Quantum Deformations of Relativistic Symmetries*

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Abstract

We discussed quantum deformations of $D = 4$ Lorentz and Poincaré algebras. In the case of Poincaré algebra it is shown that almost all classical r-matrices of S. Zakrzewski classification correspond to twisted deformations of Abelian and Jordanian types. A part of twists corresponding to the r-matrices of Zakrzewski classification are given in explicit form.

1 Introduction

The quantum deformations of relativistic symmetries are described by Hopf-algebraic deformations of Lorentz and Poincaré algebras. Such quantum deformations are classified by Lorentz and Poincaré Poisson structures. These Poisson structures given by classical r-matrices were classified already some time ago by S. Zakrzewski in [1] for the Lorentz algebra and in [2] for the Poincaré algebra. In the case of the Lorentz algebra a complete list of classical r-matrices involves the four independent formulas and the corresponding quantum deformations in different forms were already discussed in literature (see [3, 4, 5, 6, 7]). In the case of Poincaré algebra the total list of the classical r-matrices, which satisfy the homogeneous classical Yang-Baxter equation, consists of 20 cases which have various numbers of free parameters. Analysis of these twenty solutions shows that each of them can be presented as a sum of subordinated r-matrices which almost all are of Abelian and Jordanian types. A part of twists corresponding to the r-matrices of Zakrzewski classification are given in explicit form.

2 Preliminaries

Let r be a classical r-matrix of a Lie algebra $\mathfrak{g}$, i.e. $r \in \overset{2}{\wedge}\,\mathfrak{g}$ and r satisfies to the classical Yang–Baxter equation (CYBE)

$$[r^{12},\, r^{13} + r^{23}] + [r^{13},\, r^{23}] \;=\; \Omega\;, \tag{2.1}$$

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where Ω is $\mathfrak{g}$-invariant element, $\Omega \in (\overset{3}{\wedge}\ \mathfrak{g})_{\mathfrak{g}}$. We consider two types of the classical r-matrices and corresponding twists.

Let the classical r-matrix $r = r_{_A}$ has the form

$$r_{_A} \ = \ \sum_{i=1}^{n} x_i \wedge y_i \ , \tag{2.2}$$

where all elements x_i, y_i $(i = 1, \ldots, n)$ commute among themselves. Such an r-matrix is called of Abelian type. The corresponding twist is given as follows

$$F_{r_{_A}} \ = \ \exp \frac{r_{_A}}{2} = \exp\Big(\frac{1}{2} \sum_{i=1}^{n} x_i \wedge y_i\Big) \ . \tag{2.3}$$

This twisting two-tensor $F := F_{r_{_A}}$ satisfies the cocycle equation

$$F^{12}(\Delta \otimes \mathrm{id})(F) \ = \ F^{23}(\mathrm{id} \otimes \Delta)(F) \ , \tag{2.4}$$

and the "unital" normalization condition

$$(\epsilon \otimes \mathrm{id})(F) \ = \ (\mathrm{id} \otimes \epsilon)(F) \ = \ 1 \ . \tag{2.5}$$

The twisting element F defines a deformation of the universal enveloping algebra $U(\mathfrak{g})$ considered as a Hopf algebra. The new deformed coproduct and antipode are given as follows

$$\Delta^{(F)}(a) \ = \ F\Delta(a)F^{-1} \ , \qquad S^{(F)}(a) = uS(a)u^{-1} \tag{2.6}$$

for any $a \in U(\mathfrak{g})$, where $\Delta(a)$ is a co-product before twisting, and $u = \sum_i f_i^{(1)} S(f_i^{(2)})$ if $F = \sum_i f_i^{(1)} \otimes f_i^{(2)}$.

Let the classical r-matrix $r = r_{_J}(\xi)$ has the form[1]

$$r_{_J}(\xi) \ = \ \xi \left(\sum_{\nu=0}^{n} x_\nu \wedge y_\nu \right) \ , \tag{2.7}$$

where the elements x_ν, y_ν $(\nu = 0, 1, \ldots, n)$ satisfy the relations[2]

$$\begin{array}{lll} [x_0,\, y_0] \ = \ y_0 \ , & [x_0,\, x_i] \ = \ (1 - t_i)x_i \ , & [x_0,\, y_i] \ = \ t_i y_i \ , \\ [x_i,\, y_j] \ = \ \delta_{ij} y_0 \ , & [x_i,\, x_j] \ = \ [y_i,\, y_j] \ = \ 0 \ , & [y_0,\, x_j] \ = \ [y_0,\, y_j] \ = \ 0 \ , \end{array} \tag{2.8}$$

$(i, j = 1, \ldots, n)$, $(t_i \in \mathbb{C})$. Such an r-matrix is called of Jordanian type. The corresponding twist is given as follows [8, 9]

$$F_{r_{_J}} \ = \ \exp\left(\xi \sum_{i=1}^{n} x_i \otimes y_i \, e^{-2t_i\sigma} \right) \exp(2x_{_0} \otimes \sigma) \ , \tag{2.9}$$

[1] Here entering the parameter deformation ξ is a matter of convenience.

[2] It is easy to verify that the two-tensor (2.7) indeed satisfies the homogenous classical Yang-Baxter equation (2.1) (with $\Omega = 0$), if the elements x_ν, y_ν $(\nu = 0, 1, \ldots, n)$ are subject to the relations (2.8).

where $\sigma := \frac{1}{2}\ln(1+\xi y_0)$.[3]

Let r be an arbitrary r-matrix of $\mathfrak{g}$. We denote a support of r by $\mathrm{Sup}(r)$[4]. The following definition is useful.

Definition 2.1 *Let r_1 and r_2 be two arbitrary classical r-matrices. We say that r_2 is subordinated to r_1, $r_1 \succ r_2$, if $\delta_{r_1}(\mathrm{Sup}(r_2)) = 0$, i.e.*

$$\delta_{r_1}(x) \;:=\; [x\otimes 1 + 1\otimes x,\, r_1] \;=\; 0\;, \qquad \forall x \in \mathrm{Sup}(r_2)\;. \tag{2.10}$$

If $r_1 \succ r_2$ then $r = r_1 + r_2$ is also a classical r-matrix (see [15]). The subordination enables us to construct a correct sequence of quantizations. For instance, if the r-matrix of Jordanian type (2.7) is subordinated to the r-matrix of Abelian type (2.2), $r_A \succ r_J$, then the total twist corresponding to the resulting r-matrix $r = r_A + r_J$ is given as follows

$$F_r \;=\; F_{r_J}\, F_{r_A}. \tag{2.11}$$

The further definition is also useful.

Definition 2.2 *A twisting two-tensor $F_r(\xi)$ of a Hopf algebra, satisfying the conditions (2.4) and (2.5), is called locally r-symmetric if the expansion of $F_r(\xi)$ in powers of the parameter deformation ξ has the form*

$$F_r(\xi) \;=\; 1 + c\,r + \mathcal{O}(\xi^2)\ldots \tag{2.12}$$

where r is a classical r-matrix, and c is a numerical coefficient, $c \neq 0$.

It is evident that the Abelian twist (2.3) is globally r-symmetric and the twist of Jordanian type (2.9) does not satisfy the relation (2.12), i.e. it is not locally r-symmetric.

3 Quantum deformations of Lorentz algebra

The results of this section in different forms were already discussed in literature (see [3, 4, 5, 6, 7]).

The classical canonical basis of the $D = 4$ Lorentz algebra, $\mathfrak{o}(3,1)$, can be described by anti-Hermitian six generators $(h,\, e_{\pm},\, h',\, e'_{\pm})$ satisfying the following non-vanishing commutation relations[5]:

$$[h,\, e_{\pm}] \;=\; \pm e_{\pm}\,, \qquad [e_{+},\, e_{-}] \;=\; 2h\;, \tag{3.1}$$

$$[h,\, e'_{\pm}] \;=\; \pm e'_{\pm}\;, \qquad [h',\, e_{\pm}] \;=\; \pm e'_{\pm}\;, \qquad [e_{\pm},\, e'_{\mp}] \;=\; \pm 2h'\;, \tag{3.2}$$

$$[h',\, e'_{\pm}] \;=\; \mp e_{\pm}\;, \qquad [e'_{+},\, e'_{-}] \;=\; -2h\;, \tag{3.3}$$

and moreover

$$x^{*} \;=\; -x \qquad (\forall\, x \,\in\, \mathfrak{o}(3,1))\;. \tag{3.4}$$

[3] The corresponding twists for Lie algebras $\mathfrak{sl}(n)$, $\mathfrak{so}(n)$ and $\mathfrak{sp}(2n)$ were firstly constructed in the papers [10, 11, 12, 13].

[4] The support $\mathrm{Sup}(r)$ is a subalgebra of $\mathfrak{g}$ generated by the elements $\{x_i, y_i\}$ if $r = \sum_i x_i \wedge y_i$.

[5] Since the real Lie algebra $\mathfrak{o}(3,1)$ is standard realification of the complex Lie $\mathfrak{sl}(2,\mathbb{C})$ these relations are easy obtained from the defining relations for $\mathfrak{sl}(2,\mathbb{C})$, i.e. from (3.1).

A complete list of classical r-matrices which describe all Poisson structures and generate quantum deformations for $\mathfrak{o}(3,1)$ involve the four independent formulas [1]:

$$r_1 \;=\; \alpha\, e_+ \wedge h \ , \tag{3.5}$$

$$r_2 \;=\; \alpha\,(e_+ \wedge h - e'_+ \wedge h') + 2\beta\, e'_+ \wedge e_+ \ , \tag{3.6}$$

$$r_3 \;=\; \alpha\,(e'_+ \wedge e_- + e_+ \wedge e'_-) \,+\, \beta\,(e_+ \wedge e_- \,-\, e'_+ \wedge e'_-) - 2\gamma\, h \wedge h' \ , \tag{3.7}$$

$$r_4 \;=\; \alpha\big(e'_+ \wedge e_- + e_+ \wedge e'_- - 2h \wedge h'\big) \pm e_+ \wedge e'_+ \ . \tag{3.8}$$

If the universal R-matrices of the quantum deformations corresponding to the classical r-matrices (3.5)–(3.8) are unitary then these r-matrices are anti-Hermitian, i.e.

$$r_j^* \;=\; -r_j \qquad (j = 1,2,3,4)\ . \tag{3.9}$$

Therefore the $*$-operation (3.4) should be lifted to the tensor product $\mathfrak{o}(3,1) \otimes \mathfrak{o}(3,1)$. There are two variants of this lifting: *direct* and *flipped*, namely,

$$(x \otimes y)^* \;=\; x^* \otimes y^* \qquad (* - \mathrm{direct})\ , \tag{3.10}$$

$$(x \otimes y)^* \;=\; y^* \otimes x^* \qquad (* - \mathrm{flipped})\ . \tag{3.11}$$

We see that if the "direct" lifting of the $*$-operation (3.4) is used then all parameters in (3.5)–(3.8) are pure imaginary. In the case of the "flipped" lifting (3.11) all parameters in (3.5)–(3.8) are real.

The first two r-matrices (3.5) and (3.6) satisfy the homogeneous CYBE and they are of Jordanian type. If we assume (3.10), the corresponding quantum deformations were described detailed in the paper [6] and they are entire defined by the twist of Jordanian type:

$$F_{r_1} \;=\; \exp\,(h \otimes \sigma)\ , \quad \sigma \;=\; \frac{1}{2}\ln(1 + \alpha e_+) \tag{3.12}$$

for the r-matrix (3.5), and

$$F_{r_2} \;=\; \exp\Big(\frac{\imath\beta}{\alpha^2}\,\sigma \wedge \varphi\Big)\,\exp\,(h \otimes \sigma - h' \otimes \varphi)\ , \tag{3.13}$$

$$\sigma \;=\; \frac{1}{2}\ln\big[(1 + \alpha e_+)^2 + (\alpha e'_+)^2\big]\,, \quad \varphi \;=\; \arctan\frac{\alpha e'_+}{1 + \alpha e_+} \tag{3.14}$$

for the r-matrix (3.6). It should be recalled that the twists (3.12) and (3.13) are not locally r-symmetric. A locally r-symmetric twist for the r-matrix (3.5) was obtained in [14] and it has the following complicated formula:

$$F'_{r_1} \;=\; \exp\Big(\frac{1}{2}\Delta(h) - \frac{1}{2}\Big(h\frac{\sinh\alpha e_+}{\alpha e_+} \otimes e^{-\alpha e_+} + e^{\alpha e_+} \otimes h\frac{\sinh\alpha e_+}{\alpha e_+}\Big)\frac{\alpha\Delta(e_+)}{\sinh\alpha\Delta(e_+)}\Big), \tag{3.15}$$

where Δ is a primitive coproduct.

The last two r-matrices (3.7) and (3.8) satisfy the non-homogeneous (modified) CYBE and they can be easily obtained from the solutions of the complex algebra $\mathfrak{o}(4,\mathbb{C}) \simeq \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ which describes the complexification of $\mathfrak{o}(3,1)$. Indeed, let us introduce the complex basis of Lorentz algebra ($\mathfrak{o}(3,1) \simeq \mathfrak{sl}(2;\mathbb{C}) \oplus \overline{\mathfrak{sl}}(2,\mathbb{C})$) described by two commuting sets of complex generators:

$$H_1 \;=\; \frac{1}{2}\,(h + \imath h')\;, \qquad E_{1\pm} \;=\; \frac{1}{2}\,(e_\pm + \imath e'_\pm)\;, \tag{3.16}$$

$$H_2 \;=\; \frac{1}{2}\,(h - \imath h')\;, \qquad E_{2\pm} \;=\; \frac{1}{2}\,(e_\pm - \imath e'_\pm)\;, \tag{3.17}$$

which satisfy the relations (compare with (3.1))

$$[H_k,\, E_{k\pm}] \;=\; \pm E_{k\pm}\;, \qquad [E_{k+},\, E_{k-}] \;=\; 2H_k \qquad (k=1,2)\;. \tag{3.18}$$

The $*$-operation describing the real structure acts on the generators H_k, and $E_{k\pm}$ ($k=1,2$) as follows

$$H_1^* \;=\; -H_2\;, \qquad E_{1\pm}^* \;=\; -E_{2\pm}\;, \qquad H_2^* \;=\; -H_1\;, \qquad E_{2\pm}^* \;=\; -E_{1\pm}\;. \tag{3.19}$$

The classical r-matrix r_3, (3.7), and r_4, (3.8), in terms of the complex basis (3.16), (3.17) take the form

$$\begin{aligned} r_3 \;&=\; r'_1 + r''_1\;, \\ r'_3 \;&:=\; 2(\beta+\imath\alpha)E_{1+}\wedge E_{1-} + 2(\beta-\imath\alpha)E_{2+}\wedge E_{2-}\;, \\ r''_3 \;&:=\; 4\imath\gamma\, H_2\wedge H_1\;, \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} r_4 \;&=\; r'_4 + r''_4\;, \\ r'_4 \;&:=\; 2\imath\alpha(E_{1+}\wedge E_{1-} - E_{2+}\wedge E_{2-} - 2H_1\wedge H_2)\;, \\ r''_4 \;&:=\; 4\imath\lambda\, E_{1+}\wedge E_{2+} \end{aligned} \tag{3.21}$$

For the sake of convenience we introduce parameter[6] λ in r''_4. It should be noted that r'_3, r''_3 and r'_4, r''_4 are themselves classical r-matrices. We see that the r-matrix r'_3 is simply a sum of two standard r-matrices of $\mathfrak{sl}(2;\mathbb{C})$, satisfying the anti-Hermitian condition $r^* = -r$. Analogously, it is not hard to see that the r-matrix r_4 corresponds to a Belavin-Drinfeld triple [15] for the Lie algebra $\mathfrak{sl}(2;\mathbb{C}) \oplus \overline{\mathfrak{sl}}(2,\mathbb{C})$). Indeed, applying the Cartan automorphism $E_{2\pm} \to E_{2\mp}$, $H_2 \to -H_2$ we see that this is really correct (see also [16]).

We firstly describe quantum deformation corresponding to the classical r-matrix r_3 (3.20). Since the r-matrix r''_3 is Abelian and it is subordinated to r'_3 therefore the algebra $\mathfrak{o}(3,1)$ is firstly quantized in the direction r'_3 and then an Abelian twist corresponding to the r-matrix r''_3 is applied. We introduce the complex notations $z_\pm := \beta \pm \imath\alpha$. It should be noted that $z_- = z_+^*$ if the parameters α and β are real, and $z_- = -z_+^*$ if the parameters α and β are pure imaginary. From structure of the classical r-matrix r'_3 it follows that a quantum deformation $U_{r'_1}(\mathfrak{o}(3,1))$ is a combination of two q-analogs of $U(\mathfrak{sl}(2;\mathbb{C}))$ with the parameter q_{z_+} and q_{z_-}, where $q_{z_\pm} := \exp z_\pm$. Thus $U_{r'_3}(\mathfrak{o}(3,1)) \cong U_{q_{z_+}}(\mathfrak{sl}(2;\mathbb{C}))\otimes U_{q_{z_-}}(\overline{\mathfrak{sl}}(2;\mathbb{C}))$ and the standard generators $q_{z_+}^{\pm H_1}$, $E_{1\pm}$ and $q_{z_-}^{\pm H_2}$, $E_{2\pm}$ satisfy

[6] We can reduce this parameter λ to $\pm\frac{1}{2}$ by automorphism of $\mathfrak{o}(4,\mathbb{C})$.

the following non-vanishing defining relations

$$q_{z_+}^{H_1} E_{1\pm} \;=\; q_{z_+}^{\pm 1} E_{1\pm}\, q_{z_+}^{H_1}, \qquad [E_{1+},\, E_{1-}] \;=\; \frac{q_{z_+}^{2H_1} - q_{z_+}^{-2H_1}}{q_{z_+} - q_{z_+}^{-1}}, \tag{3.22}$$

$$q_{z_-}^{H_2} E_{2\pm} \;=\; q_{z_-}^{\pm 1} E_{2\pm}\, q_{z_-}^{H_2}, \qquad [E_{2+},\, E_{2-}] \;=\; \frac{q_{z_-}^{2H_2} - q_{z_-}^{-2H_2}}{q_{z_-} - q_{z_-}^{-1}}\;. \tag{3.23}$$

In this case the co-product $\Delta_{r_1'}$ and antipode $S_{r_1'}$ for the generators $q_{z_+}^{\pm H_1}$, $E_{1\pm}$ and $q_{z_-}^{\pm H_2}$, $E_{2\pm}$ can be given by the formulas:

$$\Delta_{r_1'}(q_{z_+}^{\pm H_1}) \;=\; q_{z_+}^{\pm H_1} \otimes q_{z_+}^{\pm H_1}, \qquad \Delta_{r_1'}(E_{1\pm}) \;=\; E_{1\pm} \otimes q_{z_+}^{H_1} + q_{z_+}^{-H_1} \otimes E_{1\pm}\;, \tag{3.24}$$

$$\Delta_{r_1'}(q_{z_-}^{\pm H_2}) \;=\; q_{z_-}^{\pm H_2} \otimes q_{z_-}^{\pm H_2}, \qquad \Delta_{r_1'}(E_{2\pm}) \;=\; E_{2\pm} \otimes q_{z_-}^{H_2} + q_{z_-}^{-H_2} \otimes E_{2\pm}\;, \tag{3.25}$$

$$S_{r_1'}(q_{z_+}^{\pm H_1}) \;=\; q_{z_+}^{\mp H_1}, \qquad S_{r_1'}(E_{1\pm}) \;=\; -q_{z_+}^{\pm 1} E_{1\pm}\;, \tag{3.26}$$

$$S_{r_1'}(q_{z_-}^{\pm H_2}) \;=\; q_{z_-}^{\mp H_2}, \qquad S_{r_1'}(E_{2\pm}) \;=\; -q_{z_-}^{\pm 1} E_{2\pm}\;. \tag{3.27}$$

The $*$-involution describing the real structure on the generators (3.8) can be adapted to the quantum generators as follows

$$(q_{z_+}^{\pm H_1})^* \;=\; q_{z_+^*}^{\mp H_2}, \quad E_{1\pm}^* \;=\; -E_{2\pm}\;, \quad (q_{z_-}^{\pm H_2})^* \;=\; q_{z_-^*}^{\mp H_1}, \quad E_{2\pm}^* \;=\; -E_{1\pm}\;, \tag{3.28}$$

and there exit two $*$-liftings: *direct* and *flipped*, namely,

$$(a \otimes b)^* \;=\; a^* \otimes b^* \qquad (* - \text{direct})\;, \tag{3.29}$$

$$(a \otimes b)^* \;=\; b^* \otimes a^* \qquad (* - \text{flipped}) \tag{3.30}$$

for any $a \otimes b \in U_{r_3'}(\mathfrak{o}(3,1)) \otimes U_{r_3'}(\mathfrak{o}(3,1))$, where the $*$-direct involution corresponds to the case of the pure imaginary parameters α, β and the $*$-flipped involution corresponds to the case of the real deformation parameters α, β. It should be stressed that the Hopf structure on $U_{r_3'}(\mathfrak{o}(3,1))$ satisfy the consistency conditions under the $*$-involution

$$\Delta_{r_3'}(a^*) \;=\; (\Delta_{r_3'}(a))^*, \qquad S_{r_3'}((S_{r_3'}(a^*))^*) \;=\; a \quad (\forall x \in U_{r_3'}(\mathfrak{o}(3,1))\;. \tag{3.31}$$

Now we consider deformation of the quantum algebra $U_{r_3'}(\mathfrak{o}(3,1))$ (secondary quantization of $U(\mathfrak{o}(3,1)))$ corresponding to the additional r-matrix r_3'', (3.20). Since the generators H_1 and H_2 have the trivial coproduct

$$\Delta_{r_3'}(H_k) \;=\; H_k \otimes 1 + 1 \otimes H_k \quad (k = 1, 2)\;, \tag{3.32}$$

therefore the unitary two-tensor

$$F_{r_3''} := q_{\imath\gamma}^{H_1 \wedge H_2} \qquad (F_{r_3''}^* \;=\; F_{r_1''}^{-1}) \tag{3.33}$$

satisfies the cocycle condition (2.4) and the "unital" normalization condition (2.5). Thus the complete deformation corresponding to the r-matrix r_3 is the twisted deformation of $U_{r_3'}(\mathfrak{o}(3,1))$, i.e. the resulting coproduct Δ_{r_3} is given as follows

$$\Delta_{r_3}(x) \;=\; F_{r_1''} \Delta_{r_1'}(x) F_{r_3''}^{-1} \quad (\forall x \in U_{r_1'}(\mathfrak{o}(3,1))\;. \tag{3.34}$$

and in this case the resulting antipode S_{r_3} does not change, $S_{r_3} = S_{r'_3}$. Applying the twisting two-tensor (3.33) to the formulas (3.24) and (3.25) we obtain

$$\Delta_{r_3}(q_{z_+}^{\pm H_1}) \;=\; q_{z_+}^{\pm H_1} \otimes q_{z_+}^{\pm H_1}, \quad \Delta_{r'_1}(q_{z_-}^{\pm H_2}) \;=\; q_{z_-}^{\pm H_2} \otimes q_{z_-}^{\pm H_2}, \tag{3.35}$$

$$\Delta_{r_3}(E_{1+}) \;=\; E_{1+} \otimes q_{z_+}^{H_1} q_{\imath\gamma}^{H_2} + q_{z_+}^{-H_1} q_{\imath\gamma}^{-H_2} \otimes E_{1+}\ , \tag{3.36}$$

$$\Delta_{r_3}(E_{1-}) \;=\; E_{1-} \otimes q_{z_+}^{H_1} q_{\imath\gamma}^{-H_2} + q_{z_+}^{-H_1} q_{\imath\gamma}^{H_2} \otimes E_{1-}\ , \tag{3.37}$$

$$\Delta_{r_3}(E_{2+}) \;=\; E_{2+} \otimes q_{z_-}^{H_2} q_{\imath\gamma}^{-H_1} + q_{z_-}^{-H_2} q_{\imath\gamma}^{H_1} \otimes E_{2+}\ , \tag{3.38}$$

$$\Delta_{r_3}(E_{2-}) \;=\; E_{2-} \otimes q_{z_-}^{H_2} q_{\imath\gamma}^{H_1} + q_{z_-}^{-H_2} q_{\imath\gamma}^{-H_1} \otimes E_{2-}\ . \tag{3.39}$$

Next, we describe quantum deformation corresponding to the classical r-matrix r_4 (3.21). Since the r-matrix $r'_4(\alpha) := r'_4$ is a particular case of $r_3(\alpha,\beta,\gamma) := r_3$, namely $r'_4(\alpha) = r_3(\alpha, \beta = 0, \gamma = \alpha)$, therefore a quantum deformation corresponding to the r-matrix r'_4 is obtained from the previous case by setting $\beta = 0, \gamma = \alpha$, and we have the following formulas for the coproducts $\Delta_{r'_4}$:

$$\Delta_{r'_4}(q_{\xi}^{\pm H_k}) \;=\; q_{\xi}^{\pm H_k} \otimes q_{\xi}^{\pm H_k} \qquad (k = 1, 2)\ , \tag{3.40}$$

$$\Delta_{r'_4}(E_{1+}) \;=\; E_{1+} \otimes q_{\xi}^{H_1+H_2} + q_{\xi}^{-H_1-H_2} \otimes E_{1+}\ , \tag{3.41}$$

$$\Delta_{r'_4}(E_{1-}) \;=\; E_{1-} \otimes q_{\xi}^{H_1-H_2} + q_{\xi}^{-H_1+H_2} \otimes E_{1-}\ , \tag{3.42}$$

$$\Delta_{r'_4}(E_{2+}) \;=\; E_{2+} \otimes q_{\xi}^{-H_1-H_2} + q_{\xi}^{H_1+H_2} \otimes E_{2+}\ , \tag{3.43}$$

$$\Delta_{r'_4}(E_{2-}) \;=\; E_{2-} \otimes q_{\xi}^{H_1-H_2} + q_{\xi}^{-H_1+H_2} \otimes E_{2-}\ , \tag{3.44}$$

where we set $\xi := \imath\alpha$.

Consider the two-tensor

$$F_{r''_4} := \ \exp_{q^2}\left(\lambda E_{1+} q_{\xi}^{H_1+H_2} \otimes E_{2+} q_{\xi}^{H_1+H_2}\right)\ . \tag{3.45}$$

Using properties of q-exponentials (see [17]) is not hard to verify that $F_{r''_4}$ satisfies the cocycle equation (2.4). Thus the quantization corresponding to the r-matrix r_4 is the twisted q-deformation $U_{r'_4}(\mathfrak{o}(3,1))$. Explicit formulas of the co-products $\Delta_{r_4}(\cdot) = F_{r''_4} \Delta_{r'_4}(\cdot) F_{r''_4}^{-1}$ and antipodes $S_{r_4}(\cdot)$ in the complex and real Cartan-Weyl bases of $U_{r'_4}(\mathfrak{o}(3,1))$ will be presented in the outgoing paper [7].

4 Quantum deformations of Poincare algebra

The Poincaré algebra $\mathcal{P}(3,1)$ of the 4-dimensional space-time is generated by 10 elements: the six-dimensinal Lorentz algebra $\mathfrak{o}(3,1)$ with the generators $M_i,\ N_i$ $(i = 1, 2, 3)$:

$$[M_i,\, M_j] \;=\; \imath\epsilon_{ijk}\, M_k, \;\; [M_i,\, N_j] \;=\; \imath\epsilon_{ijk}\, N_k, \;\; [N_i,\, N_j] \;=\; -\imath\epsilon_{ijk}\, M_k, \tag{4.1}$$

and the four-momenta $P_j,\ P_0$ $(j = 1, 2, 3)$ with the standard commutation relations:

$$[M_j,\, P_k] \;=\; \imath\epsilon_{jkl}\, P_l\ , \qquad [M_j,\, P_0] \;=\; 0\ , \tag{4.2}$$

$$[N_j,\, P_k] \;=\; -\imath\delta_{jk}\, P_0\ , \qquad [N_j,\, P_0] \;=\; -\imath P_j\ . \tag{4.3}$$

The physical generators of the Lorentz algebra, M_i, N_i ($i = 1, 2, 3$), are related with the canonical basis $h, h', e_\pm, e'_\pm$ as follows

$$h \;=\; \imath N_3 \;, \qquad e_\pm \;=\; \imath(N_1 \pm M_2), \tag{4.4}$$

$$h' \;=\; -\imath M_3 \;, \qquad e'_\pm \;=\; \imath(\pm N_2 - M_1). \tag{4.5}$$

The subalgebra generated by the four-momenta P_j, P_0 ($j = 1, 2, 3$) will be denoted by $\mathbf{P}$ and we also set $P_\pm := P_0 \pm P_3$.

S. Zakrzewski has shown in [2] that each classical r-matrix, $r \in \mathcal{P}(3,1) \wedge \mathcal{P}(3,1)$, has a decomposition

$$r = a + b + c \;, \tag{4.6}$$

where $a \in \mathbf{P} \wedge \mathbf{P}$, $b \in \mathfrak{o}(3,1) \wedge \mathbf{P}$, $c \in \mathfrak{o}(3,1) \wedge \mathfrak{o}(3,1)$ satisfy the following relations

$$[[c, c]] \;=\; 0 \;, \tag{4.7}$$

$$[[b, c]] \;=\; 0 \;, \tag{4.8}$$

$$2[[a, c]] + [[b, b]] \;=\; t\Omega \quad (t \in \mathbb{R}) \;, \tag{4.9}$$

$$[[a, b]] \;=\; 0 \;. \tag{4.10}$$

Here $[[\cdot, \cdot]]$ means the Schouten bracket. Moreover a total list of the classical r-matrices for the case $c \neq 0$ and also for the case $c = 0$, $t = 0$ was found.[7] It was shown that there are fifteen solutions for the case $c = 0$, $t = 0$, and six solutions for the case $c \neq 0$ where there is only one solution for $t \neq 0$. Thus Zakrzewski found twenty r-matrices which satisfy the homogeneous classical Yang-Baxter equation ($t = 0$ in (4.9)). Analysis of these twenty solutions shows that each of them can be presented as a sum of subordinated r-matrices which almost all are of Abelian and Jordanian types. Therefore these r-matrices correspond to twisted deformations of the Poincaré algebra $\mathcal{P}(3,1)$. We present here r-matrices only for the case $c \neq 0$, $t = 0$:

$$r_1 \;=\; \gamma h' \wedge h + \alpha(P_+ \wedge P_- - P_1 \wedge P_2) \;, \tag{4.11}$$

$$r_2 \;=\; \gamma e'_+ \wedge e_+ + \beta_1(e_+ \wedge P_1 - e'_+ \wedge P_2 + h \wedge P_+) + \beta_2 h' \wedge P_+ \;, \tag{4.12}$$

$$r_3 \;=\; \gamma e'_+ \wedge e_+ + \beta(e_+ \wedge P_1 - e'_+ \wedge P_2 + h \wedge P_+) + \alpha P_1 \wedge P_+ \;, \tag{4.13}$$

$$r_4 \;=\; \gamma(e'_+ \wedge e_+ + e_+ \wedge P_1 + e'_+ \wedge P_2 - P_1 \wedge P_2) + P_+ \wedge (\alpha_1 P_1 + \alpha_2 P_2) \;, \tag{4.14}$$

$$r_5 \;=\; \gamma_1(h \wedge e_+ - h' \wedge e'_+) + \gamma_2 e_+ \wedge e'_+ \;. \tag{4.15}$$

The first r-matrix r_1 is a sum of two subordinated Abelian r-matrices

$$\begin{aligned} r_1 \;&:=\; r'_1 + r''_1 \;, \quad r'_1 \succ r''_1 \;, \\ r'_1 \;&=\; \alpha(P_+ \wedge P_- - P_1 \wedge P_2) \;, \qquad r''_1 \;:=\; \gamma h' \wedge h \;. \end{aligned} \tag{4.16}$$

Therefore the total twist defining quantization in the direction to this r-matrix is the ordered product of two the Abelian twits

$$F_{r_1} \;=\; F_{r''_1} F_{r'_1} \;=\; \exp\bigl(\gamma h' \wedge h\bigr)\, \exp\bigl(\alpha(P_+ \wedge P_- - P_1 \wedge P_2)\bigr) \;. \tag{4.17}$$

[7]Classification of the r-matrices for the case $c = 0$, $t \neq 0$ is an open problem up to now.

The second r-matrix r_2 is a sum of three subordinated r-matrices where two of them are of Abelian type and one is of Jordanian type

$$
\begin{aligned}
r_2 &:= r_3' + r_2'' + r_2''' \ , \quad r_2' \succ r_2'' \succ r_2''' \ , \\
r_2' &:= \beta_1(e_+ \wedge P_1 - e_+' \wedge P_2 + h \wedge P_+) \ , \\
r_2'' &:= \gamma e_+' \wedge e_+ \ , \quad r_2''' := \beta_2 h' \wedge P_+ \ .
\end{aligned}
\tag{4.18}
$$

Corresponding twist is given by the following formulas

$$
F_{r_2} = F_{r_2'''} F_{r_2''} F_{r_2'} \ ,
\tag{4.19}
$$

where

$$
\begin{aligned}
F_{r_2'} &= \exp\bigl(\beta_1(e_+ \otimes P_1 - e_+' \otimes P_2)\bigr)\exp(2h \otimes \sigma_+) \ , \\
F_{r_2''} &= \exp(\gamma e_+' \wedge e_+) \ , \quad F_{r_2'''} = \exp(\beta_2 h' \wedge \sigma_+) \ .
\end{aligned}
\tag{4.20}
$$

Here and below we set $\sigma_+ := \frac{1}{2}\ln(1 + \beta_1 P_+)$.

The third r-matrix r_3 is a sum of two subordinated r-matrices where one is of Abelian type and another is a more complicated r-matrix which we call mixed Jordanian-Abelian type

$$
\begin{aligned}
r_3 &:= r_3' + r_3'' \ , \quad r_3' \succ r_3'' \ , \\
r_3' &:= \beta_1(e_+ \wedge P_1 - e_+' \wedge P_2 + h \wedge P_+) + \alpha P_1 \wedge P_+ \ , \\
r_3'' &:= \gamma e_+' \wedge e_+ \ .
\end{aligned}
\tag{4.21}
$$

Corresponding twist is given by the following formulas

$$
F_{r_3} = F_{r_3''} F_{r_3'} \ ,
\tag{4.22}
$$

where

$$
\begin{aligned}
F_{r_3'} &= \exp\bigl(\beta_1(e_+ \otimes P_1 - e_+' \otimes P_2)\bigr)\exp(\alpha P_1 \wedge \sigma_+)\exp(2h \otimes \sigma_+) \ , \\
F_{r_3''} &= \exp(\gamma e_+' \wedge e_+) \ .
\end{aligned}
\tag{4.23}
$$

The fourth r-matrix r_4 is a sum of two subordinated r-matrices of Abelian type

$$
\begin{aligned}
r_4 &:= r_4' + r_4'' \ , \quad r_4' \succ r_4'' \ , \\
r_4' &:= P_+ \wedge (\alpha_1 P_1 + \alpha_+ P_2) \ , \\
r_4'' &:= \gamma(e_+' - P_1) \wedge (e_+ + P_2) \ .
\end{aligned}
\tag{4.24}
$$

Corresponding twist is given by the following formulas

$$
F_{r_4} = F_{r_4''} F_{r_4'} \ ,
\tag{4.25}
$$

where

$$
\begin{aligned}
F_{r_4'} &= \exp\bigl((P_+ \otimes (\alpha_1 P_1 + \alpha_2 P_2)\bigr) \ , \\
F_{r_4''} &= \exp\bigl(\gamma(e_+' - P_1) \wedge (e_+ + P_2)\bigr) \ .
\end{aligned}
\tag{4.26}
$$

The fifth r-matrix r_5 is the r-matrix of the Lorentz algebra, (3.6), and the corresponding twist is given by the formula (3.13).

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