Liquidity, the Mundell-Tobin Effect, and the Friedman Rule*

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Abstract

We investigate how a positive relation between inflation and capital investment (the Mundell-Tobin effect) affects optimal monetary policy in a framework that combines overlapping generations and new Monetarist models. We find that inflation rates above the Friedman rule are optimal if and only if there is a Mundell-Tobin effect. In the absence of the Mundell-Tobin effect, the Friedman rule is optimal. With a Mundell-Tobin effect, increasing inflation above the Friedman rule leads to a first-order welfare gain from increasing capital investment, and only to a second-order welfare loss from reducing consumption in markets where liquidity matters. We also show that different implementations of the Friedman rule have different welfare implications.

Keywords: *New monetarism, overlapping generations, optimal monetary policy*

JEL codes: E4, E5

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1 Introduction

The Friedman rule for the optimal conduct of monetary policy is the most significant doctrine in monetary theory. Friedman (1969) argues that optimal monetary policy should equate the private opportunity costs of holding money (the nominal interest rate) to its social costs (which are zero). The debate about the optimality of the Friedman rule has continued ever since Friedman proposed it.[1] In this paper, we contribute to this debate by studying whether inflation should be set above the Friedman rule to increase capital investment. This mechanism is referred to as the Mundell-Tobin effect (Mundell (1963) and Tobin (1965)). The Mundell-Tobin effect predicts that an increase in inflation (i.e., a decrease in the return on nominal assets) increases capital investment, as agents substitute away from nominal assets into capital. Since nominal assets are typically more liquid than capital, higher inflation rates also imply that agents hold less liquid assets. Thus, higher inflation rates reduce consumption if certain trades can only be settled with liquid assets. This implies that there is a tradeoff between consumption and capital investment due to the Mundell-Tobin effect. The effect of this tradeoff on optimal monetary policy has been studied previously in the literature. Models focusing on intra-generational trade, e.g. the literature building on Lagos and Wright (2005), typically find that the Friedman rule simultaneously delivers the first-best consumption and capital accumulation even if there is a Mundell-Tobin effect, while models focusing on inter-generational trade, e.g. overlapping generations (OLG) models, find that the Friedman rule is never optimal. This suggests that the roles of money in intra- and inter-generational trade call for different monetary policies, particularly in the presence of a Mundell-Tobin effect. Thus, the goal of this paper is to understand how the tradeoff between consumption and capital investment affects optimal monetary policy in an economy where money is used both for inter- and intra-generational trade. To answer this question, we build an OLG model that also incorporates intra-generational trade à la LW. Importantly, money is more liquid than capital, but capital is more efficient to provide for old-age consumption. The frictions in our model are directly linked to the Mundell-Tobin effect: i.e., there may or may not be a Mundell-Tobin effect in our model depending on parameters, which allows us to study how the presence of a Mundell-Tobin

[1] The Friedman rule has been found to be optimal by Friedman himself in a model with money in the utility (Friedman, 1969), but also in a variety of other monetary models such as cash-in-advance (Grandmont and Younes, 1973; Lucas and Stokey, 1987), spatial separation (Townsend, 1980), and New Monetarism (Lagos and Wright, 2005). Explanations for optimal inflation rates above the Friedman rule include: incomplete taxation (Aruoba and Chugh (2010), Finocchiaro et al. (2018)); theft or socially undesirable activities financed by cash (Williamson (2012), Sanches and Williamson (2010)); labor market frictions (Carlsson and Westermark, 2016); or pecuniary exernalities (Brunnermeier and Sannikov, 2016). In New Keynesian models with sticky prices, a constant price level is typically optimal as this eliminates inefficiencies from price adjustment costs. See Schmitt-Grohé and Uribe (2010) or Fuerst (2010) for an overview on the literature about optimal inflation rates.

effect affects the results.

We find that if capital investment is socially efficient, inflation rates above the Friedman rule are optimal if and only if there is a Mundell-Tobin effect at the Friedman rule. In this case, increasing inflation above the Friedman rule leads to a first-order welfare gain from increasing capital investment, and only to a second-order welfare loss from reducing consumption resulting from intra-generational trade, and the optimal inflation rate lies between the Friedman rule and the population growth rate. Without a Mundell-Tobin effect, this mechanism is not at play and the Friedman rule is optimal. We also show that a Mundell-Tobin effect is more likely to occur at the Friedman rule if agents' demand for goods traded against liquid assets is elastic and if capital is liquid.

From the policymaker's point of view, the fundamental tradeoff in our model is that the Friedman rule delivers efficiency in intra-generational trade, but a constant price level is optimal regarding inter-generational trade. The reason for the latter point is that there are two ways to provide for old-age consumption: accumulation of capital when young, or transfers from young to old agents. The inherent return of such transfers is equal to the population growth rate (which we normalize to 1), while the return on capital is larger than that if capital investment is socially efficient; thus, the planner prefers capital accumulation for old-age consumption. However, if some intra-generational trade can only be settled with money, some old-age consumption must be financed through transfers from young to old agents as the money stock must be passed from old to young agents. Setting the inflation rate equal to one reflects the social return of using money (and thus intergenerational transfers) to acquire old-age consumption, but it inefficiently lowers intra-generational trade. Thus, there is no single money growth rate that allows for efficiency regarding both roles of money, and instead the optimal money growth rate depends on how monetary policy is implemented, the importance of money in intra-generational trade, and how elastic the agent's demand for consumption from these trades is. These results show that using models with infinitely-lived agents misses some important intergenerational aspects of monetary policy.

Model summary. In our model, each period is divided into two subperiods, called CM and DM. Agents are born at the beginning of the CM and live until the end of the CM of the following period; i.e., they are alive for three subperiods.[2] There are two assets in the economy, productive capital and fiat money. In the main body of the paper, we assume a linear return to capital that is high enough to make capital investment socially efficient.[3] When agents are born, they

[2]It is not crucial for our results that agents live for only three subperiods, but this is the easiest way of integrating OLG aspects into LW while keeping the model tractable.

[3]In the appendix, we also consider a case with linear return but socially inefficient capital investment, and a case

immediately learn whether they will be a buyer or a seller during the DM. During the first CM of their lives, all agents can work at linear disutility and accumulate capital and fiat money. In the DM, sellers can work at linear disutility and produce a DM good. Buyers cannot work during the DM, but they get concave utility from consuming the DM good. With some probability, buyers are relocated during the DM. If they are relocated, they can only use fiat money to settle trades, because we assume that capital is immobile and cannot be moved to different locations. If buyers are not relocated, they can use money and capital to purchase goods from sellers. Sellers are never relocated. During the final CM of their lives, buyers return to their original location and have access to all their remaining assets. Both buyers and sellers receive concave utility from consuming during the final CM of their lives. Monetary policy is implemented by paying transfers to either young or old agents. The relocation shock creates a basic liquidity/return tradeoff between money and capital. Since capital pays a weakly higher return than money, it is better suited as a store of value. However, because buyers can use money in the DM even if they are relocated, money is more liquid than capital, and the liquidity of capital decreases in the probability of relocation. As mentioned above, this setup blends standard OLG and LW frameworks - as in standard LW models, buyers and sellers trade with each other during the DM, and as in standard OLG models, young and old agents trade with each other during the CM. Using this setup allows us to identify the different roles of money in inter- and intra-generational trade and to study how this affects capital accumulation and optimal monetary policy.

We first study a benchmark case where all buyers are relocated, meaning that capital is perfectly illiquid. We show that in this case, CM consumption levels are independent of monetary policy and at the first-best level, and running the Friedman rule allows to implement the first-best consumption levels in the DM, but keeps the level of capital accumulation strictly below first best. We identify two channels through which inflation affects capital accumulation: on the one hand, sellers are aware that they can sell less goods in the DM at higher inflation rates, so they accumulate more capital at higher inflation rates to provide for their CM consumption (*seller channel*); on the other hand, buyers hold less capital for CM consumption and tax payments if the real tax payment decreases, which it does when inflation increases (*transfer channel*). The transfer channel is only active when old agents are taxed, so there is always a Mundell-Tobin effect when monetary policy is implemented by taxing the young agents, and it turns out that a constant money stock is optimal in this case. For any deflationary policy, the welfare loss from the reduction in capital accumulation is larger than the gains from increasing the consumption levels in the DM. If old

with concave returns to capital. We show that if capital is socially inefficient, a constant money stock is optimal, and that our main results go through if returns to capital are concave.

agents are taxed instead, whether or not there is a Mundell-Tobin effect depends on the relative strength of the two channels, which is determined by the agents' preferences. If the elasticity of DM consumption is above one, there is a Mundell-Tobin effect, while if it is below one, there is a reverse Mundell-Tobin effect. From this, it follows that the Friedman rule is optimal if the old are taxed and the elasticity of DM consumption is below one; if the elasticity is above one, the optimal money growth rate lies somewhere between the Friedman rule and one, and it is an increasing function of the elasticity of DM consumption in that interval. We also show that for any deflationary policy, welfare is higher if monetary policy is implemented over old buyers only. Next, we analyze the full model with partially liquid capital. By running the Friedman rule, the monetary authority is able to perfectly insure agents against the relocation shock, but then all DM trades are made with money, even though capital would be accepted in some of them. This adds a third channel through which inflation affects capital accumulation, which we call the *liquidity channel*. The higher the liquidity of capital, the more willingly buyers switch to accumulating capital instead of real balances if the return on money decreases. The Mundell-Tobin effect is more likely to occur at the Friedman rule if capital is liquid, and in turn this makes it less likely that the Friedman rule is the optimal monetary policy, even if the old are taxed. In the limit when capital is fully liquid, the Friedman rule is never optimal.

Existing literature. Aruoba and Wright (2003) include capital in a LW framework and finds that capital accumulation is independent of monetary policy if capital is fully illiquid. In our model, this is not true: because the OLG framework allows us to drop quasilinear preferences, capital accumulation is affected by monetary policy even with fully illiquid capital. Lagos and Rocheteau (2008) show that the Mundell-Tobin effect exists in LW models when capital is liquid. However, the Friedman rule still delivers the first-best outcome in their model. In Aruoba et al. (2011), capital is both a liquid asset and an input in DM production. They show that capital accumulation is affected more strongly by inflation if there is price-taking in the DM. Andolfatto et al. (2016) show that if taxes cannot be enforced and therefore the Friedman rule is not feasible, the first-best allocation can be implemented with a cleverly designed mechanism even if the capital stock is too small. In Wright et al. (2018, 2019), the authors study models where capital is traded in frictional markets, and they show that if money is needed to purchase capital, a reverse Mundell-Tobin effect may occur. In our model, a reverse Mundell-Tobin effect may also occur for some parameters, but due to preferences, not frictional markets. Gomis-Porqueras et al. (2020) show that there is a hump-shaped relationship between inflation and aggregate capital, as inflation affects capital accumulation negatively on the extensive margin by reducing the number of firms, besides the usual positive effect on the intensive margin. Probably the paper most closely related to ours is Matsuoka and Watanabe (2019), who build on Williamson (2012). They also study an economy with relocation, money, and capital, but agents live forever and CM-utility is quasilinear. In contrast to our model, they find that the Friedman rule is always optimal even if there is a Mundell-Tobin effect. In Matsuoka and Watanabe (2019) it is never efficient to finance CM-consumption with capital. Thus if the monetary authority runs the Friedman rule it can always achieve efficiency without capital investments. In our model, it is socially efficient to finance CM-consumption with capital investments. Thus, with a Mundell-Tobin effect, there is a welfare cost of running the Friedman rule due to lower socially useful capital investments. This justifies optimal deviations from the Friedman rule in our model but not in theirs. A few papers in the LW literature find optimal deviations from the Friedman rule due to the Mundell-Tobin effect - e.g. Venkateswaran and Wright (2013), Geromichalos and Herrenbrueck (2017), Wright et al. (2018), or Altermatt (2019a). These papers usually exhibit frictions that lead to underinvestment at the Friedman rule (e.g., limited pledgeability, taxes, or wage bargaining), but do not affect the presence of a Mundell-Tobin effect itself. If the frictions are shut down, the Mundell-Tobin effect still exists, but the Friedman rule becomes optimal.

In the OLG literature following Wallace (1980), Azariadis and Smith (1996) show that if there is private information about an agent's type, a Mundell-Tobin effect exists for low levels of inflation, while a reverse Mundell-Tobin effect exists for high levels of inflation. In models with relocation shocks, Smith (2002, 2003) and Schreft and Smith (2002) have claimed to show that the Friedman rule is suboptimal because of the Mundell-Tobin effect.[4] However, OLG models typically find deviations from the Friedman rule to be optimal even without the Mundell-Tobin effect, as in Weiss (1980), Abel (1987), or Freeman (1993).[5] Bhattacharya et al. (2005) and Haslag and Martin (2007) build on these results to show that for optimal deviations from the Friedman rule found in Smith (2002) and the other papers mentioned, the Mundell-Tobin effect is not necessary. Instead, standard properties of OLG models, in particular the presence of inter-generational transfers, are sufficient to find these deviations to be optimal. The debate whether the Mundell-Tobin effect itself can render deviations from the Friedman rule optimal in an OLG environment thus remained unsettled.[6] By transferring the basic OLG-structure into a LW model, we can study these issues

[4]See also Schreft and Smith (1997), which focuses on positive inflation rates, but endogenizes the return on capital.

[5]Matsuoka (2011) shows that the Friedman rule becomes optimal again with a monopolistic banking sector.

[6]There is a further complication in the welfare analysis of OLG models due to the absence of a representative agent. Freeman (1993) shows that the Friedman rule is typically Pareto optimal, but not maximizing steady state utility in OLG models. In this paper, we are going to focus on steady-state optimality, but we also discuss Pareto optimality in Appendix C.

more carefully. The combined model gives us more flexibility by introducing additional heterogeneity with buyers and sellers (besides young and old generations) and an additional market (DM). As a result we may or may not observe a Mundell-Tobin effect at the Friedman rule (depending on preferences and the liquidity of capital) and we are able to investigate whether the Mundell-Tobin effect is a necessary condition for the suboptimality of the Friedman rule. Within this framework, we can show that if a Mundell-Tobin effect arises, deviations from the Friedman rule are always optimal. The richer model also allows us to investigate the different channels through which inflation affects capital investments, and we can separate liquidity and store of value properties of money and capital.

A combination of OLG and LW structures has first been studied by Zhu (2008). In his model, agents do not know their type during the first CM when they are able to accumulate assets. Therefore, the Friedman rule can be suboptimal for some parameters, as it makes saving relatively cheap and reduces the sellers' willingness to produce in the DM. In contrast to this, our model follows Altermatt (2019b) by assuming that each agent knows their type. Hiraguchi (2017) extends the model of Zhu (2008) by including capital and shows that the Friedman rule remains suboptimal in this case. In another recent paper that combines OLG and LW, Huber and Kim (2020) show that the Friedman rule can be suboptimal if old agents face a higher disutility of labor than young agents. We replicate this result in the case of socially inefficient capital investment which we discuss in Appendix A.

We think that our paper contributes to the existing literature in a number of important ways. First, it is able to reconcile Smith (2002) with Bhattacharya et al. (2005) and Haslag and Martin (2007), by showing that even in a model with an OLG structure, the Friedman rule can be optimal for some parameters, but that it is never optimal when there is a Mundell-Tobin effect. This supports the claims by (Smith, 2002, 2003) that the Mundell-Tobin effect can be the source of optimal deviations from the Friedman rule. Second, our paper shows that in an LW model with intergenerational aspects and strictly concave CM-utility, capital accumulation is affected by monetary policy even if capital is illiquid, and capital accumulation can be inefficiently low at the Friedman rule due to the Mundell-Tobin effect. Third, we made some advances in understanding the frictions that arise in a model that combines OLG and LW, most importantly by showing that the timing of monetary policy implementation matters for welfare in these models.

Outline. The rest of this paper is organized as follows. In Section 2, the environment and the planner's solution is explained. In Section 3, we present the market outcome for perfectly liquid capital, and in Section 4, we discuss the market outcome for perfectly illiquid capital and monetary policy implementation. Section 5 presents the results of the full model, and finally, Section 6 concludes.

2 The model

Our model combines the Lagos and Wright (2005) environment and an overlapping generations (OLG) structure with relocation shocks such as Townsend (1987) or Smith (2002). Time is discrete and continues forever. Each period is divided into two subperiods: First, the centralized market (CM) takes place, followed by the decentralized market (DM). There are two distinct locations, which we sometimes call islands. The two locations are completely symmetric, and everything described happens simultaneously on both islands.[7] A new generation of agents is born at the beginning of each period, consisting of a unit mass of buyers and a unit mass of sellers on each island. An agent born in period t lives until the end of the CM in period $t+1$. In the CM of period 0 there is also a unit mass of sellers called the "initial old" on each island. Figure 1 gives an overview of the sequence of subperiods and the lifespans of generations. There is also a single monetary authority in charge of monetary policy on both islands.

Both buyers and sellers are able to produce a general good x during the first CM of their life at linear disutility h, whereas incurring disutility h yields h units of x; during the second CM of their life, buyers and sellers both receive utility from consuming x . During the DM, sellers can produce special goods q at linear disutility; buyers receive utility from consuming q. A fraction π of buyers are relocated during the DM, meaning that they are transferred to the other island without the ability to communicate with their previous location. Sellers are not relocated. During the CM, relocated buyers return to their original location for the final CM of their life.[8] Relocation occurs randomly, so for an individual buyer, the probability of being relocated is π. Buyers learn at the beginning of the DM whether they are relocated. Buyers' preferences are

$$\mathbb{E}_t\{-h_t^b + \pi \left[u(q_t^m) + \beta U(x_{t+1}^m)\right] + (1-\pi)\left[u(q_t^b) + \beta U(x_{t+1}^b)\right]\}, \tag{1}$$

where the expectation is taken over π, and superscript m denotes consumption of relocated buyers (movers). Buyers discount the second period of their life by $\beta \in (0,1)$, get linear disutility h from producing x during the CM when young, and gain utility $u(q)$ from consuming q in the DM

[7] To simplify notation, we state market-clearing and other conditions in terms of a single island throughout the paper, unless otherwise stated.

[8] In Smith (2002), agents live for two periods and relocation occurs during the second period, meaning that all assets they cannot spend during that period are wasted from their point of view. Our model crucially differs from Smith (2002) in that regard, as our agents have access to all their assets during the final period of their life.

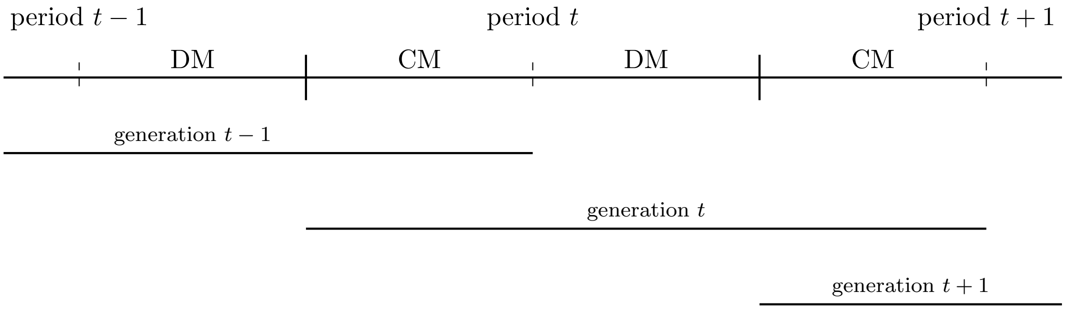

Figure 1: Timeline with lifespans of generations.

and $U(x)$ from consuming x in the CM when old, with $U(0) = u(0) = 0$, $u'(q) > 0 > u''(q)$, $U'(x) > 0 > U''(x)$, $U'(0) = u'(0) = \infty$, and $\bar{q} = u(\bar{q})$ for some $\bar{q} > 0$.[9] Sellers' preferences are

$$-h_t^s - q_t^s + \beta U(x_{t+1}^s). \tag{2}$$

Sellers also discount the second period of their life by β and get utility $U(x)$ from consuming in the CM. They get linear disutility from producing in the CM and the DM.

The monetary authority issues a stock of fiat money M_t per island, which it can produce without cost. ϕ_t denotes the value of M_t in terms of x_t, which implies gross inflation is ϕ_t/ϕ_{t+1}. The growth rate of M_t is $\gamma_t = M_t/M_{t-1}$. Newly-printed fiat money is distributed to young or old buyers via lump-sum transfers (or lump-sum taxes if $\gamma < 1$).[10] We denote transfers to young buyers as τ^y, and transfers to old buyers as τ^o. We use the indicator variable $\mathcal{I}$ to denote which generation receives transfers. If $\mathcal{I} = 1$ ($\mathcal{I} = 0$), young buyers (old buyers) get transfers, which means the monetary authority sets τ^y (τ^o) such that the money growth rate γ_t can be implemented, while $\tau^o = 0$ ($\tau^y = 0$). In period 0, the monetary authority issues M_0 to initially old sellers.

During the CM, agents can also invest general goods into capital. We assume a linear return on capital, so one unit of k delivers R goods in the CM of the following period and fully depreciates after production.[11] In contrast to money, capital is immobile, i.e., it is impossible to move it to other locations during the DM. Further, agents cannot verify claims on capital.

[9]We also assume strictly convex marginal utility in the CM, i.e. $U'''(x) > 0$ to simplify some proofs in Section 5. Most commonly used utility functions satisfy this assumption.

[10]As we will show in this paper, the exact timing of the lump-sum taxes is irrelevant for consumption allocations, but not for welfare. Assuming that only buyers are taxed is without loss of generality.

[11]The capital in our model can also be interpreted as a storage technology. In Appendix B, we consider a more general production function with endogenous R and show that our main results go through.

2.1 Planner's problem

We focus on maximizing expected steady-state welfare of a representative generation.[12] We denote total CM labor as H and total capital as K, and we impose that $H \geq K$, i.e., that steady-state consumption cannot be financed through a pre-existing capital stock. Then, the planner's problem is

$$\max_{H,K,q^b,q^m,q^s,x^b,x^m,x^s} -H - q^s + \pi\left[u(q^m) + \beta U(x^m)\right] + (1-\pi)\left[u(q^b) + \beta U(x^b)\right] + \beta U(x^s) \tag{3}$$

$$s.t. \qquad \pi q^m + (1-\pi)q^b = q^s \tag{4}$$

$$H \geq K \tag{5}$$

$$\pi x^m + (1-\pi)x^b + x^s + K = RK + H, \tag{6}$$

where the first and third constraints are the resource constraints in the DM and CM, respectively. It is easy to see that first-best level of DM consumption q^* is given by

$$q^b = q^m = q^s = q^* \quad \text{solving} \quad u'(q^*) = 1. \tag{7}$$

To finance CM consumption, the planner has two options: Either young agents work for the old and CM consumption is financed by transfers, or young agents work to invest in capital and consume the returns when old. Denoting λ as the Lagrange multiplier for (5), the FOCs for CM consumption and capital are

$$x^b = x^m = x^s = x^* \quad \text{solving} \quad \beta U'(x^*) = 1 - \lambda$$

$$R - 1 \leq R\lambda \quad \text{with equality if} \quad K > 0.$$

Suppose first (5) binds, which happens if $R > 1$. Then, the planner wants to use capital, so agents only work to build up the capital stock. x^*, H^* and K^* are

$$R\beta U'(x^*) = 1 \tag{8}$$

$$H^* = K^* = \frac{2x^*}{R}. \tag{9}$$

If instead $R \leq 1$, the planner only uses intergenerational transfers and capital investment is zero. H^* and x^* are then given by

$$\beta U'(x^*) = 1 \tag{10}$$

[12] Steady-state welfare places an equal weight on the utility of each generation. Since there is an infinite number of future generations but only a single initial old generation, this welfare objective asymptotically ignores the welfare of the initial old. For this and other reasons, this welfare measure is commonly used in the OLG literature, including in the papers we want to compare our results to, such as Smith (2002) and Haslag and Martin (2007). We also believe that focusing on steady-state welfare allows us to compare our results more easily to papers with infinitely-lived agents, such as Lagos and Rocheteau (2008). To complete our analysis, we discuss Pareto-welfare and the effects of policy changes on the agents who are old at the time of the policy change in Appendix C.

$$H^* = 2x^*. \tag{11}$$

To have a meaningful tradeoff between money and capital we assume in the following that CM consumption should be financed with capital, i.e., that investing in capital is socially efficient and more specifically that

$$R\beta = 1, \tag{12}$$

which ensures $R > 1$ and has the added benefit of simplifying the definition of the Friedman rule, as we will discuss in a moment.[13]

Before moving on, let us introduce two important concepts. First, we define the Friedman Rule (FR) as the inflation rate for which the opportunity cost of holding money is zero. This is true for $\phi/\phi_{+1} = 1/R$.[14] Second, we define a Mundell-Tobin effect (MT-E) as a positive relation between capital and inflation, i.e., as $\partial K/\partial\left(\phi/\phi_{+1}\right) > 0$, and a reverse MT-E as $\partial K/\partial\left(\phi/\phi_{+1}\right) < 0$.

2.2 Market outcomes

In the DM, q is sold in competitive manner.[15] As is standard in the LW literature, we assume that frictions such as anonymity and the lack of a record-keeping technology hinder credit in the DM, so all trades have to be settled immediately. Therefore, buyers have to transfer assets to sellers in order to purchase q. Further, because k cannot be transported to other locations and claims on k are not verifiable, relocated buyers can only use m to settle trades. Nonrelocated buyers can use m and k to purchase q. The relocation shock makes k less liquid than m and thus introduces a basic liquidity-return tradeoff between m and k, with $1 - \pi$ representing the liquidity of capital.[16] In the DM, all buyers face the same nominal price p_t regardless of their means of payment. In the CM, x is also sold competitively. Because the problem is symmetric, we focus on one location for the remainder of the analysis.

[13] For completeness, Appendix A studies an economy with $R < 1$.

[14] The remainder of the paper applies to any $R > 1$. However, by assuming $R\beta = 1$ the inflation rate at the FR also equals β, which is the standard definition of the FR in LW models.

[15] Zhu (2008) studies an economy with bilateral meetings and ex-ante uncertainty about an agent's type in a model that is otherwise similar to ours, and shows that these frictions can make deviations from the FR optimal under some conditions. By assuming fixed types and competitive markets, we want to highlight that our results stem from different frictions than those present in Zhu (2008).

[16] One could also use search frictions to microfound the liquidity of k. E.g., we could assume DM trade is bilateral, and buyers meet a seller who accepts m and k with probability $1 - \pi$, while the probability of meeting a seller who accepts only m is π. Our results go through qualitatively under this alternative specification, but the analysis in Section 5 is slightly more complicated. Further, using a relocation shock allows us to compare our results more directly to Smith (2002) and Haslag and Martin (2007).

Buyer's lifetime problem

A buyer's value function at the beginning of his life is

$$
\begin{aligned}
V^b = \max_{h_t, q_t^m, q_t^b, x_{t+1}^m, x_{t+1}^b} \quad & -h_t + \pi\big(u(q_t^m) + \beta U(x_{t+1}^m)\big) + (1-\pi)\big(u(q_t^b) + \beta U(x_{t+1}^b)\big) \\
s.t. \quad & h_t + \mathcal{I}\tau_t^y = \phi m_t + k_t^b \\
& p_t q_t^m \leq m_t \\
& p_t q_t^b \leq m_t + \frac{R k_t^b}{\phi_{t+1}} \\
& x_{t+1}^m = \phi_{t+1} m_t + R k_t^b - \phi_{t+1} p_t q_t^m + (1-\mathcal{I})\tau_{t+1}^o \\
& x_{t+1}^b = \phi_{t+1} m_t + R k_t^b - \phi_{t+1} p_t q_t^b + (1-\mathcal{I})\tau_{t+1}^o .
\end{aligned}
$$

Variables with superscript b indicate decisions of buyers prior to learning about relocation, or those of nonrelocated buyers, depending on the context. The first constraint is the budget constraint for the portfolio choice when young. The second constraint denotes that relocated buyers cannot spend more than their money holdings during the DM, and the third constraint denotes that nonrelocated buyers cannot spend more than their total wealth for consumption during the DM.[17] The fourth and fifth constraints denote that buyers use all remaining resources for consumption when old.

We only consider inflation rates where k (weakly) dominates m in terms of return, i.e. $\phi_{t+1}/\phi_t \leq R$. In this case, the second constraint always holds at equality, as there is no reason for buyers to save m for the CM if k pays a higher return. We assume throughout the paper that if agents are indifferent between m and k, they only hold k. Further, we restrict our attention to cases where the third constraint does not bind.[18] After simplification, the buyer's problem is

$$
\begin{aligned}
V^b = \max_{m_t, k_t^b, q_t^b} \quad & \mathcal{I}\tau_t^y - \phi_t m_t - k_t^b + \pi\left(u\left(\frac{m_t}{p_t}\right) + \beta U(R k_t^b + (1-\mathcal{I})\tau_{t+1}^o)\right) \\
& + (1-\pi)\big(u(q_t^b) + \beta U(\phi_{t+1} m_t + R k_t^b - \phi_{t+1} p_t q_t^b + (1-\mathcal{I})\tau_{t+1}^o)\big).
\end{aligned} \tag{13}
$$

Seller's lifetime problem

A seller's value function at the beginning of his life is

$$
V^s = \max_{h_t, q_t^s, x_{t+1}^s} \quad - h_t^s - q_t^s + \beta U(x_{t+1}^s)
$$

[17] The purchasing power of capital is scaled by R/ϕ_{t+1} to ensure that buyers give up the same amount of CM consumption by paying with capital and money.

[18] This constraint may bind if τ^o is large enough to finance a buyer's optimal CM consumption, such that he optimally spends all his funds in the DM. Note that this constraint only becomes relevant for the analysis in Section 5, and never binds if $\mathcal{I} = 1$. With $\mathcal{I} = 0$, it never binds for $\gamma \leq 1$, but it may bind for sufficiently large γ.

$$\begin{aligned} s.t. \quad h_t^s &= k_t^s \\ x_{t+1}^s &= Rk_t^s + \phi_{t+1} p_t q_t^s \end{aligned}$$

where we imposed that sellers do not accumulate m in the first CM, which is true for $\phi_{t+1}/\phi_t \leq R$. The first constraint denotes sellers work to accumulate k, and the second constraint denotes that a seller's CM consumption is equal to the return on k plus his revenue from selling q in the DM. After simplification, the seller's problem is

$$V^s = \max_{q_t^s, k_t^s} \quad -k_t^s - q_t^s + \beta U(Rk_t^s + \phi_{t+1} p_t q_t^s). \tag{14}$$

We now proceed by first studying the corner cases of $\pi = 0$ and $\pi = 1$, i.e., perfectly liquid and illiquid k, respectively. Then we will analyze the full model with $\pi \in (0, 1)$.

3 Equilibrium with perfectly liquid capital

Suppose $\pi = 0$, which implies that no relocation occurs and k is perfectly liquid as all buyers can use k during the DM. As m and k are equally liquid and safe in this case, agents only hold the asset with the higher rate of return. For $\phi_{t+1}/\phi_t \leq R$, k has a weakly higher return, so we abstract from m and monetary policy in this section. Given this, the buyer's problem from equation (13) becomes

$$V^b = \max_{k_t^b, q_t^b} \quad -k_t^b + u(q_t^b) + \beta U((k_t^b - \rho_t q_t^b)R),$$

where ρ_t denotes the price of q_t in terms of k_t.[19] The first-order conditions are

$$q^b: \quad u'(q^b) = \rho_t \beta R U'((k_t^b - \rho_t q_t^b)R) \tag{15}$$

$$k^b: \quad 1 = \beta R U'((k_t^b - \rho_t q_t^b)R). \tag{16}$$

The seller's problem is only affected by the change in notation. Solving equation (14) yields

$$q^s: \quad 1 = \beta R \rho_t U'((k_t^s + \rho_t q_t^s)R) \tag{17}$$

$$k^s: \quad 1 = \beta R U'((k_t^s + \rho_t q_t^s)R). \tag{18}$$

We see from equations (16) and (18) that $x^b = x^s = x^*$. Combining equations (17) and (18) gives $\rho_t = 1$ assuming optimal capital holdings of sellers are interior, which means that DM prices are such that sellers are indifferent between working in the CM or the DM.[20] Then, combining this

[19] Since p denotes the price of q in terms of m, we have to change notation here as p is undefined if m is not held in equilibrium.

[20] To guarantee that sellers are willing to work in both markets at the first-best, utility functions have to be such that sellers want to consume at least as much in the CM as they receive from selling q^* at $\rho = 1$. Thus we need $U'(q^* R) \geq 1/(\beta R)$. We assume this holds for the remainder of the paper.

with equations (15) and (16) yields

$$u'(q^b) = 1,$$

so $q^b = q^s = q^*$. Furthermore, it is easily confirmed that $H = H^*$ and $K = K^*$. Thus, perfectly liquid capital allows to implement the planner's solution.

4 Equilibrium with perfectly illiquid capital

Consider now $\pi = 1$, which implies k is perfectly illiquid as all buyers are relocated. Then, for $\phi_{t+1}/\phi_t \leq R$, buyers face no tradeoff between holding m and k, as only m allows them to acquire q, while k (weakly) dominates in terms of providing x.

With $\pi = 1$, the buyer's lifetime value function (13) simplifies to

$$V^b = \max_{m_t^b, k_t^b} \quad \mathcal{I}\tau^y - \phi_t m_t^b - k_t^b + u\left(\frac{m_t^b}{p_t}\right) + \beta U(Rk_t^b + (1-\mathcal{I})\tau^o).$$

Solving this problem yields two first-order conditions:

$$m_t : \qquad p_t\phi_t = u'\left(\frac{m_t^b}{p_t}\right) \tag{19}$$

$$k_t : \qquad \frac{1}{\beta R} = U'(Rk_t^b + (1-\mathcal{I})\tau^o), \tag{20}$$

while solving the seller's problem (14) yields the following first-order conditions:

$$q^s : \quad 1 = \phi_{t+1} p_t \beta_t U'(Rk_t^s + p_t q_t^s \phi_{t+1}) \tag{21}$$

$$k^b : \quad 1 = \beta R U'(Rk_t^s + p_t q_t^s \phi_{t+1}). \tag{22}$$

We first derive the stationary equilibrium when $\mathcal{I} = 1$. In a stationary equilibrium we must have: $q^m = q^s$ (DM clearing), $m_t^b = M_t$ (money market clearing) and $\phi/\phi_{+1} = \gamma$ i.e., the inflation rate must equal the growth rate of the money supply since the real value of money is constant over time, implying $\phi M = \phi_{+1}M_{+1}$. Furthermore the real value of transfers to young buyers is $\tau^y = \phi(M - M_{-1}) = \frac{\gamma-1}{\gamma}\phi M$. Using this, we can then define a stationary equilibrium with $\pi = 1$ as a list of eight variables $\{h^b, h^s, k^b, k^s, \phi_{+1}M, q^m, x^b, x^s\}$ solving:

$$u'(q^m) = \gamma R \tag{23}$$

$$x^m = x^s = x^* \tag{24}$$

$$\phi_{+1}M = q^m R \tag{25}$$

$$k^{b,\mathcal{I}=1} = \frac{x^*}{R} \tag{26}$$

$$h^{b,\mathcal{I}=1} = q^m R + \frac{x^*}{R} \tag{27}$$

$$k^s = \frac{x^*}{R} - q^m \tag{28}$$

$$h^s = k^s, \tag{29}$$

and total capital investment and labor supply with $\mathcal{I} = 1$ are

$$K^{\mathcal{I}=1} = k^{b,\mathcal{I}=1} + k^s = \frac{2x^*}{R} - q^m \tag{30}$$

$$H^{\mathcal{I}=1} = h^{b,\mathcal{I}=1} + h^s = \frac{2x^*}{R} + q^m(R-1). \tag{31}$$

Next, we derive the stationary equilibrium when $\mathcal{I} = 0$. The real value of transfers to old buyers is $\tau^o = \phi_{+1}M(\gamma - 1)$. Thus under stationarity $\tau^y = \tau^o = \tau$ for a given γ, and equilibrium consumption levels are unaffected by the different monetary policy regimes. The only changes in the equilibrium allocation affect the labor supply (equation (27)) and capital accumulation of buyers (equation (26)). These now are

$$h^{b,\mathcal{I}=0} = \gamma q^m R + \frac{x^*}{R} - q^m(\gamma - 1) \tag{32}$$

$$k^{b,\mathcal{I}=0} = \frac{x^*}{R} - q^m(\gamma - 1), \tag{33}$$

and thus aggregate capital and labor supply are

$$K^{\mathcal{I}=0} = \frac{2x^*}{R} - \gamma q^m \tag{34}$$

$$H^{\mathcal{I}=0} = \frac{2x^*}{R} + \gamma q^m(R-1). \tag{35}$$

Comparing these with equations (30) and (31) leads us to our first proposition.

Proposition 1. *If $\gamma > 1$, $K^{\mathcal{I}=0} < K^{\mathcal{I}=1}$ and $H^{\mathcal{I}=0} > H^{\mathcal{I}=1}$, while if $\gamma < 1$, $K^{\mathcal{I}=0} > K^{\mathcal{I}=1}$ and $H^{\mathcal{I}=0} < H^{\mathcal{I}=1}$. For $\gamma = 1$, the two equilibria coincide.*

To put this in words, if monetary policy is inflationary ($\gamma > 1$, so $\tau > 0$) total work can be kept lower if monetary policy is implemented over young buyers, whereas the opposite is true if monetary policy is deflationary ($\gamma < 1$, so $\tau < 0$). The reason for this is that $R > 1$; if $\tau > 0$, it is better to receive it when young and invest it in k, whereas if $\tau < 0$, it is better to use the return on k to pay it when old instead of paying it directly from labor income when young.[21]

Next, we characterize a FR equilibrium where $\gamma = 1/R$:

Proposition 2. *For $\pi = 1$, $q^m = q^*$ and $x^s = x^m = x^*$ at the FR. However, $H^{\mathcal{I}=1}|_{FR} > H^{\mathcal{I}=0}|_{FR} > H^*$.*

[21]One could also consider giving a transfer to all buyers, both young and old. But with a linear return on capital, it is strictly better to make transfers to either only young or old buyers. As discussed here, which of those is better depends on the sign of $\gamma - 1$.

It can easily be seen from equation (23) that $q^m = q^*$ for $\gamma = 1/R$, and equation (24) shows that $x^s = x^b = x^*$ always holds. Thus, consumption is efficient at the FR. K and H however are not efficient, so the FR equilibrium does not allow to implement the first-best allocation. From (30) and (34) we see that $K^{\mathcal{I}=1}|_{FR} = 2x^*/R - q^*$ and $K^{\mathcal{I}=0}|_{FR} = 2x^*/R - q^*/R$. Thus capital investment is too low in both monetary policy regimes compared to the first-best level. This is not surprising, as sellers can only be compensated with money, which implies that their CM consumption is partially financed through transfers from young to old agents - old sellers enter the CM with money and use it to purchase consumption goods from young buyers. At the first best, all CM consumption is financed with capital investment, so these inter-generational transfers imply an inefficiency. The inefficiency shows up in H, which is too high compared to the first best:

$$H^{\mathcal{I}=1}|_{FR} = \frac{2x^*}{R} + q^*(R-1) > H^{\mathcal{I}=0}|_{FR} = \frac{2x^*}{R} + \frac{q^*(R-1)}{R} > H^*.$$

Since $R > 1$, it can easily be seen that implementing the FR by taxing old buyers is more efficient - it allows to achieve the same consumption levels at strictly lower hours worked.[22] Proposition 2 shows that even though consumption is at the first-best level at the FR, there is still a welfare loss from hours worked, so it is not obvious that the FR is welfare-maximizing.

Next, we investigate the effects of inflation on H and K for $\mathcal{I} = \{0, 1\}$. For this, denote the coefficient of relative risk aversion as $\eta(q) = -qu''(q)/u'(q)$, and the elasticity of DM consumption with respect to inflation as $|\varepsilon_{q^m}|$.

Proposition 3. $|\varepsilon_{q^m}| = \frac{1}{\eta(q^m)}$ *with* $\pi = 1$. *For* $\mathcal{I} = 0$ *we have*

1. *If* $|\varepsilon_{q^m}| < 1$*:* $\frac{\partial H^{\mathcal{I}=0}}{\partial \gamma} > 0$ *and* $\frac{\partial K^{\mathcal{I}=0}}{\partial \gamma} < 0$*: Reverse MT-E.*

2. *If* $|\varepsilon_{q^m}| > 1$*:* $\frac{\partial H^{\mathcal{I}=0}}{\partial \gamma} < 0$ *and* $\frac{\partial K^{\mathcal{I}=0}}{\partial \gamma} > 0$*: MT-E.*

3. *If* $|\varepsilon_{q^m}| = 1$*:* $\frac{\partial H^{\mathcal{I}=0}}{\partial \gamma} = 0$ *and* $\frac{\partial K^{\mathcal{I}=0}}{\partial \gamma} = 0$*: No MT-E.*

With $\mathcal{I} = 1$, $\frac{\partial H^{\mathcal{I}=0}}{\partial \gamma} < 0$ *and* $\frac{\partial K^{\mathcal{I}=0}}{\partial \gamma} > 0$ *and thus there is an MT-E* $\forall |\varepsilon_{q^m}|$.

The proof to this proposition can be found in Appendix D. There are two channels through which γ affects K when $\pi = 1$. First, equation (28) shows that k^s is decreasing in q^m - and since q^m is decreasing in γ from equation (23), k^s is increasing in γ. The intuition is that sellers accumulate less k if they expect to sell more q. This is the first channel through which capital accumulation is affected by the inflation rate, and it is active independent of the tax regime. Since it affects the sellers' capital accumulation, we call it the *seller channel.* Second, equation (33) shows that when

[22] Remember we assume that if agents are indifferent between m and k, they accumulate k. This implies that agents only use k to save for old age at the FR, not m.

$\mathcal{I} = 0$, k^b depends on γ and on q^m, which is decreasing in γ. Since equation (26) shows that k^b is independent of γ for $\mathcal{I} = 1$, this channel is only active when $\mathcal{I} = 0$, so we call this the *transfer channel.* The effect of γ on k^b through the transfer channel has two components: on the one hand, higher γ decreases ϕ. On the other hand, higher γ increases the nominal value of the transfer. For $\gamma < 1$, both effects go in the same direction, as higher γ decreases ϕ, but increases the nominal value of the (negative) transfer. Thus, $\tau^o < 0$ increases and k^b decreases in γ. For $\gamma > 1$, either effect can dominate, depending on the elasticity of DM consumption.[23]

The effect on aggregate capital accumulation is given by the net effect of the two channels. With $\mathcal{I} = 1$, only the seller channel is active, and since the effect of γ on K through the seller channel is positive, there is always an MT-E in this case. For $\mathcal{I} = 0$, the aggregate effect depends on $|\varepsilon_{q^m}|$, as this governs both the sign of the transfer channel at higher γ and the relative strength of the two channels. If $|\varepsilon_{q^m}| = 1$, the increase in k^s with γ through the seller channel is exactly offset by a decrease in k^b from the transfer channel, so K remains constant and there is no MT-E. With $|\varepsilon_{q^m}| > 1$, the (positive) effect through the seller channel is strong while the (negative) effect from the transfer channel is weak, so there is an MT-E on aggregate. The reason that this happens for high $|\varepsilon_{q^m}|$ is that in this case, buyers reduce q^m by a lot if γ increases, so in turn k^s is reacting strongly to changes in γ. Strong changes in q^m also imply that ϕ decreases strongly as γ increases, which weakens the negative effect on capital accumulation from the transfer channel. The contrary is true for $|\varepsilon_{q^m}| < 1$: q^m changes very little as γ varies, implying that k^s also varies very little with inflation. On the other hand, higher γ leaves ϕ almost unchanged, so k^b reacts strongly to changes in γ through the transfer channel. On aggregate, the negative effect from the transfer channel dominates, such that there is a reverse MT-E.

Proposition 3 also shows that an MT-E is always correlated with a negative effect on aggregate labor supply. Independent of $\mathcal{I}$, CM clearing implies

$$H = K + 2x^* - KR = 2x^* - K(R - 1). \tag{36}$$

Since CM consumption is independent of inflation, H and K are negatively related for $R > 1$, so if there is an MT-E (reverse MT-E), H decreases in γ (increases in γ).

Understanding this, we can derive the optimal monetary policy.

Proposition 4. *With $\pi = 1$, optimal inflation γ^* is*

[23] Specifically, if $|\varepsilon_{q^m}| > \frac{\gamma}{\gamma - 1}$, the effect through the transfer channel of γ on k^b is positive, so a positive correlation is more likely for high $|\varepsilon_{q^m}|$ and high γ.

- $\gamma^* = 1/R$ *under* $\mathcal{I} = 0$ *if* $|\varepsilon_{q^m}| \leq 1$*;*
- $\gamma^* = \frac{|\varepsilon_{q^m}|}{|\varepsilon_{q^m}|+R-1} \in (1/R, 1)$ *under* $\mathcal{I} = 0$ *if* $|\varepsilon_{q^m}| > 1$*;*
- $\gamma^* = 1$ *under* $\mathcal{I} = 1$.

The optimal monetary policy consists of setting $\mathcal{I}^* = 0$ *and* $\gamma^* = 1/R$ *for* $|\varepsilon_{q^m}| \leq 1$*, and* $\gamma^* = \frac{|\varepsilon_{q^m}|}{|\varepsilon_{q^m}|+R-1}$ *for* $|\varepsilon_{q^m}| > 1$*. The first-best allocation is not achievable with* $\pi = 1$.

Proposition 4 states our first main result: Inflation rates above the FR are optimal if and only if there is an MT-E. The proof to this proposition can be found in Appendix D. The intuition is as follows: We know from Proposition 2 that the FR allows to achieve q^*. Thus if $\partial H/\partial\gamma \geq 0$ (Cases 1 and 3 from Proposition 3) the FR must be optimal. Higher inflation would decrease q^m while weakly increasing H, so higher inflation rates clearly decrease welfare. If $\partial H/\partial\gamma < 0$, there is a policy tradeoff: Increasing γ reduces q^m, but also reduces H. Thus, by an envelope argument, the optimal γ must be above the FR: The marginal costs of decreasing q^m are zero at the FR, but the benefits of decreasing H are positive. This is what happens in Case 2 from Proposition 3 and with $\mathcal{I} = 1$. By how much γ can be increased above the FR to further increase welfare then depends on $|\varepsilon_{q^m}|$ and on $\mathcal{I}$. We know from Proposition 1 that with $\mathcal{I} = 0$, implementing $\gamma < 1$ and thus higher q^m is relatively cheaper, so γ^* is increasing in $|\varepsilon_{q^m}|$ and approaching 1 as $|\varepsilon_{q^m}| \Rightarrow \infty$. With $\mathcal{I} = 1$ however, $\gamma < 1$ is too costly, so $\gamma^* = 1$ in this case.

Note that $\gamma = 1$ implies the return on money equals the return on intergenerational transfers which the social planner faces; note also that financing x with m implies intergenerational transfers, as only young agents are willing to sell x against m. With $\pi = 1$, buyers are only able to compensate sellers with m for q, so unless the DM is completely shut down, sellers inevitably end up with m when they enter the second CM of their life. Setting $\gamma = 1$ leads to the correct price for providing x with intergenerational transfers. On the other hand, setting $\gamma = 1/R$ is the only way to reach q^*. This analysis points out the fundamental policy tradeoff in our model: Efficiency in the DM (i.e., in intra-generational trade) requires a different γ than efficiency in the CM (i.e., in inter-generational trade), and the optimal inflation rate depends on the monetary policy regime and on how elastic DM consumption is.

Haslag and Martin (2007) have shown that $\gamma = 1$ is typically optimal in an OLG model, independent of the MT-E. We can confirm that $\gamma = 1$ is the optimal monetary policy for $\mathcal{I} = 1$, independent of all other parameters. However, we have also shown that an MT-E still exists in this case. More generally, we can show that implementing monetary policy in a less costly way, namely by taxing agents only once they are old, allows to shut down the MT-E completely for certain parameters, and that in these cases, the FR is the optimal monetary policy.[24]

Proposition 4 also states that there is an ordering of the two ways of implementing monetary policy. We know that for $\gamma = 1$, both regimes are equivalent in terms of allocations and we also know that if $\mathcal{I} = 1$, $\gamma^* = 1$ is optimal. This allocation is always feasible, but not optimal, if $\mathcal{I} = 0$. Thus, we can conclude that the optimal monetary policy is to set $\mathcal{I}^* = 0$, and to set $\gamma^* = 1/R$ for $|\varepsilon_{q^m}| \leq 1$ and to set $\gamma^* = \frac{|\varepsilon_{q^m}|}{|\varepsilon_{q^m}|+R-1} \in (1/R, 1)$ for $|\varepsilon_{q^m}| > 1$.

As we discussed earlier, this analysis focuses on steady-state welfare. This means γ^* maximizes welfare for all generations born in or after the period when γ^* is implemented. To complete our analysis, in Appendix C, we derive how changes in γ affect welfare of all generations, including the welfare of buyers and sellers who are old in the period when γ^* is implemented. Specifically, we discuss how central banks can implement γ^* in a Pareto-efficient way, given that the current γ is different from γ^*. We show there that if monetary policy is implemented efficiently, i.e., if $\mathcal{I} = 0$ for $\gamma \leq 1$ and $\mathcal{I} = 1$ for $\gamma > 1$, any $\gamma \in [\frac{1}{R}, 1]$ is Pareto-optimal for $|\varepsilon_{q^m}| < 1$, while any $\gamma \in [\frac{1}{R}, \gamma^*]$ is Pareto-optimal for $|\varepsilon_{q^m}| \geq 1$. Since most central banks are currently targeting positive inflation rates, this result is encouraging: A decrease from a positive inflation target to γ^* leads to a Pareto-improvement in welfare if $|\varepsilon_{q^m}| \geq 1$. With $|\varepsilon_{q^m}| < 1$, the central bank can lower the inflation target from a positive rate to $\gamma = 1$, but not to γ^* without hurting old agents. The analysis also shows that the FR is always Pareto-optimal.

5 Equilibrium with partially liquid capital

Assume now $\pi \in (0, 1)$, which implies partially liquid k. Uncertainty about relocation introduces a clear tradeoff: m provides insurance against the relocation shock, while k offers a higher return if $\gamma > 1/R$. At low γ, acquiring m for insurance implies only a small loss of return, thus making m relatively more attractive and reducing investment in k. For higher γ, insurance is more costly and thus k more attractive. This tradeoff adds a third channel through which inflation affects capital accumulation. Since this channel depends on π, we call it the *liquidity channel.*

Sellers face no uncertainty with $\pi \in (0, 1)$ as the aggregate share of relocated buyers on each

[24] While this works nicely in our model, it would not do the trick in pure OLG models. The difference is that relocation occurs during the final stage of an agent's life in models such as Smith (2002) or Haslag and Martin (2007). The reason that taxing the old is strictly cheaper in our model is that all agents know they have access to their capital when they have to pay the tax, and can thus fully pay the tax via capital investment. In pure OLG models, only non-relocated agents have access to their capital during the final stage of their life.

island is known, so equations (21) and (22) still hold. This implies that $p_t = R/\phi_{t+1}$ still holds. Given this, solving the buyers' problem (13) yields[25]

$$m_t: \quad \frac{\phi_t}{\phi_{t+1}} = \pi \frac{1}{R} u'\left(\frac{\phi_{t+1} m_t}{R}\right) + (1-\pi)\beta U'(\phi_{t+1} m_t + R(k_t^b - q_t^b) + (1-\mathcal{I})\tau^o) \tag{37}$$

$$k_t^b: \quad \frac{1}{\beta R} = \pi U'(R k_t^b) + (1-\pi) U'(\phi_{t+1} m_t + R(k_t^b - q_t^b) + (1-\mathcal{I})\tau^o) \tag{38}$$

$$q_t^b: \quad u'(q_t^b) = \beta R U'(\phi_{t+1} m_t + R(k_t^b - q_t^b) + (1-\mathcal{I})\tau^o). \tag{39}$$

As before, we first derive the equilibrium when $\mathcal{I} = 1$. Money market clearing $m_t = M_t$ and stationarity $\phi/\phi_{+1} = \gamma$ are identical to before but market clearing in the DM is now:

$$\pi q^m + (1-\pi) q^b = q^s. \tag{40}$$

With (40), (25), (28) and (29) and the definitions and first-order conditions derived above we can define a stationary equilibrium with partially liquid capital as a list of eleven variables $\{q^m$, q^b, q^s, x^b, x^m, x^s, $\phi_{+1}M$, k^b, h^b, k^s, $h^s\}$ solving:

$$\pi u'(q^m) + (1-\pi) u'(q^b) = \gamma R \tag{41}$$

$$\pi U'(x^m) + (1-\pi) U'(x^b) = \frac{1}{\beta R} \tag{42}$$

$$u'(q^b) = \beta R U'(x^b) \tag{43}$$

$$x^m = x^b + R(q^b - q^m) \tag{44}$$

$$x^s = x^* \tag{45}$$

$$k^{b,\mathcal{I}=1} = \frac{x^m}{R} \tag{46}$$

$$h^{b,\mathcal{I}=1} = q^m R + k^b. \tag{47}$$

Aggregate labor supply and capital investments are given by:

$$K^{\mathcal{I}=1} = \frac{x^m}{R} + \frac{x^*}{R} - q^s \tag{48}$$

$$H^{\mathcal{I}=1} = \frac{x^m}{R} + q^m R + \frac{x^*}{R} - q^s. \tag{49}$$

With $\mathcal{I} = 0$ equations (46)-(49) are replaced by

$$h^{b,\mathcal{I}=0} = \gamma q^m R + k^b \tag{50}$$

$$k^{b,\mathcal{I}=0} = \frac{x^m}{R} - q^m(\gamma - 1) \tag{51}$$

$$K^{\mathcal{I}=0} = \frac{x^m}{R} - q^m(\gamma - 1) + \frac{x^*}{R} - q^s \tag{52}$$

$$H^{\mathcal{I}=0} = \frac{x^m}{R} + \gamma q^m R - (\gamma - 1) q^m + \frac{x^*}{R} - q^s. \tag{53}$$

[25] Remember that we ignore cases where nonrelocated buyers spend everything in the DM as this may only happen for values of γ and $\mathcal{I}$ which are clearly suboptimal.

We will now interpret equilibrium outcomes with a number of propositions. The proofs to all of these can be found in Appendix D.

Proposition 5. *At the FR, $q^m = q^b = q^s = q^*$ and $x^m = x^b = x^s = x^*$.*

The allocation at the FR is identical to the FR allocation with $\pi = 1$. As we know from the previous section, this allows for optimal consumption levels, but $H > H^*$. Importantly, only m is used in the DM even though k is partially liquid.

Proposition 6. *For $\gamma > \frac{1}{R}$, $q^m < q^s < q^b < q^*$, $x^m > x^* > x^b$, and $\frac{\partial q^m}{\partial \gamma}, \frac{\partial q^b}{\partial \gamma}, \frac{\partial q^s}{\partial \gamma}, \frac{\partial x^b}{\partial \gamma} < 0$ but $\frac{\partial x^m}{\partial \gamma} > 0$. Further, $X = x^* + \pi x^m + (1-\pi)x^b$ increases in γ.*

Deviations from the FR introduce consumption risk for buyers and their consumption deviates from first best in both markets. $q^m < q^b$ as relocated buyers can only use m to purchase q. This then implies that $x^m > x^b$ since all buyers acquire the same portfolio. Nonrelocated buyers consume less than first-best in both markets because the low return on m makes them unwilling to accumulate enough assets to purchase first-best consumption levels. With $\pi \in (0,1)$, x^m and total CM consumption X are increasing in γ, while x^b is decreasing.

Proposition 7. *With $\mathcal{I} = 1$, there is an MT-E for all parameters. At the FR, $\frac{\partial H^{\mathcal{I}=1}}{\partial \gamma} < 0$, but for sufficiently high γ or sufficiently low π, a further increase in γ increases H.*

As with $\pi = 1$, K always increases in γ if $\mathcal{I} = 1$. The effect is even stronger with $\pi < 1$, as the liquidity channel is active in addition to the seller channel. Contrary to the model with $\pi = 1$, an MT-E does not always imply $\partial H/\partial \gamma < 0$ anymore. This is because X now increases in γ, as shown in Proposition 6. CM clearing implies

$$H = X - K(R-1), . \tag{54}$$

As both K and X increase in γ when there is an MT-E, there are two opposing effects on H. Which effect is stronger is ambiguous in general, but at the FR the effect through K dominates and H always decreases in γ.

Proposition 8. *With $\mathcal{I} = 0$, we have at the FR:*

1. *If $|\varepsilon_{q^m}|_{FR} < 1$: $\frac{\partial H^{\mathcal{I}=0}}{\partial \gamma} > 0$ and $\frac{\partial K^{\mathcal{I}=0}}{\partial \gamma} < 0$: Reverse MT-E.*
2. *If $|\varepsilon_{q^m}|_{FR} > 1$: $\frac{\partial H^{\mathcal{I}=0}}{\partial \gamma} < 0$ and $\frac{\partial K^{\mathcal{I}=0}}{\partial \gamma} > 0$: MT-E.*
3. *If $|\varepsilon_{q^m}|_{FR} = 1$: $\frac{\partial H^{\mathcal{I}=0}}{\partial \gamma} = 0$ and $\frac{\partial K^{\mathcal{I}=0}}{\partial \gamma} = 0$: No MT-E.*

where $|\varepsilon_{q^m}|_{FR} = \frac{\xi}{\eta(q^*)}$ and $\xi \in (1,\infty)$ with $\xi = 1$ if $\pi = 1$, $\xi \to \infty$ as $\pi \to 0$, and ξ monotonically decreasing in π at the FR. Away from the FR, $\frac{\partial K^{\mathcal{I}=0}}{\partial \gamma} > 0$ if $|\varepsilon_{q^m}| > \hat{\varepsilon}$ and $\frac{\partial H^{\mathcal{I}=0}}{\partial \gamma} < 0$ if $|\varepsilon_{q^m}| > \tilde{\varepsilon}$, where $\hat{\varepsilon} < 1 < \tilde{\varepsilon}$.

At the FR, conditions for an MT-E look identical as with $\pi = 1$. However, $|\varepsilon_{q^m}|_{FR}$ is now not only a function of $\eta(q^*)$, but also of ξ, which in turn depends on the liquidity of capital π. Since $\xi = 1$ for $\pi = 1$ and ξ is decreasing in π, the proposition shows that with lower π, there are more values of $\eta(q^*)$ for which an MT-E occurs at the FR, and if $\pi \to 0$ there is always an MT-E at the FR. The reason is the liquidity channel. At the FR, all q is purchased with m. But if k is relatively liquid (π is low) a lot of these trades could be settled with k. Thus if γ is marginally above the FR and thus k dominates m in return, there is a strong incentive for buyers to substitute m for k. The lower π, the bigger this effect, and the more likely that it dominates the agents' desire to smooth DM consumption between states which is captured by $\eta(q)$. Away from the FR, things are more complicated. In general, the MT-E occurs when $|\varepsilon_{q^m}|$ is above some threshold $\hat{\varepsilon} < 1$, but there is no simple representation of $|\varepsilon_{q^m}|$ as a function of $\eta(q)$ in this case, so it is difficult to make general statements. The threshold for whether γ affects H positively or negatively is given by $\tilde{\varepsilon} > 1$, so away from the FR γ does not generally have opposite effects on H and K.

Proposition 9. *For $\mathcal{I} = 1$, $\gamma^* = 1$ independent of other parameters. The lower π, the larger is the welfare loss at the FR relative to γ^*.*

This result is not surprising, as we have shown in Proposition 4 that $\gamma = 1$ is optimal even if $\pi = 1$ with $\mathcal{I} = 1$. With $\pi < 1$, the MT-E effect is generally stronger due to the liquidity channel, so there was no reason to expect $\gamma < 1$ to be optimal. $\gamma > 1$ is not optimal either, as the additional distortions in q^m are larger than the benefits from increased K. Further, the proposition also shows that the FR is especially costly for low π. Welfare at the FR is independent of π, while welfare at $\gamma = 1$ is decreasing in π. The reason is straightforward: as established, at the FR all q is purchased with m, which implies a large share of x^s is financed through intergenerational transfers. This is unavoidable if $\pi = 1$, but the lower π, the more this can be avoided.

Proposition 10. *For $\mathcal{I} = 0$ and $|\varepsilon_{q^m}|_{FR} \leq 1$, the FR is optimal. For $|\varepsilon_{q^m}|_{FR} > 1$, $\gamma^* = \frac{|\varepsilon_{q^m}|}{|\varepsilon_{q^m}|+R-1} \in (1/R, 1)$. When the optimal money growth rate is chosen, welfare is strictly higher with $\mathcal{I} = 0$.*

This proposition shows that our main result goes through with uncertainty about relocation: The FR is still optimal if and only if there is no MT-E at the FR. However, the conditions for whether or not there is an MT-E at the FR now also depend on π. To put the proposition in other words, for a given π, there is a threshold on risk aversion ξ, with the FR being optimal if and only if $\eta(q^*) > \xi$, with ξ decreasing in π. This shows that the liquidity channel makes it less likely that the FR is optimal, and when $\pi \to 0$, the FR is never optimal. This follows directly from Proposition 8: If there is an MT-E at the FR, the welfare gain from reducing H dominates the welfare loss from suboptimal DM and CM consumption. The result that $\mathcal{I} = 0$ is better for welfare follows again from the fact that allocations coincide for $\gamma = 1$, so the optimal allocation for $\mathcal{I} = 1$ is always feasible with $\mathcal{I} = 0$, but generally not optimal.

As in the model with $\pi = 1$, the fundamental policy tradeoff is that setting $\gamma = \frac{1}{R}$ allows for efficiency in the DM, but misrepresents the cost of using money, and thus intergenerational transfers, to provide for CM consumption. While this is a relatively small issue if capital is illiquid and money is the only way to provide for DM consumption, the welfare loss from running the FR increases with the liquidity of capital, as most DM trades could be made with capital if it is relatively liquid. Proposition 10 shows that the premise behind Smith (2002) was correct: the MT-E is enough to make deviations from the FR optimal. In our model, the FR is optimal if and only if there is no MT-E at the FR, with higher liquidity of capital and lower risk aversion of buyers making it more likely that there is one.

6 Conclusion

We have added a market which requires liquid assets to trade to an OLG model with relocation shocks in order to study whether the Mundell-Tobin effect can make deviations from the Friedman rule optimal in a model where money serves a role in both intra- and inter-generational trade. We have shown that the Friedman rule is optimal if and only if there is no Mundell-Tobin effect at the Friedman rule, and that a Mundell-Tobin effect is more likely to occur if capital is relatively liquid, risk aversion of buyers is low, and if monetary policy is implemented by taxing the young. If the Friedman rule is not optimal, the optimal money growth rate lies somewhere between the Friedman rule and a constant money stock. While the Friedman rule allows for first-best consumption levels in the DM, it misrepresents the cost of using intergenerational transfers to provide for CM consumption during old age. These costs are correctly represented by a constant money stock. We have also shown that for any deflationary policy, taxing old agents is strictly better than taxing young agents when there is a productive investment opportunity in the economy.

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Appendix A An economy with $R < 1$

Suppose now that $R < 1$. From the planner's problem, we know this makes capital accumulation socially inefficient. This implies that any $\gamma > 1/R$ cannot be optimal as it makes k more attractive than m to provide for CM consumption. Thus we restrict the analysis to $\gamma < 1/R$, which implies agents do not accumulate k and the relocation shock becomes irrelevant. To simplify notation we set $\pi = 0$ here, but the results are equivalent for any $\pi \in [0, 1]$. Then, the buyer's problem is

$$V^b = \max_{m_t^b, q_t^b} \quad \mathcal{I}\tau^y - \phi_t m_t^b + u(q_t^b) + \beta U\left[\phi_{t+1}(m_t^b - p_t q_t^b) + (1-\mathcal{I})\tau^o\right],$$

with the FOCs

$$U'(x_t^b) = \frac{\phi_t}{\phi_{t+1}}\frac{1}{\beta}$$
$$u'(q_t^b) = \phi_t p_t.$$

Meanwhile, the seller's problem is

$$V^s = \max_{m_t^s, q_t^s} \quad -\phi_t m_t^s - q_t^s + \beta U\left[\phi_{t+1}(m_t^s + p_t q_t^s)\right].$$

Solving this yields

$$U'(x_t^s) = \frac{1}{p_t \phi_{t+1}}\frac{1}{\beta}$$
$$U'(x_t^s) = \frac{\phi_t}{\phi_{t+1}}\frac{1}{\beta}, \tag{55}$$

which implies $p_t = 1/\phi_t$.[26] In a stationary equilibrium, we have $\phi_t/\phi_{t+1} = \gamma$ as always, $q^s = q^b$, $m^b + m^s = M$, and

$$u'(q^b) = 1$$
$$U'(x^b) = U'(x^s) = \frac{\gamma}{\beta}$$
$$H = x^s + x^b,$$

where the third equation follows directly from the CM resource constraint with $K = 0$. This shows that contrary to the model with $R > 1$, the monetary policy regime is irrelevant here. Interestingly, $q^b = q^*$ independent of γ *as long as sellers work in both markets.* The reason is that in order to keep sellers indifferent between working in the DM and the CM, p_t adjusts to changes in γ in a way that incentivizes buyers to always purchase q^*. This result resembles a finding in Huber and Kim (2020). Further, it can easily be seen that for $\gamma = 1$, $x^b = x^s = x^*$ and $H = H^*$, showing that a constant money supply allows to implement the first best allocation for $R < 1$.

[26] As in the main body of the paper, we focus on cases where sellers always work in both markets. Otherwise, $m_t^s \geq 0$ may bind and thus (55) does not hold. Here, this holds if $U'\left(\frac{q^s}{\gamma}\right) \geq \frac{\gamma}{\beta}$.

Appendix B An economy with endogenous R

Suppose now that CM goods are produced according to $F(K,H) = f(K) + H$, with $f(0) = 0$, $f'(K) > 0 > f''(K)$, and $f'(0) > 1$. Otherwise the model remains unchanged, but for simplicity we focus on $\pi = 1$ here.

B.1 Social planner's problem

As in the main body of the paper, we focus on maximizing expected steady-state welfare of a representative generation. The planner's problem is equivalent to (3), but with $f(K)$ replacing RK in the CM resource constraint:

$$\max_{H,K,q^m,q^s,x^m,x^s} -H - q^s + u(q^m) + \beta U(x^m) + \beta U(x^s)$$

$$\begin{aligned} s.t. \qquad \pi q^m + (1-\pi)q^b &= q^s \\ H &\geq K \\ \pi x^m + (1-\pi)x^b + x^s + K &= f(K) + H. \end{aligned}$$

The DM is unaffected, so q^* is still given by

$$q^m = q^s = q^* \quad \text{solving} \quad u'(q^*) = 1,$$

while the FOCs for K and CM consumption are

$$\begin{aligned} x^m = x^s = x^* \quad \text{solving} \quad \beta U'(x^*) &= 1 - \lambda \\ f'(K)\lambda &\geq f'(K) - 1. \end{aligned}$$

Thus, if the constraint on H binds, x^* and K^* are jointly determined by

$$\begin{aligned} \beta f'(K^*)U'(x^*) &= 1 \\ f(K^*) &= 2x^*, \end{aligned}$$

while $H^* = K^*$. This case exists if K^* solving the above equations is such that $f'(K^*) \geq 1$. Call this a case with *productive capital.* In this case, as in the case with linear $R > 1$, all CM consumption is optimally financed with capital. If this is violated, we are in a case with *unproductive capital*, where some CM consumption is optimally financed through transfers from young to old agents and optimality is given by

$$\begin{aligned} x^m = x^s = x^* \quad \text{solving} \quad \beta U'(x^*) &= 1 \\ f'(K^*) &= 1 \\ H^* &= 2x^* + K^* - f(K^*) \end{aligned}$$

B.2 Market equilibrium

We assume that competitive firms operate in the CM and produce CM goods according to $X = F(K, H)$. Denote the factor payments to H and K as ω and R, respectively. Firm optimization implies $\omega_t = 1\ \forall t$ and $R_t = f'(K_t)$. Firm profits are denoted by Δ_t. We assume that profits are paid to young buyers.[27] Given this, buyers solve

$$V^b = \max_{m_t^b, k_t^b, q_t^m} \quad \Delta_t + \mathcal{I}\tau_t^y - \phi_t m_t - k_t^b + u\left(q_t^m\right) + \beta U(Rk_t^b + \phi_{t+1}(m_t^b - p_t q_t^m) + (1-\mathcal{I})\tau_t^o)$$

$$s.t. \quad p_t q_t^m \leq m_t^b.$$

Denoting $\phi_t \lambda_t$ as the Lagrange multiplier on the liquidity constraint, the FOCs are

$$\begin{aligned}
U'(x_{t+1}^m) &= \frac{\phi_t}{\phi_{t+1}} \frac{1}{\beta}(1-\lambda_t) \\
U'(x_{t+1}^m) &= \frac{1}{\beta R} \\
u'(q_t^m) &= p_t \left(\phi_{t+1}\beta U'(x_{t+1}^m) + \phi_t \lambda_t\right),
\end{aligned}$$

Meanwhile, the seller's problem is

$$V^s = \max_{m_t^s, k_t^s, q_t^s} \quad -k_t^s - \phi_t m_t^s - q_t^s + \beta U(Rk_t^s + \phi_{t+1}(m_t^s + p_t q_t^s))$$

$$s.t. \quad m_t^s \geq 0,$$

and the corresponding FOCs are

$$\begin{aligned}
U'(x_{t+1}^m) &\leq \frac{\phi_t}{\phi_{t+1}} \frac{1}{\beta} \\
U'(x_{t+1}^m) &= \frac{1}{\beta R} \\
U'(x_{t+1}^s) &= \frac{1}{p_t \phi_{t+1}} \frac{1}{\beta},
\end{aligned}$$

where the first condition holds with equality if $m_t^s > 0$. Combining the second and third condition then yields $p_t = \frac{R}{\phi_{t+1}}$, where we maintain our assumption that parameters are such that sellers want to work in both markets. Then, in a stationary equilibrium, we have $\frac{\phi_t}{\phi_{t+1}} = \gamma$, $q^s = q^m$, $m_t^b + m_t^s = M_t$, $\Delta = f(k^b + k^s) - R(k^b + k^s)$, $\tau^y = \tau^o = (\gamma - 1)\phi_{+1} M$, and

$$R = f'(k_t^s + k_t^b) \geq \frac{1}{\gamma}. \tag{56}$$

A stationary equilibrium is given by $\left\{h^b, h^s, k^b, k^s, \phi_{+1} m^b, \phi_{+1} m^s, R, q^m, x^m, x^s\right\}$. If equation (56) does not bind, they solve

$$R = f'(k^s + k^b) \tag{57}$$

[27] We allocate profits to young buyers mainly because we want to abstract from further interactions between inflation and capital accumulation. It also simplifies the buyers' problem slightly because the buyer's utility is linear in labor supply.

$$q^m = \frac{\phi_{+1} m^b}{R} \tag{58}$$

$$u'(q^m) = \gamma R \tag{59}$$

$$m^s = 0 \tag{60}$$

$$x^m = R(k^b + (1 - \mathcal{I})(\gamma - 1)q^m) \tag{61}$$

$$x^s = R(k^s + q^m) \tag{62}$$

$$U'(x^s) = U'(x^m) = \frac{1}{\beta R} \tag{63}$$

$$h^b = k^b + \gamma R q^m - f(k^b + k^s) + R(k^b + k^s) - \mathcal{I}(\gamma - 1)Rq^m \tag{64}$$

$$h^s = k^s. \tag{65}$$

Call this an *unconstrained case*. If instead (56) is binding, the economy is in a *constrained case*, and the list of variables solves

$$R = \frac{1}{\gamma} = f'(k^s + k^b) \tag{66}$$

$$u'(q^m) = 1 \tag{67}$$

$$x^m = \frac{k^b}{\gamma} + \phi_{+1} m^b - \frac{q^m}{\gamma} + (1 - \mathcal{I})(\gamma - 1)\phi_{+1}(m^s + m^b) \tag{68}$$

$$x^s = \frac{k^s}{\gamma} + \phi_{+1} m^s + \frac{q^m}{\gamma} \tag{69}$$

$$U'(x^s) = U'(x^m) = \frac{\gamma}{\beta} \tag{70}$$

$$h^b = k^b + \frac{\phi_{+1} m^b}{\gamma} - f(k^b + k^s) + R(k^b + k^s) - \mathcal{I}(\gamma - 1)\phi_{+1}(m^b + m^s) \tag{71}$$

$$h^s = k^s + \frac{\phi_{+1} m^s}{\gamma}. \tag{72}$$

Note that there is an indeterminacy in the constrained case as only K and total portfolios of buyers and sellers are determined, but not their composition as agents are indifferent between m and k. Further, there is some $\bar{\gamma}$ at which (56) is just satisfied. For any $\gamma > \bar{\gamma}$ ($\gamma < \bar{\gamma}$), the economy is in the unconstrained (constrained case), so by choosing γ, the central bank can choose in which case the economy is.

Before we proceed, let us discuss the interpretation of the FR in this model. As discussed in Footnote 14, in LW models, the FR is typically defined as the return on a fictional illiquid asset, which due to the quasilinearity of these models is given by $1/\beta$, but that is not true here - with $\pi = 1$, K is such an illiquid asset, but it is not always priced such that $R = 1/\beta$ as just shown. The traditional definition of the FR is that the opportunity cost of holding money should be zero, which is true whenever $1/\gamma = R$.[28] But with endogenous R, this implies that as long as (56) is binding, any γ can be interpreted as the FR.

[28] And as just discussed, this also imples that the return on money equals the return on an illiquid asset (K), so

For our analysis, suppose first that the economy is in a constrained case at $\gamma = 1$, implying that $\bar{\gamma} > 1$. Note that this implies that from the planner's perspective, we are in the case with unproductive capital, as some CM consumption is financed through transfers but $f'(K) = 1$, so if all CM consumption were to be financed with capital investment, we would have $f'(K) < 1$. Next, note from equation (66) there is always an MT-E in the constrained case, as γ directly determines K. Note also that in the constrained case, $\gamma = 1$ delivers K^* and $x^m = x^s = x^*$ from the case with unproductive capital in the social planner's problem, while $q^m = q^*$ is always satisfied here. Further, at $\gamma = 1$, H is given by

$$H = h^b + h^s = 2K^* + \phi_{+1}(m^b + m^s) - f(K^*),$$

and we know that

$$x^m + x^s = 2x^* = K^* + \phi_{+1}(m^b + m^s).$$

Combining these two equations shows that $H = H^*$ at $\gamma = 1$ for $\mathcal{I} = \{0, 1\}$, confirming that a constant money stock allows to implement the first-best if capital is not productive enough to finance all CM consumption. This is the same result as we found in Appendix A with linear $R < 1$, where capital investment was never efficient. Thus, as long as capital investment is inefficient *at the margin*, which is the case if equation (56) binds for $\forall\gamma \leq 1$, $\gamma = 1$ delivers the first best.

Suppose now that (56) is nonbinding for $\gamma = 1$, which implies that $\bar{\gamma} < 1$. Note that in this case, we do not know whether capital is productive or unproductive from the social planner's perspective, but as we will show, this is irrelevant for the points we want to make. To proceed, we interpret the cutoff at $\bar{\gamma}$ as the FR in this case - similar to our baseline model, this is the highest inflation rate for which there is no opportunity cost of holding money. At $\bar{\gamma}$, we have $u'(q^m) = 1$, while $U'(x^s) = U'(x^m) = \frac{1}{\beta R} = \frac{\gamma}{\beta}$. Thus, $q^m = q^*$, and $x^m = x^s$ is optimal *given* K. We also have $R = 1/\bar{\gamma} > 1$. We will now proceed as follows: First, we determine whether or not $\bar{\gamma}$ is optimal conditional on whether or not there is an MT-E at $\bar{\gamma}$. After this, we will discuss the conditions under which there is an MT-E in the unconstrained case.

In the unconstrained case, the optimal inflation rate solves

$$\begin{aligned} \max_{\gamma \geq \bar{\gamma}} \quad W &= -H + u(q^m) - q^m + 2\beta U(x^m) \\ &= -K + f(K) + u(q^m) - q^m + 2(\beta U(x^m) - x^m), \end{aligned} \tag{73}$$

where we used $x^m = x^s$ from equation (63), and the CM resource constraint $H = 2x^m + K - f(K)$.

in a sense, this is still in line with the definition in LW models. However, this changes once one sets $\pi \in (0, 1)$, which we do not discuss here.

Differentiating (73) with respect to γ yields

$$\begin{aligned}\frac{\partial W}{\partial \gamma} &= \frac{\partial K}{\partial \gamma}(f'(K)-1) + \frac{\partial q^m}{\partial \gamma}(u'(q^m)-1) + 2\frac{\partial x^m}{\partial \gamma}(\beta U'(x^m)-1) \\ &= \frac{\partial K}{\partial \gamma}(R-1) + \frac{\partial q^m}{\partial \gamma}(\gamma R-1) - 2\frac{\partial x^m}{\partial \gamma}\frac{R-1}{R},\end{aligned} \tag{74}$$

where the second step follows from using the equilibrium conditions. At $\bar{\gamma} = 1/R$, the middle term is zero (implying that the welfare loss from changes in q^m is second order) and we are left with

$$\frac{\partial W}{\partial \gamma}|_{\bar{\gamma}} = \frac{\partial K}{\partial \gamma}(R-1) - 2\frac{\partial x^m}{\partial \gamma}\frac{R-1}{R}. \tag{75}$$

The sign of this depends on partial derivatives of K and x^m with respect to γ. If there is an MT-E, we have $\partial K/\partial\gamma > 0$ and $\partial x/\partial\gamma < 0$, while the opposite is true if there is a reverse MT-E. Thus, we can conclude that our main results from the paper hold with endogenous R: If $\bar{\gamma} < 1$, raising γ above $\bar{\gamma}$ is optimal if and only if there is an MT-E. If there is neither an MT-E nor a reverse MT-E, (75) is equal to zero, so $\bar{\gamma}$ is optimal in the unconstrained case.

Suppose now that there is a reverse MT-E. In this case, (75) is negative at any $\gamma \geq \bar{\gamma}$. However, we cannot conclude from this that $\gamma < \bar{\gamma}$ is optimal, as the economy moves into the constrained case for inflation rates below $\bar{\gamma}$. Instead, the optimal γ in the constrained case solves

$$\begin{aligned}\max_{\gamma \leq \bar{\gamma}} \quad W &= -H + u(q^*) - q^* + 2\beta U(x) \\ &= -K + f(K) + u(q^*) - q^* + 2(\beta U(x) - x),\end{aligned} \tag{76}$$

and differentiating yields

$$\frac{\partial W}{\partial \gamma} = \frac{\partial K}{\partial \gamma}(\frac{1}{\gamma} - 1) - 2\frac{\partial x}{\partial \gamma}(1-\gamma).$$

Thus $\frac{\partial W}{\partial \gamma} > 0$ for any $\gamma < 1$ because there is always an MT-E in the constrained case, so $\frac{\partial K}{\partial \gamma} > 0$ while $\frac{\partial x}{\partial \gamma} < 0$. Therefore, the optimal MP in the constrained case is $\bar{\gamma}$ if $\bar{\gamma} < 1$. This implies that if $\bar{\gamma} < 1$ and there is a reverse MT-E at $\gamma = \bar{\gamma}$, $\gamma^* = \bar{\gamma}$.

To sum up: If $\bar{\gamma} < 1$, $\gamma^* = \bar{\gamma}$ if there is no MT-E. If there is an MT-E, $\gamma^* > \bar{\gamma}$. This mirrors the results discussed in the main body of this paper. If instead $\bar{\gamma} > 1$, $\gamma^* = 1$ independent of other parameters. This mirrors the results discussed in Appendix A.

What is now left to do is to determine when there is an MT-E. To answer this, we can combine (57) with (61) and (62), which yields

$$R = f'\left(\frac{2x^m}{R} - q^m[1 + (1-\mathcal{I})(\gamma - 1)]\right).$$

Note that from (57), $\frac{\partial R}{\partial K} < 0$, so if we can determine the sign of $\frac{\partial R}{\partial \gamma}$ we also know the sign of $\frac{\partial K}{\partial \gamma}$. Taking the derivative with respect to γ then yields

$$\frac{\partial R}{\partial \gamma} = f''(K)\left[\frac{\partial x^m}{\partial \gamma}\frac{2}{R} - \frac{2x^m}{R^2}\frac{\partial R}{\partial \gamma} - \frac{\partial q^m}{\partial \gamma}[1+(1-\mathcal{I})(\gamma-1)] - (1-\mathcal{I})q^m\right].$$

Next, we can get the derivatives of q^m and x^m with respect to γ from equations (59) and (63) and insert them in the above term to get

$$\frac{\partial R}{\partial \gamma} = f''(K)\left[-\frac{2}{U''(x^m)\beta R^3}\frac{\partial R}{\partial \gamma} - \frac{2x^m}{R^2}\frac{\partial R}{\partial \gamma} - \frac{1}{u''(q^m)}\left(R+\gamma\frac{\partial R}{\partial \gamma}\right)[1+(1-\mathcal{I})(\gamma-1)] - (1-\mathcal{I})q^m\right].$$

Finally, rearranging yields

$$\frac{\partial R}{\partial \gamma} = \frac{-f''(K)\left[\frac{R}{u''(q^m)}[1+(1-\mathcal{I})(\gamma-1)] + (1-\mathcal{I})q^m\right]}{1+f''(K)\left[\frac{2}{U''(x^m)\beta R^3} + \frac{2x^m}{R^2} + \frac{\gamma}{u''(q^m)}[1+(1-\mathcal{I})(\gamma-1)]\right]},$$

which can also be written as

$$\frac{\partial R}{\partial \gamma} = \frac{-f''(K)\left[\frac{R}{u''(q^m)}[1+(1-\mathcal{I})(\gamma-1)] + (1-\mathcal{I})q^m\right]}{1+f''(K)\left[\frac{\gamma}{u''(q^m)}[1+(1-\mathcal{I})(\gamma-1)] - \frac{2x^m}{R^2}\left(\frac{1}{\eta(x^m)}-1\right)\right]},$$

with $\eta(x^m) = -x^m U''(x^m)/U'(x^m)$. So if $\frac{1}{\eta(x^m)} \geq 1$, the denominator is unambiguously positive, and the sign of $\frac{\partial R}{\partial \gamma}$ is equal to the sign of the numerator. For $\mathcal{I} = 1$, the numerator is negative, while for $\mathcal{I} = 0$, it can be written as

$$-f''(K)\left[-q^m\left(\frac{1}{\eta(q^m)}-1\right)\right],$$

so the sign is determined by $\eta(q^m)$. Thus, when $\frac{1}{\eta(x^m)} \geq 1$, the existence conditions for an MT-E are exactly the same as in our baseline model. If instead $\frac{1}{\eta(x^m)} < 1$, the sign of the denominator is ambiguous, and whether or not there is an MT-E depends on additional parameters.

Appendix C Pareto welfare with $\pi = 1$

In this section, we discuss whether changes in the inflation rate lead to Pareto-improvements for all generations in the model with $\beta R = 1$ and $\pi = 1$. This is different from our analysis in the main body of the paper, where we focused on steady-state welfare. While steady-state welfare ignores the welfare of the initial old, a Pareto-improvement in welfare is only possible if the initial old are not made worse off by a policy change.

Throughout this section, we assume that the central bank unexpectedly changes the money growth rate from γ^{old} to γ^{new} in period $t = 1$. This implies that the economy was at the steady state determined by γ^{old} in period t, while it changes to the new steady-state determined by γ^{new}

in period $t+1$. Thus, agents of generation t become the initial old that live through the transition, whereas agents of generations $j \geq t+1$ live in the new steady state, which means our steady-state welfare analysis applies to them. To determine how the welfare of the initial old changes, we need to figure out whether their CM consumption increases or decreases.

CM consumption of the initial old buyers is given by

$$x_{t+1}^b = Rk_t^b + (1-\mathcal{I})\tau(\gamma), \tag{77}$$

with $\tau(\gamma) = \phi_{t+1}M_t(\gamma-1) = q_{t+1}R(\gamma-1)$, while CM consumption of initial old sellers is given by

$$x_{t+1}^s = Rk_t^s + \phi_{t+1}M_t. \tag{78}$$

Since capital accumulation takes places in period t (i.e., before the new policy is implemented), k_t^b and k_t^s are unaffected by the policy change, so to determine how the change affects consumption of these agents, we need to determine its effect on $\tau(\gamma)$ and $\phi_{t+1}M_t$. From (25) we know in any steady state $\phi_{t+1}M_t = \phi_{+1}M = q(\gamma)R$. Thus, the initial old buyers' welfare increases with the policy change when $(1-\mathcal{I})\tau(\gamma^{new}) - (1-\mathcal{I})\tau(\gamma^{old}) \equiv \Delta x^b \geq 0$, while initial old sellers' welfare increases when $q(\gamma^{new})R - q(\gamma^{old})R \equiv \Delta x^s \geq 0$.

First consider $\mathcal{I} = 1$. In this case, consumption of initial old buyers is unaffected. They received transfers (paid taxes) in period t and their CM consumption is fully determined by their capital holdings, which are chosen before the policy change takes place. For initial old sellers, it is easy to see that their welfare is increasing (decreasing) if $\gamma^{new} < \gamma^{old}$ ($\gamma^{new} > \gamma^{old}$) since DM-consumption is decreasing in inflation. This is intuitive: Lower γ implies a higher value of money, and since the initial old sellers hold the money stock at the time of the policy change, it is clear that their welfare increases if and only if the value of money increases. The consumption changes for initial old buyers/sellers are summarized in the following table.

$\mathcal{I} = 1$	Δx^b	Δx^s
$\gamma^{old} > \gamma^{new}$	$= 0$	> 0
$\gamma^{old} < \gamma^{new}$	$= 0$	< 0

Since the effect of the policy change on the initial old only depends on whether γ is increased or decreased, we can focus on the case $\gamma^{new} = \gamma^* = 1$, as in this case we know the welfare of future generations is increasing after the policy change. This allows us to state the following proposition:

Proposition 11. *With $\mathcal{I} = 1$, changing monetary policy from any γ to $\gamma^* = 1$ leads to a Pareto-improvement in welfare if and only if $\gamma > \gamma^*$. Any $\frac{1}{R} \leq \gamma \leq 1$ is Pareto-optimal.*

From our analysis in the main body of the paper, we know that $\gamma = 1$ maximizes steady-state welfare for $\mathcal{I} = 1$. From the analysis here, we learned that a change in γ is Pareto-improving for the initial old only if $\gamma^{new} < \gamma^{old}$, so it can easily be seen that any $\gamma > \gamma^* = 1$ is not Pareto-optimal, and reducing inflation weakly increases welfare for all agents in this case. Conversely, this implies that if taxes are raised on young buyers, any $\frac{1}{R} \leq \gamma \leq 1$ is Pareto optimal, since increasing the welfare of future generations would require to increase the inflation rate in this case, but this lowers the welfare of initial old sellers.

For $\mathcal{I} = 0$, both initial old buyers and sellers are affected by a policy change. We have already established that initial old sellers benefit from a decrease in γ. To find the effect of a change in γ on the welfare of initial old buyers, we can take the derivative of the real tax payment with respect to γ:

$$\frac{\partial \tau}{\partial \gamma} = R\left(\frac{\partial q}{\partial \gamma}(\gamma - 1) + q\right).$$

Since $\frac{\partial q}{\partial \gamma} < 0$, $\frac{\partial \tau}{\partial \gamma} > 0$ for $\gamma < 1$. For $\gamma > 1$, the real value of the transfer is increasing in γ if

$$\begin{aligned}
q &> -\frac{\partial q}{\partial \gamma}(\gamma - 1) \\
1 &> -\frac{\partial q}{\partial \gamma}\left(\frac{\gamma - 1}{q}\right) \\
\gamma &> |\varepsilon_q|(\gamma - 1) \\
\frac{\gamma}{\gamma - 1} &> |\varepsilon_q|.
\end{aligned}$$

From this, the next proposition follows:

Proposition 12. *With $\mathcal{I} = 0$, marginally decreasing γ leads to a Pareto-improvement in welfare if and only if $\gamma > 1$ and $\frac{\gamma}{\gamma - 1} < |\varepsilon_q|$, while an increase in γ never leads to a Pareto-improvement in welfare.*

This shows that if monetary policy is implemented over old buyers, improving welfare for all initial old agents and future generations simultaneously is only possible for high inflation rates and / or high values of $|\varepsilon_q|$, as only then do initial old buyers also benefit from a decrease in γ.[29] Initial old sellers always benefit from a decrease in γ, and future generations benefit from a decrease in γ whenever $\gamma > \gamma^*$, so if the condition for initial old buyers is satisfied those for the other agents are always satisfied too.

Proposition 12 shows that unless $|\varepsilon_q| \to \infty$, the interests of initial old buyers and sellers go in opposite directions for realistic inflation rates - initial old sellers always benefit from a decrease

[29] In particular, with $|\varepsilon_q| < 1$, $\frac{\partial \tau}{\partial \gamma} > 0$ for any $\gamma \geq 1$, so a decrease in γ never increases the initial old buyers' welfare and thus any γ is Pareto-optimal.

in inflation as this increases the value of the money they are holding, whereas initial old buyers benefit from an increase in inflation as this increases the real value of the transfer they receive (unless inflation is very high or buyers have a very high elasticity of DM-consumption). Thus, Pareto-improvements are generally not possible for moderate inflation rates.

While strict Pareto-improvements are only possible in extreme cases if monetary policy is implemented over old buyers, policy changes might still be feasible if the benefits of one group of agents are large enough to compensate those agents that are losing. In particular, we want to analyze whether the net gain of initial old agents is positive, as we consider lump-sum transfers/taxes from initial old sellers to initial old buyers to be relatively easy to implement. Under this criteria, a policy change leads to a Pareto-improvement if $\delta \geq 0$, with

$$\begin{aligned}\delta = \Delta x^b + \Delta x^s &= q(\gamma^{new})R(\gamma^{new} - 1) - q(\gamma^{old})R(\gamma^{old} - 1) + q(\gamma^{new})R - q(\gamma^{old})R \\ &= q(\gamma^{new})R\gamma^{new} - q(\gamma^{old})R\gamma^{old} \\ &= u'^{-1}(R\gamma^{new})R\gamma^{new} - u'^{-1}(R\gamma^{old})R\gamma^{old},\end{aligned}$$

where we made use of equations (23), (25), and the fact that $\tau = \phi_{+1}M(\gamma - 1)$. From this, it can be seen that the sign of δ will depend on $\eta(q)$. To be able to proceed, we assume for the remainder of this Appendix that buyers' marginal utility in the DM is given by

$$u'(q) = q^{-\eta}, \tag{79}$$

i.e., that buyers have constant relative risk aversion η.[30] Under these preferences,

$$\delta = R^{1-\frac{1}{\eta}} \left((\gamma^{new})^{1-\frac{1}{\eta}} - (\gamma^{old})^{1-\frac{1}{\eta}} \right).$$

Thus, the sign of δ depends on η and the change in γ in the following way:

if	$\gamma^{new} > \gamma^{old}$	$\gamma^{new} < \gamma^{old}$
$\eta > 1$	$\delta > 0$	$\delta < 0$
$\eta < 1$	$\delta < 0$	$\delta > 0$
$\eta = 1$	$\delta = 0$	$\delta = 0$

Again, as the effects of the policy change only depend on whether γ increases or decreases for a given η, we can focus on the case of $\gamma^{new} = \gamma^*$ to ensure that the policy change increases the welfare of future generations. Given this, we are now ready to summarize:

[30]We need CRRA utility now because otherwise, elasticities for different levels of DM consumption could be different, which renders it impossible to make any general statement about the sign of δ.

Proposition 13. *With $\mathcal{I} = 0$, $u'(q) = q^{-\eta}$, and transfers between initial old buyers and sellers, changing monetary policy from any γ to γ^* leads to a Pareto-improvement in welfare if $\gamma > \gamma^*$ and $\eta \leq 1$ or if $\gamma < \gamma^*$ and $\eta \geq 1$. With $\eta > 1$, any γ is Pareto-optimal and with $\eta \leq 1$, any $\frac{1}{R} \leq \gamma \leq \gamma^*$ is Pareto-optimal.*

The proposition shows that for $\eta > 1$, Pareto-improvements are possible if $\gamma < \gamma^*$, but that is of little practical relevance since $\gamma^* = \frac{1}{R}$ for $\eta \geq 1$, so a Pareto-improvement is only possible if the money growth rate lies below the FR initially. In contrast, for $\eta \leq 1$, any $\gamma < \gamma^*$ maximizes Pareto-welfare, whereas Pareto-improvements are possible if the current money growth rate lies above γ^* - so if risk aversion is not too high, changing the money growth rate to γ^* leads to a Pareto-improvement in welfare. We think this is the most relevant case, as central banks in developed economies currently target positive inflation rates, which according to our analysis are Pareto-dominated by γ^*.

For $\eta > 1$, these results seem extreme, as any $\gamma > \gamma^*$ is Pareto-optimal according to our analysis - even hyperinflationary regimes. However, we have shown in the main body of the paper that implementing monetary policy over old buyers is efficient if and only if $\gamma \leq 1$, whereas if the inflation rate is positive and thus monetary policy consists of making transfers, it is better to make these transfers to young buyers. Thus, if monetary policy is implemented efficiently ($\mathcal{I} = 1$ for $\gamma > 1$ and $\mathcal{I} = 0$ for $\gamma < 1$), $\frac{1}{R} \leq \gamma \leq 1$ is Pareto-optimal for $\eta > 1$, while $\frac{1}{R} \leq \gamma \leq \gamma^*$ is Pareto-optimal for $\eta \leq 1$.

Appendix D Proofs

D.1 Proof of Proposition 3

Proof. We begin with the MT-E in both regimes. When $\mathcal{I} = 1$,

$$K^{\mathcal{I}=1} = \frac{2x^*}{R} - q^m. \tag{30}$$

Since q^m is decreasing in inflation, we always get an MT-E with $\mathcal{I} = 1$:

$$\frac{\partial K^{\mathcal{I}=1}}{\partial \gamma} = -\frac{\partial q^m}{\partial \gamma} > 0. \tag{80}$$

When $\mathcal{I} = 0$,

$$K^{\mathcal{I}=0} = \frac{2x^*}{R} - q^m(\gamma - 1) - q^m = \frac{2x^*}{R} - q^m\gamma, \tag{34}$$

with the first derivative:

$$\frac{\partial K^{\mathcal{I}=0}}{\partial \gamma} = -(\gamma\frac{\partial q^m}{\partial \gamma} + q^m) = \frac{u'(q^m)}{-u''(q^m)} - q^m = q^m\big(\frac{1}{\eta(q^m)} - 1\big). \tag{81}$$

Furthermore, $\frac{1}{\eta(q^m)} = |\varepsilon_{q^m}|$, since

$$|\varepsilon_{q^m}| = -\frac{dq^m/q^m}{d\gamma/\gamma} = -\frac{\gamma}{q^m}\frac{\partial q^m}{\partial \gamma} = -\frac{u'(q^m)}{q^m u''(q^m)}, \tag{82}$$

as $\frac{\partial q^m}{\partial \gamma} = \frac{R}{u''(q^m)}$ and $u'(q^m) = \gamma R$ from equation (23). Thus $\frac{\partial K^{\mathcal{I}=0}}{\partial \gamma} > 0$ if $|\varepsilon_{q^m}| = \frac{1}{\eta(q^m)} > 1$.

Next we turn to the effects on total labor supply. In both monetary policy regimes, total labor supply is the sum of capital investments K and the work of buyers to acquire real balances. If $\mathcal{I} = 1$, total labor supply is:

$$H^{\mathcal{I}=1} = \gamma q^m R + \frac{x^*}{R} - q^m R(\gamma - 1) + \frac{x^*}{R} - q^m = \frac{2x^*}{R} + q^m(R-1). \tag{31}$$

Buyers acquire real balances $\phi M = \gamma q^m R$ and get a transfer of $\tau = q^m R(\gamma - 1))$. So in this case the wealth effects of inflation on the holdings of real balances are canceled out and the effect of inflation on total labor supply must be negative:

$$\frac{\partial H^{\mathcal{I}=1}}{\partial \gamma} = \frac{\partial q^m}{\partial \gamma}(R-1) < 0. \tag{83}$$

With $\mathcal{I} = 0$ instead, total labor supply in the CM is:

$$H^{\mathcal{I}=0} = \gamma q^m R + \frac{x^*}{R} - q^m(\gamma - 1) + \frac{x^*}{R} - q^m = K^{\mathcal{I}=0} + \gamma q^m R = \frac{2x^*}{R} + (R-1)\gamma q^m. \tag{35}$$

Total labor supply is the sum of buyer real balances $\gamma q^m R$ and total capital investments. The effects of real balances on $K^{\mathcal{I}=0}$ and the real balance holdings $\gamma q^m R$ simplify to $(R-1)\gamma q^m$. Thus the sign of the derivative of γq^m, which is determined by $\eta(q^m)$, will also determine the sign of the derivative of total labor supply:

$$\frac{\partial H^{\mathcal{I}=0}}{\partial \gamma} = (R-1)(\frac{\partial q}{\partial \gamma}\gamma + q^m) = q^m(R-1)\big(1 - \frac{1}{\eta(q^m)}\big). \tag{84}$$

Thus we must have $\frac{\partial K^{\mathcal{I}=0}}{\partial \gamma} > 0$, $\frac{\partial H^{\mathcal{I}=0}}{\partial \gamma} < 0$ for $|\varepsilon_{q^m}| = \frac{1}{\eta(q^m)} > 1$, and $\frac{\partial K^{\mathcal{I}=0}}{\partial \gamma} < 0$, $\frac{\partial H^{\mathcal{I}=0}}{\partial \gamma} > 0$ for $|\varepsilon_{q^m}| = \frac{1}{\eta(q^m)} < 1$.

■

D.2 Proof of Proposition 4

Proof. Welfare of a representative generation with $\pi = 1$ can be written as

$$V^g = -H + u(q^m) - q^m + 2\beta U(x^*). \tag{85}$$

Because CM consumption is independent of γ for $\pi = 1$, γ affects welfare through DM consumption and aggregate labor supply:

$$\frac{\partial V^g}{\partial \gamma} = -\frac{\partial H}{\partial \gamma} + \frac{\partial q^m}{\partial \gamma}(u'(q^m) - 1). \tag{86}$$

In cases 1 and 3 from Proposition 3, $\frac{\partial H}{\partial \gamma} \geq 0$. Thus, the FR must be the optimal monetary policy since it maximizes welfare in the DM ($q^m = q^*$) from (23). Higher γ would decrease q^m while weakly increasing H.[31]

In case 2 and with $\mathcal{I} = 1$, $\frac{\partial H}{\partial \gamma} < 0$. Thus, the optimal γ must lie above the FR due to the envelope theorem. $\frac{\partial H}{\partial \gamma}$ is given by (84) and the optimal inflation rate $\gamma^* < 1$ solves

$$-q^m(R-1) - \frac{\partial q^m}{\partial \gamma}\gamma^*(R-1) + \frac{\partial q^m}{\partial \gamma}(\gamma^* R - 1) = 0$$

$$\leftrightarrow$$

$$-\frac{\partial q^m}{\partial \gamma}\frac{\gamma^*}{q^m} = |\varepsilon_{q^m}| = \frac{(R-1)\gamma^*}{1-\gamma^*}. \tag{87}$$

The interior solution solves

$$\gamma^* = \frac{|\varepsilon_{q^m}|}{|\varepsilon_{q^m}| + R - 1} \quad \in (1/R, 1). \tag{88}$$

If $\mathcal{I} = 1$, $\frac{\partial H}{\partial \gamma}$ is given by (83) and γ^* solves

$$-\frac{\partial q}{\partial \gamma}(R-1) + \frac{\partial q}{\partial \gamma}(\gamma^* R - 1) = 0. \tag{89}$$

Thus $\gamma^* = 1$.

■

D.3 Proof of Proposition 5

Proof. We proof this by contradiction using the first four optimality conditions (41) to (44) which we repeat for convenience.

$$\pi u'(q^m) + (1-\pi)u'(q^b) = \gamma R \tag{41}$$

$$\pi U'(x^m) + (1-\pi)U'(x^b) = \frac{1}{\beta R} \tag{42}$$

$$u'(q^b) = \beta R U'(x^b) \tag{43}$$

$$x^m = x^b + R(q^b - q^m). \tag{44}$$

At the FR (41) becomes:

$$\pi u'(q^m) + (1-\pi)u'(q^b) = 1.$$

Suppose $q^b > q^m$. From (44) this implies $x^m > x^b$. From the decreasing marginal utility of the utility functions and the weighted average formulation of the first and the second condition we

[31] For inflation rates below the FR, our derivation of results is incorrect, because we assumed $\gamma \geq \frac{1}{R}$. It looks like further decreasing γ is welfare-increasing if $|\varepsilon_{q^m}| < 1$, but this is incorrect, as inflation below the FR leads to a regime switch where nobody accumulates capital. This clearly reduces aggregate welfare. Thus, $\gamma = \frac{1}{R}$ is a corner solution for $|\varepsilon_{q^m}| < 1$.

must have $u'(q^m) > 1$ and $u'(q^b) < 1$ and $U'(x^m) > \frac{1}{\beta R}$ and $U'(x^b) < \frac{1}{\beta R}$. But this violates (43), the left-hand side would be < 1 and the right-hand side > 1. The opposite contradiction follows for assuming $q^b < q^m$. Thus we must have $q^b = q^m$ and $x^m = x^b$ and then $q^b = q^m = q^*$ and $x^m = x^b = x^*$ follow from (41) and (42). From $q^b = q^s$, it follows that all DM trades are made with m, and from that it follows that the allocation must be identical to an economy with $\pi = 1$. ■

D.4 Proof of Proposition 6

Proof. We first show that for $\gamma > 1/R$ we must have $q^* > q^b > q^m$ and $x^m > x^* > x^b$. The steps are the same as in the previous proof. Suppose that $q^b = q^m$ and $x^m = x^b$ from (44). This implies $u'(q) > 1$ from (41) and $\beta R U'(x) = 1$. Thus it violates (43). Similarly $q^b < q^m$ and $x^m < x^b$ imply $u'(q^b) > \gamma R > 1$ and $U'(x^b) < \frac{1}{\beta R}$ which also violates (43). Thus only $q^b > q^m$ and $x^m > x^b$ is possible, implying $u'(q^b) < \gamma R$ and $U'(x^b) > \frac{1}{\beta R}$. In this case (43) can hold if $u'(q^b) \in (1, \gamma R)$ which also implies $q^b < q^*$. Since $q^m < q^b$, q^m must also be below q^* and $x^m > x^*$ and $x^b < x^*$ follows from $U'(x^b) > \frac{1}{\beta R}$ and $U'(x^m) < \frac{1}{\beta R}$.

To find the effects of inflation on the consumption levels we differentiate (41) to (44) with respect to γ:

$$\begin{aligned}\pi u''(q^m)\frac{\partial q^m}{\partial \gamma} + (1-\pi)u''(q^b)\frac{\partial q^b}{\partial \gamma} &= R\\ \pi U''(x^m)\frac{\partial x^m}{\partial \gamma} + (1-\pi)U''(x^b)\frac{\partial x^b}{\partial \gamma} &= 0\\ u''(q^b)\frac{\partial q^b}{\partial \gamma} &= \beta R U''(x^b)\frac{\partial x^b}{\partial \gamma}\\ \frac{\partial x^m}{\partial \gamma} &= \frac{\partial x^b}{\partial \gamma} + R(\frac{\partial q^b}{\partial \gamma} - \frac{\partial q^m}{\partial \gamma}).\end{aligned} \tag{90}$$

Solving this for the partial effects yields

$$\frac{\partial q^b}{\partial \gamma} = A\frac{\partial x^b}{\partial \gamma} < 0 \tag{91}$$

$$\frac{\partial x^m}{\partial \gamma} = -B\frac{\partial x^b}{\partial \gamma} > 0 \tag{92}$$

$$\frac{\partial q^m}{\partial \gamma} = \frac{(RA+1+B)}{R}\frac{\partial x^b}{\partial \gamma} < 0 \tag{93}$$

$$\frac{\partial x^b}{\partial \gamma} = \frac{R}{C} < 0, \tag{94}$$

where $A = \frac{\beta R U''(x^b)}{u''(q^b)}$ and $B = \frac{(1-\pi)U''(x^b)}{\pi U''(x^m)}$ are positive and $C = \pi u''(q^m)\frac{RA+1+B}{R} + (1-\pi)u''(q^b)A$ is negative because $u(\cdot)$ and $U(\cdot)$ are strictly concave. The effect on q^s must also be negative because it is a weighted average of consumption of relocated and non-relocated buyers from (40).

The partial effect of γ on $X = x^* + \pi x^m + (1-\pi)x^b$ is

$$\frac{\partial X}{\partial \gamma} = \pi\frac{\partial x^m}{\partial \gamma} + (1-\pi)\frac{\partial x^b}{\partial \gamma} = -(\pi(1+B)-1)\frac{\partial x^b}{\partial \gamma}, \tag{95}$$

which is positive since $\frac{\partial x^b}{\partial \gamma} < 0$ and $\pi(1+B) > 1$ for inflation rates above the FR as we show in the proof of Proposition 7. ■

D.5 Proof of Proposition 7

Proof. With $\mathcal{I} = 1$, aggregate capital investment and labor supply are

$$K^{\mathcal{I}=1} = \frac{x^m}{R} + \frac{x^*}{R} - q^s \tag{48}$$

$$H^{\mathcal{I}=1} = \frac{x^m}{R} + q^m R + \frac{x^*}{R} - q^s = K^{\mathcal{I}=1} + q^m R. \tag{49}$$

Differentiating $K^{\mathcal{I}=1}$ with respect to inflation γ yields

$$\frac{\partial K^{\mathcal{I}=1}}{\partial \gamma} = -\frac{\partial q^m}{\partial \gamma}\mathcal{X}_1 > 0, \tag{96}$$

with $\mathcal{X}_1 = \frac{B+RA+\pi(1+B)}{B+RA+1}$.

Already a visual inspection of (48) tells us that there must be an MT-E. From the proof of Proposition 6, x^m increases with γ and q^m and q^b and thus also $q^s = \pi q^m + (1-\pi)q^b$ decrease with γ. The derivative confirms this, as all terms in $\mathcal{X}_1$ are positive.

From (49), the effect of γ on H is the sum of a positive MT-E and a negative effect on $q^m R$. This can be written as

$$\frac{\partial H^{\mathcal{I}=1}}{\partial \gamma} = -\frac{\partial q^m}{\partial \gamma}\mathcal{X}_1 + R\frac{\partial q^m}{\partial \gamma} = \frac{\partial q^m}{\partial \gamma}\big(R - \mathcal{X}_1\big), \tag{97}$$

so the effect of an increase in γ on H depends on $R \lesseqgtr \mathcal{X}_1$.

From the definition of B in the Proof to proposition 6, we know

$$\pi(1+B) = \pi + (1-\pi)\frac{U''(x^b)}{U''(x^m)} \geq 1,$$

since $x^m \geq x^b$ and $U'''(x) > 0$.[32] This implies $\mathcal{X}_1 \geq 1$. $\mathcal{X}_1 = 1$ if either $\pi = 1$ or if $x^m = x^b = x^*$ (at the FR), and strictly larger otherwise. From the partial derivatives, increasing γ from the FR increases the spread between DM-consumption x^m/x^b. Therefore $\pi(1+B)$ and $\mathcal{X}_1$ must rise with γ, while they are decreasing in π. Thus, the effect of γ on H is always negative at the FR. Away from the FR, higher γ and lower π make it more likely that there is a positive effect of inflation on aggregate labor supply.

■

[32] $U''(x^b) < U''(x^m)$, but since both second derivatives are negative ($U''(x^b), U''(x^m) < 0$), $U''(x^b)/U''(x^m) > 1$.

D.6 Proof of Proposition 8

Proof. With $\mathcal{I} = 0$, aggregate capital investment and labor supply are

$$K^{\mathcal{I}=0} = \frac{x^m}{R} + \frac{x^*}{R} - q^s - q^m(\gamma - 1) \tag{52}$$

$$H^{\mathcal{I}=0} = \frac{x^m}{R} + \gamma q^m R - (\gamma - 1)q^m + \frac{x^*}{R} - q^s = K^{\mathcal{I}=0} + \gamma q^m R. \tag{53}$$

Differentiating K with respect to γ yields

$$\frac{\partial K^{\mathcal{I}=0}}{\partial \gamma} = -\Big(\frac{\partial q^m}{\partial \gamma}\gamma \mathcal{X}_0 + q^m\Big), \tag{98}$$

with $\mathcal{X}_0 = \frac{\pi(1+B)-1}{\gamma(RA+1+B)} + 1$. $\mathcal{X}_0 = 1$ for $\pi = 1$ (since $B = 0$ for $\pi = 1$) and at the FR (since $B = \frac{1-\pi}{\pi}$ at the FR). The condition for an MT-E is

$$-\frac{\partial q^m}{\partial \gamma} > \frac{q^m}{\gamma \mathcal{X}_0} \tag{99}$$

$$|\varepsilon_{q^m}| > \frac{1}{\mathcal{X}_0} = \hat{\varepsilon}, \tag{100}$$

using the definition of the elasticity of DM-consumption with respect to inflation: $|\varepsilon_{q^m}| = -\frac{\gamma}{q^m}\frac{\partial q^m}{\partial \gamma}$.

At the FR $\mathcal{X}_0 = 1$, so there is an MT-E if $|\varepsilon_{q^m}|_{FR} > 1$. Using (93) and (94) with $q^m = q^b = q^*$ and $x^m = x^b = x^*$, the derivative of q^m with respect to γ and the elasticity are:

$$-\frac{\partial q^m}{\partial \gamma}|_{FR} = \frac{1}{u''(q^*)}\frac{R(\pi RA + 1)}{\pi(1 + RA)} \tag{101}$$

$$|\varepsilon_{q^m}|_{FR} = -\frac{u'(q^*)}{q^* u''(q^*)}\xi, \tag{102}$$

with $\xi = \frac{1+\pi RA}{\pi(1+RA)}$. Thus, there is an MT-E at the FR if:

$$-\frac{u'(q^*)}{q^* u''(q^*)}\xi > 1$$

$$\Rightarrow -\frac{u'(q^*)}{q^* u''(q^*)} > \frac{1}{\xi}. \tag{103}$$

$\xi \to 1$ for $\pi \to 1$, and $\xi \to \infty$ for $\pi \to 0$, and it can easily be shown that ξ is monotonically decreasing in π at the FR. This implies that for a given risk aversion, an MT-E is more likely to occur at the FR for lower π.

Combining (53) and (81) the derivative of H with respect to γ is:

$$\frac{\partial H^{\mathcal{I}=0}}{\partial \gamma} = (R-1)(\frac{\partial q^m}{\partial \gamma}\gamma + q^m) - \frac{\partial q^m}{\partial \gamma}(\mathcal{X}_0 - 1). \tag{104}$$

This is negative if

$$-\frac{\partial q^m}{\partial \gamma} > \frac{q^m(R-1)}{\gamma(R-1) - (\mathcal{X}_0 - 1)} \tag{105}$$

$$|\varepsilon_{q^m}| > \frac{\gamma(R-1)}{\gamma(R-1)-(\mathcal{X}_0-1)} = \tilde{\varepsilon}, \tag{106}$$

so there is a negative effect of γ on H for $\gamma(R-1) > \mathcal{X}_0 - 1$. Since $\mathcal{X}_0 = 1$ at the FR, $\hat{\varepsilon}|_{FR} = \tilde{\varepsilon}|_{FR} = 1$, so the conditions coincide, and an MT-E (reverse MT-E) always implies a negative (positive) effect of γ on H. Away from the FR, $\mathcal{X}_0 > 1$, so $\hat{\varepsilon} < 1$ while $\tilde{\varepsilon} > 1$, and thus there is a range of values for $|\varepsilon_{q^m}|$ for which an MT-E comes along with a positive effect of γ on H.

■

D.7 Proof of Proposition 9

Proof. With $\mathcal{I} = 1$, expected welfare of a representative generation is

$$\begin{aligned} V^{\mathcal{I}=1} &= -H^{\mathcal{I}=1} + \pi(u(q^m) + \beta U(x^m)) + (1-\pi)(u(q^b) + \beta U(x^b)) - q^s + \beta U(x^*) \qquad (107) \\ &= -\frac{x^m}{R} - q^m R - \frac{x^*}{R} + \pi(u(q^m) + \beta U(x^m)) + (1-\pi)(u(q^b) + \beta U(x^b)) + \beta U(x^*). \end{aligned}$$

Differentiating (107) with respect to γ and replacing $u'(q^m), U'(x^m)$ and $u'(q^b)$ using the optimality conditions (41), (42) and (44) yields

$$\frac{\partial V^{\mathcal{I}=1}}{\partial \gamma} = -R\frac{\partial q^m}{\partial \gamma} + \gamma R \frac{\partial q^m}{\partial \gamma} - (1-\pi)\beta R U'(x^b)\Big(\frac{\partial q^m}{\partial \gamma} - \frac{\partial q^b}{\partial \gamma} + \frac{\frac{\partial x^m}{\partial \gamma} - \frac{\partial x^b}{\partial \gamma}}{R}\Big).$$

The last bracket is equal to zero from (90), and we obtain the identical expression as with $\pi = 1$, (89):

$$\frac{\partial V^{\mathcal{I}=1}}{\partial \gamma} = \frac{\partial q^m}{\partial \gamma} R(\gamma - 1). \tag{108}$$

At the FR this expression must be positive since $\frac{\partial q^m}{\partial \gamma} < 0$ from (93) (and from proposition 8 the marginal effect on the labor supply is negative). From there expected welfare increases in γ for $\gamma < 1$ and decreases for $\gamma > 1$. Thus the unique optimum must be $\gamma^* = 1$ independent of π and the other parameters.

To show that the FR is relatively more costly at lower levels of π, we show that $V^{\mathcal{I}=1}(\gamma^*) - V^{\mathcal{I}=1}(\gamma^{FR})$, the difference in expected welfare under $\gamma^* = 1$ and$\gamma = 1/R$, decreases in π. From Proposition 5, welfare at the FR is

$$V^{\mathcal{I}=1}(\gamma^{FR}) = -\frac{2x^*}{R} - q^* R + u(q^*) + 2\beta U(x^*), \tag{109}$$

which is independent of π. Thus to show that $V^{\mathcal{I}=1}(\gamma^*) - V^{\mathcal{I}=1}(\gamma^{FR})$ decreases in π it is sufficient to show that expected welfare under $\gamma^* = 1$ is decreasing in π. From (107) $V^{\mathcal{I}=1}(\gamma^*)$ is

$$V^{\mathcal{I}=1}(\gamma^*) = -\frac{x^m}{R} - q^m R - \frac{x^*}{R} + \pi(u(q^m) + \beta U(x^m)) + (1-\pi)(u(q^b) + \beta U(x^b)) + \beta U(x^*), \tag{110}$$

where all optimality conditions (41) to (44) hold and (41) is evaluated at $\gamma^* = 1$. Since (110) is evaluated at the optimum we can ignore the indirect effects of π on the variables and directly take the partial derivative of (110) with respect to π.

$$\frac{\partial V^{\mathcal{I}=1}(\gamma^*)}{\partial \pi} = u(q^m) + \beta U(x^m) - (u(q^b) + \beta U(x^b)) < 0. \tag{111}$$

Changes in expected welfare at γ^* through π reflect differences in the utility of consumption as a relocated and a non-relocated buyer. If utility of consumption as a relocated buyer is higher, expected welfare would rise with π and vice versa. Since $q^b > q^m$ and $x^m > x^b$ it is not a priori clear where utility is higher. However, since for a non-relocated buyer the relocated allocation $\{q^m, x^m\}$ is also feasible but not chosen, it must be that utility of non-relocated buyers is higher or $u(q^b) + \beta U(x^b) > u(q^m) + \beta U(x^m)$. Thus $\frac{\partial V^{\mathcal{I}=1}(\gamma^*)}{\partial \pi} < 0$, implying that expected welfare at $\gamma^* = 1$ is decreasing in π. In turn, this shows that the welfare loss of running the FR is decreasing in π.

■

D.8 Proof of Proposition 10

Proof. Expected welfare with $\mathcal{I} = 0$ is

$$\begin{aligned} V^{\mathcal{I}=0} &= -H^{\mathcal{I}=0} + \pi(u(q^m) + \beta U(x^m)) + (1-\pi)(u(q^b) + \beta U(x^b)) - q^s + \beta U(x^*) \qquad (112) \\ &= -\frac{x^m}{R} - q^m R\gamma + (\gamma - 1)q^m - \frac{x^*}{R} + \pi(u(q^m) + \beta U(x^m)) + (1-\pi)(u(q^b) + \beta U(x^b)) + \beta U(x^*). \end{aligned}$$

Differentiating (112) with respect to γ and replacing $u'(q^m), U'(x^m)$ and $u'(q^b)$ using the optimality conditions (41), (42) and (44) yields:

$$\frac{\partial V^{\mathcal{I}=0}}{\partial \gamma} = \frac{\partial q^m}{\partial \gamma}(\gamma - 1) - (R-1)q^m - (1-\pi)\beta R U'(x^b)\Big(\frac{\partial q^m}{\partial \gamma} - \frac{\partial q^b}{\partial \gamma} + \frac{\frac{\partial x^m}{\partial \gamma} - \frac{\partial x^b}{\partial \gamma}}{R}\Big)$$

The last bracket is zero from (90) and we obtain the identical expression as with $\pi = 1$, (87):

$$\frac{\partial V^{\mathcal{I}=0}}{\partial \gamma} = \frac{\partial q^m}{\partial \gamma}(\gamma - 1) - (R-1)q^m \tag{113}$$

Since $\frac{\partial q^m}{\partial \gamma} < 0$ an increase in inflation can only be welfare improving if there is deflation ($\gamma < 1$). When is a deviation from the Friedman rule welfare improving? Evaluating (113) at the Friedman rule:

$$\frac{\partial V^{\mathcal{I}=0}}{\partial \gamma}|_{FR} = q^*(R-1)\Big(|\varepsilon_{q^m}|_{FR} - 1\Big). \tag{114}$$

Thus $\gamma > 1/R$ is optimal if $|\varepsilon_{q^m}|_{FR} > 1$ holds, which is the same condition as for an MT-E and a negative effect on H.

If $|\varepsilon_{q^m}|_{FR} > 1$ holds, the optimal inflation rate $\gamma^* > 1/R$ is given by (113) set to 0 which is exactly the same expression as for the case of fully illiquid capital (87)

$$(1 - \gamma^*) - \frac{\partial q^m}{\partial \gamma} = (R-1)q^m \tag{115}$$

$$\leftrightarrow \tag{116}$$

$$-\frac{\partial q^m}{\partial \gamma}\frac{\gamma^*}{q^m} = |\varepsilon_{q^m}| = \frac{(R-1)\gamma^*}{1-\gamma^*}. \tag{117}$$

It can easily be seen that the right-hand-side of this expression equals 1 at the FR, ∞ for $\gamma = 1$, and is strictly increasing in γ $\forall \gamma \in \{\frac{1}{R}, 1\}$. From the proof to Proposition 8, we know that $|\varepsilon_{q^m}|_{FR} = \frac{1}{\eta(q^*)}\xi$. Thus, the FR is optimal if and only if $\eta(q^*) > \xi$. Since ξ is monotonically decreasing in π and $\xi \to \infty$ for $\pi \to 0$, lower π makes it less likely for the FR to be optimal, and the FR is never optimal when $\pi \to 0$.

With partially liquid capital, the optimal inflation rate γ^* is still given by (88):

$$\gamma^* = \frac{|\varepsilon_{q^m}|}{|\varepsilon_{q^m}| + R - 1,} \quad \in (1/R, 1) \tag{88}$$

but with $|\varepsilon_{q^m}|$ now also being a function of π.

The result that welfare is higher with $\mathcal{I} = 0$, given that γ^* is chosen, follows once again from the fact that the optimal allocation under $\mathcal{I} = 1$, which is achieved by setting $\gamma^* = 1$, is feasible with $\mathcal{I} = 0$, but generally not optimal.

■