INTRODUCTION TO CALCULUS: TAYLOR SERIES APPOXIMATIONS

1. Taylor Series Approximation

Many times we encounter functional forms that are difficult to calculate at arbitrary values. So, we attempt to approximate the function in terms of other functions that are easier to calculate. Polynomial functions are some of the easiest functions to evaluate and can be easily coded up in an algorithm. From this insight was born the concept of Taylor Series Approximation. Taylor Series Approximation is used to represent functions as an infinite sum of polynomial terms that are calculated using a function's derivatives evaluated at a single point. Taylor series is frequently used with a finite number of polynomial terms to approximate a function. The higher the degree of the Taylor polynomial used, the better the approximation.

Taylor's Theorem states that any function that is infinitely differentiable can be represented as a polynomial of the following form:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}}{2!}(x-a)^2 + \dots \tag{1}$$

where $f^{(n)(a)}$ represents the nth derivative of $f(x)$ evaluated at $x = a$. Another way of writing this series is as follows:

$$f(x) = f(a) + (x-a)f^{(1)}(a) + \dots + \frac{1}{n!}(x-a)^n f^{(n)}(a) + \int_a^x \frac{f^{(n+1)}(t)(x-t)^n dt}{n!} \tag{2}$$

which stops the infinite series at a particular value $n < \infty$. Let's prove this by mathematical induction. Let's assume that it is true for $n = k$, for some k. Let's now show that if it is true for k, it has to be true for $k+1$. Consider the last integral $\int_a^x \frac{f^{(n+1)}(t)(x-t)^{n+1} dt}{n!}$. For $n = k$, we get this integral as

$$\int_a^x \frac{f^{(k+1)}(t)(x-t)^k dt}{k!} \tag{3}$$

Let us now focus on that integral term. We can expand that using *Integration by parts* rule. Integration by parts states that:

$$uv' = uv - \int u'v \tag{4}$$

where u and v are functions and u' and v' are their derivatives. The above in the definite integral form becomes:

$$\left[uv'\right]_a^x = [uv]_a^x - \int_a^x u'v \tag{5}$$

where the expression $[uv]_a^x$ is simply $u(x)v(x) - u(a)v(a)$ or the function evaluated at a and x, the two end points.

If we consider $f^{(k+1)}(t)$ to be u and $(x-t)^k$ to be v', we can integrate by parts and rewrite the second term as:

$$\int_a^x f^{(k+1)}(t)(x-t)^k dt = \frac{1}{k!}\left[f^{(k+1)}(t)\frac{(x-t)^{k+1}}{-(k+1)}\right]_a^x + \frac{1}{k!}\int_a^x \frac{(x-t)^{k+1}}{k+1} f^{(k+2)}(t)dt \tag{6}$$

Pulling out the constants and rearranging terms, we get

$$\int_a^x f^{(k+1)}(t)(x-t)^k dt = -\frac{1}{(k+1)!}\left[f^{(k+1)}(t)(x-t)^{k+1}\right]_a^x + \int_a^x \frac{(x-t)^{k+1}}{(k+1)!} f^{(k+2)}(t)dt \tag{7}$$

Evaluating the first term using the *Fundamental Theorem of Calculus*, we get

$$-\frac{1}{(k+1)!}\left[f^{(k+1)}(t)(x-t)^{k+1}\right]_a^x = \frac{1}{(k+1)!}f^{(k+1)}(a)(x-a)^{(k+1)} - \frac{1}{(k+1)!}^{(k+1)}(x)(x-x)^{(k+1)} \tag{8}$$

Substituting back in the equation we started with, we get:

$$f(x) = f(a) + (x-a)f^{(1)}(a) + \ldots + \frac{1}{(k+1)!}(x-a)^{(k+1)}f^{(k)}(a) + \int_a^x \frac{f^{(k+2)}(t)(x-t)^{k+1}dt}{(k+1)!} \tag{9}$$

which is just Equation 2, written for $n = k+1$ So, if Equation 2 is true for $n = k$, using the *Integration by parts* rule, we can prove it to be true for $n = k+1$. Let's evaluate Equation 2 for $n = 0$. For this choice, we get

$$f(x) = f(a) + \int_a^x f^{(1)}(t)1dt \tag{10}$$

since $(x-a)^0 = 1$ and $0! = 1$. This equation is simply the Fundamental Theorem of Calculus which states:

$$\int_a^x f'(t)dt = f(x) - f(a) \tag{11}$$

Therefore, the Taylor series expression is true for $n = 0$ and we have shown that it is true for $n = k+1$ if it is true for $n = k$. Therefore, the Taylor Series Expansion is true as long as we have an infinitely differentiable function.

2. Taylor Series Approximation of some common functions

Let's examine some commonly used Taylor series expansions. Let's consider

$$f(x) = \frac{1}{1-x}, x \in (-1, 1) \tag{12}$$

where x is constrained to be between -1 and 1, exclusive. We can expand this using Taylor series as

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, x \in (-1, 1) \tag{13}$$

Another really popular expansion is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{14}$$

and also

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^n}{n}, x \in (-1, 1] \tag{15}$$