

MULTILINEAR FUNCTION SERIES IN CONDITIONALLY FREE PROBABILITY WITH AMALGAMATION

MIHAI POPA

ABSTRACT. As in the cases of freeness and monotonic independence, the notion of conditional freeness is meaningful when complex-valued states are replaced by positive conditional expectations. In this framework, the paper presents several positivity results, a version of the central limit theorem and an analogue of the conditionally free R-transform constructed by means of multilinear function series.

1. Introduction

The paper addresses a topic related to conditionally free (or, shortly, using the term from [2], c-free) probability. This notion was developed in the '90's (see [1], [2]) as an extension of freeness within the framework of $*$ -algebras endowed with not one, but two states. Namely, given a family of unital algebras $\{\mathfrak{A}_i\}_{i \in \mathcal{I}}$, each \mathfrak{A}_i endowed with two expectations $\varphi_i, \psi_i : \mathfrak{A}_i \longrightarrow \mathbb{C}$, their c-free product is the triple $(\mathfrak{A}, \varphi, \psi)$, where:

- (i) $\mathfrak{A} = \ast_{i \in \mathcal{I}} \mathfrak{A}_i$ is the free product of the algebras \mathfrak{A}_i .
- (ii) $\psi = \ast_{i \in \mathcal{I}} \psi_i$ and $\varphi = \ast_{(\psi_i)_{i \in \mathcal{I}}} \varphi_i$ are expectations given by the relations
 - (a) $\psi(a_1 \cdots a_n) = 0$
 - (b) $\varphi(a_1 \cdots a_n) = \varphi_{\varepsilon(1)}(a_1) \cdots \varphi_{\varepsilon(n)}(a_n)$

for all $a_j \in \mathfrak{A}_{\varepsilon(j)}$, $j = 1, \dots, n$ such that $\psi_{\varepsilon(j)}(a_j) = 0$ and $\varepsilon(1) \neq \cdots \neq \varepsilon(n)$.

A key result is that if the \mathfrak{A}_i are $*$ -algebras and φ_i, ψ_i are positive functionals, then φ and ψ are also positive.

In [6], the positivity of the free product maps φ, ψ is proved for the case when φ_1, φ_2 are positive conditional expectations in a common C^* -subalgebra, but ψ_1, ψ_2 remain positive \mathbb{C} -valued maps. A more general situation is indeed discussed (see Theorem 3, Section 6, from [6]), but the question if φ, ψ are positive for $\varphi_{1,2}, \psi_{1,2}$ arbitrary positive conditional expectations is left unanswered.

A first answer was given in [8], where we showed that for \mathfrak{A} a $*$ -algebra, the analogous construction with both φ and ψ valued in a C^* -subalgebra \mathfrak{B} of \mathfrak{A} still retains the positivity. The present paper further develops this result (see Theorem 2.3) and also demonstrates the use of multilinear function series in c-free setting.

2000 *Mathematics Subject Classification.* Primary 45L53; Secondary 46L08.

Key words and phrases. conditional freeness, conditional expectation, R-transform, multilinear function series.

In [2] is constructed a c -free version of Voiculescu's R -transform, which we will call the cR -transform, with the property that ${}^cR_{X+Y} = {}^cR_X + {}^cR_Y$ if X and Y are c -free elements from the algebra \mathfrak{A} relative to φ and ψ (i.e. the relations (a) and (b) from the definition of the c -free product hold true for the subalgebras generated by X and Y .)

The apparatus of multilinear function series is used in recent work of K. Dykema ([3] and [4]) to construct suitable analogues for the R and S -transforms in the framework of freeness with amalgamation. We will show that this construction is also appropriate for the cR -transform mentioned above. The techniques used differ from the ones of [3], the Fock space type construction being substituted by combinatorial techniques similar to [2] and [7]. Particularly, Theorems 3.3 and 3.6 contain new (shorter) proofs of the results 6.1–6.13 from [3].

The paper is structured in four sections. In Section 2 are stated the basic definitions and are proved the main positivity results. Section 3 describes the construction and the basic property of the multilinear function series cR -transform and Section 4 treats the central limit theorem and a related positivity property.

2. Definitions and positivity results

Definition 2.1. Let $\{\mathfrak{A}_i\}_{i \in \mathcal{I}}$ be a family of algebras, all containing the subalgebra \mathfrak{B} . Suppose \mathfrak{D} is a subalgebra of \mathfrak{B} and $\Psi_i : \mathfrak{A}_i \rightarrow \mathfrak{D}$ and $\Phi_i : \mathfrak{A}_i \rightarrow \mathfrak{B}$ are conditional expectations, $i \in \mathcal{I}$. We say that the triple $(\mathfrak{A}, \Phi, \Psi) = \ast_{i \in \mathcal{I}} (\mathfrak{A}_i, \Phi_i, \Psi_i)$ is the conditionally free product with amalgamation over $(\mathfrak{B}, \mathfrak{D})$, or shortly, the c -free product, of the triples $(\mathfrak{A}_i, \Phi_i, \Psi_i)_{i \in \mathcal{I}}$ if

- (1) \mathfrak{A} is the free product with amalgamation over \mathfrak{B} of the family $(\mathfrak{A}_i)_{i \in \mathcal{I}}$
- (2) $\Psi = \ast_{i \in \mathcal{I}} \Psi_i$ and $\Phi = \ast_{(\Psi_i)_{i \in \mathcal{I}}} \Phi_i$ are determined by the relations

$$\begin{aligned} \Psi(a_1 a_2 \dots a_n) &= 0 \\ \Phi(a_1 a_2 \dots a_n) &= \Phi(a_1) \Phi(a_2) \dots \Phi(a_n), \end{aligned}$$

for all $a_i \in \mathfrak{A}_{\varepsilon(i)}$, $\varepsilon(i) \in \mathcal{I}$ such that $\varepsilon(1) \neq \varepsilon(2) \neq \dots \neq \varepsilon(n)$ and $\Psi_{\varepsilon(i)}(a_i) = 0$.

When $\mathfrak{D} = \mathbb{C}$, this definition reduces to the one given in [6]. When both \mathfrak{B} and \mathfrak{D} are equal to \mathbb{C} , this definition was given in [2].

When discussing positivity, we need a \ast -structure on our algebras. We will demand that \mathfrak{B} and \mathfrak{D} be C^\ast -algebras, while \mathfrak{A}_i and \mathfrak{A} are only required to be \ast -algebras.

The following results are slightly modified versions of Lemma 6.4 and Theorem 6.5 from [8].

Lemma 2.2. Let \mathfrak{B} be a C^\ast -algebra and $\mathfrak{A}_1, \mathfrak{A}_2$ be two \ast -algebras containing \mathfrak{B} as a \ast -subalgebra, endowed with positive conditional expectations $\Phi_j : \mathfrak{A}_j \rightarrow \mathfrak{B}$, $j = 1, 2$. If $a_1, \dots, a_n \in \mathfrak{A}_1$, $a_{n+1}, \dots, a_{n+m} \in \mathfrak{A}_2$ and $A = (A_{i,j})_{i,j} \in M_{n+m}(\mathfrak{B})$ is the matrix with the entries

$$A_{i,j} = \begin{cases} \Phi_1(a_i^\ast a_j) & \text{if } i, j \leq n \\ \Phi_1(a_i^\ast) \Phi_2(a_j) & \text{if } i \leq n, j > n \\ \Phi_2(a_i^\ast) \Phi_1(a_j) & \text{if } i > n, j \leq n \\ \Phi_2(a_i^\ast a_j) & \text{if } i, j > n \end{cases}$$

then A is positive.

Proof. The vector space $\mathfrak{E} = \mathfrak{B} \oplus \ker(\Phi_1) \oplus \ker(\Phi_2)$ has a \mathfrak{B} -bimodule structure given by the algebraic operations on \mathfrak{A}_1 and \mathfrak{A}_2 . Consider the \mathfrak{B} -sesquilinear pairing

$$\langle \cdot, \cdot \rangle : \mathfrak{E} \times \mathfrak{E} \longrightarrow \mathfrak{B}$$

determined by the relations:

$$\begin{aligned} \langle b_1, b_2 \rangle &= b_1^* b_2, \text{ for } b_1, b_2 \in \mathfrak{B} \\ \langle u_j, v_j \rangle &= \Phi_j(u_j^* v_j), \text{ for } u_j, v_j \in \ker(\Phi_j), j = 1, 2 \\ \langle u_1, u_2 \rangle &= \langle u_2, u_1 \rangle = 0 \text{ for } u_1 \in \ker(\Phi_1), \text{ and } u_2 \in \ker(\Phi_2). \\ \langle b, u_j \rangle &= \langle u_j, b \rangle = 0 \text{ for all } b \in \mathfrak{B}, u_j \in \mathfrak{A}_j \end{aligned}$$

With this notation, we have that $A_{i,j} = \langle a_i, a_j \rangle$, hence it suffices to show that $\langle a, a \rangle \geq 0$ for all $a \in \mathfrak{E}$.

Indeed, for an element $a = b + u_1 + u_2$ with $b \in \mathfrak{B}, u_j \in \ker(\Phi_j), j = 1, 2$, we have:

$$\begin{aligned} \langle a, a \rangle &= \langle b + u_1 + u_2, b + u_1 + u_2 \rangle \\ &= \langle b, b \rangle + \langle u_1, u_1 \rangle + \langle u_2, u_2 \rangle \\ &= b^* b + \Phi_1(u_1^* u_1) + \Phi_2(u_2^* u_2) \\ &\geq 0 \end{aligned}$$

□

Theorem 2.3. *Let \mathfrak{B} be a C^* -algebra and \mathfrak{D} a C^* -subalgebra of \mathfrak{B} . Suppose that $\mathfrak{A}_1, \mathfrak{A}_2$ are $*$ -algebras containing \mathfrak{B} , each endowed with two positive conditional expectations $\Phi_j : \mathfrak{A}_j \longrightarrow \mathfrak{B}$, and $\Psi_j : \mathfrak{A}_j \longrightarrow \mathfrak{D}$, $j = 1, 2$ and consider the c -free product $(\mathfrak{A}, \Phi, \Psi) = *_{i=1,2}(\mathfrak{A}_i, \Phi_i, \Psi_i)$.*

Then the maps Φ and Ψ are positive.

Proof. The positivity of Ψ is by now a classical result in the theory of free probability with amalgamation over a C^* -algebra (for example, see [9], Theorem 3.5.6). For the positivity of Φ we have to show that $\Phi(a^* a) \geq 0$ for any $a \in \mathfrak{A}$.

Any element of \mathfrak{A} can be written as

$$a = \sum_{k=1}^N s_{1,k} \cdots s_{n(k),k},$$

where $s_{j,k} \in \mathfrak{A}_{\varepsilon(j,k)}$ $\varepsilon(1,k) \neq \varepsilon(2,k) \neq \cdots \neq \varepsilon(n(k),k)$.

Writing

$$s_{(j,k)} = s_{(j,k)} - \Psi(s_{(j,k)}) + \Psi(s_{(j,k)})$$

and expanding the product, we can consider a of the form

$$a = d + \sum_{k=1}^N a_{1,k} \cdots a_{n(k),k}$$

with $d \in \mathfrak{D} \subset \mathfrak{B}$ and $a_{j,k} \in \mathfrak{A}_{\varepsilon(j,k)}$ such that $\Psi_{\varepsilon(j,k)}(a_{j,k}) = 0$ and $\varepsilon(1,k) \neq \varepsilon(2,k) \neq \cdots \neq \varepsilon(n(k),k)$.

Therefore

$$\begin{aligned}\Phi(a^*a) &= \Phi\left(d^*d + d^*\left[\sum_{k=1}^N a_{1,k} \dots a_{n(k),k}\right] + \left[\sum_{k=1}^N a_{1,k} \dots a_{n(k),k}\right]^* d + \right. \\ &\quad \left. \left[\sum_{k=1}^N a_{1,k} \dots a_{n(k),k}\right]^* \left[\sum_{k=1}^N a_{1,k} \dots a_{n(k),k}\right]\right).\end{aligned}$$

Since Φ is a conditional expectation and $d \in \mathfrak{D} \subset \mathfrak{B}$, the above equality becomes

$$\begin{aligned}\Phi(a^*a) &= d^*d + \sum_{k=1}^N d^* \Phi(a_{1,k} \dots a_{n(k),k}) + \sum_{k=1}^N \Phi(a_{n(k),k}^* \dots a_{1,k}^*) d \\ &\quad + \sum_{k,l=1}^N \Phi(a_{n(k),k}^* \dots a_{1,k}^* a_{1,l} \dots a_{n(l),l}).\end{aligned}$$

Using the definition of the conditionally free product with amalgamation over \mathfrak{B} and that $\Psi_{\varepsilon(j,k)}(a_{j,k}) = 0$ for all j, k , one further has

$$\begin{aligned}\Phi(a^*a) &= d^*d + \sum_{k=1}^N \Phi(d^* a_{1,k}) \Phi(a_{2,k}) \dots \Phi(a_{n(k),k}) \\ &\quad + \sum_{k=1}^N \Phi(a_{n(k),k}^* \dots \Phi(a_{2,k}^*) \Phi(a_{1,k}^*) d) \\ &\quad + \sum_{k,l=1}^N [\Phi(a_{n(k),k}^* \dots \Phi(a_{2,k}^*)] \Phi(a_{1,k}^* a_{1,l}) \Phi(a_{2,l}) \dots \Phi(a_{n(l),l})\end{aligned}$$

that is

$$\begin{aligned}\Phi(a^*a) &= d^*d + \sum_{k=1}^N \Phi(d^* a_{1,k}) [\Phi(a_{2,k}) \dots \Phi(a_{n(k),k})] \\ &\quad + \sum_{k=1}^N [\Phi(a_{2,k}) \dots \Phi(a_{n(k),k})]^* \Phi(a_{1,k}^* d) \\ &\quad + \sum_{k,l=1}^N [\Phi(a_{2,k}) \dots \Phi(a_{n(k),k})]^* \Phi(a_{1,k}^* a_{1,l}) [\Phi(a_{2,l}) \dots \Phi(a_{n(l),l})]\end{aligned}$$

From Lemma 2.2, the matrix $S = (\Phi(a_{1,i}^* a_{1,j}))_{i,j=1}^{N+1}$ is positive in $M_{N+1}(\mathfrak{B})$, therefore

$$S = T^*T, \text{ for some } T \in M_{N+1}(\mathfrak{B}).$$

Denote now $a_{1,N+1} = d$ and $v_k = \Phi(a_{2,k}) \dots \Phi(a_{n(k),k})$. The identity for $\Phi(a^*a)$ becomes:

$$\begin{aligned}\Phi(a^*a) &= (v_1, \dots, v_N, 1)^* T^* T(v_1, \dots, v_N, 1) \\ &\geq 0, \text{ as claimed.}\end{aligned}$$

□

Theorem 2.4. Assume that $\mathcal{I} = \bigcup_{j \in \mathfrak{J}} \mathcal{I}_j$ is a partition of \mathcal{I} . Then:

$$*_{j \in \mathfrak{J}} (*_{i \in \mathcal{I}_j} (\mathfrak{A}_i, \Phi_i, \Psi_i)) = *_{i \in \mathcal{I}} (\mathfrak{A}_i, \Phi_i, \Psi_i)$$

Proof. The proof is identical to the proofs of similar results in [6] and [2]. Consider $a_i \in \mathfrak{A}_{\varepsilon(i)}$, $1 \leq i \leq m$ such that $\varepsilon(1) \neq \varepsilon(2) \neq \dots \neq \varepsilon(m)$ and $\Psi_{\varepsilon(i)}(a_i) = 0$. Let $1 = i_0 < i_1 < \dots < i_k = m$ and $\mathfrak{J}_l = \{\varepsilon(i), i_{l-1} \leq i < i_l\}$. Since

$$(*_{j \in \mathfrak{J}_l} \Psi_j)((a_{i_{l-1}} \dots a_{i_l})) = 0,$$

it suffices to show that

$$\Phi(a_1 \dots a_m) = \prod_{l=1}^k [(*_{(\Psi_j), j \in \mathfrak{J}_l} \Phi_j)(a_{i_{l-1}} \dots a_{i_l})].$$

But

$$\Phi(a_1 \dots a_m) = \Phi_{\varepsilon(1)}(a_1) \dots \Phi_{\varepsilon(m)}(a_m)$$

while, since $\Psi_{\varepsilon(i)}(a_i) = 0$,

$$(*_{(\Psi_j), j \in \mathfrak{J}_l} \Phi_j)(a_{i_{l-1}} \dots a_{i_l}) = \Phi_{i_{l-1}}(a_{i_{l-1}}) \dots \Phi_{i_l}(a_{i_l})$$

and the conclusion follows. □

Definition 2.5. Let \mathfrak{A} be an algebra (respectively a $*$ -algebra), \mathfrak{B} a subalgebra ($*$ -subalgebra) of \mathfrak{A} and \mathfrak{D} a subalgebra ($*$ -subalgebra) of \mathfrak{B} . Suppose \mathfrak{A} is endowed with the conditional expectations $\Psi : \mathfrak{A} \longrightarrow \mathfrak{D}$ and $\Phi : \mathfrak{A} \longrightarrow \mathfrak{D}$.

- (i) The subalgebras ($*$ -subalgebras) $(\mathfrak{A}_i)_{i \in \mathcal{I}}$ of \mathfrak{A} are said to be *c-free* with respect to (Φ, Ψ) if:
 - (a) $(\mathfrak{A}_i)_{i \in \mathcal{I}}$ are free with respect to Ψ .
 - (b) if $a_i \in \mathfrak{A}_{\varepsilon(i)}$, $1 \leq i \leq m$, are such that $\varepsilon(1) \neq \dots \neq \varepsilon(m)$ and $\Psi(a_i) = 0$, then $\Phi(a_1 \dots a_m) = \Phi(a_1) \dots \Phi(a_m)$.
- (ii) The elements $(X_i)_{i \in \mathcal{I}}$ of \mathfrak{A} are said to be c-free with respect to (Φ, Ψ) if the subalgebras ($*$ -subalgebras) generated by \mathfrak{B} and X_i are c-free with respect to (Φ, Ψ) .

We will denote by $\mathfrak{B}\langle \xi \rangle$ the non-commutative algebra of polynomials in the symbol ξ and with coefficients from \mathfrak{B} (the coefficients do not commute with the symbol ξ). If \mathcal{I} is a family of indices, $\mathfrak{B}\langle \{\xi_i\}_{i \in \mathcal{I}} \rangle$ will denote the algebra of polynomials in the non-commuting variables $\{\xi_i\}_{i \in \mathcal{I}}$ and with coefficients from \mathfrak{B} . We will identify $\mathfrak{B}\langle \{\xi_i\}_{i \in \mathcal{I}} \rangle$ with the free product with amalgamation over \mathfrak{B} of the family $\{\mathfrak{B}\langle \xi_i \rangle\}_{i \in \mathcal{I}}$.

If \mathfrak{A} is a $*$ -algebra and \mathfrak{B} is with the C^* -algebra, $\mathfrak{B}\langle\xi\rangle$ will also be considered with a $*$ -algebra structure, by taking $\xi^* = \xi$. If X is a selfadjoint element from \mathfrak{A} , we define the conditional expectations

$$\Phi_X, \Psi_X : \mathfrak{B}\langle\xi\rangle \longrightarrow \mathfrak{B}$$

given by $\Phi_X(f(\xi)) = \Phi(f(X))$ and $\Psi_X(f(\xi)) = \Psi(f(X))$, for any $f(\xi) \in \mathfrak{B}\langle\xi\rangle$.

Corollary 2.6. *Suppose that \mathfrak{A} is a $*$ -algebra and X and Y are c -free selfadjoint elements of \mathfrak{A} such that the maps Φ_X, Ψ_X and Φ_Y, Ψ_Y are positive. Then the maps Φ_{X+Y} and Ψ_{X+Y} are also positive.*

Proof. The positivity of Ψ_{X+Y} is an immediate consequence of the fact that X and Y are free with amalgamation over \mathfrak{B} with respect to Ψ . It remains to prove the positivity of Φ_{X+Y} .

Since the maps $\Phi_X : \mathfrak{B}\langle\xi_1\rangle \longrightarrow \mathfrak{B}$ and $\Phi_Y : \mathfrak{B}\langle\xi_2\rangle \longrightarrow \mathfrak{B}$ are positive, from Theorem 2.3 so is

$$\Phi_X *_{(\Psi_X, \Psi_Y)} \Phi_Y : \mathfrak{B}\langle\xi_1\rangle *_{\mathfrak{B}} \mathfrak{B}\langle\xi_2\rangle = \mathfrak{B}\langle\xi_1, \xi_2\rangle \longrightarrow \mathfrak{B}$$

Remark also that

$$i_Z : \mathfrak{B}\langle\xi\rangle \ni f(\xi) \mapsto f(X+Y) \in \mathfrak{B}\langle\xi_1\rangle *_{\mathfrak{B}} \mathfrak{B}\langle\xi_2\rangle$$

is a positive \mathfrak{B} -functional.

The conclusion follows from the fact that the c -freeness of X and Y is equivalent to

$$\Phi_{X+Y} = (\Phi_X *_{(\Psi_X, \Psi_Y)} \Phi_Y) \circ i_{X+Y}.$$

□

3. Multilinear function series and the cR -transform

Let \mathfrak{A} be a $*$ -algebra containing the C^* -algebra \mathfrak{B} , endowed with a conditional expectation $\Psi : \mathfrak{A} \longrightarrow \mathfrak{B}$. If X is a selfadjoint element of \mathfrak{A} , then by the moment of order n of X we will understand the map

$$m_X^{(n)} : \underbrace{\mathfrak{B} \times \dots \times \mathfrak{B}}_{n-1 \text{ times}} \longrightarrow \mathfrak{B}$$

$$m_X^{(n)}(b_1, \dots, b_{n-1}) = \Psi(Xb_1X \dots Xb_{n-1}X)$$

If $\mathfrak{B} = \mathbb{C}$, then the moment-generating series of X

$$m_X(z) = \sum_{n=0}^{\infty} \Psi(X^n) z^n$$

encodes all the information about the moments of X . For $\mathfrak{B} \neq \mathbb{C}$, the straightforward generalization

$$\overline{m}_X(z) = \sum_{n=0}^{\infty} \Psi(X^n) z^n$$

generally fails to keep track of all the possible moments of X . A solution to this inconvenience was proposed in [3], namely the moment-generating multilinear function series of X . Before defining this notion, we will briefly recall the construction and several results on multilinear function series.

Let \mathfrak{B} be an algebra over a field K . We set $\tilde{\mathfrak{B}}$ equal to \mathfrak{B} if \mathfrak{B} is unital and to the unitization of \mathfrak{B} otherwise. For $n \geq 1$, we denote by $\mathcal{L}_n(\mathfrak{B})$ the set of all K -multilinear maps

$$\omega_n : \underbrace{\mathfrak{B} \times \cdots \times \mathfrak{B}}_{n \text{ times}} \longrightarrow \mathfrak{B}$$

A *formal multilinear function series* over \mathfrak{B} is a sequence $\omega = (\omega_0, \omega_1, \dots)$, where $\omega_0 \in \mathfrak{B}$ and $\omega_n \in \mathcal{L}_n(\mathfrak{B})$ for $n \geq 1$. According to [3], the set of all multilinear function series over \mathfrak{B} will be denoted by $Mul[[\mathfrak{B}]]$.

For $\alpha, \beta \in Mul[[\mathfrak{B}]]$, the *formal sum* $\alpha + \beta$ and the *formal product* $\alpha\beta$ are the elements from $Mul[[\mathfrak{B}]]$ defined by:

$$\begin{aligned} (\alpha + \beta)_n(b_1, \dots, b_n) &= \alpha_n(b_1, \dots, b_n) + \beta_n(b_1, \dots, b_n) \\ (\alpha\beta)_n(b_1, \dots, b_n) &= \sum_{k=0}^n \alpha_k(b_1, \dots, b_k) \beta_{n-k}(b_{k+1}, \dots, b_n) \end{aligned}$$

for any $b_1, \dots, b_n \in \mathfrak{B}$.

If $\beta_0 = 0$, then the *formal composition* $\alpha \circ \beta \in Mul[[\mathfrak{B}]]$ is defined by

$$(\alpha \circ \beta)_0 = \alpha_0$$

and, for $n \geq 1$, by

$$(\alpha \circ \beta)_n(b_1, \dots, b_n) = \sum_{k=1}^n \sum \alpha_k \left(\beta_{p_1}(b_1, \dots, b_{p_1}), \dots, \beta_{p_k}(b_{q_k+1}, \dots, b_{q_k+p_k}) \right)$$

where the second summation is done over all k -tuples $p_1, \dots, p_k \geq 1$ such that $p_1 + \cdots + p_k = n$ and $q_j = p_1 + \cdots + p_{j-1}$.

One can work with elements of $Mul[[\mathfrak{B}]]$ as if they were formal power series. The relevant properties are described in [3], Proposition 2.3 and Proposition 2.6. As in [3], we use 1 , respectively I , to denote the identity elements of $Mul[[\mathfrak{B}]]$ relative to multiplication, respectively composition. In other words, $1 = (1, 0, 0, \dots)$ and $I = (0, id_{\mathfrak{B}}, 0, 0, \dots)$. We will also use the fact that an element $\alpha \in Mul[[\mathfrak{B}]]$ has an inverse with respect to formal composition, denoted $\alpha^{(-1)}$, if and only if α has the form $(0, \alpha_1, \alpha_2, \dots)$ with α_1 an invertible element of $\mathcal{L}_1(\mathfrak{B})$.

Definition 3.1. *With the above notation, the moment-generating multilinear function series \mathcal{M}_X of X is the element of $Mul[[\mathfrak{B}]]$ such that:*

$$\begin{aligned} \mathcal{M}_{X,0} &= \Psi(X) \\ \mathcal{M}_{X,n}(b_1, \dots, b_n) &= \Psi(Xb_1X \cdots Xb_nX). \end{aligned}$$

Given an element $\alpha \in Mul[[\mathfrak{B}]]$, the multilinear function series R_α is defined by the following equation (see [3], Def 6.1):

$$R_\alpha = \left((1 + \alpha I)^{-1} \right) \circ (I + I\alpha I)^{(-1)}. \quad (3.1)$$

A key property of R is that for any $X, Y \in \mathfrak{A}$ free over \mathfrak{B} , we have

$$R_{\mathcal{M}_{X+Y}} = R_{\mathcal{M}_X} + R_{\mathcal{M}_Y}. \quad (3.2)$$

These relations were proved earlier in the particular case $\mathfrak{B} = \mathbb{C}$. One can also describe R_α by combinatorial means, via the recurrence relation

$$\begin{aligned} \alpha_n(b_1, \dots, b_n) &= \sum_{k=0}^n \sum R_{\alpha, k} \left([b_1 \alpha_{p(1)}(b_3, \dots, b_{i_1-2}) b_{i_1-1}], \dots \right. \\ &\quad \left. \dots, [b_{i(k-1)} \alpha_{p(k)}(b_{i(k-1)+1}, \dots, b_{i(k)-2}) b_{i(k)-1}] \right) b_{i(k)} \alpha_{n-i_k}(b_{i_{k+1}}, \dots, b_n) \end{aligned}$$

where the second summation is done over all $1 = i(0) < i(1) < \dots < i(k) \leq n$ and $p(k) = i(k) - i(k-1) - 2$.

Following an idea from [2], the above equation can be graphically illustrated by the picture:

$$\boxed{} = \sum \boxed{} \boxed{} \dots \boxed{} \boxed{}$$

In the case of scalar c-free probability, an analogue of the Voiculescu's R -transform is developed in [2]. In order to avoid confusions, we will denote it by cR .

The cR -transform has the property that it linearizes the c-free convolution of pairs of compactly supported measures. In particular, if X and Y are c-free elements from some algebra \mathfrak{A} , then

$${}^cR_{X+Y} = {}^cR_X + {}^cR_Y.$$

If the $*$ -algebra \mathfrak{A} is endowed with the \mathbb{C} -valued states φ, ψ and X is a selfadjoint element of \mathfrak{A} , then (see [2]), the coefficients $\{{}^cR_m\}_m \geq 0$ of cR_X are defined by the recurrence:

$$\varphi(X^n) = \sum_{k=1}^n \sum_{\substack{l(1), \dots, l(k) \geq 0 \\ l(1) + \dots + l(k) = n-k}} {}^cR_k \cdot \psi(X^{l(1)}) \dots \psi(X^{l(k-1)}) \varphi(X^{l(k)})$$

equation that can be graphically illustrated by the picture, where the dark boxes stand for the application of φ and the light ones for the application of ψ :

$$\blacksquare = \sum \boxed{} \boxed{} \dots \boxed{} \blacksquare$$

The above considerations lead to the following definition:

Definition 3.2. Let $\beta, \gamma \in \text{Mul}[[\mathfrak{B}]]$. The multilinear function series ${}^cR_{\beta, \gamma}$ is the element of $\text{Mul}[[\mathfrak{B}]]$ defined by the recurrence relation

$$\begin{aligned} \beta_n(b_1, \dots, b_n) &= \sum_{k=0}^n \sum {}^cR_{\beta, \gamma, k} \left([b_1 \gamma_{p(1)}(b_3, \dots, b_{i_1-2}) b_{i_1-1}], \dots \right. \\ &\quad \left. \dots, [b_{i(k-1)} \gamma_{p(k)}(b_{i(k-1)+1}, \dots, b_{i(k)-2}) b_{i(k)-1}] \right) b_{i(k)} \beta_{n-i_k}(b_{i_{k+1}}, \dots, b_n) \end{aligned}$$

where the second summation is done over all $1 = i(0) < i(1) < \dots < i(k) \leq n$ and $p(k) = i(k) - i(k-1) - 2$.

The following analytical description of ${}^cR_{\beta,\gamma}$ also shows that it is unique and well-defined:

Theorem 3.3. *For any $\beta, \gamma \in \text{Mul}[[\mathfrak{B}]]$,*

$${}^cR_{\beta,\gamma} = [\beta(1 + I\beta)^{-1}] \circ (I + I\gamma I)^{(-1)} \quad (3.3)$$

Before proving the theorem, remark that the right-hand side of (3.3) is well-defined and unique, since $1 + I\gamma$ is invertible with respect to the formal multiplication, $I + I\beta I$ is invertible with respect to formal composition and its inverse has 0 as first component (see [3]). We will need the following auxiliary result:

Lemma 3.4. *Let β be an element of $\text{Mul}[[\mathfrak{B}]]$ and I the identity element with respect to formal composition, $I = (0, id_{\mathfrak{B}}, 0, 0, \dots)$.*

(i) *the multilinear function series $I\beta$ is given by:*

$$\begin{aligned} (I\beta)_0 &= 0 \\ (I\beta)_n(b_1, \dots, b_n) &= b_1\beta_{n-1}(b_2, \dots, b_n) \end{aligned}$$

(ii) *the multilinear function series $I\beta I$ is given by*

$$\begin{aligned} (I\beta I)_0 &= 0 \\ (I\beta I)_1(b_1) &= 0 \\ (I\beta I)_n(b_1, \dots, b_n) &= b_1\beta_{n-2}(b_2, \dots, b_{n-1})b_n \end{aligned}$$

Proof. Since $I = (0, id_{\mathfrak{B}}, 0, \dots)$, one has:

$$(I\beta)_0 = I_0\beta_0 = 0.$$

If $n \geq 1$,

$$\begin{aligned} (I\beta)_n(b_1, \dots, b_n) &= \sum_{k=0}^n I_k(b_1, \dots, b_k)\beta_{n-k}(b_{k+1}, \dots, b_n) \\ &= I_1(b_1)\beta_{n-1}(b_2, \dots, b_n) \\ &= b_1\beta_{n-1}(b_2, \dots, b_n). \end{aligned}$$

For $I\beta I$, the same computations give:

$$\begin{aligned} (I\beta I)_0 &= (I\beta)_0 I_0 = 0 \\ (I\beta I)_1 &= (I\beta)_0 I_1(b_1) + (I\beta)_1(b_1) I_0 \\ &= 0. \end{aligned}$$

If $n \geq 2$, one has:

$$\begin{aligned} (I\beta I)_n(b_1, \dots, b_n) &= \sum_{k=0}^n (I\beta)_k(b_1, \dots, b_k) I_{n-k}(b_{k+1}, \dots, b_n) \\ &= (I\beta)_{n-1}(b_1, \dots, b_{n-1}) I_1(b_n) \\ &= b_1\beta_{n-2}(b_2, \dots, b_{n-1})b_n \end{aligned}$$

□

Proof of the Theorem 3.3: Set $\sigma = I + I\beta I$. Then

$$({}^cR_{\beta,\gamma} \circ \sigma)_n(b_1, \dots, b_n) = \sum_{k=1}^n \sum_{\substack{p_1, \dots, p_k \geq 1 \\ p_1 + \dots + p_k = n}} {}^cR_{\beta,\gamma,k} \left(\sigma_{p_1}(b_1, \dots, b_{p_1}), \dots, \right. \\ \left. \sigma_{p_k}(b_{q_k+1}, \dots, b_{q_k+p_k}) \right)$$

where $q_i = p_1 + \dots + p_{i-1}$.

From Lemma (3.4)(ii), we have that

$$\sigma_n(b_1, \dots, b_n) = (I + I\beta I)_n(b_1, \dots, b_n)$$

therefore Definition 3.2 is equivalent to

$$\beta_n(b_1, \dots, b_n) = \sum_{k=0}^n ({}^cR_{\beta,\gamma} \circ (I + I\beta I)_k(b_1, \dots, b_k)) b_{k+1} \beta_{n-k-2}(b_{k+2}, \dots, b_n)$$

Considering now Lemma 3.4(i), the above relation becomes

$$\beta_n(b_1, \dots, b_n) = \sum_{k=0}^n ({}^cR_{\beta,\gamma} \circ (I + I\beta I)_k(b_1, \dots, b_k)) (I + I\beta)_{n-k}(b_{k+1}, \dots, b_n)$$

therefore

$$\beta = [{}^cR_{\beta,\gamma} \circ (I + I\gamma I)] (1 + I\beta)$$

which is equivalent to (3.3). \square

Remark 3.5. Up to a shift in the coefficients, equation (3.3) is similar to the result in the case $\mathfrak{B} = \mathbb{C}$ from [2], Theorem 5.1.

Let X be a selfadjoint element of \mathfrak{A} . If \mathfrak{A} is endowed with two \mathfrak{B} -valued conditional expectations Φ, Ψ , the element X will have two moment-generating multilinear function series, one with respect to Ψ , that we will denote by \mathcal{M}_X , and one with respect to Φ , denoted \mathfrak{M}_X . For brevity, we will use the notation cR_X for the multilinear function series ${}^cR_{\mathcal{M}_X, \mathfrak{M}_X}$.

Theorem 3.6. *Let X and Y be two elements of \mathfrak{A} that are c -free with respect to the pair of conditional expectations (Φ, Ψ) . Then*

$${}^cR_{X+Y} = {}^cR_X + {}^cR_Y$$

Proof. Let \mathcal{A} be an algebra containing \mathfrak{B} as a subalgebra and endowed with the conditional expectations $\Phi, \Psi : \mathcal{A} \rightarrow \mathfrak{B}$. Consider the set $\mathcal{A}_0 = \mathcal{A} \setminus \mathfrak{B}$ (set difference). For $n \geq 1$ define the maps

$${}^c r : \underbrace{\mathcal{A}_0 \times \dots \times \mathcal{A}_0}_{n \text{ times}} \rightarrow \mathfrak{B}$$

given by the recurrence formula:

$$\Phi(a_1 \dots a_n) = \sum_{k=1}^n \sum_{\substack{l(1) < \dots < l(k) \\ 1 < l(1), l(k) \leq n}} {}^c r_k \left(a_1 [\Psi(a_2 \dots a_{l(1)-1})], \dots, \right. \\ \left. \dots, a_{l(k-1)} [\Psi(a_{l(k-1)+1} \dots a_{l(k)_1})], a_{l(k)} [\Phi(a_{l(k)+1} \dots a_n)] \right)$$

Note that ${}^c r_n$ is well defined, and that, for any $b_1, \dots, b_n \in \mathfrak{B}$,

$${}^c r_{n+1}(X, b_1 X, \dots, b_n X) = {}^c R_{X,n}(b_1, \dots, b_n). \quad (3.4)$$

As in Section 2, consider $\mathfrak{B}\langle \xi_i \rangle$, the noncommutative algebras of polynomials in the symbols $\xi_i, i = 1, 2$ and with coefficients from \mathfrak{B} and the conditional expectations

$$\Phi_X, \Psi_X : \mathfrak{B}\langle \xi_1 \rangle \longrightarrow \mathfrak{B}$$

given by

$$\begin{aligned} \Phi_X(f(\xi_1)) &= \Phi(f(X)) \\ \Psi_X(f(\xi_1)) &= \Psi(f(X)) \end{aligned}$$

and their analogues Φ_Y, Ψ_Y for $\mathfrak{B}\langle \xi_2 \rangle$.

On $\mathfrak{B}\langle \xi_1, \xi_2 \rangle$, identified to $\mathfrak{B}\langle \xi_1 \rangle *_{\mathfrak{B}} \mathfrak{B}\langle \xi_2 \rangle$, consider the conditional expectations Ψ_0, Φ_0, φ given by:

$$\begin{aligned} \Psi_0 &= \Psi_X * \Psi_Y \\ \Phi_0(f(\xi_1, \xi_2)) &= \Phi(f(X, Y)) \\ \varphi(a_1 a_2 \dots a_n) &= \sum_{k=1}^n \sum_{\substack{l(1) < \dots < l(k) \\ 1 < l(1), l(k) \leq n}} \rho_k \left(a_1 [\Psi_0(a_2 \dots a_{l(1)-1})], \dots, \right. \\ &\quad \left. \dots, a_{l(k-1)} [\Psi_0(a_{l(k-1)+1} \dots a_{l(k)_1})], a_{l(k)} [\varphi(a_{l(k)+1} \dots a_n)] \right) \end{aligned}$$

where a_1, \dots, a_n are elements of the set $\mathfrak{B}\langle \xi_1, \xi_2 \rangle_0 = \mathfrak{B}\langle \xi_1 \rangle \cup \mathfrak{B}\langle \xi_2 \rangle \setminus \mathfrak{B}$, and the maps

$$\rho_n : \underbrace{\mathfrak{B}\langle \xi_1, \xi_2 \rangle_0 \times \dots \times \mathfrak{B}\langle \xi_1, \xi_2 \rangle_0}_{n \text{ times}} \longrightarrow \mathfrak{B}$$

are given by:

$$\rho_n(a_1, \dots, a_n) = \begin{cases} {}^c r(a_1, \dots, a_n) & \text{if all } a_1, \dots, a_n \in \mathfrak{B}\langle \xi_1 \rangle \\ {}^c r(a_1, \dots, a_n) & \text{if all } a_1, \dots, a_n \in \mathfrak{B}\langle \xi_2 \rangle \\ 0 & \text{otherwise} \end{cases}.$$

We will show that $\varphi = \Phi_0$, in particular φ is also well-defined. Consider the element $a \in \mathfrak{B}\langle \xi_1, \xi_2 \rangle$ of the form $a = a_1 \dots a_n$ with $a_j \in \mathfrak{B}\langle \xi_{\varepsilon(j)} \rangle$, such that $\varepsilon(1) \neq \varepsilon(2) \neq \dots \neq \varepsilon(n)$ and $\Psi_0(a_j) = 0$. The computation of $\varphi(a_1 \dots a_n)$ is done via the recurrence relation above. Because of the definition of ρ and the fact that $\Psi_0 = \Psi_X * \Psi_Y$, only the term with $k = 1$ contribute at the sum, i.e.

$$\begin{aligned} \varphi(a_1 \dots a_n) &= \varphi(a_1 \varphi(a_2 \dots a_n)) \\ &= \varphi_{\varepsilon(1)}(a_1 \varphi(a_2 \dots a_n)) \\ &= \varphi_{\varepsilon(1)}(a_1) \varphi(a_2 \dots a_n) \end{aligned}$$

and the identity between φ and Φ_0 follows by induction over n .

Since $\varphi = \Phi_0$, the maps ρ_n and ${}^c r_n$ are satisfying the same recurrence relation, hence

$$\rho_n(a_1, \dots, a_n) = {}^c r(a_1, \dots, a_n).$$

In particular

$$\begin{aligned}
{}^cR_{X+Y,n}(b_1, \dots, b_n) &= {}^c r_{n+1}((X+Y)b_1(X+Y) \dots (X+Y)b_n(X+Y)) \\
&= \rho_{n+1}((X+Y)b_1(X+Y) \dots (X+Y)b_n(X+Y)) \\
&= \rho_{n+1}((X)b_1(X) \dots (X)b_n(X)) + \rho_{n+1}((Y)b_1(Y) \dots (Y)b_n(Y)) \\
&= {}^cR_{X,n}(b_1, \dots, b_n) + {}^cR_{Y,n}(b_1, \dots, b_n).
\end{aligned}$$

□

4. Central limit theorem

Consider the ordered set $\langle n \rangle = \{1, 2, \dots, n\}$ and π a partition of $\langle n \rangle$ with blocks B_1, \dots, B_m :

$$\langle n \rangle = B_1 \sqcup B_2 \sqcup \dots \sqcup B_m.$$

The blocks B_p and B_q of π are said to be *crossing* if there exist $i < j < k < l$ in $\langle n \rangle$ such that $i, k \in B_p$ and $j, l \in B_q$.

The partition π is said to be *non-crossing* if all pairs of distinct blocks of π are not crossing. We will denote by $NC_2(n)$ the set of all non-crossing partitions of $\langle n \rangle$ whose blocks contain exactly 2 elements and by $NC_{\leq s}(n)$ the set of all non-crossing partitions of $\langle n \rangle$ whose blocks contain at most s elements.

Let now γ be a non-crossing partition of $\langle n \rangle$ and B and C be two blocks of π . We say that B is interior to C if there exist two indices $i < j$ in $\langle n \rangle$ such that $i, j \in C$ and $B \subset \{i+1, \dots, j-1\}$. The block B is said to be *outer* if it is not interior to any other block of γ . In a non-crossing partition of $\langle n \rangle$, the block containing 1 is always outer.

Consider now an element X of \mathfrak{A} . Let π be a partition from $NC_2(n+1)$ ($n = \text{odd}$) and $B_1 = (1, k)$ be the block of π containing 1. We define, by recurrence, the following expressions:

$$\begin{aligned}
V_\pi(X, b_1, \dots, b_n) &= \Psi(Xb_1V_{\pi|\{2, \dots, j-1\}}(X, b_2, \dots, b_{k-2})b_{k-1}X)b_k \\
&\quad V_{\pi|\{k+1, \dots, n+1\}}(X, b_{k+1}, \dots, b_n) \\
W_\pi(X, b_1, \dots, b_n) &= \Phi(Xb_1V_{\pi|\{2, \dots, j-1\}}(X, b_2, \dots, b_{k-2})b_{k-1}X)b_k \\
&\quad W_{\pi|\{k+1, \dots, n+1\}}(X, b_{k+1}, \dots, b_n)
\end{aligned}$$

Theorem 4.1. (*Central Limit Theorem*) Let $(X_n)_{n \geq 1}$ be a sequence of c -free elements of \mathfrak{A} such that:

- (1) all X_n have the same moment-generating multilinear function series, \mathfrak{M} with respect to Φ and M with respect to Ψ .
- (2) $\Psi(X_n) = \Phi(X_n) = 0$.

Set

$$S_N = \frac{X_1 + \dots + X_N}{\sqrt{N}},$$

Then:

- (i) $\lim_{N \rightarrow \infty} {}^cR_{S_N} = (0, \mathfrak{M}_1(\cdot), 0, \dots)$

- (ii) $\lim_{N \rightarrow \infty} R_{S_N} = (0, M_1(\cdot), 0, \dots)$
- (iii) *there exist two conditional expectations $\nu : \mathfrak{B}\langle \xi \rangle \longrightarrow \mathfrak{B}$, depending only on $M_1(\cdot)$, and $\mu : \mathfrak{B}\langle \xi \rangle \longrightarrow \mathfrak{B}$, depending only on $M_1(\cdot)$ and $\mathfrak{M}_1(\cdot)$, such that*

$$\begin{aligned} \lim_{N \rightarrow \infty} \Psi_{S_N} &= \nu \\ \lim_{N \rightarrow \infty} \Phi_{S_N} &= \mu \end{aligned}$$

in the weak sense; in particular,

$$\begin{aligned} \nu(\xi b_1 \xi \dots b_n \xi) &= \sum_{\pi \in NC_2(n)} V_\pi(X_1, b_1, \dots, b_n) \\ \mu(\xi b_1 \xi \dots b_n \xi) &= \sum_{\pi \in NC_2(n)} W_\pi(X_1, b_1, \dots, b_n). \end{aligned}$$

Proof. Let X be an element of \mathfrak{A} with the same moment generating series as X_j , $j \geq 1$. As shown in [3],

$$R_{S_N} = \sum_{k=1}^N R_{\frac{X_k}{\sqrt{N}}} = N R_{\frac{X}{\sqrt{N}}}.$$

Also, from Theorem 2.4 and Theorem 3.6, it follows that

$${}^c R_{S_N} = \sum_{k=1}^N {}^c R_{\frac{X_k}{\sqrt{N}}} = N {}^c R_{\frac{X}{\sqrt{N}}}.$$

Since R and ${}^c R$ are multilinear and $M_0 = \mathfrak{M}_0 = 0$, we have that

$$\begin{aligned} \lim_{N \rightarrow \infty} {}^c R_{S_N, n} &= \lim_{N \rightarrow \infty} \frac{N}{N^{\frac{n+1}{2}}} {}^c R_{X, n} \\ &= \begin{cases} 0 & \text{if } n \neq 1 \\ \mathfrak{M}_1(\cdot) & \text{if } n = 1 \end{cases} \end{aligned}$$

and the similar relations for $R_{S_N, n}$, hence (i) and (ii) are proved.

For (iii) it suffices to check the relations for $\nu(\xi b_1 \xi \dots b_n \xi)$ and $\mu(\xi b_1 \xi \dots b_n \xi)$, which are a trivial corollary of (i), (ii), and the recurrence formulas that define R and ${}^c R$. \square

Remark 4.2. For $\mathfrak{B} = \mathbb{C}$, the theorem is a weaker version of Theorem 4.3 from [2]. If Ψ is \mathbb{C} -valued, then the result is similar to Corollary 5.1 from [6]. Also, under the assumptions that for some $a, b \in \mathfrak{B}$ we have that:

$$\begin{aligned} \lim_{N \rightarrow \infty} N \Psi(X_1 \cdots X_N) &= a \\ \lim_{N \rightarrow \infty} N \Psi(X_1 \cdots X_N) &= b \end{aligned}$$

the same techniques lead to a Poisson-type limit Theorem, similar to Corollary 2, Section 5 of [6].

In the following remaining pages we will describe the positivity of the limit functionals μ and ν in terms of Φ and Ψ . The central result is Corollary 4.4.

For simplicity, suppose that \mathfrak{B} is a unital $*$ -algebra (otherwise, we can replace \mathfrak{B} by its unitisation). Consider the symbol ξ , the $*$ -algebra $\mathfrak{B}\langle\xi\rangle$ of polynomials in ξ with coefficients from \mathfrak{B} , as defined before, and consider also the linear space $\mathfrak{B}\xi\mathfrak{B}$ generated by the set $\{b_1\xi b_2; b_1, b_2 \in \mathfrak{B}\}$ with the \mathfrak{B} -bimodule structure given by

$$a_1 b_1 \xi b_2 a_2 = (a_1 b_1) \xi (b_2 a_2)$$

for all $a_1, a_2, b_1, b_2 \in \mathfrak{B}$.

Lemma 4.3. *For any positive \mathfrak{B} -sesquilinear pairing $\langle \cdot, \cdot \rangle$ on $\mathfrak{B}\xi\mathfrak{B}$ there exists a positive conditional expectation $\varphi : \mathfrak{B}\langle\xi\rangle \longrightarrow \mathfrak{B}$ such that for any $b_1, b_2 \in \mathfrak{B}$ one has that*

$$\varphi(\xi b_1^* b_2 \xi) = \langle b_1 \xi, b_2 \xi \rangle$$

Proof. Without loss of generality, we can suppose that \mathfrak{B} is unital (otherwise we can replace \mathfrak{B} by its unitization).

Consider the Full Fock bimodule over $\mathfrak{B}\xi\mathfrak{B}$

$$\mathcal{F}\langle\xi\rangle = \mathfrak{B} \oplus \left(\bigoplus_{n \geq 1} \underbrace{\mathfrak{B}\xi\mathfrak{B} \otimes_{\mathfrak{B}} \cdots \otimes_{\mathfrak{B}} \mathfrak{B}\xi\mathfrak{B}}_{n \text{ times}} \right)$$

with the pairing given by

$$\begin{aligned} \langle a, b \rangle &= a^* b \\ \langle a_1 \xi \otimes \cdots \otimes a_n \xi, b_1 \xi \otimes \cdots \otimes b_m \xi \rangle &= \delta_{m,n} \langle a_n \xi, \langle \dots, \langle a_1 \xi, b_1 \xi \rangle b_2 \xi \rangle, \dots, b_n \xi \rangle. \end{aligned}$$

$(a, a_j, b, b_j \in \mathfrak{B}, j = 1, \dots, n)$

Note that the \mathfrak{B} -linear operators $A_1, A_2 : \mathcal{F}\langle\xi\rangle \longrightarrow \mathcal{F}\langle\xi\rangle$ described by the relations

$$\begin{aligned} A_1 b &= \xi b \\ A_1(a_1 \xi \otimes \cdots \otimes a_n \xi b) &= \xi \otimes a_1 \xi \otimes \cdots \otimes a_n \xi b \\ A_2 b &= 0 \\ A_2(a_1 \xi \otimes \cdots \otimes a_n \xi b) &= \langle \xi, a_1 \xi \rangle a_2 \xi \otimes \cdots \otimes a_n \xi b \end{aligned}$$

are self-adjoint to each other, in the sense that

$$\langle A_1 \tilde{\zeta}_1, \tilde{\zeta}_2 \rangle = \langle \tilde{\zeta}_1, A_2 \tilde{\zeta}_2 \rangle$$

for any $\tilde{\zeta}_1, \tilde{\zeta}_2 \in \mathcal{F}\langle\xi\rangle$, therefore $S = A_1 + A_2$ is selfadjoint.

Moreover, for any $a, b \in \mathfrak{B}$,

$$\begin{aligned} \langle 1, S a^* b S 1 \rangle &= \langle a S 1, b S 1 \rangle \\ &= \langle a(A_1 + A_2)1, b(A_1 + A_2)1 \rangle \\ &= \langle a \xi, b \xi \rangle \end{aligned}$$

and the conclusion follows by setting $\varphi(p(\xi)) = \langle 1, p(S)1 \rangle$ for all $p \in \mathfrak{B}\langle\xi\rangle$. \square

Corollary 4.4. *The maps μ and ν from Theorem 4.1 are positive if and only if for any $b \in \mathfrak{B}$ one has that $\Phi(Xb^*bX) \geq 0$ and $\Psi(Xb^*bX) \geq 0$.*

Proof. One implication is trivial, since, if ν and μ are positive, then

$$\Psi(Xb^*bX) = \nu(Xb^*bX) = \nu((bX)^*bX) \geq 0$$

and

$$\Phi(Xb^*bX) = \mu(Xb^*bX) = \mu((bX)^*bX) \geq 0.$$

Suppose now that $\Phi(Xb^*bX) \geq 0$ and $\Psi(Xb^*bX) \geq 0$ for all $b \in \mathfrak{B}$. We will use the same argument as in [9] and [8].

Consider the set of selfadjoint symbols $\{\xi_i\}_{i \geq 1}$. On each \mathfrak{B} -bimodule $\mathfrak{B}\xi_i\mathfrak{B}$ we have the positive \mathfrak{B} -sesquilinear pairings $\langle \cdot, \cdot \rangle_\Phi$ and $\langle \cdot, \cdot \rangle_\Psi$ determined by

$$\begin{aligned} \langle a\xi_i, b\xi_i \rangle_\Phi &= \Phi(Xa^*bX) \\ \langle a\xi_i, b\xi_i \rangle_\Psi &= \Psi(Xa^*bX). \end{aligned}$$

As shown in Lemma 4.3, the above \mathfrak{B} -sesquilinear pairings determine positive conditional expectations $\varphi_1, \psi_i : \mathfrak{A}_i \rightarrow \mathfrak{B}$, where $\mathfrak{A}_i = \mathfrak{B}\langle \xi_i \rangle$ be the $*$ -algebras of polynomials in ξ with coefficients from \mathfrak{B} , $i \geq 1$.

For $\tau : \mathfrak{B}\langle \xi \rangle \rightarrow \mathfrak{B}$ a conditional expectation, and $\lambda \geq 0$, note with $D_\lambda \tau$ the dilation with λ of τ , i.e.

$$D_\lambda \tau(\xi b_1 \xi \cdots b_n \xi) = \lambda^{n+1} \tau(\xi b_1 \xi \cdots b_n \xi)$$

Remark that if τ is positive, then $D_\lambda \tau$ is also positive.

With the notations above, consider, as in Definition 2.1, the conditionally free product $(\mathfrak{A}, \Phi, \Psi) = \ast_{i \in \mathcal{I}} (\mathfrak{A}_i, \Phi_i, \Psi_i)$. The elements $\{\xi_i\}_{i \geq 1}$ are conditionally free in \mathfrak{A} , so Theorem 4.1 implies that:

$$\begin{aligned} \mu &= \lim_{N \rightarrow \infty} \Phi_{\frac{\xi_1 + \cdots + \xi_N}{\sqrt{N}}} = D_{\frac{1}{\sqrt{N}}} \Phi_{\xi_1 + \cdots + \xi_N} \\ \nu &= \lim_{N \rightarrow \infty} \Psi_{\frac{\xi_1 + \cdots + \xi_N}{\sqrt{N}}} = D_{\frac{1}{\sqrt{N}}} \Psi_{\xi_1 + \cdots + \xi_N} \\ &= D_{\frac{1}{\sqrt{N}}} \left(\ast_{i=1}^N \Psi_{\xi_i} \right). \end{aligned}$$

We have that $\ast_{i=1}^N \Psi_{\xi_i} \geq 0$ since it is the free product of states (see, for example [9]), hence the positivity of ν .

Also, Theorem 2.4 and Corollary 2.6 imply that $\Phi_{\xi_1 + \cdots + \xi_N} \geq 0$, therefore $\mu \geq 0$. \square

Acknowledgment. This research was partially supported by the Grant 2-CEx06-11-34 of the Romanian Government. I am thankful to Marek Bożejko for presenting me the basics of c-freeness and bringing to my attention the references [2] and [6]. I thank also Hari Bercovici for his constant support and his many advices during the work on this paper.

References

1. Bożejko, M. and Speicher, R.: ψ -independent and symmetrized white noises., in: *Quantum Probability and Related Topics*, **VI** (1991), 219-236, World Scientific, Singapore.
2. Bożejko, M., Leinert, M. and Speicher, R.: Convolution and Limit Theorems for Conditionally free Random Variables., in: *Pac. J. Math.* **175** (1996), 357-388.
3. Dykema, K.: Multilinear function series and transforms in Free Probability theory, in: *Advances in Mathematics* **208** (2007) 351407.
4. Dykema, K.: On the S-transform over a Banach algebra, in: *Journal of Functional Analysis* **231** (2006) 90110
5. Lance, E. C.: *Hilbert C^* -modules. A toolkit for operator algebraists*, London Mathematical Society Lecture Note Series **210**, Cambridge University Press 1990.
6. Młotkowski, W.: Operator-valued version of conditionally free product, in: *Studia Mathematica* **153** (1) (2002)
7. Nica, A. and Speicher, R.: *Lectures on the Combinatorics of the Free Probability*, London mathematical Society Lecture Note Series 335, Cambridge University Press 2006
8. Popa, M.: A combinatorial approach to monotonic independence over a C^* -algebra Preprint, arXiv: math.OA/0612570, 01/2007
9. Speicher, R.: Combinatorial Theory of the Free Product with amalgamation and Operator-Valued Free Probability Theory, in: *Mem. AMS*, Vol **132**, No 627 (1998)

MIHAI POPA: DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY AT BLOOMINGTON,
 RAWLES HALL, 931 E 3RD ST, BLOOMINGTON, IN 47405
E-mail address: mipopa@indiana.edu