Problem 1
Evaluate the integral.
(a) $\int_0^2 \int_{-1}^1 x - y\,dy\,dx$
Solution

$$\begin{aligned}
\int_0^2 \int_{-1}^1 x - y\,dy\,dx &= \int_0^2 \left[xy - \frac{1}{2}y^2\right]_{-1}^1 dx \\
&= \int_0^2 \left(x - \frac{1}{2}\right) - \left(-x - \frac{1}{2}\right) dx \\
&= \int_0^2 2x\,dx \\
&= \left[x^2\right]_0^2 \\
&= \boxed{4}
\end{aligned}$$

(b) $\int_0^3 \int_{-2}^0 (x^2y - 2xy)\,dy\,dx$
Solution

$$\begin{aligned}
\int_0^3 \int_{-2}^0 x^2y - 2xy\,dy\,dx &= \int_0^3 \left[\frac{1}{2}x^2y^2 - xy^2\right]_{-2}^0 dx \\
&= \int_0^3 (0-0) - \left(\frac{1}{2}x^2(-2)^2 - x(-2)^2\right) dx \\
&= \int_0^3 -2x^2 + 4x\,dx \\
&= \left[-\frac{2}{3}x^3 + 2x^2\right]_0^3 \\
&= \left(-\frac{2}{3}(3)^3 + 2(3)^2\right) - (0+0) \\
&= \boxed{0}
\end{aligned}$$

(c) $\displaystyle\int_0^1\int_0^1 \frac{y}{1+xy}\,dx\,dy$
Solution

$$\begin{aligned}
\int_0^1\int_0^1 \frac{y}{1+xy}\,dx\,dy &= \int_0^1 \left[\ln|1+xy|\right]_0^1\,dy\\
&= \int_0^1 \ln|1+y|\,dy\\
u = 1+y &\qquad du = dy\\
u(0) = 1 &\qquad u(1) = 2\\
&= \int_1^2 \ln|u|\,du\\
v = \ln|u| &\qquad dw = du\\
dv = \frac{1}{u}du &\qquad w = u\\
&= \left[u\ln|u|\right]_1^2 - \int_1^2 1\,du\\
&= (2\ln 2) - (\ln 1) - [u]_1^2\\
&= 2\ln 2 - \big((2)-(1)\big)\\
&= \boxed{2\ln 2 - 1}
\end{aligned}$$

(d) $\displaystyle\int_0^1\int_1^2 xye^x\,dy\,dx$
Solution

$$\begin{aligned}
\int_0^1\int_1^2 xye^x\,dy\,dx &= \int_0^1 \left[\frac{1}{2}xy^2e^x\right]_1^2\,dx\\
&= \int_0^1 \frac{1}{2}x(2)^2e^x - \frac{1}{2}x(1)^2e^x\,dx\\
&= \int_0^1 \frac{3}{2}xe^x\,dx\\
u = \frac{3}{2}x &\qquad dv = e^x\,dx\\
du = \frac{3}{2}\,dx &\qquad v = e^x\\
&= \left[\frac{3}{2}xe^x\right]_0^1 - \int_0^1 \frac{3}{2}e^x\\
&= \frac{3}{2}e - \left[\frac{3}{2}e^x\right]_0^1\\
&= \frac{3}{2}e - \left(\frac{3}{2}e - \frac{3}{2}\right)\\
&= \boxed{\frac{3}{2}}
\end{aligned}$$

(e) $\int_{-1}^{2}\int_{0}^{\pi/2} y\sin x\,dx\,dy$

Solution

$$\begin{aligned}
\int_{-1}^{2}\int_{0}^{\pi/2} y\sin x\,dx\,dy &= \int_{-1}^{2} \left[-y\cos x\right]_0^{\pi/2}\,dy \\
&= \int_{-1}^{2} \left(-y\cos\frac{\pi}{2}\right) - \left(-y\cos 0\right)\,dy \\
&= \int_{-1}^{2} y\,dy \\
&= \left[\frac{1}{2}y^2\right]_{-1}^{2} \\
&= \frac{1}{2}(2)^2 - \frac{1}{2}(-1)^2 \\
&= \boxed{\frac{3}{2}}
\end{aligned}$$

(f) $\int_{1}^{4}\int_{1}^{e} \frac{\ln x}{xy}\,dx\,dy$

Solution

$$\int_{1}^{4}\int_{1}^{e} \frac{\ln x}{xy}\,dx\,dy$$

$$\begin{aligned}
&u = \ln x \qquad du = \frac{1}{x}\,dx \\
&u(1) = \ln 1 = 0 \qquad u(e) = \ln e = 1 \\
&= \int_{1}^{4}\int_{0}^{1} \frac{u}{y}\,du\,dy \\
&= \int_{1}^{4} \left[\frac{u^2}{2y}\right]_0^1\,dy \\
&= \int_{1}^{4} \frac{1}{2y}\,dy \\
&= \left[\frac{1}{2}\ln|y|\right]_1^4 \\
&= \left(\frac{1}{2}\ln 4\right) - \left(\frac{1}{2}\ln 1\right) \\
&= \ln 4^{1/2} - 0 \\
&= \boxed{\ln 2}
\end{aligned}$$

Problem 2

Find all values of c such that $\displaystyle\int_{-1}^{c}\int_{0}^{2} xy+1\,dy\,dx = 4+4c$

Solution

$$\begin{aligned}
\int_{-1}^{c}\int_{0}^{2} xy+1\,dy\,dx &= \int_{-1}^{c}\left[\frac{1}{2}xy^2+y\right]_0^2 dx \\
&= \int_{-1}^{c}(2x+2)-(0+0)\,dx \\
&= \left[x^2+2x\right]_{-1}^{c} \\
&= (c^2+2c)-(1-2) \\
&= c^2+2c+1
\end{aligned}$$

We know that $\displaystyle\int_{-1}^{c}\int_{0}^{2} xy+1\,dy\,dx = 4+4c$. Therefore,

$$\begin{aligned}
\int_{-1}^{c}\int_{0}^{2} xy+1\,dy\,dx = 4+4c &= c^2+2c+1 \\
4+4c &= c^2+2c+1 \\
0 &= c^2-2c-3 \\
0 &= (c-3)(c+1) \\
&\boxed{c=3 \text{ or } c=-1}
\end{aligned}$$

Problem 3 (Parts)

Evaluate the integral over the rectangle R.

(a) $\displaystyle\iint_R e^{x-y}\,dA$, $0\le x\le \ln 2$, $0\le y\le \ln 2$

Solution

For this problem, it really doesn't matter which variable we integrate with respect to first. It's just preference.

$$\begin{aligned}
\iint_R e^{x-y}\,dA &= \int_0^{\ln 2}\int_0^{\ln 2} e^{x-y}\,dy\,dx \\
&= \int_0^{\ln 2}\left[-e^{x-y}\right]_0^{\ln 2} dx \\
&= \int_0^{\ln 2}\left(-e^{x-\ln 2}\right)-\left(-e^{x}\right)\,dx \\
&= \left[-e^{x-\ln 2}+e^{x}\right]_0^{\ln 2} \\
&= \left(-e^{\ln 2-\ln 2}+e^{\ln 2}\right)-\left(-e^{0-\ln 2}+e^{0}\right) \\
&= (-1+2)-\left(-\frac{1}{2}+1\right) \\
&= \boxed{\frac{1}{2}}
\end{aligned}$$

(b) $\iint_R \left(\frac{\sqrt{x}}{y^2}\right)\,dA$, $0 \leq x \leq 4$, $1 \leq y \leq 2$

Solution

It really doesn't matter which variable we integrate with respect to first. However, integrating with respect to y first seems a little bit more ideal in this problem (I hate dealing with square roots).

$$\begin{aligned}
\iint_R \left(\frac{\sqrt{x}}{y^2}\right)\,dA &= \int_0^4 \int_1^2 \frac{\sqrt{x}}{y^2}\,dy\,dx \\
&= \int_0^4 \left[-\frac{\sqrt{x}}{y}\right]_1^2 \\
&= \int_0^4 \left(-\frac{\sqrt{x}}{2}\right) - \left(-\frac{\sqrt{x}}{1}\right)\,dx \\
&= \int_0^4 \frac{\sqrt{x}}{2}\,dx \\
&= \left[\frac{1}{3}x^{\frac{3}{2}}\right]_0^4 \\
&= \left(\frac{1}{3}(4)^{\frac{3}{2}}\right) - (0) \\
&= \boxed{\frac{8}{3}}
\end{aligned}$$

(c) $\iint_R xy\cos y\,dA$, $-1 \leq x \leq 1$, $0 \leq y \leq \pi$

Solution

It really doesn't matter which variable we integrate with respect to first. However, integrating with respect to y first seems a little bit more ideal in this problem (it's just preference).

$$\begin{aligned}
\iint_R xy\cos y\,dA &= \int_{-1}^1 \int_0^\pi xy\cos y\,dy\,dx \\
& u = xy \qquad dv = \cos y\,dy \\
& du = x\,dy \qquad v = \sin y \\
&= \int_{-1}^1 \left([xy\sin y]_0^\pi - \int_0^\pi x\sin y\,dy\right)\,dx \\
&= \int_{-1}^1 \left((0-0) - [-x\cos y]_0^\pi\right)\,dx \\
&= \int_{-1}^1 -\big(x - (-x)\big)\,dx \\
&= \int_{-1}^1 -2x\,dx \\
&= \left[-x^2\right]_{-1}^1 \\
&= \left(-(1)^2\right) - \left(-(-1)^2\right) \\
&= \boxed{0}
\end{aligned}$$

(d) $\iint_R \frac{y}{x^2y^2+1}\, dA$, $0 \leq x \leq 1$, $0 \leq y \leq 1$

Solution

It really doesn't matter which variable we integrate with respect to first. However, integrating with respect to x first seems a little bit more ideal in this problem (it's just preference).

$$\begin{aligned}
\iint_R \frac{y}{x^2y^2+1}\, dA &= \iint\limits_R \frac{y}{(xy)^2+1}\, dA \\
&= \int_0^1 \int_0^1 \frac{y}{(xy)^2+1}\, dx\, dy \\
&= \int_0^1 \left[\tan^{-1}(xy)\right]_0^1 dy \\
&= \int_0^1 \tan^{-1} y\, dy \\
&u = \tan^{-1} y \qquad dv = dy \\
&du = \frac{1}{y^2+1}\, dy \qquad v = y \\
&= \left[y \tan^{-1} y\right]_0^1 - \int_0^1 \frac{y}{y^2+1}\, dy \\
&w = y^2+1 \qquad dw = 2y\, dy \\
&w(0) = 1 \qquad w(1) = 2 \\
&= \left(\tan^{-1}(1) - 0\right) - \frac{1}{2}\int_1^2 \frac{1}{u}\, du \\
&= \frac{\pi}{4} - \frac{1}{2}\left[\ln|u|\right]_1^2 \\
&= \frac{\pi}{4} - \frac{1}{2}(\ln 2 - \ln 1) \\
&= \boxed{\frac{\pi}{4} - \frac{1}{2}\ln 2}
\end{aligned}$$

Problem 4
Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below by the square $-1 \le x \le 1, -1 \le y \le 1$.
Solution
If we interpret this problem into a double integral, we get

$$\iint_R z\,dA = \iint_R x^2 + y^2\,dA$$

where R is our region $-1 \le x \le 1, -1 \le y \le 1$. Plugging in the bounds, we get

$$\begin{aligned}
\iint_R x^2 + y^2\,dA &= \int_{-1}^{1}\int_{-1}^{1} x^2 + y^2\,dy\,dx \\
&= \int_{-1}^{1} \left[x^2 y + \frac{1}{3}y^3\right]_{-1}^{1} dx \\
&= \int_{-1}^{1} x^2 + \frac{1}{3}\,dx \\
&= \int_{-1}^{1} \left(x^2 + \frac{1}{3}\right) - \left(-x^2 - \frac{1}{3}\right) dx \\
&= \int_{-1}^{1} 2x^2 + \frac{2}{3}\,dx \\
&= \left[\frac{2}{3}x^3 + \frac{2}{3}x\right]_{-1}^{1} \\
&= \left(\frac{2}{3} + \frac{2}{3}\right) - \left(-\frac{2}{3} - \frac{2}{3}\right) \\
&= \frac{4}{3} - \left(-\frac{4}{3}\right) \\
&= \boxed{\frac{8}{3}}
\end{aligned}$$

Note that the order does not particularly matter for this problem. We just did y first because y not hahaha...
Problem 5
If $f(x, y)$ is continuous over R: $a \le x \le b$, $c \le y \le d$ and

$$F(x, y) = \int_a^x \int_c^y f(u, v)\,dv\,du$$

on the interior of R, find the second partial derivatives F_{xy} and F_{yx}
Solution
For F_{xy},
If $f(x, y)$ is continuous over R, then $f(u, v)$ is a continuous function of v when u is a constant value (frozen). Let's have $g(u, v)$ represent the antiderivative of $f(u, v)$ with respect to v. Therefore,

$$\begin{aligned}
F(x, y) &= \int_a^x \int_c^y f(u, v)\,dv\,du = \int_a^x [g(u, v)]_c^y\,du = \int_a^x g(u, y) - g(u, c)\,du \\
\implies F_x &= \frac{\partial}{\partial x}\int_a^x g(u, y) - g(u, c)\,du = g(x, y) - g(x, c) && \text{(think back to FTC!)} \\
\implies F_{xy} &= \frac{\partial}{\partial y}\big(g(x, y) - g(x, c)\big) = f(x, y) - 0 = f(x, y) && \text{(remember: anitderivatives)}
\end{aligned}$$

$$\boxed{F_{xy} = f(x, y)}$$

For F_{yx}, If $f(x,y)$ is continuous over R, then $f(u,v)$ is a continuous function of u when v is a constant value (frozen). Let's have $h(u,v)$ represent the antiderivative of $f(u,v)$ with respect to u. Therefore,

$$\begin{aligned}
F(x,y) &= \int_a^x \int_c^y f(u,v)\,dv\,du = \int_c^y \int_a^x f(u,v)\,du\,dv = \int_c^y \left[h(u,v)\right]_a^x\,dv = \int_c^y h(x,v) - h(a,v)\,du \\
\implies F_y &= \frac{\partial}{\partial y}\int_c^y h(x,v) - h(a,v)\,du = h(x,y) - h(a,y) && \text{(think back to FTC!)} \\
\implies F_{yx} &= \frac{\partial}{\partial x}\big(h(x,y) - h(a,y)\big) = f(x,y) - 0 = f(x,y) && \text{(remember: antiderivatives)} \\
&\boxed{F_{yx} = f(x,y)}
\end{aligned}$$

Note that we could not immediately conclude from $F_{xy} = f(x,y)$ that $F_{yx} = F_{xy} = f(x,y)$. This is because the symmetry of second derivatives only occurs when we *know* that F_x and F_y are both differentiable. This was briefly mentioned in Discussion 3/29.