Neoclassical Growth with Labor-Leisure Choice

Ramsey-Cass-Koopmans Model

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Overview

Budget Constraint from Lecture 2

$$c_t + a_{t+1} = (1 + r)a_t + y_t$$

Last lecture: Endogenous r

- Assets (a) was defined as capital (k) used by firms to produce output
- Agents savings choice affected capital accumulation
- Interest rate determined by marginal product of capital
- income "y_t" determined by capital choice

Today: Add labor income to y_t

- Endogenous income through labor choice
- Agent choose both investment and hours worked
- Firms use both capital and labor in production

Why Add Labor-Leisure Choice?

Limitations of Exogenous Labor Model:

- Cannot analyze policies that affect work incentives
- Missing key margin of adjustment (hours worked)
- Unrealistic assumption for short-run analysis

Benefits of Endogenous Labor Supply:

- Agents choose optimal work-leisure balance
- Can analyze tax policies on labor income
- More realistic model of labor markets
- Richer transition dynamics

Key Change: Agents now choose both consumption and labor supply

Note: This is the foundation for most modern macro models

Model Structure Overview

Agents (households):

- Choose consumption C_t and labor supply L_t
- Own capital stock K_t and rent it to firms
- Maximize lifetime utility

Firms:

- Choose capital demand K_t^d and labor demand L_t^d
- Take factor prices r_t, w_t as given
- Maximize profits each period

Market clearing:

- Capital: $K_t = K_t^d$ (agent capital supply = firm capital demand)
- Labor: $L_t = L_t^d$ (agent labor supply = firm labor demand)
- Goods: $Y_t = C_t + I_t$ (output = consumption + investment)

Preferences

Representative agent lifetime utility:

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - \ell_t)$$

where:

- $\beta \in (0,1)$ = discount factor
- C_t = consumption in period t
- $\ell_t \in [0,1]$ = labor supply in period t
- $1 - \ell_t$ = leisure in period t

Utility function properties:

- $u_c > 0$, $u_{cc} < 0$ (diminishing marginal utility of consumption)
- $u_{1-\ell} > 0$, $u_{(1-\ell)(1-\ell)} < 0$ (diminishing marginal utility of leisure)
- Time endowment normalized to 1

Technology

Aggregate production function:

$$Y_t = F(K_t, L_t)$$

Standard assumptions:

- Constant returns to scale: $F(\lambda K, \lambda L) = \lambda F(K, L)$
- Positive marginal products: $F_K > 0$, $F_L > 0$
- Diminishing returns: $F_{KK} < 0$, $F_{LL} < 0$
- Inada conditions: $\lim_{K \to 0} F_K = \infty$, $\lim_{K \to \infty} F_K = 0$

Capital accumulation:

$$K_{t+1} = (1 - \delta)K_t + I_t$$

where $\delta \in (0, 1)$ is the depreciation rate and I_t is investment.

Review of State Variables

A **state variable** is a variable whose value:

1. Carries over from period to period (has persistence)
2. Cannot be chosen freely in the current period (predetermined)
3. Summarizes relevant history for decision-making
4. Affects future constraints and opportunities

Previously we had the aggregate capital stock K_t as a state variables.

- is aggregate K_t still a state variable?

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- is aggregate K_t still a state variable? Yes
- is aggregate L_t a state variable? No

Why Capital is a State Variable

Capital (K_t) satisfies all criteria:

1. Persistence: $K_{t+1} = (1 - \delta)K_t + I_t$
   - Today's capital stock affects tomorrow's capital stock
   - Cannot be changed instantly
2. Predetermined: K_t is given at the beginning of period t
   - Result of past investment decisions
   - Agent must take K_t as given when making period t decisions
3. Summarizes relevant history
   - K_t embeds all past investment choices $\{I_0, I_1, \ldots, I_{t-1}\}$
   - Don't need to know those individual choices, just their cumulative effect (K_t)
4. Affects constraints and future opportunities:
   - Determines production capacity and wealth
   - Higher $K_t \rightarrow$ higher output potential $\rightarrow$ more resources available
   - Investment today determines future capital

Why Labor is NOT a State Variable

Labor (L_t) fails the key state variable criteria:

1. **No persistence:** L_t doesn't affect L_{t+1} directly
   - Each period, agent chooses labor supply fresh
   - No "labor accumulation equation"
2. **Freely chosen:** L_t is chosen optimally each period
   - Not predetermined by past decisions
   - Agent can choose any $L_t \in [0, 1]$ in period t
3. **No carry-over:** L_t doesn't create an asset that lasts into next period
   - Unlike capital, labor doesn't "build up" over time
   - Flow variable: labor services consumed in production each period

Economic intuition:

- **Capital**: "Slow" variable $\rightarrow$ investment decisions have long-term consequences
- **Labor**: "Fast" variable $\rightarrow$ period-by-period decisions without future constraints

Labor is a **control variable** (chosen freely each period), not a state variable (predetermined).

Agent's Dynamic Problem

State variable: K_t (capital stock at beginning of period t)
Control variables: c_t (consumption), ℓ_t (labor supply)

Value Function:

$$V(k, K) = \max_{c,\ell,k'} \left\{ u(c, 1-\ell) + \beta V(k', K') \right\}$$

subject to:

$$\begin{aligned} &k' = (1-\delta)k + rk + w\ell - c \\ &\ell \in [0,1], \quad k' \geq 0, \quad c \geq 0 \\ &K' = G(K) \end{aligned}$$

Firm's Problem

Define:

- K_t^d capital demanded in period t
- L_t^d labor demanded in period t

Competitive firm maximizes profits each period, taking price as given:

$$\max_{K_t^d, L_t^d} \left\{ F(K_t^d, L_t^d) - r_t K_t^d - w_t L_t^d \right\}$$

Definition of Recursive Competitive Equilibrium

A **Recursive Competitive Equilibrium** consists of:

1. Value function: $V(k, K)$
2. Policy functions: $g_c(k, K)$, $g_\ell(k, K)$, and $g_k(k, K)$
3. Factor price functions: $r(K, L)$ and $w(K, L)$
4. Aggregate law of motion: $G(K)$

Such that:

(i) Agent optimization: $V(k, K)$, $g_c(k, K)$, $g_\ell(k, K)$, and $g_k(k, K)$ solve the household problem

(ii) Firm optimization: Factor prices satisfy: (This implies market clearing.)

$$
\begin{aligned}
r(K, L) &= F_K(K, L) \\
w(K, L) &= F_L(K, L)
\end{aligned}
$$

(iii) Consistency:

$$K' = g_k(k, K) = G(K) \quad L = g_\ell(k, K)$$

Household First-Order Conditions

FOC with respect to consumption:

$$u_c(c, 1-\ell) = \beta V'(k', K')$$

FOC with respect to labor supply:

$$u_{1-\ell}(c, 1-\ell) = \beta V'(k', K') w(K, L)$$

Envelope condition:

$$V'(k, K) = u_c(c, 1-\ell)[r(K, L) + 1 - \delta]$$

Economic interpretation:

- **Consumption**: Marginal utility = discounted marginal value of wealth
- **Labor**: Marginal disutility of work = discounted value of wage income
- **Capital**: Marginal value today = discounted marginal value tomorrow times gross return

Euler Equations

Consumption Euler equation:

$$u_c(c, 1-\ell) = \beta u_c(c', 1-\ell')[r(K', L') + 1 - \delta]$$

Labor supply condition (intratemporal):

$$\frac{u_{1-\ell}(c, 1-\ell)}{u_c(c, 1-\ell)} = w(K, L)$$

Key insights:

- **Intertemporal**: Consumption growth depends on interest rate vs. time preference
- **Intratemporal**: Labor supply equates marginal rate of substitution to wage
- Both conditions must hold simultaneously in equilibrium

Taking Stock

With an endogenous labor choice

- aggregate capital remains the only state variable
- labor becomes a new control variable
- equilibrium consists of two prices r, w and both labor and capital markets clear

Agents optimize on two margins simultaneously:

- Intertemporal: Consumption today vs. consumption tomorrow (Euler equation)
- Intratemporal: Consumption today vs. leisure today (labor supply condition)

Example with CRRA Utility Function

Utility specification:

$$u(c, 1-\ell) = \frac{[c^{\gamma}(1-\ell)^{1-\gamma}]^{1-\theta}}{1-\theta}$$

where:

- $\gamma \in (0,1)$ = weight on consumption vs. leisure
- $\theta > 0$ = coefficient of relative risk aversion
- θ = relative risk aversion

Marginal utilities:

$$u_c = \gamma c^{\gamma(1-\theta)-1}(1-\ell)^{(1-\gamma)(1-\theta)}$$
$$u_{1-\ell} = (1-\gamma)c^{\gamma(1-\theta)}(1-\ell)^{(1-\gamma)(1-\theta)-1}$$

Labor Supply Function

Intratemporal condition:

$$\frac{u_{1-\ell}(c, 1-\ell)}{u_c(c, 1-\ell)} = F_L(K, L)$$

With CRRA utility:

$$\frac{1-\gamma}{\gamma}\frac{c}{1-\ell} = F_L(K, L)$$

Solving for labor supply:

$$\ell = 1 - \frac{(1-\gamma)c}{\gamma F_L(K, L)}$$

Key insight: Labor supply depends on consumption and the marginal product of labor, which itself depends on K and L. This creates an implicit relationship that must be solved simultaneously with other equilibrium conditions.

Cobb-Douglas Production Example

Specific production function: $F(K, L) = K^\alpha L^{1-\alpha}$
Factor prices:

$$F_K(K, L) = \alpha K^{\alpha-1} L^{1-\alpha}$$
$$F_L(K, L) = (1-\alpha) K^\alpha L^{-\alpha}$$

Labor supply condition becomes: ($\ell = L$ in equilibrium)

$$\frac{1-\gamma}{\gamma} \frac{c}{1-L} = (1-\alpha) K^\alpha L^{-\alpha}$$

Rearranging:

$$L^{1-\alpha} = \frac{(1-\gamma)c}{\gamma(1-\alpha)K^\alpha} + L$$

This implicit equation for L, which depends on aggregate capital (another equilibrium object), and must be solved numerically in general. Easiest to solve in steady state.

Steady State Analysis

In steady state: $K' = K = K^*$, $c' = c = c^*$, $L' = L = L^*$

Euler Equation: $1 = \beta[F_K(K^*, L^*) + 1 - \delta]$

Steady state conditions:

$$1 = \beta[\alpha(K^*)^{\alpha-1}(L^*)^{1-\alpha} + 1 - \delta] \tag{1}$$

$$\frac{1-\gamma}{\gamma}\frac{c^*}{1-L^*} = (1-\alpha)(K^*)^{\alpha}(L^*)^{-\alpha} \tag{2}$$

$$K^* = (1-\delta)K^* + F(K^*, L^*) - c^* \tag{3}$$

Solution procedure:

- Three equations, three unknowns
- Matlab: fslove

Effect of Patience (β)

More patient agents (β increases):
Euler Equation: (solve for interest rate)

$$F_K(K^*, L^*) = \frac{1}{\beta} - 1 + \delta$$

- Higher $\beta \rightarrow$ lower $F_K(K^*, L^*)$
- With CRS: need higher K^*/L^* ratio $\rightarrow$ higher K^*

General equilibrium effects:

- Higher $K^* \rightarrow$ higher marginal product of labor $F_L(K^*, L^*)$
- Higher $F_L \rightarrow$ higher labor supply L^* (from intratemporal condition)
- **Amplification**: Patient agents accumulate more capital AND work more
- Higher output: $Y^* = F(K^*, L^*)$ increases through both channels

Key insight: Factor complementarity creates positive feedback between capital accumulation and labor supply decisions.

Risk Aversion (Review)

Relative Risk Aversion (RRA):

- **Definition:** Measures how much an agent dislikes risk relative to their wealth level
- **Mathematical:** $\theta = -\frac{c \cdot u''(c)}{u'(c)}$
- **Economic question:** "How much extra expected return do I need to accept a risky investment?"
- **Higher θ:** More risk averse $\rightarrow$ prefer safer assets

Role of θ in the model

- High θ: Prefer smooth consumption paths
- Low θ: Willing to have volatile consumption

Intertemporal Elasticity of Substitution

Intertemporal Elasticity of Substitution (IES):

- **Definition:** Measures willingness to substitute consumption across time periods
- **Mathematical:** $\sigma = -\frac{d\ln(c_{t+1}/c_t)}{d\ln(1+r)}$
- **Economic question:** "How much do I change my consumption growth when interest rates change?"
- **Higher σ:** More willing to substitute $\rightarrow$ consumption growth more sensitive to returns

Role of σ in the model

- High σ: Consumption growth very sensitive to returns vs. impatience
- Low σ: Consumption growth less responsive to interest rates

Key Relationship with CRRA Utility: $\theta = \frac{1}{\sigma}$

- Higher risk aversion $\Leftrightarrow$ Lower intertemporal substitution

Behavioral Interpretation: Risk Aversion vs. IES

High Risk Aversion, Low IES (θ large, σ small)

- **Agent thinking:** "I really don't like uncertainty and variability"
- **Risk behavior:** "I won't invest even for high expected returns"
- **Time behavior:** "I won't change my consumption much even for high interest rates"

Low Risk Aversion, High IES (θ small, σ large)

- **Agent thinking:** "I'm comfortable with ups and downs"
- **Risk behavior:** "I'll take risks for higher expected returns"
- **Time behavior:** "I'll shift consumption across time for better interest rates"

Key insight: Both stem from the same source – **curvature of utility function**

- Highly curved utility $\rightarrow$ dislike both risk AND consumption variation over time
- Less curved utility $\rightarrow$ comfortable with both risk AND consumption flexibility

Effect of Risk Aversion (θ)

Higher risk aversion (θ increases, $\sigma = 1/\theta$ decreases):

Steady state effects:

- Steady-state conditions independent of θ
- Same (K^*, L^*, c^*) regardless of risk aversion
- Only affects transition dynamics, not destination

Transition dynamics effects: (θ in Euler equation when $c \neq c'$)

- Less willing to substitute consumption across time
- Smoother consumption profile during adjustment
- Labor supply adjusts more gradually (through consumption linkage)
- Slower convergence to steady state

Economic intuition: Risk aversion affects the *speed* of adjustment but not the *long-run target* of the economy.

How does wealth (capital stock) affect labor supply decisions?

Two competing effects:

- **Income effect**: Higher wealth $\rightarrow$ can afford more leisure $\rightarrow$ work less
- **Substitution effect**: Higher wealth $\rightarrow$ higher wages $\rightarrow$ leisure more expensive $\rightarrow$ work more

Key insight: In general equilibrium, wealth affects wages through capital accumulation:

$$K \uparrow \Rightarrow F_L(K, L) \uparrow \Rightarrow w \uparrow$$

First: How does more capital affect consumption?

How does wealth (capital stock) affect labor supply decisions?

Two competing effects:

- **Income effect**: Higher wealth $\rightarrow$ can afford more leisure $\rightarrow$ work less
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Key insight: In general equilibrium, wealth affects wages through capital accumulation:

$$K \uparrow \Rightarrow F_L(K, L) \uparrow \Rightarrow w \uparrow$$

First: How does more capital affect consumption? $c \uparrow$

- more capital increase consumption
- agents have more income so they consume more
- think about the saddle path

Income Effect of Wealth on Labor Supply

The **income effect** occurs through consumption. In the current period the agent faces the trade-off:

> *"Should I work one more hour (less leisure) to earn wage income and consume more, or should I enjoy more leisure and consume less?"*

This is a within-period decision between current consumption and current leisure.

Income Effect

$$K_t \uparrow \Rightarrow \text{wealth} \uparrow \Rightarrow c_t \uparrow \Rightarrow L_t \downarrow$$

Logic:

- Higher capital $\rightarrow$ richer $\rightarrow$ can afford higher consumption
- Higher consumption $\rightarrow$ less need to work for additional consumption

Substitution Effect on Labor Supply

The **substitution effect** occurs through wages:

$$K_t \uparrow \Rightarrow F_L(K_t, L_t) \uparrow \Rightarrow w_t \uparrow \Rightarrow L_t \uparrow$$

Logic:

- Higher capital $\rightarrow$ higher marginal product of labor $\rightarrow$ higher wage
- Higher wage $\rightarrow$ leisure more expensive $\rightarrow$ work more

Income Effects, Substitution Effect, General Equilibrium

From the labor supply function:

$$L_t = 1 - \frac{(1-\gamma)c_t}{\gamma w_t}$$

Effect of higher capital stock K_t:

1. Direct income effect (through consumption):

$$\frac{\partial L_t}{\partial c_t} = -\frac{1-\gamma}{\gamma w_t} < 0$$

Higher consumption $\rightarrow$ *less work*

2. Substitution effect (through wages):

$$\frac{\partial L_t}{\partial w_t} = \frac{(1-\gamma)c_t}{\gamma w_t^2} > 0$$

Higher wage $\rightarrow$ *more work*

3. General equilibrium linkage:

$$K_t \uparrow \Rightarrow w_t = F_L(K_t, L_t) \uparrow \text{ and } c_t \uparrow$$

Net Wealth Effect on Labor Supply

Total effect of higher capital on labor supply:

$$\frac{dL_t}{dK_t} = \frac{\partial L_t}{\partial c_t}\frac{dc_t}{dK_t} + \frac{\partial L_t}{\partial w_t}\frac{dw_t}{dK_t}$$

Substituting our expressions:

$$\frac{dL_t}{dK_t} = -\frac{1-\gamma}{\gamma w_t}\frac{dc_t}{dK_t} + \frac{(1-\gamma)c_t}{\gamma w_t^2}\frac{dw_t}{dK_t}$$

Economic interpretation:

- **First term**: Income effect (negative)
- **Second term**: Substitution effect (positive)
- **Net effect**: Depends on relative magnitudes and parameter γ

Key result: With complementary capital and labor ($F_{KL} > 0$), the substitution effect typically dominates $\rightarrow$ higher capital leads to more work.

The Parameter γ: Consumption vs. Leisure Weight

Utility function: $u(c, 1-\ell) = \frac{[c^\gamma(1-\ell)^{1-\gamma}]^{1-\theta}}{1-\theta}$
Interpretation of γ:

- γ close to 1: Strong preference for consumption $\rightarrow$ work more
- γ close to 0: Strong preference for leisure $\rightarrow$ work less
- $\gamma = 1$: Utility function from last lecture $\rightarrow L = 1$

Labor supply function:

$$L_t = 1 - \frac{(1-\gamma)c_t}{\gamma w_t}$$

Effects of higher γ:

- **Direct**: $\frac{\partial L_t}{\partial \gamma} > 0$ (more work-oriented)
- **Indirect**: Affects equilibrium c_t and w_t through general equilibrium

Steady State Effects of γ

Steady state conditions:

$$1 = \beta[F_K(K^*, L^*) + 1 - \delta] \tag{4}$$

$$\frac{1-\gamma}{\gamma}\frac{c^*}{1-L^*} = F_L(K^*, L^*) \tag{5}$$

$$c^* = F(K^*, L^*) - \delta K^* \tag{6}$$

Key insight: γ affects ALL steady-state variables (K^*, L^*, c^*) simultaneously through general equilibrium.

Higher γ (stronger preference for consumption):

1. More willing to work $\rightarrow$ higher L^* for given (K^*, c^*)
2. Higher labor supply $\rightarrow$ higher output $\rightarrow$ supports higher consumption and capital
3. General equilibrium: (K^*, L^*, c^*) all increase

Phase Diagram Setup

Challenge: System has 3 variables (K_t, c_t, L_t) but we can only draw 2D phase diagrams.

Solution: Use the intratemporal condition to eliminate labor:

$$L_t = 1 - \frac{(1-\gamma)c_t}{\gamma F_L(K_t, L_t)}$$

This gives us L_t as an implicit function of (K_t, c_t).

Reduced system: (K_t, c_t) with $L_t = L(K_t, c_t)$ determined implicitly.
Dynamic equations:

$$K_{t+1} = F(K_t, L(K_t, c_t)) - c_t + (1-\delta)K_t$$
$$u_c(c_t, 1 - L(K_t, c_t)) = \beta u_c(c_{t+1}, 1 - L(K_{t+1}, c_{t+1}))[F_K(K_{t+1}, L(K_{t+1}, c_{t+1}) + 1 - \delta]$$

Decrease in γ

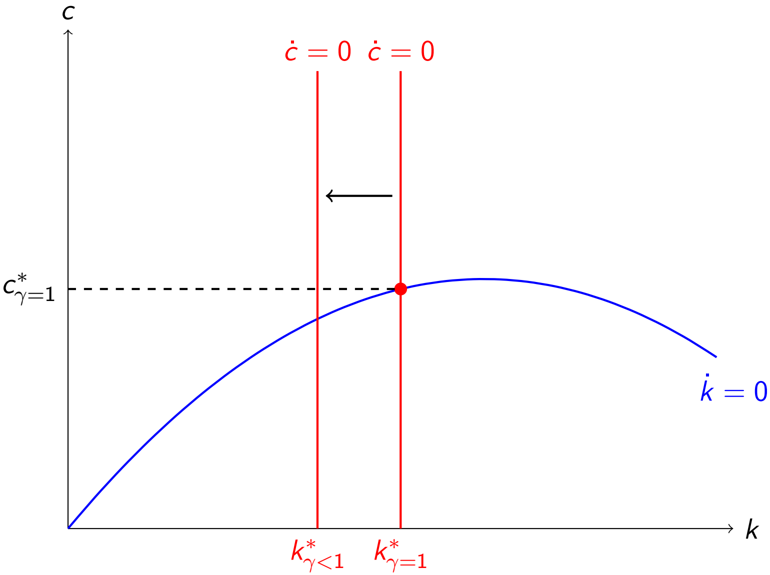
Last time: $\gamma = 1 \to L = 1$, we solved the phase diagram $\dot{K} = 0$ and $\dot{c} = 0$

Now: $\gamma < 1 \to$ endogenous labor choice

What happens to the $\dot{c} = 0$ locus? $F_K(K^*, L^*) = \frac{1}{\beta} - 1 + \delta$

- Lower $\gamma \to$ stronger preference for leisure $\to$ lower L^* for any (K^*, c^*)
- Lower $L^* \to$ lower output $F(K^*, L^*) \to$ fewer resources available
- Lower L^* also $\to$ lower $F_K(K^*, L^*)$ given K^* (factor complementarity)
- To restore $F_K(K^*, L^*) = \frac{1}{\beta} - 1 + \delta$, need lower K^*
- Lower $K^* \to$ the $\dot{c} = 0$ locus (vertical line) shifts left

Phase Diagram



Decrease in γ

What happens to the $\dot{K} = 0$ locus? $c = F(K, L(K, c)) - \delta K$

Direct Effect on Labor Supply

- Lower $\gamma \rightarrow$ stronger preference for leisure
- $\Rightarrow$ Lower L for any given (K, c) combination

Impact on the Locus For any given capital stock K:

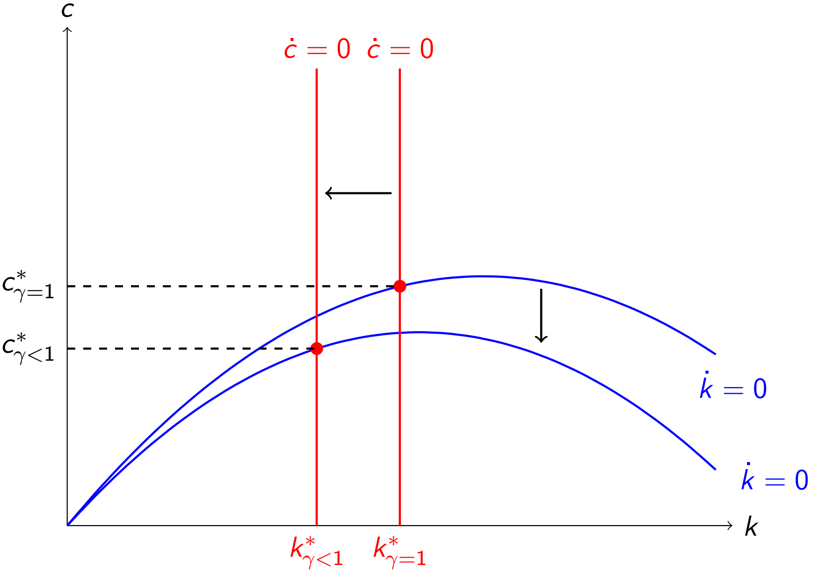
1. Lower labor supply $L \rightarrow$ lower output $F(K, L)$
2. Lower output $\rightarrow$ less available for consumption when $\dot{K} = 0$
3. The locus shifts down: lower c sustainable for each K

Peak of the Locus $\frac{\partial c}{\partial K} = 0$

- Peak occurs where $F_K(K, L) + F_L(K, L)\frac{\partial L}{\partial K} = \delta$
- With lower γ: peak shifts to lower K (leftward) and lower height (downward)

Result: Entire $\dot{K} = 0$ locus shifts **down and to the left**

Phase Diagram



Ramsey Model: With vs. Without Labor Choice

Aspect	Without Labor Choice	With Labor Choice
Choice Variables	C_t only	C_t and L_t
Optimization	1 margin: save vs. consume	2 margins: save vs. consume AND work vs. leisure
Factor Markets	Capital market only	Capital AND labor markets
Policy Analysis	Capital policies only	Capital AND labor policies
Steady State	Depends on β, ρ, δ	Also depends on labor preference γ
Dynamics	C and K adjust	C, L, and K adjust
Wealth Effects	Only through consumption	Through both consumption and wages

Key Takeaways

- **Amplification:** Labor choice creates feedback loops - capital policies affect wages, which affect work incentives
- **Complete markets:** Both factor markets (capital and labor) clear simultaneously
- **Policy relevance:** Can analyze tax policies on both capital and labor income
- **Realism:** Endogenous labor supply matches real-world adjustment patterns
- **Preference parameters matter:** Labor-leisure preference γ affects steady-state capital accumulation
- **Foundation:** Essential building block for DSGE models and heterogeneous agent frameworks