Homework #4

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```
library(ggplot2)
library(dplyr)
library(knitr)
library(grid)
library(gridExtra)
```

```
set.seed(1272)
```

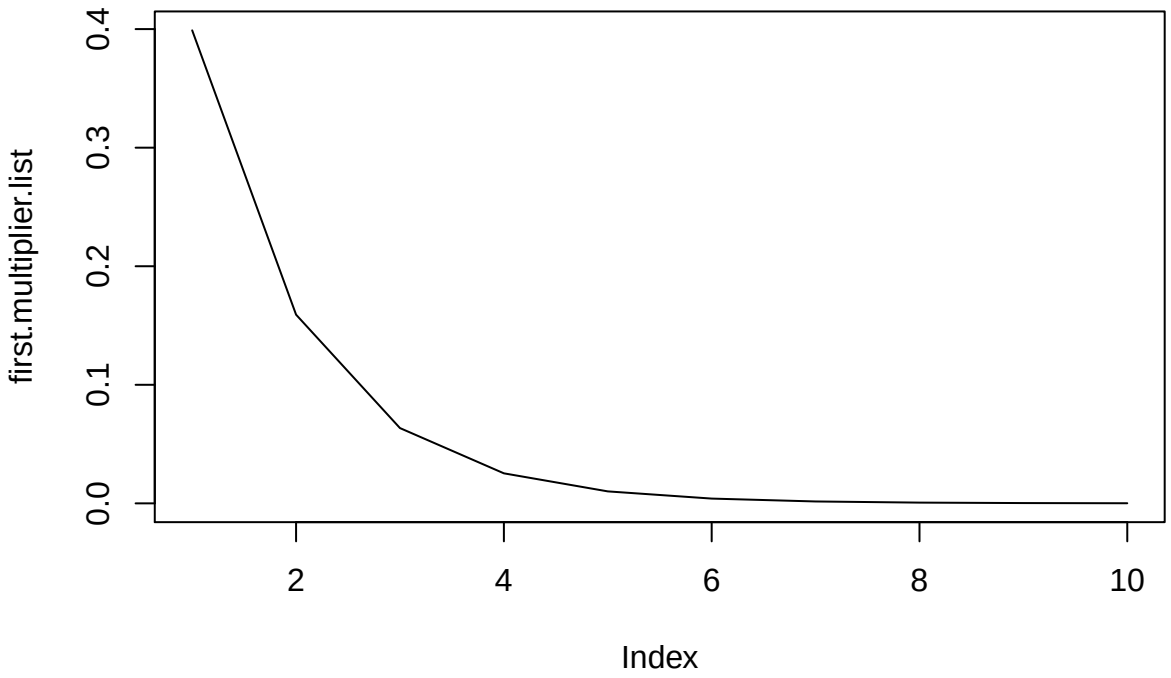
Variance reduction procedures for Monte Carlo methods

Preliminary Setup

The multiplier decreases as D increases.

```
first.multiplier <- function(D){
  return(1/((2*pi)^(D/2)))
}
```

```
D.seq <- seq(1, from=1, to=10)

first.multiplier.list <- sapply(D.seq, FUN=first.multiplier)

plot(first.multiplier.list,  type = "l")
```



Testing the exponent function.

```
exponent.function <- function(x){
  return((-1/2)*(t(x)%*%x))
}
n <- 10
D <- 2
x <- matrix(runif(D*n, min = -5, max = 5), ncol=D)

exponent.function(x)
```

```
##             [,1]       [,2]
## [1,] -62.170918  -1.373319
## [2,]  -1.373319 -30.872275
```

These don't really need to be broken out into different functions, but I found it helpful to try different inputs and see how the components of the function behave.

a) Crude Monte Carlo

First, create a function to evaluate c(x) at N samples in D dimensions (i.e., a DxN matrix).

```
cost.function.crude <- function(x, D){
  first.multiplier.sim <- first.multiplier(D)
  exponent.sim <- exponent.function(x)
  return(first.multiplier.sim * exp(exponent.sim))
}
```

```
n <- 1
D <- 2
x <- matrix(runif(D*n, min = -5, max = 5), nrow=n, ncol=D)
```

```
cost.function.crude.apply <- function(x, D){
  return(mean(apply(x, 1, FUN = cost.function.crude, D=D)))
}
cost.function.crude.apply(x, D)
```

```
## [1] 9.249131e-08
```

This appears to work (the mean is always close to 1/10). Now to move on to the crude Monte Carlo simulation of the cost function for n sizes from 1000 to 10,000 by increments of 1000.

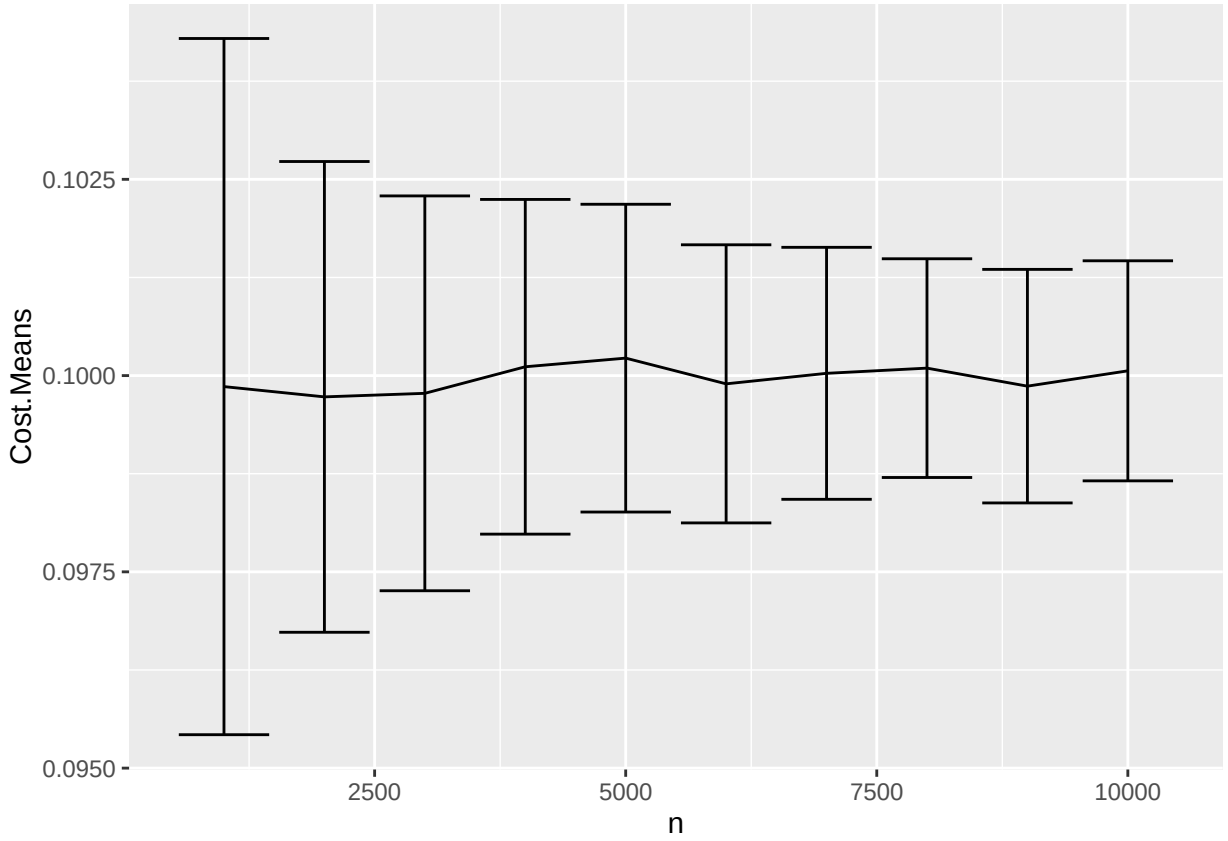
```
n.increments <- seq(from = 1000, to = 10000, by = 1000)

generate.x.list <- function(n, D){
  x <- matrix(runif(D*n, min = -5, max = 5), nrow=n, ncol=D)
  return(x)
}

x.list <- lapply(n.increments, FUN = generate.x.list, D=D)
```

For D = 1:

```
D.1 <- 1
crude.cost.sim.1 <- replicate(100, sapply(lapply(n.increments, FUN = generate.x.list, D=D.1), FUN = cos
```

```
crude.cost.sim.1.means <- apply(crude.cost.sim.1, 1, FUN = mean)
crude.cost.sim.1.sd <- apply(crude.cost.sim.1, 1, FUN = sd)
n <- n.increments
crude.cost.sim.1.results <- cbind(n, crude.cost.sim.1.means, crude.cost.sim.1.sd) %>% as.data.frame()
colnames(crude.cost.sim.1.results) <- c("n", "Cost.Means", "Cost.Standard.Deviation")
```

```
limits <- aes(ymax = Cost.Means + Cost.Standard.Deviation, ymin = Cost.Means - Cost.Standard.Deviation)
ggplot(crude.cost.sim.1.results, aes(x=n, y=Cost.Means)) + geom_line() + geom_errorbar(limits)
```




```
crude.cost.sim.1.results <- crude.cost.sim.1.results %>%
  mutate(Cost.Coeff.Varation = Cost.Standard.Deviation / Cost.Means)
```

```
kable(crude.cost.sim.1.results)
```

n	Cost.Means	Cost.Standard.Deviation	Cost.Coeff.Varation
1000	0.0998597	0.0044341	0.0444035
2000	0.0997287	0.0029983	0.0300650
3000	0.0997740	0.0025148	0.0252047
4000	0.1001121	0.0021317	0.0212933

n	Cost.Means	Cost.Standard.Deviation	Cost.Coeff.Varation
5000	0.1002223	0.0019601	0.0195579
6000	0.0998947	0.0017714	0.0177327
7000	0.1000289	0.0016050	0.0160452
8000	0.1000950	0.0013931	0.0139181
9000	0.0998653	0.0014874	0.0148941
10000	0.1000606	0.0014016	0.0140078

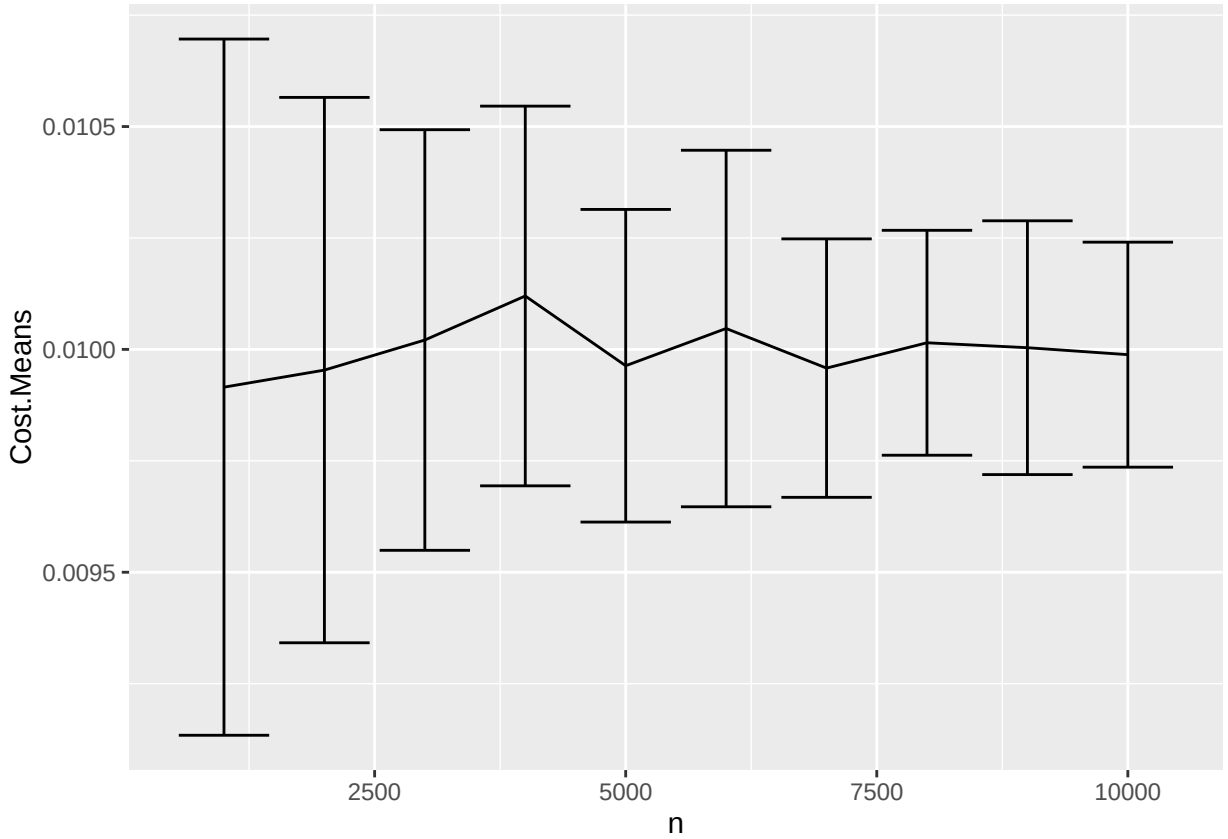
Now for D = 2:

```
D.2 = 2
crude.cost.sim.2 <- replicate(100, sapply(lapply(n.increments, FUN = generate.x.list, D = D.2), FUN = co

(target.D.2 <- (1/10)^2)
```

```
## [1] 0.01
```

```
crude.cost.sim.2.means <- apply(crude.cost.sim.2, 1, FUN = mean)
crude.cost.sim.2.sd <- apply(crude.cost.sim.2, 1, FUN = sd)

crude.cost.sim.2.results <- cbind(n, crude.cost.sim.2.means, crude.cost.sim.2.sd) %>% as.data.frame()
colnames(crude.cost.sim.2.results) <- c("n", "Cost.Means", "Cost.Standard.Deviation")
```

```
limits.2 <- aes(ymax = Cost.Means + Cost.Standard.Deviation, ymin = Cost.Means - Cost.Standard.Deviatio
ggplot(crude.cost.sim.2.results, aes(x=n, y=Cost.Means)) + geom_line() + geom_errorbar(limits.2)
```



```
crude.cost.sim.2.results <- crude.cost.sim.2.results %>%
  mutate(Cost.Coeff.Varation = Cost.Standard.Deviation / Cost.Means)
```

```
kable(crude.cost.sim.2.results)
```

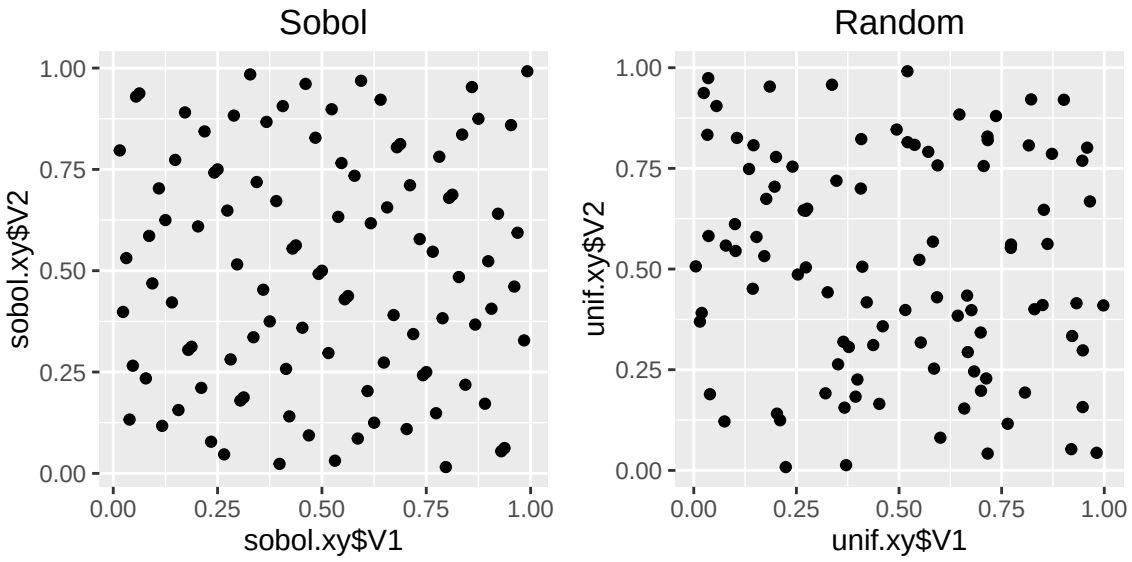
n	Cost.Means	Cost.Standard.Deviation	Cost.Coeff.Varation
1000	0.0099153	0.0007810	0.0787692
2000	0.0099536	0.0006118	0.0614684
3000	0.0100210	0.0004719	0.0470890
4000	0.0101198	0.0004260	0.0420941
5000	0.0099633	0.0003507	0.0351998
6000	0.0100468	0.0004000	0.0398108
7000	0.0099580	0.0002900	0.0291183
8000	0.0100148	0.0002523	0.0251928
9000	0.0100038	0.0002847	0.0284593
10000	0.0099882	0.0002524	0.0252733

b) Quasi-Random Numbers

Here, I use the package 'randtoolbox' to generate a Sobol sequence.

```
library(randtoolbox)
```

```
n = 100
sobol.xy <- sobol(n, dim = 2) %>% as.data.frame()
sobol <- qplot(sobol.xy$V1, sobol.xy$V2, main="Sobol")

unif.xy <- cbind(runif(n), runif(n)) %>% as.data.frame()
rand <- qplot(unif.xy$V1, unif.xy$V2, main = "Random")

grid.arrange(sobol, rand, nrow=1)
```



The random numbers (uniform distribution) form clusters where there are several random variables with close values and "dead space" where there are no random variables with the values.

The difference between the two is their discrepency. The Sobol numbers are low discrepency but are "quasi-random" - the enforcement of even distribution means that the values are not i.i.d. and therefore not truly random. The uniform distribution random numbers have the potential to be high discrepancy.

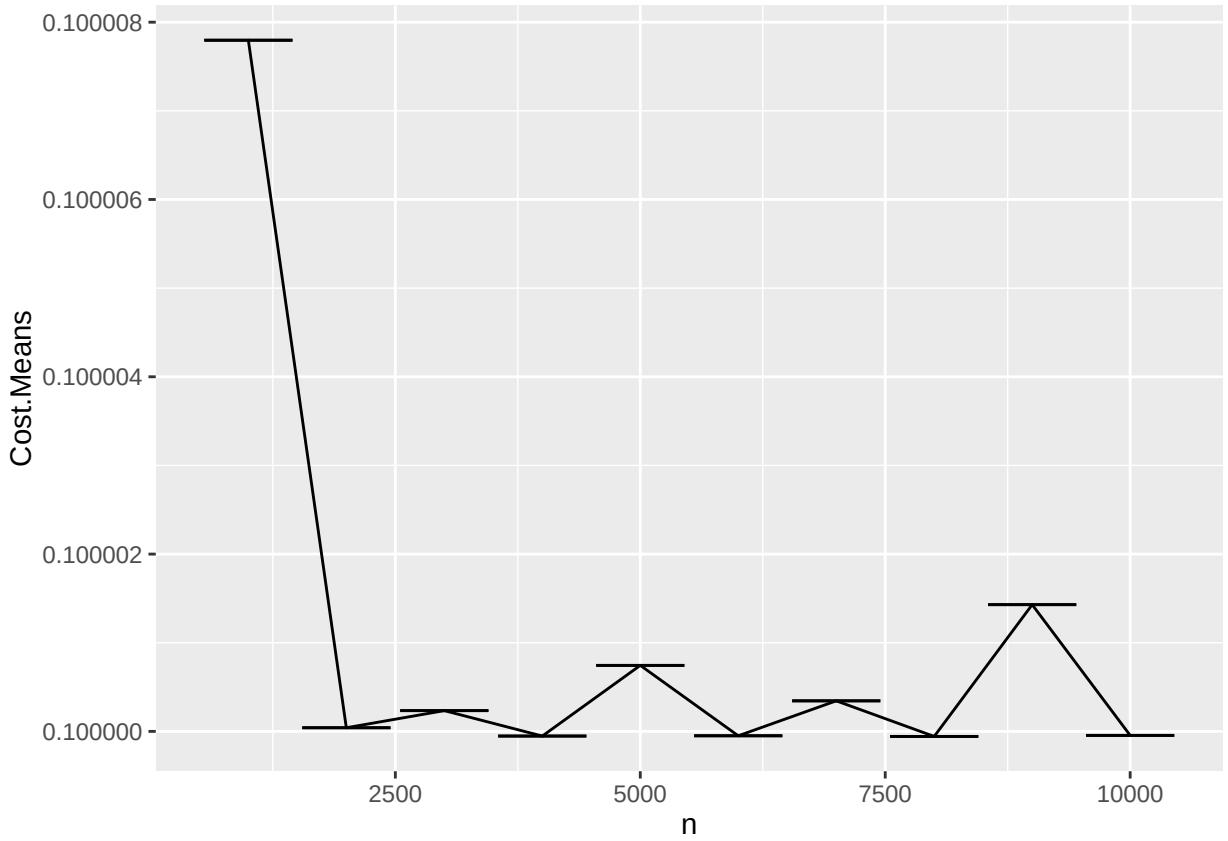
For D = 1:

```
generate.x.list.Sobol <- function(n, D){
  sobol.seq <- (sobol(n, dim = D) * 10) - 5
  x <- as.matrix(sobol.seq)
  return(x)
}
```

```
D.1 <- 1
sobol.cost.sim.1 <- replicate(100, sapply(lapply(n.increments, FUN = generate.x.list.Sobol, D=D.1), FUN
```

```
sobol.cost.sim.1.means <- apply(sobol.cost.sim.1, 1, FUN = mean)
sobol.cost.sim.1.sd <- apply(sobol.cost.sim.1, 1, FUN = sd)
n <- n.increments
sobol.cost.sim.1.results <- cbind(n, sobol.cost.sim.1.means, sobol.cost.sim.1.sd) %>% as.data.frame()
colnames(sobol.cost.sim.1.results) <- c("n", "Cost.Means", "Cost.Standard.Deviation")
```

```
ggplot(sobol.cost.sim.1.results, aes(x=n, y=Cost.Means)) + geom_line() + geom_errorbar(limits)
```



```
sobol.cost.sim.1.results <- sobol.cost.sim.1.results %>%
  mutate(Cost.Coeff.Varation = Cost.Standard.Deviation / Cost.Means)
```

```
kable(sobol.cost.sim.1.results)
```

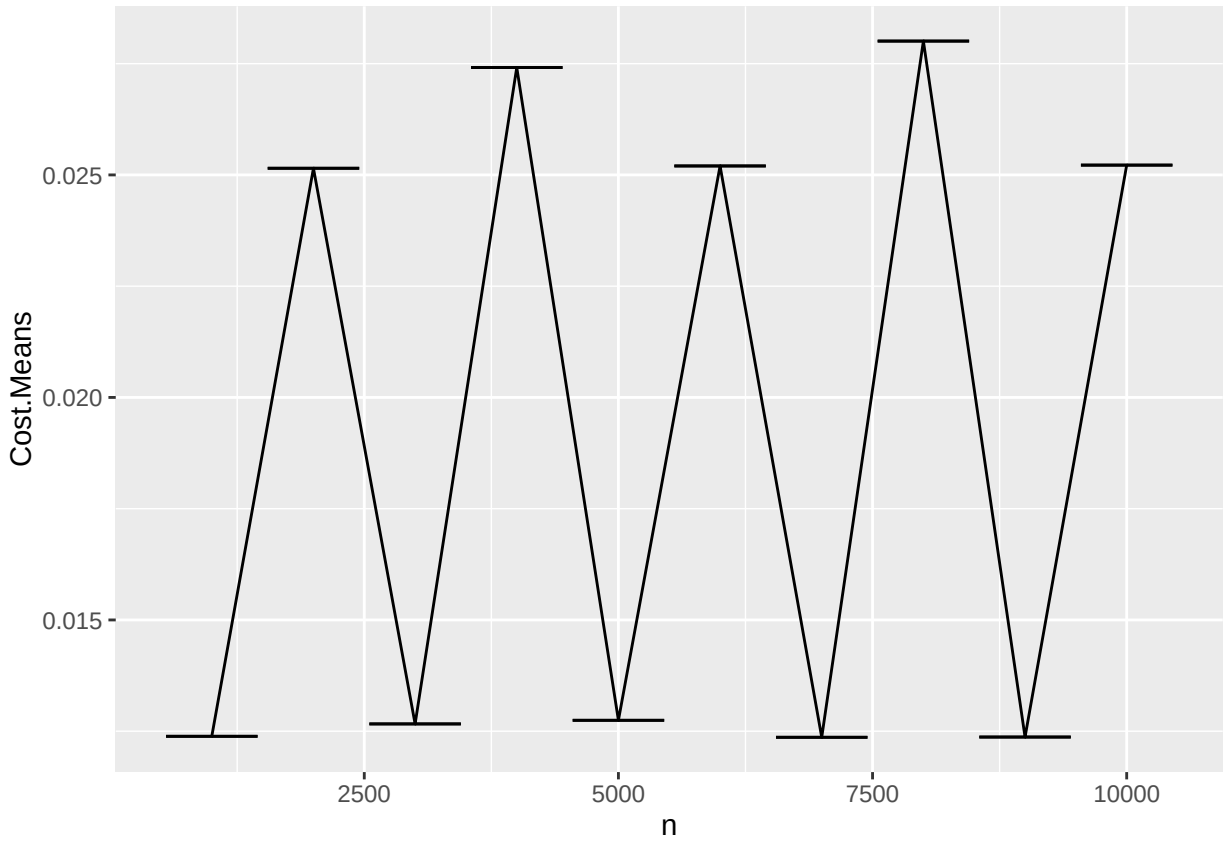
n	Cost.Means	Cost.Standard.Deviation	Cost.Coeff.Varation
1000	0.1000078	0	0
2000	0.1000000	0	0
3000	0.1000002	0	0
4000	0.0999999	0	0
5000	0.1000007	0	0
6000	0.1000000	0	0
7000	0.1000003	0	0
8000	0.0999999	0	0
9000	0.1000014	0	0
10000	0.1000000	0	0

For D = 2:

```
D.2 = 2
sobol.cost.sim.2 <- replicate(100, sapply(lapply(n.increments, FUN = generate.x.list.Sobol, D = D.2), FU
```

```
sobol.cost.sim.2.means <- apply(sobol.cost.sim.2, 1, FUN = mean)
sobol.cost.sim.2.sd <- apply(sobol.cost.sim.2, 1, FUN = sd)

sobol.cost.sim.2.results <- cbind(n, sobol.cost.sim.2.means, sobol.cost.sim.2.sd) %>% as.data.frame()
colnames(sobol.cost.sim.2.results) <- c("n", "Cost.Means", "Cost.Standard.Deviation")
```

```
ggplot(sobol.cost.sim.2.results, aes(x=n, y=Cost.Means)) + geom_line() + geom_errorbar(limits.2)
```




```
sobol.cost.sim.2.results <- sobol.cost.sim.2.results %>%
  mutate(Cost.Coeff.Varation = Cost.Standard.Deviation / Cost.Means)
```

```
kable(sobol.cost.sim.2.results)
```

n	Cost.Means	Cost.Standard.Deviation	Cost.Coeff.Varation
1000	0.0123852	0	0
2000	0.0251496	0	0
3000	0.0126636	0	0
4000	0.0274133	0	0
5000	0.0127448	0	0
6000	0.0252011	0	0
7000	0.0123633	0	0
8000	0.0280071	0	0
9000	0.0123689	0	0
10000	0.0252202	0	0

c) Antithetic Variates

```
generate.x.list.antithetic <- function(n, D){
  x <- matrix(runif(D*n, min = 0, max = 1), nrow=n, ncol=D)

  for(i in 1:nrow(x)){
```

```
    ifelse(i %% 2 == 0, x[i,] <- 1 - x[i-1,], next)
  }
  x <- (x * 10) - 5
  return(x)
}
```

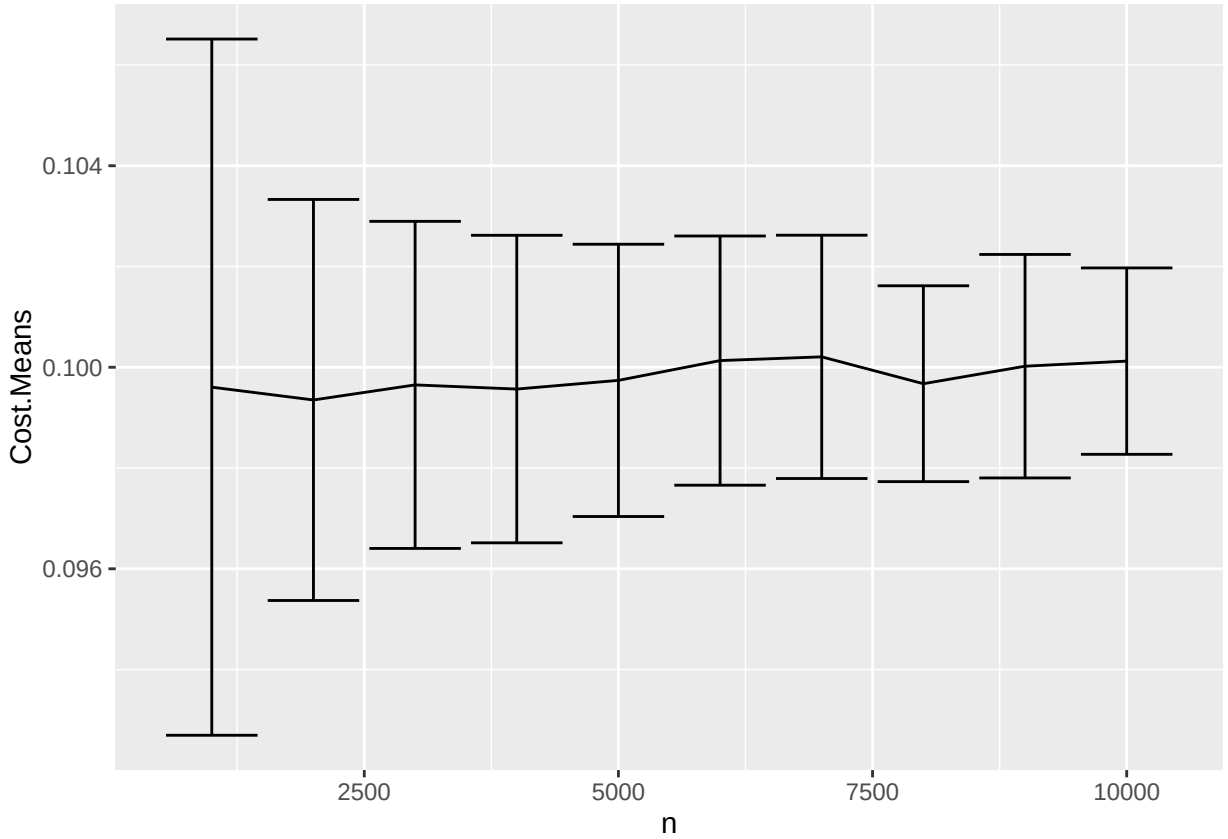
```
generate.x.list.antithetic(4, 3)
```

```
##            [,1]       [,2]       [,3]
## [1,] -2.1348339  2.4321640 -0.4028483
## [2,]  2.1348339 -2.4321640  0.4028483
## [3,] -0.7463661 -0.5485907 -3.5443062
## [4,]  0.7463661  0.5485907  3.5443062
```

```
D.1 <- 1
antithetic.cost.sim.1 <- replicate(100, sapply(lapply(n.increments, FUN = generate.x.list.antithetic, D=
```

```
antithetic.cost.sim.1.means <- apply(antithetic.cost.sim.1, 1, FUN = mean)
antithetic.cost.sim.1.sd <- apply(antithetic.cost.sim.1, 1, FUN = sd)
n <- n.increments
antithetic.cost.sim.1.results <- cbind(n, antithetic.cost.sim.1.means, antithetic.cost.sim.1.sd) %>% as
colnames(antithetic.cost.sim.1.results) <- c("n", "Cost.Means", "Cost.Standard.Deviation")
```

```
limits <- aes(ymax = Cost.Means + Cost.Standard.Deviation, ymin = Cost.Means - Cost.Standard.Deviation)
ggplot(antithetic.cost.sim.1.results, aes(x=n, y=Cost.Means)) + geom_line() + geom_errorbar(limits)
```



```
antithetic.cost.sim.1.results <- antithetic.cost.sim.1.results %>%
  mutate(Cost.Coeff.Varation = Cost.Standard.Deviation / Cost.Means)
```

```
kable(antithetic.cost.sim.1.results)
```

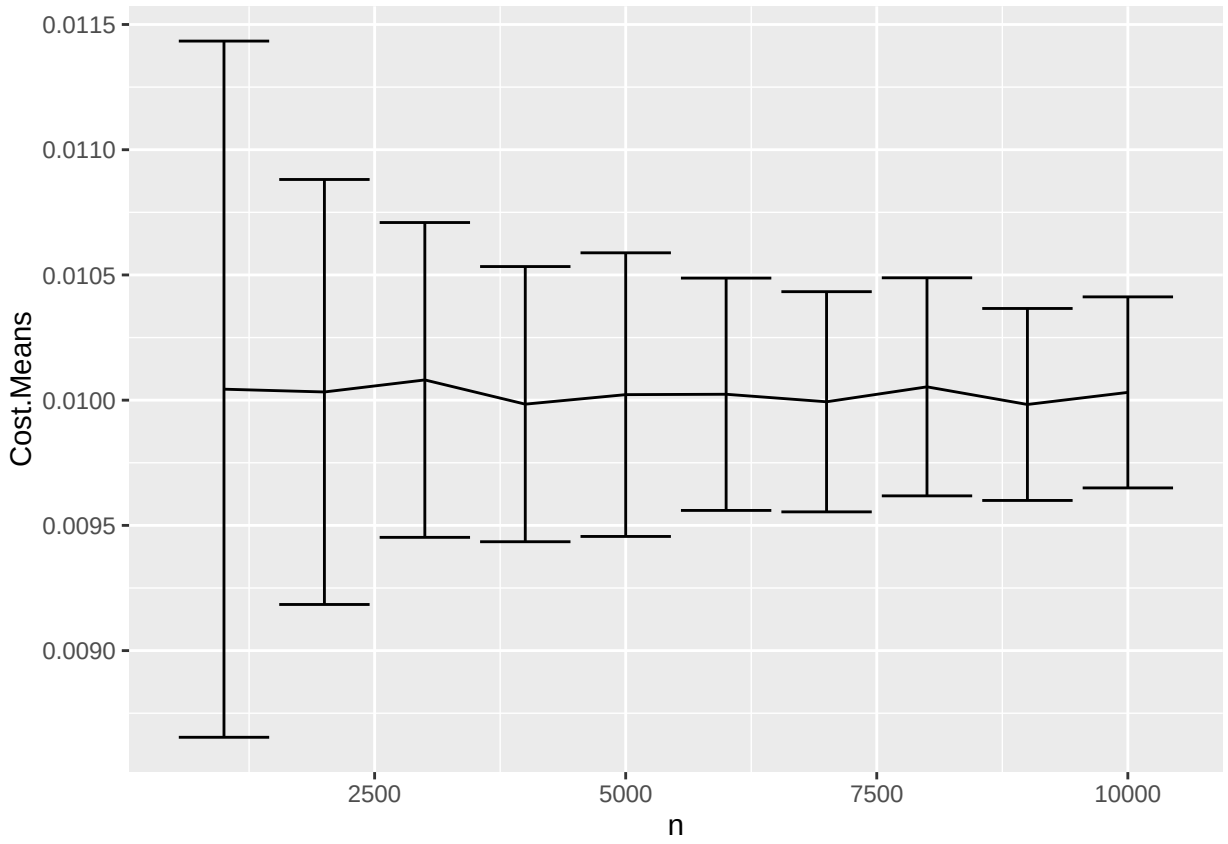
n	Cost.Means	Cost.Standard.Deviation	Cost.Coeff.Varation
1000	0.0996053	0.0069090	0.0693635
2000	0.0993512	0.0039797	0.0400569
3000	0.0996503	0.0032459	0.0325728
4000	0.0995676	0.0030517	0.0306496
5000	0.0997402	0.0027021	0.0270917
6000	0.1001326	0.0024729	0.0246960
7000	0.1002068	0.0024146	0.0240964
8000	0.0996741	0.0019436	0.0194991
9000	0.1000214	0.0022179	0.0221741
10000	0.1001223	0.0018497	0.0184745

For D = 2:

```
D.2 = 2
antithetic.cost.sim.2 <- replicate(100, sapply(lapply(n.increments, FUN = generate.x.list.antithetic, D
```

```
antithetic.cost.sim.2.means <- apply(antithetic.cost.sim.2, 1, FUN = mean)
antithetic.cost.sim.2.sd <- apply(antithetic.cost.sim.2, 1, FUN = sd)

antithetic.cost.sim.2.results <- cbind(n, antithetic.cost.sim.2.means, antithetic.cost.sim.2.sd) %>% as
colnames(antithetic.cost.sim.2.results) <- c("n", "Cost.Means", "Cost.Standard.Deviation")
```

```
limits.2 <- aes(ymax = Cost.Means + Cost.Standard.Deviation, ymin = Cost.Means - Cost.Standard.Deviatio
ggplot(antithetic.cost.sim.2.results, aes(x=n, y=Cost.Means)) + geom_line() + geom_errorbar(limits.2)
```




```
antithetic.cost.sim.2.results <- antithetic.cost.sim.2.results %>%
  mutate(Cost.Coeff.Varation = Cost.Standard.Deviation / Cost.Means)
```

```
kable(antithetic.cost.sim.2.results)
```

n	Cost.Means	Cost.Standard.Deviation	Cost.Coeff.Varation
1000	0.0100438	0.0013899	0.1383868
2000	0.0100327	0.0008485	0.0845693
3000	0.0100807	0.0006285	0.0623505
4000	0.0099841	0.0005494	0.0550251
5000	0.0100221	0.0005662	0.0564996
6000	0.0100236	0.0004638	0.0462707
7000	0.0099936	0.0004396	0.0439848
8000	0.0100532	0.0004353	0.0433040
9000	0.0099828	0.0003833	0.0383915
10000	0.0100310	0.0003814	0.0380200

d) Latin Hypercube Sampling

```
library(lhs)
```

```
## Warning: package 'lhs' was built under R version 3.2.5
```

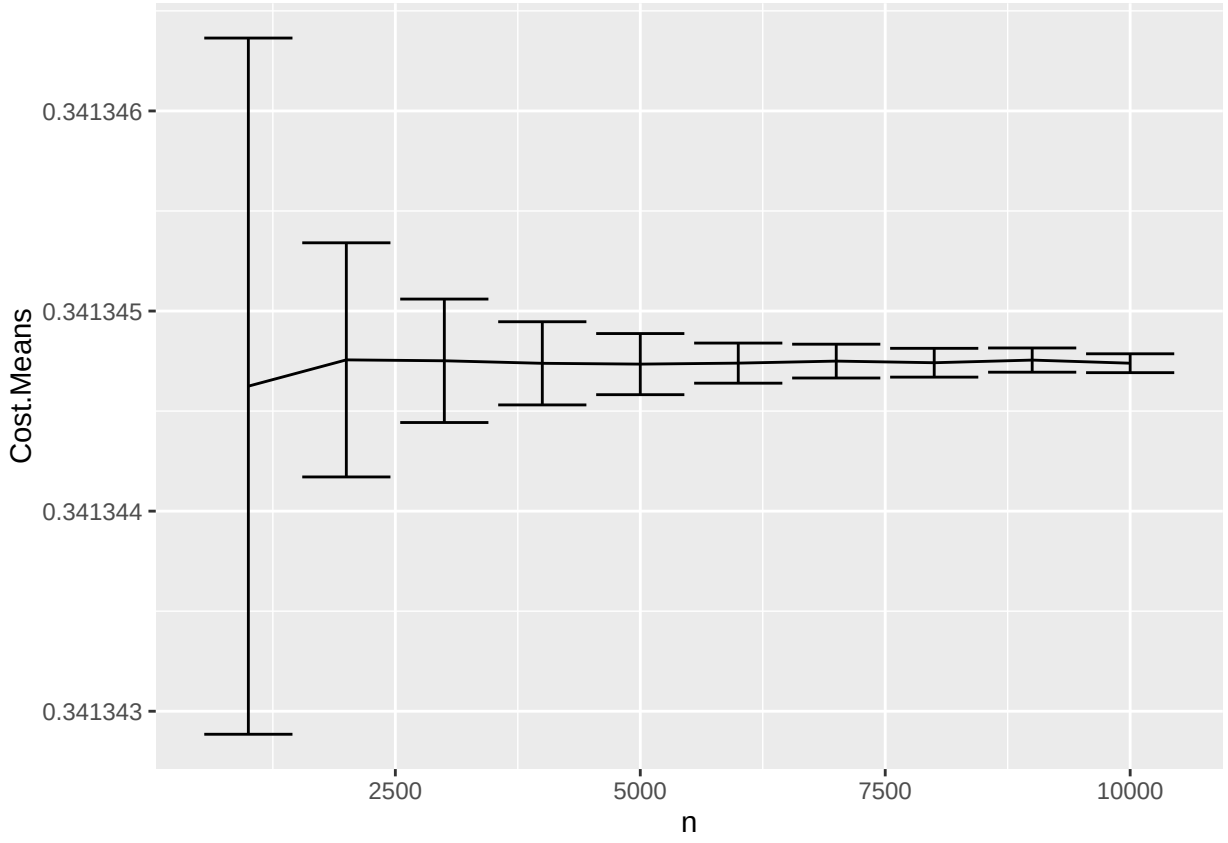
For D = 1:

```
generate.x.list.latin <- function(n, D){
  x <- as.matrix((randomLHS(n,D) * 10) - 5 )
  return(x)
}
```

```
D.1 <- 1
latin.cost.sim.1 <- replicate(100, sapply(lapply(n.increments, FUN = generate.x.list.latin, D=D.1), FUN
```

```
latin.cost.sim.1.means <- apply(latin.cost.sim.1, 1, FUN = mean)
latin.cost.sim.1.sd <- apply(latin.cost.sim.1, 1, FUN = sd)
n <- n.increments
latin.cost.sim.1.results <- cbind(n, latin.cost.sim.1.means, latin.cost.sim.1.sd) %>% as.data.frame()
colnames(latin.cost.sim.1.results) <- c("n", "Cost.Means", "Cost.Standard.Deviation")
```

```
ggplot(latin.cost.sim.1.results, aes(x=n, y=Cost.Means)) + geom_line() + geom_errorbar(limits)
```




```
latin.cost.sim.1.results <- latin.cost.sim.1.results %>%
  mutate(Cost.Coeff.Varation = Cost.Standard.Deviation / Cost.Means)
```

```
kable(latin.cost.sim.1.results)
```

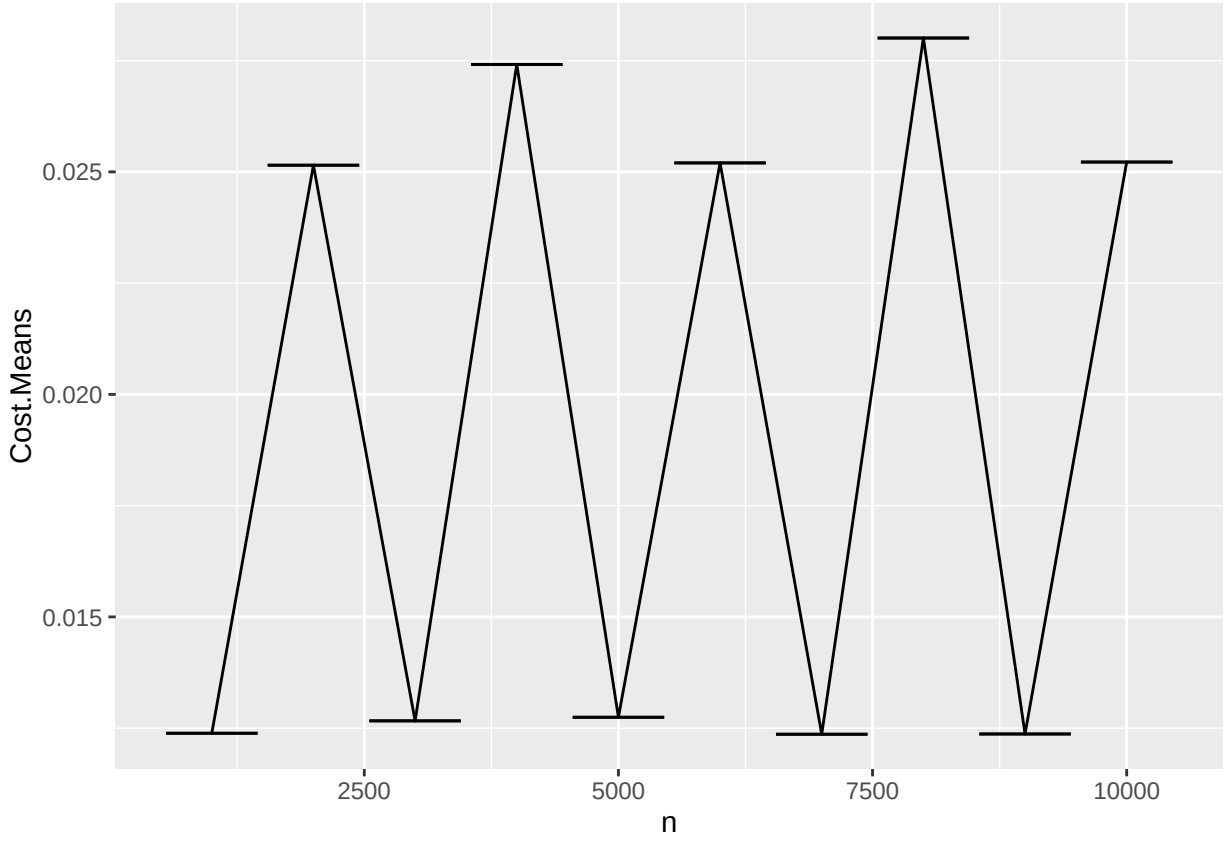
n	Cost.Means	Cost.Standard.Deviation	Cost.Coeff.Varation
1000	0.3413446	1.7e-06	5.1e-06
2000	0.3413448	6.0e-07	1.7e-06
3000	0.3413448	3.0e-07	9.0e-07
4000	0.3413447	2.0e-07	6.0e-07
5000	0.3413447	2.0e-07	4.0e-07
6000	0.3413447	1.0e-07	3.0e-07
7000	0.3413447	1.0e-07	2.0e-07
8000	0.3413447	1.0e-07	2.0e-07
9000	0.3413448	1.0e-07	2.0e-07
10000	0.3413447	0.0e+00	1.0e-07

For D = 2:

```
D.2 = 2
latin.cost.sim.2 <- replicate(100, sapply(lapply(n.increments, FUN = generate.x.list.latin, D = D.2), FU
```

```
latin.cost.sim.2.means <- apply(sobol.cost.sim.2, 1, FUN = mean)
latin.cost.sim.2.sd <- apply(sobol.cost.sim.2, 1, FUN = sd)

latin.cost.sim.2.results <- cbind(n, latin.cost.sim.2.means, latin.cost.sim.2.sd) %>% as.data.frame()
colnames(latin.cost.sim.2.results) <- c("n", "Cost.Means", "Cost.Standard.Deviation")
```

```
ggplot(latin.cost.sim.2.results, aes(x=n, y=Cost.Means)) + geom_line() + geom_errorbar(limits.2)
```



```
latin.cost.sim.2.results <- latin.cost.sim.2.results %>%
  mutate(Cost.Coeff.Varation = Cost.Standard.Deviation / Cost.Means)
```

```
kable(latin.cost.sim.2.results)
```

n	Cost.Means	Cost.Standard.Deviation	Cost.Coeff.Varation
1000	0.0123852	0	0
2000	0.0251496	0	0
3000	0.0126636	0	0
4000	0.0274133	0	0
5000	0.0127448	0	0
6000	0.0252011	0	0
7000	0.0123633	0	0
8000	0.0280071	0	0
9000	0.0123689	0	0
10000	0.0252202	0	0
##e) Imp	ortance Sampl	ing	

f) Summary

6.3

Plot the power curves for the t-test in Example 6.9 for sample sizes 10, 20, 30, 40, and 50, but omit the standard error bars. Plot the curves on the same craph, each in a different color or line type, and include a legend. Comment on the relation between power and sample size.

```
empirical.power <- function(n){
  m = 1000
  mu0 = 500
  sigma = 100
  mu <- c(seq(450, 650, 10))
  M <- length(mu)
  power <- numeric(M)

  for (i in 1:M) {
    mu1 <- mu[i]
    pvalues <- replicate(m, expr = {
      x <- rnorm(n, mean = mu1, sd = sigma)
      ttest <- t.test(x, alternative = "greater",
                      mu = mu0)
      ttest$p.value})
    power[i] <- mean(pvalues <= 0.05)
  }
  return(power)
}
```

```
n.set <- seq(from = 10, to = 50, by = 10)

power.sim <- lapply(n.set, FUN = empirical.power)
```

```r
theta <- seq(450, 650, 10)

tidy.results <- NULL

for(i in 1:length(n.set)){

  results.set <- cbind(rep(i*10, times = length(n.set)), theta,
                           unlist(power.sim[i]))
  tidy.results <- rbind(results.set, tidy.results)
}
```

```
## Warning in cbind(rep(i * 10, times = length(n.set)), theta,
## unlist(power.sim[i])): number of rows of result is not a multiple of vector
## length (arg 1)

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## Warning in cbind(rep(i * 10, times = length(n.set)), theta,
## unlist(power.sim[i])): number of rows of result is not a multiple of vector
## length (arg 1)
```
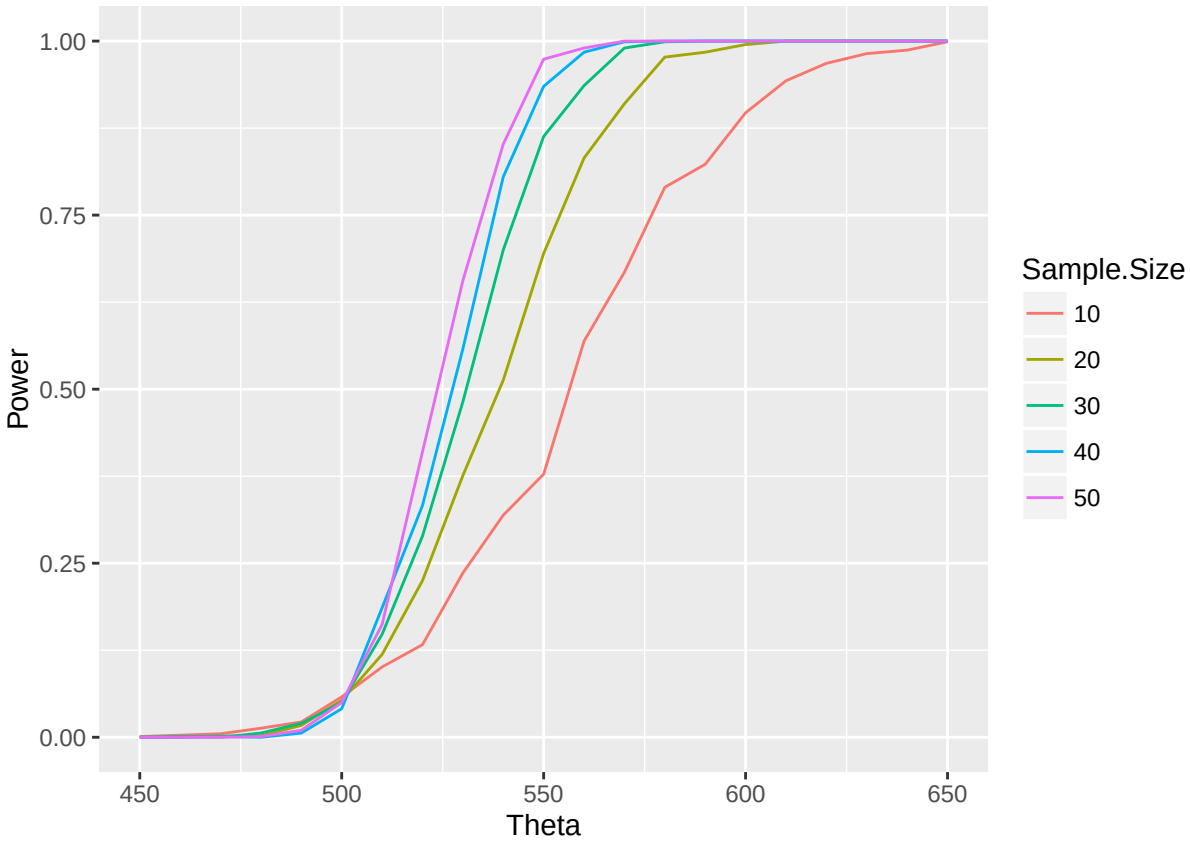

```r
tidy.results <- as.data.frame(tidy.results)
colnames(tidy.results) <- c("Sample.Size", "Theta", "Power")
tidy.results$Sample.Size <- as.factor(tidy.results$Sample.Size)
```

```r
ggplot(tidy.results) + geom_line(aes(x = Theta, y = Power, group = Sample.Size, color = Sample.Size))
```



6.4

Suppose that $X_1, ..., X_n$ are a random sample from a lognormal distribution with unknown parameters. Construct a 95% confidence interval for the parameter u. Use a Monte Carlo method to obtain an empirical estimate of the the confidence level.

7.1

7.4