A NOTE ABOUT THE $\{K_i(z)\}_{i=1}^{\infty}$ FUNCTIONS

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In the article [10], A. Petojević verified useful properties of the $K_i(z)$ functions which generalize Kurepa's [1] left factorial function. In this note, we present simplified proofs of two of these results and we answer the open question stated in [10]. Finally, we discuss the differential transcendency of the $K_i(z)$ functions.

A. Petojević [**7,** p. 3.] considered the family of functions:

$$(1)\quad {}_vM_m(s;a,z)=\sum_{k=1}^{v}(-1)^{k-1}\binom{z+m+1-k}{m+1}\mathcal{L}[s;{}_2F_1(a,k-z,m+2;1-t)],$$

for $\Re(z)>v{-}m{-}2$, where $v\in\mathbf{N}$ is a positive integer; $m\in\{-1,0,1,2,\ldots\}$ is an integer; s,a,z are complex variables; $\mathcal{L}[s;F(t)]$ is Laplace transform and ${}_2F_1(a,b,c;x)$ is the hypergeometric function ($|x|<1$). Đ. Kurepa has considered in the articles [**1,** p. 151.] and [**2,** p. 297.] a complex function defined by the integral:

$$(2)\qquad K(z)=\int_0^{\infty}e^{-t}\,\frac{t^z-1}{t-1}\,dt,$$

for $\Re(z)>0$. Especially, for Kurepa's function $K(z)$, it is true that $K(z)={}_1M_0(1;1,z)$, for $\Re(z)>0$, according to [**10**]. For various of values of parameters v,m,s,a,z from (1), different special functions, as presented in [**10**], are obtained. A. Petojević has considered in the article [**10,** p. 1640.] the following sequence of functions:

$$(3)\qquad K_i(z)=\frac{{}_1M_0(1;1,z+i-1)-{}_1M_0(1;1,i-1)}{{}_1M_{-1}(1;1,i)},$$

for $i\in\mathbf{N}$ and $\Re(z)>-i$. On the basis of the definition in (3), the following representation via Kurepa's function is true:

$$(4)\qquad K_i(z)=\frac{1}{(i-1)!}\Big(K(z+i-1)-K(i-1)\Big),$$

for $i\in\mathbf{N}$ and $\Re(z)>-i+1$. Note that $K(0)=0$ [**2,** p. 297.] and therefore $K_1(z)=K(z)$ for $\Re(z)>0$. Analytical and differential–algebraic properties of Kurepa's function $K(z)$ are considered in articles [$\mathbf{1-12}$] and in many other articles. *On the basis of well-known statements for* Kurepa*'s function $K(z)$, using representation* (4)*, in many cases we can get simple proofs for analogous statements for $K_i(z)$ functions.* For example, it is a well-known fact that it is possible to analytically continue Kurepa's function to a meromorphic function with simple poles at integer points $z=-1$ and $z=-m$, ($m\geq 3$) [**2,** p. 303.], [**3,** p. 474.]. Residues of Kurepa's function at these poles have the following form [**2**]:

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$$\operatorname*{res}_{z\,=\,-1} K(z) = -1 \quad \text{and} \quad \operatorname*{res}_{z\,=\,-m} K(z) = \sum_{k=2}^{m-1} \frac{(-1)^{k-1}}{k!}, \ \ (m \geq 3). \tag{5}$$

For KUREPA's function $K(z)$ the infinite point is an essential singularity [**3**]. Hence, on the basis of (4), each function $K_i(z)$ is meromorphic with simple poles at integer points $z = -i$ and $z = -(i+m)$, $(m \geq 2)$. On the basis of (4) we have:

$$\operatorname*{res}_{z\,=\,-(i+m)} K_i(z) = \frac{1}{(i-1)!} \cdot \operatorname*{res}_{z\,=\,-(i+m)} K(z+i-1) = \frac{1}{(i-1)!} \cdot \operatorname*{res}_{z\,=\,-(m+1)} K(z), \tag{6}$$

where $m = 0$ or $m \geq 2$. Hence:

$$\operatorname*{res}_{z\,=\,-i} K_i(z) = -\frac{1}{(i-1)!} \quad \text{and} \quad \operatorname*{res}_{z\,=\,-(i+m)} K_i(z) = \frac{1}{(i-1)!} \cdot \sum_{k=2}^{m} \frac{(-1)^{k-1}}{k!}, \ \ (m \geq 2). \tag{7}$$

For each $K_i(z)$ function the infinite point is an essential singularity. Therefore, we get Theorem **3.3.** from [**10**]. Next, it is a well-known fact that for KUREPA's function the following asymptotic relation $K(x) \sim \Gamma(x)$ is true for real x such that $x \to \infty$ and where $\Gamma(x)$ is the gamma function [**2,** p. 299.]. Hence, for fixed $i \in \mathbf{N}$ and real $x > -i+1$, on the basis of (4), we get:

$$\frac{K_i(x)}{\Gamma(x+i-1)} = \frac{1}{(i-1)!} \cdot \frac{K(i+x-1) - K(i-1)}{\Gamma(x+i-1)} \underset{x \to \infty}{\longrightarrow} \frac{1}{(i-1)!} \tag{8}$$

and

$$\frac{K_i(x)}{\Gamma(x+i)} = \frac{1}{(i-1)!} \cdot \frac{K(i+x-1) - K(i-1)}{(x+i-1)\Gamma(x+i-1)} \underset{x \to \infty}{\longrightarrow} 0. \tag{9}$$

Therefore, we get Theorem **3.6.** from [**10**]. Next we give a solution to the open problem stated in Question **3.7.** in [**10**]. Namely, the following formula in the article [**8,** p. 35.] is given:

$$K(z) = \frac{\mathrm{Ei}(1) + \mathfrak{i}\,\pi}{e} + \frac{(-1)^z \Gamma(1+z)\Gamma(-z,-1)}{e}, \tag{10}$$

for values $z \in \mathbf{C} \backslash \{-1,-2,-3,-4,\ldots\}$ and $\mathfrak{i} = \sqrt{-1}$. In the previous formula $\mathrm{Ei}(z)$ and $\Gamma(z,a)$ are exponential integral and incomplete gamma function respectively [**8**]. Then, for fixed $i \in \mathbf{N}$ and values $z \in \mathbf{C} \backslash \{-i, -i-1, -i-2, -i-3, \ldots\}$, on the basis of (4) and (10), we get:

$$\begin{aligned}
K_i(z) &= \frac{1}{(i-1)!}\Big(K(z+i-1) - K(i-1)\Big) \\
&= \frac{\mathrm{Ei}(1) + \mathfrak{i}\,\pi}{e(i-1)!} + \frac{(-1)^{z+i-1}\Gamma(1+z+i-1)\Gamma(-z-i+1,-1)}{e(i-1)!} \\
&\quad - \frac{\mathrm{Ei}(1) + \mathfrak{i}\,\pi}{e(i-1)!} - \frac{(-1)^{i-1}\Gamma(i)\Gamma(-i+1,-1)}{e(i-1)!} \\
&= (-1)^i e^{-1}\left(\Gamma(1-i,-1) - (-1)^z \frac{\Gamma(1-i-z,-1)\Gamma(i+z)}{(i-1)!}\right).
\end{aligned} \tag{11}$$

Therefore, the affirmative answer for Question **3.7.** from [**10**] is true for complex values $z \in \mathbf{C} \backslash \{-i, -i-1, -i-2, -i-3, \ldots\}$.

Finally, at the end of this note let us emphasize one differential–algebraic fact for the sequence of functions $K_i(z)$. On the basis of the formula (17) from the article **[10]**, we can conclude that each $K_i(z)$ function satisfies the following recurrence relation $(i-1)!\,K_i(z+1)-(i-1)!\,K_i(z)=\Gamma(z+i)$. The previous relation can be used to verify the differential transcendency of these functions as discussed in **[11, 12]**. Therefore, we can conclude that each $K_i(z)$ function is a differential transcendental function, i.e. it satisfies no algebraic differential equation over the field of complex rational functions.

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