# Operator Algebras Associated with Unitary Commutation Relations

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**Abstract**

We define nonselfadjoint operator algebras with generators $L_{e_1}, \ldots, L_{e_n}, L_{f_1}, \ldots, L_{f_m}$ subject to the unitary commutation relations of the form

$$L_{e_i} L_{f_j} = \sum_{k,l} u_{i,j,k,l} L_{f_l} L_{e_k}$$

where $u = (u_{i,j,k,l})$ is an $nm \times nm$ unitary matrix. These algebras, which generalise the analytic Toeplitz algebras of rank 2 graphs with a single vertex, are classified up to isometric isomorphism in terms of the matrix $u$.

# 1 Introduction

The unilateral shift on complex separable Hilbert space generates two fundamental operator algebras, namely the norm closed (unital) algebra and the weak operator topology closed algebra. The former is naturally isomorphic to the disc algebra of holomorphic functions on the unit disc, continuous to the boundary, while the latter is isomorphic to $H^\infty$. The freely noncommuting multivariable generalisations of these algebras arise from the freely noncommuting shifts $L_{e_1}, \ldots, L_{e_n}$ given by the left creation operators on the Fock space $\mathcal{F}_n = \sum_{k=0}^{\infty} \oplus (\mathbb{C}^n)^{\otimes k}$. Here the generated operator algebras, denoted $\mathcal{A}_n$ and $\mathcal{L}_n$ for the norm and weak topologies, are known as the noncommutative disc algebra and the freesemigroup algebra. They have been studied extensively with respect to operator algebra structure, representation theory and the multivariable operator theory of row contractions. See for example [2], [9].

Higher rank generalisations of these algebras arise when one considers several families of freely noncommuting generators between which there are commutation relations. In the present paper we consider a very general form of such relations, namely

$$L_{e_i} L_{f_j} = \sum_{k,l} u_{i,j,k,l} L_{f_l} L_{e_k}$$

where $L_{e_1}, \ldots, L_{e_n}$ and $L_{f_1}, \ldots, L_{f_m}$ are freely noncommuting and $u = (u_{i,j,k,l})$ is an $nm \times nm$ unitary matrix. The associated operator algebras are denoted $\mathcal{A}_u$ and $\mathcal{L}_u$ and we classify them up to various forms of isomorphism in terms of the unitary matrices $u$. Such unitary relations arose originally in the context of the general dilation theorem proven in Solel ([12], [13]) for two row contractions $[T_1 \cdots T_n]$ and $[S_1 \cdots S_m]$ satisfying the unitary commutation relations.

For $n = m = 1$, we have $u = [\alpha]$ with $|\alpha| = 1$ and $\mathcal{A}_u$ is the subalgebra of the rotation C*-algebra for the relations $uv = \alpha vu$. When $u$ is a permutation unitary matrix arising from a permutation $\theta$ in $S_{nm}$ then the relations are those associated with a single vertex rank 2 graph in the sense of Kumjian and Pask, and the algebras in this case have been considered in Kribs and Power [5] and Power [10]. In particular, in [10] it was shown that there are 9 operator algebras $\mathcal{A}_\theta$ arising from the 24 permutations in case $n = m = 2$. In contrast, we see below in Section 6 that for general 2 by 2 unitaries $u$ there are uncountably many isomorphism classes of the unitary relation algebras $\mathcal{A}_u$ expressed in terms of a nine fold real parametrisation of isomorphism types.

The algebras $\mathcal{A}_\theta$ are easily defined; they are determined by the left regular representation of the semigroup $\mathbb{F}_\theta^+$ whose generators are $e_1, \dots, e_n, f_1, \dots, f_m$ subject to the relations $e_i f_j = f_l e_k$ where $\theta(i,j) = (k,l)$. On the other hand the unitary relation algebras $\mathcal{A}_u$ are generated by creation operators on a $\mathbb{Z}_+^2$-graded Fock space $\sum_{k,l} \oplus (\mathbb{C}^n)^{\otimes k} \otimes (\mathbb{C}^m)^{\otimes l}$ with relations arising from the identification $u : \mathbb{C}^n \otimes \mathbb{C}^m \to \mathbb{C}^m \otimes \mathbb{C}^n$. In particular, $\mathcal{A}_u$ is a representation of the non-selfadjoint tensor algebra of a rank 2 correspondence (or a product system over $\mathbb{N}^2$) in the sense of [13]. See also [3]

In the main results, summarised partly in Theorem 5.10, we see that if $\mathcal{A}_u$ and $\mathcal{A}_v$ are isomorphic then the two families of generators have matching cardinalities. Furthermore, if $n \neq m$ then the algebras are isomorphic if and only if the unitaries $u, v$ in $M_{nm}(\mathbb{C})$ are unitary equivalent by a unitary $A \otimes B$ in $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$. As in [10] we term this *product unitary equivalence* (with respect to the fixed tensor product decomposition). The case $n = m$ admits an extra possibility, in view of the possibility of generator exchanging isomorphisms, namely that $u, \tilde{v}$ are product unitary equivalent, where $\tilde{v}_{i,j,k,l} = \bar{v}_{l,k,j,i}$.

The theorem is proven as follows. After some preliminaries we identify, in Section 3, the character space $M(\mathcal{A}_u)$ and the set of w*-continuous characters on $\mathcal{L}_u$. These are subsets of the closed unit ball product $\overline{\mathbb{B}}_n \times \overline{\mathbb{B}}_m$ which are associated with a variety $V_u$ in $\mathbb{C}^n \times \mathbb{C}^m$ determined by $u$. We then define the *core* $\Omega_u^0$, a closed subset of the realised character space $\Omega_u = M(\mathcal{A}_u)$, and we identify this intrinsically (algebraically) in terms of representations of $\mathcal{A}_u$

into $T_2$, the algebra of upper triangular matrices in $M_2(\mathbb{C})$. The importance of the core is that we are able to show that the interior is a minimal automorphism invariant subset on which automorphisms act transitively. This allows us to infer the existence of graded isomorphisms from general isomorphisms. To construct automorphisms we first review, in Section 4, Voiculescu's construction of a unitary action of the Lie group $U(1,n)$ on the Cuntz algebra $\mathcal{O}_n$ and the operator algebras $\mathcal{A}_n$ and $\mathcal{L}_n$. This provides, in particular, unitary automorphisms $\Theta_\alpha$, for $\alpha \in \mathbb{B}_n$, which act transitively on the interior ball, $\mathbb{B}_n$, of the character space of $\mathcal{A}_n$. For these explicit unitary automorphisms of the $e_i$-generated copy of $\mathcal{A}_n$ in $\mathcal{A}_u$, we establish unitary commutation relations for the tuples $\Theta_\alpha(L_{e_1}), \ldots, \Theta_\alpha(L_{e_n})$ and $L_{f_1}, \ldots, L_{f_m}$, when $(\alpha, 0)$ is a point in the core. This enables us to define natural unitary automorphisms of $\mathcal{A}_u$ itself, and in Theorem 4.8 the relative interior of the core is identified as an automorphism invariant set in the Gelfand space $\Omega_u$. In Section 5 we determine the graded and bigraded isomorphisms in terms of product unitary equivalence. To do this we observe that such isomorphisms induce an origin preserving biholomorphic map between the cores $\Omega_u^0$ and $\Omega_v^0$ and that these maps, by a generalised Schwarz's Lemma, are implemented by a product unitary. We then prove the main classification theorem.

In Section 6 we analyse in detail the case $n = m = 2$ and consider the special case of permutation unitaries.

Finally, in Section 7 we show that the algebra $\mathcal{A}_u$ is contained in a tensor algebra $\mathcal{T}_+(X)$, associated with a correspondence $X$ as in [7]. Moreover, at least when $n \neq m$, every automorphism of $\mathcal{A}_u$ extends to an automorphism of $\mathcal{T}_+(X)$. The advantage of the tensor algebra is that its representation theory is known ([7]) while this is not the case yet for the algebra $\mathcal{A}_u$.

# 2 Preliminaries

Fix two finite dimensional Hilbert spaces $E = \mathbb{C}^n$ and $F = \mathbb{C}^m$ and a unitary $mn \times mn$ matrix $u$. The rows and columns of $u$ are indexed by $\{1, \ldots, n\} \times \{1, \ldots, m\}$ ($u = (u_{(i,j),(k,l)})$) and when we write $u$ as an $mn \times mn$ matrix we assume that $\{1, \ldots, n\} \times \{1, \ldots, m\}$ is ordered lexicographically (so that, for example, the second row is the row indexed by $(1, 2)$). We also fix orthonormal bases $\{e_i\}$ and $\{f_j\}$ for $E$ and $F$ respectively and the matrix

$u$ is used to identify $E \otimes F$ with $F \otimes E$ through the equation

$$e_i \otimes f_j = \sum_{k,l} u_{(i,j),(k,l)} f_l \otimes e_k. \tag{1}$$

Equivalently, we write

$$f_l \otimes e_k = \sum_{i,j} \bar{u}_{(i,j),(k,l)} e_i \otimes f_j. \tag{2}$$

For every $k, l \in \mathbb{N}$, we write $X(k,l)$ for $E^{\otimes k} \otimes F^{\otimes l}$. Using succesive applications of (1), we can identify $X(k,l)$ with $E^{\otimes k_1} \otimes F^{\otimes l_1} \otimes E^{\otimes k_2} \otimes \cdots \otimes F^{\otimes l_r}$ whenever $k = \sum k_i$ and $l = \sum l_j$.

Let $\mathcal{F}(n,m,u)$ be the Fock space given by the Hilbert space direct sum

$$\sum_{k,l} X(k,l) = \sum_{k,l} E^{\otimes k} \otimes F^{\otimes l},$$

and, for $e \in E$ and $f \in F$, write $L_e$ and $L_f$ for the "shift" operators

$$L_e e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \otimes f_{j_1} \otimes f_{j_2} \otimes \cdots \otimes f_{j_l} = e \otimes e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \otimes f_{j_1} \otimes f_{j_2} \otimes \cdots \otimes f_{j_l}$$

and

$$L_f e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \otimes f_{j_1} \otimes f_{j_2} \otimes \cdots \otimes f_{j_l} = f \otimes e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \otimes f_{j_1} \otimes f_{j_2} \otimes \cdots \otimes f_{j_l}$$

where, in the last equation, we use (1) to identify the resulting vector as a vector of $E^{\otimes k} \otimes F^{\otimes (l+1)}$.

The unital semigroup generated by $\{I, L_e, L_f \ : \ e \in E, \ f \in F\}$ is denoted $\mathbb{F}_u^+$ and the algebra it generates denoted $\mathbb{C}[\mathbb{F}_u^+]$. The norm closure of $\mathbb{C}[\mathbb{F}_u^+]$ will be written $\mathcal{A}_u$ and its closure in the weak* operator topology will be written $\mathcal{L}_u$. In particular, the algebras $\mathcal{L}_\theta$ and $\mathcal{A}_\theta$ studied in [10] are the algebras $\mathcal{L}_u$ and $\mathcal{A}_u$ for $u$ which is a permutation matrix.

The results of Section 2 in [5] hold here too with minor changes. Every $A \in \mathcal{L}_u$ is the limit (in the strong operator topology) of its Cesaro sums

$$\Sigma_p(A) = \sum_{k \le p} (1 - \frac{k}{p}) \Phi_k(A)$$

where $\Phi_k(A)$ lies in $\mathcal{L}_u$ and is "supported" on $\sum_l \oplus E^{\otimes l} \otimes F^{\otimes (k-l)}$. In fact, let $Q_k$ be the projection of $\mathcal{F}(n,m,u)$ onto $\sum_l \oplus E^{\otimes l} \otimes F^{\otimes (k-l)}$, form the one-parameter unitary group $\{U_t\}$ defined by $U_t := \sum_{k=0}^{\infty} e^{ikt} Q_k$ and set $\gamma_t = AdU_t$. Then $\{\gamma_t\}_{t\in\mathbb{R}}$ is a $w^*$-continuous action of $\mathbb{R}$ on $\mathcal{L}(\mathcal{F}(n,m,u))$ that normalizes both $\mathcal{A}_u$ and $\mathcal{L}_u$ and

$$\Phi_k(a) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}\gamma_t(a)dt$$

for all $a \in \mathcal{L}(\mathcal{F}(n,m,u))$. Then $\Phi_k$ leaves $\mathcal{L}_u$ invariant.

We can define the algebra $\mathcal{R}_u$ generated by the right shifts $R_e$ and $R_f$ defined by

$$R_e e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes e_{i_k}\otimes f_{j_1}\otimes f_{j_2}\otimes\cdots\otimes f_{j_l} = e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes e_{i_k}\otimes f_{j_1}\otimes f_{j_2}\otimes\cdots\otimes f_{j_l}\otimes e$$

and

$$R_f e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes e_{i_k}\otimes f_{j_1}\otimes f_{j_2}\otimes\cdots\otimes f_{i_l} = e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes e_{i_k}\otimes f_{j_1}\otimes f_{j_2}\otimes\cdots\otimes f_{i_l}\otimes f.$$

The techniques of the proof of Proposition 2.3 of [5] can be applied here to show that the commutant of $\mathcal{R}_u$ is $\mathcal{L}_u$. Also, mapping $e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes e_{i_k}\otimes f_{j_1}\otimes f_{j_2}\otimes\cdots\otimes f_{j_l}$ to $f_{j_l}\otimes f_{j_{l-1}}\otimes\cdots\otimes f_{j_1}\otimes e_{i_k}\otimes e_{i_{k-1}}\otimes\cdots\otimes e_{i_1}$, we get a unitary operator

$$W : \mathcal{F}(n,m,u) \to \mathcal{F}(n,m,u^*)$$

implementing a unitary equivalence of $\mathcal{L}_u$ and $\mathcal{R}_{u^*}$. In fact, it is easy to check that $R_{e_i}W = WL_{e_i}$ and $R_{f_j}W = WL_{f_j}$ for every $i,j$. To see that the commutation relation in the range is given by $u^*$, apply $W$ to (2) to get (in the range of $W$) $e_k\otimes f_l = \sum_{i,j}\bar{u}_{(i,j),(k,l)} f_j\otimes e_i = \sum_{i,j}(u^*)_{(k,l),(i,j)} f_j\otimes e_i$ which is equation (1) with $u^*$ instead of $u$.

As in [5], we conclude that $(\mathcal{L}_u)' = \mathcal{R}_u$ and $(\mathcal{L}_u)'' = \mathcal{L}_u$.

# 3 The character space and its core

In the following proposition we describe the structure of the character spaces $\mathcal{M}(\mathcal{L}_u)$ and $\mathcal{M}(\mathcal{A}_u)$ (equipped with the weak* topology). Similar results were obtained in [5] for algebras defined for higher rank graphs and in [2] for analytic Toeplitz algebras. (See also [10].)

It will be convenient to write

$$V_u = \{(z,w) \in \mathbb{C}^n\times\mathbb{C}^m \ : \ z_i w_j = \sum_{k,l} u_{(i,j),(k,l)} z_k w_l \ \} \tag{3}$$

and

$$\Omega_u = V_u \cap (\overline{\mathbb{B}}_n \times \overline{\mathbb{B}}_m) \tag{4}$$

where $\mathbb{B}_n$ is the open unit ball of $\mathbb{C}^n$. We refer to $V_u$ as the *variety associated with* $u$.

**Proposition 3.1** *(1) The linear multiplicative functionals on $\mathbb{C}[\mathbb{F}_u^+]$ are in one-to-one correspondence with points $(z,w)$ in $V_u$.*

*(2) $\mathcal{M}(\mathcal{A}_u)$ is homeomorphic to $\Omega_u$.*

*(3) For $(z,w) \in \Omega_u$, write $\alpha_{(z,w)}$ for the corresponding character of $\mathcal{A}_u$. Then $\alpha_{(z,w)}$ extends to a $w^*$-continuous character on $\mathcal{L}_u$ if and only if $(z,w) \in \mathbb{B}_n \times \mathbb{B}_m$.*

**Proof.** Part (1) follows immediately from (1). Fix $\alpha \in \mathcal{M}(\mathcal{A}_u)$ and write $z_i = \alpha(L_{e_i})$, $1 \leq i \leq n$, and $w_i = \alpha(L_{f_j})$, $1 \leq j \leq m$. From the multiplicativity and linearity of $\alpha$ and (1), it follows that $(z,w) \in V_u$. Since $\alpha$ is contractive and maps $\sum_i a_i L_{e_i}$ to $\sum_i a_i z_i$, it follows that $\|z\| \leq 1$ and similarly $\|w\| \leq 1$. Thus $(z,w) \in \Omega_u$.

For the other direction, fix first $(z,w) \in \Omega_u$ with $\|z\| < 1$ and $\|w\| < 1$. It follows from the definition of $\Omega_u$ and from (1) that $(z,w)$ defines a linear and multiplicative map $\alpha$ on the algebra $\mathbb{C}[\mathbb{F}_u^+]$ such that $L_{e_i}$ is mapped into $z_i$ and $\alpha(L_{f_j}) = w_j$. Abusing notation slightly, we write $\alpha(x)$ for $\alpha(L_x)$ for every $x \in E^{\otimes k} \otimes F^{\otimes l}$. Also, for $i = (i_1, \ldots, i_k)$ and $j = (j_1, \ldots, j_l)$, we write $e_i f_j$ for $e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes f_{j_1} \otimes \cdots \otimes f_{j_l}$. These elements form an orthonormal basis for $E^{\otimes k} \otimes F^{\otimes l}$ and we now set

$$w_\alpha = \sum_{i,j} \sum_{k,l} \overline{\alpha(e_i f_j)} e_i f_j \in \mathcal{F}(X).$$

If $p_i \geq 0$ and $p_1 + \ldots + p_n = k$ then there are $\frac{k!}{p_1! \cdots p_n!}$ terms $e_{i_1} \otimes \cdots \otimes e_{i_k}$ with $\alpha(e_{i_1} \otimes \cdots \otimes e_{i_k}) = z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k}$. It follows that $\sum_k \sum_i |\alpha(e_i)|^2 = \sum_k \sum_{i=(i_1,\ldots,i_k)} |\alpha(e_{i_1})|^2 \cdots |\alpha(e_{i_k})|^2$. Thus

$$\|w_\alpha\|^2 = \sum_{i,j,k,l} |\alpha(e_i f_j)|^2 = (1 - \|z\|^2)^{-1} (1 - \|w\|^2)^{-1} < \infty$$

Note that, for every $x \in E^{\otimes k} \otimes F^{\otimes l}$,

$$\langle x, w_\alpha \rangle = \alpha(x).$$

Thus, for $e \in E$, $\langle x, L_e^* w_\alpha \rangle = \langle L_e x, w_\alpha \rangle = \alpha(e \otimes x) = \alpha(e)\alpha(x) = \langle \alpha(e) w_\alpha, x \rangle$ and, similarly $\langle x, L_f^* w_\alpha \rangle = \langle \alpha(f) w_\alpha, x \rangle$ for $f \in F$. Thus $\langle w_\alpha, L_e^* w_\alpha \rangle = \alpha(e)\alpha(w_\alpha) = \alpha(e) \sum |\alpha(e_i f_j)|^2 = \alpha(e) \|w_\alpha\|^2$. Similarly, $\langle w_\alpha, L_f^* w_\alpha \rangle = \alpha(f)\alpha(w_\alpha) = \alpha(f) \sum |\alpha(e_i f_j)|^2 = \alpha(f) \|w_\alpha\|^2$ for $f \in F$. Thus if we write $\nu_\alpha = w_\alpha / \|w_\alpha\|$ then

$$\alpha(x) = \langle L_x \nu_\alpha, \nu_\alpha \rangle$$

for every $x \in E^{\otimes k} \otimes F^{\otimes l}$ (for every $k, l$). This shows that $\alpha$ is contractive and is $w^*$-continuous. We can, therefore, extend it to an element of $\mathcal{M}(\mathcal{L}_u)$, also denoted $\alpha$.

The analysis above shows that the image of the map $\alpha \mapsto (z, w) \in \Omega_u$ defined above (on $\mathcal{M}(\mathcal{A}_u)$) contains $V_u \cap (\mathbb{B}_n \times \mathbb{B}_m)$. Since $\mathcal{M}(\mathcal{A}_u)$ is compact and the map is $w^*$-continuous, its image contains (and, thus, is equal to) $\Omega_u$. This completes the proof of (2). To complete the proof of (3), we need to show that, if $(z, w) \in \Omega_u$ and the corresponding character extends to a $w^*$-continuous character on $\mathcal{L}_u$, then $\|z\| < 1$ and $\|w\| < 1$.

For this, write $\mathcal{L}$ for the $w^*$-closed subalgebra of $\mathcal{L}_u$ generated by $\{L_e : e \in E\} \cup \{I\}$. Let $P$ be the projection of $\mathcal{F}(E, F, u)$ onto $\mathcal{F}(E) = \mathbb{C} \oplus E \oplus (E \otimes E) \oplus \cdots$. Then $P\mathcal{L}P = P\mathcal{L}_u P$ and the map $T \mapsto PTP$, is a $w^*$-continuous isomorphism of $\mathcal{L}$ onto $P\mathcal{L}_u P$. The latter algebra is unitarily equivalent to the algebra $\mathcal{L}_n$ studied in [2]. A $w^*$-continuous character of $\mathcal{L}_u$ gives rise, therefore, to a $w^*$-continuous character on $\mathcal{L}_n$. It follows from [2, Theorem 2.3] that $z \in \mathbb{B}_n$. Similarly, one shows that $w \in \mathbb{B}_m$. $\square$

To state the next result, we first write $u_{(i,j)}$ for the $n \times m$ matrix whose $k, l$-entry is $u_{(i,j),(k,l)}$. Thus, the $(i, j)$ row of $u$ provides the $n$ rows of $u_{(i,j)}$. We then compute

$$\sum_{k,l} u_{(i,j),(k,l)} z_k w_l = \sum_k (\sum_l u_{(i,j),(k,l)} w_l) z_k = \sum_k (u_{(i,j)} w)_k z_k = \langle u_{(i,j)} w, \bar{z} \rangle. \tag{5}$$

Write $E_{i,j}$ for the $n \times m$ matrix whose $i, j$-entry is 1 and all other entries are 0 (so that $\langle E_{i,j} w, \bar{z} \rangle = z_i w_j$) and write $C_{(i,j)}$ for the matrix $u_{(i,j)} - E_{i,j}$. Then the computation above yields the following.

**Lemma 3.2** *With $C_{(i,j)}$ defined as above, we have*

$$V_u = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m \ : \quad \langle C_{(i,j)} w, \bar{z} \rangle = 0, \quad \textit{for all}\, i, j\}.$$

**Definition 3.3** *The core of* $\Omega_u$ *is the subset given by*

$$\Omega_u^0 := \{(z,w) \in \overline{\mathbb{B}}_n \times \overline{\mathbb{B}}_m \ : \quad C_{(i,j)}w = 0,\ C_{(i,j)}^t z = 0 \quad \textit{for all } i,j\}.$$

Fix $(z,w) \in \Omega_u^0$. We have $u_{(i,j)}w = E_{i,j}w$ for all $i,j$. Thus, for every $k$,

$$\sum_l u_{(i,j),(k,l)}w_l = \delta_{i,k}w_j \tag{6}$$

(where $\delta_{i,k}$ is 1 if $i = k$ and 0 otherwise) and, for $a_1, a_2, \ldots, a_n$, in $\mathbb{C}$ we have $\sum_{k,l} u_{(i,j),(k,l)}a_k w_l = a_i w_j$. Hence, if we let $\tilde{w}^{(i)}$ be the vector in $\mathbb{C}^{mn}$ defined by $\tilde{w}^{(i)}_{(k,l)} = \delta_{k,i}w_l$, we get $u\tilde{w}^{(i)} = \tilde{w}^{(i)}$. Similarly, for $z$, we have

$$\sum_k u_{(i,j),(k,l)}z_k = \delta_{j,l}z_i \tag{7}$$

and for scalars $b_1, \ldots, b_m$ we have $\sum_{k,l} u_{(i,j),(k,l)}b_l z_k = b_j z_i$. Thus, writing $\tilde{z}_{(j)}$ for the vector defined by $(\tilde{z}_{(j)})_{(k,l)} = \delta_{l,j}z_k$, we have $u\tilde{z}_{(j)} = \tilde{z}_{(j)}$. The vector $\tilde{w}^{(i)}$ in $\mathbb{C}^{nm} = \mathbb{C}^n \otimes \mathbb{C}^m$ is also expressible as $\delta_i \otimes w$ where $\{\delta_1, \ldots, \delta_n\}$ is the standard basis of $\mathbb{C}^n$, and, similarly, $\tilde{z}_{(j)} = z \otimes \delta_j$. We therefore obtain Lemma 3.4 which will be useful in Section 6.

We note also the following companion formula. Suppose $(z,w) \in \Omega_u^0$. Then, as we noted above, $u\tilde{z}_{(j)} = \tilde{z}_{(j)}$ and, thus, $u^*\tilde{z}_{(j)} = \tilde{z}_{(j)}$. Writing this explicitly, we have, for all $i,j,l$,

$$\sum_k u_{(k,l),(i,j)}\bar{z}_k = \delta_{j,l}\bar{z}_i. \tag{8}$$

.

**Lemma 3.4** *Let* $(z,w)$ *be a vector in the core* $\Omega_u^0$*. Then*

$$span\{\tilde{z}_{(j)}, \tilde{w}^{(i)} \ : \ 1 \le i \le n,\ 1 \le j \le m\} \subseteq Ker(u - I).$$

*In particular,*

*(i) If the core contains a vector* $(z,w)$ *with* $z \neq 0$*, then* $dim(Ker(u-I)) \ge m$*.*

*(ii) If the core contains a vector* $(z,w)$ *with* $w \neq 0$ *then* $dim(Ker(u-I)) \ge n$*.*

*(iii) If the core contains a vector* $(z,w)$ *with* $z \neq 0$ *and* $w \neq 0$*, then* $dim(Ker(u-I)) \geq m+n-1$.

We now characterise the core in an algebraic manner in terms of representations into the algebra $T_2$ of upper triangular $2\times 2$ matrices. We remark that nest representations such as these have proven useful in the algebraic structure theory of nonself-adjoint algebra [**?**], [11].

Let $\rho : \mathbb{C}[\mathbb{F}_u^+] \to T_2$ with

$$\rho(T) = \begin{pmatrix} \rho_{1,1}(T) & \rho_{1,2}(T) \\ 0 & \rho_{2,1}(T) \end{pmatrix}$$

Then $\rho_{1,1}$ and $\rho_{2,2}$ are characters and $\rho_{1,2}$ is a linear functional that satisfies

$$\rho_{1,2}(TS) = \rho_{1,1}(T)\rho_{1,2}(S) + \rho_{1,2}(T)\rho_{2,2}(S) \tag{9}$$

for $T, S \in \mathbb{C}[\mathbb{F}_u^+]$.

We now restrict to the case where $\rho_{1,1} = \rho_{2,2}$. By Proposition 3.1(1), both are associated with a point $(z,w)$ in $V_u$. It follows from (9) that $\rho_{1,2}$ is determined by its values on $L_{e_i}$ and $L_{f_j}$. Setting $\lambda_i = \rho_{1,2}(L_{e_i})$ and $\mu_j = \rho_{1,2}(L_{f_j})$, we associate with each homomorphism $\rho$ (as discussed above) a quadruple $(z,w,\lambda,\mu)$ where $(z,w) \in V_u$ and, for every $i,j$,

$$z_i\mu_j + \lambda_i w_j = \sum_{k,l} u_{(i,j),(k,l)}(w_l\lambda_k + \mu_l z_k). \tag{10}$$

(The last equation follows from (1)). Using (5) we can write the last equation as

$$\langle u_{(i,j)}w, \bar{\lambda}\rangle + \langle u_{(i,j)}\mu, \bar{z}\rangle = z_i\mu_j + \lambda_i w_j = \langle E_{i,j}w, \bar{\lambda}\rangle + \langle E_{i,j}\mu, \bar{z}\rangle.$$

That is,

$$\langle C_{(i,j)}w, \bar{\lambda}\rangle + \langle \mu, \overline{C^t_{(i,j)}z}\rangle = 0. \tag{11}$$

The following lemma now follows from the definition of the core.

**Lemma 3.5** *A point* $(z,w) \in \Omega_u$ *lies in the core* $\Omega_u^0$ *if and only if every* $(\lambda,\mu) \in \mathbb{C}^n \times \mathbb{C}^m$ *defines a homomorphism* $\rho : \mathbb{C}[\mathbb{F}_u^+] \to T_2$ *such that*

$$\rho(L_{e_i}) = \begin{pmatrix} z_i & \lambda_i \\ 0 & z_i \end{pmatrix}$$

*and*

$$\rho(L_{f_j}) = \begin{pmatrix} w_j & \mu_j \\ 0 & w_j \end{pmatrix}$$

*for all* $i,j$.

# 4 Automorphisms of $\mathcal{L}_n$ and $\mathcal{L}_u$

We first derive the unitary automorphisms of $\mathcal{L}_n$ and $\mathcal{A}_n$ associated with $U(1,n)$. These were obtained by Voiculescu [14] in the setting of the Cuntz-Toeplitz algebra. However the automorphisms restrict to an action of $U(1,n)$ on the free semigroup algebra. The result is rather fundamental, being a higher dimensional version of the familiar Möbius automorphism group on $H^\infty$. For the reader's convenience we provide complete proofs. See also the discussion in Davidson and Pitts [2], and in [1], [10].

**Lemma 4.1** *Let $\alpha \in \mathbb{B}_n$ and write*

*(i)* $x_0 = (1 - \|\alpha\|^2)^{-1/2}$,

*(ii)* $\eta = x_0 \alpha$, *and*

*(iii)* $X_1 = (I_{\mathbb{C}^n} + \eta\eta^*)^{1/2}$.

*Then*

*(1)* $\|\eta\|^2 = |x_0|^2 - 1$,

*(2)* $X_1 \eta = x_0 \eta$, *and*

*(3)* $X_1^2 = I + \eta\eta^*$.

*In particular, the matrix* $X = \begin{pmatrix} x_0 & \eta^* \\ \eta & X_1 \end{pmatrix}$ *satisfies* $X^* J X = J$, *where* $J = \begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix}$,

**Proof.** Part (1) is an easy computation and part (3) follows from the definition of $X_1$. For (2), note that $X_1^2 \eta = (I + \eta\eta^*)\eta = \eta + \|\eta\|^2 \eta = x_0^2 \eta$ and, for every $\zeta \in \eta^\perp$, $X_1 \zeta = \zeta$. Suppose $X_1 \eta = a\eta + \zeta$ ($\zeta \in \eta^\perp$). Then $x_0^2 \eta = X_1^2 \eta = a^2 \eta + \zeta$ and it follows that $a = x_0$ (as $X_1 \geq 0$) and $\zeta = 0$. $\square$

The lemma exhibits specific matrices ($X_1$ is nonnegative) in $U(1,n)$ associated with points in the open ball. One can similarly check (see [2] or [10] for example) that the general form of a matrix $Z$ in $U(1,n)$ is $Z = \begin{pmatrix} z_0 & \eta_1^* \\ \eta_2 & Z_1 \end{pmatrix}$ where

$$\|\eta_1\|^2 = \|\eta_2\|^2 = |z_0|^2 - 1,$$
$$Z_1\eta_1 = \bar{z}_0\eta_2, \quad Z_1^*\eta_2 = z_0\eta_1,$$
$$Z_1^*Z_1 = I_n + \eta_1\eta_1^*, \quad Z_1Z_1^* = I_n + \eta_2\eta_2^*.$$

It is these equations that are equivalent to the single matrix equation $Z^*JZ = J$.

It is well known that the map $\theta_X$ defined on $\mathbb{B}_n$ by

$$\theta_X(\lambda) = \frac{X_1\lambda + \eta}{x_0 + \langle\lambda, \eta\rangle} \quad , \lambda \in \mathbb{B}_n.$$

is an automorphism of $\mathbb{B}_n$ with inverse $\theta_{X^{-1}}$. See Lemma 4.9 of [2] and Lemma 8.1 of [10] for example. We make use of this in the proof of Voiculescu's theorem below.

Let $L_1, \dots, L_n$ be the generators of the norm closed algebra $\mathcal{A}_n$ and for $\zeta \in \mathbb{C}^n$ write $L_\zeta = \sum \zeta_i L_i$. Recall that the character space $M(\mathcal{A}_n)$ is naturally identifiable with the closed ball $\bar{\mathbb{B}}_n$, with $\lambda$ in this ball providing a character $\phi_\lambda$ for which $\phi_\lambda(L_i) = \lambda_i$. The proof is a reduced version of that given above for $M(\mathcal{A}_\theta)$.

**Theorem 4.2** *Let $\alpha \in \mathbb{B}_n$ and let $X_1, x_0, \eta$ and $X$ be associated with $\alpha$ as in Lemma 4.1. Then*

*(i) there is an automorphism $\Theta_X$ of $\mathcal{L}_n$ such that*

$$\Theta_\alpha(L_\zeta) = (x_0I + L_\eta)^{-1}(L_{X_1\zeta} + \langle\zeta, \bar{\eta}\rangle I), \tag{12}$$

*(ii) the inverse automorphism $\Theta_X^{-1}$ is $\Theta_{X^{-1}}$, and $X^{-1}$ is the matrix in $U(1,n)$ associated with $-\alpha$,*

*(iii) there is a unitary $U_X$ on $\mathcal{F}_n$ such that for $a \in \mathcal{A}_n$,*

$$U_Xa\xi_0 = \Theta_\alpha(a)(x_0I + L_\eta)^{-1}\xi_0$$

*and $\Theta_X(a) = U_XaU_X^*$.*

**Proof.** Let $\mathcal{F}_n$ be the Fock space for $\mathcal{L}_n$, $I_n = I_{\mathcal{F}_n}$, and let $\tilde{L} = [I_n \;\; L_1 \cdots L_n]$ viewed as an operator from $(\mathbb{C} \oplus \mathbb{C}^n) \otimes \mathcal{F}_n = \mathcal{F}_n \oplus (\mathbb{C}^n \otimes \mathcal{F}_n)$ to $\mathcal{F}_n$. Then

$$\tilde{L}(J \otimes I)\tilde{L}^* = I_n - \tilde{L}\tilde{L}^* = I_n - (L_1L_1^* + \dots L_nL_n^*) = P_0$$

where $P_0$ is the vacuum vector projection from $\mathcal{F}_n$ to $\mathbb{C}$. Also, since $XJX = J$, we have

$$\tilde{L}(J \otimes I)\tilde{L}^* = \tilde{L}(X \otimes I_n)(J \otimes I)(X \otimes I_n)\tilde{L}^* = [Y_0 \;\; Y_1](J \otimes I)[Y_0 \;\; Y_1]^*$$

where

$$[Y_0 \;\; Y_1] = [I_n \;\; L]\begin{pmatrix} x_0 \otimes I_n & \eta^* \otimes I_n \\ \eta \otimes I_n & X_1 \otimes I_n \end{pmatrix}.$$

Thus $Y_0Y_0^* - Y_1Y_1^* = P_0$. Also

$$Y_0 = x_0 \otimes I_n + L(\eta \otimes I_n) = x_0 I_n + L_\eta,$$

$$Y_1 = \eta^* \otimes I_n + L(X_1 \otimes I_n) = \eta^* \otimes I_n + [L_{X_1e_1} \dots L_{X_1e_n}]$$

where, here, $e_1, \dots, e_n$ is the standard basis for $\mathbb{C}^n$.

The operator $V = Y_0^{-1}Y_1$ is a row isometry $[V_1 \;\cdots\; V_n]$, from $\mathbb{C}^n \otimes \mathcal{F}_n$ to $\mathcal{F}_n$ with defect 1. To see this we compute

$$I - VV^* = I - Y_0^{-1}Y_1Y_1^*Y_0^{*-1} = I - Y_0^{-1}(-P_0 + Y_0Y_0^*)Y_0^{*-1}$$

$$= I + Y_0^{-1}P_0Y_0^{*-1} - I = \xi_0'\xi_0'^*.$$

Here

$$\xi_0' = Y_0^{-1}\xi_0 = (x_0I_n + L_\eta)^{-1}\xi_0 = x_0^{-1}(\sum_{j=0}^{\infty}(x_0^{-1}L_\eta)^j\xi_0)$$

and so

$$\|\xi_0'\| = |x_0|^{-2}\sum_j |x_0|^{-2j}\|\eta\|^{2j} = \frac{1}{x_0^2 - \|\eta\|^2} = 1.$$

Considering the path $t \to t\alpha$ for $0 \le t \le 1$ and the corresponding path of partial isometries $V$ it follows from the stability of Fredholm index that the index of $V$ and $L$ coincide and so in fact $V$ is a row isometry. Thus $V_1, \dots, V_n$ are isometries with orthogonal ranges.

We now have a contractive algebra homomorphism $\mathcal{A}_n \to \mathcal{L}(\mathcal{F}_n)$ determined by the correspondence $L_{e_i} \to V_i, i = 1, \dots, n$. In fact it is an algebra endomorphism $\Theta : \mathcal{A}_n \to \mathcal{A}_n$. Indeed, for $\xi = (\xi_1, \dots, \xi_n)$ we have

$$\Theta(L_\xi) = \sum \xi_i V_i = \sum \zeta_i Y_0^{-1}Y_1(e_i \otimes I_n)$$

$$= \sum \zeta_i(x_0I_n + L_\eta)^{-1}(\eta^* \otimes I_n + [L_{X_1e_1} \dots L_{X_1e_n}])[I_n \cdots I_n]^t$$

$$= (x_0 I_n + L_\eta)^{-1}(\langle \zeta, \eta\rangle I_n + L_{X_1\zeta}).$$

Thus far we have followed Voiculescu's proof [14]. The following argument shows that $\Theta$ is an automorphism and is an alternative to the calculation suggested in [14]. The calculation shows that

$$\phi_\lambda \circ \Theta_X = \phi_{\theta_{\overline{X}}(\lambda)}.$$

We have

$$\begin{aligned}\phi_\lambda \circ \Theta_X(L_\zeta) &= \phi_\lambda((x_0 I_n + L_\eta)^{-1}(\langle \zeta, \eta\rangle I_n + L_{X_1\zeta}))\\ &= (x_0 + \langle \lambda, \eta\rangle)^{-1}(\langle \zeta, \eta\rangle + \langle X_1\zeta, \overline{\lambda}\rangle) = \phi_\mu(L_\zeta)\end{aligned}$$

where

$$\mu = \frac{\overline{X_1^*\overline{\lambda}} + \overline{\eta}}{x_0 + \langle \lambda, \overline{\eta}\rangle} = \frac{\overline{X_1}\lambda + \overline{\eta}}{x_0 + \langle \lambda, \overline{\eta}\rangle} = \theta_{\overline{X}}(\lambda).$$

Write $\Theta_X$ for the contractive endomorphism $\Theta$ of $\mathcal{A}_n$ as constructed above. It follows that the composition $\Phi = \Theta_{X^{-1}} \circ \Theta_X$ is a contractive endomorphism which, by the remarks preceding the statement of the theorem, induces the identity map on the character space, so that $\phi_\lambda = \phi_\lambda \circ \Phi^{-1}$ for all $\lambda \in \mathbb{B}_n$. Such a map must be the identity. Indeed, suppose that we have the Fourier series representation $\Phi^{-1}(L_{e_1}) = a_1 L_{e_1} + \ldots + a_n L_{e_n} + X$ where $X$ is a series with terms of total degree greater than one. It follows that

$$\lim_{t\to 0} \; t^{-1}\phi_{(t,0,\ldots,0)}(\Phi^{-1}(L_{e_1})) = a_1$$

while

$$\lim_{t\to 0} \; t^{-1}\phi_{(t,0,\ldots,0)}(L_{e_1}) = 1.$$

Since the induced map is the identity, we have $a_1 = 1$ and $a_k = 0$ for $k \geq 2$. In this way we see that the image of each $L_i$ has the form $L_i + T_i$ where $T_i$ has only terms of total degree greater than one. Since $L_i\xi_0$ is orthogonal to $T_i\xi_0$ and $\Phi^{-1}(L_i)$ is a contraction, we have $1 \geq \|\Phi^{-1}(L_i)\xi_0\|^2 = \|L_i\xi_0 + T_i\xi_0\|^2 = \|L_i\xi_0\|^2 + \|T_i\xi_0\|^2 = 1 + \|T_i\xi_0\|^2$. Thus $T_i\xi_0 = 0$ and, consequently, $T_i = 0$ and so the composition $\Phi$ is the identity map.

Finally, we show that $\Theta_\alpha$ is unitarily implemented. Define $U_X$ on $\mathcal{A}_n\xi_0$ by $U_X a\xi_0 = \Theta_X(a)\xi_0' = \Theta_X(a)(x_0 I + L_\eta)^{-1}\xi_0$ for $a \in \mathcal{A}$. Since $\Theta_X$ is an automorphism, $(U_X a)b\xi_0 = U_X ab\xi_0 = \Theta_X(a)\Theta_X(b)\xi_0' = \Theta_X(a)U_X b\xi_0$, for $a, b \in \mathcal{A}_n$, and it follows that $U_X a = \Theta_X(a)U_X$, as linear transformations on the dense space $\mathcal{A}_n\xi_0$.

Now, $V = [V_1, \ldots, V_n]$ is a row isometry with defect space spanned by $\xi_0'$. The map $U_X$ maps $\xi_i = L_i\xi_0$ to $\Theta_X(L_i)\xi_0' = V_i\xi_0'$ and, if $w = w(e_1, \ldots, e_n)$ is a word in $e_1, \ldots, e_n$ , then

$$U_X\xi_w = U_X w(L_1, \ldots, L_n)\xi_0 = \Theta_X(w(L_1, \ldots, L_n))\xi_0' = w(V_1, \ldots, V_n)\xi_0'.$$

Since $V$ is a row isometry and $\xi_0'$ is a unit wandering vector for $V$, it follows that $\{w(V_1, \ldots, V_n)\xi_0'\}$ is an orthonormal set. Thus, $U_X$ is an isometry. Since the range of $U_X$ contains $U_X\mathcal{A}_n\xi_0 = \Theta_X(\mathcal{A}_n)\xi_0' = \mathcal{A}_n(x_0I + L_\eta)^{-1}\xi_0 = \mathcal{A}_n\xi_0$ we see that $U_X$ is unitary. $\Box$

**Remark 4.3** *With the same calculations as in the proof above and slightly more notation, one can show that each invertible matrix $Z \in U(1, n)$ defines an automorphism $\Theta_Z$ and that $Z \to \Theta_Z$ is an action of $U(1, n)$ on $\mathcal{A}_n$ and, in particular, $\Theta_Z\Theta_X = \Theta_{ZX}$. Moreover, $Z \to U_Z$ is a unitary representation of $U(1, n)$ implementing this as the following calculation indicates.*

*Let $W = \begin{pmatrix} w_0 & \omega^* \\ \omega & W_1 \end{pmatrix}$ be the matrix in $U(1, n)$ associated with $\beta \in \mathbb{B}_n$ as in Lemma 4.1. Then*

$$\begin{aligned}
U_XU_Wa\xi_0 &= U_X(\Theta_\beta(a)(w_0 + L_\omega)^{-1}\xi_0) \\
&= \Theta_\alpha(\Theta_\beta(a)(w_0 + L_\omega)^{-1})(x_0I_n + L_\eta)^{-1}\xi_0 \\
&= \Theta_\alpha(\Theta_\beta(a))\Theta_\alpha((w_0 + L_\omega)^{-1})(x_0I_n + L_\eta)^{-1}\xi_0 \\
&= \Theta_{XW}(a)\Theta_\alpha((w_0 + L_\omega)^{-1})(x_0I_n + L_\eta)^{-1}\xi_0 \\
&= \Theta_{XW}(a)[w_0I_n + (x_0I_n + L_\eta)^{-1}(L_{X_1\omega} + \langle\omega, \eta\rangle I_n)]^{-1}(x_0I_n + L_\eta)^{-1}\xi_0 \\
&= \Theta_{XW}(a)[w_0x_0I_n + w_0L_\eta + L_{X_1\omega} + \langle\omega, \eta\rangle I_n)]^{-1}\xi_0 \\
&= \Theta_{XW}(a)[(w_0x_0I_n + \langle\omega, \eta\rangle)I_n + L_{\omega_0\eta+X_1\omega}]^{-1}\xi_0.
\end{aligned}$$

*One readily checks that this is the same as $U_{XW}(a)\xi_0$*

It is evident from the last theorem and its proof that the unitary automorphisms of $\mathcal{A}_n$ and $\mathcal{L}_n$ act transitively on the open subset $\mathbb{B}_n$ associated with the weak star continuous characters. We shall show that a version of this holds for the unitary relation algebras with respect to the open core of the character space. As a first step to constructing automorphisms of $\mathcal{A}_u$ we obtain unitary commutation relations for the n-tuples $[\Theta(L_{e_1}), \ldots, \Theta(L_{e_n})]$ and $[L_{f_1}, \ldots, L_{f_m}]$ for certain automorphisms $\Theta$ of the copy of $\mathcal{A}_n$ in $\mathcal{A}_u$.

**Lemma 4.4** *Suppose* $(z,w) \in \Omega_u^0 \cap (\mathbb{B}_n \times \mathbb{B}_m)$. *Write* $\alpha$ *for* $\bar{z}$ *and let* $\Theta := \Theta_\alpha$ *be as in (12). Then, for every* $1 \le i \le n$ *and* $1 \le j \le m$,

$$\Theta(L_{e_i})L_{f_j} = \sum_{k,l} u_{(i,j),(k,l)} L_{f_l} \Theta(L_{e_k}). \tag{13}$$

**Proof.** Write $Y$ for $\eta\eta^*$ and $\beta$ for $(x_0+1)^{-1}$. Since $X_1^2 = I + \eta\eta^*$, $X_1 = I + \beta\eta\eta^* = I + \beta Y$ and $Y = (Y_{i,j})$ where $Y_{i,j} = \eta_i \bar{\eta}_j = x_0^2 \bar{z}_i z_j$. We now compute

$$(X_1 e_i) f_j = e_i f_j + \sum_t \beta Y_{t,i} e_t f_j = e_i f_j + \sum_{t,k,l} \beta Y_{t,i} u_{(t,j),(k,l)} f_l e_k$$

$$= \sum_{k,l} u_{(i,j),(k,l)} f_l e_k + \sum_{t,k,l} \beta x_0^2 \bar{z}_t z_i u_{(t,j),(k,l)} f_l e_k$$

$$= \sum_{k,l} u_{(i,j),(k,l)} f_l e_k + \beta x_0^2 z_i \sum_{t,k,l} \bar{z}_t u_{(t,j),(k,l)} f_l e_k.$$

Using the core equation (8), the last expression is equal to

$$\sum_{k,l} u_{(i,j),(k,l)} f_l e_k + \beta x_0^2 z_i \sum_{k,l} \delta_{j,l} \bar{z}_k f_l e_k$$

$$= \sum_{k,l} u_{(i,j),(k,l)} f_l e_k + \beta x_0^2 z_i \sum_{k} \bar{z}_k f_j e_k$$

$$= \sum_{k,l} u_{(i,j),(k,l)} f_l e_k + \beta x_0^2 \sum_{k,l} (\delta_{j,l} z_i) \bar{z}_k f_l e_k.$$

Using the core equation (7), this is equal to

$$\sum_{k,l} u_{(i,j),(k,l)} f_l e_k + \beta x_0^2 \sum_{k,l} (\sum_t u_{(i,j),(t,l)} z_t) \bar{z}_k f_l e_k$$

$$= \sum_{k,l} u_{(i,j),(k,l)} f_l e_k + \beta x_0^2 \sum_{k,l,t} u_{(i,j),(k,l)} z_k \bar{z}_t f_l e_t$$

$$= \sum_{k,l} u_{(i,j),(k,l)} f_l e_k + \beta \sum_{k,l,t} u_{(i,j),(k,l)} Y_{t,k} f_l e_t$$

$$= \sum_{k,l} u_{(i,j),(k,l)} f_l e_k + \beta \sum_{k,l} u_{(i,j),(k,l)} f_l Y e_k$$

$$= \sum_{k,l} u_{(i,j),(k,l)} f_l X_1 e_k.$$

Thus

$$L_{X_1 e_i} L_{f_j} = \sum_{k,l} u_{(i,j),(k,l)} L_{f_l} L_{X_1 e_k}. \tag{14}$$

Next, we compute $\sum_i \bar{z}_i e_i f_j = \sum_{i,k,l} u_{(i,j),(k,l)} \bar{z}_i f_l e_k$. Using (8), this is equal to $\sum_{k,l} \delta_{j,l} \bar{z}_k f_l e_k = \sum_k \bar{z}_k f_j e_k$. Thus

$$\sum_i \bar{z}_i e_i f_j = \sum_i \bar{z}_i f_j e_i \tag{15}$$

and, hence, $L_\eta$ commutes with $L_{f_j}$. It follows that

$$L_{f_j}(x_0 I - L_\eta)^{-1} = (x_0 I - L_\eta)^{-1} L_{f_j}. \tag{16}$$

We have, using (14) and (16),

$$(x_0 I - L_\eta)^{-1} L_{X_1 e_i} L_{f_j} = \sum_{k,l} u_{(i,j),(k,l)} (x_0 I - L_\eta)^{-1} L_{f_l} L_{X_1 e_k}$$

$$= \sum_{k,l} u_{(i,j),(k,l)} L_{f_l} (x_0 I - L_\eta)^{-1} L_{X_1 e_k}.$$

Also, applying (7) and (16), we get

$$(x_0 I - L_\eta)^{-1} \langle e_i, \eta \rangle L_{f_j} = z_i L_{f_j} (x_0 I - L_\eta)^{-1}$$

$$= \sum_l \delta_{j,l} z_i L_{f_l} (x_0 I - L_\eta)^{-1}$$

$$= \sum_{k,l} u_{(i,j),(k,l)} z_k L_{f_l} (x_0 I - L_\eta)^{-1}.$$

Subtracting the last two equations, we get (13). $\square$

**Corollary 4.5** *In the notation of Lemma 4.4, for every $i, j$,*

$$L_{f_j}^* \Theta(L_{e_i}) = \sum_{k,l} u_{(i,l),(k,j)} \Theta(L_{e_k}) L_{f_l}^*.$$

**Proof.** It follows from (13) that $\Theta(L_{e_i})L_{f_l} = \sum_{k,t} u_{(i,l),(k,t)} L_{f_t}\Theta(L_{e_k})$ for every $i, l$. Thus, for $i, j, l$,

$$L_{f_j}^* \Theta(L_{e_i}) L_{f_l} L_{f_l}^* = \sum_{k,t} u_{(i,l),(k,t)} L_{f_j}^* L_{f_t} \Theta(L_{e_k}) L_{f_l}^*$$

$$= \sum_{k,t} u_{(i,l),(k,t)} \delta_{j,t} \Theta(L_{e_k}) L_{f_l}^* = \sum_k u_{(i,l),(k,j)} \Theta(L_{e_k}) L_{f_l}^*.$$

Summing over $l$, we get

$$L_{f_j}^* \Theta(L_{e_i}) (\sum_l L_{f_l} L_{f_l}^*) = \sum_{k,l} u_{(i,l),(k,j)} \Theta(L_{e_k}) L_{f_l}^*.$$

Now, $\sum_l L_{f_l} L_{f_l}^* = I - P$ where $P$ is the projection onto the subspace $\mathbb{C} \oplus E \oplus (E \otimes E) \oplus \ldots$. Note that $P$ is left invariant under the operators in the algebra generated by $\{L_{e_i} \ : \ 1 \le i \le n\}$ and, in particular, by $\Theta(L_{e_i})$. Thus $L_{f_j}^* \Theta(L_{e_i}) P = L_{f_j}^* P \Theta(L_{e_i}) P = 0 = \sum_{k,l} u_{(i,l),(k,j)} \Theta(L_{e_k}) L_{f_l}^* P$. This completes the proof of the corollary. $\Box$

**Proposition 4.6** *Suppose $(z, w) \in \Omega_u^0 \cap (\mathbb{B}_n \times \mathbb{B}_m)$. Then there is a automorphism $\tilde{\Theta}_z$ of $\mathcal{A}_u$ that is unitarily implemented and such that, for every $X \in \mathcal{A}_u$,*

$$\alpha_{(0,w)}(\tilde{\Theta}_z^{-1}(X)) = \alpha_{(z,w)}(X) \qquad (17)$$

*where $\alpha_{(z,w)}$ is the character associated with $(z, w)$ by Proposition 3.1.*

**Proof.** Let $U$ be the unitary operator implementing $\Theta$. We can view $\mathcal{F}(n, m, u)$ as the sum

$$\mathcal{F}(n, m, u) = \sum_k F^{\otimes k} \otimes \mathcal{F}(E)$$

where $\mathcal{F}(E) = \mathbb{C} \oplus E \oplus (E \otimes E) \oplus \cdots$. We now let $V$ be the unitary operator whose restriction to $F^{\otimes k} \otimes \mathcal{F}(E)$ is $I_k \otimes U$ (where $I_k$ is the identity operator on $F^{\otimes k}$). It is easy to check that, for every $f_j$,

$$V L_{f_j} V^* = L_{f_j}.$$

Now, fix $i$. We shall show, by induction, that, for every $k$ and every $\xi \in F^{\otimes k} \otimes \mathcal{F}(E)$,

$$(I_k \otimes U) L_{e_i} \xi = \Theta(L_{e_i})(I_k \otimes U)\xi. \qquad (18)$$

For $k = 0$ this is just the fact that $U$ implements $\Theta$. Suppose we know this for $k$ and fix $f_j \in F$. Then, for $\xi \in F^{\otimes k} \otimes \mathcal{F}(E)$ we have,

$$(I_{k+1} \otimes U) L_{e_i} L_{f_j} \xi = \sum_{k,l} u_{(i,j),(k,l)} (I_{k+1} \otimes U) L_{f_l} L_{e_k} \xi$$

$$= \sum_{k,l} u_{(i,j),(k,l)} L_{f_l} (I_k \otimes U) L_{e_k} \xi.$$

Applying the induction hypothesis, this is equal to $\sum_{k,l} u_{(i,j),(k,l)} L_{f_l} \Theta(L_{e_k})(I_k \otimes U)\xi$. Using (13), this is $\Theta(L_{e_i}) L_{f_j} (I_k \otimes U)\xi = \Theta(L_{e_i})(I_k \otimes U) L_{f_j} \xi$. Since $F^{\otimes(k+1)} \otimes \mathcal{F}(E)$ is spanned by elements of the form $L_{f_j}\xi$ (as above) the equality follows. From the relations of Lemma 4.4 it follows that the map $\tilde{\Theta}_z : X \to VXV^*$ defines a unitary endomorphism of $\mathcal{A}_u$. Since $\Theta$ is an automorphism of $\mathcal{A}_n$ it follows that $\tilde{\Theta}_z$ gives the desired automorphism. □

Clearly, in Proposition 4.6, we can interchange $z$ and $w$ to get the following, where $\Theta_{z,w} = \tilde{\Theta}_z \tilde{\Theta}_w$.

**Proposition 4.7** *Suppose $(z, w) \in \Omega_u^0 \cap (\mathbb{B}_n \times \mathbb{B}_m)$. Then there is a unitary automorphism $\Theta_{z,w}$ of $\mathcal{L}_u$ which is a homeomorphism with respect to the $w^*$-topologies and which restricts to an automorphism of $\mathcal{A}_u$. Moreover, for every $X \in \mathcal{L}_u$,*

$$\alpha_{(0,0)}(\Theta_{z,w}^{-1}(X)) = \alpha_{(z,w)}(X) \tag{19}$$

*where $\alpha_{(z,w)}$ is the character associated with $(z, w)$ as in Proposition 3.1.*

An automorphism $\Psi$ of $\mathcal{A}_u$, defines a map on the character space of $\mathcal{A}_u$, namely $\phi \mapsto \phi \circ \Psi^{-1}$. Thus using Proposition 3.1 we have a homeomorphism $\theta_\Psi$ of $\Omega_u$. Also, since $\Omega_u \cap (\mathbb{B}_n \times \mathbb{B}_m)$ is the interior of $\Omega_u$, $\theta_\Psi$ maps $\Omega_u \cap (\mathbb{B}_n \times \mathbb{B}_m)$ onto itself.

Similarly, if $\Psi$ is an automorphism of $\mathcal{L}_u$ which is a homeomorphism with respect to the $w^*$-topologies, then $\theta_\Psi$ is a homeomorphism of $\Omega_u \cap (\mathbb{B}_n \times \mathbb{B}_m)$.

In the following theorem we identify the relative interior of the core as the orbit of $(0, 0)$ under the group of maps $\theta_\Psi$ associated with automorphisms $\Psi$.

**Theorem 4.8** *For $(z, w) \in \mathbb{B}_n \times \mathbb{B}_m$ the following conditions are equivalent.*

*(1)* $(z, w) \in \Omega_u^0$.

(2) *There exists a completely isometric automorphism* $\Psi$ *of* $\mathcal{L}_u$ *that is a homeomorphism with respect to the* $w^*$*-topologies and restricts to an automorphism of* $\mathcal{A}_u$*, such that* $\theta_\Psi(0,0) = (z,w)$*.*

(3) *There exists an algebraic automorphism* $\Psi$ *of* $\mathcal{A}_u$ *such that* $\theta_\Psi(0,0) = (z,w)$*.*

**Proof.** The proof that (1) implies (2) follows from Proposition 4.7. Clearly (2) implies (3). It is left to show that (3) implies (1).

Given a point $(z,w) \in \Omega_u$, we saw in Lemma 3.5 that, for every $(\lambda,\mu)$ satisfying (11) there is a homomorphism $\rho_{z,w,\lambda,\mu} : \mathbb{C}[\mathbb{F}_u^+] \to T_2$. For $(z,w) = (0,0)$ equation (11) holds for every pair $(\lambda,\mu)$. Since $\rho_{0,0,\lambda,\mu}$ vanishes off a finite dimensional subspace, it is a bounded homomorphism. In fact, for every $(\lambda,\mu)$, $\|\rho_{0,0,\lambda,\mu}\| \leq 1 + \|\lambda\| + \|\mu\|$.

Given $\Psi$ and $(z,w)$ as in (3), for every $(\lambda,\mu) \in \mathbb{C}^n \times \mathbb{C}^m$, $\rho_{0,0,\lambda,\mu} \circ \Psi^{-1}$ is a homomorphism on $\mathbb{C}[\mathbb{F}_u^+]$ and, thus, it is of the form $\rho_{z,w,\lambda',\mu'}$ for some (unique) $(\lambda',\mu')$ satisfying (11). Write $\psi(\lambda,\mu) = (\lambda',\mu')$ and note that this defines a continuous map. To prove the continuity, suppose $(\lambda_n,\mu_n) \to (\lambda,\mu)$ and write $\rho_n$ for $\rho_{0,0,\lambda_n,\mu_n}$ and $\rho$ for $\rho_{0,0,\lambda,\mu}$. Then (using the estimate on the norm of $\rho_{0,0,\lambda,\mu}$) there is some $M$ such that $\|\rho_n\| \leq M$ for all $n$ and $\|\rho\| \leq M$. For every $Y \in \mathbb{C}[\mathbb{F}_u^+]$, $\rho_n(Y) \to \rho(Y)$. Now fix $X \in \mathcal{A}_u$ and $\epsilon > 0$. There is some $Y \in \mathbb{C}[\mathbb{F}_u^+]$ such that $\|X - Y\| \leq \epsilon$ and there is some $N$ such that for $n \geq N$ $\|\rho_n(Y) - \rho(Y)\| \leq \epsilon$. Thus, for such $n$, $\|\rho_n(X) - \rho(X)\| \leq (2M+1)\epsilon$. Setting $X = \Psi(L_{e_i})$, we get $\lambda_n' \to \lambda'$ and similarly for $\mu'$.

If $(z,w)$ is not in $\Omega_u^0$, then the set of all $(\lambda,\mu)$ satisfying (11) is a subspace of $\mathbb{C}^n \times \mathbb{C}^m$ of dimension strictly smaller than $n+m$ and, as is shown above, it contains the continuous image (under the injective map $\psi$) of $\mathbb{C}^n \times \mathbb{C}^m$. This is impossible. $\square$

# 5 Isomorphic algebras

In this section we shall find conditions for algebras $\mathcal{A}_u$ and $\mathcal{A}_v$ to be (isometrically) isomorphic. The characterisation also applies to the weak star closed algebras $\mathcal{L}_u$.

We start by considering a special type of isomorphism. We shall now assume that the set $\{n,m\}$ for both algebras is the same. In fact, by interchanging $E$ and $F$, we can assume that the corresponding dimensions are the same and the algebras are defined on $\mathcal{F}(n,m,u)$ and $\mathcal{F}(n,m,v)$ respectively. This assumption will be in place in the discussion below up to the end of Lemma 5.5.

The algebra $\mathcal{A}_u$ carries a natural $\mathbb{Z}_+^2$-grading, with the $(k,l)$ labeled subspace being spanned by products of the form $L_{e_{i_1}} L_{e_{i_2}} \dots L_{e_{i_k}} L_{f_{i_1}} L_{f_{i_2}} \dots L_{f_{i_l}}$. Also, the total length of such operators provides a natural $\mathbb{Z}_+$-grading. Note that an algebra isomorphism $\Psi : \mathcal{A}_u \to \mathcal{A}_v$ which respects the $\mathbb{Z}_+$-grading is determined by a linear map between the spans of the generators $L_{e_1}, \dots, L_{e_n}, L_{f_1}, \dots, L_{f_m}$. Here we use the same notation for the generators of $\mathcal{A}_u$ and $\mathcal{A}_v$. Such an isomorphism will be called *graded*.

We now consider two types of graded isomorphisms, namely, either bigraded, as in the following definition, or, in case $n = m$, bigraded after relabeling generators.

**Definition 5.1** *(i) An isomorphism $\Psi : \mathcal{A}_u \to \mathcal{A}_v$ is said to be bigraded isomorphism if there are unitary matrices $A$ $(n \times n)$ and $B$ $(m \times m)$ such that*

$$\Psi(L_{e_i}) = \sum_j a_{i,j} L_{e_j} \ , \quad \Psi(L_{f_k}) = \sum_l b_{k,l} L_{f_l}.$$

*(ii) If $m = n$ and $\Psi$ is a graded isomorphism such that*

$$\Psi(L_{e_i}) = \sum_j a_{i,j} L_{f_j} \ , \quad \Psi(L_{f_k}) = \sum_l b_{k,l} L_{e_l}$$

*for $n \times n$ unitary matrices $A$ and $B$ then we say that $\Psi$ is a graded exchange isomorphism.*

*We write $\Psi_{A,B}$ for the bigraded isomorphism (as in (i)) and $\tilde{\Psi}_{A,B}$ for the graded exchange isomorphism.*

Abusing notation, we write $\Psi(e_i) = \sum_j a_{i,j} e_j$ instead of $\Psi(L_{e_i}) = \sum_j a_{i,j} L_{e_j}$ for a bigraded isomorphism (and similarly for the other expressions).

For unitary permutation matrices the following lemma was proved in [10, Theorem 5.1(iii)].

**Lemma 5.2** *(i) If $\Psi_{A,B}$ is a bigraded isomorphism then*

$$(A \otimes B)v = u(A \otimes B) \tag{20}$$

*where $A \otimes B$ is the $mn \times mn$ matrix whose $(i,j),(k,l)$ entry is $a_{i,k} b_{j,l}$.*

*(ii) If $m = n$ and $\tilde{\Psi}_{A,B}$ is a graded exchange isomorphism then*

$$(A \otimes B)\tilde{v} = u(A \otimes B) \tag{21}$$

*where $\tilde{v}_{(i,j),(k,l)} = \bar{v}_{(l,k),(j,i)}$.*

**Proof.** Assume $\Psi = \Psi_{A,B}$ is a bigraded isomorphism. For $i, j$,

$$\Psi(e_i \otimes f_j) = (\sum_k a_{i,k} e_k) \otimes (\sum_l b_{j,l} f_l) = \sum_{k,l} (A \otimes B)_{(i,j),(k,l)} e_k \otimes f_l =$$

$$\sum_{k,l,r,t} (A \otimes B)_{(i,j),(k,l)} v_{(k,l),(r,t)} f_t \otimes e_r = \sum_{r,t} ((A \otimes B)v)_{(i,j),(r,t)} f_t \otimes e_r.$$

On the other hand,

$$\Psi(e_i \otimes f_j) = \Psi(\sum_{k,l} u_{(i,j),(k,l)} f_l \otimes e_k) = \sum_{k,l,t,r} u_{(i,j),(k,l)} b_{l,t} a_{k,r} f_t \otimes e_r =$$

$$\sum_{t,r} (u(A \otimes B))_{(i,j),(r,t)} f_t \otimes e_r.$$

This proves equation (20). A similar argument can be used to verify equation (21). $\square$

**Definition 5.3** *If $u, v$ are $mn \times mn$ unitary matrices and there exist unitary matrices $A$ and $B$ satisfying (20), we say that $u$ and $v$ are product unitary equivalent.*

Now suppose that $A$ and $B$ are unitary matrices satisfying (20). The same computation as in Lemma 5.2 shows that $W_{A,B} : E \otimes_u F \to E \otimes_v F$ defined by

$$W_{A,B}(e_i \otimes f_j) = \sum_{k,l} (A \otimes B)_{(i,j),(k,l)} e_k \otimes f_l$$

is a well defined unitary operator. Here the notation $E \otimes_u F$ indicates that this is $E \otimes F$ as a subspace of $\mathcal{F}(n, m, u)$. Similarly, one defines a unitary operator, also denoted $W_{A,B}$, from $E^{\otimes k} \otimes F^{\otimes l}$ in $\mathcal{F}(n, m, u)$ to $E^{\otimes k} \otimes F^{\otimes l}$ in $\mathcal{F}(n, m, v)$ by

$$W_{A,B}(e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes f_{j_1} \otimes \cdots \otimes f_{j_l}) =$$

$$\sum a_{i_1,r_1} \cdots a_{i_k,r_k} b_{j_1,t_1} \cdots b_{j_l,t_l} e_{r_1} \otimes \cdots \otimes e_{r_k} \otimes f_{t_1} \otimes \cdots \otimes f_{t_l}.$$

This gives a well defined unitary operator

$$W_{A,B} : \mathcal{F}(n, m, u) \to \mathcal{F}(n, m, v).$$

**Lemma 5.4** *For every $i, j$, write $Ae_i = \sum_k a_{i,k} e_k$ and $Bf_j = \sum_l b_{j,l} f_l$. Then, for $g_1, g_2, \ldots, g_r$ in $\{e_1, \ldots, e_n, f_1, \ldots, f_m\}$,*

$$W_{A,B}(g_1 \otimes g_2 \otimes \cdots \otimes g_r) = Cg_1 \otimes Cg_2 \otimes \cdots \otimes Cg_r \tag{22}$$

*where $Cg_i = Ag_i$ if $g_i \in \{e_1, \ldots, e_n\}$ and $Cg_i = Bg_i$ if $g_i \in \{f_1, \ldots, f_m\}$.*

**Proof.** If the $g_i$'s are ordered such that the first ones are from $E$ and the following vectors are from $F$, then the result is clear from the definition of $W_{A,B}$. Since we can get any other arrangement by starting with one of this kind and interchanging pairs $g_l, g_{l+1}$ successively (with $g_l \in \{e_1, \ldots, e_n\}$ and $g_{l+1} \in \{f_1, \ldots, f_m\}$), it is enough to show that that if (22) holds for a given arrangement of $e$'s and $f$'s and we apply such an interchange, then it still holds. So, we assume $g_l = e_k$, $g_{l+1} = f_s$ and we write $g' = g_1 \otimes \cdots \otimes g_{l-1}$, $g'' = g_{l+2} \otimes \cdots \otimes g_r$, $Cg' = Cg_1 \otimes \cdots \otimes Cg_{l-1}$ and $Cg'' = Cg_{l+2} \otimes \cdots \otimes Cg_r$ and compute

$$W_{A,B}(g' \otimes f_s \otimes e_k \otimes g'') = W_{A,B}(\sum_{i,j} \bar{u}_{(i,j),(k,s)} g' \otimes e_i \otimes f_j \otimes g'').$$

Using our assumption, this is equal to

$$\sum_{i,j} \bar{u}_{(i,j),(k,s)} Cg' \otimes (\sum_t a_{i,t} e_t) \otimes (\sum_q b_{j,q} f_q) \otimes Cg'' =$$

$$\sum_{i,j,t,q} \bar{u}_{(i,j),(k,s)} a_{i,t} b_{j,q} Cg' \otimes e_t \otimes f_q \otimes Cg'' =$$

$$\sum_{i,j,t,q,d,p} \bar{u}_{(i,j),(k,s)} a_{i,t} b_{j,q} v_{(t,q),(d,p)} Cg' \otimes f_p \otimes e_d \otimes Cg'' =$$

$$\sum (u^*)_{(k,s),(i,j)} (A \otimes B)_{(i,j),(t,q)} v_{(t,q),(d,p)} Cg' \otimes f_p \otimes e_d \otimes Cg'' =$$

$$\sum_{d,p} (A \otimes B)_{(k,s),(d,p)} Cg' \otimes f_p \otimes e_d \otimes Cg'' = \sum_{d,p} a_{k,d} b_{s,p} Cg' \otimes f_p \otimes e_d \otimes Cg'' =$$

$$Cg' \otimes Bf_s \otimes Ae_k \otimes Cg''$$

completing the proof. $\square$

The following lemma was proved in [10, Section 7] and it shows that the necessary conditions of Lemma 5.2 are also sufficient conditions on $A \otimes B$ for the existence of a unitarily implemented isomorphism $\Psi_{A,B}$.

**Lemma 5.5** *For unitary matrices $A, B$ satisfying (20) and $X \in \mathcal{A}_u$, the map*

$$X \mapsto W_{A,B} X W_{A,B}^*$$

*is the bigraded isomorphism $\Psi_{A,B} : \mathcal{A}_u \to \mathcal{A}_v$. Moreover $\Psi_{A,B}$ extends to a unitary isomorphism $\mathcal{L}_u \to \mathcal{L}_v$, and similar statements holds for graded exchange isomorphisms (when $m = n$).*

**Proof.** It will suffice to show the equality

$$\Psi_{A,B}(X) W_{A,B} = W_{A,B} X$$

for $X = L_{e_i}$ and for $X = L_{f_j}$. Let $X = L_{f_j}$ and apply both sides of the equation to $e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes f_{j_1} \otimes \cdots \otimes f_{j_l}$. Using Lemma 5.4, we get

$$\begin{aligned}
&\Psi_{A,B}(L_{f_j}) W_{A,B}(e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes f_{j_1} \otimes \cdots \otimes f_{j_l}) \\
&= \sum_r b_{j,r} L_{f_r}(Ae_{i_1} \otimes \cdots \otimes Ae_{i_k} \otimes Bf_{j_1} \otimes \cdots \otimes Bf_{j_l}) \\
&= Bf_j \otimes Ae_{i_1} \otimes \cdots \otimes Ae_{i_k} \otimes Bf_{j_1} \otimes \cdots \otimes Bf_{j_l} \\
&= W_{A,B}(f_j \otimes e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes f_{j_1} \otimes \cdots \otimes f_{j_l}) \\
&= W_{A,B} L_{f_j}(e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes f_{j_1} \otimes \cdots \otimes f_{j_l}).
\end{aligned}$$

This proves the equality for $X = L_{f_j}$. The proof for $X = L_{e_i}$ is similar. $\square$

At this point we drop our assumption that the set $\{n, m\}$ is the same for both algebras and write $\{n', m'\}$ for the dimensions associated with $\mathcal{A}_v$. We shall see in Proposition 5.8 (and Remark 5.11(i)) that, if the algebras are isomorphic, then necessarily $\{n, m\} = \{n', m'\}$.

Given an isomorphism $\Psi : \mathcal{A}_u \to \mathcal{A}_v$ we get a homeomorphism $\theta_\Psi : \Omega_u \to \Omega_v$ (as in the discussion preceeding Theorem 4.8). The arguments used in the proof of Theorem 4.8 to show that part (3) implies part (1) apply also to isomorphisms and thus, $\theta_\Psi(0, 0) \in \Omega_v^0$.

**Proposition 5.6** *Let $\Psi : \mathcal{A}_u \to \mathcal{A}_v$ be an (algebraic) isomorphism. Then $\theta_\Psi(\Omega_u^0) = \Omega_v^0$ and $\theta_\Psi(\Omega_u^0 \cap (\mathbb{B}_n \times \mathbb{B}_m)) = \Omega_v^0 \cap (\mathbb{B}_n \times \mathbb{B}_m)$.*

**Proof.** Fix $(z, w)$ in $\Omega_u^0$ and use Theorem 4.8 to get an automorphism $\Phi$ of $\mathcal{A}_u$ such that $\theta_\Phi(0, 0) = (z, w)$. But then $\theta_{\Psi\circ\Phi}(0, 0) = \theta_\Psi(z, w)$ and, as we noted above, this implies that $\theta_\Psi(z, w) \in \Omega_v^0$. It follows that $\theta_\Psi(\Omega_u^0) \subseteq \Omega_v^0$ and, applying this to $\Psi^{-1}$, the lemma follows. $\square$

**Lemma 5.7** *The map $\theta_\Psi$ is a biholomorphic map.*

**Proof.** The coordinate functions for $\theta_\Psi$ are $(z, w) \mapsto \alpha_{(z,w)}(\Psi^{-1}(e_i))$ (and $(z, w) \mapsto \alpha_{(z,w)}(\Psi^{-1}(f_j))$) where $\alpha_{(z,w)}$ is the character associated with $(z, w)$ by Proposition 3.1. For every $Y \in \mathbb{C}[\mathbb{F}_v^+]$, $\alpha_{(z,w)}(Y)$ is a polynomial in $(z, w)$ (for $(z, w) \in \Omega_v$) and, therefore, an analytic function. Each $X \in \mathcal{A}_v$ is a norm limit of elements in $\mathbb{C}[\mathbb{F}_v^+]$ and, thus, $\alpha_{(z,w)}(X)$ is an analytic function being a uniform limit of analytic functions on compact subsets of $\Omega_v$. Hence, for every $(z, w) \in \Omega_v$, there is a power series that converges in some, non empty, circular, neighborhood $C$ of $(z, w)$ that represents $\alpha_{(z,w)}(X)$ on $C \cap \Omega_v$. Taking for $X$ the operators $\Psi^{-1}(e_i)$ and $\Psi^{-1}(f_j)$, we see that $\theta$ is analytic. The same arguments apply to $\theta^{-1}$. $\square$

The facts in the following proposition obtained in [10] in the case of permutation matrices.

**Proposition 5.8** *Let $\Psi : \mathcal{A}_u \to \mathcal{A}_v$ be an algebraic isomorphism and let $\theta_\Psi : \Omega_u \to \Omega_v$ be the associated map between the character spaces. Suppose $\theta_\Psi(0, 0) = (0, 0)$. Then we have the following.*

*(1) $\{n, m\} = \{n', m'\}$ and we shall assume that $n = n'$ and $m = m'$ (interchanging $E$ and $F$ and changing $u$ to $u^*$ if necessary).*

*(2) There are unitary matrices $U$ $(n \times n)$ and $V$ $(m \times m)$ such that $\theta_\Psi(z, w) = (Uz, Vw)$ for $(z, w) \in \Omega_u$. (If $n = m$ it is also possible that $\theta_\Psi(z, w) = (Vw, Uz)$.)*

*(3) If $\Psi$ is an isometric isomorphism, then $\Psi$ is a bigraded isomorphism. (Or, if $m = n$, it may be a graded exchange isomorphism).*

**Proof.** The proof of Proposition 6.3 in [10] giving (1) and (2) in the permutation case is based essentially on Schwarz's lemma for holomorphic map from the unit disc. It applies without change to the case of unitary matrices.

For (3) we may assume $m = m'$ and $n = n'$. From (2) we have for each $\Phi(L_{e_i}) = L_{Ue_i} + X$ where $X$ is a sum of higher order terms. Since $\Phi(L_{e_i})$ is a contraction and $L_{Ue_i}$ is an isometry it follows, as in the proof of Voiculescu's theorem, that $X = 0$. Similarly, $\Phi(L_{f_j}) = L_{Vf_j}$ and it follows that $\Phi$ is bigraded. $\square$

Since every graded isomorphism $\Psi$ satisfies $\theta_\Psi(0,0) = (0,0)$, we conclude the following.

**Corollary 5.9** *Every graded isometric isomorphism is bigraded if $n \neq m$ and otherwise is either bigraded or is a graded exchange isomorphism.*

**Theorem 5.10** *The following statements are equivalent for unitary matrices $u, v$ in $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$.*

*(i) There is an isometric isomorphism $\Psi : \mathcal{A}_u \to \mathcal{A}_v$.*

*(ii) There is a graded isometric isomorphism from $\Psi : \mathcal{A}_u \to \mathcal{A}_v$.*

*(iii) The matrices $u, v$ are product unitary equivalent or (in case $n = m$) the matrices $u, \tilde{v}$ are product unitary equivalent, where $\tilde{v}_{(i,j),(k,l)} = \bar{v}_{(l,k),(j,i)}$.*

*(iv) There is an isometric $w^*$-continuous isomorphism $\Gamma : \L_u \to \L_v$.*

**Proof.** Given $\Psi$ in (i), let $(z, w) = \theta_\Psi(0,0)$. By Proposition 5.6 $(z, w)$ lies in the interior of $\Omega_v^0$. By Theorem 4.8 there is a completely isometric automorphism $\Phi$ of $\mathcal{A}_v$ such that $\theta_\Phi(0,0) = (z, w)$ and, therefore, $\theta_{\Phi^{-1}\circ\Psi}(0,0) = (0,0)$. By Proposition 5.8, $\Phi^{-1} \circ \Psi$ is a graded isometric isomorphism and (ii) holds. Lemma 5.2 shows that (ii) implies (iii) and Lemma 5.5 that (iii) implies (i).

Finally, (iii) implies (iv) follows from Lemma 5.5, and (iv) implies (ii) is entirely similar to (i) implies (ii). $\square$

**Remark 5.11** *The argument at the beginning of the proof of Theorem 5.10 shows that, whenever $\mathcal{A}_u$ and $\mathcal{A}_v$ are isomorphic, we have $\{n, m\} = \{n', m'\}$.*

**Theorem 5.12** *For $n \neq m$ the isometric automorphisms of $\mathcal{A}_u$ are of the form $\Psi_{A,B}\Theta_{z,w}$ where $(z, w) \in \Omega_u^0$ and $(A \otimes B)u = u(A \otimes B)$. In case $n = m$ the isometric automorphisms include, in addition, those of the form $\tilde{\Psi}_{A,B}\Theta_{z,w}$ where $(A \otimes B)\tilde{u} = u(A \otimes B)$.*

# 6 Special cases

## 6.1 The case $n = m = 2$

Even in the low dimensions $n = m = 2$ there are many isomorphism classes and special cases. Note that the product unitary equivalence class orbit $\mathcal{O}(u)$ of the $4 \times 4$ unitary matrix $u$ takes the form

$$\mathcal{O}(u) = \{(A \otimes B)u(A \otimes B)^* : A, B \in SU_2(\mathbb{C})\},$$

and so the product unitary equivalence classes are parametrised by the set of orbits, $U_4(\mathbb{C})/Ad(SU_2(\mathbb{C}) \times SU_2(\mathbb{C}))$. This set admits a 10-fold parametrisation, since, as is easily checked, $U_4(\mathbb{C})$ and $SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$ are real algebraic varieties of dimension 16 and 6 respectively. It follows that the isometric isomorphism types of the algebras $\mathcal{A}_u$ admit a 10 fold real parametrisation, with coincidences only for pairs $\mathcal{O}(u), \mathcal{O}(v)$ with $u = \tilde{v}$

We now look at some special cases in more detail. Let $d = dim Ker(u - I)$.

**Case I:** $d = 0$

For every $(z, w) \in \overline{\mathbb{B}}_2 \times \overline{\mathbb{B}}_2$, we have $(z, w) \in \Omega_u$ if and only if the vector $(z_1w_1, z_1w_2, z_2w_1, z_2w_2)^t$ lies in $Ker(u - I)$. Thus, in case I, $\Omega_u$ is as small as possible and is equal to

$$\Omega_{min} := (\overline{\mathbb{B}}_2 \times \{0\}) \cup (\{0\} \times \overline{\mathbb{B}}_2).$$

It follows from Lemma 3.4 that, in this case,

$$\Omega_u^0 = \{(0, 0)\}.$$

By Proposition 5.8 every isometric automorphism of $\mathcal{A}_u$ is graded and the isometric automorphisms of $\mathcal{A}_u$ are given by pairs $(A, B)$ of unitary matrices such that $A \otimes B$ either commutes with $u$ or intertwines $u$ and $\tilde{u}$.

**Case II:** $d = 1$

When $d = 1$ it still follows from Lemma 3.4 that

$$\Omega_u^0 = \{(0, 0)\}$$

but now it is possible for $\Omega_u$ to be larger than $\Omega_{min}$. In fact, if the non zero vector $(a, b, c, d)^t$ spanning $Ker(u - I)$ satisfies $ad \neq bc$ then $\Omega_u = \Omega_{min}$ but if $ad = bc$ then the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is of rank one and can be written as $(z_1, z_2)^t(w_1, w_2)$. Thus, $(z, w) \in V_u$ and $\Omega_u$ contains some $(z, w)$ with non zero $z$ and $w$.

Since $\Omega_u^0 = \{(0, 0)\}$, it is still true that isometric isomorphisms and automorphisms of these algebras are graded.

**Case III:** $d = 2$
When $d = 2$ it is possible that $\Omega_u^0$ will contain non zero vectors $(z, w)$ but, as Lemma 3.4 shows, it does not contain a vector with both $z \neq 0$ and $w \neq 0$. All other possibilities may occur. For example write $u_1, u_2$ and $u_3$ for the three diagonal matrices:

$$u_1 = diag(1, -1, -1, 1), \;\; u_2 = diag(1, -1, 1, -1)$$

and

$$u_3 = diag(1, 1, -1, -1).$$

Using the definition of the core, we easily see that

$$\Omega_{u_1}^0 = \{(0, 0)\}, \;\; \Omega_{u_2}^0 = \{(0, 0, w_1, 0) : \; |w_1| \leq 1\}$$

and

$$\Omega_{u_3}^0 = \{(z_1, 0, 0, 0) : \; |z_1| \leq 1\}.$$

Thus, the only isometric automorphisms of $\mathcal{A}_{u_1}$ are graded, the isometric automorphisms of $\mathcal{A}_{u_2}$ are formed by composing graded automorphisms with automorphisms of the type described in Proposition 4.7 (with $z = (0, 0)$ and $w = (w_1, 0)$). Similarly, for the automorphisms of $\mathcal{A}_{u_3}$, we use Proposition 4.6.

**Case IV:** $d = 3$
In this case we are able to obtain an explicit 2-fold parametrization of the isomorphism types of the algebra $\mathcal{A}_u$.

Every $4 \times 4$ unitary matrix $u$ with $dim(Ker(u - I)) = 3$ is determined by a unit eigenvector $x$ and its (different from 1) eigenvalue. So that $ux = \lambda x$, $\|x\| = 1$, $|\lambda| = 1$ and $\lambda \neq 1$. Suppose $u$ and $v$ are product unitary equivalent; that is

$$(A \otimes B)u = v(A \otimes B)$$

for unitary matrices $A, B$, and write $x, \lambda$ for the unit eigenvector and eigenvalue of $u$. (Of course, $x$ is determined only up to a multiple by a scalar of absolute value 1). Then $y = (A \otimes B)x$ is a unit eigenvector of $v$ with eigenvalue $\lambda$. For unit vectors $x, y$ (in $\mathbb{C}^4$) we write $x \sim y$ if there are unitary $(2 \times 2)$ matrices $A, B$ with $y = (A \otimes B)x$. For the statement of the next lemma recall that the entries of the vectors $x$ and $y$ in $\mathbb{C}^4$ are indexed by $\{(i,j) : 1 \leq i, j \leq 2\}$.

**Lemma 6.1** *For a vector $x = \{x_{(i,j)}\}$ in $\mathbb{C}^4$, write $c(x)$ for the $2 \times 2$ matrix*

$$c(x) = \begin{pmatrix} x_{(1,1)} & x_{(1,2)} \\ x_{(2,1)} & x_{(2,2)} \end{pmatrix}.$$

*Then $x \sim y$ if and only if there are unitary matrices $A, B$ such that $c(x) = Ac(y)B$. (In this case, we shall write $c(x) \sim c(y)$.)*

**Proof.** Suppose $y = (A \otimes B)x$ for some unitary matrices $A = (a_{i,j})$ and $B = (b_{i,j})$. Then $c(y)_{i,j} = y_{(i,j)} = \sum (A \otimes B)_{(i,j),(k,l)} x_{(k,l)} = \sum_{k,l} a_{i,k} b_{j,l} c(x)_{k,l} = (Ac(x)B)_{i,j}$. $\square$

Using the polar decomposition $c(x) = U|c(x)|$ and diagonalizing $|c(x)| = V \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} V^*$, we find that $c(x) \sim \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = c(y)$ where $y = (a, 0, 0, d)$ and $a, d \geq 0$. Then $a, d$ (the eigenvalues of $|c(x)|$) are uniquely determined once we choose them such that $a \leq d$ and, if $\|x\| = 1$, then $a^2 + d^2 = 1$ (so that $0 \leq a \leq 1/\sqrt{2}$ and $a$ determines $d$). In this way, we associate to each unitary matrix $u$ as above a pair $(a, \lambda)$ with $0 \leq a \leq 1/\sqrt{2}$, $\lambda \neq 1$ and $|\lambda| = 1$. Using Lemma 6.1 and the discussion preceeding it, we have the following.

**Corollary 6.2** *For every $4 \times 4$ unitary matrix $u$ with $dim(Ker(u - I)) = 3$, there are numbers $\lambda$ (with $|\lambda| = 1$ and $\lambda \neq 1$) and $a$ ($0 \leq a \leq 1/\sqrt{2}$) such that $u$ and $v$ are product unitary equivalent if and only if they have the same $a, \lambda$.*

**Proof.** Let $u$ and $v$ be unitary matrices with $dim(Ker(u - I)) = 3$ and let $(a, \lambda)$, $(b, \mu)$ be the pairs associated to $u$ and $v$ (respectively) as above. Also write $x$ for the unit eigenvector of $u$ associated to the eigenvalue $\lambda$ and let $y$ be the unit eigenvector of $v$ associated to $\mu$.

Suppose $u$ and $v$ are product unitarily equivalent. Then they are unitary equivalent and, thus, $\lambda = \mu$. Write $(A \otimes B)u = v(A \otimes B)$ for unitary matrices

$A, B$. As we saw above, $y$ can be chosen to be $(A \otimes B)x$ so that $x \sim y$ and, by Lemma 6.1, $c(x) \sim c(y)$. It follows that $a = b$.

Conversely, assume that $a = b$ and $\lambda = \mu$. Then $c(x) \sim c(y)$ and, thus, $x \sim y$ so we can write $y = (A \otimes B)x$ for some unitary matrices $A, B$. Writing $v' = (A \otimes B)u(A \otimes B)^*$, we find that $y$ is the unit eigenvector of $v'$ associated to $\lambda$. Thus $v = v'$, completing the proof. $\square$

For every $a, \lambda$ as in Corollary 6.2 we let $u(a, \lambda)$ be the following $4 \times 4$ matrix.

$$u(a,\lambda) = \begin{pmatrix} (\lambda-1)a^2+1 & 0 & 0 & (\lambda-1)a(1-a^2)^{1/2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ (\lambda-1)a(1-a^2)^{1/2} & 0 & 0 & \lambda+(1-\lambda)a^2 \end{pmatrix}.$$

It is a straightforward computation to verify that $dim(Ker(u - I)) = 3$ and that $\lambda$ is an eigenvalue of $u(a, \lambda)$ with eigenvector $(a, 0, 0, (1-a^2)^{1/2})^t$. Thus the pair associated to $u(a, \lambda)$ is $a, \lambda$ and we have

**Corollary 6.3** *Every matrix $u$ with $dim(Ker(u-I)) = 3$ is product unitary equivalent to a unique matrix of the form $u(a, \lambda)$ (with $0 \leq a \leq 1/\sqrt{2}$, $|\lambda| = 1$ and $\lambda \neq 1$).*

Using the definition of the core, we immediately get the following.

**Proposition 6.4** *If $a = 0, |\lambda| = 1, \lambda \neq 1$, then $\Omega_{u(0,\lambda)}$ is the union*

$$\{(z_1, z_2, w_1, 0) : z \in \mathbb{B}_2; |w_1| \leq 1\} \cup \{(z_1, 0, w_1, w_2) : w \in \mathbb{B}_2; |z_1| \leq 1\},$$

*and*

$$\Omega^0_{u(0,\lambda)} = \{(z_1, 0, w_1, 0) : |z_1| \leq 1; |w_1| \leq 1\}.$$

*If $a \neq 0$ then*

$$\Omega_{u(a,\lambda)} = \{(z_1, z_2, w_1, w_2) : az_1w_1 + (1-a^2)^{1/2}z_2w_2 = 0, (z, w) \in \overline{\mathbb{B}}_2 \times \overline{\mathbb{B}}_2\}$$

*and*

$$\Omega^0_{u(a,\lambda)} = \{(0, 0)\}.$$

**Proof.** The space $\Omega_{u(a,\lambda)}$ consists of points $(z, w)$ for which

$$(z_1w_1, z_1w_2, z_2w_1, z_2w_2)^t = u(a, \lambda)(z_1w_1, z_1w_2, z_2w_1, z_2w_2)^t,$$

that is, for which

$$((\lambda - 1)a^2 + 1)z_1w_1 + (\lambda - 1)a(1 - a^2)^{1/2} z_2w_2 = z_1w_1,$$
$$(\lambda - 1)a(1 - a^2)^{1/2} z_1w_1 + (\lambda + (\lambda - 1)a^2)z_2w_2 = z_2w_2.$$

If $a = 0$ this implies $z_2w_2 = 0$, while if $a \neq 0$ then $(z_1w_1, 0, 0, z_2w_2)$ is a fixed vector for $u(a, \lambda)$ and so for some scalar $\mu$ $(z_1w_1, z_2w_2) = \mu((1 - a^2)^{1/2}, -a)$. The descriptions of $\Omega_{u(a,\lambda)}$ follows.

From the definition of the core and the fact that here $C_{12} = C_{21} = 0$ and

$$C_{11} = \begin{bmatrix} (\lambda - 1) & 0 \\ 0 & (\lambda - 1)a(1 - a^2)^{1/2} \end{bmatrix},$$

$$C_{22} = \begin{bmatrix} (\lambda - 1)a(1 - a^2)^{1/2} & 0 \\ 0 & (\lambda - 1) + (\lambda - 1)a^2 \end{bmatrix},$$

we see that for $a = 0$ we have $w_2 = z_2 = 0$ while for $a \neq 0$, $z_1 = z_2 = w_1 = w_2 = 0$. $\square$

Recall that, for a $4 \times 4$ unitary matrix $v$ we defined the matrix $\tilde{v}$ by $\tilde{v}_{(i,j),(k,l)} = \bar{v}_{(l,k),(j,i)}$ and showed (Corollary 5.10) that $\mathcal{A}_u$ and $\mathcal{A}_v$ are isometrically isomorphic if and only if either $u$ and $v$ or $u$ and $\tilde{v}$ are product unitary equivalent.

Now, it is easy to check that $\widetilde{u(a, \lambda)} = u(a, \bar{\lambda})$ and so, using Proposition 3.3 and previous results, we obtain the following.

**Theorem 6.5** *Let* $0 \leq a, b \leq 1/\sqrt{2}, \quad |\lambda| = |\mu| = 1, \quad \lambda, \mu \neq 1$. *Then*

*(1)* $\mathcal{A}_{u(a,\lambda)}$ *and* $\mathcal{A}_{u(b,\mu)}$ *are isometrically isomorphic if and only if* $a = b$ *and* $\lambda$ *equals either* $\mu$ *or* $\bar{\mu}$.

*(2) When* $a \neq 0$ *the isometric automorphisms of* $\mathcal{A}_{u(0,\lambda)}$ *are all bigraded*

*(3) If* $a = 0$ *then there are isometric isomorphisms that are not graded*

**Case V:** $d = 4$

This is the case where $u = I$. We have $\Omega_u = \Omega_u^0 = \overline{\mathbb{B}}_n \times \overline{\mathbb{B}}_m$ and the isometric automorphisms are obtained by composing graded automorphisms and the automorphisms described by Proposition 4.6, Proposition 4.7.

## 6.2 Permutation unitary relation algebras

With more structure assumed for a class of unitaries $u$ it may be possible to derive an appropriately more definitive classification of the algebras $\mathcal{A}_u$. We indicate this now for the class of permutation unitaries. A fuller discussion is in [10].

Let $\theta \in S_4$, viewed as a permutation of the product set $\{1,2\} \times \{1,2\} = \{11,12,21,22\}$. Associate with $\theta$ the matrix $u_\theta = u_{(i,j),(k,l)}$ where $u_{(i,j),(k,l)} = 1$ if $(k,l) = \theta(i,j)$ and is zero otherwise. If $\tau \in S_4$ is *product conjugate* to $\theta$ in the sense that $\tau = \sigma\theta\sigma^{-1}$ with $\sigma$ in $S_2 \times S_2$, then it follows that $u_\tau$ and $u_\theta$ are product unitarily equivalent. Thus we need only consider product conjugacy classes. It turns out that these classes are the same as the product unitary equivalence classes of the matrices $u_\theta$.

It can be helpful to view a permutation $\theta$ in $S_{nm}$ as a permutation of the entries of an $n \times m$ rectangular array, since product conjugacy corresponds to conjugation through row permutations and column permutations. Considering this for $n = m = 2$ one can verify firstly that there are at most 9 isomorphism types for the algebras $\mathcal{A}_theta$ corresponding to the following permutations:

$$\theta_1 = \mathrm{id}, \theta_2 = (11,12), \theta_3 = (11,22),$$

$$\theta_{4a} = (11,22,12), \theta_{4b} = \theta_{4a}^{-1} = (11,12,22), \theta_5 = ((11,12),(21,22)),$$

$$\theta_6 = ((11,22),(12,21)), \theta_7 = (11,12,22,21), \theta_8 = (11,12,21,22).$$

The Gelfand spaces of the algebras $\mathcal{A}_\theta$ (and $\mathcal{L}_\theta$) distinguish all of these algebras except for the pairs $\{\theta_{4a}, \theta_{4b}\}$ and $\{\theta_7, \theta_8\}$. However, one can verify in both cases that neither the pair $u, v$ nor the pair $u, \tilde{v}$ are product unitary equivalent. Theorem 5.10 now applies to yield the following result from [10].

**Theorem 6.6** *For $n = m = 2$ there are 9 isometric isomorphism classes for the algebras $\mathcal{A}_\theta$ and for the algebras $\mathcal{L}_\theta$.*

To a higher rank graph $(\Lambda, d)$ in the sense of Kumjian and Pask [6] one can associate nonself-adjoint Toeplitz algebra $\mathcal{A}_\Lambda, \mathcal{L}_\Lambda$, as in Kribs and Power [5]. In the single vertex rank 2 case it is easy to see that $\mathcal{A}_\Lambda$ is equal to the algebra $\mathcal{A}_u$ for some permutation matrix $u = \theta$ in $S_{nm}$. Thus Theorem 5.10 classifies these algebras in terms of product unitary equivalence restricted to $S_{nm}$ as stated formally in the next theorem. In the rank 2 case this is a significant improvement on the results in [10] which, although covering general rank, were restricted to the case of trivial core for the character space. With $\tilde{\theta}$ the permutation for the permutation matrix $\tilde{u}_\theta$ (which corresponds to generator exchange) we have:

**Theorem 6.7** *Let $\Lambda_1$ and $\Lambda_2$ be single vertex 2-graphs with relations determined by the permutations $\theta_1$ and $\theta_2$. Then the rank 2 graph algebras $\mathcal{A}_{\Lambda_1}, \mathcal{A}_{\Lambda_2}$ are isometrically isomorphic if and only if the pair $\theta_1, \theta_2$ or the pair $\theta_1, \tilde{\theta}_2$ are product unitary equivalent*

It is natural to expect that as in the $(2,2)$ case product unitary equivalence will correspond to product conjugacy.

# 7 $\mathcal{A}_u$ as a subalgebra of a tensor algebra

Let $\mathcal{E}_n$ be the Toeplitz extension of the Cuntz algebra $O_n$ and write $H$ for the Fock space associated with $E$ (that is, $H = \mathbb{C} \oplus E \oplus (E \otimes E) \oplus \cdots$). Note that $\mathcal{E}_n$ acts naturally on $H$ ( by the "shift" or "creation" operators $L_i = L_{e_i}$, $1 \leq i \leq n$). In fact, $L_{e_1}, \ldots, L_{e_n}$ generate $\mathcal{E}_n$ as a $C^*$-algebra.

Consider also the space $\mathcal{F}(F) \otimes H = H \oplus (F \otimes H) \oplus ((F \otimes F) \otimes H) \oplus \cdots$. This space is isomorphic to $\mathcal{F}(E, F, u)$ and we write $w : \mathcal{F}(F) \otimes H \rightarrow \mathcal{F}(E, F, u)$ for the isomorphism. It will be convenient to write $w_k$ for the restriction of $w$ to the summand $F^{\otimes k} \otimes H$ (which is an isomorphism onto its image). Note that, for a fixed $k$, $\{w_k^* L_{e_i} w_k \ : \ 1 \leq i \leq n \}$ is a set of $n$ isometries with orthogonal ranges. Thus it defines a representation $\rho_k$ of $\mathcal{E}_n$ on $F^{\otimes k} \otimes H$ (with $\rho_k(L_{e_i}) = w_k^* L_{e_i} w_k$). (Note that we are using $L_{e_i}$ for the creation operators both on $H$ and on $\mathcal{F}(E, F, u)$. This should cause no confusion). We also write $\rho_\infty$ for the representation $\sum_k \oplus \rho_k$ of $\mathcal{E}_n$ on $\mathcal{F}(F) \otimes H$ (where $\rho_0$ is the representation of $\mathcal{E}_n$ on $H$).

Let $X$ be the column space $C_m(\mathcal{E}_n)$. This is a $C^*$-module over $\mathcal{E}_n$. As a vector space it is the direct sum of $m$ copies of $\mathcal{E}_n$. The right module action of $\mathcal{E}_n$ on $X$ is given by $(a_i) \cdot b = (a_i b)$ and the $\mathcal{E}_n$-valued inner product is $\langle (a_i), (b_i) \rangle = \sum_i a_i^* b_i$. For every $1 \leq i \leq n$, we write $\tilde{S}_i$ for the operator in $\mathcal{L}(X)$ defined by

$$\tilde{S}_i(a_j)_{j=1}^m = (\sum_{j,k} u_{(i,j),(k,l)} L_{e_k} a_j)_{l=1}^m.$$

Note that

$$\langle(\sum_{j,k} u_{(i,j),(k,l)} L_{e_k} a_j)_{l=1}^m, (\sum_{j',k'} u_{(i,j'),(k',l)} L_{e_{k'}} b_{j'})_{l=1}^m\rangle =$$

$$\sum_{j,j',k,k',l} \bar{u}_{(i,j),(k,l)} a_j^* L_{e_k}^* L_{e_{k'}} b_{j'} u_{(i,j'),(k',l)} =$$

$$\sum_{j,j'} (uu^*)_{(i,j'),(i,j)} a_j^* b_{j'} = \sum_j a_j^* b_j$$

$$= \langle (a_j), (b_{j'})\rangle.$$

Thus $\tilde{S}_i$ is an isometry. A similar computation shows that these isometries have orthogonal ranges and, thus, this family defines a $^*$-homomorphism $\varphi : \mathcal{E}_n \to \mathcal{L}(X)$, with $\varphi(L_{e_i}) = \tilde{S}_i$, $1 \leq i \leq n$, making $X$ a $C^*$-correspondence over $\mathcal{E}_n$ (in the sense of [8] and [7]). Once we have a correspondence we can form $X \otimes X$ and, more generally, $X^{\otimes k}$. Recall that to define $X \otimes X$ one defines the sesquilinear form $\langle x \otimes y, x' \otimes y'\rangle = \langle y, \varphi(\langle x, x'\rangle)y'\rangle$ on the algebraic tensor product and then lets $X \otimes X$ be the Hausdorff completion. The right action of $\mathcal{E}_n$ on $X \otimes X$ is $(x \otimes y) \cdot a = x \otimes (y \cdot a)$ and the left action is given by the map $\varphi_2$.

$$\varphi_2(a)(x \otimes y) = \varphi(a)x \otimes y.$$

The definition of $X^{\otimes k}$ is similar (and the left action map is denoted $\varphi_k$) For $k = 0$ we set $X^{\otimes 0} = \mathcal{E}_n$ and $\varphi_0$ is defined by left multiplication . Also write $\varphi_\infty$ for $\sum_k \oplus \varphi_k$, the left action of $\mathcal{E}_n$ on $\mathcal{F}(X)$.

One can then define the Hilbert space $X^{\otimes k} \otimes_{\mathcal{E}_n} H$ by defining the sesquilinear form $\langle x \otimes h, y \otimes k\rangle = \langle h, \langle x, y\rangle k\rangle$ ($x, y \in X^{\otimes k}$) and applying the Hausdorff completion.

Now define the map

$$v : X \otimes_{\mathcal{E}_n} H \to F \otimes H$$

by setting

$$v((a_i) \otimes h) = \sum_i f_i \otimes a_i h.$$

It is straightforward to check that this map is a well defined Hilbert space isomorphism. By induction, we also define maps $v_k : X^{\otimes k} \otimes_{\mathcal{E}_n} H \to F^{\otimes k} \otimes H$ by

$$v_{k+1}((a_j) \otimes z) = \sum_j f_j \otimes v_k((\varphi_k(a_j) \otimes I_H)z) \tag{23}$$

for $z \in X^{\otimes k} \otimes_{\mathcal{E}_n} H$ and $v_0$ is the identity map from $\mathcal{E}_n \otimes_{\mathcal{E}_n} H$ (which is isomorphic to $H$) and $F^{\otimes 0} \otimes H = H$. Assume that $v_k$ is a Hilbert space isomorphism of $X^{\otimes k} \otimes_{\mathcal{E}_n} H$ onto $F^{\otimes k} \otimes H$ and compute, for $(a_j), (b_j) \in X$ and $z, z' \in X^{\otimes k} \otimes H$,

$$\langle v_{k+1}((a_j)\otimes z), v_{k+1}((b_j)\otimes z)\rangle = \sum_{j,j'} \langle f_j \otimes v_k((\varphi_k(a_j)\otimes I_H)z), f_{j'} \otimes v_k((\varphi_k(b_{j'})\otimes I_H)z')\rangle =$$

$$\sum_j \langle v_k((\varphi_k(a_j) \otimes I_H)z), v_k((\varphi_k(b_j) \otimes I_H)z')\rangle =$$

$$\sum_j \langle z, (\varphi_k(a_j^* b_j) \otimes I_H)z')\rangle =$$

$$\langle (a_j) \otimes z, (b_j) \otimes z'\rangle.$$

Thus, by induction, each map $v_k$ is a Hilbert space isomorphism and, summing up, we get a Hilbert space isomorphism

$$v_\infty := \sum_k \oplus v_k : \mathcal{F}(X) \otimes_{\mathcal{E}_n} H \to \mathcal{F}(F) \otimes H.$$

**Lemma 7.1** *$v_\infty$ is a Hilbert space isomorphism and intertwines the actions of $\mathcal{E}_n$. That is,*

$$v_\infty \circ (\varphi_\infty(a) \otimes I_H) = \rho_\infty(a) \circ v_\infty$$

*for $a \in \mathcal{E}_n$.*

**Proof.** We show that, for every $p \geq 0$ and $a \in \mathcal{E}_n$, we have

$$v_p \circ (\varphi_p(a) \otimes I_H) = \rho_p(a) \circ v_p. \tag{24}$$

The proof will proceed by induction on $p$. For $p = 0$ this is clear so we now assume that it holds for $p$. For $1 \leq i \leq n$, $(a_j) \in X$ and $z \in X^{\otimes p} \otimes H$, we have $v_{p+1}((\varphi_{p+1}(L_{e_i}) \otimes I_H)((a_j) \otimes z)) = v_{p+1}(\varphi(L_{e_i})(a_j) \otimes z) = \sum_{l,k,j} u_{(i,j),(k,l)} f_l \otimes v_p((\varphi_p(L_{e_k} a_j) \otimes I_H)z)$. Using the induction hypothesis, this is equal to

$$\sum_{l,k,j} u_{(i,j),(k,l)} f_l \otimes \rho_p(L_{e_k})\rho_p(a_j) v_p z = \sum_{l,k,j} u_{(i,j),(k,l)} f_l \otimes w_p^* L_{e_k} w_p \rho_p(a_j) v_p z =$$

$$w_\infty^* \sum_{l,k,j} u_{(i,j),(k,l)} f_l \otimes e_k \rho_p(a_j) v_p z = w_\infty^* \sum_j e_i \otimes f_j \otimes \rho_p(a_j) v_p z =$$

$$\rho_{p+1}(L_{e_i})w_{p+1}^* \sum_j f_j \otimes \rho_p(a_j)v_p z.$$

Using the induction hypothesis again, we get $\rho_{p+1}(L_{e_i})w_{p+1}^* \sum_j f_j \otimes v_p((\varphi_p(a_j) \otimes I_H)z) = \rho_{p+1}(L_{e_i})v_{p+1}((a_j) \otimes z)$. This proves (24) for $p+1$ and the generators of $\mathcal{E}_n$. Since both $\rho_{p+1}$ and $v_{p+1}(\varphi_{p+1}(\cdot) \otimes I_H)v_{p+1}^*$ are $^*$-homomorphisms, (24) holds for $p+1$ and every $a \in \mathcal{E}_n$, completing the induction step. Thus, (24) holds for every $p$ and this implies the statement of the lemma. $\square$

Write $\delta_l$ for the vector $(a_j)$ in $X$ such that $a_l = I$ and $a_j = 0$ if $l \neq j$. The tensor algebra $\mathcal{T}_+(X)$ is generated by the operators $T_{\delta_l}$ (where $T_{\delta_l}$ is the creation operator on $\mathcal{F}(X)$ associated with $\delta_l$) and the $C^*$-algebra $\varphi_\infty(\mathcal{E}_n)$. The latter algebra is generated (as a $C^*$-algebra) by the operators $\varphi_\infty(L_i)$ where $\{L_i\}$ is the set of generators of $\mathcal{E}_n$.

We have

**Lemma 7.2** *For every $1 \leq i \leq n$ and $1 \leq j \leq m$ and $k \geq 0$,*

*(i)* $w \circ v_k \circ (\varphi_\infty(L_i) \otimes I_H) = L_{e_i} \circ w \circ v_k$.

*(ii)* $w \circ v_{k+1} \circ (T_{\delta_j} \otimes I_H) = L_{f_j} \circ w \circ v_k$.

**Proof.** Part (i) follows from (24) and part (ii) from (23) (with $\delta_j$ in place of $(a_j)$). $\square$

Recalling that $w \circ v_\infty$ is a unitary operator mapping $\mathcal{F}(X) \otimes H$ onto $\mathcal{F}(E, F, u)$, we get

**Theorem 7.3** *(1) The algebra $\mathcal{A}_u$ is unitarily isomorphic to the (norm closed) subalgebra of the tensor algebra $\mathcal{T}_+(X)$ that is generated by $\{\varphi_\infty(L_i), T_{\delta_j} \ : \ 1 \leq i \leq n,\ 1 \leq j \leq m\}$.*

*(2) The (norm closed) subalgebra of $B(\mathcal{F}(E, F, u))$ that is generated by $\{L_{e_i}, L_{e_i}^*, L_{f_j} \ : \ 1 \leq i \leq n, 1 \leq j \leq m\ \}$ is unitarily isomorphic to the tensor algebra $\mathcal{T}_+(X)$ (and contains $\mathcal{A}_u$).*

*(2) The (norm closed) subalgebra of $B(\mathcal{F}(E, F, u))$ that is generated by $\{L_{e_i}, L_{f_j}^*, L_{f_j} \ : \ 1 \leq i \leq n, 1 \leq j \leq m\ \}$ is unitarily isomorphic to a tensor algebra $\mathcal{T}_+(Y)$ (and contains $\mathcal{A}_u$).*

**Proof.** Parts (1) and (2) follow from Lemma 7.2. For part (3), note that one can interchange the roles of $E$ and $F$. More precisely, one defines the $C^*$-module $Y$ over $\mathcal{E}_m$ to be $Y = C_n(\mathcal{E}_m)$ and the left action of $\mathcal{E}_m$ on $Y$ by $\varphi_Y(L_{f_l})(b_k)_{k=1}^n = (\sum_{j,k} \bar{u}_{(i,j),(k,l)} L_{f_j} b_k)_{i=1}^n$. This makes $Y$ into a $C^*$-correspondence over $\mathcal{E}_m$ and the rest of the proof proceeds along similar lines as above. $\square$

Suppose $m = 1$. Then $X$ is the correspondence associated with the automorphism $\alpha$ of $\mathcal{E}_n$ given by mapping $T_i$ to $\sum_{j=1}^n u_{i,j} T_j$ (note that $u$, in this case, is an $n \times n$ matrix). The tensor algebra $\mathcal{T}_+(X)$ is the analytic crossed product $\mathcal{E}_n \times_\alpha \mathbb{Z}^+$ and $\mathcal{A}_u$ is unitarily isomorphic to the subalgebra of this analytic crossed product that can be written $\mathcal{A}_n \times_\alpha \mathbb{Z}^+$. One can also embed $\mathcal{A}_u$ in $\mathcal{T}_+(Y)$ (as in Corollary 7.3(3)). Here $\mathcal{E}_m$ is simply the (classical) Toeplitz algebra $\mathcal{T}$ and $Y = C_n(\mathcal{T})$ with $\varphi_Y(T_z)(b_k)_k = (\sum_k \bar{u}_{i,k} T_z b_k)_i$ (where $T_z$ is the generator of $\mathcal{T}$).

**Remark 7.4** *Since the automorphisms $\Theta_{z,w}$ and $\Psi_{A,B}$ of $\mathcal{A}_u$ are both unitarily implemented, they can be extended to $\mathcal{T}_+(X)$. It is easy to check that they map $\mathcal{T}_+(X)$ into itself and, thus, are automorphisms of $\mathcal{T}_+(X)$. Hence, at least when n $\neq m$, every automorphism of $\mathcal{A}_u$ can be extended to an automorphism of the tensor algebra $\mathcal{T}_+(X)$ that contains it (see Theorem 5.12).*

# References


[1] K.R. Davidson, *Free Semigroup Algebras : a survey.* Systems, approximation, singular integral operators, and related topics (Bordeaux, 2000), 209–240, Oper. Theory Adv. Appl. 129, Birkhauser, Basel, 2001.

[2] K.R. Davidson and D.R. Pitts, *The algebraic structure of noncommutative analytic Toeplitz algebras*, Math. Ann. **311** (1998), 275-303.

[3] N. Fowler, *Discrete product systems of Hilbert bimodules*, Pacific J. Math. **204** (2002), 335-375.

[4] E. Katsoulis, D.W. Kribs, *Isomorphisms of algebras associated with directed graphs*, Math. Ann., 330 (2004), 709-728.

[5] D.W. Kribs and S.C. Power, *The $H^\infty$ algebras of higher rank graphs*, Math. Proc. of the Royal Irish Acad., 106 (2006), 199-218.

[6] A. Kumjian and D. Pask, *Higher rank graph $C^*$ -algebras*, New York J. Math. **6** (2000), 1–20.

[7] P. Muhly and B. Solel, *Tensor algebras over $C^*$-correspondences (Representations, dilations, and $C^*$-envelopes*), J. Functional Anal. **158** (1998), 389–457.

[8] M. Pimsner, *A class of $C^*$-algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$*, in *Free Probability Theory,* D. Voiculescu, Ed., Fields Institute Communications **12**, 189-212, Amer. Math. Soc., Providence, 1997.

[9] G. Popescu, *Von Neumann inequality for* $(\mathcal{B}(\mathcal{H})^n)_1$, Math. Scand.**68** (1991), 292-304.

[10] S.C. Power, *Classifying higher rank analytic Toeplitz algebras*, preprint 2006, preprint Archive no., math.OA/0604630.

[11] B. Solel, *You can see the arrows in a quiver algebra,* J. Australian Math. Soc., 77 (2004), 111-122.

[12] B. Solel, *Regular dilations of representations of product systems*, preprint Archive no., math.OA/0504129.

[13] B. Solel, *Representations of product systems over semigroups and dilations of commuting CP maps*, J. Funct. Anal.**235** (2006), 593-618.

[14] D. Voiculescu, *Symmetries of some reduced free product $C^*$-algebras*, Lect. Notes Math. **1132**, 556-588, Springer-Verlag, New York 1985.