Infinite Horizon Consumption-Savings Problem and Intro to Dynamic Programing

Christine Braun

University of Warwick EC9A2

Review and Today

Last time

- solved a simple consumption-savings problem
- set initial wealth that agents consume over a finite period
- consumption path depends on patients vs interest rate

Today

- move to infinite horizon
- agents receive income each period
- how to solve models

Why Infinite Horizon

Why finite horizon is often unrealistic:

- Real people don't know their exact death date
- Many economic decisions (career choices, education, homeownership) are made as if life continues indefinitely
- Finite horizon models often produce unrealistic "end effects" where behavior changes dramatically near the terminal date

Examples where infinite horizon is more appropriate:

- Household savings decisions: Families plan for retirement, children's education, emergencies without a fixed endpoint
- Corporate investment: Firms make long-term investments assuming they'll operate indefinitely
- Government policy: Social security, infrastructure decisions are made with very long time horizons

Steady-State Analysis Becomes Relevant

What steady state means:

- A point where key variables (consumption, assets, income) stop changing on average. The economy's "long-run equilibrium" where temporary shocks have died out.

Why this matters economically:

- Policy analysis: We can study long-run effects of policy changes (tax rates, social security systems)
- Comparative statics: How do permanent changes in parameters (interest rates, productivity) affect long-run outcomes?
- Stability analysis: Does the economy return to steady state after shocks?

Example: If the government permanently increases unemployment benefits, steady-state analysis tells us the new long-run level of precautionary savings.

Transversality Conditions Replace Terminal Conditions

What transversality conditions are:

- Mathematical conditions that prevent “explosive” solutions where variables grow without bound.

Why this matters:

- Finite horizon: Terminal condition $a_T \geq 0$ is arbitrary - why exactly zero assets at death?
- Infinite horizon: Transversality emerges naturally from optimization - no arbitrary assumptions needed

Mathematical necessity:

- Without this condition, the optimization problem may not have a unique solution
- Prevents agents from achieving infinite utility through infinite borrowing

The Infinite Horizon Problem

Agent: Lives forever, receives income each period, chooses consumption and savings
Objective: Maximize lifetime utility

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

Constraints:

- Budget constraint: $a_{t+1} = (1+r)a_t + y_t - c_t$
- Non-negativity: $c_t \geq 0$
- Borrowing limit: $a_t \geq \underline{a}$ (natural borrowing limit)
- Transversality condition:

$$\lim_{t\to\infty} \beta^t u'(c_t)a_t = 0$$

Parameters:

- $\beta \in (0,1)$: discount factor
- $r > 0$: interest rate (constant)
- y_t: per-period income (may be stochastic)
- $u(\cdot)$: utility function (increasing, concave)

Key Features and Economic Interpretation

Constant income stream:

- Simplest case of income uncertainty: no uncertainty!
- Agent receives y every period forever

Natural borrowing limit:

$$\underline{a} = -\frac{y}{r}$$

Interpretation: Maximum debt sustainable with constant income

Economic applications:

- **Government bonds:** Country with stable tax revenue
- **Pension planning:** Retiree with fixed pension income
- **Trust funds:** Beneficiary receiving constant payments

Transversality Condition: Mathematical Intuition

The condition $\lim_{t\to\infty} \beta^t u'(c_t)a_t = 0$ means:

If $a_t > 0$ (positive wealth):

- $u'(c_t) \to 0$ as $t \to \infty$
- Marginal utility approaches zero
- Agent eventually becomes "rich enough" that extra wealth doesn't matter

If $a_t < 0$ (debt):

- When $a_t < 0$, condition becomes a *constraint* on debt growth
- Reveals fundamental limits on borrowing behavior
- Debt cannot grow faster than rate $\frac{1}{\beta} - 1$
- Prevents explosive debt paths
- Forces eventual debt repayment

Breaking Down the Condition for Debt

When agent has debt ($a_t < 0$), the transversality condition becomes:

$$\lim_{t\to\infty} \beta^t u'(c_t) a_t = 0$$

Since marginal utility is always positive ($u'(c_t) > 0$) on any feasible consumption path satisfying the budget constraint, we need:

$$\lim_{t\to\infty} \beta^t a_t = 0$$

Debt a_t must grow slower than β^{-t}

$$a_t \text{ grows slower than } \left(\frac{1}{\beta}\right)^t$$

Define subjective discount rate: $\rho = \frac{1}{\beta} - 1$

Key result: $\boxed{\text{Debt cannot grow faster than rate } \rho \text{ per period}}$

Maximum Debt Growth Rate

Economic interpretation:

- ρ: subjective discount rate (measures impatience)
- Higher ρ (more impatient) $\Rightarrow$ can sustain faster debt growth
- Lower ρ (more patient) $\Rightarrow$ must limit debt growth

Examples:

- $\beta = 0.95 \Rightarrow \rho = 5.26\%$: Debt can grow at most 5.26% per year
- $\beta = 0.99 \Rightarrow \rho = 1.01\%$: Debt can grow at most 1.01% per year
- $\beta = 0.90 \Rightarrow \rho = 11.11\%$: Debt can grow at most 11.11% per year

The Role of Patience

Impatient agents (low β, high ρ):

- Care little about future consumption
- High subjective discount rate allows more debt accumulation
- Can sustain borrowing even at moderately high interest rates
- Economic logic: "Don't mind debt growing fast since future doesn't matter much"

Patient agents (high β, low ρ):

- Care significantly about future consumption
- Low subjective discount rate severely limits debt accumulation
- Cannot sustain borrowing at high interest rates
- Economic logic: "Must be careful about debt since future matters a lot"

Paradox: More patient people are *more* constrained in their borrowing ability!

Bellman Equation

Since income is constant, the value function depends only on current assets:

$$V(a) = \max_{c \geq 0, a' \geq \underline{a}} \{u(c) + \beta V(a')\}$$

subject to:

$$a' = (1 + r)a + y - c$$

State variable: Current assets a
Choice variable: Current consumption c, next period assets a'

- can reduce to only c by substituting in the budge constraint

Key features:

- Time-invariant problem (stationary environment)
- Value function $V(a)$ independent of time
- Policy function $c^*(a)$ also time-invariant

First-Order Conditions

Interior solution requires:

$$u'(c) = \beta V'((1+r)a + y - c)$$

Envelope theorem:

$$V'(a) = \beta(1+r)V'((1+r)a + y - c^*(a))$$

where $c^*(a)$ is the optimal consumption policy. Then the **envelope condition** gives us:

$$V'(a') = (1+r)u'(c')$$

Combining FOC and envelope condition:

$$u'(c) = \beta(1+r)u'(c')$$

Key insight: Even with constant income, optimal consumption may not be constant - depends on relationship between $\beta(1+r)$ and 1.

Solving Dynamic Programing Problems

What does a solution look like?

- the solution is a set of *functions*, not a single point
- policy function: how much to consume for a given level of assets
- value function: discounted utility of all future consumption given a level of assets

Three ways to solve

- Guess and Verify (analytical)
- Value function iteration (numerical approximation)
- Policy function iteration (numerical approximation)

Before we discuss how to solve we will discuss when we have a unique solution

What Do We Mean by "Unique Solution"?

1. **Value Function Uniqueness:**
   - Is there a unique $V^*(x)$ satisfying the Bellman equation?
   - Most fundamental type of uniqueness
2. **Policy Function Uniqueness:**
   - Is there a unique optimal policy $\pi^*(x)$ for each state x?
   - Can have unique value function but multiple optimal policies
3. **Optimal Path Uniqueness:**
   - Starting from x_0, is the sequence $\{x_t\}_{t=0}^{\infty}$ unique?
   - Depends on both value and policy uniqueness

Why Does Uniqueness Matter?

Theoretical importance:

- Well-defined economic models require unique predictions
- Comparative statics analysis needs deterministic responses
- Welfare analysis requires unambiguous optimal policies

Computational implications:

- Numerical algorithms must converge to same solution
- Starting points shouldn't affect final answer
- Error bounds and convergence rates are meaningful

Policy applications:

- Government policies should have predictable effects
- Firms need unique optimal strategies
- Households should have clear decision rules

When uniqueness fails: Multiple equilibria, coordination problems, model indeterminacy.

Mathematical Foundations: Metric Spaces

Metric Space: A set X with distance function $d : X \times X \to \mathbb{R}^+$

Properties of distance function:

1. $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$ (non-negativity and identity)
2. $d(x, y) = d(y, x)$ (symmetry)
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

Example for Dynamic Programing: Space of bounded continuous functions with supremum norm (The largest absolute difference between the two functions across all possible input values):

$$d(V_1, V_2) = \|V_1 - V_2\|_\infty = \sup_{a \in \mathcal{A}} |V_1(a) - V_2(a)|$$

This gives us the mathematical framework for analyzing value functions.

Contraction Mappings

Definition: A mapping $T : X \to X$ is a contraction if there exists $\alpha \in [0, 1)$ such that:

$$d(T(x), T(y)) \leq \alpha \cdot d(x, y) \quad \forall x, y \in X$$

The constant α is called the **contraction factor**.

Intuition: Contraction mappings bring points closer together

- Take any two points in the set x and y
- After applying T, distance between $T(x)$ and $T(y)$ is smaller
- Factor $\alpha < 1$ means strict contraction

Key property: If $\alpha < 1$, repeated application of T makes points converge:

$$d(T^n(x), T^n(y)) \leq \alpha^n d(x, y) \to 0 \text{ as } n \to \infty$$

The Contraction Mapping Theorem (Banach Fixed Point Theorem)

Theorem: Let (X, d) be a complete metric space and $T : X \to X$ be a contraction mapping with factor $\alpha \in [0, 1)$. Then:

1. **Existence and Uniqueness:**

$$\exists! x^* \in X \text{ such that } T(x^*) = x^*$$

2. **Global Convergence:**

$$\forall x_0 \in X, \text{ the sequence } x_{n+1} = T(x_n) \text{ converges to } x^*$$

3. **Rate of Convergence:**

$$d(x_n, x^*) \leq \alpha^n d(x_0, x^*)$$

Simple Interpretation: No matter where you start, repeated application of T leads to the same unique fixed point.

The Bellman Operator

Consider the dynamic programming problem:

$$V(a) = \max_{c \in \mathcal{C}(a)} \left\{ u(c) + \beta V\big((1+r)a + y - c\big) \right\}$$

Define the **Bellman operator** T:

$$T(V)(a) = \max_{c \in \mathcal{C}(a)} \left\{ u(c) + \beta V\big((1+r)a + y - c\big) \right\}$$

Value function iteration: $V^{n+1} = T(V^n)$

Fixed point: True value function V^* satisfies $V^* = T(V^*)$

Goal: Show that T is a contraction mapping under appropriate conditions.

When is the Bellman Operator a Contraction?

Theorem: Under the following conditions, T is a contraction with factor β:

1. **Discount factor:** $\beta \in (0, 1)$
2. **Bounded rewards:** $\sup_c |u(c)| < \infty$
3. **Compact choice sets:** $\mathcal{C}(a)$ is compact for all a
4. **Continuity:** $u(c)$ is continuous

Contraction property:

$$\|T(V_1) - T(V_2)\|_\infty \leq \beta \|V_1 - V_2\|_\infty$$

Implications:

- Unique value function V^* exists
- Value function iteration converges to V^*
- Convergence rate is geometric with factor β

Parameter Restrictions for Uniqueness

1. **Discount Factor:**
   - $\beta \in (0, 1)$: Ensures contraction and unique value function
   - $\beta = 1$: May lead to non-uniqueness or non-existence
   - $\beta > 1$: Generally explosive, non-convergent behavior
2. **Interest Rates:**
   - $\beta(1 + r) = 1$: May create continuum of optimal consumption paths
   - $\beta(1 + r) \neq 1$: Generally ensures unique consumption dynamics
3. **Technology Parameters:**
   - Production elasticity $\alpha \in (0, 1)$: Ensures diminishing returns
   - Risk aversion $\sigma > 0$: Ensures strict concavity of utility
   - Avoid "knife-edge" parameter values that create indifference

Constraint Structure and Uniqueness

1. **Compact Choice Sets:**
   - Non-compact sets may lead to no optimal solution
   - Example: $c \in [0, \infty)$ vs. $c \in [0, (1+r)a + y]$
2. **Interior Solutions:**
   - When optimal choices are interior, first-order conditions determine unique solution
   - Corner solutions may create multiple optima
   - Inada conditions help ensure interior solutions
3. **Continuous Constraint Correspondence:**
   - $\mathcal{C}(a)$ should vary continuously with state a
   - Discontinuous constraints can create multiple local optima
   - Example: Borrowing limits that depend smoothly on income

Example: Linear Utility

Example: Consumption-savings with linear utility $u(c) = c$, assume $\beta < 1$

Bellman equation:

$$V(a) = \max_{c \in \mathcal{C}(a)} \{c + \beta V((1+r)a + y - c)\}$$

$$\mathcal{C}(a) = [0, (1+r)a + y]$$

Lets check conditions for a contraction:

1. **Discount factor:** $\beta \in (0, 1)$ ✓
2. **Bounded rewards:** $\sup_c |u(c)| < \infty$ ✓ (true in the choice set)
3. **Compact choice sets:** $\mathcal{C}(a)$ is compact for all a ✓
4. **Continuity:** $u(c)$ is continuous ✓

Result: the bellman equation for the simple consumption-savings problem with linear utility is a contraction $\rightarrow$ there exits a unique value function $V^*(a)$

Linear Utility: The Indifference Problem

Bellman equation:

$$V(a) = \max_{c \in \mathcal{C}(a)} \{c + \beta V((1+r)a + y - c)\}$$

$$\mathcal{C}(a) = [0, (1+r)a + y]$$

When $\beta(1+r) = 1$:

- Agent is indifferent between consuming today vs. tomorrow
- Any consumption path satisfying budget constraint is optimal
- Value function is unique, but policy function is not

Euler equation: $1 = \beta(1+r) = 1$ (always satisfied)
Economic interpretation: No diminishing marginal utility means no incentive to smooth consumption.

Unique value function doesn't guarantee unique policy!

Strict Concavity and Policy Uniqueness

Strict Concavity Condition: If the objective function is strictly concave in the choice variable:

$$\frac{\partial^2}{\partial c^2}[u(c) + \beta V\big((1+r)a + y - c\big)] < 0$$

Then the policy function $c^*(a)$ is unique.

Solution: Use strictly concave utility

Analytical Solution: Guess and Verify

The guess and verify method (also called the "method of undetermined coefficients") is an analytical technique for solving dynamic programming problems where you:

1. Guess the functional form of the value function based on economic intuition
2. Substitute this guess into the Bellman equation
3. Solve for the unknown parameters/coefficients
4. Verify that your solution satisfies all the necessary conditions

Log Utility Specification

Assume log utility: $u(c) = \ln(c)$

Properties:

- $u'(c) = \frac{1}{c}$ (marginal utility)
- $u''(c) = -\frac{1}{c^2}$ (diminishing marginal utility)

Euler equation becomes:

$$\frac{1}{c_t} = \beta(1+r)\frac{1}{c_{t+1}}$$

Therefore:

$$c_{t+1} = \beta(1+r)c_t$$

Consumption growth:

- If $\beta(1+r) = 1$: $c_{t+1} = c_t$ (constant consumption)
- If $\beta(1+r) > 1$: $c_{t+1} > c_t$ (growing consumption)
- If $\beta(1+r) < 1$: $c_{t+1} < c_t$ (declining consumption)

1. Guess: Value Function

Educated guess: Try value function of the form

$$V(a) = A\ln(a + B)$$

where $A > 0$ and $B > 0$ are constants to be determined.

Why this form?

- Log utility suggests log value function
- $(a + B)$ shifts the argument to handle potentially negative assets
- Linear in logs preserves analytical tractability

Marginal value of wealth:

$$V'(a) = \frac{A}{a + B}$$

Properties we expect:

- $V'(a) > 0$ (more assets are better): requires $A > 0$
- $V''(a) < 0$ (diminishing returns): $V''(a) = -\frac{A}{(a+B)^2} < 0$

2. Substituting into Bellman Equation

Substitute guess into Bellman equation:

$$A\ln(a+B) = \max_c \{\ln(c) + \beta A \ln((1+r)a + y - c + B)\}$$

First-order condition:

$$\frac{1}{c} = \frac{\beta A}{(1+r)a + y - c + B}$$

Solving for c:

$$c = \frac{(1+r)a + y + B}{1+\beta A}$$

This gives us the policy function candidate:

$$c^*(a) = \frac{(1+r)a + y + B}{1+\beta A}$$

Economic interpretation: Consumption is linear in "total wealth" $(1+r)a + y + B$.

3. Determine Constants A and B

Substitute optimal consumption back into Bellman equation and match coefficients. After substituting $c^*(a)$ and simplifying:

$$A \ln(a + B) = \ln\left(\frac{(1+r)a + y + B}{1 + \beta A}\right) + \beta A \ln\left(\frac{\beta A((1+r)a + y + B)}{1 + \beta A}\right)$$

For this to hold for all a, we need: (see extra notes for derivation)

$$A = \frac{1}{1 - \beta}$$

$$B = \frac{y}{r}$$

Final Value and Policy Functions

Value function:

$$V(a) = \frac{1}{1-\beta} \ln\left(a + \frac{y}{r}\right)$$

Policy function:

$$c^*(a) = \frac{(1+r)a + y + \frac{y}{r}}{1 + \frac{\beta}{1-\beta}}$$

$$= \frac{(1+r)a + y(1+\frac{1}{r})}{\frac{1}{1-\beta}}$$

Simplifying:

$$c^*(a) = (1-\beta)(1+r)\left[a + \frac{y}{r}\right]$$

Economic Interpretation of the Solution

Total wealth concept:

$$\text{Total wealth} = a + \frac{y}{r} = \text{financial assets} + \text{human wealth}$$

Consumption rule:

$$c^* = (1-\beta)(1+r) \times \text{total wealth}$$

Key insights:

- Agent consumes fixed fraction of total wealth each period
- Higher patience (β closer to 1) $\Rightarrow$ lower consumption rate
- Higher interest rate $\Rightarrow$ higher consumption rate
- Consumption depends on both financial and human wealth

Guess and Verify: Strengths and Limitations

Strengths:

- **Exact solutions:** No approximation error
- **Economic insight:** Clear parameter relationships
- **Analytical results:** Closed-form comparative statics
- **Fast evaluation:** No iterative computation needed t

Limitations:

- **Very restrictive:** Only works for special utility/technology
- **Guess quality:** Success depends on making good initial guess
- **Limited scope:** Can't handle general constraints or uncertainty easily
- **Algebraic complexity:** Can become very tedious
- **No general method:** Each problem requires fresh approach

Value Function Iteration

Value Function Iteration (VFI) is a numerical algorithm for solving dynamic programming problems by repeatedly applying the Bellman operator until convergence.

Main Advantages:

- General applicability - works for any well-posed dynamic programing problems
- Guaranteed convergence - mathematical certainty it will work
- Handles complexity - constraints, uncertainty, multiple state variables

Main Disadvantages:

- Computational cost - can be slow for large problems
- Approximation error - discretization introduces errors
- Curse of dimensionality - exponential growth with state variables

Basic Algorithm

1. **Discretize state space:** Create grid $\{a_1, a_2, \ldots, a_J\}$ where $\underline{a} = a_1 < a_2 < \cdots < a_J$
2. **Initial guess:** Choose $V^0(a_j)$ for all grid points
3. **Iterate:** For $j = 0, 1, 2, \ldots$:

$$V^{n+1}(a_j) = \max_c \{\ln(c) + \beta V^n((1+r)a_j + y - c)\}$$

   subject to: $c > 0$ and $(1+r)a_j + y - c \geq \underline{a}$
4. **Store policy:** $c^{n+1}(a_j) = \arg\max$ of above
5. **Check convergence:** If $\|V^{n+1} - V^n\| <$ tolerance, stop
6. **Output:** $V^*(a)$ and $c^*(a)$

Complete Algorithm

Algorithm 1 Basic VFI

```
Initialize: Grid 𝒜 = {a_1, a_2, ..., a_J}, Guess V^0(a_j), set tolerance
Set n = 0
repeat
    Calculate consumption at a_j and each a'_j: c = max((1 + r)a_j + y − 𝒜, 0)
    Find Maximum: [V^1(j), max_idx] = max(log(c) + βV^0)
    Store Asset Policy: g_a(j) = 𝒜(max_idx)
    Store Consumption Policy: g_c(j) = (1 + r)a(j) + y − g_a(j)
    Check convergence: max(|V^1 − V^0|) < tol
    Update: V^0 = V^1
    n = n + 1
until convergence
Return: value functions, policy functions
```

Initialize: Grid $\mathcal{A} = \{a_1, a_2, \ldots, a_J\}$, Guess $V^0(a_j)$, set tolerance

Set $n = 0$

repeat

Calculate consumption at a_j and each a'_j: $c = max((1+r)a_j + y - \mathcal{A}, 0)$

Find Maximum: $[V^1(j), max\_idx] = max(log(c) + \beta V^0)$

Store Asset Policy: $g_a(j) = \mathcal{A}(max\_idx)$

Store Consumption Policy: $g_c(j) = (1+r)a(j) + y - g_a(j)$

Check convergence: $max(|V^1 - V^0|) < \text{tol}$

Update: $V^0 = V^1$

$n = n + 1$

until convergence

Return: value functions, policy functions