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CSC I6716
Assignment 3

Question 1

Prove that the vector from the viewpoint of a pinhole camera to the vanishing point (in the image plane) of a set of 3D parallel lines is parallel to the direction of the parallel lines. Please show steps of your proof.

If we take a point in space $P_0 = (X_0, Y_0, Z_0)$ and a direction V = (a,b,c) then this defines the line $P = P_0 + tV$ where $P = (X_0, Y_0, Z_0) + t$ <a|b|c>. If the viewpoint of the pinhole camera does not lie on the same line then the viewpoint and the line together define the interpretation plane. The image projection of the line is the intersection of this plane with the image projection plane.

Expressing the 3D line parametrically:

$X = X_0 + ta \qquad Y = Y_0 + tb \qquad Z = Z_0 + tc$

Using the perspective equations $x = \frac{fX}{Z}$ $y = \frac{fY}{Z}$:

$$x = \frac{X_0 + ta}{Z_0 + tc} f \qquad y = \frac{Y_0 + tb}{Z_0 + tc} f$$

If we have a set of parallel lines then the vanishing point is a point at infinity in the direction of the lines. The vanishing point is defined as $t \to \infty$ where the image points become:

$$x = \frac{\frac{X_0}{t} + a}{\frac{Z_0}{t} + c} f \ , \ y = \frac{\frac{Y_0}{t} + b}{\frac{Z_0}{t} + c} f \quad \to \quad x = \frac{\frac{X_0}{\infty} + a}{\frac{Z_0}{\infty} + c} f \ , y = \frac{\frac{Y_0}{\infty} + b}{\frac{Z_0}{\infty} + c} f \quad \to \quad = \frac{a}{c} f \ , \ y = \frac{b}{c} f$$

Since the vanishing point is only defined by the direction vector, this means that the starting point can be varied and any set of parallel lines will still have the same vanishing point. Because the vector from the viewpoint of a pinhole camera shares a vanishing point with a set of 3D parallel lines therefore it is also parallel to the direction of the parallel lines.

Question 2

Show the relation between any image point $(x_{im}, y_{im})^T$ of a plane (in the form of $(x_1,x_2,x_3)_T$ in projective space) and its corresponding point $(X_w, Y_w, Z_w)^T$ on the plane in 3D space can be represented by a 3x3 matrix.

The relation between an image point of a plane and its corresponding point in 3D space is represented by $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = M_{int} M_{ext} \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{pmatrix}$ where $\begin{pmatrix} x_{im} \\ y_{im} \end{pmatrix} = \begin{pmatrix} x_1/x_3 \\ x_2/x_3 \end{pmatrix}$

and $M_{ext} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & T_x \\ r_{21} & r_{22} & r_{23} & T_y \\ r_{31} & r_{32} & r_{33} & T_z \end{bmatrix}$ and $M_{int} = \begin{bmatrix} -f_x & 0 & o_x \\ 0 & -f_y & o_y \\ 0 & 0 & 1 \end{bmatrix}$

$$M = M_{int} M_{ext} = \begin{bmatrix} -f_x r_{11} + o_x r_{31} & -f_x r_{12} + o_x r_{23} & -f_x r_{13} + o_x r_{33} & -f_x T_x + o_x T_z \\ -f_y r_{21} + o_y r_{31} & -f_y r_{22} + o_y r_{23} & -f_y r_{23} + o_y r_{33} & -f_y T_y + o_y T_z \\ r_{31} & r_{32} & r_{33} & T_z \end{bmatrix}$$

Given the equation of the plane: $n_x X_w + n_y Y_w + Z_w = d \rightarrow Z_w = d - n_x X_w - n_y Y_w$

$$\begin{bmatrix} -f_x r_{11} + o_x r_{31} & -f_x r_{12} + o_x r_{23} & -f_x r_{13} + o_x r_{33} & -f_x T_x + o_x T_z \\ -f_y r_{21} + o_y r_{31} & -f_y r_{22} + o_y r_{23} & -f_y r_{23} + o_y r_{33} & -f_y T_y + o_y T_z \\ r_{31} & r_{32} & r_{33} & T_z \end{bmatrix} \begin{pmatrix} X_w \\ Y_w \\ d - n_x X_w - n_y Y_w \\ 1 \end{pmatrix}$$
$$=$$

$$\begin{bmatrix} -f_x r_{11} X_w + o_x r_{31} X_w - f_x r_{12} Y_w + o_x r_{23} Y_w - f_x r_{13}(d - n_x X_w - n_y Y_w) + o_x r_{33}(d - n_x X_w - n_y Y_w) - f_x T_x + o_x T_z \\ -f_y r_{21} X_w + o_y r_{31} X_w - f_y r_{22} Y_w + o_y r_{23} Y_w - f_y r_{23}(d - n_x X_w - n_y Y_w) + o_y r_{33}(d - n_x X_w - n_y Y_w) - f_y T_y + o_y T_z \\ r_{31} X_w + r_{32} Y_w + r_{33}(d - n_x X_w - n_y Y_w) + T_z \end{bmatrix} =$$

$$\begin{bmatrix} -f_x r_{11} X_w + o_x r_{31} X_w - f_x r_{12} Y_w + o_x r_{23} Y_w - f_x r_{13} d + f_x r_{13} n_x X_w + f_x r_{13} n_y Y_w + o_x r_{33} d - o_x r_{33} n_x X_w - o_x r_{33} n_y Y_w - f_x T_x + o_x T_z \\ -f_y r_{21} X_w + o_y r_{31} X_w - f_y r_{22} Y_w + o_y r_{23} Y_w - f_y r_{23} d + f_y r_{23} n_x X_w + f_y r_{23} n_y Y_w + o_y r_{33} d - o_y r_{33} n_x X_w - o_y r_{33} n_y Y_w - f_y T_y + o_y T_z \\ r_{31} X_w + r_{32} Y_w + r_{33} d - r_{33} n_x X_w - r_{33} n_y Y_w + T_z \end{bmatrix} =$$

$$\begin{bmatrix} X_w(-f_x r_{11} + o_x r_{31} + f_x r_{13} n_x - o_x r_{33} n_x) + Y_w(-f_x r_{12} + o_x r_{23} + f_x r_{13} n_y - o_x r_{33} n_y) - f_x r_{13} d + o_x r_{33} d - f_x T_x + o_x T_z \\ X_w(-f_y r_{21} + o_y r_{31} + f_y r_{23} n_x - o_y r_{33} n_x) + Y_w(-f_y r_{22} + o_y r_{23} + f_y r_{23} n_y - o_y r_{33} n_y) - f_y r_{23} d + o_y r_{33} d - f_y T_y + o_y T_z \\ X_w(r_{31} - r_{33} n_x) + Y_w(r_{32} - r_{33} n_y) + r_{33} d + T_z \end{bmatrix} =$$

$$\begin{bmatrix} -f_x r_{11} + o_x r_{31} + f_x r_{13} n_x - o_x r_{33} n_x & -f_x r_{12} + o_x r_{23} + f_x r_{13} n_y - o_x r_{33} n_y & -f_x r_{13} d + o_x r_{33} d - f_x T_x + o_x T_z \\ -f_y r_{21} + o_y r_{31} + f_y r_{23} n_x - o_y r_{33} n_x & -f_y r_{21} + o_y r_{31} + f_y r_{23} n_x - o_y r_{33} n_x & -f_y r_{21} + o_y r_{31} + f_y r_{23} n_x - o_y r_{33} n_x \\ r_{31} - r_{33} n_x & r_{31} - r_{33} n_x & r_{33} d + T_z \end{bmatrix} \begin{pmatrix} X_w \\ Y_w \\ 1 \end{pmatrix}$$

$$\begin{bmatrix} -f_x(r_{11} - r_{13} n_x) + o_x(r_{31} - r_{33} n_x) & -f_x(r_{12} - r_{13} n_y) + o_x(r_{23} - r_{33} n_y) & f_x(r_{13} d - T_x) + o_x(r_{33} d + T_z) \\ -f_y(r_{21} - r_{32} n_x) + o_y(r_{22} - r_{32} n_y) & -f_y(r_{22} - r_{32} n_y) + o_y(r_{23} - r_{33} n_y) & f_y(r_{32} d - T_y) + o_y(r_{33} d + T_z) \\ r_{31} - r_{33} n_x & r_{32} - r_{33} n_y & r_{33} d + T_z \end{bmatrix} \begin{pmatrix} X_w \\ Y_w \\ 1 \end{pmatrix}$$

Question 3

Prove the Orthocenter Theorem by geometric arguments: Let T be the triangle on the image plane defined by the three vanishing points of three mutually orthogonal sets of parallel lines in space. Then the image center is the orthocenter of the triangle T (i.e., the common intersection of the three altitudes.

The Orthocenter Theorem states that given 3 mutually orthogonal sets of parallel lines in an image, a triangle T is formed on the physical image plane that is defined by the 3 vanishing points of these 3 sets of lines, then the image center is the common intersection of the 3 altitudes of T.

The vanishing points of 3 mutually orthogonal sets of parallel lines are the vertices of the triangle T. The center of projection of a camera is O and L_1 L_2 L_3 are the three sets of mutually orthogonal parallel lines, and V_1 V_2 V_3 are the three vanishing points which form the triangle T.

In the previous question it is proved that the vector from the center of projection of a camera to the vanishing point of a set of parallel lines is also parallel to their direction. Therefore OV_1 is perpendicular to V_2V_3, OV_2 is perpendicular to V_1V_3, and OV_3 is perpendicular to V_1V_2. The altitudes from each side of the triangle to the opposite vanishing point are each perpendicular to each other. If the center of the image is o then the line from the center of projection of the camera to the center of the image to the image center -- Oo is perpendicular to the image plane.

If you do not know the focal length of the camera, can you still find the image center (together with the focal length) using the Orthocenter Theorem?

You do not need the focal length information to find the image center using the Orthocenter Theorem since in the above proof we don't use any of the camera parameters to find the three vanishing points. You can then find the focal length using the vanishing points.

If you do not know the aspect ratio and the focal length of the camera, can you still find the image center using the Orthocenter Theorem?

You do need to assume that the aspect ratio is known since we need to use the orthogonal relations of the altitude. If we did not know the aspect ratio the image center will not be the correct one.

Question 4: Camera Calibration Programming Exercise

Using the camera calibration technique the Direct Parameter Method we can estimate the extrinsic and intrinsic parameters given a set of corresponding 2D and 3D points.

The Direct Parameter Method algorithm involves measuring N 3D coordinates and their corresponding image points. Here we generate 32 points on two surfaces of a cube, which can be seen plotted below. By designing a virtual camera we capture an image of the two surfaces of the cube. The assumed parameters are f = 16mm, image center = (256,256), (s_x,s_y) = (8.8/512,6.6/512), 4 meters away at a 30 degree tilt.

The Direct Parameter Method algorithm is as follows:

1. Estimate image center
2. Measure N 3D coordinates
3. Locate their corresponding image points
4. Build matrix **A** of a homogeneous system **Av** = **0**
5. Find solution **v** by computing SVD of **A**
6. Determine aspect ration α and scale $|\gamma|$
7. Recover the first 2 rows of **R** and first 2 components of T up to a sign s
8. Determine sign of s of γ
9. Compute 3rd row of **R** and enforce orthogonality constraint by SVD
10. Solve T_z and f_x using Least Square and SVD

1. Estimate image center

The image center can be estimated using the Orthocenter Theorem, which states that given 3 mutually orthogonal sets of parallel lines in an image – a triangle T can be formed that is defined by the 3 vanishing points of these 3 sets of lines. The image center is the common intersection of the 3 altitudes of T.

2. Measure N 3D coordinates

The N 3D coordinates are generated as 32 points on 2 surfaces of a cube. Fig. 1 below displays the 32 generated 3D points.

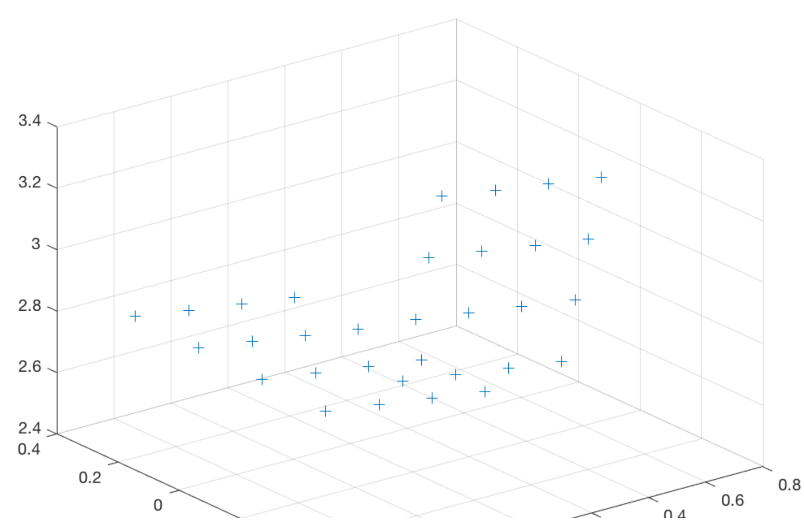

Figure 1. Plot of 32 3D points

3. Locate their corresponding image points

The corresponding image points are found by designing a virtual camera with parameters: f = 16mm, image center = (256,256), (s_x,s_y) = (8.8/512,6.6/512), 4 meters away at a 30 degree tilt. Fig. 2 below displays the 32 corresponding 2D images points.

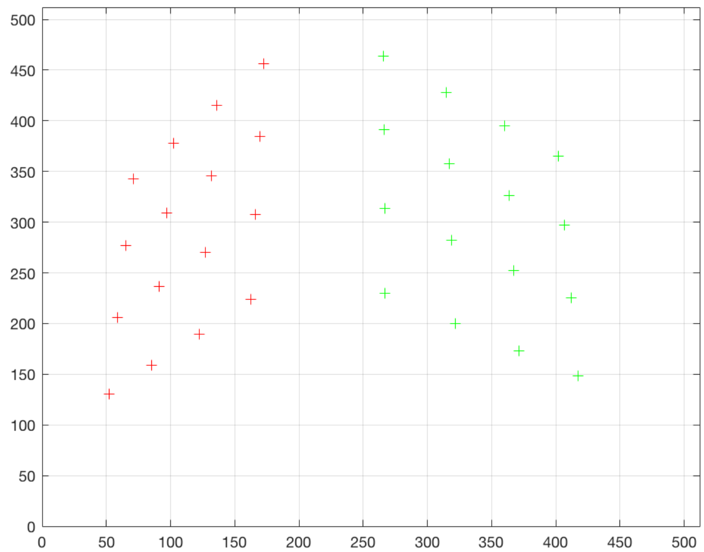

Figure 2. Plot of corresponding 32 image points

4. Build matrix A of a homogeneous system Av = 0

The transformation from the world coordinate system to the image coordinate system is defined as:

$$x' = x_{im} - o_x = -\frac{f_x(r_{11}X_w + r_{12}Y_w + r_{13}Z_w + T_x)}{r_{31}X_w + r_{32}Y_w + r_{33}Z_w + T_z}$$

$$y' = y_{im} - o_y = -\frac{f_y\left(r_{21}X_w + r_{22}Y_w + r_{23}Z_w + T_y\right)}{r_{31}X_w + r_{32}Y_w + r_{33}Z_w + T_z}$$

which leads to:

$$\frac{f_y\left(r_{21}X_w + r_{22}Y_w + r_{23}Z_w + T_y\right)}{y'} = \frac{f_x(r_{11}X_w + r_{12}Y_w + r_{13}Z_w + T_x)}{x'}$$

$$\rightarrow x' f_y\left(r_{21}X_w + r_{22}Y_w + r_{23}Z_w + T_y\right) = y' f_x(r_{11}X_w + r_{12}Y_w + r_{13}Z_w + T_x)$$

$$\rightarrow x'\left(r_{21}X_w + r_{22}Y_w + r_{23}Z_w + T_y\right) = y'\alpha(r_{11}X_w + r_{12}Y_w + r_{13}Z_w + T_x) \quad \text{where } a = \frac{f_x}{f_y}$$

$\rightarrow\ x_iX_ir_{21} + x_iY_ir_{22} + x_iZ_ir_{23} + x_iT_y - y_iX_i(\alpha r_{11}) - y_iY_i(\alpha r_{12}) - y_iZ_i(\alpha r_{13}) - y_i(\alpha T_x) = 0$
$\rightarrow\ x_iX_iv_1 + x_iY_iv_2 + x_iZ_iv_3 + x_iv_4 - y_iX_iv_5 - y_iY_iv_6 - y_iZ_iv_7 - y_iv_8 = 0$

Therefore, the homogeneous system of equations **Av = 0** has A:

$$A = [\, x_iX_i \quad x_iY_i \quad x_iZ_i \quad x_i \quad -y_iX_i \quad -y_iY_i \quad -y_iZ_i \quad -y_i \,]$$

5. Find solution v by computing SVD of A

In order to solve for the 8 unknowns in **v** the SVD of A can be calculated:

$$\mathbf{A} = \mathbf{UDV}^T$$

where the solution is proportional to the 8th row of $\mathbf{V}^T$.

6. Determine aspect ration α and scale |γ|

The scale $|\boldsymbol{\gamma}| = \sqrt{\boldsymbol{v}_1^2 + \boldsymbol{v}_2^2 + \boldsymbol{v}_3^2}$

The aspect ratio $\boldsymbol{\alpha} = \dfrac{\sqrt{v_5^2 + v_6^2 + v_7^2}}{\sqrt{v_1^2 + v_2^2 + v_3^2}}$

7. Recover the first 2 rows of R and first 2 components of T up to a sign s

Since $v_1 = r_{21}$, $v_2 = r_{22}$, $v_3 = r_{23}$, $v_5 = \alpha r_{11}$, $v_6 = \alpha r_{12}$, $v_7 = ar_{13}$, and the aspect ratio is found - the first 2 rows of R can easily be determined from vector **v**.

Since $v_4 = T_y$ and $v_8 = \alpha T_x$, the first 2 components of T can also be easily determined from **v** and the aspect ratio.

8. Determine sign of s of γ

In order to determine sign s:

Xc is calculated such that $Xc = r_{11}X_w + r_{12}Y_w + r_{13}Z_w + T_x$ by assuming s is positive for the first 3D point. If Xc has the same sign as the corresponding image point then the sign guess is correct and *s* is positive – otherwise sign *s* is negative.

The first 2 rows of R and the first 2 components of T are the recalculated by multiplying by sign s.

9. Compute 3rd row of R and enforce orthogonality constraint by SVD

The orthogonality constraint of R is enforced by using the SVD where:

$$\boldsymbol{UDV} = SVD(\boldsymbol{R})$$

$$new\ \boldsymbol{R} = \boldsymbol{UV^T}$$

10. Solve T_z and f_x using Least Square and SVD

In order to solve for T_z and f_x the following system of equation is formed from:

$$xT_z + (r_{11}X_w + r_{12}Y_w + r_{13}Z_w + T_x)f_x = -x(r_{31}X_w + r_{32}Y_w + r_{33}Z_w)$$

$$[x \quad r_{11}X_w + r_{12}Y_w + r_{13}Z_w + T_x]\begin{pmatrix} T_z \\ f_x \end{pmatrix} = [-x(r_{31}X_w + r_{32}Y_w + r_{33}Z_w)]$$

The solution can be found using Least Squares:

$$\begin{pmatrix} T_z \\ f_x \end{pmatrix} = (A^TA)^{-1}A^Tb$$

The SVD can be used to find the pseudo-inverse.
Once T_z and f_x are found f_y can be found since $f_y = \frac{f_x}{a}$

Once all the unknowns have been calculated the projective matrix can be formed (consisting of the intrinsic and extrinsic parameters) and used to re-project the 3D coordinates to image coordinates.

Fig. 3 below shows the re-projection, which can be compared to the original plot in Fig. 2 of the image points. The re-projection is identical to that of Fig. 2.

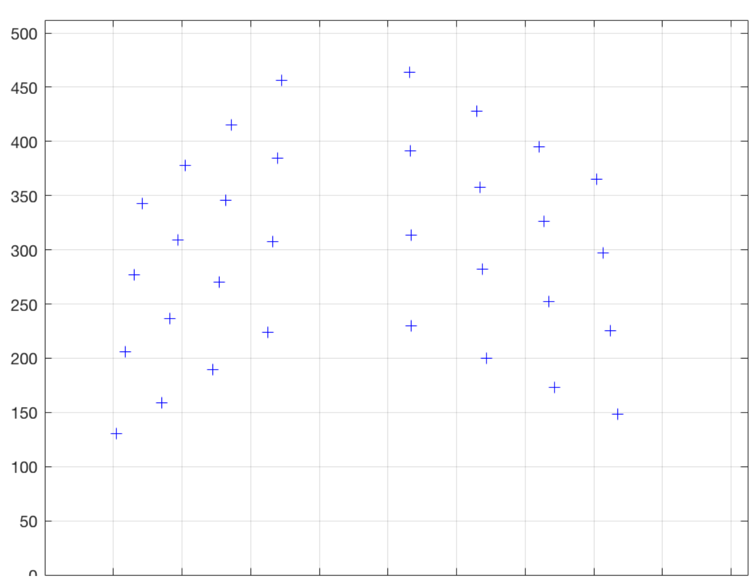

Figure 3. Re-projected image points

Comparison of calculated values with original known parameters:

	Calculated Parameter	**Known Parameter**
Aspect ratio	0.7500	0.7500
fx	930.09091	930.09091
fy	1.2412e+3	1.2412e+3