Week 9 - Central Limit Theorem & Generating Functions

Christian Thieme

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1.The price of one share of stock in the Pilsdorff Beer Company (see Exercise 8.2.12) is given by Y_n on the *n*th day of the year. Finn observes that the differences $X_n = Y_n + 1 - Y_n$ appear to be independent random variables with a common distribution having mean $\mu = 0$ and variance σ^2=1/4. If $Y_1 = 100$, estimate the probability that Y_{365} is:

. a) ≥ 100

We can solve all three of these using the `pnorm` function from R. The `pnorm` is the function that calculates the c.d.f.

```
variance <- 365*(1/4)
standard_dev <- sqrt(variance)
Y1 <- 100

pnorm(Y1 - 100, mean = 0, sd = standard_dev, lower.tail = FALSE)
```

```
## [1] 0.5
```

. b) ≥ 110

```
variance <- 365*(1/4)
standard_dev <- sqrt(variance)
Y1 <- 110

pnorm(Y1-100, mean = 0, sd = standard_dev, lower.tail = FALSE)
```

```
## [1] 0.1475849
```

. c) ≥ 120

```
variance <- 365*(1/4)
standard_dev <- sqrt(variance)
Y1 <- 120

pnorm(Y1-100, mean = 0, sd = standard_dev, lower.tail = FALSE)
```

```
## [1] 0.01814355
```

2. Calculate the expected value and variance of the binomial distribution using the moment generating function.

To solve, we'll let p be the probability of success and 1-p be the probability of failure for the binomial distribution:

$p(X_j) = (\frac{n}{j})p^j(1-p)^{n-j}$

The moment generating function is:

$g(t) = E(e^{tX}) \quad = \quad \sum_{j=1}^{\infty} e^{tx_j} p(x_j)$

so together we get:

$g(t) = \sum_{j=0}^{n} e^{tj}(\frac{n}{j})p^j q^{n-j} \quad = \quad \sum_{j=0}^{n} (\frac{n}{j})(pe^t)^j q^{n-j} \quad = \quad (pe^t + q)^n$

The expected value is:

$g'(0) = n(pe^t + q)^{n-1} pe^t \quad = \quad np \quad (t = 0)$

The variance is:

$g''(0) = n(n-1)p^2 + np \quad = \quad np(1-p)$

3. Calculate the expected value and variance of the exponential distribution using the moment generating function.

$g(t) = \int_0^{\infty} e^{tx} \lambda e^{-\lambda e} dx$

$g(t) = \frac{\lambda e^{(t-\lambda)z}}{t-\lambda} | \frac{\infty}{0}$

$g(t) = \frac{\lambda}{\lambda - t}$

$g(t) = \frac{\lambda}{(\lambda - t)^2}$

$g(t) = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$

$g''(t) = \frac{2\lambda}{(\lambda - t)^3}$

$g''(0) = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$

Expected Value:

$\mu = g'(0) = \lambda^{-1}$

Variance:

$\sigma^2 = g''(0) - g'(0)^2 = \frac{2}{\lambda^2} = \frac{1}{\lambda^2} = \lambda^{-2}$