Week 3 - Eigenvalues and Eigenvectors

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Problem Set 1:

1. What is the rank of the matrix A?

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 5 & 4 & -2 & -3 \end{bmatrix}$$

To find the rank of matrix A, we need to reduce the matrix into reduced row-echelon form and see how many pivot columns there are. We can get the rank from the pivot variables because every pivot variable is associated with a linearly independent vector in the column space.The number of basis vectors required to span the column space is equal to the number of pivot variables in a matrix. I'll now reduce matrix A using the `echelon` function from the `matlib` library.

```
library(matlib)
A <- matrix(c(1,-1,0,5,2,0,1,4,3,1,-2,-2,4,3,1,-3), nrow = 4)
matlib::echelon(A)
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    0    0    0
## [2,]    0    1    0    0
## [3,]    0    0    1    0
## [4,]    0    0    0    1
```

Now that matrix A has been reduced to row-echelon form, we can count the pivot variables, which will be the counting the 1's along the diagonal. In our case, there are 4 pivot columns, meaning the rank of this matrix is 4.

2. Given an mxn matrix where m > n, what can be the maximum rank? The minimum rank, assuming that the matrix is non-zero?

Given the information that the rows are greater than the columns, our maximum rank can only be less than or equal to n - the number of columns (rank <= n). This calculation is telling us the number of linearly independent column (or row) vectors in the matrix. Had the question not said that m > n, than rank would have to be less than or equal the min(m,n). This is because you can't have more than min(m,n) row/column vectors to span the space - for example if you have a 3 x 5 matrix, your max rank would be 3, meaning that two of the column vectors would not be linearly independent. Assuming that a matrix is non-zero, the minimum rank of a matrix would be 1. This is because if you even have one non-zero value, you are creating single vector that would be linearly independent within the matrix.

3. What is the rank of matrix B?

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 2 & 4 & 2 \end{bmatrix}$$

As we did in question 1, we'll reduce the following matrix to reduced row-echelon form. Once this is complete, we will look at the pivot entries or non-zero rows to determine the rank. This will tell us how many linearly independent rows are in the matrix.

```
B <- matrix(c(1,3,2,2,6,4,1,3,2), nrow = 3)
matlib::echelon(A)
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    0    0    0
## [2,]    0    1    0    0
## [3,]    0    0    1    0
## [4,]    0    0    0    1
```

Based on the reduced matrix above, the rank is 1 for matrix B.

Problem Set 2:

Compute the eigenvalues and eigenvectors of the matrix A. You'll need to show your work. You'll need to write out the characteristic polynomial and show your solution.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \tag{3}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \qquad \det(\lambda I_n - A) = 0$$

$$\lambda_{I_3} - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} \lambda-1 & -2 & -3 \\ 0 & \lambda-4 & -5 \\ 0 & 0 & \lambda-6 \end{bmatrix}$$

use rule of Sarus to determine determinant

$$\begin{bmatrix} \lambda-1 & -2 & -3 & \lambda-1 & -2 \\ 0 & \lambda-4 & -5 & 0 & \lambda-4 \\ 0 & 0 & \lambda-6 & 0 & 0 \end{bmatrix} = (\lambda-1)(\lambda-4)(\lambda-6) + 0 + 0 - 0 - 0 - 0$$

$\boxed{\text{Eigenvalues}} = \lambda = 1$
$\lambda = 4$
$\lambda = 6$

$\boxed{\text{Characteristic Polynomial}}$

$(\lambda-1)(\lambda-4)(\lambda-6)$

$\lambda^2 - 4\lambda - \lambda + 4(\lambda-6)$

$\lambda^3 - 5\lambda^2 + 4\lambda - 6\lambda^2 + 30\lambda - 24$

$p(\lambda) = \lambda^3 - 11\lambda^2 + 34\lambda - 24$

$\boxed{\text{Eigenvectors}}$

$\lambda = 1$

$$\begin{bmatrix} 0 & -2 & -3 \\ 0 & -3 & -5 \\ 0 & 0 & -5 \end{bmatrix}$$

rref

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$v_2 = 0$
$v_3 = 0$
$v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow E_1 = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$

$\lambda = 4$

$$\begin{bmatrix} 3 & -2 & -3 \\ 0 & 0 & -5 \\ 0 & 0 & -2 \end{bmatrix}$$

rref

$$\begin{bmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$v_1 - \frac{2}{3} v_2 = 0$
$v_1 = -\frac{2}{3} v_2$
$v_3 = 0$

$E_4 = \text{Span}\left(\begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix}\right)$

$\lambda = 6$

$$\begin{bmatrix} 5 & -2 & -3 \\ 0 & 2 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

rref

$$\begin{bmatrix} 1 & 0 & -8/5 \\ 0 & 1 & -5/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$v_1 - 8/5 v_3 = 0$
$v_2 = -5/2 v_3 = 0$
$v_1 = -8/5 v_3$
$v_2 = -5/2 v_3$

$E_6 = \text{Span}\left(\begin{bmatrix} \frac{8}{5} \\ \frac{5}{2} \\ 1 \end{bmatrix}\right)$