Dynamics of Open Quantum Systems I, Oscillation and Decay

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Abstract

A finite-dimensional quantum system is in contact with a reservoir, which has a stationary state but otherwise dissipative dynamics. We give an expansion of the propagator of the system-reservoir interacting dynamics as a sum of a main part plus a remainder. The main part consist of oscillatory and decaying terms with explicit frequencies and decay rates. The remainder is small in the system-reservoir coupling constant, uniformly for all times. Our approach is based on Mourre theory for the spectral analysis of the generator of the full dynamics. This allows for weak regularity of admissible interaction operators and has important consequences in applications.

1 Setup and assumptions

We consider a bipartite Hilbert space

$$\mathcal{H} = \mathcal{H}_{\mathrm{S}} \otimes \mathcal{H}_{\mathrm{R}}, \tag{1.1}$$

where $\dim \mathcal{H}_{\mathrm{S}} < \infty$, and a family of self-adjoint operators

$$L_\lambda = L_0 + \lambda I, \tag{1.2}$$

where both L_0 and I are self-adjoint, $\lambda \in \mathbb{R}$ and L_0 is of the form

$$L_0 = L_{\mathrm{S}} \otimes \mathbb{1}_{\mathrm{R}} + \mathbb{1}_{\mathrm{S}} \otimes L_{\mathrm{R}}. \tag{1.3}$$

This is the setup describing the composition of a system (S) plus reservoir (R) arrangement, in which $L_{\rm S}$ and $L_{\rm R}$ generate the free (non interacting) dynamics of the individual components, I is an interaction operator and λ is a coupling constant.

We suppose that L_0 has finitely many eigenvalues e, of finite multiplicity m_e, possibly embedded in continuous spectrum, which can cover parts or all of $\mathbb{R}$. The set of eigenvalues e of L_0 is denoted by $\mathcal{E}_0$ and the associated orthogonal eigenprojection is P_e. We are going to impose a regularity condition (*c.f.* (A1) below) which implies the following picture for small λ: All eigenvalues of L_λ lie inside an $O(\lambda)$ neighbourhood of $\mathcal{E}_0$. Moreover, within such a neighbourhood around any given $e\in\mathcal{E}_0$, either L_λ does not have any eigenvalues for $\lambda\neq 0$ (we say e is unstable), or L_λ does have some eigenvalues, with summed multiplicity m_e' not exceeding m_e (we say e is stable if $m_e'=m_e$, and partially stable if $0<m_e'<m_e$). In the analytic perturbation theory of isolated eigenvalues, the summed multiplicity m_e' would always equal m_e, but for embedded eigenvalues, it is generically strictly less than m_e.

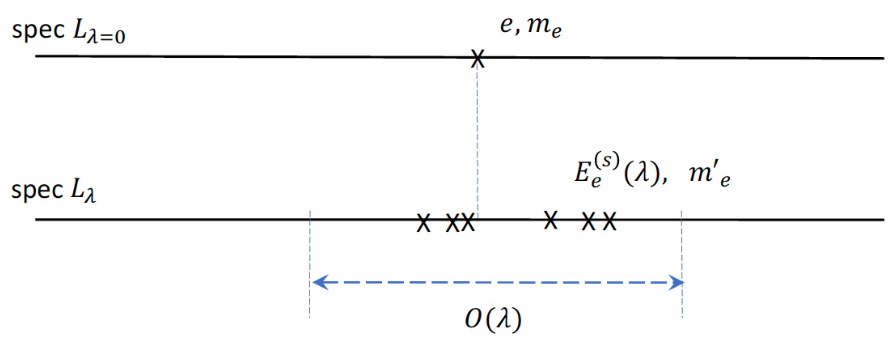

Fig. 1: An eigenvalue e of L_0 splits into eigenvalues $E_e^{(s)}(\lambda)$, $s=1,\ldots,m_e'$, of L_λ.

Fig. 1 gives a graphical depiction of the situation. The main goal of this paper is to find an expression of probability amplitudes (in the quantum mechanical sense), that is, of overlaps

$$F_{\phi,\psi}(t)=\langle\phi, e^{{\rm i}tL_\lambda}\psi\rangle, \tag{1.4}$$

for suitable vectors $\phi,\psi\in\mathcal{H}$. In particular, we try to elucidate the effect of the instability of eigenvalues on the dynamics. The key phenomenon we will uncover is this: the overlaps split into

$$F_{\phi,\psi}(t)=F^{\rm qsta}(t)+F^{\rm diss}(t)+R(t), \tag{1.5}$$

where $F^{\rm qsta}(t)$ is a quasi-static contribution, $F^{\rm diss}(t)$ is a dissipative contribution decaying as $|F^{\rm diss}(t)|\le C/(1+t^2)$, and the remainder satisfies $|R(t)|\le C|\lambda|^{1/4}$ for all $t\ge 0$. We find the detailed structure of $F^{\rm qsta}(t)$. It consists of oscillating terms and terms decaying at rates $\propto 1/\lambda^2$, see Theorem 2.1.

Let us now introduce the assumptions and discuss their meaning. We start with an assumption to simplify the bookkeeping. It is not essential for our method to work and could be removed at the cost of a more cumbersome presentation.

(A1) For $\lambda \neq 0$ small enough, all eigenvalues of L_λ are simple.

We call these eigenvalues $E_e^{(s)}(\lambda)$, with $s = 1, \ldots, m_e' \leq m_e$, and

$$\lim_{\lambda \to 0} E_e^{(s)}(\lambda) = e, \qquad s = 1, \ldots, m_e'. \tag{1.6}$$

The next assumption is a key characteristic for the physical situation we want to describe. We assume that the reservoir dynamics has a single stationary state and that the coupled dynamics is dissipative on the orthogonal complement of this state.

(A2) The reservoir dynamics has a unique stationary state $\Omega_{\rm R} \in \mathcal{H}_{\rm R}$, that is, ${\rm Ker} L_{\rm R} = \mathbb{C}\Omega_{\rm R}$. We denote

$$P_{\rm R} = \mathbb{1}_{\rm S} \otimes |\Omega_{\rm R}\rangle\langle\Omega_{\rm R}| \qquad \text{and} \qquad P_{\rm R}^\perp = \mathbb{1}_{\mathcal{H}} - P_{\rm R}. \tag{1.7}$$

On the orthogonal complement the full, coupled dynamics is dissipative in the sense that there exists a $\lambda_* > 0$ and a dense set $\mathcal{D} \subset \mathcal{H}$ such that $\forall \lambda$ with $0 \leq |\lambda| < \lambda_*$ and $\forall \phi, \psi \in \mathcal{D}$,

$$\max_{0 \leq j \leq 2} \sup_{z \in \mathbb{C}_-} \left| \partial_z^j \langle \phi, R_z^{P_{\rm R}}(\lambda) \psi \rangle \right| \;\leq\; C_1(\phi, \psi) < \infty, \tag{1.8}$$

$$\sup_{z \in \mathbb{C}_-} \left| \partial_\lambda \langle \phi, R_z^{P_{\rm R}}(\lambda) \psi \rangle \right| \;\leq\; C_1(\phi, \psi) < \infty. \tag{1.9}$$

Here $\mathbb{C}_- = \{z \in \mathbb{C} : {\rm Im} z < 0\}$ is the open lower complex half plane and

$$R_z^{P_{\rm R}}(\lambda) = (P_{\rm R}^\perp L_\lambda P_{\rm R}^\perp - z)^{-1} \restriction_{{\rm Ran} P_{\rm R}^\perp}$$

is the reduced resolvent. In (1.8), (1.9), C_1 is well defined (finite) on $\mathcal{D} \times \mathcal{D}$.

Discussion of Assumption (A2).

(i) The estimate (1.9) is of technical nature, but the estimate (1.8) is key as it implies that the dynamics generated by $\bar{L}_\lambda \equiv P_{\rm R}^\perp L_\lambda P_{\rm R}^\perp \restriction_{{\rm Ran} P_{\rm R}^\perp}$ is *dissipative*, meaning that $\lim_{t \to \infty} \langle \phi, e^{{\rm i} t \bar{L}_\lambda} \psi \rangle = 0$. More precisely, it follows from (1.8), with $0 \leq j \leq 2$ replaced by $0 \leq j \leq k$, that

$$\left| \langle \phi, e^{{\rm i} t \bar{L}_\lambda} \psi \rangle \right| \leq \frac{C}{(1 + t^2)^{k/2}}, \tag{1.10}$$

provided $(\bar{L}_\lambda + {\rm i})\phi \in \mathcal{D}$ and $(\bar{L}_\lambda + {\rm i})\psi \in \mathcal{D}$. Specifically, (1.8) implies time decay $\propto t^{-2}$ ($k = 2$) of overlaps (1.10). We prove (1.10) at the end of this section.

(ii) We argue that (1.8) is a natural assumption. Namely, (1.8) for $\lambda = 0$ is implied by

$$\max_{0 \leq j \leq 2} \sup_{z \in \mathbb{C}_-} \left| \partial_z^j \langle \phi, (L_{\rm R} - z)^{-1} P_{\rm R}^\perp \psi \rangle_{\mathcal{H}_{\rm R}} \right| \leq C_1(\phi, \psi) < \infty, \tag{1.11}$$

where we understand here (*c.f.* (1.7)) $P_{\rm R}=|\Omega_{\rm R}\rangle\langle\Omega_{\rm R}|$, $P_{\rm R}^\perp=\mathbb{1}_{\rm R}-P_{\rm R}$. This is readily seen by writing

$$R_z^{P_{\rm R}}(0)=(L_0-z)^{-1}P_{\rm R}^\perp=\sum_{e\in\mathcal{E}_0}P_{{\rm S},e}\otimes(L_{\rm R}-e-z)^{-1}P_{\rm R}^\perp,$$

where $P_{{\rm S},e}$ is the spectral projection of $L_{\rm S}$ associated to e, so that $\sum_{e\in\mathcal{E}_0}P_{{\rm S},e}=\mathbb{1}_{\rm S}$. In turn, (1.11) is a natural assumption on the dynamics of a reservoir, since that dynamics should be dissipative away from the stationary state. In concrete applications, one starts with (1.8) for $\lambda=0$ and then proves its validity for small $\lambda\neq 0$ by perturbation theory [19].

(iii) In some recent works on the dynamics on open quantum systems [15, 18], the assumption (1.8) is replaced by the condition that $z\mapsto\langle\phi,R_z^{P_{\rm R}}(\lambda)\psi\rangle$ have a *meromorphic continuation* from z in the lower complex plane across the real axis into the upper plane. This is a much stronger condition than (1.8). In applications to open quantum systems, this difference means that the reservoir correlation function has to decay exponentially quickly in time for the meromorphic situation, while under the present assumption, the decay only needs to be polynomial, *c.f.* [19].

(iv) Without loss of generality, we may assume that $C_1(\phi,\psi)=C_1(\psi,\phi)$ and that $C_1(\phi,\psi)=C_1(P_{\rm R}^\perp\phi,P_{\rm R}^\perp\psi)$ in (1.8), (1.9).

As is well known (see e.g. Proposition 4.1 of [6]) if A is a self-adjoint operator and for each vector ϕ in some dense set, there exists a constant $C(\phi)$ such that

$$\liminf_{\epsilon\to0_+}\sup_{x\in(a,b)}|\langle\phi,(A-x+\mathrm{i}\epsilon)^{-1}\phi\rangle|\le C(\phi),$$

then the spectrum of A in the interval (a,b) is purely absolutely continuous. Thus the estimate (1.8) with $j=0$ implies that the spectrum of $P_{\rm R}^\perp L_\lambda P_{\rm R}^\perp$ acting on ${\rm Ran}P_{\rm R}^\perp$ is purely absolutely continuous and for $\lambda=0$, this implies that the spectrum of L_0 reduced to ${\rm Ran}P_{\rm R}^\perp$ is purely absolutely continuous. On the finite dimensional part ${\rm Ran}P_{\rm R}\cong\mathcal{H}_{\rm S}$, the operator L_0 is (identified with) $L_{\rm S}$ which has pure point spectrum $\mathcal{E}_0$. Therefore L_0 has absolutely continuous spectrum except for the eigenvalues $\mathcal{E}_0$, the same as those of $L_{\rm S}$ and the eigenprojection P_e of L_0 associated to e is given by

$$P_e=(P_{{\rm S},e}\otimes\mathbb{1}_{\rm R})P_{\rm R}\equiv P_{{\rm S},e}\otimes P_{\rm R},\tag{1.12}$$

where $P_{{\rm S},e}$ is the eigenprojection associated to e as an eigenvalue of $L_{\rm S}$.

Instability of embedded eigenvalues and how to track them. If e is an *isolated* eigenvalue of L_0 then by standard analytic perturbation theory [13] L_λ has eigenvalues $E_e^{(s)}(\lambda)$ close to e, for λ small. Those eigenvalues coincide with the eigenvalues of the operator

$$eP_e+\lambda P_eIP_e-\lambda^2P_eI(L_0-e)^{-1}P_e^\perp IP_e+O(\lambda^3).\tag{1.13}$$

Each term in the expansion (1.13) is self-adjoint. If e is an *embedded* eigenvalue of L_0 then the reduced resolvent $(L_0 - e)^{-1} P_e^\perp$ in (1.13) is not defined as a bounded operator. However, its regularization $(L_0 - e + \mathrm{i}\epsilon)^{-1} P_e^\perp$ certainly is, for any $\epsilon > 0$. We may hope that in a sense, the perturbation expansion (1.13) stays valid also for embedded eigenvalues, upon regularizing the resolvent and taking $\epsilon \to 0_+$. But due to the regularization, the operator $(L_0 - e + \mathrm{i}\epsilon)^{-1} P_e^\perp$ is not self-adjoint any longer! So according to (1.13), the second order corrections to the embedded eigenvalue would become complex numbers. This is, however, not compatible with L_λ being self-adjoint. We may then intuit that for small nonzero λ, the number m_e' of eigenvalues of L_λ close to e might be strictly reduced, $m_e' < m_e$, and that this reduction is accounted for by the existence of complex eigenvalues of the so-called level shift operator

$$\Lambda_e = -P_e I P_e^\perp (L_0 - e + \mathrm{i}0_+)^{-1} I P_e. \tag{1.14}$$

Here, $(L_0 - e + \mathrm{i}0_+)^{-1}$ is the limit of $(L_0 - e + \mathrm{i}\epsilon)^{-1}$ as $\epsilon \to 0_+$, taken in the sense of the operator norm in the expression (1.14). This mechanism has the following precise formulation.

Let Q be an orthogonal projection on $\mathcal{H}$ and set $Q^\perp = \mathbb{1} - Q$. The *Feshbach map* applied to an operator A on $\mathcal{H}$ is defined by

$$\mathfrak{F}(A; Q) = Q\big(A - AQ^\perp (Q^\perp A Q^\perp \restriction_{\mathrm{Ran}Q^\perp})^{-1} A\big) Q, \tag{1.15}$$

where it is assumed that $Q^\perp A Q^\perp \restriction_{\mathrm{Ran}Q^\perp}$ is invertible. The Feshbach map satisfies the following *isospectrality property*: Let $a \in \mathbb{C}$ be in the resolvent set of the operator $Q^\perp A Q^\perp \restriction_{\mathrm{Ran}Q^\perp}$, so that $\mathfrak{F}(A - a; Q)$ is well defined. Then the isospectrality property says that a is an eigenvalue of A if and only if zero is an eigenvalue of $\mathfrak{F}(A - a; Q)$. The redeeming quality of this mapping is that $\mathfrak{F}(A - a; Q)$ acts on $\mathrm{Ran}Q$, a space smaller than $\mathcal{H}$ (reduction in dimension). Consider now, for $z \in \mathbb{C}_-$,

$$\mathfrak{F}(L_\lambda - z; P_e) = P_e\Big(e - z + \lambda I - \lambda^2 I R_z^{P_e}(\lambda) P_e^\perp I\Big) P_e, \tag{1.16}$$

where $R_z^{P_e}(\lambda) \equiv (P_e^\perp L_\lambda P_e^\perp - z)^{-1} \restriction_{\mathrm{Ran}P_e^\perp}$ is the reduced resolvent. The isospectrality is not of any good use to analyze the spectrum of L_λ directly, since $R_z^{P_e}(\lambda)$ is not defined for real z. However, one can show (see Theorem A1 of [14] and also Theorem 3.1 below) that the condition (A2), together with the assumption that

(A3) $$IP_e \text{ is a bounded operator and } \mathrm{Ran} IP_e \subset \mathcal{D}, \tag{1.17}$$

imply that the derivatives of order up to two of

$$z \mapsto P_e I R_z^{P_e}(\lambda) P_e^\perp I P_e \tag{1.18}$$

are bounded uniformly in $\{z \in \mathbb{C}_- : |\mathrm{Re}z - e| \le g/2\}$. Here,

$$g = \min\big\{|e - e'| : e, e' \in \mathcal{E}_0,\ e \neq e'\big\} > 0 \tag{1.19}$$

denotes the minimal gap of the eigenvalues of L_0 (which is the same as that of $L_{\rm S}$). The Feshbach map (1.16) is then well defined (by continuity) for $z\in\mathbb{R}$, $|z-e|\le g/2$. Now the isospectrality property can be extended to real values of z. Namely, one can show (see Theorem 5.1 below and also [7, 2, 14]) that any $E\in\mathbb{R}$ is an eigenvalue of L_λ if and only if zero is an eigenvalue of $\mathfrak{F}(L_\lambda-E;P_e)$, and that

$$\dim\mathrm{Ker}(L_\lambda-E)=\dim\mathrm{Ker}\big(\mathfrak{F}(L_\lambda-E;P_e)\big). \tag{1.20}$$

Due to (1.16), eigenvalues of $\mathfrak{F}(L_\lambda-E;P_e)$ are located in an $O(\lambda)$ neighbourhood of $\mathcal{E}_0$, and hence by isospectrality, so are those of L_λ. The multiplicity of E as an eigenvalue of L_λ is controlled by (1.20). We assume now

(A4)
$$P_eIP_e=0, \tag{1.21}$$

as this condition does not alter the emergence of complex eigenvalues (because P_eIP_e is self-adjoint). We could dispense with the condition (1.21) by a simple modification of our arguments. According to (1.14), (1.16), (1.21),

$$\mathfrak{F}(L_\lambda-E;P_e)=(e-E)P_e+\lambda^2\Lambda_e+O(\lambda^3). \tag{1.22}$$

As it acts on the m_e-dimensional space $\mathrm{Ran}P_e$, the operator Λ_e has m_e generally complex eigenvalues. Since Λ_e it is a dissipative operator, meaning that

$$\mathrm{Im}\Lambda_e=\lim_{\epsilon\to0_+}P_eI\frac{\epsilon P_e^\perp}{(L_0-e)^2+\epsilon^2}IP_e\ge0, \tag{1.23}$$

its eigenvalues have non-negative imaginary parts.[1] The isospectrality of the Feshbach map (1.20) and the expansion (1.22) show that for each eigenvalue $E_e^{(s)}(\lambda)$ of L_λ, there is an eigenvalue $a_e^{(s)}\in\mathbb{R}$ of Λ_e such that

$$E_e^{(s)}(\lambda)=e+\lambda^2a_e^{(s)}+O(\lambda^3). \tag{1.24}$$

However, Λ_e may have real eigenvalues without L_λ having any eigenvalues close to e (for λ small, nonzero). This is so since the $O(\lambda^3)$ term in (1.22) may cause the spectrum of (1.22) to be non-real. In this case, L_λ does not have any eigenvalues close to e, according to (1.20). To simplify the analysis, we do not consider this higher order effect. Instead, we assume that the real eigenvalues of Λ_e are in *bijection* with the eigenvalues $E_e^{(s)}(\lambda)$ of L_λ close to e. One way to ensure this is to impose the condition

(A5) The eigenvalues of Λ_e are simple and Λ_e has exactly m_e' real eigenvalues.

We recall that m_e', defined before (1.6), is the number of (distinct and simple) eigenvalues of L_λ close to e, for small $\lambda\neq0$. Assuming Λ_e to have purely simple spectrum is done for

[1] It is sometimes useful to note that the *real* eigenvalues of Λ_e are automatically semi-simple as they lie on the boundary of the numerical range of Λ_e, see *e.g.* Proposition 3.2 of [7].

convenience of the presentation. This restriction can be removed easily and our approach still works, as long as Λ_e is diagonalizable. In some models, it can happen though that Λ_e is not diagonalizable at so-called exceptional points of parameters; one then expects a qualitatively different behaviour of the dynamics (Jordan blocks of Λ_e cause polynomial corrections to exponential decay in time). We do not further explore this interesting aspect here.

The condition (A5) without the simplicity assumption is also called the *Fermi Golden Rule Condition.* It ensures that stability or instability of eigenvalues of L_λ is detected at the lowest order (λ^2) in the perturbation. In terms of dynamical properties, the condition (A5) means that metastable states have life-times of $O(\lambda^{-2})$, as shown in Theorem 2.1.

How to link stable and unstable eigenvalues to the dynamics. We have seen above that the (partially) stable eigenvalues of L_λ are the real eigenvalues of the level shift operators Λ_e and that the Λ_e may have eigenvalues $a_e^{(s)}$ with strictly positive imaginary part. We introduced the level shift operators Λ_e as the second order (λ^2) contributions in the Feshbach map, (1.22). The isospectrality property of the Feshbach map then linked Λ_e to the spectrum of L_λ. As it turns out, the Feshbach map is also an ingredient in the block decomposition of an operator H acting on $\mathcal{H} = \mathrm{Ran}Q \oplus \mathrm{Ran}Q^\perp$, where Q is an orthogonal projection. More precisely, one can readily verify that for any operator H such that $\mathfrak{F}(H;Q)$ exists, the following identity holds (see Section 5, in particular (5.5)),

$$H = \begin{pmatrix} \mathbb{1} & QHQ^\perp R_z^Q \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathfrak{F}(H;Q) & 0 \\ 0 & Q^\perp H Q^\perp \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ R_z^Q Q^\perp H Q & \mathbb{1} \end{pmatrix}. \tag{1.25}$$

The first component in the 2×2 decomposition of (1.25) is that of $\mathrm{Ran}Q$, the second one that of $\mathrm{Ran}Q^\perp$. Choosing in (1.25) $Q = P_e$ and $H = L_\lambda - z$ for $z \in \mathbb{C}\backslash\mathbb{R}$ gives the decomposition

$$(L_\lambda - z)^{-1} = [\mathfrak{F}(L_\lambda - z; P_e)]^{-1} + \mathfrak{B}(z) + R_z^{P_e}(\lambda), \tag{1.26}$$

where $\mathcal{B}(z)$ is a term of $O(\lambda)$, see (3.3). The first term on the right side is the inverse of the Feshbach map, an operator acting on $\mathrm{Ran}P_e$ and the last term acts on $\mathrm{Ran}P_e^\perp$. The resolvent, (1.26), is linked to the propagator via the Fourier-Laplace transform, for $w > 0$,

$$e^{\mathrm{i}tL_\lambda} = \frac{-1}{2\pi\mathrm{i}} \int_{\mathbb{R}-\mathrm{i}w} e^{\mathrm{i}tz}(L_\lambda - z)^{-1} dz. \tag{1.27}$$

Using (1.26) in (1.27) provides a link between the Feshbach map and the dynamics. We then expand the (inverse) of the first term on the right side of (1.26), $\mathfrak{F}(L_\lambda - z; P_e) = e - z - \lambda^2\Lambda_e + O(|\lambda|^3 + \lambda^2|z-e|)$. For z close to e, this links $[\mathfrak{F}(L_\lambda - z; P_e)]^{-1}$ to $(e - z - \lambda^2\Lambda_e)^{-1}$, which has poles $z = e - \lambda^2 a_e^{(s)}$ at the eigenvalues of $e - \lambda^2\Lambda_e$. Upon "integration around" those poles, in accordance with (1.27), one can extract the dynamical factors $e^{\mathrm{i}t(e-\lambda^2 a_e^{(s)})}$. This procedure works locally, that is, for z close to a fixed e. So we will subdivide the integration contour in (1.27) into regions close to $e \in \mathcal{E}_0$, for each e, and apply the Feshbach map with the appropriate P_e. On the complement, where z is away from all eigenvalues of L_0, a similar analysis is done using the Feshbach map with projection $P_{\rm R}$. This is the outline of the idea, and we refer to Section 3 for the detailed analysis.

Proof of (1.10)**.** Let $\phi, \psi \in {\rm Ran} P_{\rm R}^\perp$ and use the resolvent representation (Cauchy formula or Fourier-Laplace transform),

$$\langle\phi, e^{{\rm i}t\bar L_\lambda}\psi\rangle = \frac{-1}{2\pi{\rm i}}\int_{\mathbb{R}-{\rm i}w} e^{{\rm i}tz}\langle\phi, R_z^{P_{\rm R}}(\lambda)\psi\rangle dz, \tag{1.28}$$

where $w>0$ is arbitrary. We have

$$\begin{aligned} R_z^{P_{\rm R}}(\lambda) &= (z+{\rm i})^{-1}[-\mathbb{1} + R_z^{P_{\rm R}}(\lambda)\,(\bar L_\lambda+{\rm i})] \\ &= -(z+{\rm i})^{-1} - (z+{\rm i})^{-2}(\bar L_\lambda+{\rm i}) + (z+{\rm i})^{-2} R_z^{P_{\rm R}}(\lambda)\,(\bar L_\lambda+{\rm i})^2. \end{aligned} \tag{1.29}$$

Since $\int_{\mathbb{R}} \frac{e^{{\rm i}tx}}{(x+{\rm i}(1-w))^k}dx = 0$ for $t>0$ and $w<1$, the relations (1.28) and (1.29) imply

$$\langle\phi, e^{{\rm i}t\bar L_\lambda}\psi\rangle = \frac{-1}{2\pi{\rm i}}\int_{\mathbb{R}-{\rm i}w} \frac{e^{{\rm i}tz}}{(z+{\rm i})^2}\langle(\bar L_\lambda+{\rm i})\phi, R_z^{P_{\rm R}}(\lambda)(\bar L_\lambda+{\rm i})\psi\rangle dz, \tag{1.30}$$

for $\phi,\psi\in{\rm Dom}(\bar L_\lambda)$. We now use $e^{{\rm i}tz} = ({\rm i}t)^{-1}\partial_z e^{{\rm i}tz}$ and integrate in (1.30) by parts k times,

$$\langle\phi, e^{{\rm i}tL_\lambda}\psi\rangle \ = \ \frac{1}{({\rm i}t)^k}\frac{-1}{2\pi{\rm i}}\int_{\mathbb{R}-{\rm i}w} e^{{\rm i}tz}\partial_z^k\big\{(z+{\rm i})^{-2}\langle(\bar L_\lambda+{\rm i})\phi, R_z^{P_{\rm R}}(\lambda)(\bar L_\lambda+{\rm i})\psi\rangle\big\}dz. \tag{1.31}$$

It now follows from (1.31) and (1.8), with $0\le j\le 2$ replaced by $0\le j\le k$, that

$$\big|\langle\phi, e^{{\rm i}t\bar L_\lambda}\psi\rangle\big| \le \frac{C_1\big((\bar L_\lambda+{\rm i})\phi, (\bar L_\lambda+{\rm i})\psi\big)}{(1+t^2)^{k/2}}, \tag{1.32}$$

provided $(\bar L_\lambda+{\rm i})\phi\in\mathcal{D}$ and $(\bar L_\lambda+{\rm i})\phi\in\mathcal{D}$. In this way, smoothness of the resolvent gives rise to dissipation in the dynamics. This proves (1.10).

2 Main result

The simplicity of the spectrum assumed in (A5) implies that Λ_e is diagonalizable,

$$\Lambda_e = \sum_{s=1}^{m_e} a_e^{(s)} Q_e^{(s)}, \tag{2.1}$$

where $Q_e^{(s)}$ is the rank-one spectral projection associated to the eigenvalue $a_e^{(s)}$. Given $e\in\mathcal{E}_0$, we partition the indices $s=1,\ldots,m_e$ into the oscillating and decaying classes

$$\mathcal{S}_e^{\rm osc} = \big\{s : a_e^{(s)}\in\mathbb{R}\big\} \qquad \text{and} \qquad \mathcal{S}_e^{\rm dec} = \big\{s : {\rm Im} a_e^{(s)} > 0\big\}. \tag{2.2}$$

The following is our main result.

Theorem 2.1 (Resonance expansion of propagator) *There is a constant $c_0 > 0$ such that if $0 < |\lambda| < c_0$, then for all $\phi, \psi \in \mathcal{D}$, $t \geq 0$, we have*

$$\begin{aligned}\langle \phi, e^{\mathrm{i}tL_\lambda}\psi\rangle &= \sum_{e\in\mathcal{E}_0}\Big[\sum_{s\in\mathcal{S}_e^{\mathrm{dec}}} e^{\mathrm{i}t(e+\lambda^2 a_e^{(s)})}\big\langle\phi, Q_e^{(s)}\psi\big\rangle + \sum_{s\in\mathcal{S}_e^{\mathrm{osc}}} e^{\mathrm{i}tE_e^{(s)}(\lambda)}\big\langle\phi, Q_e^{(s)}\psi\big\rangle\Big]\\ &\quad +\langle\phi, P_{\mathrm{R}}^\perp e^{\mathrm{i}tP_{\mathrm{R}}^\perp L_\lambda P_{\mathrm{R}}^\perp}P_{\mathrm{R}}^\perp\psi\rangle + R(\lambda,t),\end{aligned} \tag{2.3}$$

where

$$\big|R(\lambda,t)\big| \leq C|\lambda|^{1/4}\,K(\phi,\psi), \tag{2.4}$$

for constants C and K independent of $t \geq 0$.

If interested in $\langle\phi, e^{-\mathrm{i}tL_\lambda}\psi\rangle$, $t \geq 0$, then it suffices to take the complex conjugate of (2.3). The choice of the sign in the exponent on the left side of (2.3) is motivated by the application of our result to the reduced system dynamics, [19].

Discussion of Theorem 2.1

(i) The projection operators $Q_e^{(s)}$ in (2.3) satisfy $Q_e^{(s)}P_e = P_eQ_e^{(s)} = Q_e^{(s)}$, where $P_e = P_{\mathrm{S},e}\otimes P_{\mathrm{R}}$, see (1.12). The overlaps $\langle\phi, Q_e^{(s)}\psi\rangle$ thus give the dynamics of the states ϕ, ψ projected onto the finite-dimensional space $\mathrm{Ran}P_\Omega$. The dynamics on the infinite-dimensional $\mathrm{Ran}P_{\mathrm{R}}^\perp$ is simply given by $e^{\mathrm{i}tP_{\mathrm{R}}^\perp L_\lambda P_{\mathrm{R}}^\perp}$.

(ii) On the finite-dimensional part, there are terms oscillating in time, $\propto e^{\mathrm{i}tE_e^{(s)}(\lambda)}$, and terms decaying in time, $\propto e^{\mathrm{i}t(e+\lambda^2 a_e^{(s)})}$. The decay is exponential, with rates $\lambda^2\mathrm{Im}a_e^{(s)} > 0$, since $|e^{\mathrm{i}t(e+\lambda^2a_e^{(s)})}| = e^{-\lambda^2 t\mathrm{Im}a_e^{(s)}}$. On the infinite-dimensional part, the dynamics is also decaying, but only at a polynomial rate $1/t^2$, as per (1.10).

(iii) The asymptotic dynamics for times larger than all the life times $(\lambda^2\mathrm{Im}a_e^{(s)})^{-1}$ and large enough so that the dissipative term $\langle\phi, P_{\mathrm{R}}^\perp e^{\mathrm{i}tP_{\mathrm{R}}^\perp L_\lambda P_{\mathrm{R}}^\perp}P_{\mathrm{R}}^\perp\psi\rangle \sim t^{-2}$ has decayed (*c.f.* (1.10)), is given by $\langle\phi, e^{\mathrm{i}tL_\lambda}\psi\rangle \approx \sum_{e,s} e^{\mathrm{i}tE_e^{(s)}}\langle\phi, Q_e^{(s)}\psi\rangle + O(|\lambda|^{1/4})$, where the sum is over all eigenvalues $E_e^{(s)}(\lambda)$ of L_λ. If $E_e^{(s)}(\lambda) = 0$ is the only eigenvalue of L_λ, then the dynamics relaxes to the final state $Q_e^{(s)}$ modulo an error $O(|\lambda|^{1/4})$.

(iv) The quasi-static and dissipative parts mentioned in (1.5) are the sums and the term $\langle\phi, P_{\mathrm{R}}^\perp e^{\mathrm{i}tP_{\mathrm{R}}^\perp L_\lambda P_{\mathrm{R}}^\perp}P_{\mathrm{R}}^\perp\psi\rangle$ of (2.3), respectively. The picture (1.5) is reminiscent of properties of solutions to dispersive partial differential equations, which split into a dispersive wave plus a part converging towards an invariant manifold [20, 23]. In our case, after the decay of the dissipative term, the dynamics first approaches the quasi-static manifold spanned by the $Q_e^{(s)}$ for all e, s. The orbits stay close to the quasi-static manifold for times up to $\sim 1/\lambda^2$, after which the dynamics moves to the final, invariant manifold spanned by the $Q_e^{(s)}$ with $e \in \mathcal{E}_0$ and $s \in \mathcal{S}_e^{\mathrm{osc}}$.

(v) The approach we take to prove Theorem 2.1 is similar to [14]. In that paper, an expansion of overlaps of the form (1.4) was given as a sum of a main part plus a remainder, similar to (2.3). The remainder of [14] converges to zero in the limit of large times, but it is not shown to be small in λ. In the current work, the remainder is small in λ for all times (2.4), but it does not converge to zero for large times. This means, incidentally, that the main term of the current work is simpler the one of [14]. Indeed, the current main term is a second order approximation (λ^2) of that of [14]. The result of [14] is not suitable for proving the validity of the Markovian approximation, but the current result here is.

Uncorrelated initial states. For vectors of the form $\psi = \psi_{\rm S}\otimes\Omega_{\rm R}$, $\phi = \phi_{\rm S}\otimes\Omega_{\rm R}$ we have $K(\phi,\psi)=\|\phi_{\rm S}\|\,\|\psi_{\rm S}\|$ (see (2.15) below). The term $\langle\phi, P_{\rm R}^\perp e^{{\rm i}tP_{\rm R}^\perp L_\lambda P_{\rm R}^\perp}P_{\rm R}^\perp\psi\rangle$ in (2.3) vanishes and the inner product in $\langle\phi, Q_e^{(s)}\psi\rangle$ can be seen as that of the system Hilbert space alone, because the $Q_e^{(s)}$ act on ${\rm Ran}P_{\rm R}=\mathcal{H}_{\rm S}\otimes\mathbb{C}\Omega_{\rm R}$ which is identified with $\mathcal{H}_{\rm S}$. We introduce the operators

$$M(\lambda)=\bigoplus_{e\in\mathcal{E}_0} M_e,\qquad M_e=\sum_{s\in\mathcal{S}_e^{\rm dec}}\big(e+\lambda^2 a_e^{(s)}\big)Q_e^{(s)}+\sum_{s\in\mathcal{S}_e^{\rm osc}} E_e^{(s)}(\lambda)Q_e^{(s)},\tag{2.5}$$

where each M_e acts on ${\rm Ran}P_{{\rm S},e}$, the eigenspace of $L_{\rm S}$ associated to the eigenvalue e. Again, the operator $M(\lambda)$ is viewed as an operator acting on the system Hilbert space $\mathcal{H}_{\rm S}$ alone. By construction, the operators $M(\lambda)$ and $L_{\rm S}$ commute, and $M(0)=L_{\rm S}$. The functional calculus gives

$$e^{{\rm i}tM(\lambda)}=\bigoplus_{e\in\mathcal{E}_0}e^{{\rm i}tM_e}=\bigoplus_{e\in\mathcal{E}_0}\Big[\sum_{s\in\mathcal{S}_e^{\rm dec}}e^{{\rm i}t(e+\lambda^2a_e^{(s)})}Q_e^{(s)}+\sum_{s\in\mathcal{S}_e^{\rm osc}}e^{{\rm i}tE_e^{(s)}(\lambda)}Q_e^{(s)}\Big].\tag{2.6}$$

We then obtain the following result directly from (2.3).

Corollary 2.2 *There is a constant c_0 such that if $0<|\lambda|<c_0$, then for all $\psi_{\rm S},\phi_{\rm S}\in\mathcal{H}_{\rm S}$, $t\geq 0$,*

$$\Big|\langle\phi_{\rm S}\otimes\Omega_{\rm R}, e^{{\rm i}tL_\lambda}\psi_{\rm S}\otimes\Omega_{\rm R}\rangle_{\mathcal{H}_{\rm S}\otimes\mathcal{H}_{\rm R}}-\langle\phi_{\rm S}, e^{{\rm i}tM(\lambda)}\psi_{\rm S}\rangle_{\mathcal{H}_{\rm S}}\Big|\leq C|\lambda|^{1/4}\,\|\phi_{\rm S}\|\,\|\psi_{\rm S}\|.\tag{2.7}$$

This corollary is the starting point for a proof that the Markovian approximation is valid for all times, which we give in [19]. We show that $M(\lambda)$ gives rise to a completely positive trace-preserving semigroup acting on system density matrices, which coincides with the well-known Gorini-Kossakovski-Sudarshan-Lindblad semigroup [1, 4, 18].

Parameter dependence. We take some care in estimating c_0, C and K appearing in Theorem 2.1 and Corollary 2.2 in terms of various parameters, as we explain now. The analysis is based on a perturbation theory (λ small) and the corrections of the spectrum are governed by the level shift operators Λ_e, whose spectral decomposition is given in (2.1). To quantify the perturbation theory, we introduce the quantities

$$\begin{aligned}
a &= \min_{e,s}\left\{\mathrm{Im} a_e^{(s)} : a_e^{(s)} \notin \mathbb{R}\right\} \qquad \text{(smallest nonzero imaginary part)} && (2.8)\\
\alpha &= \max_{e,s} |a_e^{(s)}| = \max_e \mathrm{spr}(\Lambda_e) \qquad \text{(maximal spectral radius)} && (2.9)\\
\delta &= \min_{e,s,s'}\left\{|a_e^{(s)} - a_e^{(s')}| : s\neq s'\right\} \qquad \text{(gap in spectrum of the } \Lambda_e\text{)} && (2.10)\\
\kappa &= \max_{e,s} \|Q_e^{(s)}\| \qquad \text{(size of biggest spectral projection)} && (2.11)
\end{aligned}$$

We also define

$$\varkappa_1 = \max_{m,n} C_1\big(I(\varphi_m\otimes\Omega_{\mathrm{R}}), I(\varphi_n\otimes\Omega_{\mathrm{R}})\big), \tag{2.12}$$

where $C_1(\cdot,\cdot)$ is given in (1.8) and where $\{\varphi_m\}_{m=1}^{\dim\mathcal{H}_{\mathrm{S}}}$ is an orthonormal eigenbasis of L_{S}. The parameter $\varkappa_1$ is well defined since due to (1.17), the vectors $I\varphi_m\otimes\Omega_{\mathrm{R}}$ are in $\mathcal{D}$. We combine all these constants and the spectral gap g of L_0 (see (1.19)) into the following two effective constants,

$$\begin{aligned}
\varkappa_0 &= \max\Big[1, 1/g, \kappa/a, \alpha\kappa, \varkappa_1(1+\varkappa_1)(1+g+1/g),\\
&\qquad \varkappa_1(1+\varkappa_1^4)\kappa\max\left\{1, \tfrac{1+\alpha}{\delta}, 1/a, \tfrac{1+\kappa}{a\delta}, \kappa^2\left(\tfrac{\kappa}{a}(1+\kappa^3/\delta^3)(1+1/a)+1/\delta^2\right)\right\}\Big] && (2.13)\\
\lambda_0 &= \frac{\min\Big[1, a, \delta/\kappa^2, \|IP_{\mathrm{R}}\|, g^{3/2}\Big]}{\max\Big[1, \varkappa_1\kappa(1+\varkappa_1\kappa/\delta), \alpha, \varkappa_1\Big]}. && (2.14)
\end{aligned}$$

Furthermore, for vectors $\phi,\psi\in\mathcal{D}$ such that $\bar{L}_\lambda\phi, \bar{L}_\lambda\phi\in\mathcal{D}$, with $\bar{L}_\lambda = P_{\mathrm{R}}^\perp L_\lambda P_{\mathrm{R}}^\perp$, we define

$$\begin{aligned}
K(\phi,\psi) &= \|\phi\|\,\|\psi\| + \max_{j=1,2} C_j(\phi,\psi)\\
&\quad + \max_{j=1,2}\big(\max_m C_j(I\varphi_m\otimes\Omega_{\mathrm{R}},\phi)\max_m C_j(I\varphi_m\otimes\Omega_{\mathrm{R}},\psi)\big)\\
&\quad + \mathfrak{S}_{\phi\leftrightarrow\psi}\|\phi\|\max_{j,m} C_j(I\varphi_m\otimes\Omega_{\mathrm{R}},\psi)\\
&\quad + \mathfrak{S}_{\phi\leftrightarrow\psi}\Big(\|\phi\| + \max_m C_1(I\varphi_m\otimes\Omega_{\mathrm{R}},\phi)\Big)\\
&\qquad \times\Big(\|P_{\mathrm{R}}I\|\|P_{\mathrm{R}}^\perp\psi\| + \max_m C_1\big(I\varphi_m\otimes\Omega_{\mathrm{R}}, (\bar{L}_\lambda+\mathrm{i})P_{\mathrm{R}}^\perp\psi\big)\Big). && (2.15)
\end{aligned}$$

In (2.15) we use the notation

$$\mathfrak{S}_{\phi\leftrightarrow\psi}E(\phi,\psi) = E(\phi,\psi) + E(\psi,\phi) \tag{2.16}$$

for any function E of ϕ and ψ.

Proposition 2.3 (Parameter dependence of c_0, C, K) *The constants c_0, C of Theorem 2.1 and Corollary 2.2 can be written as $c_0 = c'\lambda_0^{4/3}$ and $C = C'\varkappa_0$, where c', C' do not depend on $\lambda, g, a, \alpha, \kappa, \delta$ and λ_0, $\varkappa_0$ are given in* (2.14) *and* (2.13)*, respectively. The constant K in* (2.4)*, which is a function on $\mathcal{D} \times \mathcal{D}$, is given by* (2.15).

Remark. The parameter $\varkappa_1$, (2.12), is defined in terms of the function C_1, (1.8). The latter describes the dissipative nature of the interacting dynamics and, *a priori*, depends on λ. However, the dissipation is due to the nature of the environment alone and so in applications, one can bound $\varkappa_1$ uniformly in λ, for small enough λ (see [19]). See also the point (ii) in the discussion of condition (A2) above. In this sense, $\varkappa_1$ and thus λ_0, are independent of λ.

Connections with earlier work. It is intuitively plausible that eigenvectors of L_0 associated to an unstable eigenvalue e, describing bound states of a quantum system for $\lambda = 0$, turn into 'almost-bound states' for small $\lambda \neq 0$. This mechanism is known as metastability in the theory of Schrödinger operators, where the Hilbert space would not have the structure (1.1), but rather, L_0 would be the kinetic energy operator of N particles and I would be an (interaction) potential. In that setup, initial states close to the original eigenstate stay bound (e.g. spatially localized) for a long time, but eventually decay for large times, see *e.g.* [22, 10, 11] and references therein.

More recently, a dynamical resonance theory for metastability has been developed for open quantum systems [16, 14, 15, 18]. This is the setup described by (1.1)-(1.3) and L_0 is the generator of dynamics of an uncoupled system-reservoir complex, for instance a few spins (qubits) for the system plus a field of oscillators for the reservoir. Eigenstates of the uncoupled system become typically unstable when the spins start to interact with the bosonic excitations. The coupling leads to irreversible effects, such as decoherence and thermalization. The dynamics of open quantum systems is an important topic in the study of quantum theory and the literature on phenomenological models used in physics, chemistry and biology is huge and constantly growing [3, 21]. Often the analysis of open systems is analyzed by perturbation theory in which the control of the remainder is not mathematically controllable (say the Born, Markov, rotating wave approximations for the Markovian approximation). The approach of [16, 14, 15, 18] is mathematically rigorous. It is based on the algebraic and spectral framework of [12, 2, 9, 17] developed for a proof of 'return to equilibrium'. The realization that this framework can be used to give a detailed analysis of the evolution of the reduced densty matrix, beyond the asymptotic property of convergence to equilibrium, came in [16]. Later in [14, 15, 18] is was shown that one can extract the main term of the dynamics and show that it is a completely positive trace preserving semigroup. Technique-wise, the paper [14] is closest to the present work. However, as explained in discussion point (iv) after Theorem 2.1, the results of [14] cannot be used to show the Markovian approximation. On the other hand, the validity of the Markovian approximation was proven in [15, 18], but only under an analyticity assumption as explained in discussion point (iii) after Assumption (A2) above. In the present work, we make a substantial change to the methods of [14] to develop a somewhat general time-dependent perturbation theory which allows us to show (in [19]) the validity of the Markovian approximation under the weak regularity condition (A2). This improvement was mentioned before in [18]. It is important as it allows for the treatment of reservoirs with polynomially decaying correlations instead of exponentially decaying ones, as was needed in [15, 18].

3 Proof of Theorem 2.1

We are presenting the core strategy in Section 3.1. It involves estimates which we derive in Sections 3.3 and 3.4. We then collect the estimates and implement the outlined strategy in Section 3.5. As we explained after Theorem 2.1, our approach is similar to that of [14]. However, in order to be able to show that the remainder is small in λ for all times, (2.4), we need to substantially modify the analysis of [14]. The main alteration is the introduction of two new energy scales η and ϑ which allow for a more detailed estimation of the resolvent. See also Fig. 2 and Fig. 4.

Notational convention. For $X, Y \geq 0$, we write $X \prec Y$ to mean that there is a constant C independent of the coupling constant λ as well as the parameters $g, a, \alpha, \delta, \kappa$, such that $X \leq CY$. (Recall the definitions (1.19), (2.8)-(2.11) of those parameters.)

3.1 Strategy

Given an orthogonal projection Q on $\mathcal{H}$ we set $Q^\perp = \mathbb{1} - Q$ and we denote the resolvent and reduced resolvent operators by

$$R_z(\lambda) = (L_\lambda - z)^{-1} \qquad \text{and} \qquad R_z^Q(\lambda) = (Q^\perp L_\lambda Q^\perp - z)^{-1} \restriction_{\mathrm{Ran}Q^\perp} . \tag{3.1}$$

The resolvent representation of the propagator is

$$\langle \phi, e^{\mathrm{i}tL}\psi \rangle = \frac{-1}{2\pi \mathrm{i}} \int_{\mathbb{R}-\mathrm{i}w} e^{\mathrm{i}tz} \langle \phi\, R_z(\lambda)\psi \rangle \, dz, \tag{3.2}$$

valid for any $w > 0$ if either of ψ or ϕ belong to $\mathrm{Dom}(L)$ (see [8]). Given an orthogonal projection Q, the resolvent has a decomposition into a sum of three parts:[2] a part acting on Ran$Q$, one acting on Ran$Q^\perp$ and one part mixing these subspaces. This decomposition reads, for $z \in \mathbb{C}\backslash\mathbb{R}$,

$$R_z(\lambda) = \mathfrak{F}(z)^{-1} + \mathfrak{B}(z) + R_z^Q(\lambda). \tag{3.3}$$

Here, $\mathfrak{F}$ is the Feshbach map (1.15),

$$\mathfrak{F}(z) \equiv \mathfrak{F}(L - z; Q) = Q(L - z - LQ^\perp R_z^Q(\lambda) L)Q. \tag{3.4}$$

[2] *Reminiscent of "... est omnis divisa in partes tres ...",* [5]

The operator $\mathfrak{B}(z)$ in (3.3) is given by

$$\mathfrak{B}(z) = -\mathfrak{F}(z)^{-1}QLR_z^Q - R_z^Q LQ\mathfrak{F}(z)^{-1} + R_z^Q LQ\mathfrak{F}(z)^{-1}QLR_z^Q. \tag{3.5}$$

Here and in what follows, it is convenient to simply write R_z^Q for $R_z^Q(\lambda)$. The existence of $\mathfrak{F}(z)^{-1}$ is automatic, this operator equals $QR_z(\lambda)Q$, see (3.3). It is elementary to establish the relation (3.3), see Section 5 for more detail.

Our **strategy** is to partition the integration contour in (3.2) into segments and apply (3.3) with suitable projections Q on each segment. To describe the partition, fix an $\eta > 0$. It will be chosen as a certain positive power of $|\lambda|$ later on, so η is a small parameter (relative to the eigenvalue gap g of L_0, (1.19)). For every eigenvalue e of L_0 define the segment (see also Fig. 2)

$$\mathcal{G}_e = \big\{x - \mathrm{i}w \ : \ |x-e| \le \eta\big\} \quad \text{and set} \quad \mathcal{G}_\infty = \big\{x - \mathrm{i}w \ : \ \mathrm{dist}(x, \mathcal{E}_0) > \eta\big\}, \tag{3.6}$$

where $w > 0$ is the parameter in (3.2) and $\mathcal{E}_0$ is the set of all eigenvalues of L_0.

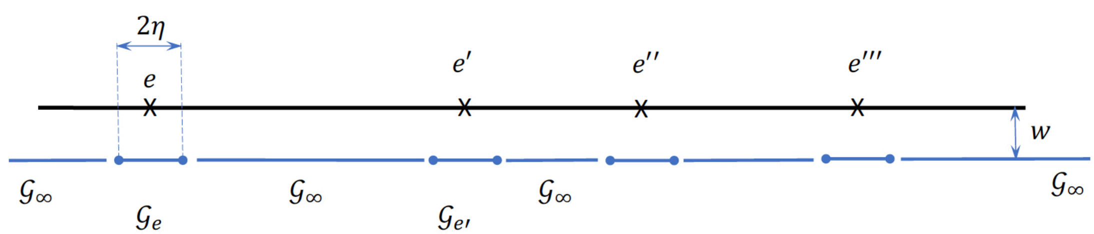

Fig. 2: Subdivision of the integration path $\mathbb{R} - \mathrm{i}w$.

We have the disjoint decomposition $\mathbb{R} - \mathrm{i}w = \mathcal{G}_\infty \bigcup_{e\in\mathcal{E}_0} \mathcal{G}_e$ and (3.2) gives

$$\langle \phi, e^{\mathrm{i}tL_\lambda}\psi\rangle = \sum_{e\in\mathcal{E}_0} J_e(t) + J_\infty(t), \quad \text{with} \quad J_\#(t) = \frac{-1}{2\pi\mathrm{i}} \int_{\mathcal{G}_\#} e^{\mathrm{i}tz} \langle \phi, \, R_z\,\psi\rangle \, dz. \tag{3.7}$$

On $\mathcal{G}_e$ we will apply the Feshbach map with projection $Q = P_e$, (1.12), and on $\mathcal{G}_\infty$ we use the projection $Q = P_{\mathrm{R}}$, (1.7). Accordingly, it is important that we have dissipative bounds on the reduced resolvents, *i.e.*, the following *local limiting absorption principles*, which follow from the global ones given in (1.8), (1.9).

Theorem 3.1 (Local limiting absorption principle) *If $\lambda^2\varkappa_1 \prec g$ then we have for all $\phi, \psi \in \mathcal{D}$ and all $e \in \mathcal{E}_0$*

$$\max_{0\le j\le 2} \sup_{\{z\in\mathbb{C}_-\,:\,|z-e|\le g/2\}} \Big|\partial_z^j \langle \phi, R_z^{P_e}(\lambda)\psi\rangle\Big| \ \le \ C_2(\phi,\psi) < \infty, \tag{3.8}$$

$$\sup_{\{z\in\mathbb{C}_-\,:\,|z-e|\le g/2\}} \Big|\partial_\lambda \langle \phi, R_z^{P_e}(\lambda)\psi\rangle\Big| \ \le \ C_2(\phi,\psi) < \infty, \tag{3.9}$$

where $C_2(\cdot,\cdot)$ is well defined (finite) on $\mathcal{D}\times\mathcal{D}$ and satisfies

$$C_2(\phi,\psi) \prec C_1(\phi,\psi) + \max\{1, 1/g^3\}\Big[\|\phi\|\,\|\psi\| \tag{3.10}$$
$$+\lambda^2 \max_m C_1(\phi, I\varphi_m\otimes\Omega_{\rm R})C_1(\psi, I\varphi_m\otimes\Omega_{\rm R}) + |\lambda|\mathfrak{S}_{\phi\leftrightarrow\psi}\|\psi\|\max_m C_1(\phi, I\varphi_m\otimes\Omega_{\rm R})\Big].$$

Furthermore, we have

$$\max_{0\le k\le 2}\ \sup_{\{z\in\mathbb{C}_-\,:\,|{\rm Re}z-e|\ge g/2\ \forall e\}} \left|\partial_z^k\langle\phi, R_z^{P_{\rm R}}\psi\rangle\right| \le C_2(\phi,\psi). \tag{3.11}$$

We give a proof of Theorem 3.1 in Section 4.1.1. Similar to $\varkappa_1$ defined in (2.12), we set

$$\varkappa_2 = \max_{m,n} C_2\big(I(\varphi_m\otimes\Omega_{\rm R}), I(\varphi_n\otimes\Omega_{\rm R})\big). \tag{3.12}$$

Using (3.10) we get the majorization

$$\varkappa_2 \prec \varkappa_1 + \max\{1, 1/g^3\}\big(|\lambda|\varkappa_1 + \|IP_{\rm R}\|\big)^2. \tag{3.13}$$

Furthermore, imposing the constraints

$$|\lambda|\varkappa_1 \prec \|IP_{\rm R}\| \quad \text{and} \quad \lambda^2\varkappa_1 \prec \min\{1, g^3\} \tag{3.14}$$

gives from (3.13) the simple bound,

$$\varkappa_2 \prec \varkappa_1. \tag{3.15}$$

3.2 Cheat sheet

We keep track of several constants during the estimates to follow. The following cheat sheet is presented for the convenience of the reader.

3.3 Analysis of $J_e(t)$

Fix an eigenvalue $e\in\mathcal{E}_0$ of $L_{\rm S}$ and choose $Q = P_e$ in the Feshbach decompositon (3.3), where P_e is the eigenprojection (1.12). The Feshbach operator (3.4) reads

$$\mathfrak{F}(z) = e - z + \lambda^2 A_e(z,\lambda), \qquad A_e(z,\lambda) = -P_e I R_z^{P_e}(\lambda) I P_e. \tag{3.16}$$

For $z=e$ and $\lambda=0$, $A_e(z,\lambda)$ reduces to the level shift operator Λ_e. We show in Lemma 4.1 below that for $z\in\mathbb{C}_-$ with $|z-e|$ not too large and λ not too large, $A_e(z,\lambda)$ has simple eigenvalues (a property inherited from Λ_e) and we denote the spectral representation by

$$A_e(z,\lambda) = \sum_{s=1}^{m_e} a_e^{(s)}(z,\lambda)Q_e^{(s)}(z,\lambda). \tag{3.17}$$

Symbol	Meaning	Definition
a	smallest nonzero imaginary part of all Λ_e	(2.8)
$a_e^{(s)}$	eigenvalues of Λ_e	(2.1)
α	maximal spectral radius of all Λ_e	(2.9)
β	inverse temperature	
$C_1(\phi,\psi)$, $C_2(\phi,\psi)$	global, local limiting absorption constants	(1.8), (3.8)
δ	minimal gap of all Λ_e	(2.10)
e	eigenvalues of $L_{\rm S}$ (or L_0)	after (1.2)
η	integration domain parameter	(3.6)
g	gap of $L_{\rm S}$	(1.19)
$\mathcal{G}_e$, $\mathcal{G}_\infty$	integration domains	(3.6)
$\varkappa_1$, $\varkappa_2$	global, local limiting absorption constants	(2.12), (3.12)
$\varkappa_0$, $\varkappa_3$, $\varkappa_4$, $\varkappa_5$		(3.142), (4.16), (4.17), (4.19)
κ	biggest norm of spectral projections of all Λ_e	(2.11)
λ	coupling constant	(1.2)
Λ_e	level shift operator	(1.14), (2.1)
$Q_e^{(s)}$	eigenprojections of Λ_e	(2.1)
$\mathcal{S}_e^{\rm osc}$, $\mathcal{S}_e^{\rm dec}$	oscillating and decaying index sets	(2.2)
ϑ	integration domain parameter	beginning Section 3.4
$\mathfrak{S}_{\phi\leftrightarrow\psi}$	symmetrizer	(2.16)
w	vertical integration offset	(3.2)
$\prec$	parameter independent majorization	beginning section 3

Here $a_e^{(s)}(z,\lambda)$ are the eigenvalues and $Q_e^{(s)}(z,\lambda)$ the rank one eigenprojections which extend by continuity to $z\in\mathbb{R}$ (Corollary 4.2).

We define the oriented contour $\mathcal{A}_e$, depicted in Fig. 3, by

$$\mathcal{A}_e = \big\{e-\eta+\mathrm{i}y : y\in[-w,0]\big\}\cup\big\{e+\eta e^{ia} : a\in[\pi,2\pi]\big\}\cup\big\{e+\eta+\mathrm{i}y : y\in[0,-w]\big\}. \quad (3.18)$$

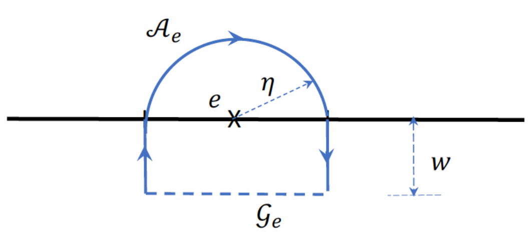

Fig. 3: The contour $\mathcal{A}_e$.

The main result of this section is

Proposition 3.2 *Suppose that*

$$|\lambda| + \eta \prec \frac{a}{\varkappa_4}, \quad \lambda^2 \varkappa_4 \prec 1 \quad \text{and} \quad \lambda^2 \frac{\alpha + |\lambda| \varkappa_4}{\eta} \prec 1. \tag{3.19}$$

Then

$$\begin{aligned} J_e(t) &= \sum_{s \in \mathcal{S}_e^{\mathrm{dec}}} e^{\mathrm{i}t(e+\lambda^2 a_e^{(s)}(e,\lambda))} \langle \phi, Q_e^{(s)}(e,\lambda)\psi\rangle + \sum_{s\in\mathcal{S}_e^{\mathrm{osc}}} e^{\mathrm{i}tE_e^{(s)}(\lambda)} \langle \phi, Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)\psi\rangle \\ &\quad -\frac{1}{2\pi \mathrm{i}} \int_{\mathcal{A}_e} e^{\mathrm{i}tz} \langle \phi, (L_{\mathrm{S}} - z)^{-1} P_{\mathrm{R}} \psi\rangle \, dz + S_e, \end{aligned} \tag{3.20}$$

where

$$\|S_e\| \prec \tag{3.21}$$

$$\begin{aligned} &\Big\{ e^{wt} \frac{w+\eta}{\eta} \Big(\frac{\eta}{g} + \lambda^2 + \varkappa_3(|\lambda| + \eta) \Big) + \lambda^2 \\ &\quad + \eta e^{wt} \Big(\varkappa_3 + \lambda^2 \varkappa_5 + \Big(\frac{\varkappa_4 \kappa}{a} + \varkappa_3 \Big) \Big(1 + \frac{\eta}{a\lambda^2} \Big) \Big) \Big\} \|\phi\| \, \|\psi\| \\ &+ \Big\{ \frac{\kappa\eta}{a|\lambda|} e^{wt} + |\lambda| \big(1 + \eta(1+\varkappa_3)\big) + \frac{(w+\eta)e^{wt}}{\eta} (|\lambda| + \lambda^2) \\ &\quad + e^{wt} \lambda^2 \eta \varkappa_5 (|\lambda| + \lambda^2) + \lambda^2 \Big(1 + (w+\eta) e^{wt} (1 + \varkappa_3) \Big) \Big\} \\ &\quad \times \Big[\max_j C_2(I\varphi_j \otimes \Omega, \phi) \max_j C_2(I\varphi_j \otimes \Omega, \psi) + \mathfrak{S}_{\phi\leftrightarrow\psi} \|\phi\| \max_j C_2(I\varphi_j \otimes \Omega_{\mathrm{R}}, \psi) \Big] \\ &+ \eta e^{wt} C_2(\phi, \psi) \end{aligned} \tag{3.22}$$

with the various constants defined in Section 3.2.

Proof of Proposition 3.2. We show (3.20) by estimating separately the three contributions coming from the decomposition (3.3) of the resolvent R_z,

$$\begin{aligned} \int_{\mathcal{G}_e} e^{\mathrm{i}tz} R_z dz &= \int_{\mathcal{G}_e} e^{\mathrm{i}tz} \mathfrak{F}(z)^{-1} dz + \int_{\mathcal{G}_e} e^{\mathrm{i}tz} \mathfrak{B}(z) dz + \int_{\mathcal{G}_e} e^{\mathrm{i}tz} R_z^Q dz \\ &\equiv D_1 + D_2 + D_3. \end{aligned} \tag{3.23}$$

The corresponding bounds are presented in (3.65), (3.83) and (3.84) below.

Estimate of D_1 in (3.23)**.** Our main estimate for D_1 is given in (3.65) below. All estimates in this section are controlled in the operator norm sense for operators on $\mathrm{Ran} P_e$. We have

$$\mathfrak{F}(z) = e - z + \lambda^2 A_e(z, \lambda), \tag{3.24}$$

where the properties of $A_e(z,\lambda) = -P_e I R_z^{P_e} I P_e$, are discussed in Section 4.2. It follows from (4.15) that

$$\mathfrak{F}(z)^{-1} = \sum_{s=1}^{m_e} \frac{Q_e^{(s)}(z,\lambda)}{e - z + \lambda^2 a_e^{(s)}(z,\lambda)}. \tag{3.25}$$

• Consider (3.25) for an $s\in\mathcal{S}_e^{\rm dec}$. The case $s\in\mathcal{S}_e^{\rm osc}$ is dealt with below. Let $z\in\mathcal{G}_e$. We show that

$$\frac{Q_e^{(s)}(z,\lambda)}{e-z+\lambda^2 a_e^{(s)}(z,\lambda)} = \frac{Q_e^{(s)}(e,\lambda)}{e-z+\lambda^2 a_e^{(s)}(e,\lambda)} + T(z,\lambda), \tag{3.26}$$

where (recall the definitions (2.8) and (2.9) of a,α)

$$\|T(z,\lambda)\| \prec \Big(\frac{\varkappa_4\kappa}{a}+\varkappa_3\Big)\Big(1+\frac{\alpha+|\lambda|\varkappa_4}{a}\Big). \tag{3.27}$$

To show (3.26), we start with the expression for $T(z,\lambda)$,

$$\begin{aligned}
T(z,\lambda) &= T'(z,\lambda)+T''(z,\lambda)\\
T'(z,\lambda) &= -\lambda^2 Q_e^{(s)}(z,\lambda)\frac{a_e^{(s)}(z,\lambda)-a_e^{(s)}(e,\lambda)}{e-z}\,\frac{e-z}{[e-z+\lambda^2 a_e^{(s)}(z,\lambda)][e-z+\lambda^2 a_e^{(s)}(e,\lambda)]}\\
T''(z,\lambda) &= \frac{Q_e^{(s)}(z,\lambda)-Q_e^{(s)}(e,\lambda)}{e-z+\lambda^2 a_e^{(s)}(e,\lambda)}.
\end{aligned}\tag{3.28}$$

Recall that $a_e^{(s)}(e,0)=a_e^{(s)}$ is an eigenvalue of Λ_e. We have

$$\begin{aligned}
&|e-z+\lambda^2 a_e^{(s)}(z,\lambda)|\\
&\quad= \big|e-z+\lambda^2 a_e^{(s)}+\lambda^2(a_e^{(s)}(e,\lambda)-a_e^{(s)})+\lambda^2\big(a_e^{(s)}(z,\lambda)-a_e^{(s)}(e,\lambda)\big)\big|\\
&\quad\ge |e-z+\lambda^2 a_e^{(s)}|-C\lambda^2\varkappa_4\big(|\lambda|+|z-e|\big),
\end{aligned}\tag{3.29}$$

where we have used (4.17) (see also Corollary 4.2). Now $|e-z+\lambda^2 a_e^{(s)}|\ge {\rm Im}(e-z+\lambda^2 a_e^{(s)})\ge\lambda^2{\rm Im}a_e^{(s)}\ge a\lambda^2$ (see (2.8)), so (3.29) yields $|e-z+\lambda^2 a_e^{(s)}(z,\lambda)|\ge a\lambda^2/2$, since $\varkappa_4\big(|\lambda|+|z-e|\big)\prec a$ by (3.19) (note that $z\in\mathcal{G}_e$ so we have $|z-e|\le\sqrt{w^2+\eta^2}\le\sqrt{2}\,\eta$ (we will take $w\to0$ at fixed η, so $w<\eta$ without loss of generality)). We get

$$\frac{1}{|e-z+\lambda^2 a_e^{(s)}(z,\lambda)|}\prec\frac{1}{a\lambda^2}. \tag{3.30}$$

Moreover, we have

$$\begin{aligned}
\Big|\frac{e-z}{e-z+\lambda^2 a_e^{(s)}(e,\lambda)}\Big| &\le 1+\lambda^2\Big|\frac{a_e^{(s)}(e,\lambda)}{e-z+\lambda^2 a_e^{(s)}(e,\lambda)}\Big|\\
&\prec 1+\frac{|a_e^{(s)}(e,\lambda)|}{a}\prec 1+\frac{\alpha+|\lambda|\varkappa_4}{a}
\end{aligned}\tag{3.31}$$

(use (2.9) and (4.17) in the last estimate). Now we estimate $T'(z,\lambda)$ given in (3.28), using the bound $\|Q_e^{(s)}(z,\lambda)\|\prec\kappa$ (see (4.30)) and (3.30), (3.31) and (4.17),

$$\|T'(z,\lambda)\|\prec\frac{\varkappa_4\kappa}{a}\Big(1+\frac{\alpha+|\lambda|\varkappa_4}{a}\Big). \tag{3.32}$$

To deal with $T''(z,\lambda)$ in (3.28) we write

$$T''(z,\lambda) = \frac{Q_e^{(s)}(z,\lambda) - Q_e^{(s)}(e,\lambda)}{e-z} \frac{e-z}{e-z+\lambda^2 a_e^{(s)}(e,\lambda)} \tag{3.33}$$

and use the bounds (4.16) and (3.31) to get

$$\|T''(z,\lambda)\| \prec \varkappa_3 \Big(1 + \frac{\alpha + |\lambda|\varkappa_4}{a}\Big). \tag{3.34}$$

The combination of (3.32) and (3.34) gives (3.27).

• Next we consider a term in (3.25) with $s \in \mathcal{S}_e^{\rm osc}$. We have to modify the above argument, since it used that the imaginary part of $a_e^{(s)}$ was strictly positive, which is not the case any more now. By the isospectrality of the Feshbach map, there is exactly one eigenvalue $E_e^{(s)}(\lambda)$ of L_λ with

$$E_e^{(s)}(\lambda) = e + \lambda^2 a_e^{(s)}\big(E_e^{(s)}(\lambda),\lambda\big). \tag{3.35}$$

Moreover, the projection $Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)$ is *orthogonal* and so it has norm one ([7], Theorem 3.8). We decompose

$$\begin{aligned}\frac{Q_e^{(s)}(z,\lambda)}{e-z+\lambda^2 a_e^{(s)}(z,\lambda)} &= \frac{Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)}{E_e^{(s)}(\lambda)-z} + \frac{Q_e^{(s)}(z,\lambda) - Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)}{e-z+\lambda^2 a_e^{(s)}(z,\lambda)} \\ &\quad + Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)\frac{E_e^{(s)}(\lambda) - e - \lambda^2 a_e^{(s)}(z,\lambda))}{(E_e^{(s)}(\lambda)-z)(e-z+\lambda^2 a_e^{(s)}(z,\lambda))}.\end{aligned} \tag{3.36}$$

Using (3.35), the second term on the right side of (3.36) becomes

$$\begin{aligned}&-\frac{Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda) - Q_e^{(s)}(z,\lambda)}{E_e^{(s)}(\lambda) - z - \lambda^2[a_e^{(s)}(E_e^{(s)}(\lambda),\lambda) - a_e^{(s)}(z,\lambda)]} \\ &\qquad = -\frac{Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda) - Q_e^{(s)}(z,\lambda)}{E_e^{(s)}(\lambda)-z} \frac{1}{1 - \lambda^2 \frac{a_e^{(s)}(E_e^{(s)}(\lambda),\lambda) - a_e^{(s)}(z,\lambda)}{E_e^{(s)}(\lambda)-z}}.\end{aligned} \tag{3.37}$$

We estimate the last fraction in (3.37) by using (4.17) as

$$\Big|\Big[1 - \lambda^2 \frac{a_e^{(s)}(E_e^{(s)}(\lambda),\lambda) - a_e^{(s)}(z,\lambda)}{E_e^{(s)}(\lambda)-z}\Big]^{-1}\Big| \le \frac{1}{1 - C\lambda^2\varkappa_4} \prec 1, \tag{3.38}$$

where the last relations holds since $\lambda^2\varkappa_4 \prec 1$ (see (3.19)). Estimates (3.38) and (4.16) show

$$\Big\|(3.37)\Big\| \prec \varkappa_3. \tag{3.39}$$

Consider now the last term on the right side of (3.36). Using (3.35), the fraction reads

$$\lambda^2 \ \frac{a_e^{(s)}(E_e^{(s)}(\lambda),\lambda) - a_e^{(s)}(z,\lambda)}{(E_e^{(s)}(\lambda)-z)^2} \frac{1}{1-\lambda^2 \ \frac{a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)-a_e^{(s)}(z,\lambda)}{E_e^{(s)}(\lambda)-z}}. \tag{3.40}$$

The second fraction in (3.40) is $\prec 1$ by (3.38). Next, we obtain from point 3. of Lemma 4.1

$$\begin{aligned}
&\Big|\frac{a_e^{(s)}(E_e^{(s)}(\lambda),\lambda) - a_e^{(s)}(z,\lambda)}{E_e^{(s)}(\lambda)-z} - \partial_z a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)\Big| \\
&= \ \Big|\int_z^{E_e^{(s)}(\lambda)} \big(\partial_\zeta a_e^{(s)}(\zeta,\lambda) - \partial_\zeta a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)\big)\frac{d\zeta}{E_e^{(s)}(\lambda)-z}\Big| \\
&= \ \Big|\int_z^{E_e^{(s)}(\lambda)} \frac{d\zeta}{E_e^{(s)}(\lambda)-z}\int_\zeta^{E_e^{(s)}(\lambda)} dw \ \partial_w^2 a_e^{(s)}(w,\lambda)\Big| \prec \varkappa_5 |E_e^{(s)}(\lambda)-z|.
\end{aligned} \tag{3.41}$$

It follows that

$$\Big|\frac{a_e^{(s)}(E_e^{(s)}(\lambda),\lambda) - a_e^{(s)}(z,\lambda)}{(E_e^{(s)}(\lambda)-z)^2} - \frac{\partial_z a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)}{E_e^{(s)}(\lambda)-z}\Big| \prec \varkappa_5. \tag{3.42}$$

Now

$$(3.40) = \lambda^2 \frac{\partial_z a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)}{E_e^{(s)}(\lambda)-z} \ \frac{1}{1-\lambda^2 \partial_z a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)} + T''', \tag{3.43}$$

where

$$\begin{aligned}
T''' \ &= \ \lambda^2\Big(\frac{a_e^{(s)}(z,\lambda) - a_e^{(s)}(z,\lambda)}{(E_e^{(s)}(\lambda)-z)^2} - \frac{\partial_z a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)}{E_e^{(s)}(\lambda)-z}\Big)\frac{1}{1-\lambda^2 \frac{a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)-a_e^{(s)}(z,\lambda)}{E_e^{(s)}(\lambda)-z}} \qquad (3.44)\\
&+\lambda^2 \frac{\partial_z a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)}{E_e^{(s)}(\lambda)-z}\Big(\frac{1}{1-\lambda^2 \frac{a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)-a_e^{(s)}(z,\lambda)}{E_e^{(s)}(\lambda)-z}} - \frac{1}{1-\lambda^2 \partial_z a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)}\Big).
\end{aligned}$$

By (4.17), $|\partial_z a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)| \prec \varkappa_4$ and so the second fraction in (3.43) is bounded $\prec 1$ since $\lambda^2\varkappa_4 \prec 1$ (see (3.19)).

The estimates (3.42) and (3.38) show that the first summand on the right side of (3.44) is $\prec \lambda^2\varkappa_5$. Next,

$$\begin{aligned}
&\frac{1}{E_e^{(s)}(\lambda)-z}\Big(\Big[1-\lambda^2 \frac{a_e^{(s)}(E_e^{(s)}(\lambda),\lambda) - a_e^{(s)}(z,\lambda)}{E_e^{(s)}(\lambda)-z}\Big]^{-1} - \Big[1-\lambda^2 \partial_z a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)\Big]^{-1}\Big) \\
&= \ \lambda^2 \frac{\partial_z a_e^{(s)}(E_e^{(s)}(\lambda),\lambda) - \frac{a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)-a_e^{(s)}(z,\lambda)}{E_e^{(s)}(\lambda)-z}}{E_e^{(s)}(\lambda)-z} \\
&\quad \times\big[1-\lambda^2 \frac{a_e^{(s)}(E_e^{(s)}(\lambda),\lambda) - a_e^{(s)}(z,\lambda)}{E_e^{(s)}(\lambda)-z}\big]^{-1}\big[1-\lambda^2\partial_z a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)\big]^{-1}
\end{aligned} \tag{3.45}$$

which has modulus $\prec \lambda^2 \varkappa_5$ (use (3.38), (3.42), (4.17)). In conclusion, we obtain

$$|T'''| \prec \lambda^2 \varkappa_5. \tag{3.46}$$

Combining (3.36), (3.39), (3.43) and (3.46), and also using that $\|Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)\| = 1$ (*c.f.* after (3.35)), gives the bound for $s \in \mathcal{S}_e^{\rm osc}$,

$$\begin{aligned}&\Big\| \frac{Q_e^{(s)}(z,\lambda)}{e-z+\lambda^2 a_e^{(s)}(z,\lambda)} - \frac{Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)}{E_e^{(s)}(\lambda)-z}\Big[1+\lambda^2 \frac{\partial_z a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)}{1-\lambda^2\partial_z a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)}\Big]\Big\| \\ &\quad \prec \varkappa_3 + \lambda^2\varkappa_5. \end{aligned} \tag{3.47}$$

- Combining (3.25), (3.26) and (3.47) shows that for every $e \in \mathcal{E}_0$,

$$\begin{aligned}\mathfrak{F}(z)^{-1} &= \sum_{s\in\mathcal{S}_e^{\rm dec}} \frac{Q_e^{(s)}(e,\lambda)}{e-z+\lambda^2 a_e^{(s)}(e,\lambda)} \qquad (3.48)\\ &\quad + \sum_{s\in\mathcal{S}_e^{\rm osc}} \frac{Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)}{E_e^{(s)}(\lambda)-z}\Big[1+\lambda^2\ \frac{\partial_z a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)}{1-\lambda^2\partial_z a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)}\Big] + \mathcal{T}(z,\lambda),\end{aligned}$$

where the remainder satisfies

$$\|\mathcal{T}(z,\lambda)\| \prec \varkappa_3 + \lambda^2\varkappa_5 + \Big(\frac{\varkappa_4\kappa}{a}+\varkappa_3\Big)\Big(1+\frac{\alpha+|\lambda|\varkappa_4}{a}\Big). \tag{3.49}$$

We now analyze D_1, see (3.23). From (3.48),

$$\begin{aligned}D_1 \equiv \int_{\mathcal{G}_e} e^{{\rm i}tz}\,\mathfrak{F}(z)^{-1}dz &= Q_e^{(s)}(e,\lambda)\sum_{s\in\mathcal{S}_e^{\rm dec}}\int_{\mathcal{G}_e} \frac{e^{{\rm i}tz}}{e-z+\lambda^2 a_e^{(s)}(e,\lambda)}\,dz \qquad (3.50)\\ + \sum_{s\in\mathcal{S}_e^{\rm osc}} Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)&\Big[1+\lambda^2 \frac{\partial_z a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)}{1-\lambda^2\partial_z a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)}\Big]\int_{\mathcal{G}_e}\frac{e^{{\rm i}tz}}{E_e^{(s)}(\lambda)-z}dz + S_1,\end{aligned}$$

where the remainder S_1 is due to the integration of $\mathcal{T}(z,\lambda)$, estimated by (take into account that $|\mathcal{G}_e| = 2\eta$)

$$\begin{aligned}\|S_1\| &\prec \eta e^{wt}\Big(\varkappa_3+\lambda^2\varkappa_5+\Big(\frac{\varkappa_4\kappa}{a}+\varkappa_3\Big)\Big(1+\frac{\alpha+|\lambda|\varkappa_4}{a}\Big)\Big)\\ &\prec \eta e^{wt}\Big(\varkappa_3+\lambda^2\varkappa_5+\Big(\frac{\varkappa_4\kappa}{a}+\varkappa_3\Big)\Big(1+\frac{\eta}{a\lambda^2}\Big)\Big).\end{aligned}$$

We used (3.19) in the second estimate. Now

$$\int_{\mathcal{G}_e}\frac{e^{{\rm i}tz}}{E_e^{(s)}(\lambda)-z}dz = -2\pi{\rm i}e^{{\rm i}tE_e^{(s)}(\lambda)} + \int_{\mathcal{A}_e}\frac{e^{{\rm i}tz}}{E_e^{(s)}(\lambda)-z}dz, \tag{3.51}$$

where $\mathcal{A}_e$ is given in (3.18). Note that by (3.35) and (4.17), $|E_e^{(s)}(\lambda)-e| = \lambda^2|a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)| \prec \lambda^2|a_e^{(s)}| + \lambda^2\varkappa_4(|E_e^{(s)}(\lambda)-e|+|\lambda|)$, which, since $\lambda^2\varkappa_4 \prec 1$ (see (3.19)), we can solve for (see (2.9)):

$$|E_e^{(s)}(\lambda)-e| \prec \lambda^2(\alpha+|\lambda|\varkappa_4) \prec \eta. \tag{3.52}$$

In the last estimate, we used that

$$\lambda^2\frac{\alpha+|\lambda|\varkappa_4}{\eta} \prec 1 \tag{3.53}$$

(see (3.19)). Thus (3.51) gives

$$\Big|\int_{\mathcal{G}_e}\frac{e^{\mathrm{i}tz}}{E_e^{(s)}(\lambda)-z}dz\Big| \prec 1+\frac{(w+\eta)e^{wt}}{\eta}. \tag{3.54}$$

Using this bound in (3.50) shows that

$$\begin{aligned}\Big\|D_1 \quad &- \sum_{s\in\mathcal{S}_e^{\mathrm{dec}}}\int_{\mathcal{G}_e}\frac{e^{\mathrm{i}tz}Q_e^{(s)}(e,\lambda)}{e-z+\lambda^2a_e^{(s)}(e,\lambda)}\,dz - \sum_{s\in\mathcal{S}_e^{\mathrm{osc}}}\int_{\mathcal{G}_e}\frac{e^{\mathrm{i}tz}Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)}{E_e^{(s)}(\lambda)-z}\,dz\,\Big\| \\ &\prec \quad \|S_1\| + \lambda^2\Big(1+\frac{(w+\eta)e^{wt}}{\eta}\Big).\end{aligned} \tag{3.55}$$

Similarly to (3.51), we also write the contour integrals in the first sum in (3.55) as

$$\int_{\mathcal{G}_e}\frac{e^{\mathrm{i}tz}}{e-z+\lambda^2a_e^{(s)}(e,\lambda)}dz = -2\pi\mathrm{i}e^{\mathrm{i}t(e+\lambda^2a_e^{(s)}(e,\lambda))} + \int_{\mathcal{A}_e}\frac{e^{\mathrm{i}tz}}{e-z+\lambda^2a_e^{(s)}(e,\lambda)}dz, \tag{3.56}$$

where $\mathcal{A}_e$ is the contour (3.18). For $z\in\mathcal{A}_e$ we have $|z-e|\geq\eta$ and so $\lambda^2|a_e^{(s)}(e,\lambda)/(e-z)| < 1$ (because we have $\lambda^2(\alpha+|\lambda|\varkappa_4)/\eta<1$). Thus the geometric series converges,

$$\frac{1}{e-z+\lambda^2a_e^{(s)}(e,\lambda)} = \frac{1}{e-z}\sum_{n\geq 0}\Big(-\frac{\lambda^2a_e^{(s)}(e,\lambda)}{e-z}\Big)^n, \tag{3.57}$$

and gives the bound

$$\Big|\frac{1}{e-z+\lambda^2a_e^{(s)}(e,\lambda)}-\frac{1}{e-z}\Big| \prec \frac{\lambda^2}{\eta}(\alpha+|\lambda|\varkappa_4). \tag{3.58}$$

With $|\mathcal{A}_e| = 2w+\pi\eta$, we obtain

$$\Big|\int_{\mathcal{A}_e}\frac{e^{\mathrm{i}tz}}{e-z+\lambda^2a_e^{(s)}(e,\lambda)}dz - \int_{\mathcal{A}_e}\frac{e^{\mathrm{i}tz}}{e-z}dz\Big| \prec e^{wt}\frac{w+\eta}{\eta}\lambda^2(\alpha+|\lambda|\varkappa_4). \tag{3.59}$$

In the same way we obtain

$$\Big|\int_{\mathcal{A}_e}\frac{e^{\mathrm{i}tz}}{E_e^{(s)}(\lambda)-z}dz-\int_{\mathcal{A}_e}\frac{e^{\mathrm{i}tz}}{e-z}dz\Big|\prec e^{wt}\frac{w+\eta}{\eta}\lambda^2(\alpha+|\lambda|\varkappa_4). \tag{3.60}$$

Then using (3.56) and (3.59) in (3.55) (and similarly for the integrals over the singularity $(E_e^{(s)}(\lambda)-z)^{-1}$) gives

$$\begin{aligned}\Big\|D_1 \quad &+ \sum_{s\in\mathcal{S}_e^{\rm dec}}\Big[2\pi\mathrm{i}\,e^{\mathrm{i}t(e+\lambda^2a_e^{(s)}(e,\lambda))}Q_e^{(s)}(e,\lambda)-\int_{\mathcal{A}_e}\frac{e^{\mathrm{i}tz}Q_e^{(s)}(e,\lambda)}{e-z}dz\Big]\\ &+\sum_{s\in\mathcal{S}_e^{\rm osc}}\Big[2\pi\mathrm{i}\,e^{\mathrm{i}tE_e^{(s)}(\lambda)}Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)-\int_{\mathcal{A}_e}\frac{e^{\mathrm{i}tz}Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)}{e-z}dz\Big]\Big\|\\ \prec \quad &\|S_1\|+\lambda^2\Big(1+\frac{(w+\eta)e^{wt}}{\eta}\big(1+\kappa(\alpha+|\lambda|\varkappa_4)\big)\Big).\end{aligned} \tag{3.61}$$

Next we replace the projections in the remaining integrals in (3.61) by their values at $\lambda=0$: By (4.16) we have $\|Q_e^{(s)}(e,\lambda)-Q_e^{(s)}(e,0)\|\prec|\lambda|\varkappa_3$ and $\|Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)-Q_e^{(s)}(e,0)\|\prec\varkappa_3(|E_e^{(s)}(\lambda)-e|+|\lambda|)\prec|\lambda|\varkappa_3[1+|\lambda|(\alpha+|\lambda|\varkappa_4)]\prec\varkappa_3(|\lambda|+\eta)$, see also (3.52), (3.19). Upon replacing the projections in the integrals of (3.61) by their values at $\lambda=0$, we thus make an error $\prec\frac{(w+\eta)e^{wt}}{\eta}\varkappa_3(|\lambda|+\eta)$. Once this replacement is made, we use the fact that $\sum_{s=1}^{m_e}Q_e^{(s)}(e,0)=P_e$ is the spectral projection of L_0 associated to the eigenvalue e and so we get from (3.61)

$$\begin{aligned}D_1 \quad = \quad &-2\pi\mathrm{i}\sum_{s\in\mathcal{S}_e^{\rm dec}}e^{\mathrm{i}t(e+\lambda^2a_e^{(s)}(e,\lambda))}Q_e^{(s)}(e,\lambda)-2\pi\mathrm{i}\sum_{s\in\mathcal{S}_e^{\rm osc}}e^{\mathrm{i}tE_e^{(s)}(\lambda)}Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)\\ &+\int_{\mathcal{A}_e}\frac{e^{\mathrm{i}tz}P_e}{e-z}dz+S_2,\end{aligned} \tag{3.62}$$

where

$$\|S_2\|\prec\|S_1\|+\lambda^2+\frac{e^{wt}(w+\eta)}{\eta}\Big(\lambda^2\big(1+\kappa(\alpha+|\lambda|\varkappa_4)\big)+\varkappa_3(|\lambda|+\eta)\Big). \tag{3.63}$$

We now further analyze the integral in (3.62). Denoting $P_{{\rm S},e}^\perp=\mathbb{1}_{\rm S}-P_{{\rm S},e}$, where $P_{{\rm S},e}$ is the spectral projection of $L_{\rm S}$ onto the eigenspace associated to e (see (1.12)), we have (as $\eta<g/2$)

$$\Big\|\int_{\mathcal{A}_e}\frac{e^{\mathrm{i}tz}}{L_{\rm S}-z}dz\,P_{{\rm S},e}^\perp\Big\|\prec\frac{e^{wt}(w+\eta)}{g}, \tag{3.64}$$

where g is the spectral gap of $L_{\rm S}$, (1.19). So we can replace $P_e=P_{{\rm S},e}\otimes P_{\rm R}$ in (3.62) by simply $P_{\rm R}$, incurring an error of size of the right hand side of (3.64),

$$\begin{aligned}D_1 \quad = \quad &-2\pi\mathrm{i}\sum_{s\in\mathcal{S}_e^{\rm dec}}e^{\mathrm{i}t(e+\lambda^2a_e^{(s)}(e,\lambda))}Q_e^{(s)}(e,\lambda)-2\pi\mathrm{i}\sum_{s\in\mathcal{S}_e^{\rm osc}}e^{\mathrm{i}tE_e^{(s)}(\lambda)}Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)\\ &+\int_{\mathcal{A}_e}\frac{e^{\mathrm{i}tz}P_{\rm R}}{L_{\rm S}-z}\,dz+S_3,\end{aligned} \tag{3.65}$$

with

$$\|S_3\| \prec \|S_1\| + \lambda^2 + \frac{e^{wt}(w+\eta)}{\eta}\Big(\frac{\eta}{g} + \lambda^2\big(1+\kappa(\alpha+|\lambda|\varkappa_4)\big) + \varkappa_3(|\lambda|+\eta)\Big). \tag{3.66}$$

Estimate of D_2 in (3.23). Our main estimate is given in (3.83) below. We have from (3.5) with $Q = P_e$,

$$\begin{aligned}\langle\phi, D_2\psi\rangle &= \int_{\mathcal{G}_e} e^{\mathrm{i}tz}\langle\phi, \mathcal{B}(z)\psi\rangle dz \qquad (3.67)\\ &= \int_{\mathcal{G}_e} e^{\mathrm{i}tz}\langle\phi,\, \{-\lambda\,\mathfrak{F}(z)^{-1}P_eIR_z^{P_e} - \lambda R_z^{P_e}IP_e\mathfrak{F}(z)^{-1} + \lambda^2 R_z^{P_e}IP_e\mathfrak{F}(z)^{-1}P_eIR_z^{P_e}\}\,\psi\rangle\, dz.\end{aligned}$$

We begin by analyzing the first term on the right side of (3.67). Using (4.15),

$$\int_{\mathcal{G}_e} e^{\mathrm{i}tz}\langle\phi,\, \mathfrak{F}(z)^{-1}\lambda P_eIR_z^{P_e}\psi\rangle dz = \lambda\sum_{s=1}^{m_e}\int_{\mathcal{G}_e} e^{\mathrm{i}tz}\frac{\langle\phi, Q_e^{(s)}(z,\lambda)P_eIR_z^{P_e}\psi\rangle}{e-z+\lambda^2 a_e^{(s)}(z,\lambda)}dz. \tag{3.68}$$

- For $s\in\mathcal{S}_e^{\mathrm{dec}}$ we have from (3.30)

$$\begin{aligned}\Big|\lambda\sum_{s=1}^{m_e}\int_{\mathcal{G}_e} e^{\mathrm{i}tz}\frac{\langle\phi, Q_e^{(s)}(z,\lambda)P_eIR_z^{P_e}\psi\rangle}{e-z+\lambda^2 a_e^{(s)}(z,\lambda)}dz\Big| &\prec \frac{|\mathcal{G}_e|}{a|\lambda|}e^{wt}\max_{z\in\mathcal{G}_e}|\langle\phi, Q_e^{(s)}(z,\lambda)P_eIR_z^{P_e}\psi\rangle|\\ &\prec \frac{\kappa\eta}{a|\lambda|}e^{wt}\|\phi\|\max_j C_2(I\varphi_j\otimes\Omega,\psi). \qquad (3.69)\end{aligned}$$

To get the last bound, we took into account that $P_e = P_eP_\Omega$, that $IP_e\subset\mathcal{D}$, so that from (4.30) and (3.8), we obtain

$$\begin{aligned}|\langle\phi, Q_e^{(s)}(z,\lambda)P_eIR_z^{P_e}\psi\rangle| &\le \sum_j |\langle\phi, Q_e^{(s)}(z,\lambda)\varphi_j\otimes\Omega\rangle|\;|\langle\varphi_j\otimes\Omega, IR_z^{P_e}\psi\rangle|\\ &\prec \kappa\|\phi\|\max_j C_2(I\varphi_j\otimes\Omega,\psi). \qquad (3.70)\end{aligned}$$

- For $s\in\mathcal{S}_e^{\mathrm{osc}}$ we use (3.35) to get

$$\frac{Q_e^{(s)}(z,\lambda)P_eIR_z^{P_e}}{e-z+\lambda^2a_e^{(s)}(z,\lambda)} = \frac{Q_e^{(s)}(z,\lambda)P_eIR_z^{P_e}}{E_e^{(s)}(\lambda)-z}\frac{1}{1-\lambda^2\frac{a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)-a_e^{(s)}(z,\lambda)}{E_e^{(s)}(\lambda)-z}}. \tag{3.71}$$

Now

$$\begin{aligned}\frac{Q_e^{(s)}(z,\lambda)P_eIR_z^{P_e}}{E_e^{(s)}(\lambda)-z} &= \frac{Q_e^{(s)}(z,\lambda)-Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)}{E_e^{(s)}(\lambda)-z}P_eIR_z^{P_e}\\ &\quad -Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)P_eI\frac{R^{P_e}_{E_e^{(s)}(\lambda)}-R_z^{P_e}}{E_e^{(s)}(\lambda)-z} + \frac{Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)P_eIR^{P_e}_{E_e^{(s)}(\lambda)}}{E_e^{(s)}(\lambda)-z}. \qquad (3.72)\end{aligned}$$

The first two terms of the right side of (3.72), when used in (3.71), are estimated by taking into account that the second factor on the right side of (3.71) is $\prec 1$ due to (3.38). However, the last term on the right side of (3.72) has a singularity in z which must be removed by integrating over $\mathcal{G}_e$. To do this, we write

$$\frac{1}{1-\lambda^2\frac{a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)-a_e^{(s)}(z,\lambda)}{E_e^{(s)}(\lambda)-z}} = \frac{1}{1-\lambda^2\partial_z a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)}$$
$$-\lambda^2 \frac{\partial_z a_e^{(s)}(E_e^{(s)}(\lambda),\lambda) - \frac{a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)-a_e^{(s)}(z,\lambda)}{E_e^{(s)}(\lambda)-z}}{[1-\lambda^2\partial_z a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)][1-\lambda^2\frac{a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)-a_e^{(s)}(z,\lambda)}{E_e^{(s)}(\lambda)-z}]}.$$

The denominator of the second summand on the right side is $\succ 1$ and its numerator is $\prec \varkappa_5|E_e^{(s)}(\lambda)-z|$ by (3.41). Thus

$$\Big|\int_{\mathcal{G}_e}\frac{e^{\mathrm{i}tz}}{E_e^{(s)}(\lambda)-z}\frac{1}{1-\lambda^2\frac{a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)-a_e^{(s)}(z,\lambda)}{E_e^{(s)}(\lambda)-z}}dz \tag{3.73}$$
$$-\frac{1}{1-\lambda^2\partial_z a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)}\int_{\mathcal{G}_e}\frac{e^{\mathrm{i}tz}}{E_e^{(s)}(\lambda)-z}dz\Big| \prec e^{wt}\lambda^2|\mathcal{G}_e|\varkappa_5 \prec e^{wt}\lambda^2\eta\varkappa_5.$$

Now $|1-\lambda^2\partial_z a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)|^{-1}\prec 1$ (see (3.19)) and so by (3.73) and (3.54), we have

$$\Big|\int_{\mathcal{G}_e}\frac{e^{\mathrm{i}tz}}{E_e^{(s)}(\lambda)-z}\frac{1}{1-\lambda^2\frac{a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)-a_e^{(s)}(z,\lambda)}{E_e^{(s)}(\lambda)-z}}dz\Big| \prec 1+\frac{(w+\eta)e^{wt}}{\eta}+e^{wt}\lambda^2\eta\varkappa_5. \tag{3.74}$$

We combine (3.74) with (3.71), (3.72) to get

$$\begin{aligned}
&\Big|\lambda\int_{\mathcal{G}_e} e^{\mathrm{i}tz}\frac{\langle\phi, Q_e^{(s)}(z,\lambda)P_eIR_z^{P_e}\psi\rangle}{e-z+\lambda^2 a_e^{(s)}(z,\lambda)}dz\Big| \\
\prec\ & |\lambda||\langle\phi, Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)P_eIR^{P_e}_{E_e^{(s)}(\lambda)}\psi\rangle|\Big(1+\frac{(w+\eta)e^{wt}}{\eta}+e^{wt}\lambda^2\eta\varkappa_5\Big) \\
&+|\lambda||\mathcal{G}_e|\max_{z\in\mathcal{G}_e}\Big|\langle\phi, \frac{Q_e^{(s)}(z,\lambda)-Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)}{E_e^{(s)}(\lambda)-z}P_eIR_z^{P_e}\psi\rangle\Big| \\
&+|\lambda||\mathcal{G}_e|\max_{z\in\mathcal{G}_e}\Big|\langle\phi, Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)P_eI\frac{R^{P_e}_{E_e^{(s)}(\lambda)}-R_z^{P_e}}{E_e^{(s)}(\lambda)-z}\psi\rangle\Big|.
\end{aligned} \tag{3.75}$$

Using (3.54) and the argument of (3.70) (with $\|Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)\|=1$ as this one is an orthogonal projection) we get $|\langle\phi, Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)P_eIR^{P_e}_{E_e^{(s)}(\lambda)}\psi\rangle|\prec\|\phi\|\max_j C_2(I\varphi_j\otimes\Omega,\psi)$. Next, by (4.16), (3.8) and again by the argument of (3.70), the last two terms on the right hand side of (3.75) has an upper bound $\prec |\lambda|\eta\|\phi\|(1+\varkappa_3)\max_j C_2(I\varphi_j\otimes\Omega,\psi)$. We thus obtain from (3.75) that for $s\in\mathcal{S}_e^{\rm osc}$,

$$\begin{aligned}&\Big|\lambda\int_{\mathcal{G}_e} e^{{\rm i}tz}\frac{\langle\phi, Q_e^{(s)}(z,\lambda)P_eIR_z^{P_e}\psi\rangle}{e-z+\lambda^2a_e^{(s)}(z,\lambda)}dz\Big| \\ &\prec\ |\lambda|\|\phi\|\max_j C_2(I\varphi_j\otimes\Omega,\psi)\Big(1+\frac{(w+\eta)e^{wt}}{\eta}+e^{wt}\lambda^2\eta\varkappa_5+\eta(1+\varkappa_3)\Big).\end{aligned}\tag{3.76}$$

- The relations (3.68), (3.69) and (3.76) give

$$\begin{aligned}\Big|\int_{\mathcal{G}_e} e^{{\rm i}tz}\langle\phi,\,\mathfrak{F}(z)^{-1}\lambda P_eIR_z^{P_e}\psi\rangle dz\Big| &\prec \|\phi\|\max_j C_2(I\varphi_j\otimes\Omega,\psi)\\ \times\Big(\frac{\kappa\eta}{a|\lambda|}e^{wt}+|\lambda|\big(1+&\frac{(w+\eta)e^{wt}}{\eta}+e^{wt}\lambda^2\eta\varkappa_5+\eta(1+\varkappa_3)\big)\Big).\end{aligned}\tag{3.77}$$

The integral $\int_{\mathcal{G}_e} e^{{\rm i}tz}\langle\phi,\lambda R_z^{P_e}IP_e\mathfrak{F}(z)^{-1}\psi\rangle dz$ in (3.67) has the same upper bound (3.77), but with ϕ and ψ interchanged.

- We now estimate the term in (3.67) involving $\lambda^2R_z^{P_e}I\mathfrak{F}(z)^{-1}IR_z^{P_e}$. We use again (3.25). Proceeding as above, we obtain from (3.30) that for $s\in\mathcal{S}_e^{\rm dec}$,

$$\lambda^2\Big|\int_{\mathcal{G}_e} e^{{\rm i}tz}\frac{\langle\phi, R_z^{P_e}IP_eQ_e^{(s)}(z,\lambda)IR_z^{P_e}\,\psi\rangle}{e-z+\lambda^2a_e^{(s)}(z,\lambda)}\,dz\Big|\prec\frac{\eta e^{wt}}{a}\max_{z\in\mathcal{G}_e}|\langle\phi, R_z^{P_e}IP_eQ_e^{(s)}(z,\lambda)IR_z^{P_e}\psi\rangle|.\tag{3.78}$$

Now

$$|\langle\phi, R_z^{P_e}IP_eQ_e^{(s)}(z,\lambda)IR_z^{P_e}\psi\rangle|\prec\|Q_e^{(s)}(z,\lambda)\|\|P_eI(R_z^{P_e})^*\phi\|\|P_eIR_z^{P_e}\psi\|\tag{3.79}$$

and

$$\begin{aligned}\|P_eI(R_z^{P_e})^*\phi\| &\prec \max_j|\langle\varphi_j\otimes\Omega, I(R_z^{P_e})^*\phi\rangle|=\max_j|\langle\phi,R_z^{P_e}I\varphi_j\otimes\Omega\rangle|\\ &\prec \max_j C_2(\phi, I\varphi_j\otimes\Omega)\end{aligned}$$

and similarly, $\|P_eIR_z^{P_e}\psi\|\prec\max_j C_2(I\varphi_j\otimes\Omega,\psi)$. Furthermore by (4.30), $\|Q_e^{(s)}(z,\lambda)\|\prec\kappa$. Combining these estimates with (3.78) yields that for all $s\in\mathcal{S}_e^{\rm dec}$,

$$\begin{aligned}\lambda^2\Big|\int_{\mathcal{G}_e} e^{{\rm i}tz}&\frac{\langle\phi, R_z^{P_e}IP_eQ_e^{(s)}(z,\lambda)IR_z^{P_e}\,\psi\rangle}{e-z+\lambda^2a_e^{(s)}(z,\lambda)}\,dz\Big|\\ &\prec\frac{\eta e^{wt}}{a}\kappa\max_j C_2(I\varphi_j\otimes\Omega,\phi)\max_j C_2(I\varphi_j\otimes\Omega,\psi).\end{aligned}\tag{3.80}$$

Next we treat the case $s \in \mathcal{S}_e^{\rm osc}$, for which we show the bound (recall that $e = E_e^{(s)}(\lambda) + \lambda^2 a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)$, (3.35))

$$\begin{aligned}\lambda^2\Big|\int_{\mathcal{G}_e} e^{\mathrm{i}tz}\frac{\langle\phi, R_z^{P_e} I P_e Q_e^{(s)}(z,\lambda) I R_z^{P_e}\psi\rangle}{E_e^{(s)}(\lambda) - z - \lambda^2[a_e^{(s)}(E_e^{(s)}(\lambda),\lambda) - a_e^{(s)}(z,\lambda)]}\, dz\Big| \\ \prec \lambda^2\Big(1 + (w+\eta)e^{wt}(1 + \tfrac{1}{\eta} + \varkappa_3) + e^{wt}\lambda^2\eta\varkappa_5\Big) \\ \times \max_j C_2(I\varphi_j\otimes\Omega,\phi)\max_j C_2(I\varphi_j\otimes\Omega,\psi)\end{aligned} \tag{3.81}$$

as follows. We proceed as in (3.71) and (3.72) and get terms with $[Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda) - Q_e^{(s)}(z,\lambda)][E_e^{(s)}(\lambda) - z]^{-1}$ and $[R^{P_e}_{E_e^{(s)}(\lambda)} - R_z^{P_e}][E_e^{(s)}(\lambda) - z]^{-1}$, all of which are controlled by bounds on the derivatives. We find that they are $\prec \lambda^2(\eta + w)e^{wt}(1+\varkappa_3)\max_j C_2(I\varphi_j\otimes\Omega,\phi)\max_j C_2(I\varphi_j\otimes\Omega,\psi)$. The only remaining term to consider is

$$\begin{aligned}&\lambda^2|\langle\phi, R^{P_e}_{E_e^{(s)}(\lambda)} I P_e Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda) I R^{P_e}_{E_e^{(s)}(\lambda)}\psi\rangle| \\ &\quad\times\Big|\int_{\mathcal{G}_e}\frac{e^{\mathrm{i}tz}}{E_e^{(s)}(\lambda) - z}\frac{1}{1 - \lambda^2\frac{a_e^{(s)}(E_e^{(s)}(\lambda),\lambda) - a_e^{(s)}(z,\lambda)}{E_e^{(s)}(\lambda) - z}}dz\Big| \\ &\quad\prec \lambda^2\Big(1 + \frac{(w+\eta)e^{wt}}{\eta} + e^{wt}\lambda^2\eta\varkappa_5\Big)\max_j C_2(I\varphi_j\otimes\Omega,\phi)\max_j C_2(I\varphi_j\otimes\Omega,\psi),\end{aligned} \tag{3.82}$$

where we have used (3.74). This yields (3.81).

We collect the estimates (3.77), (3.81) and use them in (3.67) to get

$$\begin{aligned}|\langle\phi, D_2\psi\rangle| \prec &\Big\{\frac{\kappa\eta}{a|\lambda|}e^{wt} + |\lambda|\big(1 + \eta(1+\varkappa_3)\big) + \frac{(w+\eta)e^{wt}}{\eta}(|\lambda| + \lambda^2) \\ &+ e^{wt}\lambda^2\eta\varkappa_5(|\lambda| + \lambda^2) + \lambda^2\Big(1 + (w+\eta)e^{wt}(1+\varkappa_3)\Big)\Big\} \\ &\times\Big[\max_j C_2(I\varphi_j\otimes\Omega,\phi)\max_j C_2(I\varphi_j\otimes\Omega,\psi) + \mathfrak{S}_{\phi\leftrightarrow\psi}\|\phi\|\max_j C_2(I\varphi_j\otimes\Omega_{\rm R},\psi)\Big].\end{aligned} \tag{3.83}$$

Estimate of D_3 in (3.23)**.** The bound (3.8) gives immediately

$$\big|\langle\phi, D_3\psi\rangle\big| = \Big|\int_{\mathcal{G}_e} e^{\mathrm{i}tz}\langle\phi, R_z^{P_e}\psi\rangle dz\Big| \prec \eta e^{wt} C_2(\phi,\psi). \tag{3.84}$$

The three bounds (3.65), (3.83) and (3.84), together with (3.23), show (3.20) and (3.22) and this completes the proof of Proposition 3.2. $\square$

3.4 Analysis of $J_\infty(t)$

We introduce a new parameter $\vartheta > 0$ which will be chosen suitably small (a power of $|\lambda|$) below. The goal of this section is to show the following result.

Proposition 3.3 *If $\varkappa_1\lambda^2/\eta \prec 1$, then we have*

$$
\begin{aligned}
&\Big| J_\infty(t) + \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{G}_\infty} e^{\mathrm{i}tz} \langle \phi, (L_\mathrm{S} - z)^{-1}\psi\rangle dz - \langle \phi, P_\mathrm{R}^\perp e^{\mathrm{i}tP_\mathrm{R}^\perp L_\lambda P_\mathrm{R}^\perp} P_\mathrm{R}^\perp \psi\rangle \Big| \\
&\quad \prec \lambda^2 e^{wt} \Big[\frac{\varkappa_1}{\eta} + \frac{g\varkappa_1}{(\eta+\vartheta)^2} + \frac{g\varkappa_1^2\lambda^2}{\eta^3} \\
&\qquad\qquad + \vartheta\Big(\frac{\varkappa_1(\eta+\vartheta+|\lambda|) + \alpha\kappa}{\eta^2} + \frac{\varkappa_1}{g}\big(1/\eta + 1/g\big)\Big)\Big] \|\phi\|\,\|\psi\| \\
&\qquad + \eta e^{wt} C_1(\phi,\psi) \\
&\qquad + e^{wt}\Big(\frac{|\lambda|\vartheta}{\eta} + \frac{|\lambda|}{\eta+\vartheta} + \frac{|\lambda|^3\varkappa_1}{\eta^2}\Big[1 + \frac{\lambda^2\varkappa_1}{\eta}\Big]\Big) \mathfrak{S}_{\phi\leftrightarrow\psi} \|\phi\| \max_j C_1(I\varphi_j \otimes \Omega_\mathrm{R}, \psi) \\
&\qquad + e^{wt}\frac{\lambda^2}{\eta}\Big[1 + \lambda^2\varkappa_1(1+1/\eta)\Big] \max_j C_1(I\varphi_j\otimes\Omega_\mathrm{R},\phi) \max_j C_1(I\varphi_j\otimes\Omega_\mathrm{R},\psi) \\
&\qquad + e^{wt}\frac{|\lambda|}{\eta+\vartheta} \mathfrak{S}_{\phi\leftrightarrow\psi} \|\phi\| \Big(\|P_\mathrm{R} I\| \|P_\mathrm{R}^\perp \psi\| + \max_j C_1\big(I\varphi_j\otimes\Omega_\mathrm{R}, (\bar{L}_\lambda + \mathrm{i})\psi\big)\Big) \\
&\qquad + e^{wt}\frac{\lambda^2}{\eta} \mathfrak{S}_{\phi\leftrightarrow\psi} \max_j C_1(I\varphi_j\otimes\Omega_\mathrm{R},\phi) \Big(\|P_\mathrm{R} I\| \|P_\mathrm{R}^\perp \psi\| + \max_j C_1\big(I\varphi_j\otimes\Omega_\mathrm{R}, (\bar{L}_\lambda + \mathrm{i})\psi\big)\Big).
\end{aligned}
\tag{3.85}
$$

In the last line, we use the notation $\bar{L}_\lambda \equiv P_\mathrm{R}^\perp L_\lambda P_\mathrm{R}^\perp \restriction_{\mathrm{Ran} P_\mathrm{R}^\perp}$.

Proof of Proposition 3.3. Consider two adjacent eigenvalues $e < e'$ of L_S and set

$$
\mathcal{G}_\infty^e = \{x - \mathrm{i}w : e + \eta \le x \le e' - \eta\}. \tag{3.86}
$$

We introduce a new small parameter $\vartheta > 0$ and set $\mathcal{D}_1 = \{x - \mathrm{i}w : e+\eta \le x \le e+\eta+\vartheta\}$ and $\mathcal{D}_2 = \{x - \mathrm{i}w : e+\eta+\vartheta \le x \le e + \frac{e'-e}{2}\}$, as depicted in Fig. 4.

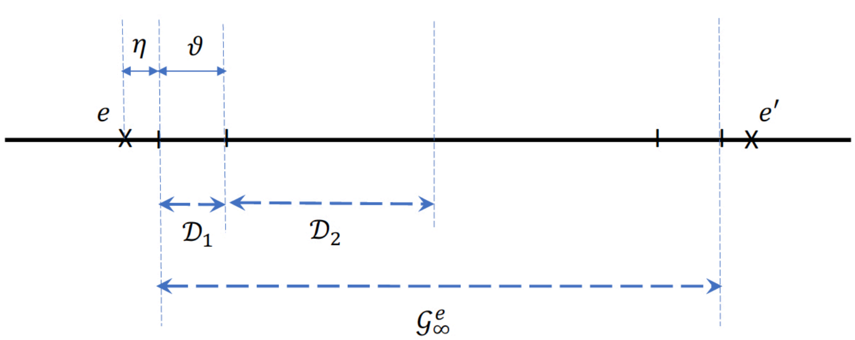

Fig. 4: Different regions for the real part of z, for $z\in\mathcal{G}^e_\infty$.

We use the Feshbach decomposition (3.3) with $Q=P_{\rm R}=|\Omega_{\rm R}\rangle\langle\Omega_{\rm R}|$ and accordingly, we get three contributions to the resolvent as in (3.23),

$$\begin{aligned}\int_{\mathcal{G}^e_\infty} e^{{\rm i}tz}R_z dz &= \int_{\mathcal{G}^e_\infty} e^{{\rm i}tz}\mathfrak{F}(z)^{-1}dz+\int_{\mathcal{G}^e_\infty} e^{{\rm i}tz}\mathfrak{B}(z)dz+\int_{\mathcal{G}^e_\infty} e^{{\rm i}tz}R^{P_{\rm R}}_z dz\\ &\equiv D_1'+D_2'+D_3'. \end{aligned}\tag{3.87}$$

Estimating D_1' of (3.87). The main bound we prove is given in (3.95). We have, in the sense of operators acting on ${\rm Ran}P_{\rm R}$,

$$\mathfrak{F}(z)=P_{\rm R}\big(L_{\rm S}-z-\lambda^2 IR^{P_{\rm R}}_z I\big)P_{\rm R}=\big(\mathbb{1}-\lambda^2P_{\rm R}IR^{P_{\rm R}}_zIP_{\rm R}(L_{\rm S}-z)^{-1}\big)(L_{\rm S}-z).\tag{3.88}$$

For $z\in\mathcal{G}^e_\infty$, $\|(L_{\rm S}-z)^{-1}\|\le 1/\eta$ and by (4.2) we get $\|\lambda^2P_{\rm R}IR^{P_{\rm R}}_zIP_{\rm R}(L_{\rm S}-z)^{-1}\|\prec\frac{\varkappa_1\lambda^2}{\eta}$ and so we get from (3.88), since $\frac{\varkappa_1\lambda^2}{\eta}\prec 1$,

$$\mathfrak{F}(z)^{-1}=(L_{\rm S}-z)^{-1}\big[\mathbb{1}+\lambda^2P_{\rm R}IR^{P_{\rm R}}_zIP_{\rm R}(L_{\rm S}-z)^{-1}+S_4\big],\qquad \|S_4\|\prec\frac{\varkappa_1^2\lambda^4}{\eta^2}.\tag{3.89}$$

We consider now $z\in\mathcal{D}_1\cup\mathcal{D}_2$ (the region closer to e than to e', see Fig. 4). By inserting $\mathbb{1}_{\rm S}=P_{{\rm S},e}+P^\perp_{{\rm S},e}$, using that $P_e=P_{{\rm S},e}\otimes P_{\rm R}$ and that $\|P^\perp_{{\rm S},e}(L_{\rm S}-z)^{-1}\|\le 2/g$ (see (1.19) for the gap g), we estimate the term in (3.89) involving the resolvent $R^{P_{\rm R}}_z$ as follows,

$$\Big\|\lambda^2(L_{\rm S}-z)^{-1}P_{\rm R}IR^{P_{\rm R}}_zIP_{\rm R}(L_{\rm S}-z)^{-1}-\frac{\lambda^2}{(e-z)^2}P_eIR^{P_{\rm R}}_zIP_e\Big\|\prec\frac{\varkappa_1\lambda^2}{g}\big(1/\eta+1/g\big).\tag{3.90}$$

By (4.2) we have $\|P_eIR^{P_{\rm R}}_zIP_e-P_eI(R^{P_{\rm R}}_e|_{\lambda=0})IP_e\|\prec\varkappa_1(|z-e|+|\lambda|)$. Thus, taking into account the definition (1.14) of the level shift operator Λ_e and using that $P^\perp_{\rm R}IP_e=P^\perp_eIP_e$, we have

$$\|P_eIR^{P_{\rm R}}_zIP_e-\Lambda_e\|\prec\varkappa_1\big(|z-e|+|\lambda|\big).\tag{3.91}$$

The reason for introducing the new small parameter ϑ (see Fig. 4), is that for $z\in\mathcal{D}_1$, the right hand side of (3.91) is bounded above by $\varkappa_1(\eta+\vartheta+|\lambda|)$. It then follows from (3.91) and (3.90) that $\forall z\in\mathcal{D}_1$,

$$\|\lambda^2(L_{\rm S}-z)^{-1}P_{\rm R}IR^{P_{\rm R}}_zIP_{\rm R}(L_{\rm S}-z)^{-1}\|\prec\frac{\lambda^2\|\Lambda_e\|}{\eta^2}+\varkappa_1\lambda^2\Big[\frac{\eta+\vartheta+|\lambda|}{\eta^2}+\frac{1}{g}\big(1/\eta+1/g\big)\Big]$$

and hence, as $\|\Lambda_e\|\prec\alpha\kappa$, we get

$$\begin{aligned}&\Big\|\int_{\mathcal{D}_1}e^{{\rm i}tz}\lambda^2(L_{\rm S}-z)^{-1}P_{\rm R}IR^{P_{\rm R}}_zIP_{\rm R}(L_{\rm S}-z)^{-1}dz\Big\| \\ &\quad\prec\ \lambda^2\vartheta e^{wt}\Big[\frac{\varkappa_1(\eta+\vartheta+|\lambda|)+\alpha\kappa}{\eta^2}+\frac{\varkappa_1}{g}\big(1/\eta+1/g\big)\Big].\end{aligned}\tag{3.92}$$

Next we find a bound for the integral on the left side of (3.92) with $\mathcal{D}_1$ replaced by $\mathcal{D}_2$. For $z \in \mathcal{D}_2$, we have $\|(L_{\rm S}-z)^{-1}\| \le (\eta+\vartheta)^{-1}$ and so it follows from (4.2) that $\|(L_{\rm S}-z)^{-1}P_{\rm R}IR_z^{P_{\rm R}}IP_{\rm R}(L_{\rm S}-z)^{-1}\| \prec \varkappa_1(\eta+\vartheta)^{-2}$. Consequently, since $|\mathcal{D}_2|<g/2$, we get

$$\Big\|\int_{\mathcal{D}_2} e^{{\rm i}tz}\lambda^2(L_{\rm S}-z)^{-1}P_{\rm R}IR_z^{P_{\rm R}}IP_{\rm R}(L_{\rm S}-z)^{-1}dz\Big\| \prec \frac{\lambda^2\varkappa_1 g e^{wt}}{(\eta+\vartheta)^2}. \tag{3.93}$$

We now combine (3.92) and (3.93) with (3.89),

$$\begin{aligned}&\Big\|\int_{\mathcal{D}_1\cup\mathcal{D}_1} e^{{\rm i}tz}\,\mathfrak{F}(z)^{-1}dz - \int_{\mathcal{D}_1\cup\mathcal{D}_2}\frac{e^{{\rm i}tz}}{L_{\rm S}-z}dz\Big\| \\ &\prec\ \lambda^2 e^{wt}\Big[\frac{g\varkappa_1}{(\eta+\vartheta)^2}+\frac{g\varkappa_1^2\lambda^2}{\eta^3}+\vartheta\Big(\frac{\varkappa_1(\eta+\vartheta+|\lambda|)+\alpha\kappa}{\eta^2}+\frac{\varkappa_1}{g}\big(1/\eta+1/g\big)\Big)\Big].\end{aligned} \tag{3.94}$$

For $z\in\mathcal{G}_\infty^e$ lying closer to e' than e (*i.e.*, $z\in\mathcal{G}_\infty^e\backslash(\mathcal{D}_1\cup\mathcal{D}_2)$) the analysis is the same, as only the distance from x to the nearest eigenvalue of $L_{\rm S}$ plays a role in the estimates. We conclude that the bound (3.94) holds with $\mathcal{D}_1\cup\mathcal{D}_2$ replaced by $\mathcal{G}_\infty^e$ in the integrals on the left side:

$$\begin{aligned}&\Big\|D_1' - \int_{\mathcal{G}_\infty^e}\frac{e^{{\rm i}tz}}{L_{\rm S}-z}dz\Big\| \\ &\prec\ \lambda^2 e^{wt}\Big[\frac{g\varkappa_1}{(\eta+\vartheta)^2}+\frac{g\varkappa_1^2\lambda^2}{\eta^3}+\vartheta\Big(\frac{\varkappa_1(\eta+\vartheta+|\lambda|)+\alpha\kappa}{\eta^2}+\frac{\varkappa_1}{g}\big(1/\eta+1/g\big)\Big)\Big].\end{aligned} \tag{3.95}$$

Estimating D_2' of (3.87)**.** Our main etimate is given in (3.102). According to (3.5), the term $\mathcal{B}(z)$ gives three contributions. The ones involving only one resolvent $R_z^{P_{\rm R}}$ are all estimated in the same way as follows. We have for all $z\in\mathbb{C}_-$ (use (1.8)),

$$\big|\langle\phi,\mathfrak{F}(z)^{-1}P_{\rm R}IR_z^{P_{\rm R}}\psi\rangle\big| \prec \max_j\big|\langle\phi,\mathfrak{F}(z)^{-1}\varphi_j\otimes\Omega_{\rm R}\rangle\big|\,C_1(I\varphi_j\otimes\Omega_{\rm R},\psi). \tag{3.96}$$

According to (3.89), the main term of $\mathfrak{F}(z)^{-1}$ is $(L_{\rm S}-z)^{-1}$ the norm of which is bounded above by $1/\eta$ for $z\in\mathcal{D}_1$ and by $1/(\eta+\vartheta)$ for $z\in\mathcal{D}_2$. Then,

$$\Big|\int_{\mathcal{D}_1\cup\mathcal{D}_2} e^{{\rm i}tz}\langle\phi,(L_{\rm S}-z)^{-1}\lambda IR_z^{P_{\rm R}}\psi\rangle dz\Big| \prec e^{wt}\Big(\frac{|\lambda|\vartheta}{\eta}+\frac{|\lambda|}{\eta+\vartheta}\Big)\|\phi\|\max_j C_1(I\varphi_j\otimes\Omega_{\rm R},\psi). \tag{3.97}$$

The terms in (3.89), which are of order two and higher in λ, are estimated as

$$\Big\|(L_{\rm S}-z)^{-1}\big[\lambda^2P_{\rm R}IR_z^{P_{\rm R}}IP_{\rm R}(L_{\rm S}-z)^{-1}+S_4\big]\Big\| \prec \frac{\lambda^2\varkappa_1}{\eta^2}\Big[1+\frac{\lambda^2\varkappa_1}{\eta}\Big] \tag{3.98}$$

for $z\in\mathcal{G}_\infty^e$. It follows from (3.89), (3.97) and (3.98) that

$$\begin{aligned}\Big|\int_{\mathcal{D}_1\cup\mathcal{D}_2} e^{{\rm i}tz}\langle\phi,\mathfrak{F}(z)^{-1}\lambda IR_z^{P_{\rm R}}\psi\rangle dz\Big| &\prec e^{wt}\Big(\frac{|\lambda|\vartheta}{\eta}+\frac{|\lambda|}{\eta+\vartheta}+\frac{|\lambda|^3\varkappa_1}{\eta^2}\Big[1+\frac{\lambda^2\varkappa_1}{\eta}\Big]\Big)\\ &\quad\times\|\phi\|\max_j C_1(I\varphi_j\otimes\Omega_{\rm R},\psi).\end{aligned} \tag{3.99}$$

The same upper bound is achieved for the term involving $R_z^{P_{\rm R}}\lambda IP_{\rm R}\mathfrak{F}(z)^{-1}$ in (3.5). To deal with the term in (3.5) involving the resolvent twice, we use the bound

$$\lambda^2\big|\langle\phi, R_z^{P_{\rm R}}I\mathfrak{F}(z)^{-1}IP_{\rm R}R_z^{P_{\rm R}}\psi\rangle\big| \prec \lambda^2\|\mathfrak{F}(z)^{-1}\|\max_j C_1(I\varphi_j\otimes\Omega_{\rm R},\phi)C_1(I\varphi_j\otimes\Omega_{\rm R},\psi). \tag{3.100}$$

From (3.89), $\|\mathfrak{F}(z)^{-1}\|\prec\frac{1}{\eta}[1+\lambda^2\varkappa_1/\eta]$ for $z\in\mathcal{G}_\infty^e$, which gives, combined with (3.100) and (3.99) the result

$$\big|\langle\phi, D_2'\psi\rangle\big| = \Big|\int_{\mathcal{G}_\infty^e} e^{{\rm i}tz}\langle\phi,\mathfrak{B}(z)\psi\rangle dz\Big| \tag{3.101}$$

$$\prec\ e^{wt}\Big(\frac{|\lambda|\vartheta}{\eta}+\frac{|\lambda|}{\eta+\vartheta}+\frac{|\lambda|^3\varkappa_1}{\eta^2}\Big[1+\frac{\lambda^2\varkappa_1}{\eta}\Big]\Big)\mathfrak{S}_{\phi\leftrightarrow\psi}\|\phi\|\max_j C_1(I\varphi_j\otimes\Omega_{\rm R},\psi)$$
$$+\,e^{wt}\frac{\lambda^2}{\eta}\Big[1+\frac{\lambda^2\varkappa_1}{\eta}\Big]\max_j C_1(I\varphi_j\otimes\Omega_{\rm R},\phi)C_1(I\varphi_j\otimes\Omega_{\rm R},\psi). \tag{3.102}$$

Estimating D_3' of (3.87). Recalling the definition (3.86) of $\mathcal{G}_\infty^e$ and using (1.8), we extend the integration by an amount of 2η to $e\le x\le e'$,

$$\langle\varphi, D_3'\psi\rangle = \int_{\mathcal{G}_\infty^e} e^{{\rm i}tz}\langle\phi, R_z^{P_{\rm R}}\psi\rangle dz = \int_e^{e'} e^{{\rm i}t(x-{\rm i}w)}\langle\phi, R_{x-{\rm i}w}^{P_{\rm R}}\psi\rangle dx + S_5, \tag{3.103}$$

with

$$\|S_5\|\prec\eta e^{wt}C_1(\phi,\psi). \tag{3.104}$$

Estimates on the infinite parts of $\mathcal{G}_\infty$. So far in this section, we have dealt with all z between any two eigenvalues of $L_{\rm S}$, see Figs. f1 and f3. Let e_+ and e_- be the largest and smallest eigenvalues of $L_{\rm S}$, respectively and set

$$\mathcal{G}_\infty^+ = \{x-{\rm i}w : x\ge e_+ +\eta\}\qquad\text{and}\qquad \mathcal{G}_\infty^- = \{x-{\rm i}w : x\le e_- -\eta\}. \tag{3.105}$$

We choose $Q=P_{\rm R}$ for the projection in the Feshbach decomposition (3.3). We use (3.88) to expand

$$\mathfrak{F}(z)^{-1} = (L_{\rm S}-z)^{-1}\sum_{n\ge0}\lambda^{2n}\big[P_{\rm R}IR_z^{P_{\rm R}}IP_{\rm R}(L_{\rm S}-z)^{-1}\big]^n, \tag{3.106}$$

which converges for $\lambda^2\varkappa_1\|(L_{\rm S}-z)^{-1}\|\prec 1$. For $z\in\mathcal{G}_\infty^+$, we have $\|(L_{\rm S}-z)^{-1}\|\le 1/\eta$ and so, since $\lambda^2\varkappa_1/\eta\prec1$, the series (3.106) converges uniformly in $z\in\mathcal{G}_\infty^+$ an uniformly in w. Splitting off the first term in the series (3.106) gives

$$\int_{\mathcal{G}_\infty^+} e^{{\rm i}tz}\mathfrak{F}(z)^{-1}dz = \int_{\mathcal{G}_\infty^+}\frac{e^{{\rm i}tz}}{L_{\rm S}-z}dz + \mathcal{I}, \tag{3.107}$$

where ($z=x-\mathrm{i}w$)

$$
\begin{aligned}
\|\mathcal{I}\| &\le e^{wt}\int_{e_++\eta}^{\infty}\big\|(L_{\rm S}-z)^{-1}\sum_{n\ge1}[\lambda^2P_{\rm R}IR_z^{P_{\rm R}}IP_{\rm R}(L_{\rm S}-z)^{-1}]^n\big\|dx\\
&\le e^{wt}\int_{e_++\eta}^{\infty}\frac{1}{x-e_+}\sum_{n\ge1}\frac{(C\varkappa_1\lambda^2)^n}{(x-e_+)^n}dx\\
&= e^{wt}\sum_{n\ge1}(C\varkappa_1\lambda^2)^n\int_\eta^\infty\frac{dx}{x^{n+1}}=e^{wt}\sum_{n\ge1}\frac{1}{n}\Big(\frac{C\varkappa_1\lambda^2}{\eta}\Big)^n\\
&\le e^{wt}\frac{C\varkappa_1\lambda^2}{\eta}\sum_{n\ge0}\Big(\frac{C\varkappa_1\lambda^2}{\eta}\Big)^n\prec e^{wt}\frac{\varkappa_1\lambda^2}{\eta},
\end{aligned}\tag{3.108}
$$

since $\frac{\varkappa_1\lambda^2}{\eta}\prec 1$. The analysis and estimates for $z\in\mathcal{G}_\infty^-$ are the same. We conclude that

$$
\Big\|\int_{\mathcal{G}_\infty^+\cup\mathcal{G}_\infty^-}e^{\mathrm{i}tz}\mathfrak{F}(z)^{-1}dz-\int_{\mathcal{G}_\infty^+\cup\mathcal{G}_\infty^-}\frac{e^{\mathrm{i}tz}}{L_{\rm S}-z}dz\Big\|\prec e^{wt}\frac{\varkappa_1\lambda^2}{\eta}.\tag{3.109}
$$

Next we need to estimate $\int_{\mathcal{G}_\infty^+\cup\mathcal{G}_\infty^-}e^{\mathrm{i}tz}\langle\phi,\mathfrak{B}(z)\psi\rangle dz$, where $\mathfrak{B}(z)$ is the sum of three terms according to (3.5). In each term, we use (3.106) to split off the main part. Accordingly, we have

$$
\begin{aligned}
\mathfrak{B}(z)=&-\lambda(L_{\rm S}-z)^{-1}P_{\rm R}IR_z^{P_{\rm R}}-\lambda R_z^{P_{\rm R}}IP_{\rm R}(L_{\rm S}-z)^{-1}\\
&+\lambda^2R_z^{P_{\rm R}}IP_{\rm R}(L_{\rm S}-z)^{-1}IR_z^{P_{\rm R}}+\mathfrak{B}_2(z),
\end{aligned}\tag{3.110}
$$

which defines the quantity $\mathfrak{B}_2(z)$. Proceeding as in the derivation of (3.108), we get

$$
\begin{aligned}
&\Big|\int_{\mathcal{G}_\infty^+\cup\mathcal{G}_\infty^-}e^{\mathrm{i}tz}\langle\phi,\mathfrak{B}_2(z)\psi\rangle dz\Big|\\
&\quad\prec\ e^{wt}\frac{\varkappa_1|\lambda|^3}{\eta}\mathfrak{S}_{\phi\leftrightarrow\psi}\|\phi\|\max_j C_1(I\varphi_j\otimes\Omega_{\rm R},\psi)\\
&\qquad+e^{wt}\frac{\varkappa_1\lambda^4}{\eta}\max_j C_1(I\varphi_j\otimes\Omega,\psi)\max_j C_1(I\varphi_j\otimes\Omega,\phi).
\end{aligned}\tag{3.111}
$$

Now we analyze the integral associated to the first term in (3.110), which is given by $-\lambda\int_{\mathcal{G}_\infty^+}e^{\mathrm{i}tz}\langle\phi,(L_{\rm S}-z)^{-1}P_{\rm R}IR_z^{P_{\rm R}}\psi\rangle dz$. We split the integration domain into $e_++\eta\le x\le e_++\eta+\vartheta$ and $x\ge e_++\eta+\vartheta$. On the compact domain, the integral has the bound $\prec e^{wt}\frac{|\lambda|\vartheta}{\eta}\|\phi\|\max_j C_1(I\varphi_j\otimes\Omega_{\rm R},\psi)$, where the factor $1/\eta$ is due to the resolvent $(L_{\rm S}-z)^{-1}$ and ϑ is the length of the interval of integration. On the infinite integration domain, we need to control the integrability for large x. Denoting $\bar L_\lambda\equiv P_{\rm R}^\perp L_\lambda P_{\rm R}^\perp\restriction_{{\rm Ran}P_{\rm R}^\perp}$, we have

$$
R_z^{P_{\rm R}}=(z+\mathrm{i})^{-1}\big[-\mathbb{1}+R_z^{P_{\rm R}}(\bar L_\lambda+\mathrm{i})\big]\tag{3.112}
$$

so that $(z = x - \mathrm{i}w)$

$$\Big|\int_{e_+ + \eta+\vartheta}^\infty e^{\mathrm{i}tz}\langle\phi,(L_{\mathrm{S}}-z)^{-1}P_{\mathrm{R}} I R_z^{P_{\mathrm{R}}}\psi\rangle dz\Big|$$
$$\prec e^{wt}\|\phi\|\Big(\|P_{\mathrm{R}} I\|\|P_{\mathrm{R}}^\perp\psi\| + \max_j C_1\big(I\varphi_j\otimes\Omega_{\mathrm{R}}, (\bar L_\lambda+\mathrm{i})\psi\big)\Big)$$
$$\times\int_{e_++\eta+\vartheta}^\infty \frac{dx}{|e_+-z||z+\mathrm{i}|}. \tag{3.113}$$

The last integral is $\prec(\eta+\vartheta)^{-1}$, thus

$$\Big|\lambda\int_{\mathcal{G}_\infty^+} e^{\mathrm{i}tz}\langle\phi,(L_{\mathrm{S}}-z)^{-1}P_{\mathrm{R}} I R_z^{P_{\mathrm{R}}}\psi\rangle dz\Big|$$
$$\prec e^{wt}\frac{|\lambda|\vartheta}{\eta}\|\phi\|\max_j C_1(I\varphi_j\otimes\Omega_{\mathrm{R}},\psi)$$
$$+ e^{wt}\frac{|\lambda|}{\eta+\vartheta}\|\phi\|\Big(\|P_{\mathrm{R}} I\|\|P_{\mathrm{R}}^\perp\psi\| + \max_j C_1\big(I\varphi_j\otimes\Omega_{\mathrm{R}}, (\bar L_\lambda+\mathrm{i})\psi\big)\Big). \tag{3.114}$$

Of course, we get the same upper bound for $-\lambda\int_{\mathcal{G}_\infty^+} e^{\mathrm{i}tz}\langle\phi, R_z^{P_{\mathrm{R}}} I P_{\mathrm{R}}(L_{\mathrm{S}}-z)^{-1}\psi\rangle dz$ (see (3.110)), with ϕ and ψ exchanged on the right side of (3.114). Finally, we estimate

$$\Big|\int_{\mathcal{G}_\infty^+} e^{\mathrm{i}tz}\langle\phi, R_z^{P_{\mathrm{R}}} I P_{\mathrm{R}}(L_{\mathrm{S}}-z)^{-1} I R_z^{P_{\mathrm{R}}}\psi\rangle dz\Big|$$
$$\prec e^{wt}\max_j C_1(I\varphi_j\otimes\Omega_{\mathrm{R}},\phi)\Big(\|P_{\mathrm{R}} I\|\|P_{\mathrm{R}}^\perp\psi\| + \max_j C_1\big(I\varphi_j\otimes\Omega_{\mathrm{R}}, (\bar L_\lambda+\mathrm{i})\psi\big)\Big)$$
$$\times\int_{e_++\eta}^\infty\frac{dx}{|e_+-z||z+\mathrm{i}|}. \tag{3.115}$$

The last integral is $\prec 1/\eta$. Collecting the bounds (3.111), (3.114) and (3.115) and using them in (3.110), we obtain

$$\Big|\int_{\mathcal{G}_\infty^+\cup\mathcal{G}_\infty^-}\langle\phi,\mathfrak{B}(z)\psi\rangle dz\Big|$$
$$\prec e^{wt}\frac{\varkappa_1|\lambda|^3+|\lambda|\vartheta}{\eta}\mathfrak{S}_{\phi\leftrightarrow\psi}\|\phi\|\max_j C_1(I\varphi_j\otimes\Omega_{\mathrm{R}},\psi)$$
$$+ e^{wt}\frac{\varkappa_1\lambda^4}{\eta}\max_j C_1(I\varphi_j\otimes\Omega,\psi)\max_j C_1(I\varphi_j\otimes\Omega,\phi)$$
$$+ e^{wt}\frac{|\lambda|}{\eta+\vartheta}\mathfrak{S}_{\phi\leftrightarrow\psi}\|\phi\|\Big(\|P_{\mathrm{R}} I\|\|P_{\mathrm{R}}^\perp\psi\| + \max_j C_1\big(I\varphi_j\otimes\Omega_{\mathrm{R}}, (\bar L_\lambda+\mathrm{i})\psi\big)\Big)$$
$$+ e^{wt}\frac{\lambda^2}{\eta}\mathfrak{S}_{\phi\leftrightarrow\psi}\max_j C_1(I\varphi_j\otimes\Omega_{\mathrm{R}},\phi)\Big(\|P_{\mathrm{R}} I\|\|P_{\mathrm{R}}^\perp\psi\| + \max_j C_1\big(I\varphi_j\otimes\Omega_{\mathrm{R}}, (\bar L_\lambda+\mathrm{i})\psi\big)\Big). \tag{3.116}$$

Next, extending the integration domain by 2η similar to (3.103), we estimate

$$\int_{\mathcal{G}_\infty^+\cup\mathcal{G}_\infty^-} e^{\mathrm{i}tz}\langle\phi, R_z^{P_{\mathrm{R}}}\psi\rangle dz = \int_{(-\infty,e_-]\cup[e_+,\infty)} e^{\mathrm{i}t(x-\mathrm{i}w)}\langle\phi, R_{x-\mathrm{i}w}^{P_{\mathrm{R}}}\psi\rangle dx + S_6, \tag{3.117}$$

where

$$\|S_6\| \prec \eta e^{wt} C_1(\phi,\psi). \tag{3.118}$$

Finally we note that $\mathcal{G}_\infty = \cup_{e<e_+}\mathcal{G}_\infty^e \cup \mathcal{G}_\infty^+ \cup \mathcal{G}_\infty^-$ and that

$$\frac{-1}{2\pi\mathrm{i}}\int_{\mathbb{R}-\mathrm{i}w} e^{\mathrm{i}tz}\langle\phi, R_z^{P_{\mathrm{R}}}\psi\rangle dz = \langle\phi, P_{\mathrm{R}}^\perp e^{\mathrm{i}tP_{\mathrm{R}}^\perp L_\lambda P_{\mathrm{R}}^\perp} P_{\mathrm{R}}^\perp\psi\rangle. \tag{3.119}$$

Combining the estimates (3.95), (3.102), (3.103), (3.109), (3.116) and (3.117) yields (3.85). This completes the proof of Proposition 3.3. □

3.5 Combining the estimates: end of the proof of Theorem 2.1.

We combine the estimates (3.20) and (3.85) (and (3.119)) into

$$\begin{aligned}\Big|\sum_{e\in\mathcal{E}_0} J_e(t) + J_\infty(t) &- \sum_{e\in\mathcal{E}_0}\sum_{s\in\mathcal{S}_e^{\mathrm{dec}}} e^{\mathrm{i}t(e+\lambda^2 a_e^{(s)}(e,\lambda))}\big\langle\phi, Q_e^{(s)}(e,\lambda)\psi\big\rangle\\
&-\sum_{e\in\mathcal{E}_0}\sum_{s\in\mathcal{S}_e^{\mathrm{osc}}} e^{\mathrm{i}tE_e^{(s)}(\lambda)}\big\langle\phi, Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)\psi\big\rangle - \langle\phi, P_{\mathrm{R}}^\perp e^{\mathrm{i}tP_{\mathrm{R}}^\perp L_\lambda P_{\mathrm{R}}^\perp} P_{\mathrm{R}}^\perp\psi\rangle\\
&+\frac{1}{2\pi\mathrm{i}}\int_\Gamma e^{\mathrm{i}tz}\langle\phi, (L_{\mathrm{S}}-z)^{-1}P_{\mathrm{R}}\psi\rangle\, dz\Big| \prec S(w,\phi,\psi),\end{aligned} \tag{3.120}$$

where the contour Γ appearing in (3.120) is given by $\Gamma = \bigcup_e \mathcal{A}_e \cup \mathcal{G}_\infty$. It is represented in Fig. 5.

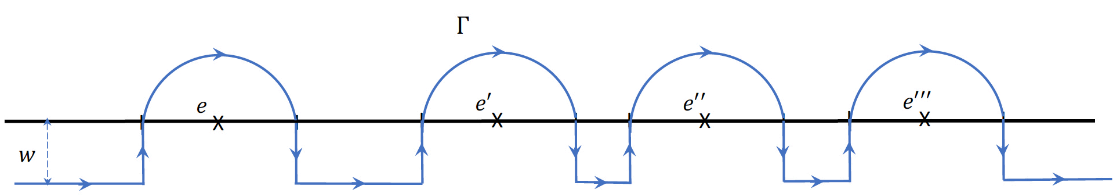

Fig. 5: The contour Γ appearing in (3.120).

The error term $S(w,\phi,\psi)$ in (3.120) is the sum of the error terms in (3.20) and (3.85), satisfying

$$
\begin{aligned}
&S(w,\phi,\psi) \prec \qquad (3.121)\\
&\quad \Big\{ e^{wt}\frac{w+\eta}{\eta}\big(\eta/g+\varkappa_3(|\lambda|+\eta)\big) + \eta e^{wt}\Big(\varkappa_3+\lambda^2\varkappa_5+\Big(\frac{\varkappa_4\kappa}{a}+\varkappa_3\Big)\Big(1+\frac{\eta}{a\lambda^2}\Big)\Big)\\
&\quad + \lambda^2 e^{wt}\Big(\frac{\varkappa_1}{\eta}+\frac{g\varkappa_1}{(\eta+\vartheta)^2}+\frac{g\varkappa_1^2\lambda^2}{\eta^3}+\vartheta\Big[\frac{\varkappa_1(\eta+\vartheta+|\lambda|)+\alpha\kappa}{\eta^2}+\frac{\varkappa_1}{g}(1/\eta+1/g)\Big]\Big)\Big\}\|\phi\|\,\|\psi\|\\
&\quad + \Big\{\frac{\kappa\eta}{a|\lambda|}e^{wt}+|\lambda|\big(1+\eta(1+\varkappa_3)\big)+\frac{(w+\eta)e^{wt}}{\eta}(|\lambda|+\lambda^2)\\
&\qquad + e^{wt}\lambda^2\eta\varkappa_5(|\lambda|+\lambda^2)+\lambda^2\big(1+(w+\eta)e^{wt}(1+\varkappa_3)\big)\Big\}\\
&\qquad \times\Big[\max_m C_2(I\varphi_m\otimes\Omega_{\mathrm{R}},\phi)\max_m C_2(I\varphi_m\otimes\Omega_{\mathrm{R}},\psi)+\mathfrak{S}_{\phi\leftrightarrow\psi}\|\phi\|\max_m C_2(I\varphi_m\otimes\Omega_{\mathrm{R}},\psi)\Big]\\
&\quad + \eta e^{wt}\big\{C_1(\phi,\psi)+C_2(\phi,\psi)\big\}\\
&\quad + e^{wt}\Big(\frac{|\lambda|\vartheta}{\eta}+\frac{|\lambda|}{\eta+\vartheta}+\frac{|\lambda|^3\varkappa_1}{\eta^2}\Big[1+\frac{\lambda^2\varkappa_1}{\eta}\Big]\Big)\mathfrak{S}_{\phi\leftrightarrow\psi}\|\phi\|\max_m C_1(I\varphi_m\otimes\Omega_{\mathrm{R}},\psi)\\
&\quad + e^{wt}\frac{\lambda^2}{\eta}\Big[1+\lambda^2\varkappa_1(1+1/\eta)\Big]\max_m C_1(I\varphi_m\otimes\Omega_{\mathrm{R}},\phi)\max_m C_1(I\varphi_m\otimes\Omega_{\mathrm{R}},\psi).\\
&\quad + e^{wt}\frac{|\lambda|}{\eta+\vartheta}\mathfrak{S}_{\phi\leftrightarrow\psi}\|\phi\|\Big(\|P_{\mathrm{R}}I\|\|P_{\mathrm{R}}^{\perp}\psi\|+\max_m C_1\big(I\varphi_m\otimes\Omega_{\mathrm{R}},(\bar{L}_\lambda+\mathrm{i})P_{\mathrm{R}}^{\perp}\psi\big)\Big)\\
&\quad + e^{wt}\frac{\lambda^2}{\eta}\mathfrak{S}_{\phi\leftrightarrow\psi}\max_m C_1(I\varphi_m\otimes\Omega_{\mathrm{R}},\phi)\Big(\|P_{\mathrm{R}}I\|\|P_{\mathrm{R}}^{\perp}\psi\|+\max_m C_1\big(I\varphi_m\otimes\Omega_{\mathrm{R}},(\bar{L}_\lambda+\mathrm{i})P_{\mathrm{R}}^{\perp}\psi\big)\Big).
\end{aligned}
$$

Since Γ does not enclose any of the eigenvalues of L_{S}, we have $\int_\Gamma \frac{e^{\mathrm{i}tz}}{L_{\mathrm{S}}-z}dz=0$, that is, the last term on the left side of (3.120) vanishes. Upon taking $w\to 0$, we obtain from (3.120), (3.7)

$$
\begin{aligned}
&\Big|\langle\phi,e^{\mathrm{i}tL_\lambda}\psi\rangle-\sum_{e\in\mathcal{E}_0}\sum_{s\in\mathcal{S}_e^{\mathrm{dec}}} e^{\mathrm{i}t(e+\lambda^2 a_e^{(s)}(e,\lambda))}\big\langle\phi,Q_e^{(s)}(e,\lambda)\psi\big\rangle \qquad (3.122)\\
&\quad -\sum_{e\in\mathcal{E}_0}\sum_{s\in\mathcal{S}_e^{\mathrm{osc}}} e^{\mathrm{i}tE_e^{(s)}(\lambda)}\big\langle\phi,Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda)\psi\big\rangle-\langle\phi,P_{\mathrm{R}}^{\perp}e^{\mathrm{i}tP_{\mathrm{R}}^{\perp}L_\lambda P_{\mathrm{R}}^{\perp}}P_{\mathrm{R}}^{\perp}\psi\rangle\Big|\prec\varkappa(0)K(\phi,\psi),
\end{aligned}
$$

with

$$
\begin{aligned}
K(\phi,\psi) &= \|\phi\|\,\|\psi\|+\max_{j=1,2}C_j(\phi,\psi)\\
&\quad +\max_{j=1,2}\big(\max_m C_j(I\varphi_m\otimes\Omega_{\mathrm{R}},\phi)\max_m C_j(I\varphi_m\otimes\Omega_{\mathrm{R}},\psi)\big)\\
&\quad +\mathfrak{S}_{\phi\leftrightarrow\psi}\|\phi\|\max_{j,m}C_j(I\varphi_m\otimes\Omega_{\mathrm{R}},\psi)\\
&\quad +\mathfrak{S}_{\phi\leftrightarrow\psi}\Big(\|\phi\|+\max_m C_1(I\varphi_m\otimes\Omega_{\mathrm{R}},\phi)\Big)\\
&\qquad \times\Big(\|P_{\mathrm{R}}I\|\|P_{\mathrm{R}}^{\perp}\psi\|+\max_m C_1\big(I\varphi_m\otimes\Omega_{\mathrm{R}},(\bar{L}_\lambda+\mathrm{i})P_{\mathrm{R}}^{\perp}\psi\big)\Big) \qquad (3.123)
\end{aligned}
$$

and

$$
\begin{aligned}
\varkappa(0) &= \eta/g + \varkappa_3(|\lambda|+\eta) + \varkappa_5\eta(\lambda^2+\eta/a)(\varkappa_4\kappa/a+\varkappa_3)\\
&\quad +\lambda^2\Big[1+\frac{\varkappa_1}{\eta}+\frac{g\varkappa_1}{(\eta+\vartheta)^2}+\frac{g\varkappa_1^2\lambda^2}{\eta^3}+\vartheta\Big(\frac{\varkappa_1(\eta+\vartheta+|\lambda|)+\alpha\kappa}{\eta^2}+\frac{\varkappa_1}{g}\big(1/\eta+1/g\big)\Big)\Big]\\
&\quad +\frac{\kappa\eta}{a|\lambda|}+|\lambda|\big(1+\eta(1+\varkappa_3)\big)+\lambda^2\eta\varkappa_5(1+\lambda^2)+\lambda^2\eta(1+\varkappa_3)\\
&\quad +\frac{\lambda^2}{\eta}\big(1+|\lambda|\varkappa_1/\eta\big)\big(1+\lambda^2\varkappa_1/\eta\big)+|\lambda|\vartheta/\eta+\frac{|\lambda|}{\eta+\vartheta}. \qquad (3.124)
\end{aligned}
$$

Next we use (4.16) and (4.17) to get

$$
\|Q_e^{(s)}(e,\lambda)-Q_e^{(s)}\| \prec \varkappa_3|\lambda| \qquad \text{and} \qquad |a_e^{(s)}(e,\lambda)-a_e^{(s)}| \prec \varkappa_4|\lambda|, \qquad (3.125)
$$

where $a_e^{(s)}$ and $Q_e^{(s)}$ are the spectral data of Λ_e, (2.1). For (e,s) such that $a_e^{(s)}\notin\mathbb{R}$ we have

$$
\begin{aligned}
\Big|e^{\mathrm{i}t(e+\lambda^2a_e^{(s)}(e,\lambda))}-e^{\mathrm{i}t(e+\lambda^2a_e^{(s)})}\Big| &= e^{-t\lambda^2\mathrm{Im}a_e^{(s)}}\Big|e^{\mathrm{i}t\lambda^2(a_e^{(s)}(e,\lambda)-a_e^{(s)})}-1\Big|\\
&= e^{-t\lambda^2\mathrm{Im}a_e^{(s)}}\Big|\int_0^{t\lambda^2(a_e^{(s)}(e,\lambda)-a_e^{(s)})}e^{\mathrm{i}z}dz\Big|, \qquad (3.126)
\end{aligned}
$$

where the integration path is the straight line linking the endpoints. On this path, $|\mathrm{Im}z|\le t\lambda^2|a_e^{(s)}(e,\lambda)-a_e^{(s)}|\le\frac12 t\lambda^2\mathrm{Im}a_e^{(s)}$, as $|\lambda|\varkappa_4\prec a$ (see (3.19)). Thus the integral is bounded above by $t\lambda^2|a_e^{(s)}(e,\lambda)-a_e^{(s)}|e^{\frac12 t\lambda^2\mathrm{Im}a_e^{(s)}}$, which gives

$$
\Big|e^{\mathrm{i}t(e+\lambda^2a_e^{(s)}(e,\lambda))}-e^{\mathrm{i}t(e+\lambda^2a_e^{(s)})}\Big| \prec \frac{|\lambda|\varkappa_4}{a}(\tfrac12 t\lambda^2\mathrm{Im}a_e^{(s)})e^{-\frac12 t\lambda^2\mathrm{Im}a_e^{(s)}}\prec\frac{|\lambda|\varkappa_4}{a}, \qquad (3.127)
$$

uniformly in $t\ge0$. For (e,s) such that $a_e^{(s)}\in\mathbb{R}$, we cannot replace $e^{\mathrm{i}tE_e^{(s)}(\lambda)}$ by $e^{\mathrm{i}t(e+\lambda^2a_e^{(s)})}$ in a manner uniform in time, as $\sup_{t\ge0}|e^{itE_e^{(s)}(\lambda)}-e^{\mathrm{i}t(e+\lambda^2a_e^{(s)})}|=2$, even though $E_e^{(s)}(\lambda)-a_e^{(s)}$ is of order λ^2. We conclude that we can replace, for the *decaying* terms in (3.122), the $a_e^{(s)}(e,\lambda)$ by $a_e^{(s)}$ and the $Q_e^{(s)}(e,\lambda)$ by $Q_e^{(s)}$ and by doing this, we incur an error $\prec|\lambda|(\varkappa_3+\varkappa_4/a)$, uniformly in time $t\ge0$. Furthermore, denoting the eigenprojection of L_λ associated to the eigenvalue $E_e^{(s)}(\lambda)$ by $\Pi_{e,\lambda}$, we have (in the strong sense) $\Pi_{e,\lambda}=\lim_{\varepsilon\to0_+}\mathrm{i}\varepsilon(L_\lambda-E_e^{(s)}(\lambda)+\mathrm{i}\varepsilon)^{-1}$ and by (3.3)

$$
P_e\Pi_{e,\lambda}P_e=\lim_{\varepsilon\to0_+}\mathrm{i}\varepsilon\big[\mathfrak{F}(L_\lambda-E_e^{(s)}(\lambda)+\mathrm{i}\varepsilon;P_e)\big]^{-1}. \qquad (3.128)
$$

Taking into account (3.16) and (3.17),

$$
P_e\Pi_{e,\lambda}P_e=\lim_{\varepsilon\to0_+}\mathrm{i}\varepsilon\sum_{s'=1}^{m_e}\frac{Q_e^{(s')}(E_e^{(s)}(\lambda)-\mathrm{i}\varepsilon,\lambda)}{e-E_e^{(s)}(\lambda)+\mathrm{i}\varepsilon+\lambda^2a_e^{(s')}(E_e^{(s)}(\lambda)-\mathrm{i}\varepsilon,\lambda)}. \qquad (3.129)
$$

According to (3.35) we have $e - E_e^{(s)}(\lambda) = -\lambda^2 a_e^{(s)}(E_e^{(s)}(\lambda),\lambda)$ and it is readily seen that the limit $\varepsilon \to 0_+$ vanishes for the terms $s' \neq s$ in the sum, because the denominator stays bounded. We conclude that

$$P_e \Pi_{e,\lambda} P_e = Q_e^{(s)}(E_e^{(s)}(\lambda),\lambda). \tag{3.130}$$

Combining (3.130) with (4.16) gives

$$\|P_e \Pi_{e,\lambda} P_e - Q_e^{(s)}\| \prec \varkappa_3\big(|E_e^{(s)}(\lambda) - e| + |\lambda|\big) \prec |\lambda|\varkappa_3\big(1 + |\lambda|\alpha\big). \tag{3.131}$$

To arrive at the last bound in (3.131), we notice that by (3.35) and (4.17), $|E_e^{(s)}(\lambda) - e| \prec \lambda^2(\alpha + \varkappa_4(|E_e^{(s)}(\lambda) - e| + |\lambda|))$, which we can solve since $\lambda^2\varkappa_4 \prec 1$ (see (3.19)) to yield $|E_e^{(s)}(\lambda) - e| \prec \lambda^2(\alpha + |\lambda|\varkappa_4)$.

Making the replacements in the decaying terms as discussed after (3.127) and using (3.131), we then obtain from (3.122) that

$$\begin{aligned}\Big|\langle\phi, e^{\mathrm{i}tL_\lambda}\psi\rangle &- \sum_{e\in\mathcal{E}_0}\sum_{s\in\mathcal{S}_e^{\mathrm{dec}}} e^{\mathrm{i}t(e+\lambda^2 a_e^{(s)})}\big\langle\phi, Q_e^{(s)}\psi\big\rangle \\ &- \sum_{e\in\mathcal{E}_0}\sum_{s\in\mathcal{S}_e^{\mathrm{osc}}} e^{\mathrm{i}tE_e^{(s)}(\lambda)}\big\langle\phi, Q_e^{(s)}\psi\big\rangle - \langle\phi, P_{\mathrm{R}}^\perp e^{\mathrm{i}tP_{\mathrm{R}}^\perp L_\lambda P_{\mathrm{R}}^\perp} P_{\mathrm{R}}^\perp\psi\rangle\Big| \prec \varkappa'(0)K(\phi,\psi),\end{aligned} \tag{3.132}$$

where

$$\varkappa'(0) = \varkappa(0) + |\lambda|\big(\varkappa_3(1+|\lambda|\alpha) + \varkappa_4/a\big). \tag{3.133}$$

We choose

$$\eta = |\lambda|^{1+\epsilon}, \quad \vartheta = |\lambda|^{1-\epsilon'} \tag{3.134}$$

for $\epsilon', \epsilon > 0$ to be determined below. Then it follows from (3.124), (3.133) that

$$\begin{aligned}\varkappa'(0) \prec\ & |\lambda|^{3+\epsilon} + |\lambda|^{1-\epsilon-\epsilon'} + |\lambda|^{\epsilon'} + |\lambda|^{1+\epsilon}/g + |\lambda|^\epsilon\kappa/a + |\lambda|^{1-2\epsilon-\epsilon'}\alpha\kappa \\ & +\varkappa_1\Big(|\lambda|^{1-\epsilon} + |\lambda|^{1-2\epsilon} + |\lambda|^{2-2\epsilon-2\epsilon'} + |\lambda|^{2-\epsilon-\epsilon'} + g(|\lambda|^{2\epsilon'} + |\lambda|^{1-3\epsilon}\varkappa_1) + \\ & \qquad + |\lambda|^{2-3\epsilon}\varkappa_1 + |\lambda|^{2-\epsilon-\epsilon'}/g + |\lambda|^{3-\epsilon'}/g^2\Big) \\ & +\varkappa_3\Big(|\lambda| + |\lambda|^{2+\epsilon} + \varkappa_5|\lambda|^{3+\epsilon} + |\lambda|^{2+2\epsilon}\varkappa_5/a + \lambda^2\alpha\Big) \\ & +\varkappa_4\Big(\lambda^{3+\epsilon}\varkappa_5\kappa/a + \lambda^{2+2\epsilon}\varkappa_5\kappa/a^2 + |\lambda|/a\Big) \\ & +\varkappa_5\Big(|\lambda|^{3+\epsilon} + |\lambda|^{5+\epsilon}\Big).\end{aligned} \tag{3.135}$$

Taking $|\lambda| \le 1$ and $\epsilon = \epsilon' = 1/4$, we obtain from (3.135),

$$\begin{aligned}\varkappa'(0) \prec\ & |\lambda|^{1/4}\max\Big\{1, 1/g, \kappa/a, \alpha\kappa, \varkappa_1^2, g\varkappa_1(1+\varkappa_1), \varkappa_1/g^2, \varkappa_3(1+\alpha+\varkappa_5+\varkappa_5/a), \\ & \qquad \varkappa_4\big(1/a + \varkappa_5\kappa(1+1/a)/a\big), \varkappa_5\Big\}.\end{aligned} \tag{3.136}$$

We further bound the maximum in (3.136). Namely, using (4.18) and (4.19) we get

$$\begin{aligned}
\varkappa_3(1+\varkappa_5+\varkappa_5/a) &\prec \varkappa_2(1+\varkappa_2^2)\frac{\kappa}{\delta}\big(1+\alpha+1/a\big) && (3.137)\\
\varkappa_4/a &\prec \varkappa_2(1+\varkappa_2)\frac{\kappa}{a}(1+\kappa/\delta) && (3.138)\\
\varkappa_4\varkappa_5\frac{\kappa}{a}(1+1/a) &\prec \varkappa_2^2(1+\varkappa_2^3)\frac{\kappa^4}{a}(1+\kappa^3/\delta^3)(1+1/a) && (3.139)\\
\varkappa_5 &\prec \varkappa_2(1+\varkappa_2)(1+\kappa^2/\delta^2)\kappa. && (3.140)
\end{aligned}$$

According to (3.15) we have $\varkappa_2 \prec \varkappa_1$ and the sum of the right hand sides of (3.137) - (3.140) is bounded above by $\varkappa_1(1+\varkappa_1^4)\kappa \max\big\{1, \frac{1+\alpha}{\delta}, 1/a, \frac{1+\kappa}{a\delta}, \kappa^2\big(\frac{\kappa}{a}(1+\kappa^3/\delta^3)(1+1/a)+1/\delta^2\big)\big\}$. We use this latter bound in (3.136) to obtain,

$$\varkappa'(0) \prec |\lambda|^{1/4}\varkappa_0 \qquad (3.141)$$

with

$$\begin{aligned}
\varkappa_0 = \max\Big\{&1, 1/g, \kappa/a, \alpha\kappa, \varkappa_1(1+\varkappa_1)(1+g+1/g),\\
&\varkappa_1(1+\varkappa_1^4)\kappa\max\big\{1, \tfrac{1+\alpha}{\delta}, 1/a, \tfrac{1+\kappa}{a\delta}, \kappa^2\big(\tfrac{\kappa}{a}(1+\kappa^3/\delta^3)(1+1/a)+1/\delta^2\big)\big\}\Big\}. \qquad (3.142)
\end{aligned}$$

We combine (3.142) and (3.132) into

$$\begin{aligned}
\Big|\langle\phi, e^{\mathrm{i}tL_\lambda}\psi\rangle &- \sum_{e\in\mathcal{E}_0}\sum_{s\in\mathcal{S}_e^{\mathrm{dec}}} e^{\mathrm{i}t(e+\lambda^2 a_e^{(s)})}\big\langle\phi, Q_e^{(s)}\psi\big\rangle \qquad (3.143)\\
&-\sum_{e\in\mathcal{E}_0}\sum_{s\in\mathcal{S}_e^{\mathrm{osc}}} e^{\mathrm{i}tE_e^{(s)}(\lambda)}\big\langle\phi, Q_e^{(s)}\psi\big\rangle - \langle\phi, P_{\mathrm{R}}^\perp e^{\mathrm{i}tP_{\mathrm{R}}^\perp L_\lambda P_{\mathrm{R}}^\perp}P_{\mathrm{R}}^\perp\psi\rangle\Big| \prec |\lambda|^{1/4}\varkappa_0\, K(\phi,\psi).
\end{aligned}$$

Along the way, in getting to (3.143), we have made several smallness conditions on λ. They come from (3.19): $|\lambda|\varkappa_4 \prec a$, $|\lambda|^{3/4}(\alpha+|\lambda|\varkappa_4) \prec 1$, from Proposition 3.3: $|\lambda|^{3/4}\varkappa_1 \prec 1$, from Theorem 3.1: $\lambda^2\varkappa_1 \prec g$, from Lemma 4.1: $|\lambda|\varkappa_1\kappa^2 \prec \delta$ and from estimate (3.14): $|\lambda|\varkappa_1 \prec \|IP_{\mathrm{R}}\|$, $\lambda^2\varkappa_1 \prec \min\{1, g^4\}$. They are summarized as follows (see also (4.18))

$$\begin{aligned}
|\lambda|\varkappa_1\kappa(1+\varkappa_1\kappa/\delta) &\prec \min\{1, a\} && (3.144)\\
|\lambda|^{3/4}(\alpha+\varkappa_1) &\prec 1 && (3.145)\\
|\lambda|\varkappa_1 &\prec \min\{\delta/\kappa^2, \|IP_{\mathrm{R}}\|\} && (3.146)\\
\lambda^2\varkappa_1 &\prec g^3. && (3.147)
\end{aligned}$$

A sufficient condition for (3.144)-(3.147) to hold is

$$|\lambda|^{3/4}\max\Big[\varkappa_1\kappa(1+\varkappa_1\kappa/\delta), \alpha, \varkappa_1, \sqrt{\varkappa_1}\Big] \prec \min\Big[1, a, \delta/\kappa^2, \|IP_{\mathrm{R}}\|, g^{3/2}\Big]. \qquad (3.148)$$

We may replace $\sqrt{\varkappa_1}$ by 1 on the left side of (3.148) and still have a sufficient condition for (3.144)-(3.147) to hold. This concludes the proof of Theorem 2.1, including that of Proposition 2.3. □

4 Proof of Theorem 3.1 and properties of $A_e(z,\lambda)$

4.1 Regularity of the resolvent, proof of Theorem 3.1

In this section, we simply write R_z^Q for $R_z^Q(\lambda)$. We also identify the range of $P_{\rm R}$ with $\mathcal{H}_{\rm S}$, *c.f.* (1.7), so that an operator $P_{\rm R}AP_{\rm R}$ is viewed as an operator on $\mathcal{H}_{\rm S}$. For any $\phi,\psi\in\mathcal{H}_{\rm S}$, $z\in\mathbb{C}_-$, $k=0,\ldots,3$ we have (recall that the φ_m are an orthonormal eigenbasis of $L_{\rm S}$, see after (2.12))

$$\begin{aligned}\left|\langle\phi,\partial_z^k P_{\rm R}IR_z^{P_{\rm R}}IP_{\rm R}\psi\rangle\right| &= \Big|\sum_{m,n}\langle\phi,\varphi_m\rangle\ \langle\varphi_n,\psi\rangle\partial_z^k\langle\varphi_m\otimes\Omega_{\rm R},(IR_z^{P_{\rm R}}I)\varphi_n\otimes\Omega_{\rm R}\rangle\Big|\\ &\le \varkappa_1\sum_{m,n}|\langle\phi,\varphi_m\rangle|\ |\langle\psi,\varphi_n\rangle|\prec\varkappa_1\|\phi\|\ \|\psi\|,\end{aligned}\tag{4.1}$$

where $\varkappa_1$ is defined in (2.12). We conclude that

$$\max_{0\le k\le 2}\sup_{z\in\mathbb{C}_-}\|\partial_z^kP_{\rm R}IR_z^{P_{\rm R}}IP_{\rm R}\|\prec\varkappa_1\quad\text{and similarly}\quad\sup_{z\in\mathbb{C}_-}\|\partial_\lambda P_{\rm R}IR_z^{P_{\rm R}}IP_{\rm R}\|\prec\varkappa_1.\tag{4.2}$$

4.1.1 Proof of Theorem 3.1.

The proof follows [14], where it is shown that the suprema in (3.8), (3.9) and (3.11) are finite without giving a specific bound (3.10). The key idea is to relate $R_z^{P_e}$ and $R_z^{P_{\rm R}}$. Introducing a new operator $K=P_e^\perp LP_e^\perp+\mathrm{i}P_e$ we have

$$R_z^{P_e}=P_e^\perp(K-z)^{-1}P_e^\perp.\tag{4.3}$$

An application of the Feshbach map (5.4) with projection $Q=P_{\rm R}$ yields

$$(K-z)^{-1}=\begin{pmatrix}\mathbb{1}&0\\-\lambda R_z^{P_{\rm R}}P_{\rm R}^\perp I\bar P_e&\mathbb{1}\end{pmatrix}\begin{pmatrix}\mathfrak{F}_z^{-1}&0\\0&R_z^{P_{\rm R}}\end{pmatrix}\begin{pmatrix}\mathbb{1}&-\lambda\bar P_eIP_{\rm R}^\perp R_z^{P_{\rm R}}\\0&\mathbb{1}\end{pmatrix},\tag{4.4}$$

where $\bar P_e=P_e^\perp P_{\rm R}=\mathbb{1}[L_{\rm S}\neq e]\otimes P_{\rm R}$. The operator $\mathfrak{F}_z$ and its inverse are given by (use that $P_e^\perp P_{\rm R}^\perp=P_{\rm R}^\perp$ and $P_{\rm R}^\perp P_e=0$, so $R_z^{P_{\rm R}}=(P_{\rm R}^\perp KP_{\rm R}^\perp-z)^{-1}\restriction_{{\rm Ran}P_{\rm R}^\perp}$),

$$\begin{aligned}\mathfrak{F}_z &= (i-z)P_e\oplus\bar P_e(L_{\rm S}-z-\lambda^2P_{\rm R}IR_z^{P_{\rm R}}IP_{\rm R})\bar P_e &(4.5)\\ \mathfrak{F}_z^{-1} &= (i-z)^{-1}P_e\oplus\bar P_e(L_{\rm S}-z)^{-1}\big(\mathbb{1}-\lambda^2P_{\rm R}IR_z^{P_{\rm R}}IP_{\rm R}(L_{\rm S}-z)^{-1}\big)^{-1}\bar P_e. &(4.6)\end{aligned}$$

Combining (4.3) with (4.4) yields four terms when estimating $\langle\phi,R_z^{P_e}\psi\rangle$. Those terms arise when we multiply out the matrices in (4.4). One of the terms is $\langle\phi,\mathfrak{F}_z^{-1}\psi\rangle$. For $|z-e|\le g/2$ and since $\|P_{\rm R}IR_z^{P_{\rm R}}IP_{\rm R}\|\prec\varkappa_1$ by (4.2), we obtain from (4.5) that

$$\|\mathfrak{F}_z^{-1}\|\prec\max\{1,1/g\},\tag{4.7}$$

provided $\lambda^2\varkappa_1/g\prec 1$. Thus

$$|\langle\phi,\mathfrak{F}_z^{-1}\psi\rangle|\prec\max\{1,1/g\}\|\phi\|\ \|\psi\|.\tag{4.8}$$

Another term we have to estimate is

$$\begin{aligned}|\langle\phi, \lambda R_z^{P_{\rm R}} I\bar{P}_e \mathfrak{F}_z^{-1}\psi\rangle| &\prec |\lambda|\, \|\mathfrak{F}_z^{-1}\|\, \|\psi\| \max_m C_1(\phi, I\varphi_m\otimes\Omega_{\rm R})\\ &\prec |\lambda| \max\{1,1/g\}\mathfrak{S}_{\phi\leftrightarrow\psi}\|\psi\| \max_m C_1(\phi, I\varphi_m\otimes\Omega_{\rm R}).\end{aligned} \tag{4.9}$$

A third term is of similar form and has the same upper bound (4.9). The fourth term to estimate is

$$\begin{aligned}&|\langle\phi, R_z^{P_{\rm R}}\psi\rangle| + \lambda^2|\langle\phi, R_z^{P_{\rm R}} I\bar{P}_e\mathfrak{F}_z^{-1}\bar{P}_e I R_z^{P_{\rm R}}\psi\rangle|\\ &\quad\prec\ C_1(\phi,\psi) + \lambda^2\max\{1,1/g\}\max_m C_1(\phi, I\varphi_m\otimes\Omega_{\rm R})C_1(\psi, I\varphi_m\otimes\Omega_{\rm R}).\end{aligned} \tag{4.10}$$

Collecting the estimates (4.8), (4.9), (4.10) shows the bound (3.8) for $j=0$. To get a bound for the derivatives $|\partial_z^j\langle\phi, R_z^{P_e}\psi\rangle|$ we again use the representation (4.3) and (4.4). The z derivatives are affecting the terms $\mathfrak{F}_z^{-1}$ and the reduced resolvents $R_z^{P_{\rm R}}$ in (4.4). The derivatives of $R_z^{P_{\rm R}}$ are controlled using (1.8). The derivatives of $\mathfrak{F}_z^{-1}$ are dealt with a repeated application of the formula $\partial_z\mathfrak{F}_z^{-1} = -\mathfrak{F}_z^{-1}(\partial_z\mathfrak{F}_z)\mathfrak{F}_z^{-1}$, then using (4.7) and $\|\partial_z^j\mathfrak{F}_z\|\prec 1+\lambda^2\varkappa_1\prec 1$. We get $\|\partial_z^j\mathfrak{F}_z^{-1}\|\prec\max\{1,1/g^{j+1}\}$. The bound (3.8) for all j then readily follows.

To prove the bound (3.9) we proceed in the same manner, using (4.3) and (4.4). The λ derivative of the resolvent $R_z^{P_{\rm R}}$ is controlled by (1.9) and we use (*c.f.* (4.5)) $\|\partial_\lambda\mathfrak{F}_z\| = \|\lambda\bar{P}_e I R_z^{P_{\rm R}} I\bar{P}_e - \lambda^2\bar{P}_e I(\partial_\lambda R_z^{P_{\rm R}})I\bar{P}_e\| \prec (|\lambda|+\lambda^2)\varkappa_1\prec 1$. The bound (3.9) follows.

Finally, to show (3.11), we use the Feshbach representation (5.4) with $Q=P_{\rm R}$. We proceed in the same way as above in this proof to get the result. □

4.2 The operators $A_e(z,\lambda)$

For every $e\in\mathcal{E}_0$ and $z\in\mathbb{C}_-$ we set (*c.f.* (3.16))

$$A_e(z,\lambda) = -P_e I R_z^{P_e}(\lambda) I P_e \equiv -P_e I R_z^{P_e} I P_e. \tag{4.11}$$

Starting from (3.8), (3.9), the bounds

$$\max_{0\le k\le 2}\sup_{\{z\in\mathbb{C}_-:|z-e|\le g/2\}}\|\partial_z^k A_e(z,\lambda)\|\prec\varkappa_2,\qquad \sup_{\{z\in\mathbb{C}_-:|z-e|\le g/2\}}\|\partial_\lambda A_e(z,\lambda)\|\prec\varkappa_2, \tag{4.12}$$

where $\varkappa_2$ is given in (3.12), are derived just as in (4.2) above. Next, $\forall z,\zeta\in\mathbb{C}_-$ with $|z-e|\le g/2$,

$$\|A_e(z,\lambda)-A_e(\zeta,\lambda)\| = \Big\|\int_z^\zeta \partial_w A_e(w,\lambda)dw\Big\|\prec|z-\zeta|\varkappa_2, \tag{4.13}$$

where the integral is over the straight line linking z and ζ. Let z_n be a sequence in $\mathbb{C}_-$ with $|z_n-e|\le g/2$, converging to some $x\in\mathbb{R}$ (so $|x-e|\le g/2$). Then by (4.13),

$A_e(z_n, \lambda)$ is Cauchy and thus converges to a limit which we call $A_e(x, \lambda)$. Using (4.13) it is easy to see that the limit $A_e(x, \lambda)$ is independent of the sequence z_n. We note that the level shift operator (1.14) equals $\Lambda_e = A_e(e, 0)$. Similarly to (4.13) we derive $\|A_e(z, \lambda) - A_e(z, 0)\| \prec \lambda^2 \varkappa_1 \varkappa_2$ and hence

$$\|A_e(z, \lambda) - \Lambda_e\| \prec \varkappa_2 \big(|z - e| + |\lambda|\big). \tag{4.14}$$

The bound (4.14) is the starting point for conventional perturbation theory. Recall the definitions of the spectral gap of the level shift operators δ and the maximal operator norm κ given in (2.10) and (2.11), respectively.

Lemma 4.1 *Suppose that $z \in \mathbb{C}_-$ and $|z - e|, |\lambda| \prec \frac{\delta}{\varkappa_2 \kappa} \min\{1, 1/\kappa\}$. Then*

1. *All eigenvalues of $A_e(z, \lambda)$ are simple. Call them $a_e^{(s)}(z, \lambda)$, $s = 1, \ldots, m_e$. Each $a_{e,s}(z, \lambda)$ satisfies $|a_{e,s}(z, \lambda) - a_e^{(s)}| < \delta/2$ for exactly one eigenvalue $a_e^{(s)}$ of Λ_e. In particular, we have the diagonal form*

$$A_e(z, \lambda) = \sum_{s=1}^{m_e} a_e^{(s)}(z, \lambda) Q_e^{(s)}(z, \lambda), \tag{4.15}$$

   where the eigenvalues $a_e^{(s)}(z, \lambda)$ are simple and the (Riesz, rank one) eigenprojections are denoted by $Q_e^{(s)}(z, \lambda)$.

2. *On the domain of z, λ determined by the constraint stated at the beginning of the lemma, the functions, $a_e^{(s)}(z, \lambda)$ and $Q_e^{(s)}(z, \lambda)$ are analytic in z and differentiable in λ. Moreover, we have*

$$\begin{aligned} \big\|Q_e^{(s)}(z, \lambda) - Q_e^{(s)}(z', \lambda')\big\| &\prec \varkappa_3 \big(|z - z'| + |\lambda - \lambda'|\big) & (4.16) \\ |a_e^{(s)}(z, \lambda) - a_e^{(s)}(z', \lambda')| &\prec \varkappa_4 \big(|z - z'| + |\lambda - \lambda'|\big), & (4.17) \end{aligned}$$

   where

$$\varkappa_3 = \varkappa_2 \kappa^2/\delta, \qquad \varkappa_4 = \varkappa_2 \kappa (1 + \varkappa_2 \kappa/\delta). \tag{4.18}$$

3. *On the domain of z, λ determined by the constraint stated at the beginning of the lemma, we have*

$$|\partial_z^2 a_e^{(s)}(z, \lambda)| \prec \varkappa_5 \equiv \varkappa_2 \kappa [1 + \frac{\varkappa_2 \kappa}{\delta}(1 + \varkappa_2 \kappa/\delta)]. \tag{4.19}$$

The previous result leads readily to the fact that one can extend the eigenvalues and eigenprojections as functions of z continuously to the real axis:

Corollary 4.2 *The maps $z \mapsto Q_e^{(s)}(z, \lambda)$ and $z \mapsto a_e^{(s)}(z, \lambda)$ extend by continuity to $z = x \in \mathbb{R}$ provided $|x - e|$, $|\lambda| \prec \frac{\delta}{\varkappa_2 \kappa} \min\{1, 1/\kappa\}$. Moreover, the estimates* (4.16) *and* (4.17) *are valid if either or both of z, z' are real.*

Proof of Corollary 4.2. Let z_n be a sequence in $\mathbb{C}_-$ with $|z_n - e| < \frac{\delta}{\varkappa_2 \kappa} \min\{1, 1/\kappa\}$ and $z_n \to x \in \mathbb{R}$. Then (4.16) shows that $Q_e^{(s)}(z_n, \lambda)$ is a Cauchy sequence, so it converges. Next, again by (4.16), $\|Q_e^{(s)}(z,\lambda) - Q_e^{(s)}(x,\lambda)\| = \lim_n \|Q_e^{(s)}(z,\lambda) - Q_e^{(s)}(z_n,\lambda)\|$ exists and is $\prec \varkappa_3 |z - x|$. If $z = x' \in \mathbb{R}$ then approximate it by $z_n' \in \mathbb{C}_-$ and the above argument works the same. The argument is also the same for the $a_e^{(s)}(x, \lambda)$. $\square$

Proof of the Lemma 4.1. Proof of 1. It follows from (2.1) that

$$\|(\Lambda_e - \zeta)^{-1}\| \prec \frac{\kappa}{d(\zeta, \mathrm{spec}(\Lambda_e))}, \tag{4.20}$$

where $d(\cdot,\cdot)$ denotes the distance function. For $|z - e| + |\lambda| \prec \frac{d(\zeta, \mathrm{spec}(\Lambda_e))}{\varkappa_2 \kappa}$ the Neumann series

$$(A_e(z,\lambda) - \zeta)^{-1} = (\Lambda_e - \zeta)^{-1} \sum_{n \geq 0} \left[\big(\Lambda_e - A_e(z,\lambda)\big)(\Lambda_e - \zeta)^{-1} \right]^n \tag{4.21}$$

converges, so the ζ satisfying $d(\zeta, \mathrm{spec}(\Lambda_e)) \prec \varkappa_2 \kappa(|z-e| + |\lambda|)$ belong to the resolvent set of $A_e(z,\lambda)$. Moreover, (4.21) gives the bounds

$$\|(A_e(z,\lambda) - \zeta)^{-1}\| \prec \frac{\kappa}{d(\zeta, \mathrm{spec}(\Lambda_e))}, \tag{4.22}$$

$$\|(A_e(z,\lambda) - \zeta)^{-1} - (\Lambda_e - \zeta)^{-1}\| \prec \big(|z-e| + |\lambda|\big) \frac{\varkappa_2 \kappa^2}{[d(\zeta, \mathrm{spec}(\Lambda_e))]^2}. \tag{4.23}$$

Let $\mathcal{C}_e^{(s)}$ be the circle centered at $a_e^{(s)}$ with radius $\delta/2$, then $d(\zeta, \mathrm{spec}(\Lambda_e)) = \delta/2$ for $\zeta \in \mathcal{C}_e^{(s)}$ and so $\mathcal{C}_e^{(s)}$ belongs to the resolvent set of $A_e(z,\lambda)$. Thus the following integral is well defined and equals the spectral projection,

$$Q_e^{(s)}(z,\lambda) = \frac{-1}{2\pi \mathrm{i}} \oint_{\mathcal{C}_e^{(s)}} (A_e(z,\lambda) - \zeta)^{-1} d\zeta. \tag{4.24}$$

We have

$$\begin{aligned} \|Q_e^{(s)}(z,\lambda) - Q_e^{(s)}\| &= \frac{1}{2\pi} \Big\| \oint_{\mathcal{C}_e^{(s)}} \left[(A_e(z,\lambda) - \zeta)^{-1} - (\Lambda_e - \zeta)^{-1} \right] d\zeta \Big\| \\ &\leq \frac{\delta}{2} \max_{\zeta \in \mathcal{C}_e^{(s)}} \|(A_e(z,\lambda) - \zeta)^{-1} - (\Lambda_e - \zeta)^{-1}\| < 1, \end{aligned} \tag{4.25}$$

since $|z-e| + |\lambda| \prec \delta/(\varkappa_2 \kappa^2)$. By standard perturbation theory [13], the ranks of $Q_e^{(s)}(z,\lambda)$ and $Q_e^{(s)}$ are the same (both $= 1$ by assumption (A5)) and hence $A_e(z,\lambda)$ has exactly one eigenvalue inside $\mathcal{C}_e^{(s)}$. This shows point 1.

We now give a proof of 2. Using (4.12) and (4.22) we obtain

$$\begin{aligned} \|\partial_z Q_e^{(s)}(z,\lambda)\| &= \frac{1}{2\pi} \Big\| \oint_{\mathcal{C}_e^{(s)}} (A_e(z,\lambda) - \zeta)^{-1} \{\partial_z A_e(z,\lambda)\} (A_e(z,\lambda) - \zeta)^{-1} d\zeta \Big\| \\ &\leq \delta \varkappa_2 \max_{\zeta \in \mathcal{C}_e^{(s)}} \|(A_e(z,\lambda) - \zeta)^{-1}\|^2 \prec \frac{\varkappa_2 \kappa^2}{\delta}. \end{aligned} \tag{4.26}$$

Taking the z derivative twice (or the λ derivative) and proceeding as in (4.26) yields the estimates

$$\big\|\partial_z^2 Q_e^{(s)}(z,\lambda)\big\| \prec \frac{\varkappa_2\kappa^2}{\delta}(1+\varkappa_2\kappa/\delta), \qquad \big\|\partial_\lambda Q_e^{(s)}(z,\lambda)\big\| \prec \frac{\varkappa_2\kappa^2}{\delta}. \tag{4.27}$$

We combine (4.26) and (4.27) to obtain

$$\begin{aligned}
\big\|Q_e^{(s)}(z,\lambda)-Q_e^{(s)}(z',\lambda')\big\| &\leq \big\|Q_e^{(s)}(z,\lambda)-Q_e^{(s)}(z',\lambda)\big\|+\big\|Q_e^{(s)}(z',\lambda)-Q_e^{(s)}(z',\lambda')\big\| \\
&= \Big\|\int_z^{z'}\partial_\zeta Q_e^{(s)}(\zeta,\lambda)d\zeta\Big\|+\Big\|\int_\lambda^{\lambda'}\partial_\mu Q_e^{(s)}(z',\mu)d\mu\Big\| \\
&\prec \frac{\varkappa_2\kappa^2}{\delta}\big(|z-z'|+|\lambda-\lambda'|\big),
\end{aligned} \tag{4.28}$$

which shows (4.16). To show (4.17), we note that

$$a_e^{(s)}(z,\lambda)=\operatorname{tr}(A_e(z,\lambda)Q_e^{(s)}(z,\lambda)) \tag{4.29}$$

and so, using (4.12), (4.26) and (see (4.24) and (4.22))

$$\|Q_e^{(s)}(z,\lambda)\| \prec \kappa, \tag{4.30}$$

we obtain $|\partial_z a_e^{(s)}(z,\lambda)| = \big|\operatorname{tr}\big(\{\partial_z A_e(z,\lambda)\}Q_e^{(s)}(z,\lambda)+A_e(z,\lambda)\{\partial_z Q_e^{(s)}(z,\lambda)\}\big)\big| \prec \varkappa_2\kappa(1+\varkappa_2\kappa/\delta)$. Proceeding in the same way we find $|\partial_\lambda a_e^{(s)}(z,\lambda)| \prec \varkappa_2\kappa(1+\varkappa_2\kappa/\delta)$. By integrating these bounds similarly to what we did in (4.28), we get (4.17).

We finally prove point 3. The relation (4.29) yields

$$\begin{aligned}
\partial_z^2 a_e^{(s)}(z,\lambda) &= \operatorname{tr}\Big(\{\partial_z^2 A_e(z,\lambda)\}Q_e^{(s)}(z,\lambda)+2\{\partial_z A_e(z,\lambda)\}\{\partial_z Q_e^{(s)}(z,\lambda)\} \\
&\quad +A_e(z,\lambda)\{\partial_z^2 Q_e^{(s)}(z,\lambda)\}\Big).
\end{aligned}$$

The bound on the second derivative given in point 3. now follows from (4.12), (4.30), (4.26) and (4.27). This completes the proof of Lemma 4.1. □

5 Feshbach decomposition of the resolvent

Let Q be an orthogonal projection on a Hilbert space $\mathcal{H}$. In the decomposition $\mathcal{H} = \operatorname{Ran}Q \oplus \operatorname{Ran}Q^\perp$ an operator $\mathcal{O}$ has the block decomposition

$$\mathcal{O} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{5.1}$$

where $A = Q\mathcal{O}Q\restriction_{\operatorname{Ran}Q}$, $B = QAQ^\perp \restriction_{\operatorname{Ran}Q^\perp}$ and so on. We want to find the block decomposition of $\mathcal{O}^{-1}$,

$$\mathcal{O}^{-1} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}, \tag{5.2}$$

assuming that D^{-1} exists. Multiplying blockwise the equation $\mathcal{O}\mathcal{O}^{-1} = \mathbb{1}$ yields the four equations

$$AX + BZ = \mathbb{1}_Q, \quad AY = -BW, \quad CX = -DZ, \quad CY + DW = \mathbb{1}_{Q^\perp}. \tag{5.3}$$

Thus $Z = -D^{-1}CX$ and $W = D^{-1} - D^{-1}CY$. Then $AX + BZ = AX - BD^{-1}CX = \mathbb{1}_Q$, so $X = (A - BD^{-1}C)^{-1}$. Also, $AY = -BW = -BD^{-1} + BD^{-1}CY$ so $(A - BD^{-1}C)Y = -BD^{-1}$ and hence $Y = -XBD^{-1}$. In conclusion,

$$\begin{aligned} X &= (A - BD^{-1}C)^{-1} \\ Y &= -XBD^{-1} \\ Z &= -D^{-1}CX \\ W &= D^{-1} - D^{-1}CXBD^{-1}. \end{aligned}$$

In the case $\mathcal{O} = H - z$, for a self-adjoint H (like $H = L_\lambda$) and $z \notin \mathbb{R}$, we have indeed that $D = Q^\perp(H - z)Q^\perp \restriction_{\mathrm{RanQ}^\perp}$ is invertible. Also, $A - BD^{-1}C = \mathfrak{F}(H - z; Q)$ and hence the sum of X, Y, Z and W give the right hand side of (3.3).

Theorem 5.1 (Weak isospectrality of the Feshbach map) *Let H be a self-adjoint operator and let $E \in \mathbb{R}$ and suppose that the function $z \mapsto QHQ^\perp R_z^Q Q^\perp HQ$ is continuously differentiable on $z \in \mathbb{C}_- \cup \{E\}$. Denote the value of $\mathfrak{F}(H - z; Q)$ at $z = E$ by $\mathfrak{F}(H - E; Q)$. Then we have the following:*

(1) E is an eigenvalue of H if and only if zero is an eigenvalue of $\mathfrak{F}(H - E; Q)$.

(2) If $H\Phi = E\Phi$, then $\varphi = Q\Phi$ satisfies $\mathfrak{F}(H - E; Q)\varphi = 0$.

(3) If $\mathfrak{F}(H - E; Q)\varphi = 0$ then the limit of $R_z^Q Q^\perp HQ\varphi$ as $z \in \mathbb{C}_-$, $z \to E$, exists. Denote it by $R_E^Q Q^\perp HQ\varphi$. Then $\Phi = \varphi - R_E^Q Q^\perp HQ\varphi$ satisfies $H\Phi = E\Phi$.

Note: It is easy to see that the correspondence $\Phi \leftrightarrow \varphi$ in Theorem 5.1 is a bijection between the kernels of $H - E$ and of $\mathfrak{F}(H - E; Q)$.

Proof of Theorem 5.1. The implication $\Rightarrow$ in (1) together with (2) is not hard to prove, see Proposition B2 of [14]. We give now a proof of $\Leftarrow$ in (1) and (3). One may write the resolvent using the above components in matrix form as

$$(H - z)^{-1} = \begin{pmatrix} \mathbb{1} & 0 \\ -R_z^Q Q^\perp HQ & \mathbb{1} \end{pmatrix} \begin{pmatrix} \left[\mathfrak{F}(H - z; Q)\right]^{-1} & 0 \\ 0 & R_z^Q \end{pmatrix} \begin{pmatrix} \mathbb{1} & -QHQ^\perp R_z^Q \\ 0 & \mathbb{1} \end{pmatrix} \tag{5.4}$$

where $R_z^Q = (Q^\perp HQ^\perp - z)^{-1} \restriction_{\mathrm{Ran}Q^\perp}$. This relation can actually be verified simply by multiplying out the matrices. As the two outside matrices in (5.4) are both invertible, we obtain for $z \in \mathbb{C}$ such that R_z^Q exists:

$$\begin{aligned} H - z &= \begin{pmatrix} \mathbb{1} & QHQ^\perp R_z^Q \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathfrak{F}(H - z; Q) & 0 \\ 0 & Q^\perp(H - z)Q^\perp \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ R_z^Q Q^\perp HQ & \mathbb{1} \end{pmatrix} \\ &= \begin{pmatrix} \mathfrak{F}(H - z; Q) + QHP^\perp R_z^Q P^\perp HQ & QHQ^\perp \\ Q^\perp HQ & Q^\perp(H - z)Q^\perp \end{pmatrix}. \end{aligned} \tag{5.5}$$

To discuss domain questions, assume for simplicity that $\dim Q < \infty$ and that $Q^\perp H Q$ is bounded. Then the relation (5.5) holds when applied to vectors of the form $\begin{pmatrix}\varphi\\ \chi\end{pmatrix}$ with $\chi \in \mathrm{Dom}(Q^\perp H Q^\perp)$. Moreover, each single product operation of the matrices in the threefold matrix product in (5.5) makes sense individually (is well defined, when the right side is applied to vectors of the indicated form).

Due to the assumption of the theorem, $QHQ^\perp R_z^Q Q^\perp HQ$ has a limit when $z = E - \mathrm{i}\epsilon$ with $\epsilon \to 0_+$. Call this limit $QHQ^\perp(\bar{H} - E)^{-1}Q^\perp HQ$ (the overlined $\bar{H}$ indicates that H is restricted to the range of $Q^\perp$). Then $\lim_{\epsilon\to 0_+}\mathfrak{F}(H-E-\mathrm{i}\epsilon; Q) = Q(H-E-HQ^\perp(\bar{H}-E)^{-1}Q^\perp H)Q \equiv \mathfrak{F}(H-E;Q)$, where the last symbol is again a definition. Suppose that $\varphi = Q\varphi$ is such that $\mathfrak{F}(H-E;Q)\varphi = 0$. For $\epsilon > 0$, define $\chi_\epsilon = -Q^\perp R^Q_{E-\mathrm{i}\epsilon}Q^\perp HQ\varphi$. Then applying equation (5.5) with $z = E - \mathrm{i}\epsilon$ gives

$$(H-E+\mathrm{i}\epsilon)(\varphi+\chi_\epsilon) = \mathfrak{F}(H-E+\mathrm{i}\epsilon;Q)\varphi \longrightarrow 0 \quad \text{as } \epsilon \to 0_+. \tag{5.6}$$

We show below that χ_ϵ has a limit χ as $\epsilon \to 0_+$. It then follows from (5.6) that $\lim_{\epsilon\to 0_+}(H-E)(\varphi+\chi_\epsilon) = 0$. Since H is a closed operator, the vector $\varphi+\chi$ is in the domain of H and $(H-E)(\varphi+\chi) = 0$.

We now show that χ_ϵ is Cauchy. This part of the analysis is inspired by [DJ], Theorem 3.8. Denote by p the spectral projection of $\mathfrak{F}(H-E;Q)$ associated to the eigenvalue 0. In particular, $p\varphi = \varphi$. Since $\mathfrak{F}(H-E;Q)$ is a dissipative operator and 0 is on the boundary of its numerical range, 0 is a semisimple eigenvalue and p is the orthogonal projection onto the kernel of $\mathfrak{F}(H-E;Q)$, in particular, $p^* = p$. First we show that $z \mapsto pHR_z^Q Hp$ is a continuously differentiable function of $z \in \mathbb{C}_- \cup \mathbb{C}_+ \cup \{E\}$. (Note: $QHR^Q_z HQ$, without the restriction to the range of p, is not even continuous at E, because the limits coming from $\mathbb{C}_+$ and $\mathbb{C}_-$ do not coincide.) We point out that $z \mapsto QHQ^\perp R_z^Q Q^\perp HQ$ is continuously differentiable on $z \in \mathbb{C}_+ \cup \{E\}$, which follows by taking the adjoint in the assumption of the theorem. Next, we have $p\mathfrak{F}(H-E;Q)p = 0$ and hence $p(H-E)p = pHR^Q_{E-\mathrm{i}0_+}Hp$, and taking the adjoint gives $p(H-E)p = pHR^Q_{E+\mathrm{i}0_+}Hp$. So we have

$$pHR^Q_{E-\mathrm{i}0_+}Hp = pHR^Q_{E-\mathrm{i}0_-}Hp = p(H-E)p, \tag{5.7}$$

which shows that $z \mapsto pHR_z^QHp$ is continuous at $z = E$ and its value at $z = E$ is self-adjoint. Next, let $\gamma(\tau)$, $\tau \in [-1,1]$, be a smooth curve in $\mathbb{C}_- \cup \{E\}$ which is tangent to the point $E \in \mathbb{C}$, satisfying $\gamma(0) = E$ and $\gamma'(0) = 1$. Then $\tau \mapsto pHR^Q_{\gamma(\tau)}Hp$ is continuously differentiable at $\tau = 0$ and $\frac{d}{d\tau}|_{\tau=0}\,\mathrm{Im}\,pHR^Q_{\gamma(\tau)}Hp = \mathrm{Im}\,pH(R^Q_{E-\mathrm{i}0_+})^2Hp$. On the other hand, this last derivative has to be equal to zero for the following reason. For all τ we have $\mathrm{Im}\,pHR^Q_{\gamma(\tau)}Hp \le 0$ and from (5.7), $\mathrm{Im}\,pHR^Q_{\gamma(\tau)}Hp = 0$ at $\tau = 0$. If $\frac{d}{d\tau}|_{\tau=0}\,\mathrm{Im}\,pHR^Q_{\gamma(\tau)}Hp$ was > 0 or < 0 this would contradict the fact that $\mathrm{Im}\,pHR^Q_{\gamma(\tau)}Hp \le 0$. Since this derivative vanishes, we get $\mathrm{Im}\,pH(R^Q_{E-\mathrm{i}0_+})^2Hp = 0$, so $pH(R^Q_{E-\mathrm{i}0_+})^2Hp$ is self-adjoint, which means that $pH(R^Q_{E-\mathrm{i}0_+})^2Hp = pH(R^Q_{E-\mathrm{i}0_-})^2Hp$. So $z \mapsto pHR_z^QHp$ is continuously differentiable on $\mathbb{C}_- \cup \mathbb{C}_+ \cup \{E\}$.

We now use this regularity property to show that χ_ϵ converges. Let $\epsilon, \epsilon' > 0$. We have

$$\left\|(R^Q_{E-\mathrm{i}\epsilon} - R^Q_{E-\mathrm{i}\epsilon'})HQ\varphi\right\|^2 = \left\langle \varphi, QH(R^Q_{E+\mathrm{i}\epsilon} - R^Q_{E+\mathrm{i}\epsilon'})(R^Q_{E-\mathrm{i}\epsilon} - R^Q_{E-\mathrm{i}\epsilon'})HQ\varphi \right\rangle. \qquad (5.8)$$

From the resolvent identity, $R^Q_{E-\mathrm{i}\epsilon}R^Q_{E+\mathrm{i}\epsilon'} = \frac{1}{\mathrm{i}(\epsilon'+\epsilon)}(R^Q_{E-\mathrm{i}\epsilon} - R^Q_{E+\mathrm{i}\epsilon'})$ and similarly for the other three terms on the right side of (5.8). Therefore, (also using that $pQ = Qp = p$),

$$\begin{aligned} &\left\|(R^Q_{E+\mathrm{i}\epsilon} - R^Q_{E+\mathrm{i}\epsilon'})HQ\varphi\right\|^2 && (5.9)\\ &= \left\langle \varphi, pH\frac{R^Q_{E-\mathrm{i}\epsilon}-R^Q_{E+\mathrm{i}\epsilon}}{2\mathrm{i}\epsilon}Hp\varphi\right\rangle - \left\langle \varphi, pH\frac{R^Q_{E-\mathrm{i}\epsilon}-R^Q_{E+\mathrm{i}\epsilon'}}{\mathrm{i}(\epsilon'+\epsilon)}Hp\varphi\right\rangle \\ &\quad - \left\langle \varphi, pH\frac{R^Q_{E-\mathrm{i}\epsilon'}-R^Q_{E+\mathrm{i}\epsilon}}{\mathrm{i}(\epsilon+\epsilon')}Hp\varphi\right\rangle + \left\langle \varphi, pH\frac{R^Q_{E-\mathrm{i}\epsilon'}-R^Q_{E+\mathrm{i}\epsilon'}}{2\mathrm{i}\epsilon'}Hp\varphi\right\rangle. && (5.10)\end{aligned}$$

Since $z \mapsto pHR^Q_z Hp$ is a continuously differentiable function of $z \in \mathbb{C}_- \cup \mathbb{C}_+ \cup \{E\}$, the right hand side of (5.10) converges to zero as $\epsilon, \epsilon' \to 0$. Thus χ_ϵ converges. $\square$

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