The decomposition method and Maple procedure for finding first integrals of nonlinear PDEs of any order with any number of independent variables

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Abstract

In present paper we propose seemingly new method for finding solutions of some types of nonlinear PDEs in closed form. The method is based on decomposition of nonlinear operators on sequence of operators of lower orders. It is shown that decomposition process can be done by iterative procedure(s), each step of which is reduced to solution of some auxiliary PDEs system(s) for one dependent variable. Moreover, we find on this way the explicit expression of the first-order PDE(s) for first integral of decomposable initial PDE. Remarkably that this first-order PDE is linear if initial PDE is linear in its highest derivatives.

The developed method is implemented in Maple procedure, which can really solve many of different order PDEs with different number of independent variables. Examples of PDEs with calculated their general solutions demonstrate a potential of the method for automatic solving of nonlinear PDEs.

1 Introduction

Nonlinear partial differential equations (PDEs) play very important role in many fields of mathematics, physics, chemistry, and biology, and numerous applications. If for nonlinear ordinary differential equations (ODEs) one can observe incontestable progress in their automatic solving, the situation for nonlinear PDEs seems as nearly hopeless one.

Despite the fact that various methods for solving nonlinear PDEs have been developed in 19-20 centuries as the suitable groups of transformations, such as point or contact transformations, differential substitutions, and Backlund transformations etc., the most powerful method for explicit integration of *second-order* nonlinear PDEs in *two independent variables* remains the method of Darboux [1]-[4]. The original Darboux method (as already Darboux stated in [1]) is extendable in principle to equations of all orders in an arbitrary number of independent variables, even to systems of equations; however, in [1]-[2] and subsequent papers by many authors, the detailed calculations were performed only for a single second-order equation with one dependent and two independent variables.

The Darboux method was refined in recent years into more precise and efficient (although not completely algorithmic) form [5]-[8] and references therein. Nevertheless this approaches suffer from high complexity and necessitate to use some tricks.

There were some partially successful attempts to extend modern variants of the Darboux method based on Laplace cascade method on higher-order PDEs and PDEs in the space of more than two independent variables [10]-[13] but they suffer from high complexity too.

There is an original approach to the problem, based on the special type of local change of variables which leads to the order reduction of initial PDE, proposed in [14], which is suitable for high dimensions problems but of very special class though.

In present paper we propose seemingly new method for finding solutions of some types of nonlinear PDEs in closed form. The method is based on decomposition of nonlinear operators on sequence of operators of lower orders. It is shown that decomposition process can be done by iterative procedure(s), each step of which is reduced to solution of some auxiliary PDEs system(s) for one dependent variable. Moreover, we find on this way the explicit expression of the first-order PDE(s) for first integral of decomposable initial PDE. Remarkably that this first-order PDE is linear if initial PDE is linear in its highest derivatives.

The developed method is implemented in Maple procedure, which can really solve many of different order PDEs with different number of independent variables. Examples of PDEs with calculated their general solutions demonstrate a potential of the method for automatic solving of nonlinear PDEs.

2 Bases of the method

2.1 Decomposable PDEs

The simplest second-order non-linear PDE for $w = w(t, x)$

$$\frac{\partial^2 w}{\partial t \partial x} = \frac{1}{w} \frac{\partial w}{\partial t} \frac{\partial w}{\partial x} \tag{1}$$

can be easily transformed to the following decomposed form

$$\frac{\partial}{\partial t} \ln\left(\frac{1}{w} \frac{\partial w}{\partial x}\right) = 0\,, \tag{2}$$

from which we can without difficulty obtain the general solution to PDE (1) in two steps. First step gives us

$$\frac{1}{w} \frac{\partial w}{\partial x} = \frac{d \ln(G(x))}{dx}\,, \tag{3}$$

where $G(x)$ is an arbitrary function. And then, solving the equation (3) on the second step, we obtain

$$w(t,x) = F(t)G(x)\,, \tag{4}$$

where $F(t)$ is one more arbitrary function.

The main observations on analyzing the grounds of solvability of the PDE (1) *by the above* method are that

1. The PDE (1) is *"decomposable"*, i.e., it can be represented as a composition of successive differential operators of type (5) (not necessarily linear). It is clear that such type of decomposition can be done for some PDEs of any order and with any number of independent variables in the following manner

$$\begin{aligned} D_1(w) &= u_1\,,\\ D_2(u_1) &= u_2\,,\\ \ldots&\ldots\,,\\ D_n(u_{n-1}) &= 0\,, \end{aligned} \tag{5}$$

where $\vec{x} = (x_1, \ldots, x_m)$, $w = w(\vec{x})$, $u_i = u_i(\vec{x})$ and

$$D_i(u) = V_i(\vec{x}, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_m})\,.$$

Assuming that V_i are arbitrary functions, and eliminating u_i by successive substitutions in system (5), we get a family of PDEs for w of nth order

$$D_n(D_{n-1}(\ldots D_1(w)\ldots)) = 0\,. \tag{6}$$

which are "decomposable" and *in principle* their solutions *general* or *particular* can be obtained by integration of split system (5). The PDE (6) is nonlinear if at least one of the operators D_i is nonlinear. Not all PDEs admit such representation. And in positive cases such representation is not unique in general.

Note that as a matter of fact D_i need not be the first-order differential operators. So the composition procedure for nth order PDE, when $n > 2$ can be as follows

$$\begin{aligned} D_1^{n_1}(w) &= u\,,\\ D_2^{n_2}(u) &= 0\,, \end{aligned} \tag{7}$$

where n_1, n_2 are integers and $n_1 + n_2 = n$, $w = w(\vec{x})$, $u = u(\vec{x})$, and $(k \leq j)$

$$D_i^j(u) = V_i(\vec{x}, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial^k u}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}|_{k_1+\cdots+k_m=k\leq j}}, \ldots, \frac{\partial^j u}{\partial x_m^j})\,.$$

The late representation allows us to carry out the PDE's decomposition or order reduction *gradually* bit by bit.

We have to stress here that in general representations (5) and (7) may have different meaning. For example, some PDEs do not admit representation (5) but permit the form (7) with both solvable DEs.

2. Each step of the solving process for decomposed PDE is faced with the necessity to solve differential equation $D_i(u_{i-1}) = u_i$ (or $D_i^j(u_{i-1}) = u_i$), so all such DEs must be solvable. Note that only first step $D_n(u_{n-1}) = 0$ is *free from arbitrary functions*.

So one of the PDEs solving strategies may be as follows. First of all we try to decompose given PDE. In order to do so we have to solve corresponding auxiliary nonlinear PDE system for unknown functions V_i, it is sufficient to find a particular solution here. And, if it is successful, then, deciding between the variants, try to solve each arising DE from the chain (5). Main obstacle here, beginning at the second step is just mentioned necessity to solve DEs with arbitrary functions. There are sufficiently narrow circle of solvable (in sense of the general solutions) DEs with an arbitrary function as a parameter.

Another (classification) approach can be based on the usage of only solvable DEs. That is, we can form a composition of successive *solvable* differential operators and as a result obtain a families of solvable PDEs. Such a way leads to extensive nontrivial families for different types of nonlinear PDEs which general solutions can be expressed in closed form. But on this way we encounter a difficulty to circumscribe such families integrally and are forced to consider particular subfamilies. Nevertheless it yields extensive field of PDEs for methods testing [15].

2.2 Decomposition algorithm for decomposable PDEs

For nth order PDE, when $n > 2$ there are some slightly different approaches which are dictated by goals of the problem. If the goal is to *decompose* given nonlinear operator then we have to use the scheme (7) with $n_1 = 1$, $n_2 = n - 1$. And conversely we have to use the scheme (7) with $n_1 = n - 1$, $n_2 = 1$ if the goal is to *solve* given PDE. The last procedure in some features resembles the well-known technics of reducing ODEs order, e.g., by first integral method. Of course, it is possible to use intermediate cases.

All above cases can be treated by the same way as we consider below but each of them leads to auxiliary PDEs systems of different order, viz $n_2 + 1$, with corresponding calculation complexity.

In sequel we will consider for shortness only the case with $n_1 = n-1$, $n_2 = 1$, as more practical for PDEs solving.

Let us consider the decomposition of type (7) with $D_1^{n-1}(w)$ as a solution of the following equation with respect of $u = u(\vec{x})$

$$J(u, \vec{x}, w, \frac{\partial w}{\partial x_1}, \ldots, \frac{\partial^k w}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}|_{k_1+\cdots+k_m=k\leq n-1}}, \ldots, \frac{\partial^{n-1} w}{\partial x_m^{n-1}}) = 0 \quad (8)$$

and

$$D_2(u) = V(\vec{x}, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_m})\,. \quad (9)$$

If substitute $u = D_1^{n-1}(w)$ into (9) we obtain decomposable n-th order PDE

$$V(\vec{x}, U_0, U_{x_1}, \ldots, U_{x_m}) = 0\,, \quad (10)$$

where (we use below the following notation $w = W_0$ and $\frac{\partial^k w}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}} = W_{k_1,\ldots,k_m}$)

$$D_1^{n-1}(w) = U_0\,, \tag{11}$$

$$-\frac{\frac{\partial J}{\partial x_i} + \sum \frac{\partial J}{\partial W_{k_1,\ldots,k_m}} W_{k_1^*,\ldots,k_m^*}}{\frac{\partial J}{\partial u}} = U_{x_i} \qquad (i = 1, \ldots, m)\,, \tag{12}$$

where $k_j^* = k_j + 1$ if $j = i$ and $k_j^* = k_j$ otherwise, and it is supposed that differentiations in sum are carried out on all indexed W's which are involved in J.

Here we can introduce U_0 and $U_{x_1}, \ldots, U_{x_m}$ as new independent variables if express m variables from the set $\{W_{k_1,\ldots,k_m}\}$ with $k_1 + \cdots + k_m = n$ using linear system (12).

Assuming that given PDE of order n

$$F(\vec{x}, w, \frac{\partial w}{\partial x_1}, \ldots, \frac{\partial^k w}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}|_{k_1+\cdots+k_m=k\leq n}}, \ldots, \frac{\partial^n w}{\partial x_m^n}) = 0 \tag{13}$$

is decomposable, we receive, that after substitution of the new variables, *left-hand side* of given PDE must turn into (10) with some V.

Left-hand side of given PDE expressed in new variables is the first-order differential expression with respect to

$$J(U_0, \vec{x}, W_0, W_{1,0,\ldots,0}, \ldots, W_{k_1,\ldots,k_m}|_{k_1+\cdots+k_m=k\leq n-1}, \ldots, W_{0,0,\ldots,n-1})$$

and must not depend on all indexed W's, that is derivatives of F expressed in new variables with respect to all indexed W's are equal to zero. Sequence of such derivatives of F equated to zero form a *second-order* PDE system for J. So a solution (particular as well) the PDE system gives possible expression of differential operator $D_1^{n-1}(w)$ through (8) and differential operator $D_2(u)$ by substituting the solution of J into *left-hand side* of given PDE expressed in new variables.

Of course, there are problems where a operator decomposition is required only. But in most cases obtained decomposition is intended for finding solutions for given PDE. If in obtained decomposition the corresponding PDE $D_2(u) = 0$ is solvable, then substitution of obtained u into J expressed in original variables gives us *a first integral* (see its definition in the next subsection) of given PDE. It is easy to see that for decomposable PDEs the first integral is a differential equation, so we can try to solve it or to find a first integral for this new DE (or decompose it) by the scheme described above until we come to the first-order DE.

Remarkably that in the approach under consideration the finding of first integrals can be done more directly and effectively.

2.3 Differential equation for first integral of decomposable PDEs

The first integral I of the PDE is an expression, involving *one* arbitrary function, which is equivalent in some sense to the given PDE. The first integral vanishes on the set of solutions of given PDE. And (in accordance with [4]) all differential consequences of the equation $I = 0$ coincide with respective differential consequences of given PDE (e.g., elimination of the arbitrary function leads to the given PDE).

Our goal here is to find PDE for first integral of a decomposable PDE. To do so we first of all have to take into account that $u(\vec{x})$ is the solution of the corresponding PDE

$$V(\vec{x}, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_m}) = 0\,,$$

so $u(\vec{x})$ depends only on $\vec{x}$ but in no way on indexed W's. Secondly, the dependent variable in this case, namely

$$J(u(\vec{x}), \vec{x}, W_0, W_{1,0,\dots,0}, \dots, W_{k_1,\dots,k_m}|_{k_1+\dots+k_m=k\le n-1}, \dots, W_{0,0,\dots,n-1})$$

of given PDE (13) expressed in new variables do not to depend on $U_{x_1}, \dots, U_{x_m}$ and is *a first integral* of given PDE.

If now consider $u(\vec{x})$ as an unknown function, we can denote the first integral as

$$I(\vec{x}, W_0, W_{1,0,\dots,0}, \dots, W_{k_1,\dots,k_m}|_{k_1+\dots+k_m=k\le n-1}, \dots, W_{0,0,\dots,n-1}) =$$
$$J(u(\vec{x}), \vec{x}, W_0, W_{1,0,\dots,0}, \dots, W_{k_1,\dots,k_m}|_{k_1+\dots+k_m=k\le n-1}, \dots, W_{0,0,\dots,n-1})$$

and instead of (12) in the form

$$\frac{\partial J}{\partial x_i} + \sum \frac{\partial J}{\partial W_{k_1,\dots,k_m}} W_{k_1^*,\dots,k_m^*} = -U_{x_i}\frac{\partial J}{\partial u} \qquad (i = 1, \dots, m)$$

we arrive to the following system

$$\frac{\partial I}{\partial x_i} + \sum \frac{\partial I}{\partial W_{k_1,\dots,k_m}} W_{k_1^*,\dots,k_m^*} = 0 \qquad (i = 1, \dots, m)\,. \tag{14}$$

If express m variables from the set $\{W_{k_1,\dots,k_m}\}$ with $k_1 + \dots + k_m = n$ (at least one of which is actual for given PDE - note that there are some variants here as a rule, so we can obtain *some* consistent PDEs on this stage) using linear system (14) and substitute them into given PDE (13) we receive a *first-order* (even *linear* if PDE (13) is linear in its highest derivatives) PDE with respect to first integral I. And it remains only to solve this PDE(s) to find a first integral of given PDE.

Note, given PDE is decomposable iff exists a solution of such first-order PDE(s).

3 Examples

To facilitate necessary calculations in the process of finding first integrals I have implemented above described method in *prototype* of *Maple* procedure *reduce_PDE_order* (see Appendix). The input data of the procedure are given

PDE of any order and dependent variable of the PDE with any number of independent variables. The procedure tries to find first integral(s) of the input linear or nonlinear PDE.

The *Maple* built-in procedure *pdsolve* is used inside my procedure to solve *the first-order* PDE for first integral. As different *Maple* versions have different PDE solving abilities so the output depends on *Maple* version. In the following examples I refer to *Maple* 11.

The procedure *reduce_PDE_order* is able to find first integrals for many known and unknown linear and nonlinear PDEs. Here we give examples of PDEs for which it is possible to find finally their *general* solutions. More examples one can find in collection of solvable nonlinear PDEs [15].

3.1 Second-order PDE with two independent variables

For PDE ($w = w(t,x)$)

$$\frac{\partial^2 w}{\partial t \partial x} - \frac{a}{w}\left(\frac{\partial w}{\partial x}\right)^2 - \left(\frac{1}{w}\frac{\partial w}{\partial t} + b + \frac{c}{w}\right)\frac{\partial w}{\partial x} - \frac{c}{2aw}\frac{\partial w}{\partial t} - kw - \frac{bc}{2a} - \frac{c^2}{4aw} = 0 \quad (15)$$

with $a \neq 0$ and $4ak - b^2 \neq 0$ the procedure *reduce_PDE_order* outputs the following first integral

$$I = F1\left[x, \frac{t\sqrt{4ak-b^2} - 2\arctan\left(\frac{c + 2a\frac{\partial w}{\partial x} + bw}{w\sqrt{4ak-b^2}}\right)}{\sqrt{4ak-b^2}}\right]$$

with arbitrary function $F1$.

The ODE $I = 0$ can be solved and one obtains (after some hand simplifications and edition) the following *general* solution to (15)

$$w(t,x) =$$

$$-\frac{c}{2a}\{\int \exp\left[\frac{1}{2a}\int \frac{\exp(t\sqrt{b^2-4ak})F(x)(b+\sqrt{b^2-4ak}) - \sqrt{b^2-4ak}+b}{1+\exp(t\sqrt{b^2-4ak})F(x)}dx\right]dx +$$

$$G(t)\}\exp\left[-\frac{1}{2a}\int \frac{\exp(t\sqrt{b^2-4ak})F(x)(b+\sqrt{b^2-4ak}) - \sqrt{b^2-4ak}+b}{1+\exp(t\sqrt{b^2-4ak})F(x)}dx\right],$$

where $F(x)$ and $G(t)$ are arbitrary functions.

3.2 Second-order PDE with four independent variables

For PDE

$$
\begin{aligned}
&A_1\frac{\partial^2 w}{\partial x_1\partial x_4}+A_2\frac{\partial^2 w}{\partial x_2\partial x_4}+A_3\frac{\partial^2 w}{\partial x_3\partial x_4}+C_0+B_1\frac{\partial w}{\partial x_4}+\\
&C_1(A_1\frac{\partial w}{\partial x_1}+A_2\frac{\partial w}{\partial x_2}+A_3\frac{\partial w}{\partial x_3}+B_1 w+B_0)+\\
&C_2(A_1\frac{\partial w}{\partial x_1}+A_2\frac{\partial w}{\partial x_2}+A_3\frac{\partial w}{\partial x_3}+B_1 w+B_0)^2=0\,,
\end{aligned}
\tag{16}
$$

where $w = w(x_1, x_2, x_3, x_4)$ and A_i, B_i, C_i are constants, the procedure *reduce_PDE_order* outputs the following first integral

$$
I = F1\left[x_1,x_2,x_3,x_4+\frac{2\arctan\left(\dfrac{2C_2(A_1\frac{\partial w}{\partial x_1}+A_2\frac{\partial w}{\partial x_2}+A_3\frac{\partial w}{\partial x_3}+B_1w+B_0)+C_1}{\sqrt{4C_0C_2-C_1^2}}\right)}{\sqrt{4C_0C_2-C_1^2}}\right]
$$

with arbitrary function $F1$.

The PDE $I = 0$ can be solved and one obtains the following *general* solution to (16)

$$
\begin{aligned}
&w(x_1,x_2,x_3,x_4)=\\
&-\frac{1}{2A_1C_2}\exp(-\frac{B_1x_1}{A_1})\int_c^{x_1}\exp(\frac{B_1\xi}{A_1})(2B_0C_2+C_1+\tan[\frac{1}{2}x_4\sqrt{4C_0C_2-C_1^2}\\
&+G(\xi,(A_2\xi+A_1x_2-A_2x_1),(A_3\xi+A_1x_3-A_3x_1))]\sqrt{4C_0C_2-C_1^2})d\xi\\
&+\exp(-\frac{B_1x_1}{A_1})F[(A_1x_2-A_2x_1),(A_1x_3-A_3x_1),x_4]\,,
\end{aligned}
$$

where $F(t_1, t_2, t_3)$ and $G(t_1, t_2, t_3)$ are arbitrary functions, c is arbitrary constant.

3.3 Third order PDE with two independent variables

For PDE ($w = w(t, x)$)

$$
w^2\frac{\partial^3 w}{\partial t\partial x^2}-2w\frac{\partial^2 w}{\partial t\partial x}\frac{\partial w}{\partial x}+2\frac{\partial w}{\partial t}\left(\frac{\partial w}{\partial x}\right)^2-w\frac{\partial w}{\partial t}\frac{\partial^2 w}{\partial x^2}-aw^3=0 \tag{17}
$$

the procedure *reduce_PDE_order* outputs the following first integrals

$$
I_1 = F1\left[t,\frac{1}{w^2}(w\frac{\partial^2 w}{\partial t\partial x}-\frac{\partial w}{\partial t}\frac{\partial w}{\partial x}-axw^2),\frac{1}{w^2}\left[ax^2w^2+2w(\frac{\partial w}{\partial t}-x\frac{\partial^2 w}{\partial t\partial x})+2x\frac{\partial w}{\partial t}\frac{\partial w}{\partial x}\right]\right]
$$

and

$$I_2 = F1\left[x, \frac{1}{w^2}\left[w\frac{\partial^2 w}{\partial x^2} - atw^2 - \left(\frac{\partial w}{\partial x}\right)^2\right]\right]$$

with arbitrary function $F1$.

We can form some PDEs from I_1 and to solve them we can repeat the process of order reduction with the procedure *reduce_PDE_order*. The ODE $I_2 = 0$ can be solved directly and one obtains in any way the following *general* solution to (17)

$$w(t,x) = F(t)\exp\left(\frac{atx^2}{2} - xH(t) + x\int G(x)dx - \int xG(x)dx\right),$$

where $F(t)$, $H(t)$ and $G(x)$ are arbitrary functions.

3.4 Fourth order PDE with two independent variables

For PDE ($w = w(t,x)$)

$$w^3\frac{\partial^4 w}{\partial t^2 \partial x^2} - 2w^2\left(\frac{\partial^3 w}{\partial t^2 \partial x}\frac{\partial w}{\partial x} + \frac{\partial^3 w}{\partial t \partial x^2}\frac{\partial w}{\partial t}\right) - 2\left(w\frac{\partial^2 w}{\partial t \partial x} - 2\frac{\partial w}{\partial t}\frac{\partial w}{\partial x}\right)^2 +$$

$$2\left[w\frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial w}{\partial x}\right)^2\right]\left(\frac{\partial w}{\partial t}\right)^2 - w\frac{\partial^2 w}{\partial t^2}\left[w\frac{\partial^2 w}{\partial x^2} - 2\left(\frac{\partial w}{\partial x}\right)^2\right] = 0 \qquad (18)$$

the procedure *reduce_PDE_order* outputs the following first integrals

$$I_1 = F1(t, \frac{1}{w^3}\left[w^2\frac{\partial^3 w}{\partial t^2 \partial x} - 2w\frac{\partial w}{\partial t}\frac{\partial^2 w}{\partial t \partial x} - w\frac{\partial^2 w}{\partial t^2}\frac{\partial w}{\partial x} + 2\frac{\partial w}{\partial x}\left(\frac{\partial w}{\partial t}\right)^2\right],$$

$$\frac{1}{w^3}\left[\left(\frac{\partial^2 w}{\partial t^2} - x\frac{\partial^3 w}{\partial t^2 \partial x}\right)w^2 + \left[\left(2\frac{\partial w}{\partial t}\frac{\partial^2 w}{\partial t \partial x} + \frac{\partial^2 w}{\partial t^2}\frac{\partial w}{\partial x}\right)x - \left(\frac{\partial w}{\partial t}\right)^2\right]w - 2x\frac{\partial w}{\partial x}\left(\frac{\partial w}{\partial t}\right)^2\right])$$

and

$$I_2 = F1(x, \frac{1}{w^3}\left[w^2\frac{\partial^3 w}{\partial t \partial x^2} - 2w\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial t \partial x} - \frac{\partial w}{\partial t}\frac{\partial^2 w}{\partial x^2}w + 2\frac{\partial w}{\partial t}\left(\frac{\partial w}{\partial x}\right)^2\right],$$

$$\frac{1}{w^3}\left[\left(\frac{\partial^2 w}{\partial x^2} - t\frac{\partial^3 w}{\partial t \partial x^2}\right)w^2 + \left[\left(2\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial t \partial x} + \frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial t}\right)t - \left(\frac{\partial w}{\partial x}\right)^2\right]w - 2t\frac{\partial w}{\partial t}\left(\frac{\partial w}{\partial x}\right)^2\right])$$

with arbitrary function $F1$.

The wealth of first integrals here allows us to operate with them in many different ways. Apart from aforesaid subsequent order reduction we can, for example, from

$$\frac{1}{w^3}\left[w^2\frac{\partial^3 w}{\partial t\partial x^2}-2w\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial t\partial x}-\frac{\partial w}{\partial t}\frac{\partial^2 w}{\partial x^2}w+2\frac{\partial w}{\partial t}\left(\frac{\partial w}{\partial x}\right)^2\right]=F(x)$$

and

$$\frac{1}{w^3}[\left(\frac{\partial^2 w}{\partial x^2}-t\frac{\partial^3 w}{\partial t\partial x^2}\right)w^2+\left[\left(2\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial t\partial x}+\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial t}\right)t-\left(\frac{\partial w}{\partial x}\right)^2\right]w-$$
$$2t\frac{\partial w}{\partial t}\left(\frac{\partial w}{\partial x}\right)^2]=G(x)\,,$$

where $F(x)$ and $G(x)$ are arbitrary functions, algebraically eliminate mixed derivative and obtain the following ODE

$$w\frac{\partial^2 w}{\partial x^2}-\left(\frac{\partial w}{\partial x}\right)^2+\left[tF(x)-G(x)\right]w^2=0\,,$$

which gives the *general* solution to (18)

$$w(t,x)=H(t)\exp\left[t\int xF(x)\,dx-tx\int F(x)\,dx+\right.$$
$$\left.x\int G(x)\,dx-\int xG(x)\,dx+xK(t)\right]\,,$$

where $F(x)$, $H(t)$, $G(x)$ and $K(t)$ are arbitrary functions.

4 Conclusion

The method have considered above is efficient enough for solving decomposable PDEs of relatively high order with many independent variables. The main limitation here is concerned with abilities to solve corresponding auxiliary first-order PDEs for first integrals.

An adaptability of the method to PDEs which are not decomposable but which general solutions can be expressed in closed form remains unsolved yet. But it can be shown on examples that there are some ways to extend the method for some types of such PDEs. These approaches deserve further thorough study in another publication.

References

[1] G. Darboux, Sur les equations aux derivees partieles du second ordre. Ann. Sci. Ecole Norm. Sup. 1870, v. 7, pp. 163-173.

[2] G. Darboux, Lecons sur la theorie generale des surfaces. v.II. Paris: Hermann, 1915.

[3] E. Goursat, Lecons sur l'integration des equations aux derivees partieles du second ordre a deux variables independantes. V.I,II. Paris: Hermann, 1896, 1898.

[4] A.R. Forsyth, Theory of differential equations. Part 4. Partial differential equations, vol. 6, Dover Press, New York, 1959.

[5] M. Juras, Generalized Laplace invariants and classical integration methods for second-order scalar hyperbolic partial differential equations in the plane, *Differential Geometry and Applications: Proc., Conf. Brno (Czech Republic), 28 Aug.-1 Sept. 1995*, Brno: Masaric Univ., 1966, pp. 275-284.

[6] M. Juras, Geometric aspects of second-order scalar hyperbolic partial differential equations in the plane, Ph.D. thesis, 1997, Utah State University, USA.

[7] V.V. Sokolov, A.V. Zhiber, On the Darboux integrable hyperbolic equations. Phys. Lett. A, v. 208, pp. 303-308, 1995.

[8] A.V. Zhiber, V.V. Sokolov, Exact integrable Liouville type hyperbolic equations [in Russian], Uspekhi Mat. Nauk, Vol. 56, No. 1, pp. 64-104, 2001.

[9] S.P.Tsarev, On Darboux integrable nonlinear partial differential equations, Proc. Steklov Institute of Mathematics, v. 225, p. 372-381, 1999.

[10] J. Le Roux. Extensions de la methode de Laplace aux equations lineaires aux derivees partielles dordre superieur au second. *Bull. Soc. Math. de France*, v. 27, p. 237262, 1899. A digitized copy is obtainable from http://www.numdam.org/

[11] U. Dini, Sopra una classe di equazioni a derivate parziali di secondordine con un numero qualunque di variabili. *Atti Acc. Naz. dei Lincei. Mem. Classe fis., mat., nat.* (ser. 5) v. 4, 1901, p. 121178. Also Opere v. III, p. 489566.

[12] U. Dini, Sopra una classe di equazioni a derivate parziali di secondordine. *Atti Acc. Naz. dei Lincei. Mem. Classe fis., mat., nat.* (ser. 5) v. 4, 1902, p. 431467. Also Opere v. III, p. 613660.

[13] S.P. Tsarev, On factorization and solution of multidimensional linear partial differential equations. http://arxiv.org/abs/cs.SC/0609075, 2006.

[14] V.M. Boyko, W.I. Fushchych, Lowering of order and general solutions of some classes of partial differential equations, Reports on Math. Phys., V. 41, No. 3, pp. 311-318, 1998.

[15] Yu.N. Kosovtsov, The general solutions of some nonlinear second and third order PDEs with constant and nonconstant parameters. http://arxiv.org/abs/math-ph/0609003 , 2006.

5 Appendix. *Maple* procedure *reduce_PDE_order*

```
reduce_PDE_order:=proc(pde,unk)
local B,W,N,NN,ARG,acargs,i,M,pde0,DN,IND,IND2,IND3,IND4,ARGS,SUB,SUB0,
Z0,Bargs,EQS,XXX,WW,BB,PP,pdeI,IV,s,AN;
    option `Copyright (c) 2006-2007 by Yuri N. Kosovtsov. All rights reserved.`;
N:=PDETools[difforder](op(1,[selectremove(has,indets(pde,function),unk)]));
NN:=op(1,[selectremove(has,op(1,[selectremove(has,indets(pde,function),unk)]),diff)]);
ARG:=[op(unk)];
acargs:={};
for i from 1 to nops(ARG) do
if PDETools[difforder](NN,op(i,ARG))=0 then else acargs:=acargs union {op(i,ARG)}
fi; od;
acargs:=convert(acargs,list);
M:=op(0,unk)(op(acargs));
if type(pde,equation)=true then
pde0:=lhs(subs(unk=M,pde))-rhs(subs(unk=M,pde)) else pde0:=subs(unk=M,pde)
fi;
DN:=[seq(seq(i,i=1..nops(acargs)),j=1..N)];
IND:=seq(op(combinat[choose](DN,i)),i=1..N);
IND2:=seq(op(combinat[choose](DN,i)),i=1..N-2);
IND3:=op(combinat[choose](DN,N-1));
IND4:=op(combinat[choose](DN,N));
ARGS:=op(unk),M,seq(convert(D[op(op(i,[IND2]))](op(0,unk))
(op(acargs)),diff),i=1..nops([IND2]));
SUB:={M=W[0],seq(convert(D[op(op(i,[IND]))](op(0,unk))
(op(acargs)),diff)=W[op(op(i,[IND]))],i=1..nops([IND]))};
SUB0:={W[0]=op(0,unk)(op(ARG)),
seq(W[op(op(i,[IND]))]=subs(M=op(0,unk)(op(ARG)),
convert(D[op(op(i,[IND]))](op(0,unk))(op(acargs)),diff)),i=1..nops([IND]))};
Z0:=B(ARGS,seq(convert(D[op(op(i,[IND3]))](op(0,unk))(op(acargs)),diff),
i=1..nops([IND3])));
Bargs:=op(indets(subs(SUB,Z0),name));
EQS:=convert(subs(SUB,{seq(diff(Z0,op(i,acargs))=0,i=1..nops(acargs))}),diff);
XXX:={seq(W[op(op(i,[IND4]))],i=1..nops([IND4]))};
WW:=select(type,indets(subs(SUB,pde0)), 'name') intersect
{seq(W[op(op(i,[IND4]))],i=1..nops([IND4]))};
BB:=select(has,combinat[choose](XXX, nops(acargs)),WW);
PP:={};
pdeI:={seq({subs(subs(solve(EQS,op(i,BB)),subs(SUB,pde0)))},i=1..nops(BB))};
IV:={seq(W[op(op(i,[IND4]))],i=1..nops([IND4]))};
for s from 1 to nops(pdeI) do
try
AN:=pdsolve(op(s,pdeI),{B},ivars=IV);
```

```
for i from 1 to nops(AN) do
if op(0,lhs(op(i,AN)))=B then
PP:=PP union {rhs(op(i,AN))}
fi;
od;
catch:
end try;
od;
PP:=subs(SUB0,PP);
RETURN(PP);
end proc:
```

Calling Sequence: *reduce_PDE_order*(**PDE**, $f(\vec{x})$);

PDE - partial differential equation;
$f(\vec{x})$ - indeterminate function with its arguments.