Lagrangian quantum field theory in momentum picture

IV. Commutation relations for free fields

Bozhidar Z. Iliev * † ‡

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*Laboratory of Mathematical Modeling in Physics, Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Boul. Tzarigradsko chaussée 72, 1784 Sofia, Bulgaria

†E-mail address: bozho@inrne.bas.bg

‡URL: http://theo.inrne.bas.bg/~bozho/

Contents

Abstract

Possible (algebraic) commutation relations in the Lagrangian quantum theory of free (scalar, spinor and vector) fields are considered from mathematical view-point. As sources of these relations are employed the Heisenberg equations/relations for the dynamical variables and a specific condition for uniqueness of the operators of the dynamical variables (with respect to some class of Lagrangians). The paracommutation relations or some their generalizations are pointed as the most general ones that entail the validity of all Heisenberg equations. The simultaneous fulfillment of the Heisenberg equations and the uniqueness requirement turn to be impossible. This problem is solved via a redefinition of the dynamical variables, similar to the normal ordering procedure and containing it as a special case. That implies corresponding changes in the admissible commutation relations. The introduction of the concept of the vacuum makes narrow the class of the possible commutation relations; in particular, the mentioned redefinition of the dynamical variables is reduced to normal ordering. As a last restriction on that class is imposed the requirement for existing of an effective procedure for calculating vacuum mean values. The standard bilinear commutation relations are pointed as the only known ones that satisfy all of the mentioned conditions and do not contradict to the existing data.

1. Introduction

The main subject of this paper is an analysis of possible (algebraic) commutation relations in the Lagrangian quantum theory[1] of free fields. These relations are considered only from mathematical view-point and physical consequence of them, like the statistics of many-particle systems, are not investigated.

The canonical quantization method finds its origin in the classical Hamiltonian mechanics [9,10] and naturally leads to the canonical (anti)commutation relations [3,11,12]. These relations can be obtained from different assumptions (see, e.g., [1,13–15]) and are one of the basic corner stones of the present-day quantum field theory.

Theoretically there are possible also non-canonical commutation relations. The best known example of them being the so-called paracommutation relations [16–18]. But, however, it seems no one of the presently known particles/fields obeys them.

In the present work is shown how different classes of commutation relations, understood in a broad sense as algebraic connections between creation and/or annihilation operators, arise from the Lagrangian formalism, when applied to three types of Lagrangians describing free scalar, spinor and vector fields. Their origin is twofold. One one hand, a requirement for uniqueness of the dynamical variables (that can be calculated from Lagrangians leading to identical Euler-Lagrange equation) entails a number of specific commutation relations. On another hand, any one of the so-called Heisenberg relations/equations [3,11], implies corresponding commutation relations; for example, the paracommutation relations arise from the Heisenberg equations regarding the momentum operator, when 'charge symmetric' Lagrangian is employed.[2] The combination of the both methods leads to strong, generally incompatible, restrictions on the admissible types of commutation relations.

The introduction of the concept of vacuum, combined with the mentioned uniqueness of the operators of the dynamical variables, changes the situation and requires a redefinition of these operators in a way similar to the one known as the normal ordering [1,3,11,12], which is its special case. Some natural assumptions reduce the former to the letter one; in particular, in that way are excluded the paracommutation relations. However, this does not reduce the possible commutation relations to the canonical ones. Further, the requirement to be available an effective procedure for calculating vacuum mean (expectation) values, to which reduce all predictable results in the theory, puts new restriction, whose only realistic solution at the time being seems to be the standard canonical (anti)commutation relations.

The layout of the work is as follows.

Sect. 2 gives an idea of the momentum picture of motion and discusses the relations between the creation and annihilation operators in it and in Heisenberg picture. In Sect. 3 are reviewed some basic results from [13–15], part of which can be found also in papers like [1, 3,11,12]. In particular, the explicit expression of the dynamical variables via the creation

[1] In this paper we considered only the Lagrangian (canonical) quantum field theory in which the quantum fields are represented as operators, called field operators, acting on some Hilbert space, which in general is unknown if interacting fields are studied. These operators are supposed to satisfy some equations of motion, from them are constructed conserved quantities satisfying conservation laws, etc. From the view-point of present-day quantum field theory, this approach is only a preliminary stage for more or less rigorous formulation of the theory in which the fields are represented via operator-valued distributions, a fact required even for description of free fields. Moreover, in non-perturbative directions, like constructive and conformal field theories, the main objects are the vacuum mean (expectation) values of the fields and from these are reconstructed the Hilbert space of states and the acting on it fields. Regardless of these facts, the Lagrangian (canonical) quantum field theory is an inherent component of the most of the ways of presentation of quantum field theory adopted explicitly or implicitly in books like [1–8]. Besides, the Lagrangian approach is a source of many ideas for other directions of research, like the axiomatic quantum field theory [3,7,8].

[2] Ordinary [3,11], the commutation relations are postulated and the validity of the Heisenberg relations is then verified. We follow the opposite method by postulating the Heisenberg equations and, then, looking for commutation relations that are compatible with them.

and annihilation operators are presented (without assuming some commutation relations or normal ordering) and it is pointed to the existence of a family of such variables for a given system of Euler-Lagrange equations for free fields. The last fact is analyzed in Sect. 4, where a number of its consequences, having a sense of commutation relations, are drawn. The Heisenberg relations and the commutation relations between the dynamical variables are reviewed and analyzed in Sect. 5. It is pointed that the letter should be consequences from the former ones. Arguments are presented that the Heisenberg equation concerning the angular momentum operator should be split into two independent ones, representing its 'orbital' and 'spin' parts, respectively.

Sect. 6 contains a method for assigning commutation relations to the Heisenberg equations. It is shown that the Heisenberg equation involving the 'orbital' part of the angular momentum gives rise to a differential, not algebraic, commutation relation and the one concerning the 'spin' part of the angular momentum implies a complicated integro-differential connections between the creation and annihilation operators. Special attention is paid to the paracommutation relations, whose particular kind are the ordinary ones, which ensure the validity of the Heisenberg equations concerning the momentum operator. Partially is analyzed the problem for compatibility of the different types of commutation relations derived. It is proved that some generalization of the paracommutation relations ensures the fulfillment of all of the Heisenberg relations.

Sect. 7 is devoted to consequences from the commutation relations derived in Sect. 6 under the conditions for uniqueness of the dynamical variables presented in Sect. 4. Generally, these requirements are incompatible with the commutation relations. To overcome the problem, it is proposed a redefinition of the dynamical variables via a method similar to (and generalizing) the normal ordering. This, of course, entails changes in the commutation relations, the new versions of which happen to be compatible with the uniqueness conditions and ensure the validity of the Heisenberg relations.

The concept of the vacuum is introduced in Sect. 8. It reduces (practically) the redefinition of the operators of the dynamical variables to the one obtained via the normal ordering procedure in the ordinary quantum field theory, but, without additional suppositions, does not reduce the commutation relations to the standard bilinear ones. As a last step in specifying the commutation relations as much as possible, we introduce the requirement the theory to supply an effective way for calculating vacuum mean values of (anti-normally ordered) products of creation and annihilation operators to which are reduced all predictable results, in particular the mean values of the dynamical variables. The standard bilinear commutation relation seems to be the only ones know at present that survive that last condition, however their uniqueness in this respect is not investigated.

Sect. 9 deals with the same problems as described above but for systems containing at least two different quantum fields. The main obstacle is the establishment of commutation relations between creation/annihilation operators concerning different fields. Argument is presented that they should contain commutators or anticommutators of these operators. The major of corresponding commutation relations are explicitly written and the results obtained turn to be similar to the ones just described, only in 'multifield' version.

Section 10 closes the paper by summarizing its main results.

The books [1–3] will be used as standard reference works on quantum field theory. Of course, this is more or less a random selection between the great number of (text)books and papers on the theme to which the reader is referred for more details or other points of view. For this end, e.g., [4, 12, 19] or the literature cited in [1–4, 12, 19] may be helpful.

Throughout this paper $\hbar$ denotes the Planck's constant (divided by 2π), c is the velocity of light in vacuum, and i stands for the imaginary unit. The superscripts $\dagger$ and $\top$ mean respectively Hermitian conjugation and transposition (of operators or matrices), the superscript $*$

denotes complex conjugation, and the symbol $\circ$ denotes compositions of mappings/operators.

By δ_{fg}, or δ_f^g or δ^{fg} ($:= 1$ for $f = g$, $:= 0$ for $f = g$) is denoted the Kronecker δ-symbol, depending on arguments f and g, and $\delta^n(y)$, $y \in \mathbb{R}^n$, stands for the n-dimensional Dirac δ-function; $\delta(y) := \delta^1(y)$ for $y \in \mathbb{R}$.

The Minkowski spacetime is denoted by M. The Greek indices run from 0 to $\dim M - 1 = 3$. All Greek indices will be raised and lowered by means of the standard 4-dimensional Lorentz metric tensor $\eta^{\mu\nu}$ and its inverse $\eta_{\mu\nu}$ with signature $(+\,-\,-\,-)$. The Latin indices $a, b, \dots$ run from 1 to $\dim M - 1 = 3$ and, usually, label the spacial components of some object. The Einstein's summation convention over indices repeated on different levels is assumed over the whole range of their values.

At last, we ought to give an explanation why this work appears under the general title "Lagrangian quantum field theory in momentum picture" when in it all considerations are done, in fact, in Heisenberg picture with possible, but not necessary, usage of the creation and annihilation operators in momentum picture. First of all, we essentially employ the obtained in [13–15] expressions for the dynamical variables in momentum picture for three types of Lagrangians. The corresponding operators in Heisenberg picture, which in fact is used in this paper, can be obtained via a direct calculation, as it is partially done in, e.g., [1] for one of the mentioned types of Lagrangians. The important point here is that in Heisenberg picture it suffice to be used only the standard Lagrangian formalism, while in momentum picture one has to suppose the commutativity between the components of the momentum operator and the validity of the Heisenberg relations for it (see below equations (2.6) and (2.7)). Since for the analysis of the commutation relations we intend to do the fulfillment of these relations is not necessary (they are subsidiary restrictions on the Lagrangian formalism), the Heisenberg picture of motion is the natural one that has to be used. For this reason, the expression for the dynamical variables obtained in [13–15] will be used simply as their Heisenberg counterparts, but expressed via the creation and annihilation operators in momentum picture. The only real advantage one gets in this way is the more natural structure of the orbital angular momentum operator. As the commutation relations considered below are algebraic ones, it is inessential in what picture of motion they are written or investigated.

2. The momentum picture

Since the momentum picture of motion will be used only partially in this work, below is presented only its definition and the connection between the creation/annihilation operators in it and in Heisenberg picture. Details concerning the momentum picture can be found in [20, 21] and in the corresponding sections devoted to it in [13–15].

Let us consider a system of quantum fields, represented in Heisenberg picture of motion by field operators $\tilde{\varphi}_i(x)\colon \mathcal{F} \to \mathcal{F}$, $i = 1, \dots, n \in \mathbb{N}$, acting on the system's Hilbert space $\mathcal{F}$ of states and depending on a point x in Minkowski spacetime M. Here and henceforth, all quantities in Heisenberg picture will be marked by a tilde (wave) "$\tilde{\ }$" over their kernel symbols. Let $\tilde{\mathcal{P}}_\mu$ denotes the system's (canonical) momentum vectorial operator, defined via the energy-momentum tensorial operator $\tilde{\mathcal{T}}^{\mu\nu}$ of the system, viz.

$$\tilde{\mathcal{P}}_\mu := \frac{1}{c} \int\limits_{x^0=\text{const}} \tilde{\mathcal{T}}_{0\mu}(x)\,\mathrm{d}^3\boldsymbol{x}. \tag{2.1}$$

Since this operator is Hermitian, $\tilde{\mathcal{P}}_\mu^\dagger = \tilde{\mathcal{P}}_\mu$, the operator

$$\mathcal{U}(x, x_0) = \exp\Bigl(\frac{1}{\mathrm{i}\hbar} \sum_\mu (x^\mu - x_0^\mu)\,\tilde{\mathcal{P}}_\mu\Bigr), \tag{2.2}$$

where $x_0 \in M$ is arbitrarily fixed and $x \in M$,[3] is unitary, i.e. $\mathcal{U}^\dagger(x_0, x) := (\mathcal{U}(x, x_0))^\dagger = \mathcal{U}^{-1}(x, x_0) := (\mathcal{U}(x, x_0))^{-1}$ and, via the formulae

$$\tilde{\mathcal{X}} \mapsto \mathcal{X}(x) = \mathcal{U}(x, x_0)(\tilde{\mathcal{X}}) \tag{2.3}$$

$$\tilde{\mathcal{A}}(x) \mapsto \mathcal{A}(x) = \mathcal{U}(x, x_0) \circ (\tilde{\mathcal{A}}(x)) \circ \mathcal{U}^{-1}(x, x_0), \tag{2.4}$$

realizes the transition to the *momentum picture*. Here $\tilde{\mathcal{X}}$ is a state vector in system's Hilbert space of states $\mathcal{F}$ and $\tilde{\mathcal{A}}(x)\colon \mathcal{F} \to \mathcal{F}$ is (observable or not) operator-valued function of $x \in M$ which, in particular, can be polynomial or convergent power series in the field operators $\tilde{\varphi}_i(x)$; respectively $\mathcal{X}(x)$ and $\mathcal{A}(x)$ are the corresponding quantities in momentum picture. In particular, the field operators transform as

$$\tilde{\varphi}_i(x) \mapsto \varphi_i(x) = \mathcal{U}(x, x_0) \circ \tilde{\varphi}_i(x) \circ \mathcal{U}^{-1}(x, x_0). \tag{2.5}$$

Notice, in (2.2) the multiplier $(x^\mu - x_0^\mu)$ is regarded as a real parameter (in which $\tilde{\mathcal{P}}_\mu$ is linear). Generally, $\mathcal{X}(x)$ and $\mathcal{A}(x)$ depend also on the point x_0 and, to be quite correct, one should write $\mathcal{X}(x, x_0)$ and $\mathcal{A}(x, x_0)$ for $\mathcal{X}(x)$ and $\mathcal{A}(x)$, respectively. However, in the most situations in the present work, this dependence is not essential or, in fact, is not presented at all. For that reason, we shall *not* indicate it explicitly.

The momentum picture is most suitable in quantum field theories in which the components $\tilde{\mathcal{P}}_\mu$ of the momentum operator commute between themselves and satisfy the Heisenberg relations/equations with the field operators, i.e. when $\tilde{\mathcal{P}}_\mu$ and $\tilde{\varphi}_i(x)$ satisfy the relations:

$$[\tilde{\mathcal{P}}_\mu, \tilde{\mathcal{P}}_\nu]_{\_} = 0 \tag{2.6}$$

$$[\tilde{\varphi}_i(x), \tilde{\mathcal{P}}_\mu]_{\_} = \mathrm{i}\hbar \partial_\mu \tilde{\varphi}_i(x). \tag{2.7}$$

Here $[A, B]_\pm := A \circ B \pm B \circ A$, $\circ$ being the composition of mappings sign, is the commutator/anticommutator of operators (or matrices) A and B.

However, the fulfillment of the relations (2.6) and (2.7) will not be supposed in this paper until Sect. 6 (see also Sect. 5).

Let $a_s^\pm(\boldsymbol{k})$ and $a_s^{\dagger\pm}(\boldsymbol{k})$ be the creation/annihilation operators of some free particular field (see Sect. 3 below for a detailed explanation of the notation). We have the connections

$$\left.\begin{aligned} \tilde{a}_s^\pm(\boldsymbol{k}) &= \mathrm{e}^{\pm\frac{1}{\mathrm{i}\hbar}x^\mu k_\mu}\,\mathcal{U}^{-1}(x, x_0) \circ a_s^\pm(\boldsymbol{k}) \circ \mathcal{U}(x, x_0) \\ \tilde{a}_s^{\dagger\pm}(\boldsymbol{k}) &= \mathrm{e}^{\pm\frac{1}{\mathrm{i}\hbar}x^\mu k_\mu}\,\mathcal{U}^{-1}(x, x_0) \circ a_s^{\dagger\pm}(\boldsymbol{k}) \circ \mathcal{U}(x, x_0) \end{aligned}\right\} \quad k_0 = \sqrt{m^2c^2 + \boldsymbol{k}^2} \tag{2.8}$$

whose explicit form is

$$\left.\begin{aligned} \tilde{a}_s^\pm(\boldsymbol{k}) &= \mathrm{e}^{\pm\frac{1}{\mathrm{i}\hbar}x_0^\mu k_\mu} a_s^\pm(\boldsymbol{k}) \\ \tilde{a}_s^{\dagger\pm}(\boldsymbol{k}) &= e^{\pm\frac{1}{\mathrm{i}\hbar}x_0^\mu k_\mu} a_s^{\dagger\pm}(\boldsymbol{k}) \end{aligned}\right\} \quad k_0 = \sqrt{m^2c^2 + \boldsymbol{k}^2}. \tag{2.9}$$

Further it will be assumed $\tilde{a}_s^\pm(\boldsymbol{k})$ and $\tilde{a}_s^{\dagger\pm}(\boldsymbol{k})$ to be defined in Heisenberg picture, independently of $a_s^\pm(\boldsymbol{k})$ and $a_s^{\dagger\pm}(\boldsymbol{k})$, by means of the standard Lagrangian formalism. What concerns the operators $a_s^\pm(\boldsymbol{k})$ and $a_s^{\dagger\pm}(\boldsymbol{k})$, we shall regard them as *defined* via (2.9); this makes them independent from the momentum picture of motion. The fact that the so-defined operators $a_s^\pm(\boldsymbol{k})$ and $a_s^{\dagger\pm}(\boldsymbol{k})$ coincide with the creation/annihilation operators in momentum picture (under the conditions (2.6) and (2.7)) will be inessential in the almost whole text.

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[3] The notation x_0, for a fixed point in M, should not be confused with the zeroth covariant coordinate $\eta_{0\mu}x^\mu$ of x which, following the convention $x_\nu := \eta_{\nu\mu}x^\mu$, is denoted by the same symbol x_0. From the context, it will always be clear whether x_0 refers to a point in M or to the zeroth covariant coordinate of a point $x \in M$.

3. Lagrangians, Euler-Lagrange equations and dynamical variables

In [13–15] we have investigated the Lagrangian quantum field theory of respectively scalar, spin $\frac{1}{2}$ and vector free fields. The main Lagrangians from which it was derived are respectively (see *loc. cit.* or, e.g. [1, 3, 11, 12]):

$$\tilde{\mathcal{L}}'_{\mathrm{sc}} = \tilde{\mathcal{L}}'_{\mathrm{sc}}(\tilde{\varphi}, \tilde{\varphi}^\dagger) = -\frac{1}{1+\tau(\tilde{\varphi})} m^2 c^4 \tilde{\varphi}(x) \circ \tilde{\varphi}^\dagger(x) + \frac{1}{1+\tau(\tilde{\varphi})} c^2 \hbar^2 (\partial_\mu \tilde{\varphi}(x)) \circ (\partial^\mu \tilde{\varphi}^\dagger(x)) \tag{3.1a}$$

$$\begin{aligned} \tilde{\mathcal{L}}'_{\mathrm{sp}} = \tilde{\mathcal{L}}'_{\mathrm{sp}}(\tilde{\psi}, \tilde{\tilde{\psi}}) = &-\frac{1}{2}\mathrm{i}\hbar c \{ \tilde{\tilde{\psi}}^\top(x) C^{-1}\gamma^\mu \circ (\partial_\mu \tilde{\psi}(x)) \\ &- (\partial_\mu \tilde{\tilde{\psi}}^\top(x)) C^{-1}\gamma^\mu \circ \tilde{\psi}(x)\} + mc^2 \tilde{\tilde{\psi}}^\top(x) C^{-1} \circ \tilde{\psi}(x) \end{aligned} \tag{3.1b}$$

$$\begin{aligned} \tilde{\mathcal{L}}'_{\mathrm{v}} = \tilde{\mathcal{L}}'_{\mathrm{v}}(\tilde{\mathcal{U}}, \tilde{\mathcal{U}}^\dagger) = &\frac{m^2c^4}{1+\tau(\tilde{\mathcal{U}})} \tilde{\mathcal{U}}^\dagger_\mu \circ \tilde{\mathcal{U}}^\mu \\ &+ \frac{c^2\hbar^2}{1+\tau(\tilde{\mathcal{U}})} \{-(\partial_\mu \tilde{\mathcal{U}}^\dagger_\nu) \circ (\partial^\mu \tilde{\mathcal{U}}^\nu) + (\partial_\mu \tilde{\mathcal{U}}^{\mu\dagger}) \circ (\partial_\nu \tilde{\mathcal{U}}^\nu)\} \end{aligned} \tag{3.1c}$$

Here it is used the following notation: $\tilde{\varphi}(x)$ is a scalar field, a tilde (wave) over a symbol means that it is in Heisenberg picture, the dagger $\dagger$ denotes Hermitian conjugation, $\tilde{\psi} := (\tilde{\psi}_0, \tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3)$ is a 4-spinor field, $\tilde{\tilde{\psi}} := C\bar{\tilde{\psi}}^\top := C(\tilde{\psi}^\dagger\gamma^0)$ is its charge conjugate with γ^μ being the Dirac gamma matrices and the matrix C satisfies the equations $C^{-1}\gamma^\mu C = -\gamma^\mu$ and $C^\top = -C$, U_μ is a vector field, m is the field's mass (parameter) and the function

$$\tau(A) := \begin{cases} 1 & \text{for } A^\dagger = A \text{ (Hermitian operator)} \\ 0 & \text{for } A^\dagger \neq A \text{ (non-Hermitian operator)} \end{cases}, \tag{3.2}$$

with $A\colon \mathcal{F} \to \mathcal{F}$ being an operator on the systems Hilbert space $\mathcal{F}$ of states, takes care of is the field charged (non-Hermitian) or neutral (Hermitian, uncharged). Since a spinor field is a charged one, we have $\tau(\tilde{\psi}) = 0$; sometimes below the number $0 = \tau(\tilde{\psi})$ will be written explicitly for unification of the notation.

We have explored also the consequences from the 'charge conjugate' Lagrangians

$$\tilde{\mathcal{L}}''_{\mathrm{sc}} = \tilde{\mathcal{L}}''_{\mathrm{sc}}(\tilde{\varphi}, \tilde{\varphi}^\dagger) := \tilde{\mathcal{L}}'_{\mathrm{sc}}(\tilde{\varphi}^\dagger, \tilde{\varphi}) \tag{3.3a}$$
$$\tilde{\mathcal{L}}''_{\mathrm{sp}} = \tilde{\mathcal{L}}''_{\mathrm{sp}}(\tilde{\psi}, \tilde{\tilde{\psi}}) := \tilde{\mathcal{L}}'_{\mathrm{sp}}(\tilde{\tilde{\psi}}, \tilde{\psi}) \tag{3.3b}$$
$$\tilde{\mathcal{L}}''_{\mathrm{v}} = \tilde{\mathcal{L}}''_{\mathrm{v}}(\tilde{\mathcal{U}}, \tilde{\mathcal{U}}^\dagger) := \tilde{\mathcal{L}}'_{\mathrm{v}}(\tilde{\mathcal{U}}^\dagger, \tilde{\mathcal{U}}), \tag{3.3c}$$

as well as from the 'charge symmetric' Lagrangians

$$\tilde{\mathcal{L}}'''_{\mathrm{sc}} = \tilde{\mathcal{L}}'''_{\mathrm{sc}}(\tilde{\varphi}, \tilde{\varphi}^\dagger) := \frac{1}{2}(\tilde{\mathcal{L}}'_{\mathrm{sc}} + \tilde{\mathcal{L}}''_{\mathrm{sc}}) = \frac{1}{2}\{\tilde{\mathcal{L}}'_{\mathrm{sc}}(\tilde{\varphi}, \tilde{\varphi}^\dagger) + \tilde{\mathcal{L}}'_{\mathrm{sc}}(\tilde{\varphi}^\dagger, \tilde{\varphi})\} \tag{3.4a}$$
$$\tilde{\mathcal{L}}'''_{\mathrm{sp}} = \tilde{\mathcal{L}}'''_{\mathrm{sp}}(\tilde{\psi}, \tilde{\tilde{\psi}}) := \frac{1}{2}(\tilde{\mathcal{L}}'_{\mathrm{sp}} + \tilde{\mathcal{L}}''_{\mathrm{sp}}) = \frac{1}{2}\{\tilde{\mathcal{L}}'_{\mathrm{sp}}(\tilde{\psi}, \tilde{\tilde{\psi}}) + \tilde{\mathcal{L}}'_{\mathrm{sp}}(\tilde{\tilde{\psi}}, \tilde{\psi})\} \tag{3.4b}$$
$$\tilde{\mathcal{L}}'''_{\mathrm{v}} = \tilde{\mathcal{L}}'''_{\mathrm{v}}(\tilde{\mathcal{U}}, \tilde{\mathcal{U}}^\dagger) := \frac{1}{2}(\tilde{\mathcal{L}}'_{\mathrm{v}} + \tilde{\mathcal{L}}''_{\mathrm{v}}) = \frac{1}{2}\{\tilde{\mathcal{L}}'_{\mathrm{v}}(\tilde{\mathcal{U}}, \tilde{\mathcal{U}}^\dagger) + \tilde{\mathcal{L}}'_{\mathrm{v}}(\tilde{\mathcal{U}}^\dagger, \tilde{\mathcal{U}})\}. \tag{3.4c}$$

It is essential to be noted, for a massless, $m = 0$, vector field to the Lagrangian formalism are added as subsidiary conditions the Lorenz conditions

$$\partial^\mu \tilde{\mathcal{U}}_\mu = 0 \quad \partial^\mu \tilde{\mathcal{U}}^\dagger_\mu = 0 \tag{3.5}$$

on the solutions of the corresponding Euler-Lagrange equations. Besides, if the opposite is not stated explicitly, no other restrictions, like the (anti)commutation relations, are supposed to be imposed on the above Lagrangians. And a technical remark, for convenience, the fields $\tilde{\varphi}$, $\tilde{\psi}$ and $\tilde{\mathcal{U}}$ and their charge conjugate $\tilde{\varphi}^\dagger$, $\tilde{\bar{\psi}}$ and $\tilde{\mathcal{U}}^\dagger$, respectively, are considered as independent field variables.

Let $\tilde{\mathcal{L}}'$ denotes any one of the Lagrangians (3.1) and $\tilde{\mathcal{L}}''$ (resp. $\tilde{\mathcal{L}}'''$) the corresponding to it Lagrangian given via (3.3) (resp. (3.4)). Physically the difference between $\tilde{\mathcal{L}}'$ and $\tilde{\mathcal{L}}''$ is that the particles for $\tilde{\mathcal{L}}'$ are antiparticles for $\tilde{\mathcal{L}}''$ and *vice versa*. Both of the Lagrangians $\tilde{\mathcal{L}}'$ and $\tilde{\mathcal{L}}''$ are *not* charge symmetric, i.e. the arising from them theories are not invariant under the change particle$\leftrightarrow$antiparticle (or, in mathematical terms, under some of the changes $\tilde{\varphi} \leftrightarrow \tilde{\varphi}^\dagger$, $\tilde{\psi} \leftrightarrow \tilde{\bar{\psi}}$, $\tilde{\mathcal{U}} \leftrightarrow \tilde{\mathcal{U}}^\dagger$) unless some additional hypotheses are made. Contrary to this, the Lagrangian $\tilde{\mathcal{L}}'''$ *is* charge symmetric and, consequently, the formalism on its base is invariant under the change particle$\leftrightarrow$antiparticle.[4]

The Euler-Lagrange equations for the Lagrangians $\tilde{\mathcal{L}}'$, $\tilde{\mathcal{L}}''$ and $\tilde{\mathcal{L}}'''$ happen to coincide [13–15]:[5]

$$\frac{\partial\tilde{\mathcal{L}}'}{\partial\chi} - \frac{\partial}{\partial x^\mu}\Big(\frac{\partial\tilde{\mathcal{L}}'}{\partial(\partial_\mu\chi)}\Big) \equiv \frac{\partial\tilde{\mathcal{L}}''}{\partial\chi} - \frac{\partial}{\partial x^\mu}\Big(\frac{\partial\tilde{\mathcal{L}}''}{\partial(\partial_\mu\chi)}\Big) \equiv \frac{\partial\tilde{\mathcal{L}}'''}{\partial\chi} - \frac{\partial}{\partial x^\mu}\Big(\frac{\partial\tilde{\mathcal{L}}'''}{\partial(\partial_\mu\chi)}\Big) = 0, \qquad (3.6)$$

where $\chi = \tilde{\varphi}, \tilde{\varphi}^\dagger, \tilde{\psi}, \tilde{\bar{\psi}}, \tilde{\mathcal{U}}, \tilde{\mathcal{U}}^\dagger$ for respectively scalar, spinor and vector field.

Since the creation and annihilation operators are defined only on the base of Euler-Lagrange equations [1, 3, 11–15], we can assert that these operators are identical for the Lagrangians $\tilde{\mathcal{L}}'$, $\tilde{\mathcal{L}}''$ and $\tilde{\mathcal{L}}'''$. We shall denote these operators by $a_s^\pm(\boldsymbol{k})$ and $a_s^{\dagger\,\pm}(\boldsymbol{k})$ with the convention that $a_s^+(\boldsymbol{k})$ (resp. $a_s^{\dagger\,+}(\boldsymbol{k})$) creates a particle (resp. antiparticle) with 4-momentum $(\sqrt{m^2c^2+\boldsymbol{k}^2}, \boldsymbol{k})$, polarization s (see below) and charge $(-q)$ (resp. $(+q)$)[6] and $a_s^{\dagger\,-}(\boldsymbol{k})$ (resp. $a_s^-(\boldsymbol{k})$) annihilates/destroys such a particle (resp. antiparticle). Here and henceforth $\boldsymbol{k} \in \mathbb{R}^3$ is interpreted as (anti)particle's 3-momentum and the values of the polarization index s depend on the field considered: $s = 1$ for a scalar field, $s = 1$ or $s = 1, 2$ for respectively massless ($m = 0$) or massive ($m \neq 0$) spinor field, and $s = 1, 2, 3$ for a vector field.[7] Since massless vector field's modes with $s = 3$ may enter only in the spin and orbital angular momenta operators [15], we, for convenience, shall assume that the polarization indices $s, t, \dots$ take the values from 1 to $2j+1-\delta_{0m}(1-\delta_{0j})$, where $j = 0, \frac{1}{2}, 1$ is the spin for scalar, spinor and vector field, respectively, and $\delta_{0m} := 1$ for $m = 0$ and $\delta_{0m} := 0$ for $m \neq 0$;[8] if the value $s = 3$ is important when $j = 1$ and $m = 0$, it will be commented/considered separately. Of course, the creation and annihilation operators are different for different fields; one should write, e.g., ${}_j a_s^\pm(\boldsymbol{k})$ for $a_s^\pm(\boldsymbol{k})$, but we shall not use such a complicated notation and will assume the dependence on j to be an implicit one.

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[4] Besides, under the same assumptions, the Lagrangian $\tilde{\mathcal{L}}'''$ does not admit quantization via anticommutators (commutators) for integer (half-integer) spin field, while $\tilde{\mathcal{L}}'$ and $\tilde{\mathcal{L}}''$ do not make difference between integer and half-integer spin fields.

[5] Rigorously speaking, the Euler-Lagrange equations for the Lagrangian (3.4b) are identities like $0 = 0$ — see [22]. However, bellow we shall handle this exceptional case as pointed in [14].

[6] For a neutral field, we put $q = 0$.

[7] For convenience, in [14], we have set $s = 0$ if $m = 0$ and $s = 1, 2$ if $m \neq 0$ for a spinor field. For a massless vector field, one may set $s = 1, 2$, thus eliminating the 'unphysical' value $s = 3$ for $m = 0$ — see [1, 11, 15]. In [13], for a scalar field, the notation $\varphi_0^\pm(\boldsymbol{k})$ and $\varphi_0^{\dagger\,\pm}(\boldsymbol{k})$ is used for $a_1^\pm(\boldsymbol{k})$ and $a_1^{\dagger\,\pm}(\boldsymbol{k})$, respectively.

[8] In this way the case $(j, s, m) = (1, 3, 0)$ is excluded from further considerations; if $(j, m) = (1, 0)$ and $q = 0$, the case considered further in this work corresponds to an electromagnetic field in Coulomb gauge, as the modes with $s = 3$ are excluded [15]. However, if the case $(j, s, m) = (1, 3, 0)$ is important for some reasons, the reader can easily obtain the corresponding results by applying the ones from [15].

The following settings will be frequently used throughout this chapter:

$$j := \begin{cases} 0 & \text{for scalar field} \\ \frac{1}{2} & \text{for spinor field} \\ 1 & \text{for vector field} \end{cases} \qquad \tau := \begin{cases} 1 & \text{for } q = 0 \text{ (neutral (Hermitian) field)} \\ 0 & \text{for } q \neq 0 \text{ (charged (non-Hermitian) field)} \end{cases} \tag{3.7}$$

$$\varepsilon := (-1)^{2j} = \begin{cases} +1 & \text{for integer } j \text{ (bose fields)} \\ -1 & \text{for half-integer } j \text{ (fermi fields)} \end{cases}$$

$$[A,B]_{\pm} := [A,B]_{\pm 1} := A \circ B \pm B \circ A, \tag{3.8}$$

where A and B are operators on the system's Hilbert space $\mathcal{F}$ of states.

The dynamical variables corresponding to $\tilde{\mathcal{L}}'$, $\tilde{\mathcal{L}}''$ and $\tilde{\mathcal{L}}'''$ are, however, completely different, unless some additional conditions are imposed on the Lagrangian formalism [13–15]. In particular, the momentum operators $\tilde{\mathcal{P}}_\mu^\omega$, charge operators $\tilde{\mathcal{Q}}^\omega$, spin operators $\tilde{\mathcal{S}}_{\mu\nu}^\omega$ and orbital operators $\tilde{\mathcal{L}}_{\mu\nu}^\omega$, where $\omega = {}', {}'', {}'''$, for these Lagrangians are [13–15]:

$$\tilde{\mathcal{P}}'_\mu = \frac{1}{1+\tau} \sum_{s=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{k}\, k_\mu|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}} \{a_s^{\dagger +}(\boldsymbol{k}) \circ a_s^-(\boldsymbol{k}) + \varepsilon a_s^{\dagger -}(\boldsymbol{k}) \circ a_s^+(\boldsymbol{k})\} \tag{3.9a}$$

$$\tilde{\mathcal{P}}''_\mu = \frac{1}{1+\tau} \sum_{s=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{k}\, k_\mu|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}} \{a_s^+(\boldsymbol{k}) \circ a_s^{\dagger -}(\boldsymbol{k}) + \varepsilon a_s^-(\boldsymbol{k}) \circ a_s^{\dagger +}(\boldsymbol{k})\} \tag{3.9b}$$

$$\tilde{\mathcal{P}}'''_\mu = \frac{1}{2(1+\tau)} \sum_{s=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{k}\, k_\mu|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}} \{[a_s^{\dagger +}(\boldsymbol{k}), a_s^-(\boldsymbol{k})]_\varepsilon + [a_s^+(\boldsymbol{k}), a_s^{\dagger -}(\boldsymbol{k})]_\varepsilon\} \tag{3.9c}$$

$$\tilde{\mathcal{Q}}' = +q \sum_{s=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{k} \{a_s^{\dagger +}(\boldsymbol{k}) \circ a_s^-(\boldsymbol{k}) - \varepsilon a_s^{\dagger -}(\boldsymbol{k}) \circ a_s^+(\boldsymbol{k})\} \tag{3.10a}$$

$$\tilde{\mathcal{Q}}'' = -q \sum_{s=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{k} \{a_s^+(\boldsymbol{k}) \circ a_s^{\dagger -}(\boldsymbol{k}) - \varepsilon a_s^-(\boldsymbol{k}) \circ a_s^{\dagger +}(\boldsymbol{k})\} \tag{3.10b}$$

$$\tilde{\mathcal{Q}}''' = \frac{1}{2} q \sum_{s=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{k} \{[a_s^{\dagger +}(\boldsymbol{k}), a_s^-(\boldsymbol{k})]_\varepsilon - [a_s^+(\boldsymbol{k}), a_s^{\dagger -}(\boldsymbol{k})]_\varepsilon\} \tag{3.10c}$$

$$\begin{aligned} \tilde{\mathcal{S}}'_{\mu\nu} = \frac{(-1)^{j-1/2} j\hbar}{1+\tau} \sum_{s,s'=1}^{2j+1-\delta_{0m}(1-\delta_{1j})} \int \mathrm{d}^3\boldsymbol{k} \{ & \sigma_{\mu\nu}^{ss',-}(\boldsymbol{k}) a_s^{\dagger +}(\boldsymbol{k}) \circ a_{s'}^-(\boldsymbol{k}) \\ & + \sigma_{\mu\nu}^{ss',+}(\boldsymbol{k}) a_s^{\dagger -}(\boldsymbol{k}) \circ a_{s'}^+(\boldsymbol{k})\} \end{aligned} \tag{3.11a}$$

$$\begin{aligned} \tilde{\mathcal{S}}''_{\mu\nu} = \varepsilon \frac{(-1)^{j-1/2} j\hbar}{1+\tau} \sum_{s,s'=1}^{2j+1-\delta_{0m}(1-\delta_{1j})} \int \mathrm{d}^3\boldsymbol{k} \{ & \sigma_{\mu\nu}^{ss',+}(\boldsymbol{k}) a_{s'}^+(\boldsymbol{k}) \circ a_s^{\dagger -}(\boldsymbol{k}) \\ & + \sigma_{\mu\nu}^{ss',-}(\boldsymbol{k}) a_{s'}^-(\boldsymbol{k}) \circ a_s^{\dagger +}(\boldsymbol{k})\} \end{aligned} \tag{3.11b}$$

$$\begin{aligned} \tilde{\mathcal{S}}'''_{\mu\nu} = \frac{(-1)^{j-1/2} j\hbar}{2(1+\tau)} \sum_{s,s'=1}^{2j+1-\delta_{0m}(1-\delta_{1j})} \int \mathrm{d}^3\boldsymbol{k} \{ & \sigma_{\mu\nu}^{ss',-}(\boldsymbol{k}) [a_s^{\dagger +}(\boldsymbol{k}), a_{s'}^-(\boldsymbol{k})]_\varepsilon \\ & + \sigma_{\mu\nu}^{ss',+}(\boldsymbol{k}) [a_s^{\dagger -}(\boldsymbol{k}), a_{s'}^+(\boldsymbol{k})]_\varepsilon\} \end{aligned} \tag{3.11c}$$

$$
\begin{aligned}
\tilde{\mathcal{L}}'_{\mu\nu} = & x_{0\,\mu}\,\tilde{\mathcal{P}}'_{\nu} - x_{0\,\nu}\,\tilde{\mathcal{P}}'_{\mu} \\
& + \frac{(-1)^{j-1/2} j\hbar}{1+\tau} \sum_{s,s'=1}^{2j+1-\delta_{0m}(1-\delta_{1j})} \int \mathrm{d}^3\boldsymbol{k} \{ l^{ss',-}_{\mu\nu}(\boldsymbol{k}) a^{\dagger\,+}_{s}(\boldsymbol{k}) \circ a^{-}_{s'}(\boldsymbol{k}) \\
& \qquad + l^{ss',+}_{\mu\nu}(\boldsymbol{k}) a^{\dagger\,-}_{s}(\boldsymbol{k}) \circ a^{+}_{s'}(\boldsymbol{k}) \} \\
& + \frac{\mathrm{i}\hbar}{2(1+\tau)} \sum_{s=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{k} \Big\{ a^{\dagger\,+}_{s}(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a^{-}_{s}(\boldsymbol{k}) \\
& - \varepsilon a^{\dagger\,-}_{s}(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a^{+}_{s}(\boldsymbol{k}) \Big\} \Big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}
\end{aligned}
\tag{3.12a}
$$

$$
\begin{aligned}
\tilde{\mathcal{L}}''_{\mu\nu} = & x_{0\,\mu}\,\tilde{\mathcal{P}}''_{\nu} - x_{0\,\nu}\,\tilde{\mathcal{P}}''_{\mu} \\
& + \varepsilon \frac{(-1)^{j-1/2} j\hbar}{1+\tau} \sum_{s,s'=1}^{2j+1-\delta_{0m}(1-\delta_{1j})} \int \mathrm{d}^3\boldsymbol{k} \{ l^{ss',+}_{\mu\nu}(\boldsymbol{k}) a^{+}_{s'}(\boldsymbol{k}) \circ a^{\dagger\,-}_{s}(\boldsymbol{k}) \\
& \qquad + l^{ss',-}_{\mu\nu}(\boldsymbol{k}) a^{-}_{s'}(\boldsymbol{k}) \circ a^{\dagger\,+}_{s}(\boldsymbol{k}) \} \\
& + \frac{\mathrm{i}\hbar}{2(1+\tau)} \sum_{s=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{k} \Big\{ a^{+}_{s}(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a^{\dagger\,-}_{s}(\boldsymbol{k}) \\
& - \varepsilon a^{-}_{s}(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a^{\dagger\,+}_{s}(\boldsymbol{k}) \Big\} \Big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}
\end{aligned}
\tag{3.12b}
$$

$$
\begin{aligned}
\tilde{\mathcal{L}}'''_{\mu\nu} = & x_{0\,\mu}\,\tilde{\mathcal{P}}'''_{\nu} - x_{0\,\nu}\,\tilde{\mathcal{P}}'''_{\mu} \\
& + \frac{(-1)^{j-1/2} j\hbar}{2(1+\tau)} \sum_{s,s'=1}^{2j+1-\delta_{0m}(1-\delta_{1j})} \int \mathrm{d}^3\boldsymbol{k} \{ l^{ss',-}_{\mu\nu}(\boldsymbol{k}) [a^{\dagger\,+}_{s}(\boldsymbol{k}), a^{-}_{s'}(\boldsymbol{k})]_{\varepsilon} \\
& \qquad + l^{ss',+}_{\mu\nu}(\boldsymbol{k}) [a^{\dagger\,-}_{s}(\boldsymbol{k}), a^{+}_{s'}(\boldsymbol{k})]_{\varepsilon} \} \\
& + \frac{\mathrm{i}\hbar}{4(1+\tau)} \sum_{s=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{k} \Big\{ a^{\dagger\,+}_{s}(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a^{-}_{s}(\boldsymbol{k}) \\
& - \varepsilon a^{-}_{s}(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a^{\dagger\,+}_{s}(\boldsymbol{k}) + a^{+}_{s}(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a^{\dagger\,-}_{s}(\boldsymbol{k}) \\
& - \varepsilon a^{\dagger\,-}_{s}(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a^{+}_{s}(\boldsymbol{k}) \Big\} \Big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}} .
\end{aligned}
\tag{3.12c}
$$

Here we have used the following notation: $(-1)^{n+1/2} := (-1)^n \mathrm{i}$ for all $n \in \mathbb{N}$ and $\mathrm{i} := +\sqrt{-1}$,

$$
\begin{aligned}
A(\boldsymbol{k}) k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} \circ B(\boldsymbol{k}) := & -\Big(k_\mu \frac{\partial A(\boldsymbol{k})}{\partial k^\nu} \Big) \circ B(\boldsymbol{k}) + \Big(A(\boldsymbol{k}) \circ k_\mu \frac{\partial B(\boldsymbol{k})}{\partial k^\nu} \Big) \\
& = k_\mu \Big(A(\boldsymbol{k}) \overleftrightarrow{\frac{\partial}{\partial k^\nu}} \circ B(\boldsymbol{k}) \Big)
\end{aligned}
\tag{3.13}
$$

for operators $A(\boldsymbol{k})$ and $B(\boldsymbol{k})$ having C^1 dependence on $\boldsymbol{k}$,[9] and $\sigma^{ss',\pm}_{\mu\nu}(\boldsymbol{k})$ and $l^{ss',\pm}_{\mu\nu}(\boldsymbol{k})$ are

[9] More generally, if $\omega\colon \{\mathcal{F} \to \mathcal{F}\} \to \{\mathcal{F} \to \mathcal{F}\}$ is a mapping on the operator space over the system's Hilbert space, we put $A \overleftrightarrow{\omega} \circ B := -\omega(A) \circ B + A \circ \omega(B)$ for any $A, B\colon \mathcal{F} \to \mathcal{F}$. Usually [2,12], this notation is used for $\omega = \partial_\mu$.

some functions of $\boldsymbol{k}$ such that[10]

$$\begin{aligned}
&\sigma_{\mu\nu}^{ss',\pm}(\boldsymbol{k}) = -\sigma_{\nu\mu}^{ss',\pm}(\boldsymbol{k}) \quad l_{\mu\nu}^{ss',\pm}(\boldsymbol{k}) = -l_{\nu\mu}^{ss',\pm}(\boldsymbol{k}) \\
&\sigma_{\mu\nu}^{ss',\pm}(\boldsymbol{k}) = l_{\nu\mu}^{ss',\pm}(\boldsymbol{k}) = 0 \qquad \text{for } j=0 \text{ (scalar field)} \\
&\sigma_{\mu\nu}^{ss',-}(\boldsymbol{k}) = -\sigma_{\mu\nu}^{ss',+}(\boldsymbol{k}) =: \sigma_{\mu\nu}^{ss'}(\boldsymbol{k}) = -\sigma_{\mu\nu}^{s's}(\boldsymbol{k}) = -\sigma_{\nu\mu}^{ss'}(\boldsymbol{k}) \qquad \text{for } j=1 \text{ (vector field)} \\
&l_{\mu\nu}^{ss',-}(\boldsymbol{k}) = -l_{\mu\nu}^{ss',+}(\boldsymbol{k}) =: l_{\mu\nu}^{ss'}(\boldsymbol{k}) = -l_{\mu\nu}^{s's}(\boldsymbol{k}) = -l_{\nu\mu}^{ss'}(\boldsymbol{k}) \qquad \text{for } j=1 \text{ (vector field).}
\end{aligned} \tag{3.14}$$

A technical remark must be make at this point. The equations (3.9)–(3.12) were derived in [13–15] under some additional conditions, represented by equations (2.6) and (2.7), which are considered bellow in Sect. 5 and ensure the effectiveness of the momentum picture of motion [21] used in [13–15]. However, as it is partially proved, e.g., in [1], when the quantities (3.9)–(3.12) are expressed via the Heisenberg creation and annihilation operators (see (2.9)), they remain valid, up to a phase factor, and without making the mentioned assumptions, i.e. these assumptions are needless when one works entirely in Heisenberg picture. For this reason, we shall consider (3.9)–(3.12) as pure consequence of the Lagrangian formalism.

We should emphasize, in (3.11) and (3.12) with $\tilde{\mathcal{S}}_{\mu\nu}^{\omega}$ and $\tilde{\mathcal{L}}_{\mu\nu}^{\omega}$, $\omega = {}^{\prime}, {}^{\prime\prime}, {}^{\prime\prime\prime}$, are denoted the spin and orbital, respectively, operators for $\tilde{\mathcal{L}}^{\omega}$, which are the spacetime-independent parts of the spin and orbital, respectively, angular momentum operators [14, 23]; if the last operators are denoted by $\underline{\tilde{\mathcal{S}}}_{\mu\nu}^{\omega}$ and $\underline{\tilde{\mathcal{L}}}_{\mu\nu}^{\omega}$, the total angular momentum operator of a system with Lagrangian $\tilde{\mathcal{L}}^{\omega}$ is [23]

$$\tilde{\mathcal{M}}_{\mu\nu}^{\omega} = \underline{\tilde{\mathcal{L}}}_{\mu\nu}^{\omega} + \underline{\tilde{\mathcal{S}}}_{\mu\nu}^{\omega} = \tilde{\mathcal{L}}_{\mu\nu}^{\omega} + \tilde{\mathcal{S}}_{\mu\nu}^{\omega}, \qquad \omega = {}^{\prime}, {}^{\prime\prime}, {}^{\prime\prime\prime} \tag{3.15}$$

and $\underline{\tilde{\mathcal{S}}}_{\mu\nu}^{\omega} = \tilde{\mathcal{S}}_{\mu\nu}^{\omega}$ (and hence $\underline{\tilde{\mathcal{L}}}_{\mu\nu}^{\omega} = \tilde{\mathcal{L}}_{\mu\nu}^{\omega}$) iff $\underline{\tilde{\mathcal{S}}}_{\mu\nu}^{\omega}$ is a conserved operator or, equivalently, iff the system's canonical energy-momentum tensor is symmetric.[11]

Going ahead (see Sect. 6), we would like to note that the expressions (3.9c) and, consequently, the Lagrangian $\tilde{\mathcal{L}}'''$ are the base from which the paracommutation relations were first derived [16].

And a last remark. Above we have expressed the dynamical variables in *Heisenberg picture via the creation and annihilation operators in momentum picture*. If one works entirely in Heisenberg picture, the operators (2.9), representing the creation and annihilation operators in Heisenberg picture, should be used. Besides, by virtue of the equations

$$(a_s^{\pm}(\boldsymbol{k}))^{\dagger} = a_s^{\dagger\,\mp}(\boldsymbol{k}) \quad (a_s^{\dagger\,\pm}(\boldsymbol{k}))^{\dagger} = a_s^{\mp}(\boldsymbol{k}) \tag{3.16}$$

$$\big(\tilde{a}_s^{\pm}(\boldsymbol{k})\big)^{\dagger} = \tilde{a}_s^{\dagger\,\mp}(\boldsymbol{k}) \quad \big(\tilde{a}_s^{\dagger\,\pm}(\boldsymbol{k})\big)^{\dagger} = \tilde{a}_s^{\mp}(\boldsymbol{k}), \tag{3.17}$$

some of the relations concerning $a_s^{\dagger\,\pm}(\boldsymbol{k})$, e.g. the Euler-Lagrange and Heisenberg equations, are consequences of the similar ones regarding $a_s^{\pm}(\boldsymbol{k})$. In view of (2.9), we shall consider (3.9)–(3.12) as obtained form the corresponding expressions in Heisenberg picture by making the replacements $\tilde{a}_s^{\pm}(\boldsymbol{k}) \mapsto a_s^{\pm}(\boldsymbol{k})$ and $\tilde{a}_s^{\dagger\,\pm}(\boldsymbol{k}) \mapsto a_s^{\dagger\,\pm}(\boldsymbol{k})$. So, (3.9)–(3.12) will have, up to a phase factor, a sense of dynamical variables in Heisenberg picture expressed via the creation/annihilation operators in momentum picture.

[10] For the explicit form of these functions, see [13–15]; see also equation (6.57) below.

[11] In [14,23] the spin and orbital operators are labeled with an additional left superscript $\circ$, which, for brevity, is omitted in the present work as in it only these operators, not $\underline{\tilde{\mathcal{S}}}_{\mu\nu}^{\omega}$ and $\underline{\tilde{\mathcal{L}}}_{\mu\nu}^{\omega}$, will be considered. Notice, the operators $\underline{\tilde{\mathcal{S}}}_{\mu\nu}^{\omega}$ and $\underline{\tilde{\mathcal{L}}}_{\mu\nu}^{\omega}$ are, generally, time-dependent while the orbital and spin ones are conserved, as a result of which the total angular momentum is a conserved operator too [14, 23].

4. On the uniqueness of the dynamical variables

Let $\mathcal{D} = \mathcal{P}_\mu, \mathcal{Q}, \mathcal{S}_{\mu\nu}, \mathcal{L}_{\mu\nu}$ denotes some dynamical variable, viz. the momentum, charge, spin, or orbital operator, of a system with Lagrangian $\mathcal{L}$. Since the Euler-Lagrange equations for the Lagrangians $\mathcal{L}'$, $\mathcal{L}''$ and $\mathcal{L}'''$ coincide (see (3.6)), we can assert that any field satisfying these equations admits at least three classes of conserved operators, viz. $\mathcal{D}'$, $\mathcal{D}''$ and $\mathcal{D}''' = \frac{1}{2}\big(\mathcal{D}' + \mathcal{D}''\big)$. Moreover, it can be proved that the Euler-Lagrange equations for the Lagrangian

$$\mathcal{L}_{\alpha,\beta} := \alpha\,\mathcal{L}' + \beta\,\mathcal{L}'' \qquad \alpha + \beta \neq 0 \tag{4.1}$$

do not depend on $\alpha, \beta \in \mathbb{C}$ and coincide with (3.6). Therefore there exists a two parameter family of conserved dynamical variables for these equations given via

$$\mathcal{D}_{\alpha,\beta} := \alpha\,\mathcal{D}' + \beta\,\mathcal{D}'' \qquad \alpha + \beta \neq 0. \tag{4.2}$$

Evidently $\mathcal{L}''' = \mathcal{L}_{\frac{1}{2},\frac{1}{2}}$ and $\mathcal{D}''' = \mathcal{D}_{\frac{1}{2},\frac{1}{2}}$. Since the Euler-Lagrange equations (3.6) are linear and homogeneous (in the cases considered), we can, without a lost of generality, restrict the parameters $\alpha, \beta \in \mathbb{C}$ to such that

$$\alpha + \beta = 1, \tag{4.3}$$

which can be achieved by an appropriate renormalization (by a factor $(\alpha+\beta)^{-1/2}$) of the field operators. Thus any field satisfying the Euler-Lagrange equations (3.6) admits the family $\mathcal{D}_{\alpha,\beta}$, $\alpha + \beta = 1$, of conserved operators. Obviously, this conclusion is valid if in (4.1) we replace the particular Lagrangians $\mathcal{L}'$ and $\mathcal{L}''$ (see (3.1) and (3.3)) with any two Lagrangians (of one and the same field variables) which lead to identical Euler-Lagrange equations. However, the essential point in our case is that $\mathcal{L}'$ and $\mathcal{L}''$ do not differ only by a full divergence, as a result of which the operators $\mathcal{D}_{\alpha,\beta}$ are different for different pairs (α, β), $\alpha + \beta = 1$.[12]

Since one expects a physical system to possess uniquely defined dynamical characteristics, e.g. energy and total angular momentum, and the Euler-Lagrange equations are considered (in the framework of Lagrangian formalism) as the ones governing the spacetime evolution of the system considered, the problem arises when the dynamical operators $\mathcal{D}_{\alpha,\beta}$, $\alpha+\beta = 1$, are independent of the particular choice of α and β, i.e. of the initial Lagrangian one starts off. Simple calculation show that the operators (4.2), under the condition (4.3), are independent of the particular values of the parameters α and β if and only if

$$\mathcal{D}' = \mathcal{D}''. \tag{4.4}$$

Some consequences of the condition(s) (4.4) will be considered below, as well as possible ways for satisfying these restrictions on the Lagrangian formalism.

Combining (3.9)–(3.12) with (4.4), for respectively $\mathcal{D} = \mathcal{P}_\mu, \mathcal{Q}, \mathcal{S}_{\mu\nu}, \mathcal{L}_{\mu\nu}$, we see that a free scalar, spinor or vector field has a uniquely defined dynamical variables if and only if the following equations are fulfilled:

$$\begin{split}\sum_{s=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{k}\, k_\mu\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}} \big\{ a_s^{\dagger\,+}(\boldsymbol{k}) \circ a_s^-(\boldsymbol{k}) - \varepsilon a_s^-(\boldsymbol{k}) \circ a_s^{\dagger\,+}(\boldsymbol{k}) \\ - a_s^+(\boldsymbol{k}) \circ a_s^{\dagger\,-}(\boldsymbol{k}) + \varepsilon a_s^{\dagger\,-}(\boldsymbol{k}) \circ a_s^+(\boldsymbol{k}) \big\} = 0\end{split} \tag{4.5}$$

[12] Note, no commutativity or some commutation relations between the field operators and their charge (or Hermitian) conjugate are presupposed, i.e., at the moment, we work in a theory without such relations and normal ordering.

$$q \times \sum_{s=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{k} \, \{ a_s^{\dagger\,+}(\boldsymbol{k}) \circ a_s^-(\boldsymbol{k}) - \varepsilon a_s^-(\boldsymbol{k}) \circ a_s^{\dagger\,+}(\boldsymbol{k}) \\ + a_s^+(\boldsymbol{k}) \circ a_s^{\dagger\,-}(\boldsymbol{k}) - \varepsilon a_s^{\dagger\,-}(\boldsymbol{k}) \circ a_s^+(\boldsymbol{k}) \} = 0 \quad (4.6)$$

$$\sum_{s,s'=1}^{2j+1-\delta_{0m}(1-\delta_{1j})} \int \mathrm{d}^3\boldsymbol{k} \, \{ \sigma_{\mu\nu}^{ss',-}(\boldsymbol{k}) a_s^{\dagger\,+}(\boldsymbol{k}) \circ a_{s'}^-(\boldsymbol{k}) - \varepsilon \sigma_{\mu\nu}^{ss',-}(\boldsymbol{k}) a_{s'}^-(\boldsymbol{k}) \circ a_s^{\dagger\,+}(\boldsymbol{k}) \\ - \varepsilon \sigma_{\mu\nu}^{ss',+}(\boldsymbol{k}) a_{s'}^+(\boldsymbol{k}) \circ a_s^{\dagger\,-}(\boldsymbol{k}) + \sigma_{\mu\nu}^{ss',+}(\boldsymbol{k}) a_s^{\dagger\,-}(\boldsymbol{k}) \circ a_{s'}^+(\boldsymbol{k}) \} = 0 \quad (4.7)$$

$$\sum_{s,s'=1}^{2j+1-\delta_{0m}(1-\delta_{1j})} \int \mathrm{d}^3\boldsymbol{k} \, \{ l_{\mu\nu}^{ss',-}(\boldsymbol{k}) a_s^{\dagger\,+}(\boldsymbol{k}) \circ a_{s'}^-(\boldsymbol{k}) - \varepsilon l_{\mu\nu}^{ss',-}(\boldsymbol{k}) a_{s'}^-(\boldsymbol{k}) \circ a_s^{\dagger\,+}(\boldsymbol{k}) \\ - \varepsilon l_{\mu\nu}^{ss',+}(\boldsymbol{k}) a_{s'}^+(\boldsymbol{k}) \circ a_s^{\dagger\,-}(\boldsymbol{k}) + l_{\mu\nu}^{ss',+}(\boldsymbol{k}) a_s^{\dagger\,-}(\boldsymbol{k}) \circ a_{s'}^+(\boldsymbol{k}) \} \\ + \frac{1}{2} \sum_{s=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{k} \, \Big\{ a_s^{\dagger\,+}(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a_s^-(\boldsymbol{k}) + \varepsilon a_s^-(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a_s^{\dagger\,+}(\boldsymbol{k}) \\ - a_s^+(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a_s^{\dagger\,-}(\boldsymbol{k}) - \varepsilon a_s^{\dagger\,-}(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a_s^+(\boldsymbol{k}) \Big\} \Big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}} = 0. \quad (4.8)$$

In (4.6) is retained the constant factor q as in the neutral case it is equal to zero and, consequently, the equation (4.6) reduces to identity.

Since the Euler-Lagrange equations do not impose some restrictions on the creation and annihilation operators, the equations (4.5)–(4.8) can be regarded as subsidiary conditions on the Lagrangian formalism and can serve as equations for (partial) determination of the creation and annihilation operators. The system of integral equations (4.5)–(4.8) is quite complicated and we are not going to investigate it in the general case. Below we shall restrict ourselves to analysis of only those solutions of (4.5)–(4.8), if any, for which the integrands in (4.5)–(4.8) vanish. This means that we shall replace the system of *integral* equations (4.5)–(4.8) with respect to creation and annihilation operators with the following system of *algebraic* equations (do not sum over s and s' in (4.12) and (4.13)!):

$$a_s^{\dagger\,+}(\boldsymbol{k}) \circ a_s^-(\boldsymbol{k}) - \varepsilon a_s^-(\boldsymbol{k}) \circ a_s^{\dagger\,+}(\boldsymbol{k}) - a_s^+(\boldsymbol{k}) \circ a_s^{\dagger\,-}(\boldsymbol{k}) + \varepsilon a_s^{\dagger\,-}(\boldsymbol{k}) \circ a_s^+(\boldsymbol{k}) = 0 \quad (4.9)$$

$$a_s^{\dagger\,+}(\boldsymbol{k}) \circ a_s^-(\boldsymbol{k}) - \varepsilon a_s^-(\boldsymbol{k}) \circ a_s^{\dagger\,+}(\boldsymbol{k}) + a_s^+(\boldsymbol{k}) \circ a_s^{\dagger\,-}(\boldsymbol{k}) - \varepsilon a_s^{\dagger\,-}(\boldsymbol{k}) \circ a_s^+(\boldsymbol{k}) = 0 \quad \text{if } q \neq 0 \quad (4.10)$$

$$\Big\{ a_s^{\dagger\,+}(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a_s^-(\boldsymbol{k}) + \varepsilon a_s^-(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a_s^{\dagger\,+}(\boldsymbol{k}) \\ - a_s^+(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a_s^{\dagger\,-}(\boldsymbol{k}) - \varepsilon a_s^{\dagger\,-}(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a_s^+(\boldsymbol{k}) \Big\} \Big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}} = 0 \quad (4.11)$$

$$\sum_{s,s'} \{ \sigma_{\mu\nu}^{ss',-}(\boldsymbol{k}) a_s^{\dagger\,+}(\boldsymbol{k}) \circ a_{s'}^-(\boldsymbol{k}) - \varepsilon \sigma_{\mu\nu}^{ss',-}(\boldsymbol{k}) a_{s'}^-(\boldsymbol{k}) \circ a_s^{\dagger\,+}(\boldsymbol{k}) \\ - \varepsilon \sigma_{\mu\nu}^{ss',+}(\boldsymbol{k}) a_{s'}^+(\boldsymbol{k}) \circ a_s^{\dagger\,-}(\boldsymbol{k}) + \sigma_{\mu\nu}^{ss',+}(\boldsymbol{k}) a_s^{\dagger\,-}(\boldsymbol{k}) \circ a_{s'}^+(\boldsymbol{k}) \} = 0 \quad (4.12)$$

$$\sum_{s,s'}\{l^{ss',-}_{\mu\nu}(\boldsymbol{k})a^{\dagger\,+}_{s}(\boldsymbol{k})\circ a^{-}_{s'}(\boldsymbol{k}) - \varepsilon l^{ss',-}_{\mu\nu}(\boldsymbol{k})a^{-}_{s'}(\boldsymbol{k})\circ a^{\dagger\,+}_{s}(\boldsymbol{k})$$
$$- \varepsilon l^{ss',+}_{\mu\nu}(\boldsymbol{k})a^{+}_{s'}(\boldsymbol{k})\circ a^{\dagger\,-}_{s}(\boldsymbol{k}) + l^{ss',+}_{\mu\nu}(\boldsymbol{k})a^{\dagger\,-}_{s}(\boldsymbol{k})\circ a^{+}_{s'}(\boldsymbol{k})\} = 0 \quad (4.13)$$

Here: $s = 1,\dots,2j+1-\delta_{0m}(1-\delta_{0j})$ in (4.9)–(4.11) and $s,s' = 1,\dots,2j+1-\delta_{0m}(1-\delta_{1j})$ in (4.12) and (4.13). (Notice, by virtue of (3.14), the equations (4.12) and (4.13) are identically valid for $j = 0$, i.e. for scalar fields.) Since all polarization indices enter in (4.5) and (4.6) on equal footing, we do not sum over s in (4.9)–(4.11). But in (4.12) and (4.13) we have retain the summation sign as the modes with definite polarization cannot be singled out in the general case. One may obtain weaker versions of (4.9)–(4.13) by summing in them over the polarization indices, but we shall not consider these conditions below regardless of the fact that they also ensure uniqueness of the dynamical variables.

At first, consider the equations (4.9)–(4.11). Since for a neutral field, $q = 0$, we have $a^{\dagger\,\pm}_{s}(\boldsymbol{k}) = a^{\pm}_{s}(\boldsymbol{k})$, which physically means coincidence of field's particles and antiparticles, the equations (4.9)–(4.11) hold identically in this case.

Let consider now the case $q \neq 0$, i.e. the investigated field to be charged one. Using the standard notation (cf. (3.8))

$$[A,B]_{\eta} := A\circ B + \eta B\circ A, \tag{4.14}$$

for operators A and B and $\eta\in\mathbb{C}$, we rewrite (4.9) and (4.10) as

$$[a^{\dagger\,+}_{s}(\boldsymbol{k}), a^{-}_{s}(\boldsymbol{k})]_{-\varepsilon} - [a^{+}_{s}(\boldsymbol{k}), a^{\dagger\,-}_{s}(\boldsymbol{k})]_{-\varepsilon} = 0 \tag{4.9$'$}$$
$$[a^{\dagger\,+}_{s}(\boldsymbol{k}), a^{-}_{s}(\boldsymbol{k})]_{-\varepsilon} + [a^{+}_{s}(\boldsymbol{k}), a^{\dagger\,-}_{s}(\boldsymbol{k})]_{-\varepsilon} = 0 \quad \text{if } q\neq 0, \tag{4.10$'$}$$

which are equivalent to

$$[a^{\dagger\,\pm}_{s}(\boldsymbol{k}), a^{\mp}_{s}(\boldsymbol{k})]_{-\varepsilon} = 0 \quad \text{if } q\neq 0. \tag{4.15}$$

Differentiating (4.15) and inserting the result into (4.11), one can verify that (4.11) is tantamount to

$$\Big\{\Big[a^{\dagger\,+}_{s}(\boldsymbol{k}), \Big(k_\mu\frac{\partial}{\partial k^\nu} - k_\nu\frac{\partial}{\partial k^\mu}\Big)\circ a^{-}_{s}(\boldsymbol{k})\Big]_{-\varepsilon}$$
$$- \Big[a^{+}_{s}(\boldsymbol{k}), \Big(k_\mu\frac{\partial}{\partial k^\nu} - k_\nu\frac{\partial}{\partial k^\mu}\Big)\circ a^{\dagger\,-}_{s}(\boldsymbol{k})\Big]_{-\varepsilon}\Big\}\Big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}} = 0 \quad \text{if } q\neq 0, \tag{4.16}$$

Consider now (4.12) and (4.13). By means of the shorthand (4.14), they read

$$\sum_{s,s'}\{\sigma^{ss',-}_{\mu\nu}(\boldsymbol{k})[a^{\dagger\,+}_{s}(\boldsymbol{k}), a^{-}_{s'}(\boldsymbol{k})]_{-\varepsilon} + \sigma^{ss',+}_{\mu\nu}(\boldsymbol{k})[a^{\dagger\,-}_{s}(\boldsymbol{k}), a^{+}_{s'}(\boldsymbol{k})]_{-\varepsilon}\} = 0 \tag{4.17}$$
$$\sum_{s,s'}\{l^{ss',-}_{\mu\nu}(\boldsymbol{k})[a^{\dagger\,+}_{s}(\boldsymbol{k}), a^{-}_{s'}(\boldsymbol{k})]_{-\varepsilon} + l^{ss',+}_{\mu\nu}(\boldsymbol{k})[a^{\dagger\,-}_{s}(\boldsymbol{k}), a^{+}_{s'}(\boldsymbol{k})]_{-\varepsilon}\} = 0. \tag{4.18}$$

For a scalar field, $j = 0$, these conditions hold identically, due to (3.14). But for $j \neq 0$ they impose new restrictions on the formalism. In particular, for vector fields, $j = 1$ and $\varepsilon = +1$ they are satisfied iff (see (3.14))

$$[a^{\dagger\,+}_{s}(\boldsymbol{k}), a^{-}_{s'}(\boldsymbol{k})]_{-\varepsilon} - [a^{\dagger\,-}_{s}(\boldsymbol{k}), a^{+}_{s'}(\boldsymbol{k})]_{-\varepsilon} - [a^{\dagger\,+}_{s'}(\boldsymbol{k}), a^{-}_{s}(\boldsymbol{k})]_{-\varepsilon} + [a^{\dagger\,-}_{s'}(\boldsymbol{k}), a^{+}_{s}(\boldsymbol{k})]_{-\varepsilon} = 0. \tag{4.19}$$

One can satisfy (4.17) and (4.18) if the following generalization of (4.15) holds

$$[a^{\dagger\,\pm}_{s}(\boldsymbol{k}), a^{\mp}_{s'}(\boldsymbol{k})]_{-\varepsilon} = 0. \tag{4.20}$$

For spin $j=\frac{1}{2}$ (and hence $\varepsilon=-1$ – see (3.7)), the conditions (4.12) and (4.13) cannot be simplified much, but, if one requires the vanishment of the operator coefficients after $\sigma_{\mu\nu}^{ss',\pm}(\boldsymbol{k})$ and $l_{\mu\nu}^{ss',\pm}(\boldsymbol{k})$, one gets

$$a_s^{\dagger\,\pm}(\boldsymbol{k})\circ a_{s'}^{\mp}(\boldsymbol{k})=0 \qquad j=\frac{1}{2} \quad \varepsilon=-1. \tag{4.21}$$

Excluding some special cases, e.g. neutral scalar field ($q=0$ and $j=0$), the equations (4.15) and (4.21) are unacceptable from many viewpoints. The main of them is that they are incompatible with the ordinary (anti)commutation relations (see, e.g., e.g. [1,11,12,18] or Sect. 6, in particular, equations (6.13) bellow); for example, (4.21) means that the acts of creation and annihilation of (anti)particles with identical characteristics should be mutually independent, which contradicts to the existing theory and experimental data.

Now we shall try another way for achieving uniqueness of the dynamical variables for free fields. Since in (4.9)–(4.13) naturally appear (anti)commutators between creation and annihilation operators and these (anti)commutators vanish under the standard normal ordering [1,11,12,18], one may suppose that the normally ordered expressions of the dynamical variables may coincide. Let us analyze this method.

Recall [1, 3, 11, 12], the normal ordering operator $\mathcal{N}$ (for free field theory) is a linear operator on the operator space of the system considered such that to a product (composition) $c_1\circ\dots\circ c_n$ of $n\in\mathbb{N}$ creation and/or annihilation operators $c_1,\dots\, c_n$ it assigns the operator $(-1)^f c_{\alpha_1}\circ\cdots c_{\alpha_n}$. Here $(\alpha_1,\dots,\alpha_n)$ is a permutation of $(1,\dots,n)$, all creation operators stand to the left of all annihilation ones, the relative order between the creation/annihilation operators is preserved, and f is equal to the number of transpositions among the fermion operators ($j=\frac{1}{2}$) needed to be achieved the just-described order ("normal order") of the operators $c_1\circ\dots\circ c_n$ in $c_{\alpha_1}\circ\cdots c_{\alpha_n}$.[13] In particular this means that

$$\begin{aligned}
&\mathcal{N}\big(a_s^{+}(\boldsymbol{k})\circ a_t^{\dagger\,-}(\boldsymbol{p})\big)=a_s^{+}(\boldsymbol{k})\circ a_t^{\dagger\,-}(\boldsymbol{p}) &&\mathcal{N}\big(a_s^{\dagger\,+}(\boldsymbol{k})\circ a_t^{-}(\boldsymbol{p})\big)=a_s^{\dagger\,+}(\boldsymbol{k})\circ a_t^{-}(\boldsymbol{p})\\
&\mathcal{N}\big(a_s^{-}(\boldsymbol{k})\circ a_t^{\dagger\,+}(\boldsymbol{p})\big)=\varepsilon a_t^{\dagger\,+}(\boldsymbol{p})\circ a_s^{-}(\boldsymbol{k}) &&\mathcal{N}\big(a_s^{\dagger\,-}(\boldsymbol{k})\circ a_t^{+}(\boldsymbol{p})\big)=\varepsilon a_t^{+}(\boldsymbol{p})\circ a_s^{\dagger\,-}(\boldsymbol{k})
\end{aligned} \tag{4.22}$$

and, consequently, we have

$$\mathcal{N}\big([a_s^{\dagger\,\pm}(\boldsymbol{k}),a_t^{\mp}(\boldsymbol{p})]_{-\varepsilon}\big)=0 \quad \mathcal{N}\big([a_s^{\pm}(\boldsymbol{k}),a_t^{\dagger\,\mp}(\boldsymbol{p})]_{-\varepsilon}\big)=0, \tag{4.23}$$

due to $\varepsilon:=(-1)^{2j}=\pm1$ (see (3.7)). (In fact, below only the equalities (4.22) and (4.23), not the general definition of a normal product, will be applied.)

Applying the normal ordering operator to (4.9′), (4.10′), (4.17) and (4.18), we, in view of (4.23), get the identity $0=0$, which means that the conditions (4.9), (4.10), (4.12) and (4.13) are identically satisfied after normal ordering. This is confirmed by the application of $\mathcal{N}$ to (3.9) and (3.10), which results respectively in (see (4.22))

$$\begin{aligned}
&\mathcal{N}(\tilde{\mathcal{P}}'_\mu)=\mathcal{N}(\tilde{\mathcal{P}}''_\mu)\\
&=\frac{1}{1+\tau}\sum_{s=1}^{2j+1-\delta_{0m}(1-\delta_{0j})}\int \mathrm{d}^3\boldsymbol{k}k_\mu|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}\{a_s^{\dagger\,+}(\boldsymbol{k})\circ a_s^{-}(\boldsymbol{k})+a_s^{+}(\boldsymbol{k})\circ a_s^{\dagger\,-}(\boldsymbol{k})\}
\end{aligned} \tag{4.24}$$

---

[13] We have slightly modified the definition given in [1,3,11,12] because no (anti)commutation relations are presented in our exposition till the moment. In this paper we do not concern the problem for elimination of the 'unphysical' operators $a_3^{\pm}(\boldsymbol{k})$ and $a_3^{\dagger\,\pm}(\boldsymbol{k})$ from the spin and orbital momentum operators when $j=1$; for details, see [15], where it is proved that, for an electromagnetic field, $j=1$ and $q=0$, one way to achieve this is by adding to the number f above the number of transpositions between $a_s^{\pm}(\boldsymbol{k})$, $s=1,2$, and $a_3^{\pm}(\boldsymbol{k})$ needed for getting normal order.

$$\mathcal{N}(\tilde{\mathcal{Q}}') = \mathcal{N}(\tilde{\mathcal{Q}}'') = \frac{1}{1+\tau} \sum_{s=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{k} \{ a_s^{\dagger\,+}(\boldsymbol{k}) \circ a_s^-(\boldsymbol{k}) - a_s^+(\boldsymbol{k}) \circ a_s^{\dagger\,-}(\boldsymbol{k}) \}. \tag{4.25}$$

Therefore the normal ordering ensures the uniqueness of the momentum and charge operators, if we redefine them respectively as

$$\tilde{\mathcal{P}}_\mu := \mathcal{N}(\tilde{\mathcal{P}}'_\mu) \quad \tilde{\mathcal{Q}} := \mathcal{N}(\tilde{\mathcal{Q}}'). \tag{4.26}$$

Putting $\omega_{\mu\nu} := k_\mu \frac{\partial}{\partial k^\nu} - k_\nu \frac{\partial}{\partial k^\mu}$ and using (4.22), one can verify that

$$\begin{aligned}
\mathcal{N}\big(a_s^+(\boldsymbol{k}) \overleftrightarrow{\omega_{\mu\nu}} \circ a_s^{\dagger\,-}(\boldsymbol{k})\big) &= a_s^+(\boldsymbol{k}) \overleftrightarrow{\omega_{\mu\nu}} \circ a_s^{\dagger\,-}(\boldsymbol{k}) \\
\mathcal{N}\big(a_s^{\dagger\,+}(\boldsymbol{k}) \overleftrightarrow{\omega_{\mu\nu}} \circ a_s^-(\boldsymbol{k})\big) &= a_s^{\dagger\,+}(\boldsymbol{k}) \overleftrightarrow{\omega_{\mu\nu}} \circ a_s^-(\boldsymbol{k}) \\
\mathcal{N}\big(a_s^-(\boldsymbol{k}) \overleftrightarrow{\omega_{\mu\nu}} \circ a_s^{\dagger\,+}(\boldsymbol{k})\big) &= -\varepsilon a_s^{\dagger\,+}(\boldsymbol{k}) \overleftrightarrow{\omega_{\mu\nu}} \circ a_s^-(\boldsymbol{k}) \\
\mathcal{N}\big(a_s^{\dagger\,-}(\boldsymbol{k}) \overleftrightarrow{\omega_{\mu\nu}} \circ a_s^+(\boldsymbol{k})\big) &= -\varepsilon a_s^+(\boldsymbol{k}) \overleftrightarrow{\omega_{\mu\nu}} \circ a_s^{\dagger\,-}(\boldsymbol{k}).
\end{aligned} \tag{4.27}$$

As a consequence of these equalities, the action of $\mathcal{N}$ on the l.h.s. of (4.11) vanishes. Combining this result with the mentioned fact that the normal ordering converts (4.12) and (4.13) into identities, we see that the normal ordering procedure ensures also uniqueness of the spin and orbital operators if we redefine them respectively as:

$$\begin{aligned}
\tilde{\mathcal{S}}_{\mu\nu} &:= \mathcal{N}(\tilde{\mathcal{S}}'_{\mu\nu}) := \mathcal{N}(\tilde{\mathcal{S}}''_{\mu\nu}) = \frac{(-1)^{j-1/2} j\hbar}{1+\tau} \\
&\times \sum_{s,s'=1}^{2j+1-\delta_{0m}(1-\delta_{1j})} \int \mathrm{d}^3\boldsymbol{k} \big\{ \sigma_{\mu\nu}^{ss',-}(\boldsymbol{k}) a_s^{\dagger\,+}(\boldsymbol{k}) \circ a_{s'}^-(\boldsymbol{k}) + \varepsilon \sigma_{\mu\nu}^{ss',+}(\boldsymbol{k}) a_{s'}^+(\boldsymbol{k}) \circ a_s^{\dagger\,-}(\boldsymbol{k}) \big\}
\end{aligned} \tag{4.28}$$

$$\begin{aligned}
\tilde{\mathcal{L}}_{\mu\nu} &:= \mathcal{N}(\tilde{\mathcal{L}}'_{\mu\nu}) := \mathcal{N}(\tilde{\mathcal{L}}''_{\mu\nu}) = x_{0\,\mu} \tilde{\mathcal{P}}_\nu - x_{0\,\nu} \tilde{\mathcal{P}}_\mu + \frac{(-1)^{j-1/2} j\hbar}{1+\tau} \\
&\times \sum_{s,s'=1}^{2j+1-\delta_{0m}(1-\delta_{1j})} \int \mathrm{d}^3\boldsymbol{k} \big\{ l_{\mu\nu}^{ss',-}(\boldsymbol{k}) a_s^{\dagger\,+}(\boldsymbol{k}) \circ a_{s'}^-(\boldsymbol{k}) + \varepsilon l_{\mu\nu}^{ss',+}(\boldsymbol{k}) a_{s'}^+(\boldsymbol{k}) \circ a_s^{\dagger\,-}(\boldsymbol{k}) \big\} \\
&+ \frac{\mathrm{i}\hbar}{2(1+\tau)} \sum_{s=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{k} \Big\{ a_s^{\dagger\,+}(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a_s^-(\boldsymbol{k}) \\
&+ a_s^+(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a_s^{\dagger\,-}(\boldsymbol{k}) \Big\} \Big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}},
\end{aligned} \tag{4.29}$$

where (3.14) was applied.

5. Heisenberg relations

The conserved operators, like momentum and charge operators, are often identified with the generators of the corresponding transformations under which the action operator is invariant [1, 3, 11, 12]. This leads to a number of commutation relations between the components of these operators and between them and the field operators. The relations of the letter set are known/referred as the *Heisenberg relations* or *equations*. Both kinds of commutation relations are from pure geometric origin and, consequently, are completely external to the Lagrangian formalism; one of the reasons being that the mentioned identification is, in general, unacceptable and may be carried out only on some subset of the system's Hilbert space of states [23, 24]. Therefore their validity in a pure Lagrangian theory is questionable and should be verified [11]. However, the considered relations are weaker conditions than the identification of the corresponding operators and there are strong evidences that these relations should be valid in a realistic quantum field theory [1,11]; e.g., the commutativity between the momentum and charge operators (see below (5.18)) expresses the experimental fact that the 4-momentum and charge of any system are simultaneously measurable quantities.

It is known [1, 11], in a pure Lagrangian approach, the field equations, which are usually identified with the Euler-Lagrange, [14] are the only restrictions on the field operators. Besides, these equations do not determine uniquely the field operators and the letter can be expressed through the creation and annihilation operators. Since the last operators are left completely arbitrary by a pure Lagrangian formalism, one is free to impose on them any system of *compatible* restrictions. The best known examples of this kind are the famous canonical (anti)commutation relations and their generalization, the so-called paracommutation relations [16, 18]. In general, the problem for compatibility of such subsidiary to the Lagrangian formalism system of restrictions with, for instance, the Heisenberg relations is open and requires particular investigation [11]. For example, even the canonical (anti)commutation relations for electromagnetic field in Coulomb gauge are incompatible with the Heisenberg equation involving the (total) angular momentum operator unless the gauge symmetry of this field is taken into account [11, § 84]. However, the (para)commutation relations are, by construction, compatible with the Heisenberg relations regarding momentum operator (see [16] or below Subsect. 6.1). The ordinary approach is to be imposed a system of equations on the creation and annihilation operators and, then, to be checked its compatibility with, e.g., the Heisenberg relations. In the next sections we shall investigate the opposite situation: assuming the validity of (some of) the Heisenberg equations, the possible restrictions on the creation and annihilation operators will be explored. For this purpose, below we briefly review the Heisenberg relations and other ones related to them.

Consider a system of quantum fields $\tilde{\varphi}_i(x)$, $i = 1, \dots, N \in \mathbb{N}$, where $\tilde{\varphi}_i(x)$ denote the components of all fields (and their Hermitian conjugates), and $\tilde{\mathcal{P}}_\mu$, $\tilde{\mathcal{Q}}$ and $\tilde{\mathcal{M}}_{\mu\nu}$ be its momentum, charge and (total) angular momentum operators, respectively. The Heisenberg relations/equations for these operators are [1, 3, 11, 12]

$$[\tilde{\varphi}_i(x), \tilde{\mathcal{P}}_\mu]_{\_} = \mathrm{i}\hbar \frac{\partial \tilde{\varphi}_i(x)}{\partial x^\mu} \tag{5.1}$$

$$[\tilde{\varphi}_i(x), \tilde{\mathcal{Q}}]_{\_} = e(\tilde{\varphi}_i) q \tilde{\varphi}_i(x) \tag{5.2}$$

$$[\tilde{\varphi}_i(x), \tilde{\mathcal{M}}_{\mu\nu}]_{\_} = \mathrm{i}\hbar \{ x_\mu \partial_\nu \tilde{\varphi}_i(x) - x_\nu \partial_\mu \tilde{\varphi}_i(x) \} + \mathrm{i}\hbar \sum_{i'} I^{j}_{i'\mu\nu} \tilde{\varphi}_{i'}(x). \tag{5.3}$$

Here: $q = \text{const}$ is the fields' charge, $e(\tilde{\varphi}_i) = 0$ if $\tilde{\varphi}_i^\dagger = \tilde{\varphi}_i$, $e(\tilde{\varphi}_i) = \pm 1$ if $\tilde{\varphi}_i^\dagger \neq \tilde{\varphi}_i$ with $e(\tilde{\varphi}_i) + e(\tilde{\varphi}_i^\dagger) = 0$, and the constants $I^{i'}_{i\mu\nu} = -I^{i'}_{i\nu\mu}$ characterize the transformation properties of the field operators under 4-rotations. (If $\varepsilon(\tilde{\varphi}_i) \neq 0$, it is a convention whether to put $\varepsilon(\tilde{\varphi}_i) = +1$ or $\varepsilon(\tilde{\varphi}_i) = -1$ for a fixed i.)

We would like to make some comments on (5.3). Since its r.h.s. is a sum of two operators, the first (second) characterizing the pure orbital (spin) angular momentum properties of the system considered, the idea arises to split (5.3) into two independent equations, one involving the orbital angular momentum operator and another concerning the spin angular momentum operator. This is supported by the observation that, it seems, no process is known for transforming orbital angular momentum into spin one and v.v. (without destroying the

[14] Recall, there are Lagrangians whose classical Euler-Lagrange equations are identities. However, their correct and rigorous treatment [22] reveals that they entail field equations which are mathematically correct and physically sensible.

system). So one may suppose the existence of operators $\tilde{\mathcal{M}}^{\text{or}}_{\mu\nu}$ and $\tilde{\mathcal{M}}^{\text{sp}}_{\mu\nu}$ such that

$$[\tilde{\varphi}_i(x),\, \tilde{\mathcal{M}}^{\text{or}}_{\mu\nu}]_{\_} = \mathrm{i}\hbar\{x_\mu\partial_\nu\,\tilde{\varphi}_i(x) - x_\nu\partial_\mu\,\tilde{\varphi}_i(x)\} \tag{5.4}$$

$$[\tilde{\varphi}_i(x),\, \tilde{\mathcal{M}}^{\text{sp}}_{\mu\nu}]_{\_} = \mathrm{i}\hbar\sum_{i'} I^{i'}_{i\mu\nu}\,\tilde{\varphi}_{i'}(x) \tag{5.5}$$

$$\tilde{\mathcal{M}}_{\mu\nu} = \tilde{\mathcal{M}}^{\text{or}}_{\mu\nu} + \tilde{\mathcal{M}}^{\text{sp}}_{\mu\nu}. \tag{5.6}$$

However, as particular calculations demonstrate [5,14,15], neither the spin (resp. orbital) nor the spin (resp. orbital) angular momentum operator is a suitable candidate for $\tilde{\mathcal{M}}^{\text{sp}}_{\mu\nu}$ (resp. $\tilde{\mathcal{M}}^{\text{or}}_{\mu\nu}$). If we assume the validity of (5.1), then equations (5.4) and (5.5) can be satisfied if we choose

$$\tilde{\mathcal{M}}^{\text{or}}_{\mu\nu}(x) = \tilde{\mathcal{L}}^{\text{ext}}_{\mu\nu} := x_\mu\,\tilde{\mathcal{P}}_\nu - x_\nu\,\tilde{\mathcal{P}}_\mu \tag{5.7}$$

$$\tilde{\mathcal{M}}^{\text{sp}}_{\mu\nu}(x) = \tilde{\mathcal{M}}^{(0)}_{\mu\nu}(x) := \tilde{\mathcal{M}}_{\mu\nu} - \tilde{\mathcal{L}}^{\text{ext}}_{\mu\nu} = \tilde{\mathcal{S}}_{\mu\nu} + \tilde{\mathcal{L}}_{\mu\nu} - \{x_\mu\,\tilde{\mathcal{P}}_\nu - x_\nu\,\tilde{\mathcal{P}}_\mu\} \tag{5.8}$$

with $\tilde{\mathcal{M}}_{\mu\nu}$ satisfying (5.3). These operators are *not* conserved ones. Such a representation is in agreement with the equations (3.12), according to which the operator (5.7) enters additively in the expressions for the orbital operator.[15] The physical sense of the operator (5.7) is that it represents the orbital angular momentum of the system due to its movement as a whole. Respectively, the operator (5.8) describes the system's angular momentum as a result of its internal movement and/or structure.

Since the spin (orbital) angular momentum is associated with the structure (movement) of a system, in the operator (5.8) are mixed the spin and orbital angular momenta. These quantities can be separated completely via the following representations of the operators $\mathcal{M}^{\text{or}}_{\mu\nu}$ and $\mathcal{M}^{\text{sp}}_{\mu\nu}$ in momentum picture (when (5.1) holds)

$$\mathcal{M}^{\text{or}}_{\mu\nu} = x_\mu\,\mathcal{P}_\nu - x_\mu\,\mathcal{P}_\mu + \mathcal{L}^{\text{int}}_{\mu\nu} \tag{5.9}$$

$$\mathcal{M}^{\text{sp}}_{\mu\nu} = \mathcal{M}_{\mu\nu} - (x_\mu\,\mathcal{P}_\nu - x_\mu\,\mathcal{P}_\mu) - \mathcal{L}^{\text{int}}_{\mu\nu}, \tag{5.10}$$

where $\mathcal{L}^{\text{int}}_{\mu\nu}$ describes the 'internal' orbital angular momentum of the system considered and depends on the Lagrangian we have started off. Generally said, $\mathcal{L}^{\text{int}}_{\mu\nu}$ is the part of the orbital angular momentum operator containing derivatives of the creation and annihilation operators. In particular, for the Lagrangians $\mathcal{L}'$, $\mathcal{L}''$ and $\mathcal{L}'''$ (see Sect. 3), the explicit forms of the operators (5.9) and (5.10) respectively are:

$$\begin{split}\mathcal{M}'^{\,\text{or}}_{\mu\nu} =& x_\mu\,\mathcal{P}'_\nu - x_\nu\,\mathcal{P}'_\mu \\ &+ \frac{\mathrm{i}\hbar}{2(1+\tau)}\sum_{s=1}^{2j+1-\delta_{0m}(1-\delta_{0j})}\int \mathrm{d}^3\boldsymbol{k}\Big\{a^{\dagger\,+}_s(\boldsymbol{k})\Big(k_\mu\frac{\overleftrightarrow{\partial}}{\partial k^\nu} - k_\nu\frac{\overleftrightarrow{\partial}}{\partial k^\mu}\Big)\circ a^-_s(\boldsymbol{k}) \\ &- \varepsilon a^{\dagger\,-}_s(\boldsymbol{k})\Big(k_\mu\frac{\overleftrightarrow{\partial}}{\partial k^\nu} - k_\nu\frac{\overleftrightarrow{\partial}}{\partial k^\mu}\Big)\circ a^+_s(\boldsymbol{k})\Big\}\Big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}\end{split} \tag{5.11a}$$

$$\begin{split}\mathcal{M}''^{\,\text{or}}_{\mu\nu} =& x_\mu\,\mathcal{P}''_\nu - x_\nu\,\mathcal{P}''_\mu \\ &+ \frac{\mathrm{i}\hbar}{2(1+\tau)}\sum_{s=1}^{2j+1-\delta_{0m}(1-\delta_{0j})}\int \mathrm{d}^3\boldsymbol{k}\Big\{a^+_s(\boldsymbol{k})\Big(k_\mu\frac{\overleftrightarrow{\partial}}{\partial k^\nu} - k_\nu\frac{\overleftrightarrow{\partial}}{\partial k^\mu}\Big)\circ a^{\dagger\,-}_s(\boldsymbol{k}) \\ &- \varepsilon a^-_s(\boldsymbol{k})\Big(k_\mu\frac{\overleftrightarrow{\partial}}{\partial k^\nu} - k_\nu\frac{\overleftrightarrow{\partial}}{\partial k^\mu}\Big)\circ a^{\dagger\,+}_s(\boldsymbol{k})\Big\}\Big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}\end{split} \tag{5.11b}$$

[15] This is evident in the momentum picture of motion, in which x_μ stands for $x_{0\,\mu}$ in (3.12) — see [13–15].

$$
\begin{aligned}
\mathcal{M}'''^{\,\mathrm{or}}_{\mu\nu} =& x_\mu \mathcal{P}'''_\nu - x_\nu \mathcal{P}'''_\mu \\
&+ \frac{\mathrm{i}\hbar}{4(1+\tau)} \sum_{s=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{k} \Big\{ a_s^{\dagger\,+}(\boldsymbol{k}) \Big(k_\mu \frac{\overleftrightarrow{\partial}}{\partial k^\nu} - k_\nu \frac{\overleftrightarrow{\partial}}{\partial k^\mu} \Big) \circ a_s^-(\boldsymbol{k}) \\
&- \varepsilon a_s^-(\boldsymbol{k}) \Big(k_\mu \frac{\overleftrightarrow{\partial}}{\partial k^\nu} - k_\nu \frac{\overleftrightarrow{\partial}}{\partial k^\mu} \Big) \circ a_s^{\dagger\,+}(\boldsymbol{k}) + a_s^+(\boldsymbol{k}) \Big(k_\mu \frac{\overleftrightarrow{\partial}}{\partial k^\nu} - k_\nu \frac{\overleftrightarrow{\partial}}{\partial k^\mu} \Big) \circ a_s^{\dagger\,-}(\boldsymbol{k}) \\
&- \varepsilon a_s^{\dagger\,-}(\boldsymbol{k}) \Big(k_\mu \frac{\overleftrightarrow{\partial}}{\partial k^\nu} - k_\nu \frac{\overleftrightarrow{\partial}}{\partial k^\mu} \Big) \circ a_s^+(\boldsymbol{k}) \Big\} \Big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}} .
\end{aligned}
\tag{5.11c}
$$

$$
\begin{aligned}
\mathcal{M}'^{\,\mathrm{sp}}_{\mu\nu} = \frac{(-1)^{j-1/2} j\hbar}{1+\tau} \sum_{s,s'=1}^{2j+1-\delta_{0m}(1-\delta_{1j})} \int \mathrm{d}^3\boldsymbol{k} \{ & (\sigma_{\mu\nu}^{ss',-}(\boldsymbol{k}) + l_{\mu\nu}^{ss',-}(\boldsymbol{k})) a_s^{\dagger\,+}(\boldsymbol{k}) \circ a_{s'}^-(\boldsymbol{k}) \\
&+ (\sigma_{\mu\nu}^{ss',+}(\boldsymbol{k}) + l_{\mu\nu}^{ss',+}(\boldsymbol{k})) a_s^{\dagger\,-}(\boldsymbol{k}) \circ a_{s'}^+(\boldsymbol{k}) \}
\end{aligned}
\tag{5.12a}
$$

$$
\begin{aligned}
\mathcal{M}''^{\,\mathrm{sp}}_{\mu\nu} = \varepsilon \frac{(-1)^{j-1/2} j\hbar}{1+\tau} \sum_{s,s'=1}^{2j+1-\delta_{0m}(1-\delta_{1j})} \int \mathrm{d}^3\boldsymbol{k} \{ & (\sigma_{\mu\nu}^{ss',+}(\boldsymbol{k}) + l_{\mu\nu}^{ss',+}(\boldsymbol{k})) a_{s'}^+(\boldsymbol{k}) \circ a_s^{\dagger\,-}(\boldsymbol{k}) \\
&+ (\sigma_{\mu\nu}^{ss',-}(\boldsymbol{k}) + \sigma_{\mu\nu}^{ss',-}(\boldsymbol{k})) a_{s'}^-(\boldsymbol{k}) \circ a_s^{\dagger\,+}(\boldsymbol{k}) \}
\end{aligned}
\tag{5.12b}
$$

$$
\begin{aligned}
\mathcal{M}'''^{\,\mathrm{sp}}_{\mu\nu} = \frac{(-1)^{j-1/2} j\hbar}{2(1+\tau)} \sum_{s,s'=1}^{2j+1-\delta_{0m}(1-\delta_{1j})} \int \mathrm{d}^3\boldsymbol{k} \{ & (\sigma_{\mu\nu}^{ss',-}(\boldsymbol{k}) + l_{\mu\nu}^{ss',-}(\boldsymbol{k})) [a_s^{\dagger\,+}(\boldsymbol{k}), a_{s'}^-(\boldsymbol{k})]_\varepsilon \\
&+ (\sigma_{\mu\nu}^{ss',+}(\boldsymbol{k}) + l_{\mu\nu}^{ss',+}(\boldsymbol{k})) [a_s^{\dagger\,-}(\boldsymbol{k}), a_{s'}^+(\boldsymbol{k})]_\varepsilon \}.
\end{aligned}
\tag{5.12c}
$$

Obviously (see Sect. 2), the equations (5.12) have the same form in Heisenberg picture in terms of the operators (2.9) (only tildes over $\mathcal{M}$ and a must be added), but the equations (5.11) change substantially due to the existence of derivatives of the creation and annihilation operators in them [13–15]:

$$
\begin{aligned}
\tilde{\mathcal{M}}'^{\,\mathrm{or}}_{\mu\nu} =& \frac{\mathrm{i}\hbar}{2(1+\tau)} \sum_{s=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{k} \Big\{ \tilde{a}_s^{\dagger\,+}(\boldsymbol{k}) \Big(k_\mu \frac{\overleftrightarrow{\partial}}{\partial k^\nu} - k_\nu \frac{\overleftrightarrow{\partial}}{\partial k^\mu} \Big) \circ \tilde{a}_s^-(\boldsymbol{k}) \\
&- \varepsilon \tilde{a}_s^{\dagger\,-}(\boldsymbol{k}) \Big(k_\mu \frac{\overleftrightarrow{\partial}}{\partial k^\nu} - k_\nu \frac{\overleftrightarrow{\partial}}{\partial k^\mu} \Big) \circ \tilde{a}_s^+(\boldsymbol{k}) \Big\} \Big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}
\end{aligned}
\tag{5.13a}
$$

$$
\begin{aligned}
\tilde{\mathcal{M}}''^{\,\mathrm{or}}_{\mu\nu} =& \frac{\mathrm{i}\hbar}{2(1+\tau)} \sum_{s=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{k} \Big\{ \tilde{a}_s^+(\boldsymbol{k}) \Big(k_\mu \frac{\overleftrightarrow{\partial}}{\partial k^\nu} - k_\nu \frac{\overleftrightarrow{\partial}}{\partial k^\mu} \Big) \circ \tilde{a}_s^{\dagger\,-}(\boldsymbol{k}) \\
&- \varepsilon \tilde{a}_s^-(\boldsymbol{k}) \Big(k_\mu \frac{\overleftrightarrow{\partial}}{\partial k^\nu} - k_\nu \frac{\overleftrightarrow{\partial}}{\partial k^\mu} \Big) \circ \tilde{a}_s^{\dagger\,+}(\boldsymbol{k}) \Big\} \Big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}
\end{aligned}
\tag{5.13b}
$$

$$
\begin{aligned}
\tilde{\mathcal{M}}_{\mu\nu}^{\prime\prime\prime\,\mathrm{or}} = & \frac{\mathrm{i}\hbar}{4(1+\tau)} \sum_{s=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{k} \Big\{ \tilde{a}_s^{\dagger\,+}(\boldsymbol{k}) \Big(k_\mu \frac{\overleftrightarrow{\partial}}{\partial k^\nu} - k_\nu \frac{\overleftrightarrow{\partial}}{\partial k^\mu} \Big) \circ \tilde{a}_s^{-}(\boldsymbol{k}) \\
& - \varepsilon \tilde{a}_s^{-}(\boldsymbol{k}) \Big(k_\mu \frac{\overleftrightarrow{\partial}}{\partial k^\nu} - k_\nu \frac{\overleftrightarrow{\partial}}{\partial k^\mu} \Big) \circ \tilde{a}_s^{\dagger\,+}(\boldsymbol{k}) + \tilde{a}_s^{+}(\boldsymbol{k}) \Big(k_\mu \frac{\overleftrightarrow{\partial}}{\partial k^\nu} - k_\nu \frac{\overleftrightarrow{\partial}}{\partial k^\mu} \Big) \circ \tilde{a}_s^{\dagger\,-}(\boldsymbol{k}) \\
& - \varepsilon \tilde{a}_s^{\dagger\,-}(\boldsymbol{k}) \Big(k_\mu \frac{\overleftrightarrow{\partial}}{\partial k^\nu} - k_\nu \frac{\overleftrightarrow{\partial}}{\partial k^\mu} \Big) \circ \tilde{a}_s^{+}(\boldsymbol{k}) \Big\} \Big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}} .
\end{aligned}
\tag{5.13c}
$$

From (5.13) and (5.12) is clear that the operators $\tilde{\mathcal{M}}_{\mu\nu}^{\mathrm{or}}$ and $\tilde{\mathcal{M}}_{\mu\nu}^{\mathrm{sp}}$ so defined are *conserved* (contrary to (5.7) and (5.8)) and do not depend on the validity of the Heisenberg relations (5.1) (contrary to expressions (5.11) in momentum picture).

The problem for whether the operators (5.12) and (5.13) satisfy the equations (5.4) and (5.5), respectively, will be considered in Sect. 6.

There is an essential difference between (5.4) and (5.5): the equation (5.5) depends on the particular properties of the operators $\tilde{\varphi}_i(x)$ under 4-rotations via the coefficients $I_{i\mu\nu}^{i'}$ (see (5.25) below), while (5.4) does not depend on them. This is explicitly reflected in (5.11) and (5.12): the former set of equations is valid independently of the geometrical nature of the fields considered, while the latter one depends on it via the 'spin' ('polarization') functions $\sigma_{\mu\nu}^{ss',\pm}(\boldsymbol{k})$ and $l_{\mu\nu}^{ss',\pm}(\boldsymbol{k})$. Similar remark concerns (5.3), on one hand, and (5.1) and (5.2), on another hand: the particular form of (5.3) essentially depends on the geometric properties of $\tilde{\varphi}_i(x)$ under 4-rotations, the other equations being independent of them.

It should also be noted, the relation (5.3) does not hold for a canonically quantized electromagnetic field in Coulomb gauge unless some additional terms it its r.h.s., reflecting the gauge symmetry of the field, are taken into account [11, § 84].

As it was said above, the relations (5.1)–(5.3) are from pure geometrical origin. However, the last discussion, concerning (5.4)–(5.8), reveals that the terms in braces in (5.3) should be connected with the momentum operator in the (pure) Lagrangian approach. More precisely, on the background of equations (3.11a)–(3.12c), the Heisenberg relation (5.3) should be replaced with

$$
[\tilde{\varphi}_i(x), \tilde{\mathcal{M}}_{\mu\nu}]_{\_} = x_\mu[\tilde{\varphi}_i(x), \tilde{\mathcal{P}}_\nu]_{\_} - x_\nu[\tilde{\varphi}_i(x), \tilde{\mathcal{P}}_\mu]_{\_} + \mathrm{i}\hbar \sum_j I_{i\mu\nu}^{i'} \tilde{\varphi}_{i'}(x), \tag{5.14}
$$

which is equivalent to (5.3) if (5.1) is true. An advantage of the last equation is that it is valid in any picture of motion (in the same form) while (5.3) holds only in Heisenberg picture.[16] Obviously, (5.14) is equivalent to (5.5) with $\tilde{\mathcal{M}}_{\mu\nu}^{\mathrm{sp}}$ defined by (5.8).

The other kind of geometric relations mentioned at the beginning of this section are connected with the basic relations defining the Lie algebra of the Poincaré group [7, pp. 143–147], [8, sect. 7.1]. They require the fulfillment of the following equations between the components $\tilde{\mathcal{P}}_\mu$ of the momentum and $\tilde{\mathcal{M}}_{\mu\nu}$ of the angular momentum operators [3, 5, 7, 8]:

$$
[\tilde{\mathcal{P}}_\mu, \tilde{\mathcal{P}}_\nu]_{\_} = 0 \tag{5.15}
$$

$$
[\tilde{\mathcal{M}}_{\mu\nu}, \tilde{\mathcal{P}}_\lambda]_{\_} = -\mathrm{i}\hbar(\eta_{\lambda\mu}\tilde{\mathcal{P}}_\nu - \eta_{\lambda\nu}\tilde{\mathcal{P}}_\mu). \tag{5.16}
$$

$$
[\tilde{\mathcal{M}}_{\varkappa\lambda}, \tilde{\mathcal{M}}_{\mu\nu}]_{\_} = -\mathrm{i}\hbar\{\eta_{\varkappa\mu}\tilde{\mathcal{M}}_{\lambda\nu} - \eta_{\lambda\mu}\tilde{\mathcal{M}}_{\varkappa\nu} - \eta_{\varkappa\nu}\tilde{\mathcal{M}}_{\lambda\mu} + \eta_{\lambda\nu}\tilde{\mathcal{M}}_{\varkappa\mu}\}. \tag{5.17}
$$

We would like to pay attention to the minus sign in the multiplier $(-\mathrm{i}\hbar)$ in (5.16) and (5.17) with respect to the above references, where $\mathrm{i}\hbar$ stands instead of $-\mathrm{i}\hbar$ in these equations. When

[16] In other pictures of motion, generally, additional terms in the r.h.s. of (5.3) will appear, i.e. the functional form of the r.h.s. of (5.3) is not invariant under changes of the picture of motion, contrary to (5.14).

(a representation of) the Lie algebra of the Poincaré group is considered, this difference in the sign is insignificant as it can be absorbed into the definition of $\tilde{\mathcal{M}}_{\mu\nu}$. However, the change of the sign of the angular momentum operator, $\tilde{\mathcal{M}}_{\mu\nu} \mapsto -\tilde{\mathcal{M}}_{\mu\nu}$, will result in the change $\mathrm{i}\hbar \mapsto -\mathrm{i}\hbar$ in the r.h.s. of (5.3). This means that equations (5.15), (5.16) and (5.3), when considered together, require a suitable choice of the signs of the multiplier $\mathrm{i}\hbar$ in their right hand sides as these signs change simultaneously when $\tilde{\mathcal{M}}_{\mu\nu}$ is replaced with $-\tilde{\mathcal{M}}_{\mu\nu}$. Since equations (5.3), (5.16) and (5.17) hold, when $\tilde{\mathcal{M}}_{\mu\nu}$ is defined according to the Noether's theorem and the ordinary (anti)commutation relations are valid [13–15], we accept these equations in the way they are written above.

To the relations (5.15)–(5.17) should be added the equations [3, p. 78]

$$[\tilde{\mathcal{Q}}, \tilde{\mathcal{P}}_\mu]_{\_} = 0 \tag{5.18}$$

$$[\tilde{\mathcal{Q}}, \tilde{\mathcal{M}}_{\mu\nu}]_{\_} = 0, \tag{5.19}$$

which complete the algebra of observables and express, respectively, the translational and rotational invariance of the charge operator $\tilde{\mathcal{Q}}$; physically they mean that the charge and momentum or the charge and angular momentum are simultaneously measurable quantities.

Since the spin properties of a system are generally independent of its charge or momentum, one may also expect the validity of the relations[17]

$$[\tilde{\mathcal{S}}_{\mu\nu}, \tilde{\mathcal{P}}_\mu]_{\_} = 0 \tag{5.20}$$

$$[\tilde{\mathcal{S}}_{\mu\nu}, \tilde{\mathcal{Q}}]_{\_} = 0. \tag{5.21}$$

But, as the spin describes, in a sense, some of the rotational properties of the system, equality like $[\tilde{\mathcal{S}}_{\mu\nu}, \tilde{\mathcal{L}}_{\varkappa\lambda}]_{\_} = 0$ is not likely to hold. Indeed, the considerations in [13–15] reveal that (5.20) and (5.21), but not the last equation, are true in the framework of the Lagrangian formalism with added to it standard (anti)commutation relations. Notice, if (5.20) and (5.21) hold, then, respectively, (5.16) and (5.19) are equivalent to

$$[\tilde{\mathcal{L}}_{\mu\nu}, \tilde{\mathcal{P}}_\lambda]_{\_} = -\mathrm{i}\hbar(\eta_{\lambda\mu}\tilde{\mathcal{P}}_\nu - \eta_{\lambda\nu}\tilde{\mathcal{P}}_\mu). \tag{5.22}$$

$$[\tilde{\mathcal{Q}}, \tilde{\mathcal{L}}_{\mu\nu}]_{\_} = 0. \tag{5.23}$$

It is intuitively clear, not all of the commutation relations (5.1)–(5.3) and (5.15)–(5.21) are independent: if $\tilde{\mathcal{D}}$ denotes some of the operators $\tilde{\mathcal{P}}_\mu$, $\tilde{\mathcal{Q}}$, $\tilde{\mathcal{M}}_{\mu\nu}$, $\tilde{\mathcal{S}}_{\mu\nu}$ or $\tilde{\mathcal{L}}_{\mu\nu}$ and the commutators $[\tilde{\varphi}_i(x), \tilde{\mathcal{D}}]_{\_}$, $i = 1, \dots, N$, are known, then, in principle, one can calculate the commutators $[\Gamma(\tilde{\varphi}_1(x), \dots, \tilde{\varphi}_N(x)), \tilde{\mathcal{D}}]_{\_}$, where $\Gamma(\tilde{\varphi}_1(x), \dots, \tilde{\varphi}_N(x))$ is, for example, any function/functional bilinear in $\tilde{\varphi}_1(x), \dots, \tilde{\varphi}_N(x)$; to prove this fact, one should apply the identity $[A, B \circ C]_{\_} = [A, B]_{\_} \circ C + B \circ [A, C]_{\_}$ a suitable number of times. In particular, if $\tilde{\mathcal{D}}_1$ and $\tilde{\mathcal{D}}_2$ denote any two (distinct) operators of the dynamical variables, and $[\tilde{\varphi}_i(x), \tilde{\mathcal{D}}_1]_{\_}$ is known, then the commutator $[\tilde{\mathcal{D}}_1, \tilde{\mathcal{D}}_2]_{\_}$ can be calculated explicitly. For this reason, we can expect that:

(i) Equation (5.1) implies (5.15), (5.16), (5.18), (5.20) and (5.22).

(ii) Equation (5.2) implies (5.18), (5.19), (5.21), and (5.23).

(iii) Equation (5.3) implies (5.16), (5.17), and (5.19).

Besides, (5.3) may, possibly, entail equations like (5.17) with S or L for M, with an exception of $\tilde{\mathcal{M}}_{\mu\nu}$ in the l.h.s., i.e.

$$\begin{aligned} [\tilde{\mathcal{S}}_{\varkappa\lambda}, \tilde{\mathcal{M}}_{\mu\nu}]_{\_} &= -\mathrm{i}\hbar\{\eta_{\varkappa\mu}\tilde{\mathcal{S}}_{\lambda\nu} - \eta_{\lambda\mu}\tilde{\mathcal{S}}_{\varkappa\nu} - \eta_{\varkappa\nu}\tilde{\mathcal{S}}_{\lambda\mu} + \eta_{\lambda\nu}\tilde{\mathcal{S}}_{\varkappa\mu}\} \\ [\tilde{\mathcal{L}}_{\varkappa\lambda}, \tilde{\mathcal{M}}_{\mu\nu}]_{\_} &= -\mathrm{i}\hbar\{\eta_{\varkappa\mu}\tilde{\mathcal{L}}_{\lambda\nu} - \eta_{\lambda\mu}\tilde{\mathcal{L}}_{\varkappa\nu} - \eta_{\varkappa\nu}\tilde{\mathcal{L}}_{\lambda\mu} + \eta_{\lambda\nu}\tilde{\mathcal{L}}_{\varkappa\mu}\} \end{aligned} \tag{5.24}$$

---

[17] Recall, $\tilde{\mathcal{S}}_{\mu\nu}$ (resp. $\tilde{\mathcal{L}}_{\mu\nu}$) is the conserved spin (resp. orbital) operator, not the generally non-conserved spin (resp. orbital) angular momentum operator [23].

The validity of assertions (i)–(iii) above for free scalar, spinor and vector fields, when respectively

$$\tilde{\varphi}_i(x) \mapsto \tilde{\varphi}(x),\, \tilde{\varphi}^\dagger(x) \quad I^{i'}_{i\mu\nu} \mapsto I_{\mu\nu} = 0 \quad e(\tilde{\varphi}) = -e(\tilde{\varphi}^\dagger) = +1 \tag{5.25a}$$

$$\tilde{\varphi}_i(x) \mapsto \tilde{\psi}(x),\, \tilde{\breve{\psi}}(x) \quad I^{i'}_{i\mu\nu} \mapsto I_{\psi\mu\nu} = I_{\breve{\psi}\mu\nu} = -\frac{\mathrm{i}}{2}\sigma_{\mu\nu} \quad e(\tilde{\psi}) = -e(\tilde{\breve{\psi}}) = +1 \tag{5.25b}$$

$$\tilde{\varphi}_i(x) \mapsto \tilde{\mathcal{U}}_\mu(x),\, \tilde{\mathcal{U}}^\dagger_\mu(x) \quad I^{i'}_{i\mu\nu} \mapsto I^\sigma_{\rho\mu\nu} = I^{\dagger\sigma}_{\rho\mu\nu} = \delta^\sigma_\mu \eta_{\nu\rho} - \delta^\sigma_\nu \eta_{\mu\rho} \quad e(\tilde{\mathcal{U}}_\mu) = -e(\tilde{\mathcal{U}}^\dagger_\mu) = +1, \tag{5.25c}$$

where $\sigma^{\mu\nu} := \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]_{\_}$ with γ^μ being the Dirac γ-matrices [1, 25], is proved in [13–15], respectively. Besides, in *loc. cit.* is proved that equations (5.24) hold for scalar and vector fields, but not for a spinor field.[18]

Thus, we see that the Heisenberg relations (5.1)–(5.3) are stronger than the commutation relations (5.15)–(5.23), when imposed on the Lagrangian formalism as subsidiary restrictions.

6. Types of possible commutation relations

In a broad sense, by a *commutation relation* we shall understand any *algebraic* relation between the creation and annihilation operators imposed as subsidiary restriction on the Lagrangian formalism. In a narrow sense, the *commutation relations* are the equations (6.13), with $\varepsilon = -1$, written below and satisfied by the bose creation and annihilation operators. As *anticommutation relations* are known the equations (6.13), with $\varepsilon = +1$, written below and satisfied by the fermi creation and annihilation operators. The last two types of relations are often referred as the *bilinear commutation relations* [18]. Theoretically are possible also *trilinear commutation relations*, an example being the *paracommutation relations* [16, 18] represented below by equations (6.18) (or (6.20)).

Generally said, the commutation relations should be postulated. Alternatively, they could be derived from (equivalent to them) different assumptions added to the Lagrangian formalism. The purpose of this section is to be explored possible classes of commutation relations, which follow from some natural restrictions on the Lagrangian formalism that are consequences from the considerations in the previous sections. Special attention will be paid on some consequences of the charge symmetric Lagrangians as the free fields possess such a symmetry [1, 3, 11, 12].

As pointed in Sect 3, the Euler-Lagrange equations for the Lagrangians $\tilde{\mathcal{L}}'$, $\tilde{\mathcal{L}}''$ and $\tilde{\mathcal{L}}'''$ coincide and, in quantum field theory, the role of these equations is to be singled out the independent degrees of freedom of the fields in the form of creation and annihilation operators $a^\pm_s(\boldsymbol{k})$ and $a^{\dagger\,\pm}_s(\boldsymbol{k})$ (which are identical for $\tilde{\mathcal{L}}'$, $\tilde{\mathcal{L}}''$ and $\tilde{\mathcal{L}}'''$). Further specialization of these operators is provided by the commutation relations (in broad sense) which play a role of field equations in this situation (with respect to the mentioned operators).

Before proceeding on, we would like to simplify our notation. As a spin variable, s say, is always coupled with a 3-momentum one, $\boldsymbol{k}$ say, we shall use the letters l, m and n to denote pairs like $l = (s, \boldsymbol{k})$, $m = (t, \boldsymbol{p})$ and $n = (r, \boldsymbol{q})$. Equipped with this convention, we shall write, e.g., $a^\pm_l$ for $a^\pm_s(\boldsymbol{k})$ and $a^{\dagger\,\pm}_l$ for $a^{\dagger\,\pm}_s(\boldsymbol{k})$. We set $\delta_{lm} := \delta_{st}\delta^3(\boldsymbol{k} - \boldsymbol{p})$ and a summation sign like $\sum_l$ should be understood as $\sum_s \int \mathrm{d}^3\boldsymbol{k}$, where the range of the polarization variable s will be clear from the context (see, e.g., (3.9)–(3.12)).

[18] The problem for the validity of assertions (i)–(iii) or equations (5.24) in the general case of arbitrary fields (Lagrangians) is not a subject of the present work.

6.1. Restrictions related to the momentum operator

First of all, let us examine the consequences of the Heisenberg relation (5.1) involving the momentum operator. Since in terms of creation and annihilation operators it reads [1,13–15]

$$[a_s^{\pm}(\boldsymbol{k}),\, \mathcal{P}_\mu]_{\_} = \mp k_\mu a_s^{\pm}(\boldsymbol{k}) \quad [a_s^{\dagger\,\pm}(\boldsymbol{k}),\, \mathcal{P}_\mu]_{\_} = \mp k_\mu a_s^{\dagger\,\pm}(\boldsymbol{k}) \qquad k_0 = \sqrt{m^2c^2 + \boldsymbol{k}^2}, \tag{6.1}$$

the field equations in terms of creation and annihilation operators for the Lagrangians (3.1), (3.3) and (3.4) respectively are (see [13–15] or (6.1) and (3.9)):

$$\sum_{t=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int q_\mu\big|_{q_0=\sqrt{m^2c^2+\boldsymbol{q}^2}} \big\{ \big[a_s^{\pm}(\boldsymbol{k}), a_t^{\dagger\,+}(\boldsymbol{q}) \circ a_t^{-}(\boldsymbol{q}) + \varepsilon a_t^{\dagger\,-}(\boldsymbol{q}) \circ a_t^{+}(\boldsymbol{q})\big]_{\_} \\ \pm (1+\tau) a_s^{\pm}(\boldsymbol{k}) \delta_{st} \delta^3(\boldsymbol{k}-\boldsymbol{q}) \big\} \,\mathrm{d}^3\boldsymbol{q} = 0 \tag{6.2a}$$

$$\sum_{t=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int q_\mu\big|_{q_0=\sqrt{m^2c^2+\boldsymbol{q}^2}} \big\{ \big[a_s^{\dagger\,\pm}(\boldsymbol{k}), a_t^{\dagger\,+}(\boldsymbol{q}) \circ a_t^{-}(\boldsymbol{q}) + \varepsilon a_t^{\dagger\,-}(\boldsymbol{q}) \circ a_t^{+}(\boldsymbol{q})\big]_{\_} \\ \pm (1+\tau) a_s^{\dagger\,\pm}(\boldsymbol{k}) \delta_{st} \delta^3(\boldsymbol{k}-\boldsymbol{q}) \big\} \,\mathrm{d}^3\boldsymbol{q} = 0 \tag{6.2b}$$

$$\sum_{t=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int q_\mu\big|_{q_0=\sqrt{m^2c^2+\boldsymbol{q}^2}} \big\{ \big[a_s^{\pm}(\boldsymbol{k}), a_t^{+}(\boldsymbol{q}) \circ a_t^{\dagger\,-}(\boldsymbol{q}) + \varepsilon a_t^{-}(\boldsymbol{q}) \circ a_t^{\dagger\,+}(\boldsymbol{q})\big]_{\_} \\ \pm (1+\tau) a_s^{\pm}(\boldsymbol{k}) \delta_{st} \delta^3(\boldsymbol{k}-\boldsymbol{q}) \big\} \,\mathrm{d}^3\boldsymbol{q} = 0 \tag{6.3a}$$

$$\sum_{t=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int q_\mu\big|_{q_0=\sqrt{m^2c^2+\boldsymbol{q}^2}} \big\{ \big[a_s^{\dagger\,\pm}(\boldsymbol{k}), a_t^{+}(\boldsymbol{q}) \circ a_t^{\dagger\,-}(\boldsymbol{q}) + \varepsilon a_t^{-}(\boldsymbol{q}) \circ a_t^{\dagger\,+}(\boldsymbol{q})\big]_{\_} \\ \pm (1+\tau) a_s^{\dagger\,\pm}(\boldsymbol{k}) \delta_{st} \delta^3(\boldsymbol{k}-\boldsymbol{q}) \big\} \,\mathrm{d}^3\boldsymbol{q} = 0 \tag{6.3b}$$

$$\sum_{t=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int q_\mu\big|_{q_0=\sqrt{m^2c^2+\boldsymbol{q}^2}} \big\{ \big[a_s^{\pm}(\boldsymbol{k}), [a_t^{\dagger\,+}(\boldsymbol{q}), a_t^{-}(\boldsymbol{q})]_\varepsilon + [a_t^{+}(\boldsymbol{q}), a_t^{\dagger\,-}(\boldsymbol{q})]_\varepsilon\big]_{\_} \\ \pm (1+\tau) a_s^{\pm}(\boldsymbol{k}) \delta_{st} \delta^3(\boldsymbol{k}-\boldsymbol{q}) \big\} \,\mathrm{d}^3\boldsymbol{q} = 0 \tag{6.4a}$$

$$\sum_{t=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int q_\mu\big|_{q_0=\sqrt{m^2c^2+\boldsymbol{q}^2}} \big\{ \big[a_s^{\dagger\,\pm}(\boldsymbol{k}), [a_t^{\dagger\,+}(\boldsymbol{q}), a_t^{-}(\boldsymbol{q})]_\varepsilon + [a_t^{+}(\boldsymbol{q}), a_t^{\dagger\,-}(\boldsymbol{q})]_\varepsilon\big]_{\_} \\ \pm (1+\tau) a_s^{\dagger\,\pm}(\boldsymbol{k}) \delta_{st} \delta^3(\boldsymbol{k}-\boldsymbol{q}) \big\} \,\mathrm{d}^3\boldsymbol{q} = 0, \tag{6.4b}$$

where j and ε are given via (3.7), the generalized commutation function $[\cdot,\cdot]_\varepsilon$ is defined by (4.14), and the polarization indices take the values

$$s,t = 1,\dots,2j+1-\delta_{0m}(1-\delta_{0j}) = \begin{cases} 1 & \text{for } j=0 \text{ or for } j=\frac{1}{2} \text{ and } m=0 \\ 1,2 & \text{for } j=\frac{1}{2} \text{ and } m\neq 0 \text{ or for } j=1 \text{ and } m=0 \\ 1,2,3 & \text{for } j=1 \text{ and } m\neq 0 \end{cases}. \tag{6.5}$$

The "b" versions of the equations (6.2)–(6.4) are consequences of the "a" versions and the equalities

$$(a_l^{\pm})^\dagger = a_l^{\dagger\,\mp} \quad (a_l^{\dagger\,\pm}) = a_l^{\mp} \tag{6.6}$$

$$\big([A,B]_\eta\big)^\dagger = \eta[A^\dagger, B^\dagger]_\eta \qquad \text{for } [A,B]_\eta = \eta[B,A]_\eta \qquad \eta=\pm 1. \tag{6.7}$$

Applying (6.2)–(6.4) and the identity

$$[A, B\circ C]_{\_} = [A,B]_\eta \circ C - \eta B \circ [A,C]_\eta \qquad \text{for } \eta = \pm 1 \tag{6.8}$$

for the choice $\eta = -1$, one can prove by a direct calculation that

$$\begin{aligned} &[\tilde{\mathcal{P}}_\mu, \tilde{\mathcal{P}}_\nu]_{\_} = 0 \quad [\tilde{\mathcal{Q}}, \tilde{\mathcal{P}}_\mu]_{\_} = 0 \quad [\tilde{\mathcal{S}}_{\mu\nu}, \tilde{\mathcal{P}}_\lambda]_{\_} = 0 \\ &[\tilde{\mathcal{L}}_{\mu\nu}, \tilde{\mathcal{P}}_\lambda]_{\_} = -\mathrm{i}\hbar\{\eta_{\lambda\mu}\tilde{\mathcal{P}}_\nu - \eta_{\lambda\nu}\tilde{\mathcal{P}}_\mu\} \quad [\tilde{\mathcal{M}}_{\mu\nu}, \tilde{\mathcal{P}}_\lambda]_{\_} = -\mathrm{i}\hbar\{\eta_{\lambda\mu}\tilde{\mathcal{P}}_\nu - \eta_{\lambda\nu}\tilde{\mathcal{P}}_\mu\}, \end{aligned} \tag{6.9}$$

where the operators $\tilde{\mathcal{P}}_\mu$, $\tilde{\mathcal{Q}}$, $\tilde{\mathcal{S}}_{\mu\nu}$, $\tilde{\mathcal{L}}_{\mu\nu}$, and $\tilde{\mathcal{M}}_{\mu\nu}$ denote the momentum, charge, spin, orbital and total angular momentum operators, respectively, of the system considered and are calculated from one and the same initial Lagrangian. This result confirms the supposition, made in Sect. 5, that the assertion (i) before (5.24) holds for the fields investigated here.

Below we shall study only those solutions of (6.2)–(6.4) for which the integrands in them vanish, i.e. we shall replace the systems of *integral* equations (6.2)–(6.4) with the following systems of *algebraic* equations (see the above convention on the indices l and m and do not sum over indices repeated on one and the same level):

$$\big[a_l^\pm, a_m^{\dagger\,+} \circ a_m^- + \varepsilon a_m^{\dagger\,-} \circ a_m^+\big]_\_ \pm (1+\tau)\delta_{lm} a_l^\pm = 0 \tag{6.10a}$$

$$\big[a_l^{\dagger\,\pm}, a_m^{\dagger\,+} \circ a_m^- + \varepsilon a_m^{\dagger\,-} \circ a_m^+\big]_\_ \pm (1+\tau)\delta_{lm} a_l^{\dagger\,\pm} = 0 \tag{6.10b}$$

$$\big[a_l^\pm, a_m^+ \circ a_m^{\dagger\,-} + \varepsilon a_m^- \circ a_m^{\dagger\,+}\big]_\_ \pm (1+\tau)\delta_{lm} a_l^\pm = 0 \tag{6.11a}$$

$$\big[a_l^{\dagger\,\pm}, a_m^+ \circ a_m^{\dagger\,-} + \varepsilon a_m^- \circ a_m^{\dagger\,+}\big]_\_ \pm (1+\tau)\delta_{lm} a_l^{\dagger\,\pm} = 0 \tag{6.11b}$$

$$\big[a_l^\pm, [a_m^{\dagger\,+}, a_m^-]_\varepsilon + [a_m^+, a_m^{\dagger\,-}]_\varepsilon\big]_\_ \pm 2(1+\tau)\delta_{lm} a_l^\pm = 0 \tag{6.12a}$$

$$\big[a_l^{\dagger\,\pm}, [a_m^{\dagger\,+}, a_m^-]_\varepsilon + [a_m^+, a_m^{\dagger\,-}]_\varepsilon\big]_\_ \pm 2(1+\tau)\delta_{lm} a_l^{\dagger\,\pm} = 0. \tag{6.12b}$$

It seems, these are the most general and sensible *trilinear commutation relations* one may impose on the creation and annihilation operators.

First of all, we should mentioned that the *standard bilinear commutation relations*, viz. [1, 3, 11–15]

$$\begin{aligned} &[a_l^\pm, a_m^\pm]_{-\varepsilon} = 0 && [a_l^{\dagger\,\pm}, a_m^{\dagger\,\pm}]_{-\varepsilon} = 0 \\ &[a_l^\mp, a_m^\pm]_{-\varepsilon} = (\pm 1)^{2j+1}\tau\delta_{lm}\,\mathsf{id}_\mathcal{F} && [a_l^{\dagger\,\mp}, a_m^{\dagger\,\pm}]_{-\varepsilon} = (\pm 1)^{2j+1}\tau\delta_{lm}\,\mathsf{id}_\mathcal{F} \\ &[a_l^\pm, a_m^{\dagger\,\pm}]_{-\varepsilon} = 0 && [a_l^{\dagger\,\pm}, a_m^\pm]_{-\varepsilon} = 0 \\ &[a_l^\mp, a_m^{\dagger\,\pm}]_{-\varepsilon} = (\pm 1)^{2j+1}\delta_{lm}\,\mathsf{id}_\mathcal{F} && [a_l^{\dagger\,\mp}, a_m^\pm]_{-\varepsilon} = (\pm 1)^{2j+1}\delta_{lm}\,\mathsf{id}_\mathcal{F}, \end{aligned} \tag{6.13}$$

provide a solution of any one of the equations (6.10)–(6.12) in a sense that, due to (3.7) and (6.8), with $\eta = -\varepsilon$ any set of operators satisfying (6.13) converts (6.10)–(6.12) into identities.

Besides, this conclusion remains valid also if the normal ordering is taken into account, i.e. if, in this particular case, the changes $a_m^{\dagger\,-} \circ a_m^+ \mapsto \varepsilon a_m^+ \circ a_m^{\dagger\,-}$ and $a_m^- \circ a_m^{\dagger\,+} \mapsto \varepsilon a_m^{\dagger\,+} \circ a_m^-$ are made in (6.10)–(6.12).

Now we shall demonstrate how the trilinear relations (6.12) lead to the paracommutation relations. Equations (6.12) can be 'split' into different kinds of trilinear commutation relations into infinitely many ways. For example, the system of equations

$$\big[a_l^\pm, [a_m^+, a_m^{\dagger\,-}]_\varepsilon\big]_\_ \pm (1+\tau)\delta_{lm} a_l^\pm = 0 \tag{6.14a}$$

$$\big[a_l^\pm, [a_m^{\dagger\,+}, a_m^-]_\varepsilon\big]_\_ \pm (1+\tau)\delta_{lm} a_l^\pm = 0 \tag{6.14b}$$

$$\big[a_l^{\dagger\,\pm}, [a_m^+, a_m^{\dagger\,-}]_\varepsilon\big]_\_ \pm (1+\tau)\delta_{lm} a_l^{\dagger\,\pm} = 0 \tag{6.14c}$$

$$\big[a_l^{\dagger\,\pm}, [a_m^{\dagger\,+}, a_m^-]_\varepsilon\big]_\_ \pm (1+\tau)\delta_{lm} a_l^{\dagger\,\pm} = 0 \tag{6.14d}$$

provides an evident solution of (6.12). However, it is a simple algebra to be seen that these relations are incompatible with the standard (anti)commutation relations (6.13) and, in this sense, are not suitable as subsidiary restrictions on the Lagrangian formalism. For our purpose, the equations

$$\bigl[a_l^+, [a_m^+, a_m^{\dagger\,-}]_\varepsilon\bigr]_- + 2\delta_{lm} a_l^+ = 0 \tag{6.15a}$$

$$\bigl[a_l^+, [a_m^{\dagger\,+}, a_m^-]_\varepsilon\bigr]_- + 2\tau\delta_{lm} a_l^+ = 0 \tag{6.15b}$$

$$\bigl[a_l^-, [a_m^+, a_m^{\dagger\,-}]_\varepsilon\bigr]_- - 2\tau\delta_{lm} a_l^- = 0 \tag{6.15c}$$

$$\bigl[a_l^-, [a_m^{\dagger\,+}, a_m^-]_\varepsilon\bigr]_- - 2\delta_{lm} a_l^- = 0 \tag{6.15d}$$

and their Hermitian conjugate provide a solution of (6.12), which is compatible with (6.13), i.e. if (6.13) hold, the equations (6.15) are converted into identities.

The idea of the paraquantization is in the following generalization of (6.15)

$$\bigl[a_l^+, [a_m^+, a_n^{\dagger\,-}]_\varepsilon\bigr]_- + 2\delta_{ln} a_m^+ = 0 \tag{6.16a}$$

$$\bigl[a_l^+, [a_m^{\dagger\,+}, a_n^-]_\varepsilon\bigr]_- + 2\tau\delta_{ln} a_m^+ = 0 \tag{6.16b}$$

$$\bigl[a_l^-, [a_m^+, a_n^{\dagger\,-}]_\varepsilon\bigr]_- - 2\tau\delta_{lm} a_n^- = 0 \tag{6.16c}$$

$$\bigl[a_l^-, [a_m^{\dagger\,+}, a_n^-]_\varepsilon\bigr]_- - 2\delta_{lm} a_n^- = 0 \tag{6.16d}$$

which reduces to (6.15) for $n = m$ and is a generalization of (6.13) in a sense that any set of operators satisfying (6.13) converts (6.16) into identities, the opposite being generally not valid.[19]

Suppose that the field considered consists of a single sort of particles, e.g. electrons or photons, created by $b_l^\dagger := a_l^\dagger$ and annihilated by $b_l := a_l^{\dagger\,-}$. Then the equation Hermitian conjugated to (6.15a) reads

$$[b_l, [b_m^\dagger, b_m]_\varepsilon]_- = 2\delta_{lm} b_m. \tag{6.17}$$

This is the main relation from which the paper [16] starts. The basic *paracommutation relations* are [16–18, 26]:

$$[b_l, [b_m^\dagger, b_n]_\varepsilon]_- = 2\delta_{lm} b_n \tag{6.18a}$$

$$[b_l, [b_m, b_n]_\varepsilon]_- = 0. \tag{6.18b}$$

The first of them is a generalization (stronger version) of (6.17) by replacing the second index m with an arbitrary one, say n, and the second one is added (by "hands") in the theory as an additional assumption. Obviously, (6.18) are a solution of (6.15) and therefore of (6.12) in the considered case of a field consisting of only one sort of particles.

The equations (6.15) contain also the relativistic version of the paracommutation relations, when the existence of antiparticles must be respected [18, sec. 18.1]. Indeed, noticing that the field's particles (resp. antiparticles) are created by $b_l^\dagger := a_l^+$ (resp. $c_l^\dagger := a_l^{\dagger\,+}$) and annihilated by $b_l := a_l^{\dagger\,-}$ (resp. $c_l := a_l^-$), from (6.15) and the Hermitian conjugate to them equations, we get

$$[b_l, [b_m^\dagger, b_m]_\varepsilon]_- = 2\delta_{lm} b_m \qquad [c_l, [c_m^\dagger, c_m]_\varepsilon]_- = 2\delta_{lm} c_m \tag{6.19a}$$

$$[b_l^\dagger, [c_m^\dagger, c_m]_\varepsilon]_- = -2\tau\delta_{lm} b_m^\dagger \quad [c_l^\dagger, [b_m^\dagger, b_m]_\varepsilon]_- = -2\tau\delta_{lm} c_m^\dagger. \tag{6.19b}$$

Generalizing these equations in a way similar to the transition from (6.17) to (6.18), we obtain the *relativistic paracommutation relations* as (cf. (6.16))

$$[b_l, [b_m^\dagger, b_n]_\varepsilon]_- = 2\delta_{lm} b_n \qquad [b_l, [b_m, b_n]_\varepsilon]_- = 0 \tag{6.20a}$$

$$[c_l, [c_m^\dagger, c_n]_\varepsilon]_- = 2\delta_{lm} c_n \qquad [c_l, [c_m, c_n]_\varepsilon]_- = 0 \tag{6.20b}$$

$$[b_l^\dagger, [c_m^\dagger, c_n]_\varepsilon]_- = -2\tau\delta_{ln} b_m^\dagger \quad [c_l^\dagger, [b_m^\dagger, b_n]_\varepsilon]_- = -2\tau\delta_{ln} c_m^\dagger. \tag{6.20c}$$

[19] Other generalizations of (6.15) are also possible, but they do not agree with (6.13). Moreover, it is easy to be proved, any other (non-trivial) arrangement of the indices in (6.16) is incompatible with (6.13).

The equations (6.20a) (resp. (6.20b)) represent the paracommutation relations for the field's particles (resp. antiparticles) as independent objects, while (6.20c) describe a pure relativistic effect of some "interaction" (or its absents) between field's particles and antiparticles and fixes the paracommutation relations involving the b_l's and c_l's, as pointed in [18, p. 207] (where b_l is denoted by a_l and c_l by b_l). The relations (6.17) and (6.20) for $\varepsilon = +1$ (resp. $\varepsilon = -1$) are referred as the *parabose* (resp. *parafermi*) *commutation relations* [18]. This terminology is a natural one also with respect to the commutation relations (6.16), which will be referred as the *paracommutation relations* too.

As first noted in [16], the equations (6.13) provide a solution of (6.20) (or (6.18) in the nonrelativistic case) but the latter equations admit also an infinite number of other solutions. Besides, by taking Hermitian conjugations of (some of) the equations (6.18) or (6.20) and applying generalized Jacobi identities, like

$$\begin{aligned}
&\alpha[[A,B]_\xi,C]_\eta + \xi\eta[[A,C]_{-\alpha/\xi},B]_{-\alpha/\eta} - \alpha^2[[B,C]_{\xi\eta/\alpha},A]_{1/\alpha} = 0 \quad \alpha\xi\eta \neq 0\\
&\beta[A,[B,C]_\alpha,]_{-\beta\gamma} + \gamma[B,[C,A]_\beta,]_{-\gamma\alpha} + \alpha[C,[A,B]_\gamma,]_{-\alpha\beta} = 0 \quad \alpha,\beta,\gamma = \pm 1\\
&[[A,B]_\eta,C]_- + [[B,C]_\eta,A]_- + [[C,A]_\eta,B]_- = 0 \quad \eta = \pm 1\\
&[[A,B]_\xi,[C,D]_\eta]_- = [[A,B]_\xi,C]_-,D]_\eta + \eta[[A,B]_\xi,D]_-,C]_{1/\eta} \quad \eta \neq 0,
\end{aligned} \tag{6.21}$$

one can obtain a number of other (para)commutation relations for which the reader is referred to [16, 18, 26].

Of course, the paracommutation relations (6.16), in particular (6.18) and (6.20) as their stronger versions, do not give the general solution of the trilinear relations (6.12). For instance, one may replace (6.12) with the equations

$$\big[a_l^+,[a_m^{\dagger+},a_n^-]_\varepsilon + [a_m^+,a_n^{\dagger-}]_\varepsilon\big]_- + 2(1+\tau)\delta_{ln}a_m^+ = 0 \tag{6.22a}$$

$$\big[a_l^-,[a_m^{\dagger+},a_n^-]_\varepsilon + [a_m^+,a_n^{\dagger-}]_\varepsilon\big]_- - 2(1+\tau)\delta_{lm}a_n^- = 0. \tag{6.22b}$$

and their Hermitian conjugate, which in terms of the operators b_l and c_l introduced above read

$$[b_l,[b_m^\dagger,b_n]_\varepsilon + [c_m^\dagger,c_m]_\varepsilon]_- = 2(1+\tau)\delta_{lm}b_n \tag{6.23a}$$

$$[c_l,[b_m^\dagger,b_n]_\varepsilon + [c_m^\dagger,c_m]_\varepsilon]_- = 2(1+\tau)\delta_{lm}c_n, \tag{6.23b}$$

and supplement these relations with equations like (6.18b). Obviously, equations (6.16) convert (6.22) into identities and, consequently, the (standard) paracommutation relations (6.20) provide a solution of (6.23). On the base of (6.23) or other similar equations that can be obtained by generalizing the ones in (6.10)–(6.12), further research on particular classes of trilinear commutation relations can be done, but, however, this is not a subject of the present work.

Let us now pay attention to the fact that equations (6.10), (6.11) and (6.12) are generally different (regardless of existence of some connections between their solutions). The cause for this being that the momentum operators for the Lagrangians $\mathcal{L}'$, $\mathcal{L}''$ and $\mathcal{L}'''$ are generally different unless some additional restrictions are added to the Lagrangian formalism (see Sect. 4). A necessary and sufficient condition for (6.10)–(6.12) to be identical is

$$[a_l^\pm,[a_m^{\dagger+},a_m^-]_{-\varepsilon} - [a_m^+,a_m^{\dagger-}]_{-\varepsilon}]_- = 0, \tag{6.24}$$

which certainly is valid if the condition (4.9′), viz.

$$[a_m^{\dagger+},a_m^-]_{-\varepsilon} - [a_m^+,a_m^{\dagger-}]_{-\varepsilon} = 0, \tag{6.25}$$

ensuring the uniqueness of the momentum operator are, holds. If one adopts the standard bilinear commutation relations (6.13), then (6.25), and hence (6.24), is identically valid, but in the framework of, e.g., the paracommutation relations (6.16) (or (6.20) in other form) the equations (6.25) should be postulated to ensure uniqueness of the momentum operator and therefore of the field equations.

On the base of (6.10) or (6.11) one may invent other types of commutation relations, which will not be investigated in this paper because we shall be interested mainly in the case when (6.10), (6.11) and (6.12) are identical (see (6.24)) or, more generally, when the dynamical variables are unique in the sense pointed in Sect. 4.

6.2. Restrictions related to the charge operator

The consequences of the Heisenberg relations (5.2), involving the charge operator for a *charged* field, $q \neq 0$ (and hence $\tau = 0$ – see (3.7)), will be examined in this subsection. In terms of creation and annihilation operators it is equivalent to [1, 13–15]

$$[a_s^{\pm}(\boldsymbol{k}),\, \mathcal{Q}]_{\_} = q a_s^{\pm}(\boldsymbol{k}) \quad [a_s^{\dagger\,\pm}(\boldsymbol{k}),\, \mathcal{Q}]_{\_} = -q a_s^{\dagger\,\pm}(\boldsymbol{k}), \tag{6.26}$$

the values of the polarization indices being specified by (6.5). Substituting here (3.10), we see that, for a *charged* field, the field equations for the Lagrangians $\mathcal{L}'$, $\mathcal{L}''$ and $\mathcal{L}'''$ (see Sect. 3) respectively are:

$$\sum_{t=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{p}\{[a_s^{\pm}(\boldsymbol{k}), a_t^{\dagger\,+}(\boldsymbol{p}) \circ a_t^{-}(\boldsymbol{p}) - \varepsilon a_t^{\dagger\,-}(\boldsymbol{p}) \circ a_t^{+}(\boldsymbol{p})]_{\_} - a_s^{\pm}(\boldsymbol{k})\delta_{st}\delta^3(\boldsymbol{k}-\boldsymbol{p})\} = 0 \tag{6.27a}$$

$$\sum_{t=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{p}\{[a_s^{\dagger\,\pm}(\boldsymbol{k}), a_t^{\dagger\,+}(\boldsymbol{p}) \circ a_t^{-}(\boldsymbol{p}) - \varepsilon a_t^{\dagger\,-}(\boldsymbol{p}) \circ a_t^{+}(\boldsymbol{p})]_{\_} + a_s^{\dagger\,\pm}(\boldsymbol{k})\delta_{st}\delta^3(\boldsymbol{k}-\boldsymbol{p})\} = 0 \tag{6.27b}$$

$$\sum_{t=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{p}\{[a_s^{\pm}(\boldsymbol{k}), a_t^{+}(\boldsymbol{p}) \circ a_t^{\dagger\,-}(\boldsymbol{p}) - \varepsilon a_t^{-}(\boldsymbol{p}) \circ a_t^{\dagger\,+}(\boldsymbol{p})]_{\_} + a_s^{\pm}(\boldsymbol{k})\delta_{st}\delta^3(\boldsymbol{k}-\boldsymbol{p})\} = 0 \tag{6.28a}$$

$$\sum_{t=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{p}\{[a_s^{\dagger\,\pm}(\boldsymbol{k}), a_t^{+}(\boldsymbol{p}) \circ a_t^{\dagger\,-}(\boldsymbol{p}) - \varepsilon a_t^{-}(\boldsymbol{p}) \circ a_t^{\dagger\,+}(\boldsymbol{p})]_{\_} - a_s^{\dagger\,\pm}(\boldsymbol{k})\delta_{st}\delta^3(\boldsymbol{k}-\boldsymbol{p})\} = 0 \tag{6.28b}$$

$$\sum_{t=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{p}\{[a_s^{\pm}(\boldsymbol{k}), [a_t^{\dagger\,+}(\boldsymbol{p}), a_t^{-}(\boldsymbol{p})]_{\varepsilon} - [a_t^{+}(\boldsymbol{p}), a_t^{\dagger\,-}(\boldsymbol{p})_{\varepsilon}]_{\_} - 2a_s^{\pm}(\boldsymbol{k})\delta_{st}\delta^3(\boldsymbol{k}-\boldsymbol{p})\} = 0 \tag{6.29a}$$

$$\sum_{t=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{p}\{[a_s^{\dagger\,\pm}(\boldsymbol{k}), [a_t^{\dagger\,+}(\boldsymbol{p}), a_t^{-}(\boldsymbol{p})]_{\varepsilon} - [a_t^{+}(\boldsymbol{p}), a_t^{\dagger\,-}(\boldsymbol{p})_{\varepsilon}]_{\_} + 2a_s^{\dagger\,\pm}(\boldsymbol{k})\delta_{st}\delta^3(\boldsymbol{k}-\boldsymbol{p})\} = 0. \tag{6.29b}$$

Using (6.27)–(6.29) and (6.8), with $\eta = \varepsilon = -1$, or simply (6.26), one can easily verify the validity of the equations

$$\begin{aligned} &[\tilde{\mathcal{P}}_{\mu},\, \tilde{\mathcal{Q}}]_{\_} = 0 \quad [\tilde{\mathcal{L}}_{\mu\nu},\, \tilde{\mathcal{Q}}]_{\_} = 0 \\ &[\tilde{\mathcal{S}}_{\mu\nu},\, \tilde{\mathcal{Q}}]_{\_} = 0 \quad [\tilde{\mathcal{M}}_{\mu\nu},\, \tilde{\mathcal{Q}}]_{\_} = 0, \end{aligned} \tag{6.30}$$

where the operators $\tilde{\mathcal{P}}_\mu$, $\tilde{\mathcal{Q}}$, $\tilde{\mathcal{S}}_{\mu\nu}$, $\tilde{\mathcal{L}}_{\mu\nu}$ and $\tilde{\mathcal{M}}_{\mu\nu}$ are calculated from one and the same initial Lagrangian according to (3.9)–(3.12). This result confirms the validity of assertion (ii) before (5.24) for the fields considered.

Following the above considerations, concerning the momentum operator, we shall now replace the systems of *integral* equations (6.27)–(6.29) with respectively the following stronger systems of *algebraic* equations (by equating to zero the integrands in (6.27)–(6.29)):

$$[a_l^{\pm}, a_m^{\dagger +} \circ a_m^{-} - \varepsilon a_m^{\dagger -} \circ a_m^{+}]_{\_} - \delta_{lm} a_l^{\pm} = 0 \tag{6.31a}$$
$$[a_l^{\dagger \pm}, a_m^{\dagger +} \circ a_m^{-} - \varepsilon a_m^{\dagger -} \circ a_m^{+}]_{\_} + \delta_{lm} a_l^{\dagger \pm} = 0 \tag{6.31b}$$
$$[a_l^{\pm}, a_m^{+} \circ a_m^{\dagger -} - \varepsilon a_m^{-} \circ a_m^{\dagger +}]_{\_} + \delta_{lm} a_l^{\pm} = 0 \tag{6.32a}$$
$$[a_l^{\dagger \pm}, a_m^{+} \circ a_m^{\dagger -} - \varepsilon a_m^{-} \circ a_m^{\dagger +}]_{\_} - \delta_{lm} a_l^{\dagger \pm} = 0 \tag{6.32b}$$
$$[a_l^{\pm}, [a_m^{\dagger +}, a_m^{-}]_\varepsilon - [a_m^{+}, a_m^{\dagger -}]_\varepsilon]_{\_} - 2\delta_{lm} a_l^{\pm} = 0 \tag{6.33a}$$
$$[a_l^{\dagger \pm}, [a_m^{\dagger +}, a_m^{-}]_\varepsilon - [a_m^{+}, a_m^{\dagger -}]_\varepsilon]_{\_} + 2\delta_{lm} a_l^{\dagger \pm} = 0. \tag{6.33b}$$

These *trilinear commutation relations* are similar to (6.10)–(6.12) and, consequently, can be treated in analogous way.

By invoking (6.8), it is a simple algebra to be proved that the standard bilinear commutation relations (6.13) convert (6.31)–(6.33) into identities. Thus (6.13) are stronger version of (6.31)–(6.33) and, in this sense, any type of commutation relations, which provide a solution of (6.31)–(6.33) and is compatible with (6.13), is a suitable candidate for generalizing (6.13). To illustrate that idea, we shall proceed with (6.33) in a way similar to the 'derivation' of the paracommutation relations from (6.12).

Obviously, the equations (cf. (6.14) with $\tau = 0$, as now $q \neq 0$)

$$[a_l^{\pm}, [a_m^{+}, a_m^{\dagger -}]_\varepsilon]_{\_} + \delta_{lm} a_m^{\pm} = 0 \tag{6.34a}$$
$$[a_l^{\pm}, [a_m^{\dagger +}, a_m^{-}]_\varepsilon]_{\_} - \delta_{lm} a_m^{\pm} = 0 \tag{6.34b}$$

and their Hermitian conjugate provide a solution of (6.33), but, as a direct calculations shows, they do not agree with the standard (anti)commutation relations (6.13). A solution of (6.33) compatible with (6.13) is given by the equations (6.15), with $\tau = 0$ as the field considered is charged one — see (3.7). Therefore equations (6.16), with $\tau = 0$, also provide a compatible with (6.13) solution of (6.33), from where immediately follows that the paracommutation relations (6.20), with $\tau = 0$, convert (6.33) into identities. To conclude, we can say that the paracommutation relations (6.20), in particular their special case (6.13), ensure the simultaneous validity of the Heisenberg relations (5.1) and (5.2) for free scalar, spinor and vector fields.

Similarly to (6.22), one may generalize (6.33) to

$$[a_l^{+}, [a_m^{\dagger +}, a_n^{-}]_\varepsilon - [a_m^{+}, a_n^{\dagger -}]_\varepsilon]_{\_} - 2\delta_{ln} a_m^{+} = 0 \tag{6.35a}$$
$$[a_l^{-}, [a_m^{\dagger +}, a_n^{-}]_\varepsilon - [a_m^{+}, a_n^{\dagger -}]_\varepsilon]_{\_} - 2\delta_{lm} a_n^{-} = 0. \tag{6.35b}$$

which equations agree with (6.13), (6.15), (6.16) and (6.20), but generally do not agree with (6.22), with $\tau = 0$, unless the equations (6.16), with $\tau = 0$, hold.

More generally, we can assert that (6.33) and (6.12), with $\tau = 0$, hold simultaneously if and only if (6.15), with $\tau = 0$, is fulfilled. From here, again, it follows that the paracommutation relations ensure the simultaneous validity of (5.1) and (5.2).

Let us say now some words on the uniqueness problem for the Heisenberg equations involving the charge operator. The systems of equations (6.31)–(6.33) are identical iff

$$[a_l^{\pm}, [a_m^{\dagger +}, a_m^{-}]_{-\varepsilon} + [a_m^{+}, a_m^{\dagger -}]_{-\varepsilon}]_{\_} = 0, \tag{6.36}$$

which, in particular, is satisfied if the condition

$$[a_m^{\dagger +}, a_m^-]_{-\varepsilon} + [a_m^+, a_m^{\dagger -}]_{-\varepsilon} = 0, \tag{6.37}$$

ensuring the uniqueness of the charge operator (see (4.10′)), is valid. Evidently, equations (6.36) and (6.24) are compatible iff

$$\big[a_l^+, [a_m^{\dagger \pm}, a_m^{\mp}]_{-\varepsilon}\big]_- = 0 \quad \big[a_l^-, [a_m^{\dagger \pm}, a_m^{\mp}]_{-\varepsilon}\big]_- = 0 \tag{6.38}$$

which is a weaker form of (4.15) ensuring simultaneous uniqueness of the momentum and charge operator.

6.3. Restrictions related to the angular momentum operator(s)

It is now turn to be investigated the restrictions on the creation and annihilation operators that follow from the Heisenberg relations (5.3) concerning the angular momentum operator. They can be obtained by inserting the equations (3.11) and (3.12) into (5.3). As pointed in Sect. 5, the resulting equalities, however, depend not only on the particular Lagrangian employed, but also on the geometric nature of the field considered; the last dependence being explicitly given via (5.25) and the polarization functions $\sigma_{\mu\nu}^{ss'm\pm}(\boldsymbol{k})$ and $l_{\mu\nu}^{ss'm\pm}(\boldsymbol{k})$ (see also (3.14)).

Consider the terms containing derivatives in (5.3),

$$\tilde{\mathcal{L}}_{\mu\nu}^{\mathrm{or}} := \mathrm{i}\hbar\Big(x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}\Big)\,\tilde{\varphi}_i(x). \tag{6.39}$$

If $\underline{\tilde{\varphi}}_i(k)$ denotes the Fourier image of $\tilde{\varphi}_i(x)$, i.e.

$$\tilde{\varphi}_i(x) = \Lambda \int \mathrm{d}^4 k \mathrm{e}^{-\frac{1}{\mathrm{i}\hbar}k^\mu x_\mu}\,\underline{\tilde{\varphi}}_i(k), \tag{6.40}$$

with Λ being a normalization constant, then the Fourier image of (6.39) is

$$\underline{\tilde{\mathcal{L}}}_{\mu\nu}^{\mathrm{or}} = \mathrm{i}\hbar\Big(k_\mu \frac{\partial}{\partial k^\nu} - k_\nu \frac{\partial}{\partial k^\mu}\Big)\,\underline{\tilde{\varphi}}_i(k). \tag{6.41}$$

Comparing this expression with equations (3.12), we see that the terms containing derivatives in (3.12) should be responsible for the term (6.39) in (5.3).[20] For this reason, we shall suppose that the momentum operator $\tilde{\mathcal{M}}_{\mu\nu}$ admits a representation

$$\tilde{\mathcal{M}}_{\mu\nu} = \tilde{\mathcal{M}}_{\mu\nu}^{\mathrm{or}} + \tilde{\mathcal{M}}_{\mu\nu}^{\mathrm{sp}} \tag{6.42}$$

such that the operators $\tilde{\mathcal{M}}_{\mu\nu}^{\mathrm{or}}$ and $\tilde{\mathcal{M}}_{\mu\nu}^{\mathrm{sp}}$ satisfy the relations (5.4) and (5.5), respectively. Thus we shall replace (5.3) with the stronger system of equations (5.4)–(5.5). Besides, we shall admit that the explicit form of the operators $\tilde{\mathcal{M}}_{\mu\nu}^{\mathrm{or}}$ and $\tilde{\mathcal{M}}_{\mu\nu}^{\mathrm{sp}}$ are given via (5.13) and (5.12) for the fields investigated in the present work.

Let us consider at first the 'orbital' Heisenberg relations (5.4), which is independent of the particular geometrical nature of the fields studied. Substituting (5.13) and (6.40) into (5.4), using that $\underline{\tilde{\varphi}}_i(\pm k)$, with $k^2 = m^2c^2$, is a linear combination of $\tilde{a}_s^{\pm}(\boldsymbol{k})$ with classical, not operator-valued, functions of $\boldsymbol{k}$ as coefficients [1, 13–15] and introducing for brevity the operator

$$\omega_{\mu\nu}(k) := k_\mu \frac{\partial}{\partial k^\nu} - k_\nu \frac{\partial}{\partial k^\mu}, \tag{6.43}$$

[20] The terms proportional to the momentum operator in (3.12) disappear if the creation and annihilation operators (2.9) in Heisenberg picture are employed (see also [13–15]).

we arrive to the following *integro-differential* systems of equations:

$$\sum_{t=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{p}\big\{\big((-\omega_{\mu\nu}(p)+\omega_{\mu\nu}(q))([\tilde{a}_s^{\pm}(\boldsymbol{k}),\tilde{a}_t^{\dagger\,+}(\boldsymbol{p})\circ\tilde{a}_t^{-}(\boldsymbol{q}) - \varepsilon\tilde{a}_t^{\dagger\,-}(\boldsymbol{p})\circ\tilde{a}_t^{+}(\boldsymbol{q})]_{\_})\big)\big|_{q=p}\big\}\big|_{p_0=\sqrt{m^2c^2+\boldsymbol{p}^2}} = 2(1+\tau)\omega_{\mu\nu}(k)(\tilde{a}_s^{\pm}(\boldsymbol{k})) \tag{6.44a}$$

$$\sum_{t=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{p}\big\{\big((-\omega_{\mu\nu}(p)+\omega_{\mu\nu}(q))([\tilde{a}_s^{\dagger\,\pm}(\boldsymbol{k}),\tilde{a}_t^{\dagger\,+}(\boldsymbol{p})\circ\tilde{a}_t^{-}(\boldsymbol{q}) - \varepsilon\tilde{a}_t^{\dagger\,-}(\boldsymbol{p})\circ\tilde{a}_t^{+}(\boldsymbol{q})]_{\_})\big)\big|_{q=p}\big\}\big|_{p_0=\sqrt{m^2c^2+\boldsymbol{p}^2}} = 2(1+\tau)\omega_{\mu\nu}(k)(\tilde{a}_s^{\dagger\,\pm}(\boldsymbol{k})) \tag{6.44b}$$

$$\sum_{t=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{p}\big\{\big((-\omega_{\mu\nu}(p)+\omega_{\mu\nu}(q))([\tilde{a}_s^{\pm}(\boldsymbol{k}),\tilde{a}_t^{+}(\boldsymbol{p})\circ\tilde{a}_t^{\dagger\,-}(\boldsymbol{q}) - \varepsilon\tilde{a}_t^{-}(\boldsymbol{p})\circ\tilde{a}_t^{\dagger\,+}(\boldsymbol{q})]_{\_})\big)\big|_{q=p}\big\}\big|_{p_0=\sqrt{m^2c^2+\boldsymbol{p}^2}} = 2(1+\tau)\omega_{\mu\nu}(k)(\tilde{a}_s^{\pm}(\boldsymbol{k})) \tag{6.45a}$$

$$\sum_{t=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{p}\big\{\big((-\omega_{\mu\nu}(p)+\omega_{\mu\nu}(q))([\tilde{a}_s^{\dagger\,\pm}(\boldsymbol{k}),\tilde{a}_t^{+}(\boldsymbol{p})\circ\tilde{a}_t^{\dagger\,-}(\boldsymbol{q}) - \varepsilon\tilde{a}_t^{-}(\boldsymbol{p})\circ\tilde{a}_t^{\dagger\,+}(\boldsymbol{q})]_{\_})\big)\big|_{q=p}\big\}\big|_{p_0=\sqrt{m^2c^2+\boldsymbol{p}^2}} = 2(1+\tau)\omega_{\mu\nu}(k)(\tilde{a}_s^{\dagger\,\pm}(\boldsymbol{k})) \tag{6.45b}$$

$$\sum_{t=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{p}\big\{\big((-\omega_{\mu\nu}(p)+\omega_{\mu\nu}(q))([\tilde{a}_s^{\pm}(\boldsymbol{k}),[\tilde{a}_t^{\dagger\,+}(\boldsymbol{p}),\tilde{a}_t^{-}(\boldsymbol{q})]_{\varepsilon} + [\tilde{a}_t^{+}(\boldsymbol{p}),\tilde{a}_t^{\dagger\,-}(\boldsymbol{q})]_{\varepsilon}]_{\_})\big)\big|_{q=p}\big\}\big|_{p_0=\sqrt{m^2c^2+\boldsymbol{p}^2}} = 4(1+\tau)\omega_{\mu\nu}(k)(\tilde{a}_s^{\pm}(\boldsymbol{k})) \tag{6.46a}$$

$$\sum_{t=1}^{2j+1-\delta_{0m}(1-\delta_{0j})} \int \mathrm{d}^3\boldsymbol{p}\big\{\big((-\omega_{\mu\nu}(p)+\omega_{\mu\nu}(q))([\tilde{a}_s^{\dagger\,\pm}(\boldsymbol{k}),[\tilde{a}_t^{\dagger\,+}(\boldsymbol{p}),\tilde{a}_t^{-}(\boldsymbol{q})]_{\varepsilon} + [\tilde{a}_t^{+}(\boldsymbol{p}),\tilde{a}_t^{\dagger\,-}(\boldsymbol{q})]_{\varepsilon}]_{\_})\big)\big|_{q=p}\big\}\big|_{p_0=\sqrt{m^2c^2+\boldsymbol{p}^2}} = 4(1+\tau)\omega_{\mu\nu}(k)(\tilde{a}_s^{\dagger\,\pm}(\boldsymbol{k})), \tag{6.46b}$$

where $k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}$ is set after the differentiations are performed (see (6.43)). Following the procedure of the previous considerations, we replace the *integro-differential* equations (6.44)–(6.46) with the following *differential* ones:

$$\big\{(-\omega_{\mu\nu}^{\circ}(m)+\omega_{\mu\nu}^{\circ}(n))([\tilde{a}_l^{\pm},\tilde{a}_m^{\dagger\,+}\circ\tilde{a}_n^{-} - \varepsilon\tilde{a}_m^{\dagger\,-}\circ\tilde{a}_n^{+}]_{\_})\big\}\big|_{n=m} = 2(1+\tau)\delta_{lm}\omega_{\mu\nu}^{\circ}(l)(\tilde{a}_l^{\pm}) \tag{6.47a}$$
$$\big\{(-\omega_{\mu\nu}^{\circ}(m)+\omega_{\mu\nu}^{\circ}(n))([\tilde{a}_l^{\dagger\,\pm},\tilde{a}_m^{\dagger\,+}\circ\tilde{a}_n^{-} - \varepsilon\tilde{a}_m^{\dagger\,-}\circ\tilde{a}_n^{+}]_{\_})\big\}\big|_{n=m} = 2(1+\tau)\delta_{lm}\omega_{\mu\nu}^{\circ}(l)(\tilde{a}_l^{\dagger\,\pm}) \tag{6.47b}$$

$$\big\{(-\omega_{\mu\nu}^{\circ}(m)+\omega_{\mu\nu}^{\circ}(n))([\tilde{a}_l^{\pm},\tilde{a}_m^{+}\circ\tilde{a}_n^{\dagger\,-} - \varepsilon\tilde{a}_m^{-}\circ\tilde{a}_n^{\dagger\,+}]_{\_})\big\}\big|_{n=m} = 2(1+\tau)\delta_{lm}\omega_{\mu\nu}^{\circ}(l)(\tilde{a}_l^{\pm}) \tag{6.48a}$$
$$\big\{(-\omega_{\mu\nu}^{\circ}(m)+\omega_{\mu\nu}^{\circ}(n))([\tilde{a}_l^{\dagger\,\pm},\tilde{a}_m^{+}\circ\tilde{a}_n^{\dagger\,-} - \varepsilon\tilde{a}_m^{-}\circ\tilde{a}_n^{\dagger\,+}]_{\_})\big\}\big|_{n=m} = 2(1+\tau)\delta_{lm}\omega_{\mu\nu}^{\circ}(l)(\tilde{a}_l^{\dagger\,\pm}) \tag{6.48b}$$

$$\big\{(-\omega_{\mu\nu}^{\circ}(m)+\omega_{\mu\nu}^{\circ}(n))([\tilde{a}_l^{\pm},[\tilde{a}_m^{\dagger\,+},\tilde{a}_n^{-}]_{\varepsilon} + [\tilde{a}_m^{+},\tilde{a}_n^{\dagger\,-}]_{\varepsilon}]_{\_})\big\}\big|_{n=m} = 4(1+\tau)\delta_{lm}\omega_{\mu\nu}^{\circ}(l)(\tilde{a}_l^{\pm}) \tag{6.49a}$$
$$\big\{(-\omega_{\mu\nu}^{\circ}(m)+\omega_{\mu\nu}^{\circ}(n))([\tilde{a}_l^{\dagger\,\pm},[\tilde{a}_m^{\dagger\,+},\tilde{a}_n^{-}]_{\varepsilon} + [\tilde{a}_m^{+},\tilde{a}_n^{\dagger\,-}]_{\varepsilon}]_{\_})\big\}\big|_{n=m} = 4(1+\tau)\delta_{lm}\omega_{\mu\nu}^{\circ}(l)(\tilde{a}_l^{\dagger\,\pm}), \tag{6.49b}$$

where we have set (cf. (6.43))

$$\omega_{\mu\nu}^{\circ}(l) := \omega_{\mu\nu}(k) = k_\mu\frac{\partial}{\partial k^\nu} - k_\nu\frac{\partial}{\partial k^\mu} \qquad \text{if } l=(s,\boldsymbol{k}) \tag{6.50}$$

and $k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}$ is set after the differentiations are performed.

Remark. Instead of (6.47)–(6.49) one can write similar equations in which the operator $-\omega^{\circ}_{\mu\nu}(m)$ or $+\omega^{\circ}_{\mu\nu}(n)$ is deleted and the factor $+\frac{1}{2}$ or $-\frac{1}{2}$, respectively, is added on their right hand sides. These manipulations correspond to an integration by parts of some of the terms in (6.44)–(6.46).

The main difference of the obtained trilinear relations with respect to the previous ones considered above is that they are *partial differential* equations of first order.

The relations (6.49) agree with the equations (6.16) in a sense that if (6.16) hold, then (6.49) become identically valid. Indeed, since

$$\begin{aligned} \{(-\omega^{\circ}_{\mu\nu}(m)+\omega^{\circ}_{\mu\nu}(n))(\tilde{a}^{\pm}_{m}\delta_{ln})\}\big|_{n=m} &= -2\delta_{lm}\omega^{\circ}_{\mu\nu}(m)(\tilde{a}^{\pm}_{m}) \\ \{(-\omega^{\circ}_{\mu\nu}(m)+\omega^{\circ}_{\mu\nu}(n))(\tilde{a}^{\pm}_{n}\delta_{lm})\}\big|_{n=m} &= +2\delta_{lm}\omega^{\circ}_{\mu\nu}(m)(\tilde{a}^{\pm}_{m}), \end{aligned} \tag{6.51}$$

due to (6.50), (6.43) and the equality $\frac{\mathrm{d}\delta(x)}{\mathrm{d}x}f(x) = -\delta(x)\frac{\mathrm{d}f(x)}{\mathrm{d}x}$ for a C^1 function f, the application of the operator $(-\omega^{\circ}_{\mu\nu}(m)+\omega^{\circ}_{\mu\nu}(n))$ to (6.16) and subsequent setting $n=m$ entails (6.49). In particular, this means that the paracommutation relations (6.20) and, moreover, the standard (anti)commutation relations (6.13) convert (6.49) into identities. Therefore the 'orbital' Heisenberg relations (5.4) hold for scalar, spinor and vector fields satisfying the bilinear or para commutation relations.

It should be noted, the paracommutation relations are *not* the only trilinear commutation relations that are solutions of (6.49). As an example, we shall present the trilinear relations

$$\big[a^{+}_{l},[a^{+}_{m},a^{\dagger -}_{n}]_{\varepsilon}\big]_{-} = \big[a^{+}_{l},[a^{\dagger +}_{m},a^{-}_{n}]_{\varepsilon}\big]_{-} = -(1+\tau)\delta_{ln}a^{+}_{m} \tag{6.52a}$$

$$\big[a^{-}_{l},[a^{+}_{m},a^{\dagger -}_{n}]_{\varepsilon}\big]_{-} = \big[a^{-}_{l},[a^{\dagger +}_{m},a^{-}_{n}]_{\varepsilon}\big]_{-} = +(1+\tau)\delta_{lm}a^{+}_{n}, \tag{6.52b}$$

which reduce to (6.14) for $n=m$, do not agree with (6.13), but convert (6.49) into identities (see (6.51)). Other example is provided by the equations (6.22), which are compatible with the paracommutation relations and, as a result of (6.51), convert (6.49) into identities. *Prima facie* one may suppose that any solution of (6.12) provides a solution of (6.49), but this is not the general case. A counterexample is provided by the commutation relations

$$\big[a^{\pm}_{l},[a^{\dagger +}_{m},a^{-}_{n}]_{\varepsilon}+[a^{+}_{m},a^{\dagger -}_{n}]_{\varepsilon}\big]_{-} \pm 2(1+\tau)\delta_{ln}a^{\pm}_{m} = 0, \tag{6.53}$$

which reduce to (6.12) for $n=m$, satisfy (6.49) with $\tilde{a}^{+}_{l}$ for $\tilde{a}^{\pm}_{l}$, and do *not* satisfy (6.49) with $\tilde{a}^{-}_{l}$ for $\tilde{a}^{\pm}_{l}$ (see (6.51) and cf. (6.22)).

From (5.13) follows that the operator $\tilde{\mathcal{M}}^{\mathrm{or}}_{\mu\nu}$ is independent of the Lagrangian $\mathcal{L}'$, $\mathcal{L}''$ or $\mathcal{L}'''$ one starts off if and only if (see (4.11))

$$\big\{(-\omega^{\circ}_{\mu\nu}(m)+\omega^{\circ}_{\mu\nu}(n))\big([\tilde{a}^{\dagger +}_{m},\tilde{a}^{-}_{n}]_{-\varepsilon} - [\tilde{a}^{+}_{m},\tilde{a}^{\dagger -}_{n}]_{-\varepsilon}\big)\big\}\big|_{n=m} = 0. \tag{6.54}$$

This condition ensures the coincidence of the systems of equations (6.47), (6.48) and (6.49) too. However, the following necessary and sufficient condition for the coincidence of these systems is expressed by the weaker equations

$$\big\{(-\omega^{\circ}_{\mu\nu}(m)+\omega^{\circ}_{\mu\nu}(n))\big(\big[\tilde{a}^{\pm}_{l},[\tilde{a}^{\dagger +}_{m},\tilde{a}^{-}_{n}]_{-\varepsilon} - [\tilde{a}^{+}_{m},\tilde{a}^{\dagger -}_{n}]_{-\varepsilon}\big]_{-}\big)\big\}\big|_{n=m} = 0. \tag{6.55}$$

It is now turn to be considered the 'spin' Heisenberg relations (5.5).

Recall, the field operators φ_i for the fields considered here admit a representation [13–15]

$$\varphi_i = \Lambda \sum_{t} \int \mathrm{d}^3\boldsymbol{p}\{v^{t,+}_{i}(\boldsymbol{p})a^{+}_{t}(\boldsymbol{p}) + v^{t,-}_{i}(\boldsymbol{p})a^{-}_{t}(\boldsymbol{p})\}, \tag{6.56}$$

where Λ is a normalization constant and $v^{t,\pm}_{i}(\boldsymbol{p})$ are classical, not operator-valued, complex or real functions which are linearly independent. The particular definition of $v^{t,\pm}_{i}(\boldsymbol{p})$ depends on the geometrical nature of φ_i and can be found in [13–15] (see also [1]), where the reader can find also a number of relations satisfied by $v_i^{t,\pm}(\boldsymbol{p})$. Here we shall mention only that $v_i^{t,\pm}(\boldsymbol{p}) = 1$ for a scalar field and $v_i^{t,+}(\boldsymbol{p}) = v_i^{t,-}(\boldsymbol{p}) =: v_i^t(\boldsymbol{p}) = (v_i^t(\boldsymbol{p}))^*$ for a vector field.

The explicit form of the polarization functions $\sigma_{\mu\nu}^{ss',\pm}(\boldsymbol{k})$ and $l_{\mu\nu}^{ss',\pm}(\boldsymbol{k})$ (see Sect. 3, in particular (3.14)) through $v_i^{t,\pm}(\boldsymbol{k})$ are [13–15]:

$$\begin{aligned}
\sigma_{\mu\nu}^{ss',\pm}(\boldsymbol{k}) &= \frac{(-1)^j}{j+\delta_{j0}} \sum_{i,i'} (v_i^{s,\pm}(\boldsymbol{k}))^* I_{i'\mu\nu}^{i} v_{i'}^{t,\pm}(\boldsymbol{k}) \\
l_{\mu\nu}^{ss',\pm}(\boldsymbol{k}) &= \frac{(-1)^j}{2j+\delta_{j0}} \sum_{i} (v_i^{s,\pm}(\boldsymbol{k}))^* \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) v_i^{t,\pm}(\boldsymbol{k}),
\end{aligned} \tag{6.57}$$

with an exception that $\sigma_{0a}^{ss',\pm}(\boldsymbol{k}) = \sigma_{a0}^{ss',\pm}(\boldsymbol{k}) = 0$, $a = 1,2,3$, for a spinor field, $j = \frac{1}{2}$, [14]. Evidently, the equations (3.14) follow from the mentioned facts (see also (5.25)).

Substituting (6.56) and (5.12) into (5.5), we obtain the following systems of *integral* equations (corresponding respectively to the Lagrangians $\mathcal{L}'$, $\mathcal{L}''$ and $\mathcal{L}'''$):

$$\begin{gathered}
\frac{(-1)^{j+1} j}{1+\tau} \sum_{s,s',t} \int \mathrm{d}^3\boldsymbol{k} \int \mathrm{d}^3\boldsymbol{p} v_i^{t,\pm}(\boldsymbol{p}) \big\{ (\sigma_{\mu\nu}^{ss',-}(\boldsymbol{k}) + l_{\mu\nu}^{ss',-}(\boldsymbol{k})) [a_t^\pm(\boldsymbol{p}), a_s^{\dagger\,+}(\boldsymbol{k}) \circ a_{s'}^-(\boldsymbol{k})]_- \\
+ (\sigma_{\mu\nu}^{ss',+}(\boldsymbol{k}) + l_{\mu\nu}^{ss',+}(\boldsymbol{k})) [a_t^\pm(\boldsymbol{p}), a_s^{\dagger\,-}(\boldsymbol{k}) \circ a_{s'}^+(\boldsymbol{k})]_- \big\} = \sum_{i'} \sum_t \int \mathrm{d}^3\boldsymbol{p} I_{i\mu\nu}^{i'} v_{i'}^{t,\pm}(\boldsymbol{p}) a_t^\pm(\boldsymbol{p})
\end{gathered} \tag{6.58}$$

$$\begin{gathered}
\varepsilon \frac{(-1)^{j+1} j}{1+\tau} \sum_{s,s',t} \int \mathrm{d}^3\boldsymbol{k} \int \mathrm{d}^3\boldsymbol{p} v_i^{t,\pm}(\boldsymbol{p}) \big\{ (\sigma_{\mu\nu}^{ss',+}(\boldsymbol{k}) + l_{\mu\nu}^{ss',+}(\boldsymbol{k})) [a_t^\pm(\boldsymbol{p}), a_{s'}^+(\boldsymbol{k}) \circ a_s^{\dagger\,-}(\boldsymbol{k})]_- \\
+ (\sigma_{\mu\nu}^{ss',-}(\boldsymbol{k}) + l_{\mu\nu}^{ss',-}(\boldsymbol{k})) [a_t^\pm(\boldsymbol{p}), a_{s'}^-(\boldsymbol{k}) \circ a_s^{\dagger\,+}(\boldsymbol{k})]_- \big\} = \sum_{i'} \sum_t \int \mathrm{d}^3\boldsymbol{p} I_{i\mu\nu}^{i'} v_{i'}^{t,\pm}(\boldsymbol{p}) a_t^\pm(\boldsymbol{p})
\end{gathered} \tag{6.59}$$

$$\begin{gathered}
\frac{(-1)^{j+1} j}{2(1+\tau)} \sum_{s,s',t} \int \mathrm{d}^3\boldsymbol{k} \int \mathrm{d}^3\boldsymbol{p} v_i^{t,\pm}(\boldsymbol{p}) \big\{ (\sigma_{\mu\nu}^{ss',-}(\boldsymbol{k}) + l_{\mu\nu}^{ss',-}(\boldsymbol{k})) \big[a_t^\pm(\boldsymbol{p}), [a_s^{\dagger\,+}(\boldsymbol{k}), a_{s'}^-(\boldsymbol{k})]_\varepsilon \big]_- \\
+ (\sigma_{\mu\nu}^{ss',+}(\boldsymbol{k}) + l_{\mu\nu}^{ss',+}(\boldsymbol{k})) \big[a_t^\pm(\boldsymbol{p}), [a_s^{\dagger\,-}(\boldsymbol{k}), a_{s'}^+(\boldsymbol{k})]_\varepsilon \big]_- \big\} = \sum_{i'} \sum_t \int \mathrm{d}^3\boldsymbol{p} I_{i\mu\nu}^{i'} v_{i'}^{t,\pm}(\boldsymbol{p}) a_t^\pm(\boldsymbol{p}).
\end{gathered} \tag{6.60}$$

For the difference of all previously considered systems of *integral* equations, like (6.2)–(6.4), (6.27)–(6.29) and (6.44)–(6.46), the systems (6.58)–(6.60) cannot be replaced by ones consisting of algebraic (or differential) equations. The cause for this state of affairs is that in (6.58)–(6.60) enter polarization modes with arbitrary s and s' and, generally, one cannot 'diagonalize' the integrand(s) with respect to s and s'; moreover, for a vector field, the modes with $s = s'$ are not presented at all (see (3.14)). That is why no commutation relations can be extracted from (6.58)–(6.60) unless further assumptions are made. Without going into details, below we shall sketch the proof of the assertion that the *commutation relations* (6.16) *convert* (6.60) *into identities for massive spinor and vector fields.*[21] In particular, this entails that the paracommutation and the bilinear commutation relations provide solutions of (6.60).

Let (6.16) holds. Combining it with (6.60), we see that the latter splits into the equations

[21] The equations (6.58)–(6.60) are identities for scalar fields as for them $I_{\mu\nu} = 0$ and $v_i^{t,\pm}(\boldsymbol{k}) = 1$, which reflects the absents of spin for these fields.

$$\frac{(-1)^j j}{1+\tau} \sum_{s,t} \int \mathrm{d}^3 \boldsymbol{p} v_i^{t,+}(\boldsymbol{p}) \{ \tau(\sigma_{\mu\nu}^{st,-}(\boldsymbol{p}) + l_{\mu\nu}^{st,-}(\boldsymbol{p})) + \varepsilon(\sigma_{\mu\nu}^{ts,+}(\boldsymbol{p}) + l_{\mu\nu}^{ts,+}(\boldsymbol{p})) \} a_s^+(\boldsymbol{p}),$$

$$= \sum_{i'} I_{i\mu\nu}^{i'} \sum_s \int \mathrm{d}^3 \boldsymbol{p} v_{i'}^{s,+}(\boldsymbol{p}) a_s^+(\boldsymbol{p}) \quad (6.61\text{a})$$

$$\frac{(-1)^{j+1} j}{1+\tau} \sum_{s,t} \int \mathrm{d}^3 \boldsymbol{p} v_i^{t,-}(\boldsymbol{p}) \{ (\sigma_{\mu\nu}^{ts,-}(\boldsymbol{p}) + l_{\mu\nu}^{ts,-}(\boldsymbol{p})) + \varepsilon\tau(\sigma_{\mu\nu}^{st,+}(\boldsymbol{p}) + l_{\mu\nu}^{st,+}(\boldsymbol{p})) \} a_s^-(\boldsymbol{p}),$$

$$= \sum_{i'} I_{i\mu\nu}^{i'} \sum_s \int \mathrm{d}^3 \boldsymbol{p} v_{i'}^{s,-}(\boldsymbol{p}) a_s^-(\boldsymbol{p}). \quad (6.61\text{b})$$

Inserting here (6.57), we see that one needs the explicit definition of $v_i^{s,\pm}(\boldsymbol{k})$ and formulae for sums like $\rho_{ii'}(\boldsymbol{k}) := \sum_s v_i^{s,\pm}(\boldsymbol{k})(v_{i'}^{s,\pm}(\boldsymbol{k}))^*$, which are specific for any particular field and can be found in [13–15]. In this way, applying (5.25), (3.7) and the mentioned results from [13–15], one can check the validity of (6.61) for massive fields in a way similar to the proof of (5.3) in [13–15] for scalar, spinor and vector fields, respectively.

We shall end the present subsection with the remark that the equations (4.17) and (4.18), which together with (4.15) ensure the uniqueness of the spin and orbital operators, are sufficient conditions for the coincidence of the equations (6.58), (6.59) and (6.60).

7. Inferences

To begin with, let us summarize the major conclusions from Sect. 6. Each of the Heisenberg equations (5.1)–(5.3), the equations (5.3) being split into (5.4) and (5.5), induces in a natural way some relations that the creation and annihilation operators should satisfy. These relations can be chosen as algebraic trilinear ones in a case of (5.1) and (5.2) (see (6.10)–(6.12) and (6.31)–(6.33), respectively). But for (5.4) and (5.5) they need not to be algebraic and are differential ones in the case of (5.4) (see (6.47)–(6.49)) and integral equations in the case of (5.5) (see (6.58)–(6.60)). It was pointed that the cited relations depend on the initial Lagrangian from which the theory is derived, unless some explicitly written conditions hold (see (6.24), (6.37) and (6.55)); in particular, these conditions are true if the equations (4.9)–(4.13), ensuring the uniqueness of the corresponding dynamical operators, are valid. Since the 'charge symmetric' Lagrangians (3.4) seem to be the ones that best describe free fields, the arising from them (commutation) relations (6.12), (6.33), (6.49) and (6.60) were studied in more details. It was proved that the trilinear commutation relations (6.16) convert them into identities, as a result of which the same property possess the paracommutation relations (6.20) and, in particular, the bilinear commutation relations (6.13). Examples of trilinear commutation relations, which are neither ordinary nor para ones, were presented; some of them, like (6.14), (6.34) and (6.52), do not agree with (6.13) and other ones, like (6.16), (6.22) and (6.35), generalize (6.20) and hence are compatible with (6.13). At last, it was demonstrated that the commutators between the dynamical variables (see (5.15)–(5.23)) are uniquely defined if a Heisenberg relation for one of the operators entering in it is postulated.

The chief aim of the present section is to be explored the problem whether all of the reasonable conditions, mentioned in the previous sections and that can be imposed on the creation and annihilation operators, can hold or not hold simultaneously. This problem is suggested by the strong evidences that the relations (5.1)–(5.3) and (5.15)–(5.23), with a possible exception of (5.3) (more precisely, of (5.5)) in the massless case, should be valid in a realistic quantum field theory [1, 3, 7, 8, 11, 12]. Besides, to the arguments in *loc. cit.*, we shall add the requirement for uniqueness of the dynamical variables (see Sect. 4).

As it was shown in Sect. 6, the relations (5.1), (5.2), (5.4) and (5.5) are compatible if one starts from a charge symmetric Lagrangian (see (3.4)), which best describes a free field theory; in particular, the commutation relations (6.16) (and hence (6.20) and (6.13)) ensure their simultaneous validity.[22] For that reason, we shall investigate below only commutation relations for which (5.1), (5.2), (5.4) and (5.5) hold. It will be assumed that they should be such that the equations (6.10)–(6.12), (6.31)–(6.33), (6.47)–(6.49) and (6.58)–(6.60), respectively, hold.

Consider now the problem for the uniqueness of the dynamical variables and its consistency with the commutation relations just mentioned for a charged field. It will be assumed that this uniqueness is ensured via the equations (4.9)–(4.11).

The equation (4.15), viz.

$$[a_m^{\dagger\pm}, a_m^{\mp}]_{-\varepsilon} = 0, \tag{7.1}$$

is a necessary and sufficient conditions for the uniqueness of the momentum and charge operators (see Sect. 4 and the notation introduced at the beginning of Sect. 6). Before commenting on this relation, we would like to derive some consequences of it. Applying consequently (6.8) for $\eta = -\varepsilon$, (7.1) and the identity

$$[A, B \circ C]_+ = [A, B]_\eta \circ C - \eta B \circ [A, C]_{-\eta} \qquad \eta = \pm 1 \tag{7.2}$$

for $\eta = +\varepsilon, -\varepsilon$, we, in view of (7.1), obtain

$$\begin{aligned} &[a_m^+, [a_m^+, a_m^{\dagger -}]_\varepsilon]_- = [a_m^{\dagger -}, [a_m^+, a_m^+]_{-\varepsilon}]_+ = (1-\varepsilon)[a_m^{\dagger -}, a_m^+]_\varepsilon \circ a_m^+ \\ &[a_m^-, [a_m^{\dagger +}, a_m^-]_\varepsilon]_- = \varepsilon[a_m^{\dagger +}, [a_m^-, a_m^-]_{-\varepsilon}]_+ = \varepsilon(1-\varepsilon)[a_m^{\dagger +}, a_m^-]_\varepsilon \circ a_m^-. \end{aligned} \tag{7.3}$$

Forming the sum and difference of (6.12a), for $\tau = 0$, and (6.33a), we see that the system of equations they form is equivalent to

$$[a_l^+, [a_m^{\dagger +}, a_m^-]_\varepsilon]_- = 0 \qquad [a_l^-, [a_m^+, a_m^{\dagger -}]_\varepsilon]_- = 0 \tag{7.4a}$$

$$[a_l^+, [a_m^+, a_m^{\dagger -}]_\varepsilon]_- + 2\delta_{lm} a_l^+ = 0 \quad [a_l^-, [a_m^{\dagger +}, a_m^-]_\varepsilon]_- - 2\delta_{lm} a_l^- = 0. \tag{7.4b}$$

Combining (7.4b), for $l = m$, with (7.3), we get

$$(1-\varepsilon)[a_m^{\dagger -}, a_m^+]_\varepsilon \circ a_m^+ + 2a_m^+ = 0 \quad \varepsilon(1-\varepsilon)[a_m^{\dagger +}, a_m^-]_\varepsilon \circ a_m^- - 2a_m^- = 0. \tag{7.5}$$

Obviously, these equations reduce to

$$a_m^\pm = 0 \tag{7.6}$$

for bose fields as for them $\varepsilon = +1$ (see (3.7)). Since the operators (7.6) describe a completely unobservable field, or, more precisely, an absence of a field at all, the obtained result means that the theory considered cannot describe any really existing physical field with spin $j = 0, 1$. Such a conclusion should be regarded as a contradiction in the theory. For fermi fields, $j = \frac{1}{2}$ and $\varepsilon = -1$, the equations (7.5) have solutions different from (7.6) iff $a_m^\pm$ are degenerate operators, i.e. with no inverse ones, in which case (7.4a) is a consequence of (7.5) and (7.1) (see (6.8) and (7.3) too).

The source of the above contradiction is in the equation (7.1), which does not agree with the bilinear commutation relations (6.13) and contradicts to the existing correlation between creation and annihilation of particles with identical characteristics ($m = (t, \boldsymbol{p})$ in our case) as (7.1) can be interpreted physically as mutual independence of the acts of creation and annihilation of such particles [1, § 10.1].

At this point, there are two ways for 'repairing' of the theory. On one hand, one can forget about the uniqueness of the dynamical variables (in a sense of Sect. 4), after which

[22] The special case(s) when (5.5) may not hold for a massless field will not be considered below.

the formalism can be developed by choosing, e.g., the charge symmetric Lagrangians (3.4) and following the usual Lagrangian formalism; in fact, this is the way the parafield theory is build [16,18]. On another hand, one may try to change something at the ground of the theory in such a way that the uniqueness of the dynamical variables to be ensured automatically. We shall follow the second method. As a guiding idea, we shall have in mind that the bilinear commutation relations (6.13) and the related to them normal ordering procedure provide a base for the present-day quantum field theory, which describes sufficiently well the discovered elementary particles/fields. On this background, an extensive exploration of commutation relations which are incompatible with (6.13) is justified only if there appear some evidences for fields/particles that can be described via them. In that connection it should be recalled [17, 18], it seems that all known particles/fields are described via (6.13) and no one of them is a para particle/field.

Using the notation introduced at the beginning of Sect. 4, we shall look for a linear mapping (operator) $\mathcal{E}$ on the operator space over the system's Hilbert space $\mathcal{F}$ of states such that

$$\mathcal{E}(\mathcal{D}') = \mathcal{E}(\mathcal{D}''). \tag{7.7}$$

As it was shown in Sect. 4, an example of an operator $\mathcal{E}$ is provided by the normal ordering operator $\mathcal{N}$. Therefore an operator satisfying (7.7) always exists. To any such operator $\mathcal{E}$ there corresponds a set of dynamical variables defined via

$$\mathcal{D} = \mathcal{E}(\mathcal{D}'). \tag{7.8}$$

Let us examine the properties of the mapping $\mathcal{E}$ that it should possess due to the requirement (7.7).

First of all, as the operators of the dynamical variables should be Hermitian, we shall require

$$\big(\mathcal{E}(\mathcal{B})\big)^\dagger = \mathcal{E}(\mathcal{B}^\dagger) \tag{7.9}$$

for any operator $\mathcal{B}$, which entails

$$\mathcal{D}^\dagger = \mathcal{D}, \tag{7.10}$$

due to (3.9)–(3.12) and (7.8).

As in Sect. 4, we shall replace the so-arising integral equations with corresponding algebraic ones. Thus the equations (4.5)–(4.20) remain valid if the operator $\mathcal{E}$ is applied to their left hand sides.

Consider the general case of a charged field, $q \neq 0$. So, the analogue of (4.15) reads

$$\mathcal{E}\big([a_m^{\dagger\,\pm}, a_m^{\mp}]_{-\varepsilon}\big) = 0, \tag{7.11}$$

which equation ensures the uniqueness of the momentum and charge operators. Respectively, the condition (4.11) transforms into

$$\big\{(-\overset{\circ}{\omega}_{\mu\nu}(m) + \overset{\circ}{\omega}_{\mu\nu}(n))\big(\mathcal{E}([a_m^{\dagger\,+}, a_n^-]_{-\varepsilon}) - \mathcal{E}([a_m^+, a_n^{\dagger\,-}]_{-\varepsilon})\big)\big\}\big|_{n=m} = 0, \tag{7.12}$$

which, by means of (7.11) can be rewritten as (cf. (4.16))

$$\big\{\overset{\circ}{\omega}_{\mu\nu}(n)\big(\mathcal{E}([a_m^{\dagger\,+}, a_n^-]_{-\varepsilon}) - \mathcal{E}([a_m^+, a_n^{\dagger\,-}]_{-\varepsilon})\big)\big\}\big|_{n=m} = 0. \tag{7.13}$$

At the end, equations (4.17) and (4.18) now should be written as

$$\sum_{s,s'}\big\{\sigma_{\mu\nu}^{ss',-}(\boldsymbol{k})\,\mathcal{E}\big([a_s^{\dagger\,+}(\boldsymbol{k}), a_{s'}^-(\boldsymbol{k})]_{-\varepsilon}\big) + \sigma_{\mu\nu}^{ss',+}(\boldsymbol{k})\,\mathcal{E}\big([a_s^{\dagger\,-}(\boldsymbol{k}), a_{s'}^+(\boldsymbol{k})]_{-\varepsilon}\big)\big\} = 0 \tag{7.14}$$

$$\sum_{s,s'}\big\{l_{\mu\nu}^{ss',-}(\boldsymbol{k})\,\mathcal{E}\big([a_s^{\dagger\,+}(\boldsymbol{k}), a_{s'}^-(\boldsymbol{k})]_{-\varepsilon}\big) + l_{\mu\nu}^{ss',+}(\boldsymbol{k})\,\mathcal{E}\big([a_s^{\dagger\,-}(\boldsymbol{k}), a_{s'}^+(\boldsymbol{k})]_{-\varepsilon}\big)\big\} = 0. \tag{7.15}$$

These equations can be satisfied if we generalize (7.11) to (cf. (4.20))

$$\mathcal{E}\big([a_s^{\dagger\pm}(\boldsymbol{k}), a_{s'}^{\mp}(\boldsymbol{k})]_{-\varepsilon}\big) = 0 \tag{7.16}$$

for any s and s'. At last, the following stronger version of (7.16)

$$\mathcal{E}\big([a_m^{\dagger\pm}, a_n^{\mp}]_{-\varepsilon}\big) = 0, \tag{7.17}$$

for any $m = (t, \boldsymbol{p})$ and $n = (r, \boldsymbol{q})$, ensures the validity of (7.14) and (7.15) and thus of the uniqueness of all dynamical variables.

It is time now to call attention to the possible commutation relations. The replacement $\mathcal{D}'$, $\mathcal{D}''$, $\mathcal{D}''' \mapsto \mathcal{D} := \mathcal{E}(\mathcal{D}') = \mathcal{E}(\mathcal{D}'') = \mathcal{E}(\mathcal{D}''')$ results in corresponding changes in the whole of the material of Sect. 6. In particular, the systems of commutation relations (6.10)–(6.12), (6.31)–(6.33), (6.47)–(6.49) and (6.58)–(6.60) should be replaced respectively with:[23]

$$\big[a_l^{\pm}, \mathcal{E}(a_m^{\dagger +} \circ a_m^{-}) + \varepsilon\,\mathcal{E}(a_m^{\dagger -} \circ a_m^{+})\big]_{\_} \pm (1+\tau)\delta_{lm} a_l^{\pm} = 0 \tag{7.18}$$

$$\big[a_l^{\pm}, \mathcal{E}(a_m^{\dagger +} \circ a_m^{-}) - \varepsilon\,\mathcal{E}(a_m^{\dagger -} \circ a_m^{+})\big]_{\_} - \delta_{lm} a_l^{\pm} = 0 \tag{7.19}$$

$$\big\{(-\omega_{\mu\nu}^{\circ}(m) + \omega_{\mu\nu}^{\circ}(n))([\tilde{a}_l^{\pm}, \mathcal{E}(\tilde{a}_m^{\dagger +} \circ \tilde{a}_n^{-}) - \varepsilon\,\mathcal{E}(\tilde{a}_m^{\dagger -} \circ \tilde{a}_n^{+})]_{\_})\big\}\big|_{n=m} = 2(1+\tau)\delta_{lm}\omega_{\mu\nu}^{\circ}(l)(\tilde{a}_l^{\pm}) \tag{7.20}$$

$$\begin{aligned}\frac{(-1)^{j+1}j}{1+\tau} \sum_{s,s',t} \int \mathrm{d}^3\boldsymbol{k} \int \mathrm{d}^3\boldsymbol{p}\, v_i^{t,\pm}(\boldsymbol{p}) \big\{(\sigma_{\mu\nu}^{ss',-}(\boldsymbol{k}) + l_{\mu\nu}^{ss',-}(\boldsymbol{k}))[a_t^{\pm}(\boldsymbol{p}), \mathcal{E}(a_s^{\dagger +}(\boldsymbol{k}) \circ a_{s'}^{-}(\boldsymbol{k}))]_{\_} \\ + (\sigma_{\mu\nu}^{ss',+}(\boldsymbol{k}) + l_{\mu\nu}^{ss',+}(\boldsymbol{k}))[a_t^{\pm}(\boldsymbol{p}), \mathcal{E}(a_s^{\dagger -}(\boldsymbol{k}) \circ a_{s'}^{+}(\boldsymbol{k}))]_{\_}\big\} = \sum_{i'}\sum_{t} \int \mathrm{d}^3\boldsymbol{p} I_{i\mu\nu}^{i'} v_{i'}^{t,\pm}(\boldsymbol{p}) a_t^{\pm}(\boldsymbol{p}).\end{aligned} \tag{7.21}$$

Due to the uniqueness conditions (7.11)–(7.14), one can rewrite the terms $\mathcal{E}(a_m^{\dagger\pm} \circ a_m^{\mp})$ in (7.18)–(7.21) in a number of equivalent ways; e.g. (see (7.11))

$$\mathcal{E}(a_m^{\dagger\pm} \circ a_m^{\mp}) = \varepsilon\,\mathcal{E}(a_m^{\mp} \circ a_m^{\dagger\pm}) = \frac{1}{2}\,\mathcal{E}([a_m^{\dagger\pm}, a_m^{\mp}]_{\varepsilon}). \tag{7.22}$$

Consider the general case of a charged field, $q \neq 0$ (and hence $\tau = 0$). The system of equations (7.18)–(7.19) is then equivalent to

$$\big[a_l^{\pm}, \mathcal{E}(a_m^{\dagger\pm} \circ a_m^{\mp})\big]_{\_} = 0 \tag{7.23a}$$

$$\big[a_l^{+}, \mathcal{E}(a_m^{\dagger -} \circ a_m^{+})\big]_{\_} + \varepsilon\delta_{lm} a_l^{+} = 0 \tag{7.23b}$$

$$\big[a_l^{-}, \mathcal{E}(a_m^{\dagger +} \circ a_m^{-})\big]_{\_} - \delta_{lm} a_l^{-} = 0. \tag{7.23c}$$

These (commutation) relations ensure the simultaneous fulfillment of the Heisenberg relations (5.1) and (5.2) involving the momentum and charge operators, respectively. To ensure also the validity of (7.20), with $\tau = 0$, and, consequently, of (5.4), we generalize (7.23) to

$$\big[a_l^{\pm}, \mathcal{E}(a_m^{\dagger\pm} \circ a_n^{\mp})\big]_{\_} = 0 \tag{7.24a}$$

$$\big[a_l^{+}, \mathcal{E}(a_m^{\dagger -} \circ a_n^{+})\big]_{\_} + \varepsilon\delta_{lm} a_n^{+} = 0 \tag{7.24b}$$

$$\big[a_l^{-}, \mathcal{E}(a_m^{\dagger +} \circ a_n^{-})\big]_{\_} - \delta_{lm} a_n^{-} = 0, \tag{7.24c}$$

for any $l = (s, \boldsymbol{k})$, $m = (t, \boldsymbol{p})$ and $n = (t, \boldsymbol{q})$ (see also (6.51)). In the way pointed in Sect. 6, one can verify that (7.24) for any $l = (s, \boldsymbol{k})$, $m = (t, \boldsymbol{p})$ and $n = (r, \boldsymbol{p})$ entails (7.21) and hence (5.5). At last, to ensure the validity of all of the mentioned conditions and a

[23] To save some space, we do not write the Hermitian conjugate of the below-written equations.

suitable transition to a case of Hermitian field, for which $q=0$ and $\tau=1$ (see (3.7)), we generalize (7.24) to

$$
\begin{aligned}
&\left[a_l^+,\, \mathcal{E}(a_m^{\dagger +} \circ a_n^-)\right]_- + \tau\delta_{ln} a_m^+ = 0 &\text{(7.25a)}\\
&\left[a_l^-,\, \mathcal{E}(a_m^{\dagger -} \circ a_n^+)\right]_- - \varepsilon\tau\delta_{ln} a_m^- = 0 &\text{(7.25b)}\\
&\left[a_l^+,\, \mathcal{E}(a_m^{\dagger -} \circ a_n^+)\right]_- + \varepsilon\delta_{lm} a_n^+ = 0, &\text{(7.25c)}\\
&\left[a_l^-,\, \mathcal{E}(a_m^{\dagger +} \circ a_n^-)\right]_- - \delta_{lm} a_n^- = 0 &\text{(7.25d)}
\end{aligned}
$$

where l, m and n are arbitrary. As a result of (7.17), which we assume to hold, and $\tau a_l^{\dagger\,\pm} = \tau a_l^\pm$ (see (3.7)), the equations (7.25a) and (7.25c) (resp. (7.25b) and (7.25d)) become identical when $\tau=1$ (and hence $a_l^{\dagger\,\pm} = a_l^\pm$); for $\tau=0$ the system (7.25) reduces to (7.24). Recalling that $\varepsilon=(-1)^{2j}$ (see (3.7)), we can rewrite (7.25) in a more compact form as

$$
\begin{aligned}
&\left[a_l^\pm,\, \mathcal{E}(a_m^{\dagger\pm} \circ a_n^\mp)\right]_- + (\pm 1)^{2j+1}\tau\delta_{ln} a_m^\pm = 0 &\text{(7.26a)}\\
&\left[a_l^\pm,\, \mathcal{E}(a_m^{\dagger\mp} \circ a_n^\pm)\right]_- - (\mp 1)^{2j+1}\tau\delta_{lm} a_n^\pm = 0. &\text{(7.26b)}
\end{aligned}
$$

Since the last equation is equivalent to (see (7.17)) and use that $\varepsilon=(-1)^{2j}$)

$$
\left[a_l^\pm,\, \mathcal{E}(a_m^\pm \circ a_n^{\dagger\mp})\right]_- + (\pm 1)^{2j+1}\delta_{ln} a_m^\pm = 0, \qquad \text{(7.26b}'\text{)}
$$

it is evident that the equations (7.26a) and (7.26b) coincide for a neutral field.

Let us draw the main moral from the above considerations: the equations (7.17) are sufficient conditions for the uniqueness of the dynamical variables, while (7.26) are such conditions for the validity of the Heisenberg relations (5.1)–(5.5), in which the dynamical variables are redefined according to (7.8). So, any set of operators $a_l^\pm$ and $\mathcal{E}$, which are simultaneous solutions of (7.17) and (7.26), ensure uniqueness of the dynamical variables and at the same time the validity of the Heisenberg relations.

Consider the uniqueness problem for the solutions of the system of equations consisting of (7.17)and (7.26). Writing (7.17) as

$$
\mathcal{E}(a_m^{\dagger\pm} \circ a_n^\mp) = \varepsilon\,\mathcal{E}(a_n^\mp \circ a_m^{\dagger\pm}) = \frac{1}{2}\,\mathcal{E}([a_m^{\dagger\pm}, a_n^\mp]_\varepsilon), \qquad \text{(7.27)}
$$

which reduces to (7.22) for $n=m$, and using $\varepsilon=(-1)^{2j}$ (see (3.7)), one can verify that (7.26) is equivalent to

$$
\begin{aligned}
&\left[a_l^+,\, \mathcal{E}([a_m^+, a_n^{\dagger -}]_\varepsilon)\right]_- + 2\delta_{ln} a_m^+ = 0 &\text{(7.28a)}\\
&\left[a_l^+,\, \mathcal{E}([a_m^{\dagger +}, a_n^-]_\varepsilon)\right]_- + 2\tau\delta_{ln} a_m^+ = 0 &\text{(7.28b)}\\
&\left[a_l^-,\, \mathcal{E}([a_m^+, a_n^{\dagger -}]_\varepsilon)\right]_- - 2\tau\delta_{lm} a_n^- = 0 &\text{(7.28c)}\\
&\left[a_l^-,\, \mathcal{E}([a_m^{\dagger +}, a_n^-]_\varepsilon)\right]_- - 2\delta_{lm} a_n^- = 0. &\text{(7.28d)}
\end{aligned}
$$

The similarity between this system of equations and (6.16) is more than evident: (7.28) can be obtained from (6.16) by replacing $[\cdot,\cdot]_\varepsilon$ with $\mathcal{E}([\cdot,\cdot]_\varepsilon)$.

As it was said earlier, the bilinear commutation relations (6.13) and the identification of $\mathcal{E}$ with the normal ordering operator $\mathcal{N}$,

$$
\mathcal{E} = \mathcal{N}, \qquad \text{(7.29)}
$$

convert (7.27)–(7.28) into identities; by invoking (6.8), for $\eta=-\varepsilon$, the reader can check this via a direct calculation (see also (4.23)). However, this is not the only possible solution of (7.27)–(7.28). For example, if, in the particular case, one defines an 'anti-normal' ordering operator $\mathcal{A}$ as a linear mapping such that

$$\begin{aligned} \mathcal{A}(a_m^+ \circ a_n^{\dagger\,-}) &:= \varepsilon a_n^{\dagger\,-} \circ a_m^+ & \mathcal{A}(a_m^{\dagger\,+} \circ a_n^-) &:= \varepsilon a_n^- \circ a_m^{\dagger\,+} \\ \mathcal{A}(a_m^- \circ a_n^{\dagger\,+}) &:= a_m^- \circ a_n^{\dagger\,+} & \mathcal{A}(a_m^{\dagger\,-} \circ a_n^+) &:= a_m^{\dagger\,-} \circ a_n^+, \end{aligned} \tag{7.30}$$

then the bilinear commutation relations (6.13) and the setting $\mathcal{E} = \mathcal{A}$ provide a solution of (7.27)–(7.28); to prove this, apply (6.8) for $\eta = -\varepsilon$. Evidently, a linear combination of $\mathcal{N}$ and $\mathcal{A}$, together with (6.13), also provides a solution of (7.27)–(7.28).[24] Other solution of the same system of equations is given by $\mathcal{E} = \mathsf{id}$ and operators $a_l^\pm$ satisfying (6.16), in particular the paracommutation relations (6.20), and $a_m^{\dagger\,\pm} \circ a_n^{\mp} = \varepsilon a_n^{\mp} \circ a_m^{\dagger\,\pm}$. The problem for the general solution of (7.27)–(7.28) with respect to $\mathcal{E}$ and $a_l^\pm$ is open at present.

Let us introduce the *particle* and *antiparticle number operators* respectively by (see (7.27), (7.9) and (3.16))

$$\begin{aligned} \mathcal{N}_l &:= \frac{1}{2}\,\mathcal{E}\big([a_l^+, a_l^{\dagger\,-}]\big) = \mathcal{E}(a_l^+ \circ a_l^{\dagger\,-}) = (\mathcal{N}_l)^\dagger =: \mathcal{N}_l^\dagger \\ {}^\dagger\mathcal{N}_l &:= \frac{1}{2}\,\mathcal{E}\big([a_l^{\dagger\,+}, a_l^-]\big) = \mathcal{E}(a_l^{\dagger\,+} \circ a_l^-) = ({}^\dagger\mathcal{N}_l)^\dagger =: {}^\dagger\mathcal{N}_l^\dagger. \end{aligned} \tag{7.31}$$

As a result of the commutation relations (7.28), with $n = m$, they satisfy the equations[25]

$$[\mathcal{N}_l, a_m^+]_- = \delta_{lm} a_l^+ \tag{7.32a}$$
$$[{}^\dagger\mathcal{N}_l, a_m^+]_- = \tau \delta_{lm} a_l^+ \tag{7.32b}$$
$$[\mathcal{N}_l, a_m^{\dagger\,+}]_- = \tau \delta_{lm} a_l^{\dagger\,+} \tag{7.32c}$$
$$[{}^\dagger\mathcal{N}_l, a_m^{\dagger\,+}]_- = \delta_{lm} a_l^{\dagger\,+}. \tag{7.32d}$$

Combining (3.9)–(3.12) and (5.11)–(5.13) with (7.8), (7.27) and (7.31), we get the following expressions for the operators of the (redefined) dynamical variables:

$$\tilde{\mathcal{P}}_\mu = \frac{1}{1+\tau} \sum_l k_\mu|_{k_0 = \sqrt{m^2c^2 + \boldsymbol{k}^2}} (\mathcal{N}_l + {}^\dagger\mathcal{N}_l) \qquad l = (s, \boldsymbol{k}) \tag{7.33}$$

$$\tilde{\mathcal{Q}} = q \sum_l (-\mathcal{N}_l + {}^\dagger\mathcal{N}_l) \tag{7.34}$$

$$\tilde{\mathcal{S}}_{\mu\nu} = \frac{(-1)^{j-1/2} j\hbar}{1+\tau} \sum_{m,n} \{\varepsilon \sigma_{\mu\nu}^{mn,+} \mathcal{N}_{nm} + \sigma_{\mu\nu}^{mn,-}\, {}^\dagger\mathcal{N}_{mn})\}\big|_{\substack{m=(s,\boldsymbol{k}) \\ n=(s',\boldsymbol{k})}} \tag{7.35}$$

$$\begin{aligned} \tilde{\mathcal{L}}_{\mu\nu} = x_{0\,\mu}\, \tilde{\mathcal{P}}_\nu - x_{0\,\nu}\, \tilde{\mathcal{P}}_\mu &+ \frac{(-1)^{j-1/2} j\hbar}{1+\tau} \sum_{m,n} \{\varepsilon l_{\mu\nu}^{mn,+} \mathcal{N}_{nm} + l_{\mu\nu}^{mn,-}\, {}^\dagger\mathcal{N}_{mn})\}\big|_{\substack{m=(s,\boldsymbol{k}) \\ n=(s',\boldsymbol{k})}} \\ &+ \frac{\mathrm{i}\hbar}{2(1+\tau)} \sum_l \big\{\big(-\omega_{\mu\nu}^{\circ}(l) + \omega_{\mu\nu}^{\circ}(m)\big)(\mathcal{N}_l + {}^\dagger\mathcal{N}_l)\big\}\big|_{m=l=(s,\boldsymbol{k})} \end{aligned} \tag{7.36}$$

$$\tilde{\mathcal{M}}_{\mu\nu}^{\mathrm{sp}} = \frac{(-1)^{j-1/2} j\hbar}{1+\tau} \sum_{m,n} \{\varepsilon(\sigma_{\mu\nu}^{mn,+} + l_{\mu\nu}^{mn,+})\, \mathcal{N}_{nm} + (\sigma_{\mu\nu}^{mn,-} + l_{\mu\nu}^{mn,-})\, {}^\dagger\mathcal{N}_{mn})\}\big|_{\substack{m=(s,\boldsymbol{k}) \\ n=(s',\boldsymbol{k})}} \tag{7.37}$$

$$\tilde{\mathcal{M}}_{\mu\nu}^{\mathrm{or}} = \frac{\mathrm{i}\hbar}{2(1+\tau)} \sum_l \big\{\big(-\omega_{\mu\nu}^{\circ}(l) + \omega_{\mu\nu}^{\circ}(m)\big)(\mathcal{N}_l + {}^\dagger\mathcal{N}_l)\big\}\big|_{m=l=(s,\boldsymbol{k})}. \tag{7.38}$$

---

[24] If we admit $a_l^\pm$ to satisfy the 'anomalous" bilinear commutation relations (8.27) (see below), i.e. (6.13) with ε for $-\varepsilon$ and $(\pm 1)^{2j}$ for $(\pm 1)^{2j+1}$, then $\mathcal{E} = \mathcal{N}, \mathcal{A}$ also provides a solution of (7.27)–(7.28). However, as it was demonstrated in [13–15], the anomalous commutation relations are rejected if one works with the charge symmetric Lagrangians (3.4).

[25] The equations (7.32a) and (7.32b) correspond to (7.28a) and (7.28b), respectively, and (7.32c) and (7.32d) correspond to the Hermitian conjugate to (7.28c) and (7.28d), respectively.

Here $\omega^{\circ}_{\mu\nu}(l)$ is defined via (6.50), we have set

$$\sigma^{mn,\pm}_{\mu\nu} := \sigma^{ss',\pm}_{\mu\nu}(\boldsymbol{k}) \quad l^{mn,\pm}_{\mu\nu} := l^{ss',\pm}_{\mu\nu}(\boldsymbol{k}) \qquad \text{for } m = (s,\boldsymbol{k}) \text{ and } n = (s',\boldsymbol{k}), \tag{7.39}$$

and (see (7.27))

$$\begin{aligned} \mathcal{N}_{lm} &:= \frac{1}{2}\mathcal{E}\big([a^+_l, a^{\dagger\,-}_m]\big) = \mathcal{E}(a^+_l \circ a^{\dagger\,-}_m) = (\mathcal{N}_{ml})^\dagger =: \mathcal{N}^\dagger_{ml} \\ {}^\dagger\!\mathcal{N}_{lm} &:= \frac{1}{2}\mathcal{E}\big([a^{\dagger\,+}_l, a^-_m]\big) = \mathcal{E}(a^{\dagger\,+}_l \circ a^-_m) = ({}^\dagger\!\mathcal{N}_{ml})^\dagger =: {}^\dagger\!\mathcal{N}_{ml}{}^\dagger \end{aligned} \tag{7.40}$$

are respectively the *particle* and *antiparticle transition operators* (cf. [26, sec. 1] in a case of parafields). Obviously, we have

$$\mathcal{N}_l = \mathcal{N}_{ll} \quad {}^\dagger\!\mathcal{N}_l = {}^\dagger\!\mathcal{N}_{ll}. \tag{7.41}$$

The choice (7.29), evidently, reduces (7.33)–(7.36) to (4.24), (4.25), (4.28) and (4.29), respectively.

In terms of the operators (7.38), the commutation relations (7.28) can equivalently be rewritten as (see also (7.9))

$$[\mathcal{N}_{lm}, a^+_n]_- = \delta_{mn} a^+_l \tag{7.42a}$$
$$[{}^\dagger\!\mathcal{N}_{lm}, a^+_n]_- = \tau\delta_{mn} a^+_l \tag{7.42b}$$
$$[\mathcal{N}_{lm}, a^{\dagger\,+}_n]_- = \tau\delta_{mn} a^{\dagger\,+}_l \tag{7.42c}$$
$$[{}^\dagger\!\mathcal{N}_{lm}, a^{\dagger\,+}_n]_- = \delta_{mn} a^{\dagger\,+}_l. \tag{7.42d}$$

If $m = l$, these relations reduce to (7.32), due to (7.39).

We shall end this section with the remark that the conditions for the uniqueness of the dynamical variables and the validity of the Heisenberg relations are quite general and are not enough for fixing some commutation relations regardless of a number of additional assumptions made to reduce these conditions to the system of equations (7.27)–(7.28).

8. State vectors, vacuum and mean values

Until now we have looked on the commutation relations only from pure mathematical viewpoint. In this way, making a number of assumptions, we arrived to the system (7.27)–(7.28) of commutation relations. Further specialization of this system is, however, almost impossible without making contact with physics. For the purpose, we have to recall [1, 3, 11, 12] that the physically measurable quantities are the mean (expectation) values of the dynamical variables (in some state) and the transition amplitudes between different states. To make some conclusions from these basic assumption of the quantum theory, we must rigorously said how the states are described as vectors in system's Hilbert space $\mathcal{F}$ of states, on which all operators considered act.

For the purpose, we shall need the notion of the vacuum or, more precisely, the assumption of the existence of unique vacuum state (vector) (known also as the no-particle condition). Before defining rigorously this state, which will be denoted by $\mathcal{X}_0$, we shall heuristically analyze the properties it should possess.

First of all, the vacuum state vector $\mathcal{X}_0$ should represent a state of the field without any particles. From here two conclusions may be drawn: (i) as a field is thought as a collection of particles and a 'missing' particle should have vanishing dynamical variables, those of the vacuum should vanish too (or, more generally, to be finite constants, which can be set equal to zero by rescaling some theory's parameters) and (ii) since the operators a_l^- and $a_l^{\dagger\,-}$ are interpreted as ones that annihilate a particle characterize by $l = (s, \boldsymbol{k})$ and charge $-q$ or $+q$, respectively, and one cannot destroy an 'absent' particle, these operators should transform the vacuum into the zero vector, which may be interpreted as a complete absents of the field. Thus, we can expect that

$$\mathcal{D}(\mathcal{X}_0) = 0 \tag{8.1a}$$

$$a_l^-(\mathcal{X}_0) = 0 \quad a_l^{\dagger\,-}(\mathcal{X}_0) = 0. \tag{8.1b}$$

Further, as the operators a_l^+ and $a_l^{\dagger\,+}$ are interpreted as ones creating a particle characterize by $l = (s, \boldsymbol{k})$ and charge $-q$ or $+q$, respectively, state vectors like $a_l^+(\mathcal{X}_0)$ and $a_l^{\dagger\,+}(\mathcal{X}_0)$ should correspond to 1-particle states. Of course, a necessary condition for this is

$$\mathcal{X}_0 \neq 0, \tag{8.2}$$

due to which the vacuum can be normalize to unit,

$$\langle \mathcal{X}_0 | \mathcal{X}_0 \rangle = 1, \tag{8.3}$$

where $\langle \cdot | \cdot \rangle \colon \mathcal{F} \times \mathcal{F} \to \mathbb{C}$ is the Hermitian scalar (inner) product of $\mathcal{F}$. More generally, if $\mathcal{M}(a_{l_1}^+, a_{l_2}^{\dagger\,+}, \dots)$ is a monomial only in $i \in \mathbb{N}$ creation operators, the vector

$$\psi_{l_1 l_2 \dots} := \mathcal{M}(a_{l_1}^+, a_{l_2}^{\dagger\,+}, \dots)(\mathcal{X}_0) \tag{8.4}$$

may be expected to describe an i-particle state (with i_1 particles and i_2 antiparticles, $i_1 + i_2 = i$, where i_1 and i_2 are the number of operators a_l^+ and $a_l^{\dagger\,+}$, respectively, in $\mathcal{M}(a_{l_1}^+, a_{l_2}^{\dagger\,+}, \dots)$). Moreover, as a free field is intuitively thought as a collection of particles and antiparticles, it is natural to suppose that the vectors (8.4) form a basis in the Hilbert space $\mathcal{F}$. But the validity of this assumption depends on the accepted commutation relations; for its proof, when the paracommutation relations are adopted, see the proof of [18, p. 26, theorem I-1].

Accepting the last assumption and recalling that the transition amplitude between two states is represented via the scalar product of the corresponding to them state vectors, it is clear that for the calculation of such an amplitude is needed an effective procedure for calculation of scalar products of the form

$$\langle \psi_{l_1 l_2 \dots} | \varphi_{m_1 m_2 \dots} \rangle := \langle \mathcal{X}_0 | (\mathcal{M}(a_{l_1}^+, a_{l_2}^{\dagger\,+}, \dots))^\dagger \circ \mathcal{M}'(a_{m_1}^+, a_{m_2}^{\dagger\,+}, \dots) \mathcal{X}_0 \rangle, \tag{8.5}$$

with $\mathcal{M}$ and $\mathcal{M}'$ being monomials only in the creation operators. Similarly, for computation of the mean value of some dynamical operator $\mathcal{D}$ in a certain state, one should be equipped with a method for calculation of scalar products like

$$\langle \psi_{l_1 l_2 \dots} | \mathcal{D} \varphi_{m_1 m_2 \dots} \rangle := \langle \mathcal{X}_0 | (\mathcal{M}(a_{l_1}^+, a_{l_2}^{\dagger\,+}, \dots))^\dagger \circ \mathcal{D} \circ \mathcal{M}'(a_{m_1}^+, a_{m_2}^{\dagger\,+}, \dots) \mathcal{X}_0 \rangle. \tag{8.6}$$

Supposing, for the moment, the vacuum to be defined via (8.1), let us analyze (8.1)–(8.6). Besides, the validity of (7.27)–(7.28) will be assumed.

From the expressions (7.8) and (3.9)–(3.12) for the dynamical variables, it is clear that the condition (8.1a) can be satisfied if

$$\mathcal{E}(a_m^{\dagger\,\pm} \circ a_n^{\mp})(\mathcal{X}_0) = 0, \tag{8.7}$$

which, in view of (7.27), is equivalent to any one of the equations

$$\mathcal{E}(a_m^{\pm} \circ a_n^{\dagger\,\mp})(\mathcal{X}_0) = 0 \tag{8.8a}$$

$$\mathcal{E}([a_m^{\pm}, a_n^{\dagger\,\mp}]_{\varepsilon})(\mathcal{X}_0) = 0. \tag{8.8b}$$

Equation (8.7) is quite natural as it expresses the vanishment of all modes of the vacuum corresponding to different polarizations, 4-momentum and charge. It will be accepted hereafter.

By means of (8.8) and the commutation relations (7.28) in the form (7.42), in particular (7.32), one can explicitly calculate the action of any one of the operators (7.33)–(7.38) on the vectors (8.4): for the purpose one should simply to commute the operators $\mathcal{N}_{lm}$ (or $\mathcal{N}_l = \mathcal{N}_{ll}$) with the creation operators in (8.4) according to (7.42) (resp. (7.32)) until they act on the vacuum and, hence, giving zero, as a result of (8.8) and (7.42) (resp. (7.32)). In particular, we have the equations ($k_0 = \sqrt{m^2c^2 + \boldsymbol{k}^2}$):

$$\tilde{\mathcal{P}}_\mu\big(a_l^+(\mathcal{X}_0)\big) = k_\mu a_l^+(\mathcal{X}_0) \quad \tilde{\mathcal{P}}_\mu\big(a_l^{\dagger\,+}(\mathcal{X}_0)\big) = k_\mu a_l^{\dagger\,+}(\mathcal{X}_0) \quad l = (s, \boldsymbol{k}) \tag{8.9}$$

$$\tilde{\mathcal{Q}}\big(a_l^+(\mathcal{X}_0)\big) = -q a_l^+(\mathcal{X}_0) \quad \tilde{\mathcal{Q}}\big(a_l^{\dagger\,+}(\mathcal{X}_0)\big) = +q a_l^{\dagger\,+}(\mathcal{X}_0) \tag{8.10}$$

$$\begin{aligned}
\tilde{\mathcal{S}}_{\mu\nu}\big(a_l^+|_{l=(s,\boldsymbol{k})}(\mathcal{X}_0)\big) &= \frac{(-1)^{j-1/2} j\hbar}{1+\tau} \sum_t \{\varepsilon\sigma_{\mu\nu}^{lm,+} + \tau\sigma_{\mu\nu}^{ml,-}\}|_{m=(t,\boldsymbol{k})} a_m^+|_{m=(t,k)}(\mathcal{X}_0)\\
\tilde{\mathcal{S}}_{\mu\nu}\big(a_l^{\dagger\,+}|_{l=(s,\boldsymbol{k})}(\mathcal{X}_0)\big) &= \frac{(-1)^{j-1/2} j\hbar}{1+\tau} \sum_t \{\varepsilon\tau\sigma_{\mu\nu}^{lm,+} + \sigma_{\mu\nu}^{ml,-}\}|_{m=(t,\boldsymbol{k})} a_m^{\dagger\,+}|_{m=(t,k)}(\mathcal{X}_0)
\end{aligned} \tag{8.11}$$

$$\begin{aligned}
\tilde{\mathcal{L}}_{\mu\nu}\big(a_l^+|_{l=(s,\boldsymbol{k})}(\mathcal{X}_0)\big) = {}&(x_{0\,\mu}k_\nu - x_{0\,\nu}k_\mu)(a_l^+)(\mathcal{X}_0) - \mathrm{i}\hbar\big(\omega_{\mu\nu}^{\circ}(l)(a_l^+)\big)(\mathcal{X}_0)\\
&+ \frac{(-1)^{j-1/2} j\hbar}{1+\tau} \sum_t \{\varepsilon l_{\mu\nu}^{lm,+} + \tau l_{\mu\nu}^{ml,-}\}|_{m=(t,\boldsymbol{k})} a_m^+|_{m=(t,k)}(\mathcal{X}_0)\\
\tilde{\mathcal{L}}_{\mu\nu}\big(a_l^{\dagger\,+}|_{l=(s,\boldsymbol{k})}(\mathcal{X}_0)\big) = {}&(x_{0\,\mu}k_\nu - x_{0\,\nu}k_\mu)(a_l^{\dagger\,+})(\mathcal{X}_0) - \mathrm{i}\hbar\big(\omega_{\mu\nu}^{\circ}(l)(a_l^{\dagger\,+})\big)(\mathcal{X}_0)\\
&+ \frac{(-1)^{j-1/2} j\hbar}{1+\tau} \sum_t \{\varepsilon\tau l_{\mu\nu}^{lm,+} + l_{\mu\nu}^{ml,-}\}|_{m=(t,\boldsymbol{k})} a_m^{\dagger\,+}|_{m=(t,k)}(\mathcal{X}_0)
\end{aligned} \tag{8.12}$$

$$\begin{aligned}
\tilde{\mathcal{M}}_{\mu\nu}^{\mathrm{sp}}\big(a_l^+|_{l=(s,\boldsymbol{k})}(\mathcal{X}_0)\big) = {}&\frac{(-1)^{j-1/2} j\hbar}{1+\tau} \sum_t \{\varepsilon(\sigma_{\mu\nu}^{lm,+} + l_{\mu\nu}^{lm,+})\\
&+ \tau(\sigma_{\mu\nu}^{ml,-} + l_{\mu\nu}^{ml,-})\}|_{m=(t,\boldsymbol{k})} a_m^+|_{m=(t,k)}(\mathcal{X}_0)\\
\tilde{\mathcal{M}}_{\mu\nu}^{\mathrm{sp}}\big(a_l^{\dagger\,+}|_{l=(s,\boldsymbol{k})}(\mathcal{X}_0)\big) = {}&\frac{(-1)^{j-1/2} j\hbar}{1+\tau} \sum_t \{\varepsilon\tau(\sigma_{\mu\nu}^{lm,+} + l_{\mu\nu}^{lm,+})\\
&+ (\sigma_{\mu\nu}^{ml,-} + l_{\mu\nu}^{ml,-})\}|_{m=(t,\boldsymbol{k})} a_m^{\dagger\,+}|_{m=(t,k)}(\mathcal{X}_0)
\end{aligned} \tag{8.13}$$

$$\tilde{\mathcal{M}}_{\mu\nu}^{\mathrm{or}}\big(\tilde{a}_l^+(\mathcal{X}_0)\big) = -\mathrm{i}\hbar\big(\omega_{\mu\nu}^{\circ}(l)(\tilde{a}_l^+)\big)(\mathcal{X}_0) \quad \tilde{\mathcal{M}}_{\mu\nu}^{\mathrm{or}}\big(\tilde{a}_l^{\dagger\,+}(\mathcal{X}_0)\big) = -\mathrm{i}\hbar\big(\omega_{\mu\nu}^{\circ}(l)(\tilde{a}_l^{\dagger\,+})\big)(\mathcal{X}_0). \tag{8.14}$$

These equations and similar, but more complicated, ones with an arbitrary monomial in the creation operators for a_l^+ or $a_l^{\dagger\,+}$ are the base for the particle interpretation of the quantum theory of free fields. For instance, in view of (8.9) and (8.10), the state vectors $a_l^+(\mathcal{X}_0)$ and $a_l^{\dagger\,+}(\mathcal{X}_0)$ are interpreted as ones representing particles with 4-momentum $(\sqrt{m^2c^2 + \boldsymbol{k}^2}, \boldsymbol{k})$ and charges $-q$ and $+q$, respectively; similar multiparticle interpretation can be given to the general vectors (8.4) too.

The equations (8.9)–(8.12) completely agree with similar ones obtained in [13–15] on the base of the bilinear commutation relations (6.13).

By means of (8.7), the expression (8.6) can be represented as a linear combination of terms like (8.5). Indeed, as $\mathcal{D}$ is a linear combinations of terms like $\mathcal{E}(a_m^{\dagger\,\pm} \circ a_n^{\mp})$, by means of the relations (7.28) we can commute each of these terms with the creation (resp. annihilation) operators in the monomial $\mathcal{M}'(a_{m_1}^+, a_{m_2}^{\dagger\,+}, \dots)$ (resp. $(\mathcal{M}(a_{l_1}^+, a_{l_2}^{\dagger\,+}, \dots))^\dagger = \mathcal{M}''(a_{l_1}^{\dagger\,-}, a_{l_2}^-, \dots)$) and thus moving them to the right (resp. left) until they act on the vacuum $\mathcal{X}_0$, giving the zero vector — see (8.7). In this way the matrix elements of the dynamical variables, in particular their mean values, can be expressed as linear combinations of scalar products of the form (8.5). Therefore the supposition (8.7) reduces the computation of mean values of dynamical variables to the one of the vacuum mean value of a product (composition) of creation and annihilation operators in which the former operators stand to the right of the latter ones. (Such a product of creation and annihilation operators can be called their 'antinormal' product; cf. the properties (7.30) of the antinormal ordering operator $\mathcal{A}$.)

The calculation of such mean values, like (8.5) for states $\psi, \varphi \neq \mathcal{X}_0$, however, cannot be done (on the base of (7.27)–(7.28), (8.7) and (8.1a)) unless additional assumption are made. For the purpose one needs some kind of commutation relations by means of which the creation (resp. annihilation) operators on the r.h.s. of (8.5) to be moved to the left (resp. right) until they act on the left (resp. right) vacuum vector $\mathcal{X}_0$; as a result of this operation, the expressions between the two vacuum vectors in (8.5) should transform into a linear combination of constant terms and such with no contribution in (8.5). (Examples of the last type of terms are $\mathcal{E}(a_m^{\dagger\pm} \circ a^{\mp})$ and normally ordered products of creation and annihilation operators.) An alternative procedure may consists in defining axiomatically the values of all or some of the mean values (8.5) or, more stronger, the explicit action of all or some of the operators, entering in the r.h.s. of (8.5), on the vacuum.[26] It is clear, both proposed schemes should be consistent with the relations (7.27)–(7.28), (8.1b) and (8.7)–(8.8).

Let us summarize the problem before us: the operator $\mathcal{E}$ in (7.27)–(7.28) has to be fixed and a method for computation of scalar products like (8.5) should be given provided the vacuum vector $\mathcal{X}_0$ satisfies (8.1b), (8.2), (8.3) and (8.7). Two possible ways for exploration of this problem were indicated above.

Consider the operator $\mathcal{E}$. Supposing $\mathcal{E}(a_m^{\dagger\pm} \circ a_n^{\mp})$ to be a function only of $a_m^{\dagger\pm}$ and $a_n^{\mp}$, we, in view of (8.1b), can write $\mathcal{E}(a_m^{\dagger\pm} \circ a_n^{\mp}) = f^{\pm}(a_m^{\dagger\pm} \circ a_n^{\mp}) \circ b$ with $b = a_n^-$ (upper sign) or $b = a_m^{\dagger -}$ (lower sign) and some functions $f^{\pm}$. Applying (7.27), we obtain (do not sum over l)

$$\begin{aligned} \mathcal{E}(a_m^{\dagger +} \circ a_l^-) &= f^+(a_m^{\dagger +}, a_l^-) \circ a_l^- \qquad & \mathcal{E}(a_m^+ \circ a_l^{\dagger -}) &= f^-(a_m^+, a_l^{\dagger -}) \circ a_l^{\dagger -} \\ \mathcal{E}(a_l^- \circ a_m^{\dagger +}) &= \varepsilon f^+(a_m^{\dagger +}, a_l^-) \circ a_l^- \qquad & \mathcal{E}(a_l^{\dagger -} \circ a_m^+) &= \varepsilon f^-(a_m^+, a_l^{\dagger -}) \circ a_l^{\dagger -}. \end{aligned}$$

Since $\mathcal{E}$ is a linear operator, the expression $\mathcal{E}(a_m^{\dagger\pm} \circ a_n^{\mp})$ turns to be a linear and homogeneous function of $a_m^{\dagger\pm}$ and $a_n^{\mp}$, which immediately implies $f^{\pm}(A, B) = \lambda^{\pm} A$ for operators A and B and some constants $\lambda^{\pm} \in \mathbb{C}$. For future convenience, we assume $\lambda^{\pm} = 1$, which can be achieved via a suitable renormalization of the creation and annihilation operators.[27] Thus, the last equations reduce to

$$\mathcal{E}(a_m^{\dagger +} \circ a_l^-) = a_m^{\dagger +} \circ a_l^- \qquad \mathcal{E}(a_m^+ \circ a_l^{\dagger -}) = a_m^+ \circ a_l^{\dagger -} \tag{8.15a}$$

$$\mathcal{E}(a_l^- \circ a_m^{\dagger +}) = \varepsilon a_m^{\dagger +} \circ a_l^- \qquad \mathcal{E}(a_l^{\dagger -} \circ a_m^+) = \varepsilon a_m^+ \circ a_l^{\dagger -}. \tag{8.15b}$$

Evidently, these equations convert (7.27), (8.7) and (8.8) into identities. Comparing (8.15) and (4.22), we see that the identification

$$\mathcal{E} = \mathcal{N} \tag{8.16}$$

of the operator $\mathcal{E}$ with the normal ordering operator $\mathcal{N}$ is quite natural. However, for our purposes, this identification is not necessary as only the equations (8.15), not the general definition of $\mathcal{N}$, will be employed.

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[26] Such an approach resembles the axiomatic description of the scattering matrix [1, 7, 8].

[27] Since $\lambda^+ = 0$ or/and $\lambda^- = 0$ implies $\mathcal{D} = 0$, due to (7.8), these values are excluded for evident reasons.

As a result of (8.15), the commutation relations (7.28) now read:

$$[a_l^+, a_m^+ \circ a_n^{\dagger -}]_- + \delta_{ln} a_m^+ = 0 \tag{8.17a}$$
$$[a_l^+, a_m^{\dagger +} \circ a_n^-]_- + \tau \delta_{ln} a_m^+ = 0 \tag{8.17b}$$
$$[a_l^-, a_m^+ \circ a_n^{\dagger -}]_- - \tau \delta_{lm} a_n^- = 0 \tag{8.17c}$$
$$[a_l^-, a_m^{\dagger +} \circ a_n^-]_- - \delta_{lm} a_n^- = 0. \tag{8.17d}$$

(In a sense, these relations are 'one half' of the (para)commutation relations (6.16): the latter are a sum of the former and the ones obtained from (8.17) via the changes $a_m^+ \circ a_n^{\dagger -} \mapsto \varepsilon a_n^{\dagger -} \circ a_m^+$ and $a_m^{\dagger +} \circ a_n^- \mapsto \varepsilon a_n^- \circ a_m^{\dagger +}$; the last relations correspond to (7.28) with $\mathcal{E} = \mathcal{A}$, $\mathcal{A}$ being the antinormal ordering operator — see (7.30). Said differently, up to the replacement $a_i^{\pm} \mapsto \sqrt{2} a_l^{\pm}$ for all l, the relations (8.17) are identical with (6.16) for $\varepsilon = 0$; as noted in [26, the remarks following theorem 2 in sec. 1], this is a quite exceptional case from the view-point of parastatistics theory.) By means of (6.8) for $\eta = -\varepsilon$, one can verify that equations (8.17) agree with the bilinear commutation relations (6.13), i.e. (6.13) convert (8.17) into identities.

The equations (8.15) imply the following explicit forms of the number operators (7.31) and the transition operators (7.40):

$$\mathcal{N}_l = a_l^+ \circ a_l^{\dagger -} \quad {}^{\dagger}\mathcal{N}_l = a_l^{\dagger +} \circ a_l^- \tag{8.18}$$
$$\mathcal{N}_{lm} = a_l^+ \circ a_m^{\dagger -} \quad {}^{\dagger}\mathcal{N}_{lm} = a_l^{\dagger +} \circ a_m^-. \tag{8.19}$$

As a result of them, the equations (7.33)–(7.36) are simply a different form of writing of (4.24), (4.25), (4.28) and (4.29), respectively.

Let us return to the problem of calculation of vacuum mean values of antinormal ordered products like (8.5). In view of (8.1b) and (8.3), the simplest of them are

$$\langle \mathcal{X}_0 | \lambda \,\mathsf{id}_{\mathcal{F}}(\mathcal{X}_0) \rangle = \lambda \quad \langle \mathcal{X}_0 | \mathcal{M}^{\pm}(\mathcal{X}_0) \rangle = 0 \tag{8.20}$$

where $\lambda \in \mathbb{C}$ and $\mathcal{M}^+$ (resp. $\mathcal{M}^-$) is any monomial of degree not less than 1 only in the creation (resp. annihilation) operators; e.g. $\mathcal{M}^{\pm} = a_l^{\pm}, a_l^{\dagger \pm}, a_{l_1}^{\pm} \circ a_{l_2}^{\pm}, a_{l_1}^{\pm} \circ a_{l_2}^{\dagger \pm}$. These equations, with $\lambda = 1$, are another form of what is called the *stability of the vacuum*: if $\mathcal{X}_i$ denotes an i-particle state, $i \in \mathbb{N} \cup \{0\}$, then, by virtue of (8.20) and the particle interpretation of (8.4), we have

$$\langle \mathcal{X}_i | \mathcal{X}_0 \rangle = \delta_{i0}, \tag{8.21}$$

i.e. the only non-forbidden transition into (from) the vacuum is from (into) the vacuum. More generally, if $\mathcal{X}_{i',0}$ and $\mathcal{X}_{0,j''}$ denote respectively i'-particle and j''-antiparticle states, with $\mathcal{X}_{0,0} := \mathcal{X}_0$, then

$$\langle \mathcal{X}_{i',0} | \mathcal{X}_{0,j''} \rangle = \delta_{i'0} \delta_{0j''}, \tag{8.22}$$

i.e. transitions between two states consisting entirely of particles and antiparticles, respectively, are forbidden unless both states coincide with the vacuum. Since we are dealing with free fields, one can expect that the amplitude of a transitions from an (i'-particle + j'-antiparticle) state $\mathcal{X}_{i',j'}$ into an (i''-particle + j''-antiparticle) state $\mathcal{X}_{i'',j''}$ is

$$\langle \mathcal{X}_{i',j'} | \mathcal{X}_{i'',j''} \rangle = \delta_{i'i''} \delta_{j'j''}, \tag{8.23}$$

but, however, the proof of this hypothesis requires new assumptions (*vide infra*).

Let us try to employ (8.17) for calculation of expressions like (8.5). Acting with (8.17) and their Hermitian conjugate on the vacuum, in view of (8.1b), we get

$$\begin{aligned} &a_m^+ \circ (-a_n^{\dagger -} \circ a_l^+ + \delta_{ln} \,\mathsf{id}_{\mathcal{F}})(\mathcal{X}_0) = 0 \quad a_n^{\dagger +} \circ (a_m^- \circ a_l^{\dagger +} - \delta_{lm} \,\mathsf{id}_{\mathcal{F}})(\mathcal{X}_0) = 0 \\ &a_m^{\dagger +} \circ (-a_n^- \circ a_l^+ + \tau \delta_{ln} \,\mathsf{id}_{\mathcal{F}})(\mathcal{X}_0) = 0 \quad a_n^+ \circ (a_m^{\dagger -} \circ a_l^{\dagger +} - \tau \delta_{lm} \,\mathsf{id}_{\mathcal{F}})(\mathcal{X}_0) = 0. \end{aligned} \tag{8.24}$$

These equalities, as well as (8.17), cannot help directly to compute vacuum mean values of antinormally ordered products of creation and annihilation operators. But the equations (8.24) suggest the restrictions[28]

$$
\begin{aligned}
&a_l^{\dagger\,-} \circ a_m^{+}(\mathcal{X}_0) = \delta_{lm}\,\mathcal{X}_0 \quad && a_l^{-} \circ a_m^{\dagger\,+}(\mathcal{X}_0) = \delta_{lm}\,\mathcal{X}_0 \\
&a_l^{-} \circ a_m^{+}(\mathcal{X}_0) = \tau\delta_{lm}\,\mathcal{X}_0 \quad && a_l^{\dagger\,-} \circ a_m^{\dagger\,+}(\mathcal{X}_0) = \tau\delta_{lm}\,\mathcal{X}_0
\end{aligned}
\tag{8.25}
$$

to be added to the definition of the vacuum. These conditions convert (8.24) into identities and, in this sense agree with (8.17) and, consequently, with the bilinear commutation relations (6.13). Recall [16,18], the relations (8.25) are similar to ones accepted in the parafield theory and coincide with that for parastatistics of order $p = 1$; however, here we do not suppose the validity of the paracommutation relations (6.20) (or (6.16)). Equipped with (8.25), one is able to calculate the r.h.s. of (8.5) for any monomial $\mathcal{M}$ (resp. $\mathcal{M}'$) and monomials $\mathcal{M}'$ (resp. $\mathcal{M}$) of degree 1, $\deg \mathcal{M}' = 1$ (resp. $\deg \mathcal{M} = 1$).[29] Indeed, (8.25), (8.1b) and (8.3) entail:

$$
\begin{aligned}
&\langle \mathcal{X}_0 | a_l^{\dagger\,-} \circ a_m^{+}(\mathcal{X}_0)\rangle = \langle \mathcal{X}_0 | a_l^{-} \circ a_m^{\dagger\,+}(\mathcal{X}_0)\rangle = \delta_{lm} \\
&\langle \mathcal{X}_0 | a_l^{-} \circ a_m^{+}(\mathcal{X}_0)\rangle = \langle \mathcal{X}_0 | a_l^{\dagger\,-} \circ a_m^{\dagger\,+}(\mathcal{X}_0)\rangle = \tau\delta_{lm} \\
&\langle \mathcal{X}_0 | (\mathcal{M}(a_{l_1}^{+}, a_{l_2}^{\dagger\,+}, \cdots))^{\dagger} \circ a_m^{+}(\mathcal{X}_0)\rangle = \langle \mathcal{X}_0 | (\mathcal{M}(a_{l_1}^{+}, a_{l_2}^{\dagger\,+}, \cdots))^{\dagger} \circ a_m^{\dagger\,+}(\mathcal{X}_0)\rangle = 0 \qquad \deg \mathcal{M} \ge 2 \\
&\langle \mathcal{X}_0 | a_l^{-} \circ \mathcal{M}(a_{m_1}^{+}, a_{m_2}^{\dagger\,+}, \cdots)(\mathcal{X}_0)\rangle = \langle \mathcal{X}_0 | a_l^{\dagger\,-} \circ \mathcal{M}(a_{m_1}^{+}, a_{m_2}^{\dagger\,+}, \cdots)(\mathcal{X}_0)\rangle = 0 \qquad \deg \mathcal{M} \ge 2.
\end{aligned}
\tag{8.26}
$$

Hereof the equation (8.23) for $i' + j' = 1$ (resp. $i'' + j'' = 1$) and arbitrary i'' and j'' (resp. i' and j') follows.

However, it is not difficult to be realized, the calculation of (8.5) in cases more general than (8.20) and (8.26) is not possible on the base of the assumptions made until now.[30] At this point, one is free so set in an arbitrary way the r.h.s. of (8.5) in the mentioned general case or to add to (8.17) (and, possibly, (8.25)) other (commutation) relations by means of which the r.h.s. of (8.5) to be calculated explicitly; other approaches, e.g. some mixture of the just pointed ones, for finding the explicit form of (8.5) are evidently also possible. Since expressions like (8.5) are directly connected with observable experimental results, the only criterion for solving the problem for calculating the r.h.s. of (8.5) in the general case can be the agreement with the existing experimental data. As it is known [1,3,11,12], at present (almost?) all of them are satisfactory described within the framework of the bilinear commutation relations (6.13). This means that, from physical point of view, the theory should be considered as realistic one if the r.h.s. of (8.5) is the same as if (6.13) are valid or is reducible to it for some particular realization of an accepted method of calculation, e.g. if one accepts some commutation relations, like the paracommutation ones, which are a generalization of (6.13) and reduce to them as a special case (see, e.g., (6.20)). It should be noted, the conditions (8.1b)–(8.3) and (8.25) are enough for calculating (8.5) if (6.16), or its versions (6.17) or (6.20), are accepted (cf. [16]). The causes for that difference are replacements like $[a_m^{+}, a_n^{\dagger\,-}]_{\_} \mapsto 2a_m^{+} \circ a_n^{\dagger\,-}$, when one passes from (6.16) to (8.17); the existence of terms like $a_n^{\dagger\,-} \circ a_m^{+} a_l^{+}$ in (6.16) are responsible for the possibility to calculate (8.5).

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[28] Since the operators $a_l^{\pm}$ and $a_l^{\dagger\,\pm}$ are, generally, degenerate (with no inverse ones), we cannot say that (8.24) implies (8.25).

[29] For $\deg \mathcal{M}' = 0$ (resp. $\deg \mathcal{M}' = 0$) — see (8.20).

[30] It should be noted, the conditions (8.1b)–(8.3) and (8.25) are enough for calculating (8.5) if the relations (6.16), or their version (6.20), are accepted (cf. [16]). The cause for that difference is in replacements like $[a_m^{+}, a_n^{\dagger\,-}]_{\_} \mapsto 2a_m^{+} \circ a_n^{\dagger\,-}$, when one passes from (6.16) to (8.17); the existence of terms like $a_n^{\dagger\,-} \circ a_m^{+} \circ a_l^{+}$ in (6.16) is responsible for the possibility to calculate (8.5), in case (6.16) hold.

If evidences appear for events for which (8.5) takes other values, one should look, e.g., for other commutation relations leading to desired mean values. As an example of the last type can be pointed the following *anomalous bilinear commutation relations* (cf. (6.13))

$$\begin{aligned}
&[a_l^{\pm}, a_m^{\pm}]_{\varepsilon} = 0 && [a_l^{\dagger\pm}, a_m^{\dagger\pm}]_{\varepsilon} = 0 \\
&[a_l^{\mp}, a_m^{\pm}]_{\varepsilon} = (\pm 1)^{2j} \tau \delta_{lm} \,\mathsf{id}_{\mathcal{F}} && [a_l^{\dagger\mp}, a_m^{\dagger\pm}]_{\varepsilon} = (\pm 1)^{2j} \tau \delta_{lm} \,\mathsf{id}_{\mathcal{F}} \\
&[a_l^{\pm}, a_m^{\dagger\pm}]_{\varepsilon} = 0 && [a_l^{\dagger\pm}, a_m^{\pm}]_{\varepsilon} = 0 \\
&[a_l^{\mp}, a_m^{\dagger\pm}]_{\varepsilon} = (\pm 1)^{2j} \delta_{lm} \,\mathsf{id}_{\mathcal{F}} && [a_l^{\dagger\mp}, a_m^{\pm}]_{\varepsilon} = (\pm 1)^{2j} \delta_{lm} \,\mathsf{id}_{\mathcal{F}},
\end{aligned} \tag{8.27}$$

which should be imposed after expressions like $\mathcal{E}(a_m^{\dagger\pm} \circ a_n^{\mp})$ are explicitly calculated. These relations convert (8.17) and (8.25) into identities and by their means the r.h.s. of (8.5) can be calculated explicitly, but, as it is well known [1,3,11,12,27] they lead to deep contradictions in the theory, due to which should be rejected.[31]

At present, it seems, the bilinear commutation relations (6.13) are the only known commutation relations which satisfy all of the mentioned conditions and simultaneously provide an evident procedure for effective calculation of all expressions of the form (8.5). (Besides, for them and for the paracommutation relations the vectors (8.4) form a base, the Fock base, for the system's Hilbert space of states [18].) In this connection, we want to mention that the paracommutation relations (6.16) (or their conventional version (6.20)), if imposed as additional restrictions to the theory together with (8.17), reduce in this particular case to (6.13) as the conditions (8.25) show that we are dealing with a parafield of order $p = 1$, i.e. with an ordinary field [17,18].[32]

Ending this section, let us return to the definition of the vacuum $\mathcal{X}_0$. It, generally, depends on the adopted commutation relations. For instance, in a case of the bilinear commutation relations (6.13) it consists of the equations (8.1a)–(8.3), while in a case of the paracommutation relations (6.16) (or other ones generalizing (6.13)) it includes (8.1a)–(8.3) and (8.25).

9. Commutation relations for several coexisting different free fields

Until now we have considered commutation relations for a single free field, which can be scalar, or spinor or vector one. The present section is devoted to similar treatment of a system consisting of several, not less than two, *different free* fields. In our context, the fields may differ by their masses and/or charges and/or spins; e.g., the system may consist of charged scalar field, neutral scalar field, massless spinor field, massive spinor field and massless neural vector field. It is *a priori* evident, the commutation relations regarding only one field of the system should be as discussed in the previous sections. The problem is to be derived/postulated commutation relations concerning *different* fields. It will be shown, the developed Lagrangian formalism provides a natural base for such an investigation and makes superfluous some of the assumptions made, for example, in [17, p. B 1159, left column] or in [18, sec. 12.1], where systems of different parafields are explored.

To begin with, let us introduce suitable notation. With the indices $\alpha, \beta, \gamma = 1, 2, \dots, N$ will be distinguished the different fields of the system, with $N \in \mathbb{N}$, $N \geq 2$, being their number, and the corresponding to them quantities. Let q^{α} and j^{α} be respectively the charge

[31] As it was demonstrated in [13–15], a quantization like (8.27) contradicts to (is rejected by) the charge symmetric Lagrangians (3.4).

[32] Notice, as a result of (8.17), the relations (6.16) correspond to (7.28) for $\mathcal{E} = \mathcal{A}$, with $\mathcal{A}$ being the antinormal ordering operator (see (7.30)).

and spin of the α-th field. Similarly to (3.7), we define

$$j^\alpha := \begin{cases} 0 & \text{for scalar } \alpha\text{-th field} \\ \frac{1}{2} & \text{for spinor } \alpha\text{-th field} \\ 1 & \text{for vector } \alpha\text{-th field} \end{cases} \qquad \tau^\alpha := \begin{cases} 1 & \text{for } q^\alpha = 0 \text{ (neutral (Hermitian) field)} \\ 0 & \text{for } q^\alpha \neq 0 \text{ (charged (non-Hermitian) field)} \end{cases}$$
$$\varepsilon^\alpha := (-1)^{2j^\alpha} = \begin{cases} +1 & \text{for integer } j^\alpha \text{ (bose fields)} \\ -1 & \text{for half-integer } j^\alpha \text{ (fermi fields)} \end{cases}. \tag{9.1}$$

Suppose $\mathcal{L}^\alpha$ is the Lagrangian of the α-field. For definiteness, we assume $\mathcal{L}^\alpha$ for all α to be given by one and the same set of equations, viz. (3.1), or (3.3) or (3.4). To save some space, below the case (3.4), corresponding to charge symmetric Lagrangians, will be considered in more details; the reader can explore other cases as exercises.

Since the Lagrangian of our system of free fields is

$$\mathcal{L} := \sum_\alpha \mathcal{L}^\alpha, \tag{9.2}$$

the dynamical variables are

$$\mathcal{D} = \sum_\alpha \mathcal{D}^\alpha \tag{9.3}$$

and the corresponding system of Euler-Lagrange equations consists of the independent equations for each of the fields of the system (see (3.6) with $\mathcal{L}^\alpha$ for $\mathcal{L}$). This allows an introduction of independent creation and annihilation operators for each field. The ones for the α-th field will be denoted by $a^\pm_{\alpha,s^\alpha}(\boldsymbol{k})$ and $a^{\dagger\,\pm}_{\alpha,s^\alpha}(\boldsymbol{k})$; notice, the values of the polarization variables generally depend on the field considered and, therefore, they also are labeled with index α for the α-th field. For brevity, we shall use the collective indices l^α, m^α and n^α, with $l^\alpha := (\alpha, s^\alpha, \boldsymbol{k})$ etc., in terms of which the last operators are $a^\pm_{l^\alpha}$ and $a^{\dagger\,\pm}_{l^\alpha}$, respectively. The particular expressions for the dynamical operators $\mathcal{D}^\alpha$ are given via (3.9)–(3.12) in which the following changes should be made:

$$\begin{gathered} \tau \mapsto \tau^\alpha \quad j \mapsto j^\alpha \quad \varepsilon \mapsto \varepsilon^\alpha \quad s \mapsto s^\alpha \quad s' \mapsto s'^\alpha \\ \sigma^{ss',\pm}_{\mu\nu}(\boldsymbol{k}) \mapsto \sigma^{s^\alpha s'^\alpha,\pm}_{\mu\nu}(\boldsymbol{k}) \quad l^{ss',\pm}_{\mu\nu}(\boldsymbol{k}) \mapsto l^{s^\alpha s'^\alpha,\pm}_{\mu\nu}(\boldsymbol{k}). \end{gathered} \tag{9.4}$$

The content of sections 4 and 5 remains valid *mutatis mutandis*, viz. provided the just pointed changes (9.4) are made and the (integral) dynamical variables are understood in conformity with (9.3).

9.1. Commutation relations connected with the momentum operator. Problems and their possible solutions

In sections 6–8, however, substantial changes occur; for instance, when one passes from (6.12) or (6.15) to (6.16). We shall consider them briefly in a case when one starts from the charge symmetric Lagrangians (3.4).

The basic relations (6.12), which arise from the Heisenberg relation (5.1) concerning the momentum operator, now read (here and below, do not sum over α, and/or β and/or γ if the opposite is not indicated explicitly!)

$$\big[a^\pm_{l^\alpha}, [a^{\dagger\,+}_{m^\beta}, a^-_{m^\beta}]_{\varepsilon^\beta} + [a^+_{m^\beta}, a^{\dagger\,-}_{m^\beta}]_{\varepsilon^\beta}\big]_- \pm (1+\tau)\delta_{l^\alpha m^\beta} a^\pm_{l^\alpha} = 0 \tag{9.5a}$$
$$\big[a^{\dagger\,\pm}_{l^\alpha}, [a^{\dagger\,+}_{m^\beta}, a^-_{m^\beta}]_{\varepsilon^\beta} + [a^+_{m^\beta}, a^{\dagger\,-}_{m^\beta}]_{\varepsilon^\beta}\big]_- \pm (1+\tau)\delta_{l^\alpha m^\beta} a^{\dagger\,\pm}_{l^\alpha} = 0. \tag{9.5b}$$

It is trivial to be seen, the following generalizations of respectively (6.14) and (6.15)

$$\big[a^{\pm}_{l^\alpha}, [a^{+}_{m^\beta}, a^{\dagger -}_{m^\beta}]_{\varepsilon^\beta}\big]_- \pm (1+\tau^\beta)\delta_{l^\alpha m^\beta} a^{\pm}_{l^\alpha} = 0 \tag{9.6a}$$
$$\big[a^{\pm}_{l^\alpha}, [a^{\dagger +}_{m^\beta}, a^{-}_{m^\beta}]_{\varepsilon^\beta}\big]_- \pm (1+\tau^\beta)\delta_{l^\alpha m^\beta} a^{\pm}_{l^\alpha} = 0 \tag{9.6b}$$
$$\big[a^{\dagger\pm}_{l^\alpha}, [a^{+}_{m^\beta}, a^{\dagger -}_{m^\beta}]_{\varepsilon^\beta}\big]_- \pm (1+\tau^\beta)\delta_{l^\alpha m^\beta} a^{\dagger\pm}_{l^\alpha} = 0 \tag{9.6c}$$
$$\big[a^{\dagger\pm}_{l^\alpha}, [a^{\dagger +}_{m^\beta}, a^{-}_{m^\beta}]_{\varepsilon^\beta}\big]_- \pm (1+\tau^\beta)\delta_{l^\alpha m^\beta} a^{\dagger\pm}_{l^\alpha} = 0 \tag{9.6d}$$

$$\big[a^{+}_{l^\alpha}, [a^{+}_{m^\beta}, a^{\dagger -}_{m^\beta}]_{\varepsilon^\beta}\big]_- + 2\delta_{l^\alpha m^\beta} a^{+}_{l^\alpha} = 0 \tag{9.7a}$$
$$\big[a^{+}_{l^\alpha}, [a^{\dagger +}_{m^\beta}, a^{-}_{m^\beta}]_{\varepsilon^\beta}\big]_- + 2\tau^\beta\delta_{l^\alpha m^\beta} a^{+}_{l^\alpha} = 0 \tag{9.7b}$$
$$\big[a^{-}_{l^\alpha}, [a^{+}_{m^\beta}, a^{\dagger -}_{m^\beta}]_{\varepsilon^\beta}\big]_- - 2\tau^\beta\delta_{l^\alpha m^\beta} a^{-}_{l^\alpha} = 0 \tag{9.7c}$$
$$\big[a^{-}_{l^\alpha}, [a^{\dagger +}_{m^\beta}, a^{-}_{m^\beta}]_{\varepsilon^\beta}\big]_- - 2\delta_{l^\alpha m^\beta} a^{-}_{l^\alpha} = 0 \tag{9.7d}$$

provide a solution of (9.5) in a sense that they convert it into identity. As it was said in Sect. 6, the equations (9.6) (resp. (9.7)) for a single field, i.e. for $\beta = \alpha$, agree (resp. disagree) with the bilinear commutation relations (6.13).

The only problem arises when one tries to generalize, e.g., the relations (9.7) in a way similar to the transition from (6.15) to (6.16). Its essence is in the generalization of expressions like $[a^{\dagger\pm}_{m^\beta}, a^{\mp}_{m^\beta}]_{\varepsilon^\beta}$ and $\tau^\beta\delta_{l^\alpha m^\beta} a^{\pm}_{l^\alpha}$. When passing from (6.15) to (6.16), the indices l and m are changed so that the obtained equations to be consistent with (6.13); of course, the numbers ε and τ are preserved because this change does not concern the field regarded. But the situation with (9.7) is different in two directions:

(i) If we change the pair (m^β, m^β) in $[a^{\dagger\pm}_{m^\beta}, a^{\mp}_{m^\beta}]_{\varepsilon^\beta}$ with (m^β, n^γ), then with what the number ε^β should be replace? With ε^β, or ε^γ or with something else? Similarly, if the mentioned changed is performed, with what the multiplier τ^β in $\tau^\beta\delta_{l^\alpha m^\beta} a^{\pm}_{l^\alpha}$ should be replaced? The problem is that the numbers ε^β and τ^β are related to terms like $a^{\dagger\pm}_{m^\beta} \circ a^{\mp}_{m^\beta}$ and $a^{\pm}_{m^\beta} \circ a^{\dagger\mp}_{m^\beta}$, in the momentum operator, as a whole and we cannot say whether the index β in ε^β and τ^β originates from the first of second index m^β in these expressions.

(ii) When writing (m^β, n^γ) for (m^β, m^β) (see (i) above), then shall we replace $\delta_{l^\alpha m^\beta} a^{\pm}_{l^\alpha}$ with $\delta_{l^\alpha m^\beta} a^{\pm}_{n^\gamma}$, or $\delta_{l^\alpha n^\gamma} a^{\pm}_{m^\beta}$, or $\delta_{m^\beta n^\gamma} a^{\pm}_{l^\alpha}$? For a single field, $\gamma = \beta = \alpha$, this problem is solved by requiring an agreement of the resulting generalization (of (6.16) in the particular case) with the bilinear commutation relations (6.13). So, how shall (6.13) be generalized for several, not less than two, different fields? Obviously, here we meet an obstacle similar to the one described in (i) above, with the only change that $-\varepsilon^\beta$ should stand for ε^β.

Let b_{l^α} and c_{l^α} denote some creation or annihilation operator of the α-field. Consider the problem for generalizing the (anti)commutator $[b_{l^\alpha}, c_{l^\alpha}]_{\pm\varepsilon^\alpha}$. This means that we are looking for a replacement

$$[b_{l^\alpha}, c_{l^\alpha}]_{\pm\varepsilon^\alpha} \mapsto f^{\pm}(b_{l^\alpha}, c_{m^\beta}; \alpha, \beta), \tag{9.8}$$

where the functions $f^{\pm}$ are such that

$$f^{\pm}(b_{l^\alpha}, c_{m^\beta}; \alpha, \beta)\big|_{\beta=\alpha} = [b_{l^\alpha}, c_{l^\alpha}]_{\pm\varepsilon^\alpha}. \tag{9.9}$$

Unfortunately, the condition (9.9) is the only restriction on $f^{\pm}$ that the theory of free fields can provide. Thus the functions $f^{\pm}$, subjected to equation (9.9), become new free parameters of the quantum theory of different free fields and it is a matter of convention how to choose/fix them.

It is generally accepted [18, appendix F], the functions $f^{\pm}$ to have forms 'maximum' similar to the (anti)commutators they generalize. More precisely, the functions

$$f^{\pm}(b_{l^\alpha}, c_{m^\beta}; \alpha, \beta) = [b_{l^\alpha}, c_{m^\beta}]_{\pm\varepsilon^{\alpha\beta}} \tag{9.10}$$

where $\varepsilon^{\alpha\beta} \in \mathbb{C}$ are such that

$$\varepsilon^{\alpha\alpha} = \varepsilon^{\alpha}, \tag{9.11}$$

are usually considered as the only candidates for $f^{\pm}$. Notice, in (9.10), $\varepsilon^{\alpha\beta}$ are functions in α and β, not in l^α and/or m^β. Besides, if we assume $\varepsilon^{\alpha\beta}$ to be function only in ε^α and ε^β, then the general form of $\varepsilon^{\alpha\beta}$ is

$$\varepsilon^{\alpha\beta} = u^{\alpha\beta}\varepsilon^{\alpha} + (1 - u^{\alpha\beta})\varepsilon^{\beta} + v^{\alpha\beta}(1 - \varepsilon^{\alpha}\varepsilon^{\beta}) \qquad u^{\alpha\beta}, v^{\alpha\beta} \in \mathbb{C}, \tag{9.12}$$

due to (9.1) and (9.11). (In view of (6.13), the value $\varepsilon^{\alpha\beta} = +1$ (resp. $\varepsilon^{\alpha\beta} = -1$) corresponds to quantization via commutators (resp. anticommutators) of the corresponding fields.)

Call attention now on the numbers τ^α which originate and are associated with each term $[b_{l^\alpha}, c_{m^\alpha}]_{\pm\varepsilon^\alpha}$. With every change (9.8) one can associate a replacement

$$\tau^{\alpha} \mapsto g(b_{l^\alpha}, c_{m^\beta}; \alpha, \beta), \tag{9.13}$$

where the function g is such that

$$g(b_{l^\alpha}, c_{m^\beta}; \alpha, \beta)\big|_{\beta=\alpha} = \tau^{\alpha}. \tag{9.14}$$

Of course, the last condition does not define g uniquely and, consequently, the function g, satisfying (9.14), enters in the theory as a new free parameter. Suppose, as a working hypothesis similar to (9.10)–(9.11), that g is of the form

$$g(b_{l^\alpha}, c_{m^\beta}; \alpha, \beta) = \tau^{\alpha\beta}, \tag{9.15}$$

where $\tau^{\alpha\beta}$ are complex numbers that may depend only on α and β and are such that

$$\tau^{\alpha\alpha} = \tau^{\alpha}. \tag{9.16}$$

Besides, if we suppose $\tau^{\alpha\beta}$ to be functions only in τ^α and τ^β, then

$$\tau^{\alpha\beta} = x^{\alpha\beta}\tau^{\alpha} + y^{\alpha\beta}\tau^{\beta} + (1 - x^{\alpha\beta} - y^{\alpha\beta})\tau^{\alpha}\tau^{\beta} \qquad x^{\alpha\beta}, y^{\alpha\beta} \in \mathbb{C}, \tag{9.17}$$

as a result of (9.1) and (9.16).

Let us summarize the above discussion. If we suppose a preservation of the algebraic structure of the bilinear commutation relations (6.13) for a system of different free fields, then the replacements

$$[b_{l^\alpha}, c_{l^\alpha}]_{\pm\varepsilon^\alpha} \mapsto [b_{l^\alpha}, c_{m^\beta}]_{\pm\varepsilon^{\alpha\beta}} \qquad \varepsilon^{\alpha\alpha} = \varepsilon^{\alpha} \tag{9.18a}$$

$$\tau^{\alpha} \mapsto \tau^{\alpha\alpha} \qquad \tau^{\alpha\alpha} = \tau^{\alpha} \tag{9.18b}$$

should be made; accordingly, the relations (6.13) transform into:

$$\begin{aligned}
&[a_{l^\alpha}^{\pm}, a_{m^\beta}^{\pm}]_{-\varepsilon^{\alpha\beta}} = 0 && [a_{l^\alpha}^{\dagger\pm}, a_{m^\beta}^{\dagger\pm}]_{-\varepsilon^{\alpha\beta}} = 0 \\
&[a_{l^\alpha}^{\mp}, a_{m^\beta}^{\pm}]_{-\varepsilon^{\alpha\beta}} = \tau^{\alpha\beta}\delta_{l^\alpha m^\beta}\,\mathsf{id}_{\mathcal{F}} \times \begin{cases} 1 \\ -\varepsilon^{\alpha\beta} \end{cases} && [a_{l^\alpha}^{\dagger\mp}, a_{m^\beta}^{\dagger\pm}]_{-\varepsilon^{\alpha\beta}} = \tau^{\alpha\beta}\delta_{l^\alpha m^\beta}\,\mathsf{id}_{\mathcal{F}} \times \begin{cases} 1 \\ -\varepsilon^{\alpha\beta} \end{cases} \\
&[a_{l^\alpha}^{\pm}, a_{m^\beta}^{\dagger\pm}]_{-\varepsilon^{\alpha\beta}} = 0 && [a_{l^\alpha}^{\dagger\pm}, a_{m^\beta}^{\pm}]_{-\varepsilon^{\alpha\beta}} = 0 \\
&[a_{l^\alpha}^{\mp}, a_{m^\beta}^{\dagger\pm}]_{-\varepsilon^{\alpha\beta}} = \delta_{l^\alpha m^\beta}\,\mathsf{id}_{\mathcal{F}} \times \begin{cases} 1 \\ -\varepsilon^{\alpha\beta} \end{cases} && [a_{l^\alpha}^{\dagger\mp}, a_{m^\beta}^{\pm}]_{-\varepsilon^{\alpha\beta}} = \delta_{l^\alpha m^\beta}\,\mathsf{id}_{\mathcal{F}} \times \begin{cases} 1 \\ -\varepsilon^{\alpha\beta} \end{cases},
\end{aligned} \tag{9.19}$$

where 1 (resp. $-\varepsilon^{\alpha\beta}$) in $\left\{ {}^{1}_{-\varepsilon^{\alpha\beta}} \right.$ corresponds to the choice of the upper (resp. lower) signs. If we suppose additionally $\varepsilon^{\alpha\beta}$ (resp. $\tau^{\alpha\beta}$) to be a function only in ε^{α} and ε^{β} (resp. in τ^{α} and τ^{β}), then these numbers are defined up to two sets of complex parameters:

$$\varepsilon^{\alpha\beta} = u^{\alpha\beta}\varepsilon^{\alpha} + (1-u^{\alpha\beta})\varepsilon^{\beta} + v^{\alpha\beta}(1-\varepsilon^{\alpha}\varepsilon^{\beta}) \qquad u^{\alpha\beta}, v^{\alpha\beta} \in \mathbb{C} \tag{9.20a}$$
$$\tau^{\alpha\beta} = x^{\alpha\beta}\tau^{\alpha} + y^{\alpha\beta}\tau^{\beta} + (1-x^{\alpha\beta}-y^{\alpha\beta})\tau^{\alpha}\tau^{\beta} \qquad x^{\alpha\beta}, y^{\alpha\beta} \in \mathbb{C}. \tag{9.20b}$$

A reasonable further specialization of $\varepsilon^{\alpha\beta}$ and $\tau^{\alpha\beta}$ may be the assumption their ranges to coincide with those of ε^{α} and τ^{α}, respectively. As a result of (9.1), this supposition is equivalent to

$$v^{\alpha\beta} = -u^{\alpha\beta}, -u^{\alpha\beta}+1, u^{\alpha\beta}-1, u^{\alpha\beta} \quad u^{\alpha\beta} \in \mathbb{C} \tag{9.21a}$$
$$(x^{\alpha\beta}, y^{\alpha\beta}) = (0,0), (0,1), (1,0), (1,1). \tag{9.21b}$$

Other admissible restriction on (9.20) may be the requirement $\varepsilon^{\alpha\beta}$ and $\tau^{\alpha\beta}$ to be symmetric, viz.

$$\varepsilon^{\alpha\beta}(\varepsilon^{\alpha}, \varepsilon^{\beta}) = \varepsilon^{\beta\alpha}(\varepsilon^{\alpha}, \varepsilon^{\beta}) = \varepsilon^{\alpha\beta}(\varepsilon^{\beta}, \varepsilon^{\alpha}) \tag{9.22a}$$
$$\tau^{\alpha\beta}(\tau^{\alpha}, \tau^{\beta}) = \tau^{\beta\alpha}(\tau^{\alpha}, \tau^{\beta}) = \tau^{\alpha\beta}(\tau^{\beta}, \tau^{\alpha}), \tag{9.22b}$$

which means that the α-th and β-th fields are treated on equal footing and there is no *a priori* way to number some of them as the 'first' or 'second' one.[33] In view of (9.20), the conditions (9.22) are equivalent to

$$u^{\alpha\beta} = \frac{1}{2} \quad v^{\alpha\beta} \in \mathbb{C} \tag{9.23a}$$
$$y^{\alpha\beta} = x^{\alpha\beta}. \tag{9.23b}$$

If both of the restrictions (9.21) and (9.23) are imposed on (9.20), then the arbitrariness of the parameters in (9.20) is reduced to:

$$(u^{\alpha\beta}, u^{\alpha\beta}) = \Big(\frac{1}{2}, -\frac{1}{2}\Big), \Big(\frac{1}{2}, \frac{1}{2}\Big) \tag{9.24a}$$
$$(x^{\alpha\beta}, y^{\alpha\beta}) = (0,0), (1,1) \tag{9.24b}$$

and, for any *fixed* pair (α, β), we are left with the following candidates for respectively $\varepsilon^{\alpha\beta}$ and $\tau^{\alpha\beta}$:

$$\varepsilon^{\alpha\beta}_{+} := \frac{1}{2}(+1 + \varepsilon^{\alpha} + \varepsilon^{\beta} - \varepsilon^{\alpha}\varepsilon^{\beta}) \tag{9.25a}$$
$$\varepsilon^{\alpha\beta}_{-} := \frac{1}{2}(-1 + \varepsilon^{\alpha} + \varepsilon^{\beta} + \varepsilon^{\alpha}\varepsilon^{\beta}) \tag{9.25b}$$
$$\tau^{\alpha\beta}_{0} := \tau^{\alpha} + \tau^{\beta} \tag{9.25c}$$
$$\tau^{\alpha\beta}_{1} := \tau^{\alpha} + \tau^{\beta} - \tau^{\alpha}\tau^{\beta}. \tag{9.25d}$$

When free fields are considered, as in our case, no further arguments from mathematical or physical nature can help for choosing a particular combination $(\varepsilon^{\alpha\beta}, \tau^{\alpha\beta})$ from the four possible ones according to (9.25) for a fixed pair (α, β). To end the above considerations of $\varepsilon^{\alpha\beta}$ and $\tau^{\alpha\beta}$, we have to say that the choice

$$(\varepsilon^{\alpha\beta}, \tau^{\alpha\beta}) = (\varepsilon^{\alpha\beta}_{+}, \tau^{\alpha\beta}_{0}) = \Big(\frac{1}{2}(+1 + \varepsilon^{\alpha} + \varepsilon^{\beta} - \varepsilon^{\alpha}\varepsilon^{\beta}), \tau^{\alpha} + \tau^{\beta}\Big) \tag{9.26}$$

[33] However, nothing can prevent us to make other choices, compatible with (9.18), in the theory of free fields; for instance, one may set $\varepsilon^{\alpha\beta} = \varepsilon^{\alpha}\varepsilon^{\beta}\varepsilon^{\beta\alpha}$ and $\tau^{\alpha\beta} = \frac{1}{2}(\tau^{\alpha} + \tau^{\beta})\tau^{\beta\alpha}$.

is known as the *normal case* [18, appendix F]; in it the relative behavior of bose (resp. fermi) fields is as in the case of a single field, i.e. they are quantized via commutators (resp. anticommutators) as $(\varepsilon^{\alpha\beta},\tau^{\alpha\beta})=(+1,0)$ (resp. $(\varepsilon^{\alpha\beta},\tau^{\alpha\beta})=(-1,0)$), and the one of bose and fermi field is as in the case of a single fermi field, viz. the quantization is via commutators as $(\varepsilon^{\alpha\beta},\tau^{\alpha\beta})=(+1,0)$. All combinations between $\varepsilon^{\alpha\beta}_{\pm}$ and $\tau^{\alpha\beta}_{0,1}$ different from (9.26) are referred as *anomalous cases*. Above we supposed the pair (α,β) to be fixed. If α and β are *arbitrary*, the only essential change this implies is in (9.25), where the choice of the subscripts $+$, $-$, 0 and 1 may depend on α and β. In this general situation, the *normal case* is defined as the one when (9.26) holds for all α and β. All other combinations are referred as *anomalous cases*; such are, for instance, the ones when some fermi and bose operators satisfy anticommutation relations, e.g. (9.19) with $\varepsilon^{\alpha\beta}=-1$ for $\varepsilon^{\alpha}+\varepsilon^{\beta}=0$, or some fermi fields are subjected to commutation relations, like (9.19) with $\varepsilon^{\alpha\beta}=+1$ for $\varepsilon^{\alpha}=\varepsilon^{\beta}=-1$. For some details on this topic, see, for instance, [18, appendix F], [7, chapter 20] and [27, sect 4-4]. Fields/operators for which $\varepsilon^{\alpha\beta}=+1$ (resp. $\varepsilon^{\alpha\beta}=-1$), with $\beta\neq\alpha$, are referred as *relative parabose* (resp. *parafermi*) in the parafield theory [17, 18]. One can transfer this terminology in the general case and call the fields/operators for which $\varepsilon^{\alpha\beta}=+1$ (resp. $\varepsilon^{\alpha\beta}=-1$), with $\beta\neq\alpha$, *relative bose* (resp. *fermi*) fields/operators.

Further the relations (9.19) will be referred as the *multifield bilinear commutation relations* and it will be assumed that they represent the generalization of the bilinear commutation relations (6.13) when we are dealing with several, not less than two, different quantum fields. The particular values of $\varepsilon^{\alpha\beta}$ and $\tau^{\alpha\beta}$ in them are insignificant in the following; if one likes, one can fix them as in the normal case (9.26). Moreover, even the definition (9.19) of $\tau^{\alpha\beta}$ is completely inessential at all, as $\tau^{\alpha\beta}$ always appears in combinations like $\tau^{\alpha\beta}\delta_{l^\alpha m^\beta}$ (see (9.19) or similar relations, like (9.27), below), which are non-vanishing if $\beta=\alpha$, but then $\tau^{\alpha\alpha}=\tau^{\alpha}$; so one can freely write τ^{α} for $\tau^{\alpha\beta}$ in all such cases.

Equipped with (9.19) and (9.18), we can generalize (9.7) in different ways. For example, the straightforward generalization of (6.16) is:

$$\big[a^{+}_{l^\alpha},[a^{+}_{m^\beta},a^{\dagger\,-}_{n^\gamma}]_{\varepsilon^{\beta\gamma}}\big]_{-}+2\delta_{l^\alpha n^\gamma}a^{+}_{m^\beta}=0 \tag{9.27a}$$
$$\big[a^{+}_{l^\alpha},[a^{\dagger\,+}_{m^\beta},a^{-}_{n^\gamma}]_{\varepsilon^{\beta\gamma}}\big]_{-}+2\tau^{\alpha\gamma}\delta_{l^\alpha n^\gamma}a^{+}_{m^\beta}=0 \tag{9.27b}$$
$$\big[a^{-}_{l^\alpha},[a^{+}_{m^\beta},a^{\dagger\,-}_{n^\gamma}]_{\varepsilon^{\beta\gamma}}\big]_{-}-2\tau^{\alpha\beta}\delta_{l^\alpha m^\beta}a^{-}_{n^\gamma}=0 \tag{9.27c}$$
$$\big[a^{-}_{l^\alpha},[a^{\dagger\,+}_{m^\beta},a^{-}_{n^\gamma}]_{\varepsilon^{\beta\gamma}}\big]_{-}-2\delta_{l^\alpha m^\beta}a^{-}_{n^\gamma}=0. \tag{9.27d}$$

However, generally, the relations (9.19) do *not* convert (9.27) into identities. The reason is that an equality/identity like (cf. (6.8))

$$[b_{l^\alpha},c_{m^\beta}\circ d_{n^\gamma}]_{-}=[b_{l^\alpha},c_{m^\beta}]_{-\varepsilon^{\alpha\beta}}\circ d_{n^\gamma}+\lambda^{\alpha\beta\gamma}c_{m^\beta}\circ[b_{l^\alpha},d_{n^\gamma}]_{-\varepsilon^{\alpha\gamma}}, \tag{9.28}$$

where b_{l^α}, c_{m^β} and d_{n^γ} are some creation/annihilation operators and $\lambda^{\alpha\beta\gamma}\in\mathbb{C}$, can be valid only for

$$\lambda^{\alpha\beta\gamma}=\varepsilon^{\alpha\beta}\quad \varepsilon^{\alpha\gamma}=1/\varepsilon^{\alpha\beta}\qquad(\varepsilon^{\alpha\beta}\neq 0), \tag{9.29}$$

which, in particular, is fulfilled if $\gamma=\beta$ and $\varepsilon^{\alpha\beta}=\pm1$. So, the agreement between (9.19) and (9.27) depends on the concrete choice of the numbers $\varepsilon^{\alpha\beta}$. There exist cases when even the normal case (9.26) cannot ensure (9.19) to convert (9.27) into identities; e.g. when the α-th field and β-th fields are fermion ones and the γ-th field is a boson one. Moreover, it can be proved that (9.19) and (9.27) are compatible in the general case if unacceptable equalities like $a^{\pm}_{l}\circ a^{\pm}_{m}=0$ hold.

One may call (9.27) the *multifield paracommutation relations* as from them a corresponding generalization of (6.18) and/or (6.20) can be derived. For completeness, we shall record the multifield version of (6.20):

$$[b_{l^\alpha}, [b^\dagger_{m^\beta}, b_{n^\gamma}]_{\varepsilon^{\beta\gamma}}]_- = 2\delta_{l^\alpha m^\beta} b_{n^\gamma} \qquad [b_{l^\alpha}, [b_{m^\beta}, b_{n^\gamma}]_{\varepsilon^{\beta\gamma}}]_- = 0 \tag{9.30a}$$

$$[c_{l^\alpha}, [c^\dagger_{m^\beta}, c_{n^\gamma}]_{\varepsilon^{\beta\gamma}}]_- = 2\delta_{l^\alpha m^\beta} c_{n^\gamma} \qquad [c_{l^\alpha}, [c_{m^\beta}, c_{n^\gamma}]_{\varepsilon^{\beta\gamma}}]_- = 0 \tag{9.30b}$$

$$[b^\dagger_{l^\alpha}, [c^\dagger_{m^\beta}, c_{n^\gamma}]_{\varepsilon^{\beta\gamma}}]_- = -2\tau^{\alpha\gamma}\delta_{l^\alpha n^\gamma} b^\dagger_{m^\beta} \quad [c^\dagger_{l^\alpha}, [b^\dagger_{m^\beta}, b_{n^\gamma}]_{\varepsilon^{\beta\gamma}}]_- = -2\tau^{\alpha\gamma}\delta_{l^\alpha n^\gamma} c^\dagger_{m^\beta}. \tag{9.30c}$$

For details regarding these multifield paracommutation relations, the reader is referred to [17, 18], where the case $\tau^\alpha = \tau^\beta = \tau^{\alpha\beta} = 0$ is considered.

We leave to the reader as exercise to write down the multifield versions of the commutation relations (6.22) or (6.23), which provide examples of generalizations of (9.7) and hence of (9.19) and (9.27).

9.2. Commutation relations connected with the charge and angular momentum operators

In a case of several, not less than two, different fields, the basic trilinear commutation relations (6.33), which ensure the validity of the Heisenberg relation (5.2) concerning the charge operator, read:

$$\big[a^\pm_{l^\alpha}, [a^{\dagger\,+}_{m^\beta}, a^-_{m^\beta}]_{\varepsilon^\beta} - [a^+_{m^\beta}, a^{\dagger\,-}_{m^\beta}]_{\varepsilon^\beta}\big]_- - 2\delta_{l^\alpha m^\beta} a^\pm_{l^\alpha} = 0 \tag{9.31a}$$

$$\big[a^{\dagger\,\pm}_{l^\alpha}, [a^{\dagger\,+}_{m^\beta}, a^-_{m^\beta}]_{\varepsilon^\beta} - [a^+_{m^\beta}, a^{\dagger\,-}_{m^\beta}]_{\varepsilon^\beta}\big]_- + 2\delta_{l^\alpha m^\beta} a^{\dagger\,\pm}_{l^\alpha} = 0. \tag{9.31b}$$

Of course, these relations hold only for those fields which have non-vanishing charges, i.e. in (9.31) is supposed (see (9.1))

$$\tau^\alpha = 0 \quad \tau^\beta = 0 \qquad (\iff q^\alpha q^\beta \neq 0). \tag{9.32}$$

The problem for generalizing (9.31) for these fields is similar to the one for (9.7) in the case of non-vanishing charges, $\tau^\beta = 0$. Without repeating the discussion of Subsect. 9.1, we shall adopt the rule (9.18) for generalizing (anti)commutation relations between creation/annihilation operators of a single field. By its means one can obtain different generalizations of (9.31). For instance, the commutation relations.

$$\big[a^+_{l^\alpha}, [a^{\dagger\,+}_{m^\beta}, a^-_{n^\gamma}]_{\varepsilon^{\beta\gamma}} - [a^+_{m^\beta}, a^{\dagger\,-}_{n^\gamma}]_{\varepsilon^{\beta\gamma}}\big]_- - 2\delta_{l^\alpha n^\gamma} a^+_{m^\beta} = 0 \tag{9.33a}$$

$$\big[a^-_{l^\alpha}, [a^{\dagger\,+}_{m^\beta}, a^-_{n^\gamma}]_{\varepsilon^{\beta\gamma}} - [a^+_{m^\beta}, a^{\dagger\,-}_{n^\gamma}]_{\varepsilon^{\beta\gamma}}\big]_- - 2\delta_{l^\alpha m^\beta} a^-_{n^\gamma} = 0 \tag{9.33b}$$

and their Hermitian conjugate contain (9.31) and (6.35) as evident special cases and agree with (9.19) if $\gamma = \beta$ and $\varepsilon^{\alpha\beta}\varepsilon^{\beta\gamma} = +1$. Besides, the multifield paracommutation relations (9.27) for charged fields, $\tau^\alpha = \tau^\beta = \tau^\gamma = 0$, convert (9.33) into identities and, in this sense, (9.33) agree with (contain as special case) (9.27) for charged fields. As an example of commutation relations that do not agree with (9.27) for charged fields and, consequently, with (9.33), we shall point the following ones:

$$\big[a^\pm_{l^\alpha}, [a^+_{m^\beta}, a^{\dagger\,-}_{n^\gamma}]_{\varepsilon^{\beta\gamma}}\big]_- + \delta_{l^\alpha n^\gamma} a^\pm_{m^\beta} = 0 \tag{9.34a}$$

$$\big[a^\pm_{l^\alpha}, [a^{\dagger\,+}_{m^\beta}, a^-_{n^\gamma}]_{\varepsilon^{\beta\gamma}}-\big]_- - \delta_{l^\alpha n^\gamma} a^\pm_{m^\beta} = 0, \tag{9.34b}$$

which are a multifield generalization of (6.34).

The consideration of commutation relations originating from the 'orbital' Heisenberg equation (5.4) is analogous to the one of the same relations regarding the charge operator. The multifield version of (6.49) is:

$$\begin{multline}\big\{(-\omega^\circ_{\mu\nu}(m^\beta) + \omega^\circ_{\mu\nu}(n^\gamma))([\tilde a^\pm_{l^\alpha}, [\tilde a^{\dagger\,+}_{m^\beta}, \tilde a^-_{n^\gamma}]_{\varepsilon^{\beta\gamma}} \\ + [\tilde a^+_{m^\beta}, \tilde a^{\dagger\,-}_{n^\gamma}]_{\varepsilon^{\beta\gamma}}]_-)\big\}\big|_{n^\gamma = m^\beta} = 4(1+\tau^{\alpha\beta})\delta_{l^\alpha m^\beta}\omega^\circ_{\mu\nu}(l^\alpha)(\tilde a^\pm_{l^\alpha}) \end{multline} \tag{9.35a}$$

$$\{(-\omega^{\circ}_{\mu\nu}(m^\beta)+\omega^{\circ}_{\mu\nu}(n^\gamma))([\tilde a^{\dagger\,\pm}_{l^\alpha},[\tilde a^{\dagger\,+}_{m^\beta},\tilde a^{-}_{n^\gamma}]_{\varepsilon^{\beta\gamma}} \\ +[\tilde a^{+}_{m^\beta},\tilde a^{\dagger\,-}_{n^\gamma}]_{\varepsilon^{\beta\gamma}}]_{\_})\}\big|_{n^\gamma=m^\beta}=4(1+\tau^{\alpha\beta})\delta_{l^\alpha m^\beta}\omega^{\circ}_{\mu\nu}(l^\alpha)(\tilde a^{\dagger\,\pm}_{l^\alpha}) \quad (9.35\text{b})$$

where

$$\omega^{\circ}_{\mu\nu}(l^\alpha):=\omega_{\mu\nu}(k)=k_\mu\frac{\partial}{\partial k^\nu}-k_\nu\frac{\partial}{\partial k^\mu}\qquad \text{if } l^\alpha=(\alpha,s^\alpha,\boldsymbol{k}). \quad (9.36)$$

Applying (6.51), with m^β for m and n^γ for n, one can check that the multifield paracommutation relations (9.27) convert (9.35) into identities and hence provide a solution of (9.35) and ensure the validity of (5.4), when system of different free fields is considered. An example of a solution of (9.35) which does not agree with (9.27) is provided by the following multifield generalization of (6.52):

$$\big[a^{+}_{l^\alpha},[a^{+}_{m^\beta},a^{\dagger\,-}_{n^\gamma}]_{\varepsilon^{\beta\gamma}}\big]_{\_}=\big[a^{+}_{l^\alpha},[a^{\dagger\,+}_{m^\beta},a^{-}_{n^\gamma}]_{\varepsilon^{\beta\gamma}}\big]_{\_}=-(1+\tau^{\alpha\gamma})\delta_{l^\alpha n^\gamma}a^{+}_{m^\beta} \quad (9.37\text{a})$$

$$\big[a^{-}_{l^\alpha},[a^{+}_{m^\beta},a^{\dagger\,-}_{n^\gamma}]_{\varepsilon^{\beta\gamma}}\big]_{\_}=\big[a^{-}_{l^\alpha},[a^{\dagger\,+}_{m^\beta},a^{-}_{n^\gamma}]_{\varepsilon^{\beta\gamma}}\big]_{\_}=+(1+\tau^{\alpha\beta})\delta_{l^\alpha m^\beta}a^{+}_{n^\gamma}, \quad (9.37\text{b})$$

which provides a solution of (9.5). Notice, the evident multifield version of (6.53) agrees with (9.5), but disagrees with (9.35) when the lower signs are used.

At last, the multifield exploration of the 'spin' Heisenberg relations (5.5) is a *mutatis mutandis* (see (9.35)) version of the corresponding considerations in the second part of Subsect. 6.3. The main result here is that the multifield bilinear commutation relations (9.19), as well as their para counterparts (9.27), ensure the validity of (5.5).

9.3. Commutation relations between the dynamical variables

The aim of this subsection is to be discussed/proved the commutation relations (5.15)–(5.24) for a system of at least two different quantum fields from the view-point of the commutation relations considered in subsections 9.1 and 9.2.

To begin with, we rewrite the Heisenberg relations (5.1), (5.2) and (5.4) in terms of creation and annihilation operators for a multifield system [1, 11]:

$$[a^{\pm}_{l^\alpha},\mathcal{P}_\mu]_{\_}=\mp k_\mu a^{\pm}_{l^\alpha}\qquad [a^{\dagger\,\pm}_{l^\alpha},\mathcal{P}_\mu]_{\_}=\mp k_\mu a^{\dagger\,\pm}_{l^\alpha} \quad (9.38)$$

$$[a^{\pm}_{l^\alpha},\mathcal{Q}]_{\_}=q a^{\pm}_{l^\alpha}\qquad [a^{\dagger\,\pm}_{l^\alpha},\mathcal{Q}]_{\_}=-q a^{\dagger\,\pm}_{l^\alpha} \quad (9.39)$$

$$[\tilde a^{\pm}_{l^\alpha},\mathcal{M}^{\text{or}}_{\mu\nu}]_{\_}=\mathrm{i}\hbar\omega^{\circ}_{\mu\nu}(l^\alpha)\big(\tilde a^{\pm}_{l^\alpha}\big)\qquad [\tilde a^{\dagger\,\pm}_{l^\alpha},\mathcal{M}^{\text{or}}_{\mu\nu}]_{\_}=\mathrm{i}\hbar\omega^{\circ}_{\mu\nu}(l^\alpha)\big(\tilde a^{\dagger\,\pm}_{l^\alpha}\big), \quad (9.40)$$

where $l^\alpha=(\alpha,s^\alpha,\boldsymbol{k})$, $\omega^{\circ}(l^\alpha)$ is defined by (9.36) and $k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}$ is set in (9.38) and (9.40) (after the differentiations are performed in the last case). The corresponding version of (5.5) is more complicated and depends on the particular field considered (do not sum over s^α!):

$$\begin{aligned} f^{s^\alpha}[a^{\pm}_{\alpha,s^\alpha}(\boldsymbol{k}),\mathcal{M}^{\text{sp}}_{\mu\nu}]_{\_}&=\mathrm{i}\hbar g_\alpha\sum_{t^\alpha}\big\{{}^{\pm}\sigma^{s^\alpha t^\alpha,+}_{\mu\nu}(\boldsymbol{k})a^{+}_{\alpha,t^\alpha}(\boldsymbol{k})+{}^{\pm}\sigma^{s^\alpha t^\alpha,-}_{\mu\nu}(\boldsymbol{k})a^{-}_{\alpha,t^\alpha}(\boldsymbol{k})\big\}\\ f^{s^\alpha}[a^{\dagger\,\pm}_{\alpha,s^\alpha}(\boldsymbol{k}),\mathcal{M}^{\text{sp}}_{\mu\nu}]_{\_}&=\mathrm{i}\hbar h_\alpha\sum_{t^\alpha}\big\{{}^{\pm}\sigma^{s^\alpha t^\alpha,-}_{\mu\nu}(\boldsymbol{k})a^{\dagger\,+}_{\alpha,t^\alpha}(\boldsymbol{k})+{}^{\pm}\sigma^{s^\alpha t^\alpha,+}_{\mu\nu}(\boldsymbol{k})a^{\dagger\,-}_{\alpha,t^\alpha}(\boldsymbol{k})\big\}, \end{aligned} \quad (9.41)$$

where $f_{s^\alpha}=-1,0,+1$ (depending on the particular field), $g_\alpha:=-h_\alpha:=\frac{1}{j^\alpha+\delta_{j^\alpha 0}}(-1)^{j^\alpha+1}$ and ${}^{\pm}\sigma^{s^\alpha t^\alpha,+}_{\mu\nu}(\boldsymbol{k})$ and ${}^{\pm}\sigma^{s^\alpha t^\alpha,-}_{\mu\nu}(\boldsymbol{k})$ are some functions which strongly depend on the particular field considered, with ${}^{\pm}\sigma^{s^\alpha t^\alpha,\pm}_{\mu\nu}(\boldsymbol{k})$ being related to the spin (polarization) functions $\sigma^{s^\alpha t^\alpha,\pm}_{\mu\nu}(\boldsymbol{k})$ (see (3.14) and (3.11)).[34] As a result of (5.6), (9.40) and (9.41), one can easily write the Heisenberg relations (5.3) in a form similar to (9.38)–(9.41).

[34] If $\tilde\phi^\alpha_i(\boldsymbol{k})$ are the Fourier images of the α-th field and

$$\tilde\phi^\alpha_i(\boldsymbol{k})=\sum_{s^\alpha}\big\{v^{s^\alpha,+}_i(\boldsymbol{k})\tilde a^{+}_{\alpha,s^\alpha}(\boldsymbol{k})+v^{s^\alpha,-}_i(\boldsymbol{k})\tilde a^{-}_{\alpha,s^\alpha}(\boldsymbol{k})\big\}, \quad (9.42)$$

The commutation relations involving the momentum operator are:

$$
\begin{aligned}
&[\mathcal{P}_\mu,\, \mathcal{P}_\nu]_{\_} = 0 \quad [\mathcal{Q},\, \mathcal{P}_\mu]_{\_} = 0\\
&[\mathcal{S}_{\mu\nu},\, \mathcal{P}_\lambda]_{\_} = [\mathcal{M}^{\mathrm{sp}}_{\mu\nu},\, \mathcal{P}_\lambda]_{\_} = 0\\
&[\mathcal{L}_{\mu\nu},\, \mathcal{P}_\lambda]_{\_} = [\mathcal{M}^{\mathrm{or}}_{\mu\nu},\, \mathcal{P}_\lambda]_{\_} = [\mathcal{M}_{\mu\nu},\, \mathcal{P}_\lambda]_{\_} = -\mathrm{i}\hbar\{\eta_{\lambda\mu}\,\mathcal{P}_\nu - \eta_{\lambda\nu}\,\mathcal{P}_\mu\}.
\end{aligned}
\tag{9.45}
$$

We claim that these equations are consequences from (9.38) and the explicit expressions (3.9)–(3.12) and (5.11)–(5.13) for the operators of the dynamical variables of the free fields considered in the present work. In fact, since (9.38) implies

$$
[b^{\pm}_{l^\alpha} \circ c^{\mp}_{m^\beta},\, \mathcal{P}_\mu]_{\_} = 0 \qquad l^\alpha = (\alpha, s^\alpha, \boldsymbol{k}),\ m^\beta = (\beta, s^\beta, \boldsymbol{k}) \tag{9.46a}
$$

$$
[b^{\pm}_{l^\alpha}\, \overset{\longleftrightarrow}{\omega^{\circ}_{\mu\nu}}(l^\alpha) \circ c^{\mp}_{m^\beta},\, \mathcal{P}_\mu]_{\_} = \pm 2(k_\mu\eta_{\nu\lambda} - k_\nu\eta_{\mu\lambda}) b^{\pm}_{l^\alpha} \circ c^{\mp}_{m^\beta}, \tag{9.46b}
$$

where $b^{\pm}_{l^\alpha}, c^{\pm}_{l^\alpha} = a^{\pm}_{l^\alpha}, a^{\dagger\pm}_{l^\alpha}$ and $\overset{\longleftrightarrow}{\omega^{\circ}_{\mu\nu}}(l^\alpha)$ is defined via (9.36) and (3.13), the verification of (9.45) reduces to almost trivial algebraic calculations. Further, we assert that any system of commutation relations considered in Subsect. 9.1 entails (9.45): as these relations always imply (9.5) (or similar multifield versions of (6.10) and (6.11) in the case of the Lagrangians (3.1) or (3.3), respectively) and, on its turn, (9.5) implies (5.1), the required result follows from the last assertion and the remark that (5.1) and (9.38) are equivalent. As an additional verification of the validity of (9.45), the reader can prove them by invoking the identity (6.8) and any system of commutation relations mentioned in Subsect. 9.1, in particular (9.19) and (9.27).

The commutation relations concerning the charge operator read:

$$
\begin{aligned}
&[\mathcal{P}_\mu,\, \mathcal{Q}]_{\_} = 0 \quad [\mathcal{Q},\, \mathcal{Q}]_{\_} = 0\\
&[\mathcal{L}_{\mu\nu},\, \mathcal{Q}]_{\_} = [\mathcal{S}_{\mu\nu},\, \mathcal{Q}]_{\_} = 0\\
&[\mathcal{M}^{\mathrm{or}}_{\mu\nu},\, \mathcal{Q}]_{\_} = [\mathcal{M}^{\mathrm{sp}}_{\mu\nu},\, \mathcal{Q}]_{\_} = [\mathcal{M}_{\mu\nu},\, \mathcal{Q}]_{\_} = 0.
\end{aligned}
\tag{9.47}
$$

These equations are trivial corollaries from (3.9)–(3.12) and (5.11)–(5.13) and the observation that (9.39) implies

$$
[a^{\dagger\pm}_{l^\alpha} \circ a^{\mp}_{m^\beta},\, \mathcal{Q}]_{\_} = [a^{\pm}_{l^\alpha} \circ a^{\dagger\mp}_{m^\beta},\, \mathcal{Q}]_{\_} = 0 \qquad \text{if } q^\alpha = q^\beta, \tag{9.48}
$$

due to (6.8) for $\eta = -1$. Since any one of the systems of commutation relations mentioned in Subsect. 9.2 entails (9.31) (or systems of similar multifield versions of (6.31) and (6.32), if the Lagrangians (3.1) or (3.3) are employed), which is equivalent to (9.39), the equations (9.47) hold if some of these systems is valid. Alternatively, one can prove via a direct calculation that the commutation relations arising from the charge operator entail the validity of (9.47);

---

where $v^{s^\alpha,\pm}_i(\boldsymbol{k})$ are linearly independent functions normalize via the condition

$$
\sum_i \bigl(v^{s^\alpha,\pm}_i(\boldsymbol{k})\bigr)^* v^{t^\alpha,\pm}_i(\boldsymbol{k}) = \delta^{s^\alpha t^\alpha} f^{s^\alpha}, \tag{9.43}
$$

with $f^{s^\alpha} = 1$ for $j^\alpha = 0, \frac{1}{2}$ and $f^{s^\alpha} = 0, -1$ for $(j^\alpha, s^\alpha) = (1,3)$ or $(j^\alpha, s^\alpha) = (1,1), (1,2)$, respectively, then

$$
\begin{aligned}
{}^{+}\sigma^{s^\alpha t^\alpha,\pm}_{\mu\nu}(\boldsymbol{k}) &:= \frac{1}{g_\alpha} \sum_{i,i'} \bigl(v^{s^\alpha,+}_i(\boldsymbol{k})\bigr)^* I^{i'}_{i\mu\nu} v^{t^\alpha,\pm}_{i'}(\boldsymbol{k})\\
{}^{-}\sigma^{s^\alpha t^\alpha,\pm}_{\mu\nu}(\boldsymbol{k}) &:= \frac{1}{g_\alpha} \sum_{i,i'} \bigl(v^{s^\alpha,-}_i(\boldsymbol{k})\bigr)^* I^{i'}_{i\mu\nu} v^{t^\alpha,\pm}_{i'}(\boldsymbol{k}),
\end{aligned}
\tag{9.44}
$$

with $I^{i'}_{i\mu\nu}$ given via (5.25). Besides, $\sigma^{s^\alpha t^\alpha,\pm}_{\mu\nu}(\boldsymbol{k}) = {}^{\pm}\sigma^{s^\alpha t^\alpha,\pm}_{\mu\nu}(\boldsymbol{k})$ with an exception that $\sigma^{s^\alpha t^\alpha,\pm}_{\mu\nu}(\boldsymbol{k}) = 0$ for $j^\alpha = \frac{1}{2}$ and $(\mu,\nu) = (a,0),(0,a)$ with $a = 1,2,3$.

for the purpose the identity (6.8) and the explicit expressions for the dynamical variables via the creation and annihilation operators should be applied.

At last, consider the commutation relations involving the different angular momentum operators:

$$\begin{aligned}
&[\mathcal{P}_\lambda,\, \mathcal{S}_{\mu\nu}]_{_} = [\mathcal{P}_\lambda,\, \mathcal{M}^{\text{sp}}_{\mu\nu}]_{_} = 0\\
&[\mathcal{P}_\lambda,\, \mathcal{L}_{\mu\nu}]_{_} = [\mathcal{P}_\lambda,\, \mathcal{M}^{\text{or}}_{\mu\nu}]_{_} = [\mathcal{P}_\lambda,\, \mathcal{M}_{\mu\nu}]_{_} = +\mathrm{i}\hbar\{\eta_{\lambda\mu}\,\mathcal{P}_\nu - \eta_{\lambda\nu}\,\mathcal{P}_\mu\}\\
&[\mathcal{Q},\, \mathcal{L}_{\mu\nu}]_{_} = [\mathcal{Q},\, \mathcal{S}_{\mu\nu}]_{_} = [\mathcal{Q},\, \mathcal{M}^{\text{or}}_{\mu\nu}]_{_} = [\mathcal{Q},\, \mathcal{M}^{\text{sp}}_{\mu\nu}]_{_} = [\mathcal{Q},\, \mathcal{M}_{\mu\nu}]_{_} = 0\\
&[\mathcal{S}_{\varkappa\lambda},\, \mathcal{M}_{\mu\nu}]_{_} = -\mathrm{i}\hbar\big\{\eta_{\varkappa\mu}\,\mathcal{S}_{\lambda\nu} - \eta_{\lambda\mu}\,\mathcal{S}_{\varkappa\nu} - \eta_{\varkappa\nu}\,\mathcal{S}_{\lambda\mu} + \eta_{\lambda\nu}\,\mathcal{S}_{\varkappa\mu}\big\}\\
&[\mathcal{L}_{\varkappa\lambda},\, \mathcal{M}_{\mu\nu}]_{_} = -\mathrm{i}\hbar\big\{\eta_{\varkappa\mu}\,\mathcal{L}_{\lambda\nu} - \eta_{\lambda\mu}\,\mathcal{L}_{\varkappa\nu} - \eta_{\varkappa\nu}\,\mathcal{L}_{\lambda\mu} + \eta_{\lambda\nu}\,\mathcal{L}_{\varkappa\mu}\big\}\\
&[\mathcal{M}_{\varkappa\lambda},\, \mathcal{M}_{\mu\nu}]_{_} = -\mathrm{i}\hbar\big\{\eta_{\varkappa\mu}\,\mathcal{M}_{\lambda\nu} - \eta_{\lambda\mu}\,\mathcal{M}_{\varkappa\nu} - \eta_{\varkappa\nu}\,\mathcal{M}_{\lambda\mu} + \eta_{\lambda\nu}\,\mathcal{M}_{\varkappa\mu}\big\}.
\end{aligned} \tag{9.49}$$

(The other commutators, that can be form from the different angular momentum operators, are complicated and cannot be expressed in a 'closed' form.) The proof of these relations is based on equations like (see (9.40) and (6.8))

$$[b_{l^\alpha}\circ c_{m^\beta},\, \mathcal{M}^{\text{or}}_{\mu\nu}]_{\_} = \mathrm{i}\hbar\omega^{\circ}_{\mu\nu}(l^\alpha)\big(b_{l^\alpha}\circ c_{m^\beta}\big) \qquad l^\alpha = (\alpha, s^\alpha, \boldsymbol{k}),\ m^\beta = (\beta, s^\beta, \boldsymbol{k}), \tag{9.50}$$

with $b_{l^\alpha}, c_{l^\alpha} = a^{+}_{l^\alpha}, a^{-}_{l^\alpha}, a^{\dagger +}_{l^\alpha}, a^{\dagger -}_{l^\alpha}$, and similar, but more complicated, ones involving the other angular momentum operators. It, generally, depends on the particular field considered and will be omitted.

As it was said in Subsect. 6.3, the Heisenberg relations concerning the angular momentum operator(s) do not give rise to some (algebraic) commutation relations for the creation and annihilation operators. For this reason, the only problem is which of the commutation relations discussed in subsections 9.1 and 9.2 imply the validity of the equations (9.49) (or part of them). The general answer of this problem is not known but, however, a direct calculation by means of (9.7), if it holds, and (6.8) shows the validity of (9.49). Since (9.19) and (9.27) imply (9.7), this means that the multifield bilinear and para commutation relations are sufficient for the fulfillment of (9.49).

To conclude, let us draw the major moral of the above material: the multifield bilinear commutation relations (9.19) and the multifield paracommutation relations (9.27) ensure the validity of all 'standard' commutation relations (9.45), (9.47) and (9.49) between the operators of the dynamical variables characterizing free scalar, spinor and vector fields.

9.4. Commutation relations under the uniqueness conditions

As it was said at the end of the introduction to this section, the replacements (9.4) ensure the validity of the material of Sect. 4 in the multifield case. Correspondingly, the considerations in Sect. 7 remain valid in this case provided the changes

$$\begin{aligned}
&l \mapsto l^\alpha \quad m \mapsto m^\beta \quad n \mapsto n^\gamma\\
&\tau\delta_{lm} \mapsto \tau^{\alpha\beta}\delta_{l^\alpha m^\beta} = \tau^{\alpha}\delta_{l^\alpha m^\beta}\\
&[b_m, b_m]_\varepsilon \mapsto [b_{m^\beta}, b_{m^\beta}]_{\varepsilon^\beta} \quad [b_m, b_n]_\varepsilon \mapsto [b_{m^\beta}, b_{n^\gamma}]_{\varepsilon^{\beta\gamma}},
\end{aligned} \tag{9.51}$$

with b_m (or b_{m^β}) being any creation/annihilation operator, and, in some cases, (9.4) are made.[35] Without going into details, we shall write the final results.

The multifield version of (7.27)–(7.28) is:

$$\mathcal{E}(a^{\dagger\pm}_{m^\beta}\circ a^{\mp}_{n^\gamma}) = \varepsilon^{\beta\gamma}\,\mathcal{E}(a^{\mp}_{n^\gamma}\circ a^{\dagger\pm}_{m^\beta}) = \frac{1}{2}\,\mathcal{E}([a^{\dagger\pm}_{m^\beta}, a^{\mp}_{n^\gamma}]_{\varepsilon^{\beta\gamma}}) \tag{9.52}$$

[35] As a result of (7.11), (7.16) and (7.17), in expressions like (7.18)–(7.26) the number ε should be replace by $\varepsilon^{\alpha\beta}$, where α and β are the corresponding field indices of the creation/annihilation operators on which the operator $\mathcal{E}$ acts, i.e. $\varepsilon\,\mathcal{E}(b_m\circ b_n) \mapsto \varepsilon^{\beta\gamma}\,\mathcal{E}(b_{m^\beta}\circ b_{n^\gamma})$.

$$\left[a^{+}_{l^\alpha},\, \mathcal{E}([a^{+}_{m^\beta}, a^{\dagger\,-}_{n^\gamma}]_{\varepsilon^{\beta\gamma}})\right]_{-} + 2\delta_{l^\alpha n^\gamma} a^{+}_{m^\beta} = 0 \tag{9.53a}$$
$$\left[a^{+}_{l^\alpha},\, \mathcal{E}([a^{\dagger\,+}_{m^\beta}, a^{-}_{n^\gamma}]_{\varepsilon^{\beta\gamma}})\right]_{-} + 2\tau^{\alpha\gamma}\delta_{l^\alpha n^\gamma} a^{+}_{m^\beta} = 0 \tag{9.53b}$$
$$\left[a^{-}_{l^\alpha},\, \mathcal{E}([a^{+}_{m^\beta}, a^{\dagger\,-}_{n^\gamma}]_{\varepsilon^{\beta\gamma}})\right]_{-} - 2\tau^{\alpha\beta}\delta_{l^\alpha m^\beta} a^{-}_{n^\gamma} = 0 \tag{9.53c}$$
$$\left[a^{-}_{l^\alpha},\, \mathcal{E}([a^{\dagger\,+}_{m^\beta}, a^{-}_{n^\gamma}]_{\varepsilon^{\beta\gamma}})\right]_{-} - 2\delta_{l^\alpha m^\beta} a^{-}_{n^\gamma} = 0 \tag{9.53d}$$
$$\gamma = \beta. \tag{9.53e}$$

As one can expect, the relations (9.53a)–(9.53d) can be obtained from the multifield paracommutation relations (9.27) via the replacement $[\cdot,\cdot]_\varepsilon \mapsto \mathcal{E}([\cdot,\cdot]_{\varepsilon^{\beta\gamma}})$. It should be paid special attention on the equation (9.53e). It is due to the fact that in the expressions for the dynamical variables do not enter 'cross-field-products', like $a^{\dagger\,+}_{l^\alpha} \circ a^{-}_{m^\beta}$ for $\beta \neq \alpha$, and it corresponds to the condition (ii) in [17, p. B 1159]. The equality (9.53e) is quite important as it selects only that part of the '$\mathcal{E}$-transformed' multifield paracommutation relations (9.27) which is compatible with the bilinear commutation relations (9.19) (see (9.28) and (9.29)). Besides, (9.53e) makes (9.53a)–(9.53d) independent of the particular definition of $\varepsilon^{\alpha\beta}$ (see (9.11)).

The equations (9.52) are the only restrictions on the operator $\mathcal{E}$; examples of this operator are provided by the normal (resp. antinormal) ordering operator $\mathcal{N}$ (resp. $\mathcal{A}$), which has the properties (cf. (4.22) (resp. (7.30))

$$\begin{aligned}
&\mathcal{N}\big(a^{+}_{m^\beta} \circ a^{\dagger\,-}_{n^\gamma}\big) := a^{+}_{m^\beta} \circ a^{\dagger\,-}_{n^\gamma} && \mathcal{N}\big(a^{\dagger\,+}_{m^\beta} \circ a^{-}_{n^\gamma}\big) := a^{\dagger\,+}_{m^\beta} \circ a^{-}_{n^\gamma}\\
&\mathcal{N}\big(a^{-}_{m^\beta} \circ a^{\dagger\,+}_{n^\gamma}\big) := \varepsilon^{\beta\gamma} a^{\dagger\,+}_{n^\gamma} \circ a^{-}_{m^\beta} && \mathcal{N}\big(a^{\dagger\,-}_{m^\beta} \circ a^{+}_{n^\gamma}\big) := \varepsilon^{\beta\gamma} a^{+}_{n^\gamma} \circ a^{\dagger\,-}_{m^\beta}
\end{aligned} \tag{9.54}$$

$$\begin{aligned}
&\mathcal{A}\big(a^{+}_{m^\beta} \circ a^{\dagger\,-}_{n^\gamma}\big) := \varepsilon^{\beta\gamma} a^{\dagger\,-}_{n^\gamma} \circ a^{+}_{m^\beta} && \mathcal{A}\big(a^{\dagger\,+}_{m^\beta} \circ a^{-}_{n^\gamma}\big) := \varepsilon^{\beta\gamma} a^{-}_{n^\gamma} \circ a^{\dagger\,+}_{m^\beta}\\
&\mathcal{A}\big(a^{-}_{m^\beta} \circ a^{\dagger\,+}_{n^\gamma}\big) := a^{-}_{m^\beta} \circ a^{\dagger\,+}_{n^\gamma} && \mathcal{A}\big(a^{\dagger\,-}_{m^\beta} \circ a^{+}_{n^\gamma}\big) := a^{\dagger\,-}_{m^\beta} \circ a^{+}_{n^\gamma}.
\end{aligned} \tag{9.55}$$

The material of Sect. 8 has also a multifield variant that can be obtained via the replacements (9.51) and (9.4). Here is a brief summary of the main results found in that way.

The operator $\mathcal{E}$ should possess the properties (9.54) and, in this sense, can be identified with the normal ordering operator,

$$\mathcal{E} = \mathcal{N}. \tag{9.56}$$

As a result of this fact and $\varepsilon^{\beta\beta} = \varepsilon^{\beta}$ (see (9.11)), the commutation relations (9.53) take the final form:

$$\left[a^{+}_{l^\alpha}, a^{+}_{m^\beta} \circ a^{\dagger\,-}_{n^\beta}\right]_{-} + \delta_{l^\alpha n^\beta} a^{+}_{m^\beta} = 0 \tag{9.57a}$$
$$\left[a^{+}_{l^\alpha}, a^{\dagger\,+}_{m^\beta} \circ a^{-}_{n^\beta}\right]_{-} + \tau^{\alpha\beta}\delta_{l^\alpha n^\beta} a^{+}_{m^\beta} = 0 \tag{9.57b}$$
$$\left[a^{-}_{l^\alpha}, a^{+}_{m^\beta} \circ a^{\dagger\,-}_{n^\beta}\right]_{-} - \tau^{\alpha\beta}\delta_{l^\alpha m^\beta} a^{-}_{n^\beta} = 0 \tag{9.57c}$$
$$\left[a^{-}_{l^\alpha}, a^{\dagger\,+}_{m^\beta} \circ a^{-}_{n^\beta}\right]_{-} - \delta_{l^\alpha m^\beta} a^{-}_{n^\beta} = 0 \tag{9.57d}$$

which is the multifield version of (8.17) and corresponds, up to the replacement $a^{\pm}_{l^\alpha} \mapsto \sqrt{2} a^{\pm}_{l^\alpha}$, to (9.27) with $\varepsilon^{\beta\gamma} = 0$.

The vacuum state vector $\mathcal{X}_0$ is supposed to be uniquely defined by the following equations (cf. (8.1b)–(8.3)):

$$a^{-}_{l^\alpha}\, \mathcal{X}_0 = 0 \quad a^{\dagger\,-}_{l^\alpha}\, \mathcal{X}_0 = 0 \tag{9.58a}$$
$$\mathcal{X}_0 \neq 0 \tag{9.58b}$$
$$\langle \mathcal{X}_0 | \mathcal{X}_0 \rangle = 1 \tag{9.58c}$$
$$\begin{aligned}
&a^{\dagger\,-}_{l^\alpha} \circ a^{+}_{m^\beta}(\mathcal{X}_0) = \delta_{l^\alpha m^\beta}\, \mathcal{X}_0 && a^{-}_{l^\alpha} \circ a^{\dagger\,+}_{m^\beta}(\mathcal{X}_0) = \delta_{l^\alpha m^\beta}\, \mathcal{X}_0\\
&a^{-}_{l^\alpha} \circ a^{+}_{m^\beta}(\mathcal{X}_0) = \tau^{\alpha\beta}\delta_{l^\alpha m^\beta}\, \mathcal{X}_0 && a^{-}_{l^\alpha} \circ a^{+}_{m^\beta}(\mathcal{X}_0) = \tau^{\alpha\beta}\delta_{l^\alpha m^\beta}\, \mathcal{X}_0.
\end{aligned} \tag{9.58d}$$

The Hilbert space $\mathcal{F}$ of state vectors is a direct sum of the Hilbert spaces $\mathcal{F}^{\alpha}$ of the different fields and it is supposed to be spanned by the vectors

$$\psi_{l_1^{\alpha_1} l_2^{\alpha_2} \dots} = \mathcal{M}(a^{+}_{l_1^{\alpha_1}}, a^{+}_{l_2^{\alpha_2}}, \dots)(\mathcal{X}_0) \tag{9.59}$$

with $\mathcal{M}(a^{+}_{l_1^{\alpha_1}}, a^{+}_{l_2^{\alpha_2}}, \dots)$ being arbitrary monomial only in the creation operators.

Since (9.58a), (9.56) and (9.54) imply the multifield version of (8.7), the computation of the mean values of (8.6), with $l_1 \mapsto l_1^{\alpha_1}$ etc., of the dynamical variables is reduced to the one of scalar products like (cf. (8.5))

$$\langle \psi_{l_1^{\alpha_1} l_2^{\alpha_2} \dots} | \phi_{m_1^{\beta_1} m_2^{\beta_2} \dots} \rangle = \langle \mathcal{X}_0 | \big(\mathcal{M}(a^{+}_{l_1^{\alpha_1}}, a^{+}_{l_2^{\alpha_2}}, \dots) \big)^{\dagger} \circ \mathcal{M}'(a^{+}_{m_1^{\beta_1}}, a^{+}_{m_2^{\beta_2}}, \dots)(\mathcal{X}_0) \rangle \tag{9.60}$$

of basic vectors of the form (9.59). By means of the basic properties (9.58) of the vacuum, one is able to calculate the simplest forms of the vacuum mean values (9.60), viz. the multifield versions (see (9.51)) of (8.20) and (8.26). But more general such expression cannot be calculated by means of (9.57)–(9.58). *Prima facie* one can suppose that the multifield commutation relations (9.19), which ensure the vectors (9.59) to form a base of the system's Hilbert space of states, can help for the calculation of (9.60) in more complicated cases. In fact, this is the case which works perfectly well and covers the available experimental data. In this connection, we must mention that the applicability of (9.19) for calculation of (9.60) is ensured by the *compatibility/agreement* between (9.19) and (9.57): by means of (6.8) for $\eta = -\varepsilon^{\alpha\beta}$, one can check that (9.19) converts (9.57) into identities.[36]

The commutation relations (9.57) admit as a solution also the multifield version of the anomalous bilinear commutation relations (8.27) but it, as we said earlier, leads to contradictions and must be rejected. The existence of solutions of (9.57) different from it and (9.19) seems not to be investigated. If there appear date which do not fit into the description by means of (9.19), one should look for other, if any, solutions of (9.57) or compatible with (9.57) effective procedures for calculating vacuum mean values like (9.60).

10. Conclusion

In this paper we have investigated two sources of (algebraic) commutation relations in the Lagrangian quantum theory of free scalar, spinor and vector fields: the uniqueness of the dynamical variables (momentum, charge and angular momentum) and the Heisenberg relations/equations for them. If one ignores the former origin, which is the ordinary case, the paracommutation relations or some their generalizations seems to be the most suitable candidates for the most general commutation relations that ensure the validity of all Heisenberg equations. The simultaneous consideration of the both sources mentioned reveals, however, their incompatibility in the general case. The outlet of this situation is in the redefinition of the operators of the dynamical variables, similar to the normal ordering procedure and containing it as a special case. That operation ensures the uniqueness of the new (redefined) dynamical variables and changes the possible types of commutation relations. Again, the commutation relations, connected with the Heisenberg relations concerning the (redefined) momentum operator, entail the validity of all Heisenberg equations.

[36] Recall, equations (9.19) and (9.27), or (9.53a)–(9.53d), for $\gamma \neq \beta$ are generally incompatible. For instance, excluding some special cases, like systems consisting of only fermi (bose) fields or one fermi (bose) field and arbitrary number of bose (fermi) fields, the only operators satisfying (9.19) and (9.27) for $\gamma \neq \beta$ and having normal spin-statistics connection are such that $b_{m^{\beta}} \circ b_{n^{\gamma}} = 0$, with $\gamma \neq \beta$ and $b_{m^{\beta}}$ and $c_{n^{\gamma}}$ being any creation/annihilation operators, which, in particular, means that no states with two particles from different fields can exist.

Further constraints on the possible commutation relations follow from the definition/introduction of the concept of the vacuum (vacuum state vector). They practically reduce the redefined dynamical variables to the ones obtained via normal ordering procedure, which results in the explicit form (8.17) of the admissible commutation relations. In a sense, they happen to be 'one half' of the paracommutation ones. As a last argument in the way for finding the 'unique true' commutation relations, we require the existence of procedure for calculation of vacuum mean values of anti-normally ordered products of creation and annihilation operators, to which the mean values of the dynamical variables and the transition amplitudes between different states are reduced. We have pointed that the standard bilinear commutation relations are, at present, the only known ones that satisfy all of the conditions imposed and do not contradict to the existing experimental data.

The consideration of a system of at least two different quantum free fields meets a new problem: the general relations between creation/annihilation operators belonging to different fields turn to be undefined. The cause for this is that the commutation relations for any fixed field are well defined only on the corresponding to it Hilbert subspace of the system's Hilbert space of states and their extension on the whole space, as well as the inclusion in them of creation/annihilation operators of other fields, is a matter of convention (when free fields are concerned); formally this is reflected in the structure of the dynamical variables which are sums of those of the individual fields included in the system under consideration. We have, however, presented argument by means of which the *a priori* existing arbitrariness in the commutation relations involving different field operators can be reduced to the 'standard' one: these relations should contain either commutators or anticommutators of the creation/annihilation operators belonging to different fields. A free field theory cannot make difference between these two possibilities. Accepting these possibilities, the admissible commutation relations (9.57) for system of several different fields are considered. They turn to be corresponding multifield versions of the ones regarding a single field. Similarly to the single field case, the standard multifield bilinear commutation relations seem to be the only known ones that satisfy all of the imposed restrictions and are in agreement with the existing data.

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