Problem 1 (Parts)
Find the absolute minimum and maximum values of the function on the domain:
(a) $f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$ on the triangle bounded by $x = 0, y = 2, y = 2x$
Solution
To find our interior critical point(s), we need to find out at what point(s) does both f_x and f_y equal 0 or are undefined. Therefore,

$$\begin{aligned} f_x &= 4x - 4 = 0 \implies x = 1 \\ f_y &= 2y - 4 = 0 \implies y = 2 \\ &\text{critical point at } (1, 2) \end{aligned}$$

To find our boundary critical points, let's "plug in" our bounding lines.
For $x = 0$,

$$\begin{aligned} f(0, y) &= 2(0)^2 - 4(0) + y^2 - 4y + 1 \\ &= y^2 - 4y + 1 \\ f'(0, y) &= 2y - 4 \\ f'(0, y) &= 2y - 4 = 0 \implies y = 2 \implies x = 0 \\ &\text{critical point at } (0, 2) \end{aligned}$$

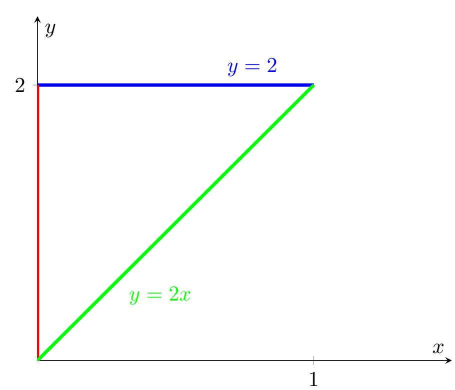
For $y = 2$,

$$\begin{aligned} f(x, 2) &= 2x^2 - 4x + (2)^2 - 4(2) + 1 \\ &= 2x^2 - 4x + 1 \\ f'(x, 2) &= 4x - 4 \\ f'(x, 2) &= 4x - 4 = 0 \implies x = 1 \implies y = 2 \\ &\text{critical point at } (1, 2) \end{aligned}$$

For $y = 2x$,

$$\begin{aligned} f(x, 2x) &= 2x^2 - 4x + (2x)^2 - 4(2x) + 1 \\ &= 6x^2 - 12x + 1 \\ f'(x, 2x) &= 12x - 12 \\ f'(x, 2x) &= 12x - 12 = 0 \implies x = 1 \implies y = 2(1) = 2 \\ &\text{critical point at } (1, 2) \end{aligned}$$

Let's find our "corners". To do that, let's graph the function.



Based on the graph, our "corners" are $(0, 0)$, $(0, 2)$, and $(1, 2)$.

Now, let's plug-n-chug all the critical points we found. The lowest value of $f(x,y)$ we get will be our absolute minimum value on the domain and the highest value of $f(x,y)$ we get will be out absolute maximum value on the domain.

$$\begin{aligned}
f(1,2) &= 2(1)^2 - 4(1) + (2)^2 - 4(2) + 1 = -5 \\
f(0,2) &= 2(0)^2 - 4(0) + (1)^2 - 4(1) + 1 = -2 \\
f(0,0) &= 2(0)^2 - 4(0) + (0)^2 - 4(0) + 1 = 1 \\
f(0,2) &= 2(0)^2 - 4(0) + (2)^2 - 4(2) + 1 = -3
\end{aligned}$$

Note: There were a few duplicate points that I omitted to save space.
Our final answer is

$$\boxed{\begin{array}{c} \text{absolute minimum: } f(1,2) = -5 \\ \text{absolute maximum: } f(0,0) = 1 \end{array}}$$

(b) $f(x,y) = x^2 + xy + y^2 - 6x$ on the rectangle $0 \leq x \leq 5, = 3 \leq y \leq 3$
Solution
To find our interior critical point(s), we need to find out at what point(s) does both f_x and f_y equal 0 or are undefined. Therefore,

$$\begin{aligned}
f_x &= 2x + y - 6 = 0 \implies y = -2x + 6 \\
f_y &= x + 2y = 0 \\
x + 2(-2x+6) &= 0 \implies -3x + 12 = 0 \implies x = 4 \implies y = -2(4) + 6 = -2 \\
&\text{critical point at } (4,-2)
\end{aligned}$$

To find our boundary critical points, let's "plug in" our bounding lines. The inequality $0 \leq x \leq 5$ turns into the lines $x = 0$ and $x = 5$. The inequality $-3 \leq y \leq 3$ turns into the lines $y = -3$ and $y = 3$.
For $x = 0$,

$$\begin{aligned}
f(0,y) &= 0^2 + (0)y + y^2 - 6(0) \\
&= y^2 \\
f'(0,y) &= 2y \\
f'(0,y) &= 2y = 0 \implies y = 0 \\
\text{critical point at } &(0,0)
\end{aligned}$$

For $x = 5$,

$$\begin{aligned}
f(5,y) &= 5^2 + (5)y + y^2 - 6(5) \\
&= y^2 + 5y - 5 \\
f'(5,y) &= 2y + 5 \\
f'(5,y) &= 2y + 5 = 0 \implies y = -\frac{5}{2} \\
\text{critical point at } &\left(5, -\frac{5}{2}\right)
\end{aligned}$$

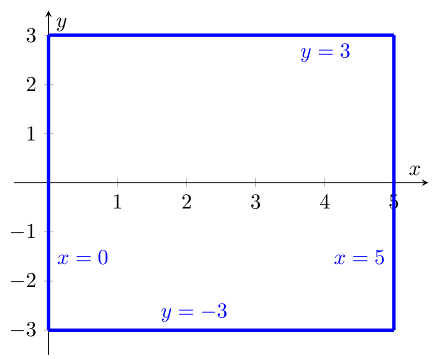
For $y = -3$,

$$\begin{aligned}
f(x,-3) &= x^2 + x(-3) + (-3)^2 - 6x \\
&= x^2 - 9x + 9 \\
f'(x,-3) &= 2x - 9 \\
f'(x,-3) &= 2x - 9 = 0 \implies x = \frac{9}{2} \\
\text{critical point at } &\left(\frac{9}{2}, -3\right)
\end{aligned}$$

For $y = 3$,

$$\begin{aligned} f(x,-3) &= x^2 + x(3) + (3)^2 - 6x \\ &= x^2 - 3x + 9 \\ f'(x,3) &= 2x - 3 \\ f'(x,3) = 2x - 3 = 0 &\implies x = \frac{3}{2} \\ \text{critical point at } &\left(\frac{3}{2}, 3\right) \end{aligned}$$

Let's find our "corners". To do that, let's graph the function. Based on the graph, our "corners" are



$(0,-3), (0,3), (5,3)$, and $(5,-3)$.
Now, let's plug-n-chug all the critical points we found. The lowest value of $f(x,y)$ we get will be our absolute minimum value on the domain and the highest value of $f(x,y)$ we get will be out absolute maximum value on the domain.

$$\begin{aligned} f(4,-2) &= (4)^2 + (4)(-2) + (-2)^2 - 6(4) = -12 \\ f(0,0) &= (0)^2 + (0)(0) + (0)^2 - 6(0) = 0 \\ f\left(5, -\frac{5}{2}\right) &= (5)^2 + (5)\left(-\frac{5}{2}\right) + \left(-\frac{5}{2}\right)^2 - 6(5) = -\frac{45}{4} \\ f\left(\frac{9}{2}, -3\right) &= \left(\frac{9}{2}\right)^2 + \left(\frac{9}{2}\right)(-3) + (-3)^2 - 6\left(\frac{9}{2}\right) = -\frac{45}{4} \\ f\left(\frac{3}{2}, 3\right) &= \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)(3) + (3)^2 - 6\left(\frac{3}{2}\right) = \frac{27}{4} \\ f(0,-3) &= (0)^2 + (0)(-3) + (-3)^2 - 6(0) = 9 \\ f(0,3) &= (0)^2 + (0)(3) + (3)^2 - 6(0) = 9 \\ f(5,3) &= (5)^2 + (5)(3) + (3)^2 - 6(5) = 19 \\ f(5,-3) &= (5)^2 + (5)(-3) + (-3)^2 - 6(5) = -11 \end{aligned}$$

Our final answer is

$$\boxed{\begin{array}{c} \text{absolute minimum: } f(4,-2) = -12 \\ \text{absolute maximum: } f(5,3) = 19 \end{array}}$$

(c) $f(x,y) = (4x - x^2)\cos y$ on the rectangle $1 \le x \le 3, -\frac{\pi}{4} \le y \le \frac{\pi}{4}$
*Solution*To find our interior critical point(s), we need to find out at what point(s) does both f_x and f_y equal

0 or are undefined. Therefore,

$$\begin{aligned} f_x &= (4-2x)\cos y = 0 \implies x = 2 \\ f_y &= -(4x-x^2)\sin y = 0 \implies y = 0 \\ &\text{critical point at } (2,0) \end{aligned}$$

Note the inequalities $1 \le x \le 3, -\frac{\pi}{4} \le y \le \frac{\pi}{4}$. This is why $x = -2, y = \pm\frac{\pi}{2}$, and $y = -\pi$ are not valid values.
To find our boundary critical points, let's "plug in" our bounding lines. The inequality $1 \le x \le 3$ turns into the lines $x = 1$ and $x = 3$. The inequality $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$ turns into the lines $y = -\frac{\pi}{4}$ and $y = \frac{\pi}{4}$.
For $x = 1$,

$$\begin{aligned} f(1,y) &= \left(4(1)-(1)^2\right)\cos y \\ &= 3\cos y \\ f'(1,y) &= -3\sin y \\ f'(1,y) &= -3\sin y = 0 \implies y = 0 \\ &\text{critical point at } (1,0) \end{aligned}$$

For $x = 3$,

$$\begin{aligned} f(3,y) &= \left(4(3)-(3)^2\right)\cos y \\ &= 9\cos y \\ f'(3,y) &= -9\sin y \\ f'(3,y) &= -9\sin y = 0 \implies y = 0 \\ &\text{critical point at } (3,0) \end{aligned}$$

For $y = -\frac{\pi}{4}$,

$$\begin{aligned} f\left(x, -\frac{\pi}{4}\right) &= (4x-x^2)\cos\left(-\frac{\pi}{4}\right) \\ &= \frac{1}{\sqrt{2}}(4x-x^2) \\ f'\left(x, -\frac{\pi}{4}\right) &= \frac{1}{\sqrt{2}}(4-2x) \\ f'\left(x, -\frac{\pi}{4}\right) &= \frac{1}{\sqrt{2}}(4-2x) = 0 \implies x = 2 \\ &\text{critical point at } \left(2, -\frac{\pi}{4}\right) \end{aligned}$$

For $y = \frac{\pi}{4}$,

$$\begin{aligned} f\left(x, \frac{\pi}{4}\right) &= (4x-x^2)\cos\left(\frac{\pi}{4}\right) \\ &= \frac{1}{\sqrt{2}}(4x-x^2) \\ f'\left(x, \frac{\pi}{4}\right) &= \frac{1}{\sqrt{2}}(4-2x) \\ f'\left(x, \frac{\pi}{4}\right) &= \frac{1}{\sqrt{2}}(4-2x) = 0 \implies x = 2 \\ &\text{critical point at } \left(2, \frac{\pi}{4}\right) \end{aligned}$$

Let's find our "corners". To do that, let's graph the function.
graphing irrational numbers is too confusing sorry! use desmos instead
Based on the graph, our "corners" are $\left(1, -\frac{\pi}{4}\right), \left(1, \frac{\pi}{4}\right), \left(3, \frac{\pi}{4}\right)$, and $\left(3, -\frac{\pi}{4}\right)$.
Now, let's plug-n-chug all the critical points we found. The lowest value of $f(x, y)$ we get will be our absolute minimum value on the domain and the highest value of $f(x, y)$ we get will be out absolute maximum value on the domain.

$$\begin{aligned}
f(2,0) &= \left(4\,(2) - (2)^2\right)\cos(0) = 4 \\
f(1,0) &= \left(4\,(1) - (1)^2\right)\cos(0) = 3 \\
f(3,0) &= \left(4\,(3) - (3)^2\right)\cos(0) = 3 \\
f\left(2, -\frac{\pi}{4}\right) &= \left(4\,(2) - (2)^2\right)\cos\left(-\frac{\pi}{4}\right) = 2\sqrt{2} \\
f\left(2, \frac{\pi}{4}\right) &= \left(4\,(2) - (2)^2\right)\cos\left(\frac{\pi}{4}\right) = 2\sqrt{2} \\
f\left(1, -\frac{\pi}{4}\right) &= \left(4\,(1) - (1)^2\right)\cos\left(-\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2} \\
f\left(1, \frac{\pi}{4}\right) &= \left(4\,(1) - (1)^2\right)\cos\left(\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2} \\
f\left(3, \frac{\pi}{4}\right) &= \left(4\,(3) - (3)^2\right)\cos\left(\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2} \\
f\left(3, -\frac{\pi}{4}\right) &= \left(4\,(3) - (3)^2\right)\cos\left(-\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}
\end{aligned}$$

Our final answer is

$$\boxed{\begin{aligned}
&\text{absolute minimum: } f\left(1, -\frac{\pi}{4}\right) = f\left(1, \frac{\pi}{4}\right) = f\left(3, \frac{\pi}{4}\right) = f\left(3, -\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2} \\
&\text{absolute maximum: } f(2,0) = 4
\end{aligned}}$$

(d) $f(x, y) = 4x - 8xy + 2y + 1$ on the triangle bounded by $x = 0, y = 0, x + y = 1$
Solution
To find our interior critical point(s), we need to find out at what point(s) does both f_x and f_y equal 0 or are undefined. Therefore,

$$\begin{aligned}
f_x &= 4 - 8y = 0 \implies y = \frac{1}{2} \\
f_y &= -8x + 2 \qquad = 0 \implies x = \frac{1}{4} \\
&\text{critical point at } \left(\frac{1}{4}, \frac{1}{2}\right)
\end{aligned}$$

To find our boundary points, let's "plug-in" our bounding lines.
For $x = 0$,

$$\begin{aligned}
f(0, y) &= 4(0) - 8(0)y + 2y + 1 \\
&= 2y + 1 \\
f'(0, y) &= 2 \\
f'(0, y) &= 2 \neq 0 \\
&\text{no critical points}
\end{aligned}$$

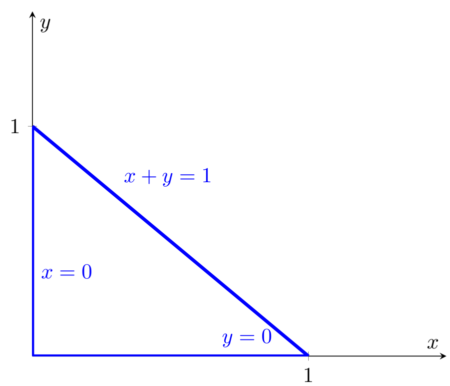
For $y = 0$,

$$\begin{aligned} f(x,0) &= 4x - 8x(0) + 2(0) + 1 \\ &= 4x + 1 \\ f'(x,0) &= 4 \\ f'(x,0) &= 4 \neq 0 \\ &\text{no critical points} \end{aligned}$$

For $x + y = 1$,

$$\begin{aligned} f(x,1-x) &= 4x - 8x(1-x) + 2(1-x) + 1 \\ &= 8x^2 - 6x + 3 \\ f'(x,1-x) &= 16x - 6 \\ f'(x,1-x) &= 16x - 6 = 0 \implies x = \frac{3}{8} \implies y = 1 - \frac{3}{8} = \frac{5}{8} \\ &\text{critical point at } \left(\frac{3}{8}, \frac{5}{8}\right) \end{aligned}$$

Let's find our "corners". To do that, let's graph the function. Based on the graph, our "corners" are



$(0,0), (1,0)$, and $(0,1)$.
Now, let's plug-n-chug all the critical points we found. The lowest value of $f(x,y)$ we get will be our absolute minimum value on the domain and the highest value of $f(x,y)$ we get will be out absolute maximum value on the domain.

$$\begin{aligned} f\left(\frac{1}{4}, \frac{1}{2}\right) &= 4\left(\frac{1}{4}\right) - 8\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right) + 1 = 2 \\ f\left(\frac{3}{8}, \frac{5}{8}\right) &= 4\left(\frac{3}{8}\right) - 8\left(\frac{3}{8}\right)\left(\frac{5}{8}\right) + 2\left(\frac{5}{8}\right) + 1 = \frac{15}{8} \\ f(0,0) &= 4(0) - 8(0)(0) + 2(0) + 1 = 1 \\ f(1,0) &= 4(1) - 8(1)(0) + 2(0) + 1 = 5 \\ f(0,1) &= 4(0) - 8(0)(1) + 2(1) + 1 = 3 \end{aligned}$$

Our final answer is

$$\boxed{\begin{array}{c} \text{absolute minimum: } f(0,0) = 1 \\ \text{absolute maximum: } f(1,0) = 5 \end{array}}$$

(e) $f(x,y) = x^2 + 2y^2 - x$ on the circle $x^2 + y^2 \leq 1$
Solution

To find our interior critical point(s), we need to find out at what point(s) does both f_x and f_y equal 0 or are undefined. Therefore,

$$f_x = 2x - 1 = 0 \implies x = \frac{1}{2}$$
$$f_y = 4y = 0 \implies y = 0$$
$$\text{critical point at } \left(\frac{1}{2}, 0\right)$$

To find our boundary points, let's "plug-in" our bounding line. For $x^2 + y^2 = 1 \implies y^2 = 1 - x^2$,

$$\begin{aligned} f(x, \pm\sqrt{1-x^2}) &= x^2 + 2(1-x^2) - x \\ &= -x^2 - x + 2 \\ f'(x, \pm\sqrt{1-x^2}) &= -2x - 1 \end{aligned}$$
$$f'(x, \pm\sqrt{1-x^2}) = -2x - 1 = 0 \implies x = -\frac{1}{2} \implies y = \pm\sqrt{1 - \left(-\frac{1}{2}\right)^2} = \pm\frac{\sqrt{3}}{2}$$
$$\text{critical point at } \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

Circles have no corners B)
Now, let's plug-n-chug all the critical points we found. The lowest value of $f(x, y)$ we get will be our absolute minimum value on the domain and the highest value of $f(x, y)$ we get will be out absolute maximum value on the domain.

$$f\left(\frac{1}{2}, 0\right) = \left(\frac{1}{2}\right)^2 + 2(0)^2 - \frac{1}{2} = -\frac{1}{4}$$
$$f\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \left(-\frac{1}{2}\right)^2 + 2\left(\frac{\sqrt{3}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{9}{4}$$
$$f\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) = \left(-\frac{1}{2}\right)^2 + 2\left(-\frac{\sqrt{3}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{9}{4}$$

Our final answer is

$$\boxed{\begin{aligned} &\text{absolute minimum: } f\left(\frac{1}{2}, 0\right) = -\frac{1}{4} \\ &\text{absolute maximum: } f\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) = f\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \frac{9}{4} \end{aligned}}$$

Problem 2

Find numbers a and b with $a \leq b$ that maximize the integral $\displaystyle\int_a^b (24 - 2x - x^2)^{1/3}\,dx$

Solution

For no particular reason, let's have a function $f(a, b) = \displaystyle\int_a^b (24 - 2x - x^2)^{1/3}\,dx$. From the problem, we know that one of our bounding lines is $a = b$. The domain of a is pretty much $(-\infty, b]$ and the domain of b is pretty much $[a, \infty)$.
To find our interior critical point(s), we need to find out at what point(s) does both f_a and f_b equal 0 or are undefined. Therefore,

$$f_a = -(24 - 2a - a^2)^{\frac{1}{3}} = 0 \implies -\big((4-a)(6+a)\big)^{\frac{1}{3}} = 0 \implies a = 4, -6$$
$$f_b = (24 - 2b - b^2)^{\frac{1}{3}} = 0 \implies \big((4-b)(6+b)\big)^{\frac{1}{3}} = 0 \implies b = 4, -6$$

Reviewing MAT 21A's FTC notes might help.
Remember: $a \leq b$, so the only pairs that work out are $a = 4, b = 4$ and $a = -6, b = 4$.
Let's test out both points

$$f(4,4) = \int_4^4 (24 - 2x - x^2)^{1/3}\,dx = 0$$
$$f(-6,4) = \int_{-6}^4 (24 - 2x - x^2)^{1/3}\,dx = (+)$$

There's no need to *actually* evaluate the integral because if you actually graph $y = (24 - 2x - x^2)^{1/3}$, you see that $y \geq 0$ on $x \in [-6, 4]$. Therefore, $f(-6,4) = \int_{-6}^4 (24 - 2x - x^2)^{1/3}\,dx$ must be a positive value (we're taking area under the curve above x-axis).
Our final answer is

$$\boxed{\begin{aligned} a &= -6 \\ b &= 4 \end{aligned}}$$

Problem 3
Find three numbers whose sum is 9 and whose sum of squares is a minimum.
Solution
Let's write these equations out. Let's call the function we're trying to minimize $S(x, y, z)$.

$$x + y + z = 9$$
$$S(x, y, z) = x^2 + y^2 + z^2$$

Three variables is a little tricky to work with, so let's solve for z.

$$z = 9 - x - y$$
$$\implies S(x, y) = x^2 + y^2 + (9 - x - y)^2$$

I think it's easier to proceed in this problem without figuring out the domain stuff. So that's what we're going to do.
To find our interior critical point(s), we need to find out at what point(s) does both S_x and S_y equal 0 or are undefined. Therefore,

$$\begin{aligned} S_x = 2x - 2(9 - x - y) &= 0 \\ S_y = 2y - 2(9 - x - y) &= 0 \\ 2x - 2y &= 0 & \text{(Subtract eqns)} \\ x &= y \\ 2x - 2(9 - x - x) &= 0 \implies 6x = 18 \implies x = 3 \implies y = 3 \\ \text{critical point at } &(3, 3) \end{aligned}$$

We're not done yet!
We need to show that at $(3, 3)$, a minimum actually occurs. Let's find $D(3, 3)$.

$$S_{xx} = 2 + 2 \implies S_{xx}(3,3) = 4$$
$$S_{yy} = 2 + 2 \implies S_{yy}(3,3) = 4$$
$$S_{xy} = 2 \implies S_{xy}(3,3) = 2$$
$$D(3,3) = (4)(4) - (2)^2 = 12$$

Since $D(3, 3) = 12 > 0$ and $S_{xx}(3, 3) = 4 > 0$, a relative minimum occurs at $(3, 3)$.
Great. Now we just need to find out third number. Remember: $x + y + z = 9$. Therefore,

$$z = 9 - x - y$$
$$z(3,3) = 9 - 3 - 3 = 3$$

Our final answer is

$$\boxed{\begin{aligned} x &= 3 \\ y &= 3 \\ z &= 3 \end{aligned}}$$

Problem 4
Find the dimensions of the rectangle box of maximum volume that can be inscribed inside the sphere $x^2 + y^2 + z^2 = 4$.
Solution
Let's write these equations out. Let's call the the function we're trying to maximize $V(x, y, z)$.

$$\begin{aligned} x^2 + y^2 + z^2 &= 4 \\ V(x, y, z) &= (2x)(2y)(2z) = 8xyz \end{aligned}$$

Note the $2x, 2y$, and $2z$. If you actually draw a box inscribed inside a sphere, you'll notice that the actual length is 2 times the x-coordinate, the actual width is 2 times the y-coordinate, and the actual height is 2 times the z-coordinate.
Three variables is a little tricky to work with, so let's solve for z.

$$\begin{aligned} z^2 &= 4 - x^2 - y^2 \\ \implies z &= \sqrt{4 - x^2 - y^2} \\ \implies V(x, y) &= 8xy\sqrt{4 - x^2 - y^2} \end{aligned}$$

Let's establish some domain restrictions. Clearly, $x, y \geq 0$. Figuring out the upper bound for x and y is a little bit more tricky but doable.
Remember that we cannot square root a negative number. Therefore,

$$\begin{aligned} 4 - x^2 - y^2 &\geq 0 \\ \implies x^2 + y^2 &\leq 4 \\ \implies 0 \leq x^2 + y^2 &\leq 4 && ((+)^2 + (+)^2 \geq 0) \end{aligned}$$

To find our interior critical point(s), we need to find out at what point(s) does both V_x and V_y equal 0 or are undefined. Therefore,

$$\begin{aligned} V_x &= \frac{32y - 16x^2y - 8y^3}{\sqrt{4 - x^2 - y^2}} = 0 \\ V_y &= \frac{32x - 16xy^2 - 8x^3}{\sqrt{4 - x^2 - y^2}} = 0 \\ &\implies y = x && \text{(look at the two eqns)} \\ 32y - 16y^3 - 8y^3 = 0 &\implies 32y - 24y^2 = 0 \implies 8y(4 - 3y^2) = 0 \\ \implies y = 0, \pm\frac{2}{\sqrt{3}} &\implies x = 0, \pm\frac{2}{\sqrt{3}} \\ &\text{critical point at } (0, 0), \left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \end{aligned}$$

Note 1: I did these derivatives by hand because typing it out is too much work. Please PM me if you want to see the work for these.
Note 2: Remember that $x, y \geq 0$, so the only valid points are $(0, 0)$ and $\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$.
To find our boundary points, let's "plug-in" our bounding line.
For $x^2 + y^2 = 4 \implies y^2 = 4 - x^2, x^2 = 4 - y^2$,

$$\begin{aligned} V\left(x, \pm\sqrt{4 - x^2}\right) &= 8xy\sqrt{4 - x^2 - (4 - x^2)} \\ &= 0 \\ &\text{no critical points} \end{aligned}$$

Based on this work, any point on the bounding line results in a volume of 0, so there's no point in testing or finding the boundary points anymore.
Let's plug-n-chug all the critical points we found. The highest value of $V(x,y)$ we get will get us the x and y values of our answer.

$$V(0,0) = 8(0)(0)\sqrt{4-0^2-0^2} = 0$$
$$V\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) = 8\left(\frac{2}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right)\sqrt{4-\left(\frac{2}{\sqrt{3}}\right)^2-\left(\frac{2}{\sqrt{3}}\right)^2} = \frac{64}{3\sqrt{3}}$$

Clearly, the x and y value of our answer should be $\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$.
Solving for z, we get

$$z\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) = \sqrt{4-\left(\frac{2}{\sqrt{3}}\right)^2-\left(\frac{2}{\sqrt{3}}\right)^2} = \frac{2}{\sqrt{3}}$$

Our final answer is

$$\boxed{\begin{aligned} x &= \frac{2}{\sqrt{3}} \\ y &= \frac{2}{\sqrt{3}} \\ z &= \frac{2}{\sqrt{3}} \end{aligned}}$$

Problem 5
Among all closed rectangular boxes of volume 27 cm^3, what is the smallest possible surface area?
Solution
Let's write these equations out. Let's call the minimizing function $A(x,y,z)$.

$$xyz = 27$$
$$A(x,y,z) = 2xy + 2xz + 2yz$$

Three variables is a little tricky to solve for, so let's solve for z.

$$\begin{aligned} z &= \frac{27}{xy} \\ \implies A(x,y) &= 2xy + 2x\left(\frac{27}{xy}\right) + 2y\left(\frac{27}{xy}\right) \\ &= 2xy + \frac{54}{y} + \frac{54}{x} \end{aligned}$$

I think it's easier to proceed in this problem without figuring out the domain stuff. So that's what we're going to do.
To find our interior critical point(s), we need to find out at what point(s) does both A_x and A_y equal 0 or are undefined. Therefore,

$$\begin{aligned} A_x &= 2y - \frac{54}{x^2} = 0 \\ A_y &= 2x - \frac{54}{y^2} = 0 \\ \implies & y = x \\ 2x - \frac{54}{x^2} &= 0 \\ 2x &= \frac{54}{x^2} \\ x^3 &= 27 \implies x = 3 \implies y = 3 \end{aligned}$$

critical point at $(3,3)$

We're not done yet!
We need to show that at $(3,3)$, a minimum actually occurs. Let's find $D(3,3)$.

$$A_{xx} = \frac{108}{x^3} \implies A_{xx}(3,3) = 4$$
$$A_{yy} = \frac{108}{y^3} \implies A_{yy}(3,3) = 4$$
$$A_{xy} = 2 \implies A_{xy}(3,3) = 2$$
$$D(3,3) = (4)(4) - (2)^2 = 12$$

Since $D(3,3) = 12 > 0$ and $A_{xx}(3,3) = 4 > 0$, a relative minimum occurs at $(3,3)$.
Great. Now we just need to find $A(3,3)$.

$$A(x,y) = 2xy + \frac{54}{y} + \frac{54}{x}$$
$$\implies A(3,3) = 2(3)(3) + \frac{54}{3} + \frac{54}{3} = 54$$

Our final answer is

$$\boxed{54 \text{ cm}^2}$$