EC9A2 Problem Set 3

The Model

Consider the Ramsey model with exogenous labor supply (normalized to 1):

Preferences:

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t) = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

where $\beta \in (0, 1)$ is the discount factor and $\sigma > 0$ is the coefficient of relative risk aversion.

Technology:

$$y_t = f(k_t) = k_t^{\alpha}$$

where $\alpha \in (0, 1)$ is the capital share.

Capital accumulation:

$$k_{t+1} = (1 - \delta)k_t + i_t$$

where $\delta \in (0, 1)$ is the depreciation rate.

Resource constraint:

$$c_t + i_t = f(k_t)$$

Combining the capital accumulation and resource constraint:

$$k_{t+1} = (1 - \delta)k_t + f(k_t) - c_t$$

Bellman equation: (for the planner's problem, i.e. we are using the resource constraint)

$$V(k) = \max_{c,k'} \{u(c) + \beta V(k')\}$$

subject to:

$$\begin{aligned} k' &= (1 - \delta)k + f(k) - c \\ c &\geq 0 \\ k' &\geq 0 \end{aligned}$$

Parameter Values:

- $\beta = 0.96$
- $\sigma = 2$
- $\alpha = 0.33$
- $\delta = 0.1$

1 Analytical Steady State

(a) Before implementing the numerical solution, derive the analytical steady state and calculate the value of steady state consumption, capital, and the capital output ratio, $k^*/f(k^*)$, at the given parameter values. (You can start from the equations you solved from in the PS2 Question 1a.)

ANSWER: From PS2 Question 1a we know that the steady state is characterized by:

$$\begin{aligned} f'(k^*) &= \rho + \delta && \text{(determines } k^*\text{)} \\ c^* &= f(k^*) - \delta k^* && \text{(determines } c^* \text{ given } k^*\text{)} \end{aligned}$$

Substituting in the marginal product of capital

$$\alpha (k^*)^{\alpha-1} = \rho + \delta$$

Solving for k^*:

$$k^* = \left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}$$

Plugging in the production function to get consumption:

$$c^* = f(k^*) - \delta k^* = (k^*)^\alpha - \delta k^*$$

And the capital-output ratio:

$$\frac{k^*}{f(k^*)} = \frac{k^*}{(k^*)^\alpha} = \frac{1}{(k^*)^{\alpha-1}} = \frac{\alpha}{\rho+\delta}$$

Plugging in the parameter values we get:

$$\begin{aligned} k^* &= 3.5329 \\ c^* &= 1.1634 \\ \frac{k^*}{f(k^*)} &= 2.3294 \end{aligned}$$

(b) Interpret the Modified Golden Rule condition $f'(k^*) = \rho + \delta$ in the context of your numerical steady state. Calculate $f'(k^*)$ and verify it equals $\rho + \delta$ where $\rho = \frac{1-\beta}{\beta}$.

ANSWER: The Modified Golden Rule condition states:

$$f'(k^*) = \rho + \delta$$

This condition characterizes the optimal steady-state capital stock by equating:

- **Left side** $f'(k^*)$: The marginal product of capital, which represents the gross return to investing in one more unit of capital

- **Right side** $\rho + \delta$: The "required return" which consists of:
  - $\rho = \frac{1-\beta}{\beta}$: The rate of time preference (impatience). This captures how much agents discount future consumption relative to present consumption.
  - δ: The depreciation rate, representing the physical loss of capital over time

For an investment in capital to be worthwhile at the margin, its gross return ($f'(k^*)$) must equal the sum of the opportunity cost of waiting (ρ) and the cost of depreciation (δ).

The marginal product of capital at the steady state:

$$\begin{aligned} f'(k^*) &= \alpha(k^*)^{\alpha-1} \\ &= 0.142 \end{aligned}$$

And the required return is:

$$\begin{aligned} \rho + \delta &= \frac{1-\beta}{\beta} + \delta \\ &= 0.142 \end{aligned}$$

So the Modified Golden Rule condition holds.

2 Setting Up Value Function Iteration

(a) Explain the value function iteration algorithm in your own words. Your explanation should include:

  (i) What the value function represents

  ANSWER: The value function $V(k)$ represents the maximum lifetime utility that an agent can achieve starting from a given capital stock k. It accounts for both immediate utility and the continuation value from all future periods.

  (ii) Why we need to discretize the state space

  ANSWER: We need to discretize the continuous state space (capital k) for computational feasibility. Capital k can take infinitely many values in the continuous interval $[k_{min}, k_{max}]$ so discretization allows us to represent $V(k)$ with a finite number of values.

  (iii) The iterative process (how we update the value function)

  ANSWER: Step 0: Initialize

  i. Choose a capital grid: $k \in \{k_1, k_2, \ldots, k_N\}$

  ii. Make an initial guess: $V_0(k_i)$ for $i = 1, \ldots, N$

For iteration n, given $V_n(k)$, compute the updated value function $V_{n+1}(k)$ by solving the maximization problem at each grid point:

$$V_{n+1}(k_i) = \max_{c,k'} \{u(c) + \beta V_n(k')\}$$

This involves:

i. For each k_i on the grid, search over feasible choices of consumption c (or equivalently, next-period capital k')
ii. For each choice, evaluate the objective function: current utility $u(c)$ plus discounted continuation value $\beta V_n(k')$
iii. Select the choice that maximizes the objective
iv. Store the maximum value as $V_{n+1}(k_i)$ and the optimal choice as the policy function

(iv) The convergence criterion

ANSWER: The convergence criterion determines when to stop iterating. We typically use:

$$\max_{i=1,\dots,N} |V_{n+1}(k_i) - V_n(k_i)| < \epsilon$$

where $\epsilon > 0$ is a small tolerance level (e.g., $\epsilon = 10^{-6}$).
What this measures:

- The criterion measures the **maximum change** in the value function across all grid points between successive iterations
- When this change becomes smaller than ϵ, the value function has essentially stopped changing
- This indicates we are very close to the true fixed point

(v) What the policy function tells us

ANSWER: The policy functions are the optimal decision rules that emerge from solving the Bellman equation. They tell us what actions to take as a function of the current state. The consumption policy function: $g_c(k)$ answers: "If I have capital k today, how much should I consume?" The capital policy function: $g_k(k)$ answers: "If I have capital k today, how much capital should I have tomorrow?"

(b) Propose a reasonable range for the capital grid: $k \in [k_{min}, k_{max}]$. Justify your choice.

ANSWER: For the capital grid $k \in [k_{min}, k_{max}]$ we can use $k_{min} = 0.25 \times k^*$ and $k_{max} = 2.0 \times k^* = 2.0$. The range $[0.25k^*, 2k^*]$ covers capital levels from 25% to 200% of the steady state. This is wide enough to capture realistic deviations from steady state and allows for analysis of substantial shocks or policy changes.

(c) Propose a reasonable initial guess for $V_0(k)$. Explain why your choice might speed convergence.

ANSWER: For an initial guess we can use:

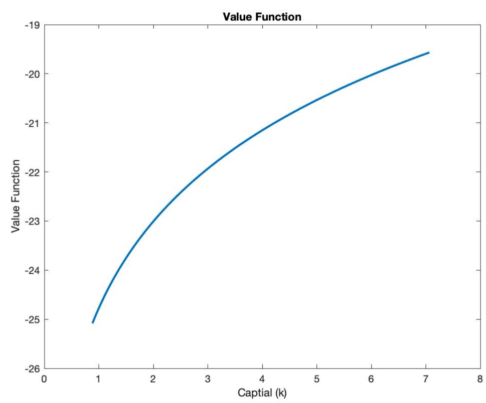
$$V_0(k) = \frac{u(f(k))}{1-\beta} = \frac{u(k^\alpha)}{1-\beta} = \frac{(k^\alpha)^{1-\sigma}}{(1-\sigma)(1-\beta)}$$

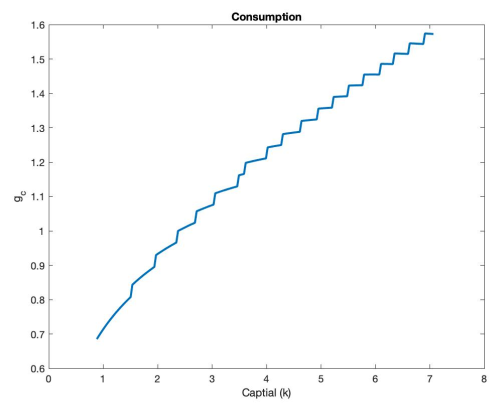
This represents the value of consuming all output forever with no investment or saving. This guess represents a feasible (though suboptimal) strategy and is a lower bound on the true value function since allowing saving and investment can only increase utility. The true value function is increasing and concave in k as is this guess. Starting with the right shape reduces the distance to the true solution. Alternatively we can use $V_0(k) = 0 \forall k$, but this guess will take more iterations to converge.

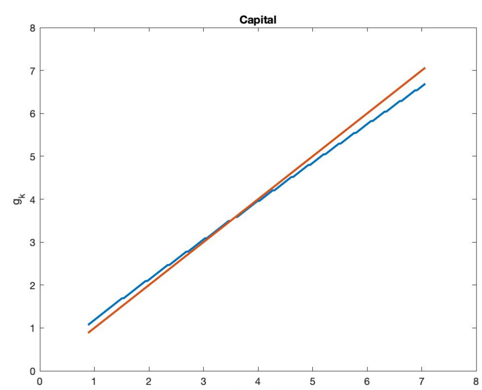
3 Implementation in MATLAB

(a) Write a MATLAB script solves the model using VFI. The script should produce, the converged value function and the policy functions. Plot the value function $V(k)$. Plot the policy functions $g_c(k)$ and $g_k(k)$. On the plot of the capital policy function, include a 45 degree line.

ANSWER







(b) What is the steady-state capital from your policy function $g_k(k)$? (Find where $g_k(k) = k$). How close is this to the analytical k^* you calculated in Problem 1?

ANSWER: Since our state space is discrete there are 4 points where $g_k(k) = k$, if we average across these values we get $k^* = 3.5395$ which is very close to the analytical value.

(c) What is the steady-state consumption from your policy function? How close is this to the analytical c^* you calculated in Problem 1?

ANSWER: If we average across the 4 corresponding consumption values we get $c^* = 1.1636$ which is very close to the analytical value.

(d) Looking at your graph for $g_k(k)$, when is the policy function above the 45 degree line and when is it below. What does this tell you about capital accumulation/decumulation?

ANSWER: The policy function is below the 45 degree line when capital is below the steady state value. This means that when we are below the steady state we save so that we have more next period than we had today. Therefore we are accumulating capital and move toward the steady state.

(e) Starting from $k_0 = 0.5k^*$, use your policy function to simulate the path of capital and consumption for 50 periods. Plot the time paths of k_t and c_t. (HINT: you will need to use the matlab function "interp1" for this.)

ANSWER: