Series Notes

MAT 21C
Winter 2022
Christina Phan

1 Strategy For Determining Series Convergence

Follow the following steps to determine the convergence of a series.

1. Check if $\lim_{n\to\infty} a_n \neq 0$. If so, use **Nth Term Test For Divergence**.
2. Check if the series is a p-series ($\sum \frac{1}{n^p}$). If so, use **P-series Test**.
3. Check if the series is a geometric series ($\sum ar^n$). If so, use **Geometric Test**.
4. Check if the series is similar to a p-series or geometric series. If so, use **Comparison Test**.
5. Check if the series can be written in the form $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$. If so, use **Alternating Series Test**.
6. Check if the series contains factorials or expressions raised to the nth power. If so, use either **Ratio Test** or **Root Test**. Use intuition to determine which is more appropriate.
7. Check if $a_n = f(x)$ is positive, decreasing, and easy to integrate. If so, use **Integral Test**.
8. As a last resort, use either **Comparison Test** or **Limit Comparison Test**. Note that $\sum a_n$ must contain only positive terms.

1.1 Absolute Convergence

For absolute convergence, recall that:

1. If $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges absolutely.
2. If $\displaystyle\sum_{n=1}^{\infty} |a_n|$ diverges but $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges conditionally.
3. If $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

2 Series Convergence Test Overview

2.1 Nth Term Test For Divergence

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

Note that if $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ may or may not converge. This test *only* proves divergence.
Example

$$\sum_{n=1}^{\infty} n^2$$

Solution
By Nth Term Test For Divergence, $\sum_{n=1}^{\infty} n^2$ diverges since $\lim_{n\to\infty} n^2 = \infty \neq 0$.

2.2 P-series Test

Given the series $\sum \frac{1}{n^p}$.

1. If $p > 1$, then $\sum \frac{1}{n^p}$ converges.
2. If $p \leq 1$, then $\sum \frac{1}{n^p}$ diverges.

Example

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{999}{1000}}}$$

Solution
By P-series Test, $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{999}{1000}}}$ diverges since $p = \frac{999}{1000} \leq 1$.

2.3 Geometric Test

Given the series $\sum a(r)^n$.

1. If $|r| < 1$, then $\sum a(r)^n$ converges.
2. If $|r| \geq 1$, then $\sum a(r)^n$ diverges.

The sum of a geometric series is $S = \frac{a}{1-r}$.
Example

$$\sum_{n=1}^{\infty} 5^{n+3} 4^n$$

Solution

$$\sum_{n=1}^{\infty} 5^{n+3} 4^n = \sum_{n=1}^{\infty} 5^3 5^n 4^n = \sum_{n=1}^{\infty} 5^3 (20)^n$$

By Geometric Test, $\displaystyle\sum_{n=1}^{\infty} 5^{n+3} 4^n$ diverges since $|r| = 20 \geq 1$.

2.4 Comparison Test

Let $\sum a_n$ and $\sum b_n$ be series with positive terms such that $a_n \leq b_n$ for all n.

1. If $\sum b_n$ converges, then $\sum a_n$ converges.
2. If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Example

$$\sum_{n=4}^{\infty} \frac{1}{(2n-1)(n-3)}$$

Solution
Let $\displaystyle\sum b_n = \sum_{n=4}^{\infty} \frac{1}{2n^2}$.
$\displaystyle 0 < \frac{1}{(2n-1)(n-3)} < \frac{1}{2n^2}$ for all n.
By P-series Test, $\displaystyle\sum_{n=4}^{\infty} \frac{1}{2n^2}$ converges since $p = 2 > 1$.
By Comparison Test, $\displaystyle\sum_{n=4}^{\infty} \frac{1}{(2n-1)(n-3)}$ converges.

2.5 Alternating Series Test

If the series fulfills all of these conditions:

1. Is an alternating series
2. $\lim_{n\to\infty} b_n = 0$
3. $\{b_n\}$ is a decreasing sequence

then the alternating series converges.

Example

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{7+2n}$$

Solution

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{7+2n}$ is an alternating series where $\{b_n\} = \frac{1}{7+2n}$.
2. $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{7+2n} = 0$
3. $\displaystyle\frac{1}{7+2(n+1)} < \frac{1}{7+2n}$, so $\{b_n\} = \frac{1}{7+2n}$ is a decreasing sequence.

By Alternating Series Test, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{7+2n}$ converges.

2.6 Ratio Test

Given the series $\sum a_n$ where $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = p$.

1. If $p < 1$, then $\sum a_n$ converges absolutely.
2. If $p > 1$, then $\sum a_n$ diverges.
3. If $p = 1$, then no conclusion is met.

Example

$$\sum_{n=0}^{\infty} \frac{(2n)!}{5n+1}$$

Solution

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(2\,(n+1))!}{5\,(n+1)+1}\,\frac{5n+1}{(2n)!}\right| \\
&= \lim_{n\to\infty}\left|\frac{(2n+2)!}{5n+6}\,\frac{5n+1}{(2n)!}\right| \\
&= \lim_{n\to\infty}\left|\frac{(2n+2)\,(2n+1)\,(2n)!}{5n+6}\,\frac{5n+1}{(2n)!}\right| \\
&= \lim_{n\to\infty}\left|\frac{(2n+2)\,(2n+1)\,(5n+1)}{5n+6}\right| \\
&= \infty > 1
\end{aligned}$$

By Ratio Test, $\displaystyle\sum_{n=0}^{\infty}\frac{(2n)!}{5n+1}$ diverges since $\displaystyle\lim_{n\to\infty}\left|\frac{(2\,(n+1))!}{5\,(n+1)+1}\,\frac{5n+1}{(2n)!}\right| = \infty > 1$.

2.7 Root Test

Given the series $\sum a_n$ where $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|} = p$.

1. If $p < 1$, then $\sum a_n$ converges absolutely.
2. If $p > 1$, then $\sum a_n$ diverges.
3. If $p = 1$, then no conclusion is met.

Example

$$\sum_{n=1}^{\infty}\left(\frac{3n+1}{4-2n}\right)^{2n}$$

Solution

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left|\left(\frac{3n+1}{4-2n}\right)^{2n}\right|^{\frac{1}{n}} = \lim_{n\to\infty}\left|\left(\frac{3n+1}{4-2n}\right)^{2}\right| = \left(-\frac{3}{2}\right)^{2} = \frac{9}{4} > 1$$

By Root Test, $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3n+1}{4-2n}\right)^{2n}$ diverges since $\displaystyle\lim_{n\to\infty}\left|\left(\tfrac{3n+1}{4-2n}\right)^{2n}\right|^{\frac{1}{n}} = \tfrac{9}{4} > 1$.

2.8 Integral Test

Let $\sum a_n$ be a series with positive terms where $a_n = f(x)$ is decreasing and continuous for $x \geq 1$.

1. If $\displaystyle\int_1^\infty f(x)\,dx$ converges, then $\sum a_n$ converges.
2. If $\displaystyle\int_1^\infty f(x)\,dx$ diverges, then $\sum a_n$ diverges.

Example

$$\sum_{n=1}^{\infty} \frac{9}{n^2+5n+4}$$

Solution
I used partial fractions for this problem. To save space, I omitted my partial fractions work from here.
Let $f(x) = \frac{9}{x^2+5x+4} = \frac{3}{x+1} - \frac{3}{x+4}$. $f(x)$ is positive and continuous for $x \geq 1$.
$\frac{9}{x^2+5x+4} > \frac{9}{(x+1)^2+5(x+1)+4}$ for all $x \geq 1$, so $f(x)$ is decreasing for $x \geq 1$.

$$\begin{aligned}
\int_1^\infty \frac{3}{x+1} - \frac{3}{x+4}\,dx &= \lim_{t\to\infty} \Big[3\ln|x+1| - 3\ln|x+4|\Big]_1^t \\
&= \lim_{t\to\infty} \Big(3\ln|t+1| - 3\ln|t+4|\Big) - \Big(3\ln|1+1| - 3\ln|1+4|\Big) \\
&= \lim_{t\to\infty} \left(3\ln\frac{t+1}{t+4} - \ln\frac{8}{125}\right) \\
&= 3\ln\left(\lim_{t\to\infty}\frac{t+1}{t+4}\right) - \ln\frac{8}{125} \\
&= 3\ln\left(\lim_{t\to\infty}\frac{1+\frac{1}{t}}{1+\frac{4}{t}}\right) - \ln\frac{8}{125} \\
&= 3\ln(1) - \ln\frac{8}{125} \\
&= -\ln\frac{8}{125}
\end{aligned}$$

By Integral Test, $\displaystyle\sum_{n=1}^{\infty}\frac{9}{n^2+5n+4}$ converges since $\displaystyle\int_1^\infty \frac{3}{x+1} - \frac{3}{x+4}\,dx$ converges.

2.9 Limit Comparison Test

Let $\sum a_n$ and $\sum b_n$ be series with positive terms where $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = p$.

1. If p is positive and finite, then either both $\sum a_n$ and $\sum b_n$ converge or both $\sum a_n$ and $\sum b_n$ diverge.
2. If $p = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $p = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Example 1

$$\sum_{n=3}^{\infty} \frac{n-1}{\sqrt{n^3+n+3}}$$

Solution 1

Let $\displaystyle\sum b_n = \sum_{n=3}^{\infty} \frac{n}{\sqrt{n^3}} = \sum_{n=3}^{\infty} \frac{1}{n^{\frac{1}{2}}}$.

$0 < \dfrac{n-1}{\sqrt{n^3+n+3}}, \dfrac{1}{n^{\frac{1}{2}}}$ for all n.

$$\lim_{n\to\infty} \frac{\frac{n-1}{\sqrt{n^3+n+3}}}{\frac{1}{n^{\frac{1}{2}}}} = \lim_{n\to\infty} \frac{n^{\frac{3}{2}} + n^{\frac{1}{2}}}{\sqrt{n^3+n+3}} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{\left(1+\frac{1}{n^2}+\frac{3}{n^3}\right)^{\frac{1}{2}}} = \frac{1+0}{\left(1+0+0\right)^{\frac{1}{2}}} = 1$$

By P-series Test, $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ diverges since $p = \dfrac{1}{2} \leq 1$.

By Limit Comparison Test, $\displaystyle\sum_{n=3}^{\infty} \frac{n-1}{\sqrt{n^3+n+3}}$ diverges since $\displaystyle\lim_{n\to\infty} \frac{\frac{n-1}{\sqrt{n^3+n+3}}}{\frac{1}{n^{\frac{1}{2}}}} = 1$ is a positive, finite value.

Example 2

$$\sum_{n=2}^{\infty} \frac{n^2 \ln n}{n^3+5}$$

Solution 2

Let $\displaystyle\sum b_n = \sum_{n=2}^{\infty} \frac{1}{n}$.

$0 < \dfrac{n^2 \ln n}{n^3+5}, \dfrac{1}{n}$ for all n.

$$\lim_{n\to\infty} \frac{\frac{n^2 \ln n}{n^3+5}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3 \ln n}{n^3+5} = \lim_{n\to\infty} \frac{\ln n}{1+\frac{5}{n^3}} = \infty$$

By P-series Test, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ diverges since $p = 1 \leq 1$.

By Limit Comparison Test, $\displaystyle\sum_{n=2}^{\infty} \frac{n^2 \ln n}{n^3+5}$ diverges since $\displaystyle\lim_{n\to\infty} \frac{\frac{n^2 \ln n}{n^3+5}}{\frac{1}{n}} = \infty$ and $\displaystyle\sum_{n=2}^{\infty} \frac{n^2 \ln n}{n^3+5}$ diverges.

Example 3

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^4}$$

Solution 3

Let $\sum b_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$.

$0 < \frac{\ln n}{n^4}, \frac{1}{n^3}$ for all n.

$$\lim_{n\to\infty} \frac{\frac{\ln n}{n^4}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{\ln n}{n} \overset{\text{L.H.}}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \frac{0}{1} = 0$$

By P-series Test, $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges since $p = 3 > 1$.

By Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{\ln n}{n^4}$ converges since $\lim_{n\to\infty} \frac{\frac{\ln n}{n^4}}{\frac{1}{n^3}} = 0$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

Test Name	Statement	Comments
Nth Term Test For Divergence	If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.	If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ may or may not converge.
P-series Test	a) $p > 1$ series converge b) $p \leq 1$ series diverge where $\sum \frac{1}{n^p}$.	Harmonic is special P-series.
Geometric Test	a) $\|r\| < 1$ series converge b) $\|r\| \geq 1$ series diverge where $\sum a(r)^n$.	Sum of a geometric series is $S = \frac{a}{1-r}$.
Comparison Test	Let $\sum a_n$ and $\sum b_n$ be series with positive terms such that $a_n \leq b_n$ for all n. a) If $\sum b_n$ converges, then $\sum a_n$ converges. b) If $\sum a_n$ diverges, then $\sum b_n$ diverges.	Use this test as a last resort.
Alternating Series Test	If the alternating series fulfills the following conditions: 1. $\lim_{n\to\infty} b_n = 0$ 2. $\{b_n\}$ is a decreasing sequence then the alternating series converges.	This test only applies to alternating series.
Ratio Test	Given the series $\sum a_n$ where $\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| = p$. a) If $p < 1$, then $\sum a_n$ converges absolutely. b) If $p > 1$, then $\sum a_n$ diverges. c) If $p = 1$, then no conclusion is met.	Use this test when a_n contains factorials or nth powers.
Root Test	Given the series $\sum a_n$ where $\lim_{n\to\infty} \sqrt[n]{\|a_n\|} = p$. a) If $p < 1$, then $\sum a_n$ converges absolutely. b) If $p > 1$, then $\sum a_n$ diverges. c) If $p = 1$, then no conclusion is met.	Use this test when a_n contains nth powers.
Integral Test	Let $\sum a_n$ be a series with positive terms where $a_n = f(x)$ is decreasing and continuous for $x \geq 1$. a) If $\int_1^\infty f(x)\,dx$ converges, then $\sum a_n$ converges. b) If $\int_1^\infty f(x)\,dx$ diverges, then $\sum a_n$ diverges.	Use this test when the integral is easy to integrate and the series fulfills all conditions.
Limit Comparison Test	Let $\sum a_n$ and $\sum b_n$ be series with positive terms where $\lim_{n\to\infty} \frac{a_n}{b_n} = p$. a) If p is positive and finite, then either both series converge or both series diverge. b) If $p = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges. c) If $p = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.	Use this test as a last resort. This test is marginally easier to use than Comparison Test, but still requires some skill in choosing $\sum b_n$.