $\{v \in V \mid \langle u_I^{e_i}, v \rangle < 0\}$ and V is a presubset of $\mathcal{V}(P)$, F_i is a facet of P by lemma 4.2.

Consider the addition of simplices in step 9: If F is a facet of P , then by lemma 2.4 the simplex $\text{conv}(\{v\} \cup \mathcal{V}(F) \setminus \{w\})$ is a facet of P .

By induction we conclude, that every simplex in \mathcal{F}' is a facet of P . Then any simplex $F \in \mathcal{F}'$ will pass the tests in steps 6–8 (use lemma 2.3 to see that the last test is passed).

This proves the theorem. \square

Theorem 5.2. *The SFP-algorithm produces the promised output.*

Proof. Let P be a smooth Fano d -polytope. Clearly, P is isomorphic to at most one polytope in the output list.

Let Q be a special embedding of P such that $\mathcal{V}(Q) = \text{ord}(P)$. We need to show that Q is in the output list. Let $\mathcal{V}(Q) = \{e_1, \dots, e_d, q_1, \dots, q_k\}$, where $q_1 \prec \dots \prec q_k$, and let $V_i = \{e_1, \dots, e_d, q_1, \dots, q_i\}$ for every $1 \leq i \leq k$.

Certainly the function **AddPoint** has been called with input $\{e_1, \dots, e_d\}$ and $\{I\}$.

By theorem 5.1 the function call **CheckSubset**($V_1, \{I\}$) returns a set \mathcal{F}_1 of $(d-1)$ -simplices which are facets of Q , $I \subset \mathcal{F}_1$. By lemma 4.5 the set V_1 passes the test in 2b in **AddPoint**. Then **AddPoint** is called recursively with input V_1 and \mathcal{F}_1 .

The call **CheckSubset**(V_1, \mathcal{F}_1) returns a subset \mathcal{F}_2 of facets of Q , and the set V_2 passes the test in 2b in **AddPoint**. So the call **AddPoint**(V_2, \mathcal{F}_2) is made.

Proceed in this way to see that the call **AddPoint**(V_k, \mathcal{F}_k) is made, and then the polytope $Q = \text{conv} V_k$ is outputted in step 1 in **AddPoint**. \square

6 Classification results and where to get them

A modified version of the SFP-algorithm has been implemented in C++, and used to classify smooth Fano d -polytopes for $d \leq 7$. On an average home computer our program needs less than one day (january 2007) to construct the classification list of smooth Fano 7-polytopes. These lists can be downloaded from the authors homepage: <http://home.imf.au.dk/oebro>

An advantage of the SFP-algorithm is that it requires almost no memory: When the algorithm has found a smooth Fano d -polytope P , it needs not consult the output list to decide whether to output the polytope P or not. The construction guarantees that $\mathcal{V}(P) = \min\{\sigma.\mathcal{V}(P) \mid \sigma \in S_d\}$ and it remains to check if $\mathcal{V}(P) = \text{ord}(P)$. Thus there is no need of storing the output list.

The table below shows the number of isomorphism classes of smooth Fano d -polytopes with n vertices.

n	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$	$d = 7$
1	1						
2							
3		1					
4		2	1				
5		1	4	1			
6		1	7	9	1		
7			4	28	15	1	
8			2	47	91	26	1
9				27	268	257	40
10				10	312	1318	643
11				1	137	2807	5347
12				1	35	2204	19516
13					5	771	26312
14					2	186	14758
15						39	4362
16						11	1013
17						1	214
18						1	43
19							5
20							2
Total	1	5	18	124	866	7622	72256

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