Problem 1
Given $f(x,y,z) = 2z^3 - 3(x^2+y^2)z + \tan^{-1} z$, find ∇f at the point $(1,1,1)$.
Solution

$$\begin{aligned}
\nabla f &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \\
&= \left\langle -6xz, -6yz, 6z^2 - 3(x^2+y^2) + \frac{1}{1+z^2} \right\rangle \\
\nabla f(1,1,1) &= \left\langle -6(1)(1), -6(1)(1), 6(1)^2 - 3(1^2+1^2) + \frac{1}{1+(1)^2} \right\rangle \\
&\boxed{= \left\langle -6, -6, \frac{1}{2} \right\rangle}
\end{aligned}$$

Problem 2 (Parts)
Find the derivative of the function at the point in the direction of $\mathbf{u}$:
Note
For the actual direction vector, I'll be notating it was $\mathbf{v}$.
(a) $f(x,y) = 2xy - 3y^2$ at $(5,5)$, $\mathbf{u} = \langle 4, 3 \rangle$
Solution
First, let's make our actual direction (unit) vector.

$$\mathbf{v} = \frac{1}{\sqrt{4^2+3^2}}\langle 4,3\rangle = \frac{1}{5}\langle 4,3\rangle$$

Now, let's set up our directional derivative and then plug in the point $(5,5)$.

$$\begin{aligned}
D_{\mathbf{v}} f &= \nabla f \cdot \mathbf{v} \\
&= \langle 2y, 2x - 6y \rangle \cdot \frac{1}{5}\langle 4,3\rangle \\
D_{\mathbf{v}} f(5,5) &= \langle 2(5), 2(5) - 6(5) \rangle \cdot \frac{1}{5}\langle 4,3\rangle \\
&= 10\left(\frac{4}{5}\right) - 20\left(\frac{3}{5}\right) \\
&\boxed{= -4}
\end{aligned}$$

(b) $f(x,y) = \dfrac{x-y}{xy+2}$ at $(1,-1)$, $\mathbf{u} = \langle 12, 5 \rangle$
Solution
First, let's make our actual direction (unit) vector.

$$\mathbf{v} = \frac{1}{\sqrt{12^2+5^2}}\langle 12,5\rangle = \frac{1}{13}\langle 12,5\rangle$$

Now, let's set up our directional derivative and then plug in the point $(1,-1)$.

$$\begin{aligned}
D_{\mathbf{v}}f &= \nabla f \cdot \mathbf{v} \\
&= \left\langle \frac{(xy+2)-y(x-y)}{(xy+2)^2}, \frac{-(xy+2)-x(x-y)}{(xy+2)^2} \right\rangle \cdot \frac{1}{13}\langle 12,5\rangle \\
&= \left\langle \frac{2+y^2}{(xy+2)^2}, \frac{-2-x^2}{(xy+2)^2} \right\rangle \cdot \frac{1}{13}\langle 12,5\rangle \\
D_{\mathbf{v}}f(1,-1) &= \left\langle \frac{2+(-1)^2}{\left((1)(-1)+2\right)^2}, \frac{-2-(1)^2}{\left((1)(-1)+2\right)^2} \right\rangle \cdot \frac{1}{13}\langle 12,5\rangle \\
&= \frac{36}{13} - \frac{15}{13} \\
&\boxed{= \frac{21}{13}}
\end{aligned}$$

(c) $f(x,y,z) = x^2 + 2y^2 - 3x^2$ at $(1,1,1)$, $\mathbf{u} = \langle 1,1,1\rangle$

Solution

First, let's make our actual direction (unit) vector.

$$\mathbf{v} = \frac{1}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}\langle 1,1,1\rangle = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

Now, let's set up our directional derivative and then plug in the point $(1,1,1)$.

$$\begin{aligned}
D_{\mathbf{v}}f &= \nabla f \cdot \mathbf{v} \\
&= \langle 2x, 4y, -6z\rangle \cdot \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle \\
D_{\mathbf{v}}f(1,1,1) &= \langle 2(1), 4(1), -6(1)\rangle \cdot \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle \\
&= 2\left(\frac{1}{\sqrt{3}}\right) + 4\left(\frac{1}{\sqrt{3}}\right) + (-6)\left(\frac{1}{\sqrt{3}}\right) \\
&\boxed{= 0}
\end{aligned}$$

(d) $f(x,y,z) = \cos xy + e^{yz} + \ln zx$ at $\left(1, 0, \frac{1}{2}\right)$, $\mathbf{u} = \langle 1,2,2\rangle$

Solution

First, let's make our actual direction (unit) vector.

$$\mathbf{v} = \frac{1}{\sqrt{1^2+2^2+2^2}}\langle 1,2,2\rangle = \frac{1}{3}\langle 1,2,2\rangle = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$$

Now, let's set up our directional derivative and then plug in the point $\left(1, 0, \frac{1}{2}\right)$.

$$\begin{aligned}
D_{\mathbf{v}}f &= \nabla f \cdot \mathbf{v} \\
&= \left\langle -y\sin xy + \frac{1}{x}, -y\sin yx + ze^{yz}, ye^{yz} + \frac{1}{z} \right\rangle \cdot \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle \\
&= \left\langle -0\sin 1(0) + \frac{1}{1}, -0\sin 0(1) + \frac{1}{2}e^{0\left(\frac{1}{2}\right)}, 0e^{0\left(\frac{1}{2}\right)} + \frac{1}{\frac{1}{2}} \right\rangle \cdot \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle \\
&= \left\langle 1, \frac{1}{2}, 2 \right\rangle \cdot \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle \\
&= 1\left(\frac{1}{3}\right) + \frac{1}{2}\left(\frac{2}{3}\right) + 2\left(\frac{2}{3}\right) \\
&\boxed{= 2}
\end{aligned}$$

Problem 3 (Parts)
Find the directions in which the function increases and decreases most rapidly at the point. Then find the derivatives of the function in those directions:
Note
There are two ways to solve this problem. In (a), I show the literal dot product computation. In (b) and (c), I show the nifty trick that the function increase and decreases most rapidly at $\pm\|\nabla f\|$. Both methods are fine. Though, the latter method is nicer imo
The proof for both methods is in lecture notes 2/28.
(a) $f(x, y) = x^2y + e^{xy}\sin y$ at $(1, 0)$
Solution
First, let's figure out what our ∇f is.

$$\begin{aligned}\nabla f &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \\ &= \langle 2xy + ye^{xy}\sin y, x^2 + xe^{xy}\cos y\rangle \\ \nabla f(1, 0) &= \langle 2(1)(0) + (0)e^{1(0)}\sin(0), 1^2 + (1)e^{1(0)}\cos(0)\rangle \\ &= \langle 0, 2\rangle\end{aligned}$$

Let's make this a unit vector.

$$\mathbf{u} = \frac{1}{\sqrt{0^2 + 2^2}}\langle 0, 2\rangle = \frac{1}{2}\langle 0, 2\rangle = \langle 0, 1\rangle$$

Recall that the function increase most rapidly at the point in the direction of ∇f and decreases most rapidly at the point in the direction of $-\nabla f$.
Therefore, $f(x, y)$ increases most rapidly in the direction of $\mathbf{u} = \langle 0, 1\rangle$ and decreases most rapidly in the direction of $-\mathbf{u} = -\langle 0, 1\rangle$.
Let's find the derivative of the function in these directions.
For $\mathbf{u}$,

$$\begin{aligned}D_{\mathbf{u}}f(1, 0) &= \nabla f(1, 0) \cdot \mathbf{u} \\ &= \langle 0, 2\rangle \cdot \langle 0, 1\rangle \\ &= 0(0) + 2(1) \\ &= 2\end{aligned}$$

For $-\mathbf{u}$

$$\begin{aligned}D_{-\mathbf{u}}f(1, 0) &= \nabla f(1, 0) \cdot (-\mathbf{u}) \\ &= \langle 0, 2\rangle \cdot \langle 0, -1\rangle \\ &= 0(0) + (-2)(1) \\ &= -2\end{aligned}$$

(b) $f(x, y, z) = \frac{x}{y} - yz$ at $(4, 1, 1)$
Solution
First, let's figure out what our ∇f is.

$$\begin{aligned}\nabla f &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \\ &= \left\langle \frac{1}{y}, -\frac{x}{y^2} - z, -y \right\rangle \\ \nabla f(4, 1, 1) &= \left\langle \frac{1}{1}, -\frac{4}{1^2} - 1, -1 \right\rangle \\ &= \langle 1, -5, -1\rangle\end{aligned}$$

Let's make this a unit vector.

$$\mathbf{u} = \frac{1}{\sqrt{1^2 + (-5)^2 + (-1)^2}} \langle 1, -5, -1 \rangle = \frac{1}{\sqrt{27}} \langle 1, -5, -1 \rangle = \frac{1}{3\sqrt{3}} \langle 1, -5, -1 \rangle$$

Recall that the function increase most rapidly at the point in the direction of ∇f and decreases most rapidly at the point in the direction of $-\nabla f$.
Therefore, $f(x, y)$ increases most rapidly in the direction of $\mathbf{u} = \frac{1}{3\sqrt{3}} \langle 1, -5, -1 \rangle$ and decreases most rapidly in the direction of $-\mathbf{u} = -\frac{1}{3\sqrt{3}} \langle 1, -5, -1 \rangle$.
Let's find the derivative of the function in these directions.
For $\mathbf{u}$,

$$\begin{aligned} D_{\mathbf{u}} f(4, 1, 1) &= \nabla f(4, 1, 1) \cdot \mathbf{u} \\ &= \|\nabla f(4, 1, 1)\| \\ &= \sqrt{1^2 + (-5)^2 + (-1)^2} \\ &= \sqrt{27} \\ &= 3\sqrt{3} \end{aligned}$$

For $-\mathbf{u}$,

$$\begin{aligned} D_{-\mathbf{u}} f(4, 1, 1) &= \nabla f(4, 1, 1) \cdot (-\mathbf{u}) \\ &= -\|\nabla f(4, 1, 1)\| \\ &= -\sqrt{1^2 + (-5)^2 + (-1)^2} \\ &= -\sqrt{27} \\ &= -3\sqrt{3} \end{aligned}$$

(c) $f(x, y, z) = \ln(x^2 + y^2 - 1) + y + 6z$ at $(1, 1, 0)$
Solution
First, let's figure out what our ∇f is.

$$\begin{aligned} \nabla f &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \\ &= \left\langle \frac{2x}{x^2 + y^2 - 1}, \frac{2y}{x^2 + y^2 - 1} + 1, 6 \right\rangle \\ \nabla f(1, 1, 0) &= \left\langle \frac{2(1)}{1^2 + 1^2 - 1}, \frac{2(1)}{1^2 + 1^2 - 1} + 1, 6 \right\rangle \\ &= \langle 2, 3, 6 \rangle \end{aligned}$$

Let's make this a unit vector.

$$\mathbf{u} = \frac{1}{\sqrt{2^2 + 3^2 + 6^2}} \langle 2, 3, 6 \rangle = \frac{1}{\sqrt{49}} \langle 2, 2, 6 \rangle = \frac{1}{7} \langle 2, 2, 6 \rangle$$

Recall that the function increase most rapidly at the point in the direction of ∇f and decreases most rapidly at the point in the direction of $-\nabla f$.
Therefore, $f(x, y)$ increases most rapidly in the direction of $\mathbf{u} = \frac{1}{7} \langle 2, 2, 6 \rangle$ and decreases most rapidly in the direction of $-\mathbf{u} = -\frac{1}{7} \langle 2, 2, 6 \rangle$.
Let's find the derivative of the function in these directions.
For $\mathbf{u}$,

$$\begin{aligned} D_{\mathbf{u}} f(1, 1, 0) &= \nabla f(1, 1, 0) \cdot \mathbf{u} \\ &= \|\nabla f(1, 1, 0)\| \\ &= \sqrt{2^2 + 2^2 + 6^2} \\ &= \sqrt{49} \\ &= 7 \end{aligned}$$

For $-\mathbf{u}$,

$$\begin{aligned} D_{-\mathbf{u}}f(1,1,0) &= \nabla f(1,1,0)\cdot(-\mathbf{u}) \\ &= -\|\nabla f(1,1,0)\| \\ &= -\sqrt{2^2+2^2+6^2} \\ &= -\sqrt{49} \\ &= -7 \end{aligned}$$

Problem 4 (Parts)
Find an equation for the tangent line to the level curve $f(x,y)=c$ at the point and then sketch the curve together with the tangent line and the vector ∇f:
(a) $x^2+y^2=4$ at $(\sqrt{2},\sqrt{2})$
Solution
First, let's figure out what our ∇f is.

$$\begin{aligned} \nabla f &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \\ &= \langle 2x, 2y\rangle \\ \nabla f\left(\sqrt{2},\sqrt{2}\right) &= \left\langle 2\sqrt{2}, 2\sqrt{2}\right\rangle \end{aligned}$$

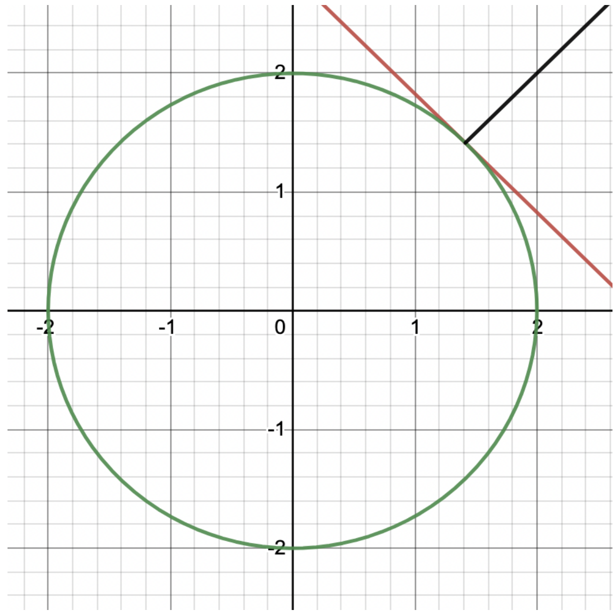
Recall that the equation for a tangent line to the level curve $f(x,y)=c$ is

$$\frac{\partial f}{\partial x}(x-x_0)+\frac{\partial f}{\partial y}(y-y_0)=0$$

Let's do some plug-n-chug :)

$$\begin{aligned} 2\sqrt{2}(x-\sqrt{2})+2\sqrt{2}(y-\sqrt{2}) &= 0 \\ 2\sqrt{2}x-4+2\sqrt{2}y-4 &= 0 \\ 2\sqrt{2}x+2\sqrt{2}y &= 8 \\ \sqrt{2}x+\sqrt{2}y &= 4 \end{aligned}$$

Below is the sketch of the curve together with the tangent line and vector ∇f.
The green curve is $x^2+y^2=4$, the red curve is the tangent line $\sqrt{2}x+\sqrt{2}y=4$, and the black "vector" is $\nabla f=\langle 2\sqrt{2}, 2\sqrt{2}\rangle$.



(b) $xy = -4$ at $(2, -2)$
Solution
First, let's figure out what our ∇f is.

$$\begin{aligned}\nabla f &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \\ &= \langle y, x \rangle \\ \nabla f(2,-2) &= \langle -2, 2 \rangle\end{aligned}$$

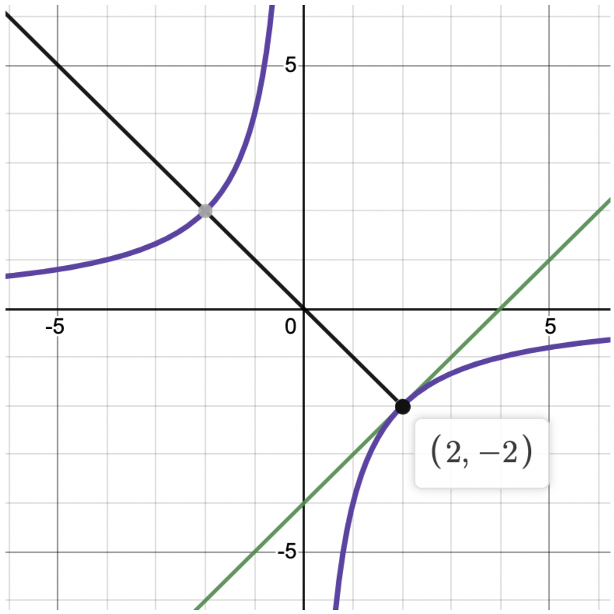
Recall that the equation for a tangent line to the level curve $f(x, y) = c$ is

$$\frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) = 0$$

Let's do some plug-n-chug :)

$$\begin{aligned}-2(x-2) + 2(y+2) &= 0 \\ -2x + 4 + 2y + 4 &= 0 \\ -2x + 2y &= -8 \\ -x + y &= -4\end{aligned}$$

Below is the sketch of the curve together with the tangent line and vector ∇f.
The green curve is $xy = -4$, the red curve is the tangent line $-x + y = -4$, and the black "vector" is $\nabla f = \langle -2, 2 \rangle$.



Problem 5 (Parts)
Given $f(x,y) = x^2 - xy + y^2 - y$, find the directions $\mathbf{u}$ and the values of $D_{\mathbf{u}}f(1,-1)$ for which:
(a) $D_{\mathbf{u}}f(1,-1)$ is largest
Solution
First, let's find ∇f.

$$\begin{aligned}\nabla f &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \\ &= \langle 2x - y, -x + 2y - 1\rangle \\ \nabla f(1,-1) &= \langle 2(1) - (-1), -1 + 2(-1) - 1\rangle \\ &= \langle 3, -4\rangle\end{aligned}$$

$D_{\mathbf{u}}f(1,-1)$ is largest when $D_{\mathbf{u}}f(1,-1) = \|\nabla f(1,-1)\|$. That is, $D_{\mathbf{u}}f(1,-1)$ is largest in the direction of $\frac{\nabla f}{\|\nabla f\|} = \frac{1}{\sqrt{3^2+(-4)^2}} \langle 3, -4\rangle = \frac{1}{5}\langle 3, 4\rangle$. Therefore, $D_{\mathbf{u}}f(1,-1) = \|\nabla f(1,-1)\| = \sqrt{3 + (-4)^2} = 5$.
(b) $D_{\mathbf{u}}f(1,-1)$ is smallest
Solution
From (a), $\nabla f(1,-1) = \langle 3, -4\rangle$.
$D_{\mathbf{u}}f(1,-1)$ is smallest when $D_{\mathbf{u}}f(1,-1) = -\|\nabla f(1,-1)\|$. That is, $D_{\mathbf{u}}f(1,-1)$ is largest in the direction of $-\frac{\nabla f}{\|\nabla f\|} = -\frac{1}{\sqrt{3^2+(-4)^2}} \langle 3, -4\rangle = -\frac{1}{5}\langle 3, 4\rangle$. Therefore, $D_{\mathbf{u}}f(1,-1) = -\|\nabla f(1,-1)\| = -\sqrt{3 + (-4)^2} = -5$.
(c) $D_{\mathbf{u}}f(1,-1) = 0$
Solution
From (a), $\nabla f(1,-1) = \langle 3, -4\rangle$.
$D_{\mathbf{u}}f(1,-1) = 0$ in the direction orthogonal to ∇f. This only occurs when $\mathbf{u} = \langle 4, 3\rangle$ or when $\mathbf{u} = \langle -4, -3\rangle$.
We're not done yet!
Making these two vectors unit vectors, we get

$$\mathbf{u} = \frac{1}{\sqrt{4^2 + 3^2}} \langle 4, 3\rangle = \frac{1}{5}\langle 4, 3\rangle$$

or

$$\mathbf{u} = \frac{1}{\sqrt{(-4)^2 + (-3)^2}} \langle -4, -3\rangle = \frac{1}{5}\langle -4, -3\rangle$$

Clearly, $D_{\mathbf{u}}f(1,-1) = 0$.

(d) $D_{\mathbf{u}}f(1,-1) = 4$
Solution
From (a), $\nabla f(1,-1) = \langle 3,-4\rangle$.
Recall that $D_{\mathbf{u}}f(1,-1) = \nabla f(1,-1)\cdot\mathbf{u}$. Therefore,

$$\begin{aligned}\nabla f(1,-1)\cdot\mathbf{u} = 4 &= \langle 3,-4\rangle\cdot\langle u_1,u_2\rangle\\ &= 3u_1 - 4u_2\\ 4 - 3u_1 &= -4u_2\\ -1 + \frac{3}{4}u_1 &= u_2\end{aligned}$$

We also know that $\mathbf{u}$ must be a unit vector. Therefore,

$$\begin{aligned}\|\mathbf{u}\| &= 1\\ \|\mathbf{u}\|^2 &=\\ u_1^2 + u_2^2 &=\\ u_1^2 + \left(-1 + \frac{3}{4}u_1\right)^2 &=\\ \frac{25}{16}u_1^2 - \frac{3}{2}u_1 &= 0\\ u_1 &= 0 \text{ or } \frac{24}{25} && \text{(thank you mathway!)}\\ u_2 &= -1 \text{ or } -\frac{7}{25} && (u_2 = -1 + \tfrac{3}{4}u_1)\end{aligned}$$

Therefore, $\mathbf{u} = \langle 0,-1\rangle$ or $\mathbf{u} = \left\langle \frac{24}{25}, -\frac{7}{25}\right\rangle$. Clearly, $D_{\mathbf{u}}f(1,-1) = 4$.
(e) $D_{\mathbf{u}}f(1,-1) = 4$
Solution
From (a), $\nabla f(1,-1) = \langle 3,-4\rangle$.
Recall that $D_{\mathbf{u}}f(1,-1) = \nabla f(1,-1)\cdot\mathbf{u}$. Therefore,

$$\begin{aligned}\nabla f(1,-1)\cdot\mathbf{u} = -3 &= \langle 3,-4\rangle\cdot\langle u_1,u_2\rangle\\ &= 3u_1 - 4u_2\\ 3 + 4u_2 &= 3u_1\\ 1 + \frac{4}{3}u_2 &= u_1\end{aligned}$$

We also know that $\mathbf{u}$ must be a unit vector. Therefore,

$$\begin{aligned}\|\mathbf{u}\| &= 1\\ \|\mathbf{u}\|^2 &=\\ u_1^2 + u_2^2 &=\\ \left(1 + \frac{4}{3}u_2\right)^2 + u_2^2 &=\\ \frac{25}{9}u_2^2 - \frac{8}{3}u_2 &= 0\\ u_2 &= 0 \text{ or } \frac{24}{25} && \text{(thank you mathway!)}\\ u_1 &= -1 \text{ or } \frac{7}{25} && (u_1 = 1 + \tfrac{4}{3}u_2)\end{aligned}$$

Therefore, $\mathbf{u} = \langle -1,0\rangle$ or $\mathbf{u} = \left\langle \frac{7}{25}, \frac{24}{25}\right\rangle$. Clearly, $D_{\mathbf{u}}f(1,-1) = -3$.

Problem 6

In what directions is the derivative of $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ at $(3,2)$ equal to 0?

Solution

First, let's figure out what ∇f is.

$$\begin{aligned}
\nabla f &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \\
&= \left\langle \frac{2x(x^2+y^2)-2x(x^2-y^2)}{(x^2+y^2)^2}, \frac{-2y(x^2+y^2)-2y(x^2-y^2)}{(x^2+y^2)^2} \right\rangle \\
&= \left\langle \frac{4xy^2}{(x^2+y^2)^2}, \frac{4x^2y}{(x^2+y^2)^2} \right\rangle \\
\nabla f(3,2) &= \left\langle \frac{4(3)(2)^2}{(3^2+2^2)^2}, -\frac{4(3)(2)^2}{(3^2+2^2)^2} \right\rangle \\
&= \langle 1, -1 \rangle \qquad \text{(thank you mathway!)}
\end{aligned}$$

The directional derivative is only equal to 0 in the direction of a vector orthogonal to ∇f. That is,

$$\nabla f(3,2) \cdot \mathbf{u} = 0$$

Therefore, $\mathbf{u}\,\langle 1,1\rangle$ or $-\mathbf{u} = -\langle 1,1\rangle$.

But we're not done yet!

We need to make $\mathbf{u}$ a unit vector. Our final answer is,

$$\begin{aligned}
\frac{\mathbf{u}}{\|\mathbf{u}\|} &= \frac{1}{\sqrt{1^2+1^2}}\langle 1,1\rangle = \frac{1}{\sqrt{2}}\langle 1,1\rangle \\
&\text{or} \\
-\frac{\mathbf{u}}{\|\mathbf{u}\|} &= -\frac{1}{\sqrt{1^2+1^2}}\langle 1,1\rangle = -\frac{1}{\sqrt{2}}\langle 1,1\rangle
\end{aligned}$$

Problem 7
Prove the derivative rules for gradients.
Solution
Constant Rule

$$\begin{aligned}\nabla(cf) &= \left\langle \frac{\partial(cf)}{\partial x}, \frac{\partial(cf)}{\partial y} \right\rangle \\ &= \left\langle c\frac{\partial f}{\partial x}, c\frac{\partial f}{\partial y} \right\rangle \\ &= c\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \\ &= c\nabla f\end{aligned}$$

□

Sum & Difference Rule

$$\begin{aligned}\nabla(f \pm g) &= \left\langle \frac{\partial(f \pm g)}{\partial x}, \frac{\partial(f \pm g)}{\partial y} \right\rangle \\ &= \left\langle \frac{\partial f}{\partial x} \pm \frac{\partial g}{\partial x}, \frac{\partial f}{\partial y} \pm \frac{\partial g}{\partial y} \right\rangle \\ &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \pm \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right\rangle \\ &= \nabla f \pm \nabla g\end{aligned}$$

□

Product Rule

$$\begin{aligned}\nabla(fg) &= \left\langle \frac{\partial(fg)}{\partial x}, \frac{\partial(fg)}{\partial y} \right\rangle \\ &= \left\langle \frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial x}, \frac{\partial f}{\partial y}g + f\frac{\partial g}{\partial y} \right\rangle && \text{(Chain Rule)} \\ &= f\left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right\rangle + g\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \\ &= f\nabla g + g\nabla f\end{aligned}$$

□

Quotient Rule
Not too sure about my method for this one!

$$\begin{aligned}\nabla\left(\frac{f}{g}\right) &= \left\langle \frac{\partial(f(g)^{-1})}{\partial x}, \frac{\partial(f(g)^{-1})}{\partial y} \right\rangle \\ &= \left\langle \frac{\partial f}{\partial x}(g)^{-1} - f(g)^{-2}\frac{\partial g}{\partial x}, \frac{\partial f}{\partial y}(g)^{-1} - f(g)^{-2}\frac{\partial g}{\partial y} \right\rangle \\ &= \left\langle \frac{\partial f}{\partial x}\frac{g}{g^2} - \frac{f}{g^2}\frac{\partial g}{\partial x}, \frac{\partial f}{\partial y}\frac{g}{g^2} - \frac{f}{g^2}\frac{\partial g}{\partial y} \right\rangle \\ &= \frac{1}{g^2}\left\langle \frac{\partial f}{\partial x}g - f\frac{\partial g}{\partial x}, \frac{\partial f}{\partial y}g - f\frac{\partial g}{\partial y} \right\rangle \\ &= \frac{1}{g^2}\left(g\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle - f\left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right\rangle\right) \\ &= \frac{1}{g^2}\left(g\nabla f - f\nabla g\right) \\ &= \frac{g\nabla f - f\nabla g}{g^2}\end{aligned}$$