Taylor Series Approximations

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Perform the following Taylor Series Expansions:

1. $f(x) = \frac{1}{(1-x)}$

$\frac{1}{(1-x)}$ is equivalent to $1 \cdot (1-x)^{-1}$

Taking the first derivative we get: $-(1-x)^{-2} \cdot (-1) = \frac{1}{(1-x)^2}$

The second derivative is: $-2(1-x)^{-3} \cdot (-1) = \frac{2}{(1-x)^3}$

The third derivative is: $-3 \cdot 2 \cdot (1-x)^{-4} \cdot (-1) = \frac{6}{(1-x)^4}$

The fourth derivative is: $-4 \cdot 3 \cdot 2 \cdot (1-x)^{-5} \cdot (-1) = \frac{24}{(1-x)^5}$

Now, evaluating the above derivatives at $x = 0$, we get:

$f(0) = \frac{1}{1-0} = 1 = 0!$

$f'(0) = \frac{1}{(1-0)^2} = 1 = 1!$

$f''(0) = \frac{2}{(1-0)^3} = 2 = 2!$

$f'''(0) = \frac{6}{(1-0)^4} = 6 = 3!$

$f''''(0) = \frac{24}{(1-0)^5} = 24 = 4!$

So:

$1 + \frac{1}{1!}x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 + \frac{24}{4!}x^4 + \ldots$

$= 1 + x + x^2 + x^3 + x^4 \ldots$

$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$

Since the above is a geometric series, it converges when $|x| < 1$

2. $f(x) = e^x$

Since the derivative of e^x is always just e^x, our first through fourth derivatives will all result in e^x.

Again, since all the derivatives are the same, we'll solve for where $x = 0$ and the result will be the same for all four derivatives:

$f(0) = e^0 = 1$

Which leads us to:

$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$

$= 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots$

$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$

3. $f(x) = ln(1+x)$

We know that the derivative of ln is $\frac{1}{x}$. So, taking the first derivative, we get:

$(1+x)^{-1}$

The second derivative would be:

$-1 \cdot (1+x)^{-2}$

The third derivative would be:

$2 \cdot (1+x)^{-3}$

Finally, the fourth derivative would be:

$-3 \cdot 2 \cdot (1+x)^{-4}$

Next, we'll evaluate each of the above derivatives at $x = 0$:

$f(0) = ln(1+0) = ln(1) = 0$

$f'(0) = (1+0)^{-1} = 1$

$f''(0) = -1 \cdot (1+0)^{-2} = -1$

$f'''(0) = 2 \cdot (1+0)^{-3} = 2$

$f''''(0) = -3 \cdot 2 \cdot (1+0)^{-4} = -6$

This leads us to:

$ln(1+x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$

$= 0 + \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}(n-1)!}{n!}(x)^n$

$= \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}(x^n)}{n}$