Foundations of Dynamic Optimization

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Course Outline

Goal: To develop both theoretical foundations and computational techniques essential for modern macroeconomic research. Building our way up to the foundations of a modern macro model.

Main tools we will learn:

- dynamic programing
- numerical methods
- implementations in Matlab

Course Roadmap

- **Week 1** Foundations of dynamic optimization and intro to Matlab
- **Week 2-3:** Neoclassical Growth Model
- **Week 3-4:** Optimization under uncertainty
- **Week 5:** Heterogeneous agent models
- **Time Permitting:** Lifecycle income process / inequality

Course Info

Office Hours: Tuesday 13-14 S2.87

TA: David Boll

Problem Sets: There are 5 problem sets

- you can work in groups of 2-4 to complete problem sets
- starting from PS2 email your work to David before Friday's sections
- PS1 - attempt problem 3 before Friday (more on this later)
- I will post answers on Mondays after David's section

Why Dynamic Optimization?

- Economic agents make decisions over time with consequences extending into the future
- Examples:
  - Households: consumption vs. saving decisions
  - Firms: investment in capital stock
  - Governments: public investment and debt policy
- Key features:
  - State variables evolve over time
  - Current decisions affect future constraints
  - Trade-offs between present and future

Basic Intertemporal Choice Problem

Agent: Lives for T periods, no income, initial wealth $W_0 > 0$

Objective: Maximize lifetime utility

$$\max \sum_{t=0}^{T-1} \beta^t u(c_t)$$

Constraints:

- Budget constraint: $W_{t+1} = (1+r)(W_t - c_t)$
- Non-negativity: $c_t \geq 0$, $W_t \geq 0$
- Initial condition: W_0 given
- Terminal condition: $W_T \geq 0$

Parameters:

- $\beta \in (0,1)$: discount factor
- $r > 0$: interest rate (constant)
- $u(\cdot)$: utility function (increasing, concave)

Economic Interpretation

Key trade-off: Consume today vs. save for future consumption

Role of interest rate:

- Each unit saved today becomes $(1 + r)$ units tomorrow
- Higher r makes saving more attractive
- Interest can potentially offset impatience $(\beta < 1)$

Goal: Solve this problem using two methods and show they yield identical results.

- Lagrangian method
- Dynamic programming approach

Static Approach: Lagrangian Method

Step 1: Derive present Value Budget Constraint. From $W_{t+1} = (1+r)W_t - c_t$, we can iterate forward:

$$W_1 = (1+r)(W_0 - c_0)$$
$$W_2 = (1+r)(W_1 - c_1) = (1+r)^2 W_0 - (1+r)^2 c_0 - (1+r)c_1$$
$$W_3 = (1+r)(W_2 - c_2) = (1+r)^3 W_0 - (1+r)^3 c_0 - (1+r)^2 c_1 - (1+r)c_2$$
$$\vdots$$

General pattern:

$$W_T = (1+r)^T W_0 - \sum_{t=0}^{T-1} (1+r)^{T-t} c_t$$

With terminal condition $W_T \geq 0$ (equality at optimum):

$$\sum_{t=0}^{T-1} (1+r)^{T-t} c_t = (1+r)^T W_0$$

Static Approach: Lagrangian Method

Dividing by $(1+r)^T$:

$$\sum_{t=0}^{T-1} \frac{c_t}{(1+r)^t} = W_0$$

This is the **present value budget constraint**.

Economic interpretation:

- LHS: Present value of consumption stream
- RHS: Initial wealth
- Constraint: Cannot consume more than initial wealth in present value terms

Key insight: This transforms the dynamic problem into a static optimization problem with a single constraint.

Static Approach: Lagrangian Method

Step 2: Form the Lagrangian

$$\mathcal{L} = \sum_{t=0}^{T-1} \beta^t u(c_t) - \lambda \left(\sum_{t=0}^{T-1} \frac{c_t}{(1+r)^t} - W_0 \right)$$

Interpretation:

- Objective: maximize discounted utility
- Constraint: present value budget constraint
- λ: shadow price of wealth (marginal utility of wealth)

Static Approach: Lagrangian Method

Step 3: First-Order Conditions

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u'(c_t) - \lambda \frac{1}{(1+r)^t} = 0$$

Therefore:

$$\beta^t u'(c_t) = \frac{\lambda}{(1+r)^t}$$

Rearranging:

$$u'(c_t) = \frac{\lambda}{\beta^t (1+r)^t} = \lambda \left(\frac{1}{\beta(1+r)} \right)^t$$

Key result: Marginal utility follows a geometric progression with ratio $\frac{1}{\beta(1+r)}$.

Static Approach: Lagrangian Method

Step 4: Derive the Euler Equation
From the FOCs:

$$\beta^t u'(c_t)(1+r)^t = \lambda$$
$$\beta^{t+1} u'(c_{t+1})(1+r)^{t+1} = \lambda$$

Equating:

$$\beta^t u'(c_t)(1+r)^t = \beta^{t+1} u'(c_{t+1})(1+r)^{t+1}$$

Simplifying:

$$\boxed{u'(c_t) = \beta(1+r)u'(c_{t+1})}$$

This is the **Euler equation** - the fundamental condition for intertemporal optimization.

Interpretation: Marginal utility of consumption today must equal discounted marginal utility of consumption tomorrow (adjusted for interest rate).

An Example: CRRA Utility

Lets assume **Constant Relative Risk Aversion** utility

$$u(c) = \frac{c^{1-\theta}}{1-\theta}, \quad u'(c) = c^{-\theta}$$

We will solve for the

- consumption path - how consumption changes over time
- consumption levels

CRRA utility: Economic Interpretation

Why "Constant Relative Risk Aversion"? $RRA = -\frac{c \cdot u''(c)}{u'(c)} = \theta$ Risk aversion is constant regardless of consumption level.

Economic Meaning of θ:

- θ measures how much people dislike risk and variability in consumption
- Higher θ: More risk averse, prefer smoother consumption over time
- Lower θ: Less risk averse, more willing to have volatile consumption

Special Cases:

- $\theta = 0$: Linear utility $u(c) = c$ (risk neutral, infinite substitution)
- $\theta = 1$: Log utility $u(c) = \ln(c)$ (unit elasticity)
- $\theta \to \infty$: No intertemporal substitution (Leontief-like)

CRRA utility: Consumption Path

Euler Equation

$$\boxed{u'(c_t) = \beta(1+r)u'(c_{t+1})}$$

Plugging in the utility function:

$$c_t^{-\theta} = \beta(1+r)c_{t+1}^{-\theta}$$

Solving for consumption growth:

$$\left(\frac{c_{t+1}}{c_t}\right)^{-\theta} = \beta(1+r)$$

Consumption Path: $\dfrac{c_{t+1}}{c_t} = [\beta(1+r)]^{1/\theta}$

Economic Intuition: Consumption Path

Flat Consumption ($\beta(1 + r) = 1$):

- Interest rate *exactly* compensates for impatience
- Agent consumes the same amount every period

Rising Consumption ($\beta(1 + r) > 1$):

- Interest rate *over-compensates* for impatience
- Agent postpones consumption
- Starts with low consumption, lets wealth grow, consumes more later

Falling Consumption ($\beta(1 + r) < 1$):

- Interest rate *under-compensates* for impatience
- Even with interest earnings, impatience dominates
- Front-loads consumption

Role of θ: High $\theta \to$ smoother consumption; Low $\theta \to$ more responsive to interest rates

CRRA utility: Solve for Consumption Levels

With geometric consumption growth: $c_t = c_0[\beta(1+r)]^{t/\theta}$
Substituting into budget constraint:

$$\sum_{t=0}^{T-1} \frac{c_0[\beta(1+r)]^{t/\theta}}{(1+r)^t} = W_0$$

$$c_0 \sum_{t=0}^{T-1} \left[\frac{\beta^{1/\theta}}{(1+r)^{1-1/\theta}}\right]^t = W_0$$

Let $\phi = \frac{\beta^{1/\theta}}{(1+r)^{1-1/\theta}}$. Then if $\phi < 1$ (geometric series):

$$c_0 \sum_{t=0}^{T-1} \phi^t = c_0 \frac{1-\phi^T}{1-\phi} = W_0$$

Therefore: $\boxed{c_0 = W_0 \frac{1-\phi}{1-\phi^T}}$

Alternative Lagrangian Approach: Period-by-Period Constraints

Instead of deriving a single present value budget constraint, we can:

- Keep the period-by-period budget constraints: $W_{t+1} = (1 + r)(W_t - c_t)$
- Introduce a Lagrange multiplier λ_t for each period $t = 0, 1, \ldots, T - 1$
- Plus a multiplier μ for the terminal condition $W_T \geq 0$

Advantage: This approach more naturally reveals the dynamic structure and connects directly to the envelope theorem used in dynamic programming.

Setting Up the Lagrangian

Form the Lagrangian with all constraints:

$$\mathcal{L} = \sum_{t=0}^{T-1} \beta^t u(c_t) + \sum_{t=0}^{T-1} \lambda_t [W_t - c_t - W_{t+1}/(1+r)] + \mu W_T$$

Variables:

- Choice variables: $\{c_0, c_1, \ldots, c_{T-1}, W_1, W_2, \ldots, W_T\}$
- Lagrange multipliers: $\{\lambda_0, \lambda_1, \ldots, \lambda_{T-1}, \mu\}$

Note: W_0 is given (not a choice variable), but $W_1, \ldots, W_T$ are chosen subject to the constraints.

First-Order Conditions: Consumption

Take the derivative with respect to c_t for $t = 0, 1, \ldots, T-1$:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u'(c_t) - \lambda_t = 0$$

Therefore:

$$\lambda_t = \beta^t u'(c_t) \quad \text{for } t = 0, 1, \ldots, T-1$$

Interpretation: The multiplier λ_t represents the marginal value (in period-0 utility terms) of relaxing the budget constraint in period t.

First-Order Conditions: Wealth

Take the derivative with respect to W_t for $t = 1, 2, \ldots, T-1$:

$$\frac{\partial \mathcal{L}}{\partial W_t} = \lambda_t - \frac{\lambda_{t-1}}{(1+r)} = 0$$

Therefore:

$$\lambda_{t-1} = (1+r)\lambda_t \quad \text{for } t = 1, 2, \ldots, T-1$$

For the terminal wealth W_T:

$$\frac{\partial \mathcal{L}}{\partial W_T} = \mu - \lambda_{T-1} = 0 \quad \Rightarrow \quad \lambda_{T-1} = \mu$$

Deriving the Euler Equation

From the FOC for consumption: $\lambda_t = \beta^t u'(c_t)$
From the FOC for wealth: $\lambda_{t-1} = (1+r)\lambda_t$

Substituting the first into the second:

$$\beta^{t-1} u'(c_{t-1}) = (1+r)\beta^t u'(c_t)$$

$$u'(c_{t-1}) = (1+r)\beta u'(c_t)$$

Shifting the time index forward by one period:

$$\boxed{u'(c_t) = \beta(1+r)u'(c_{t+1})}$$

This is exactly the same Euler equation we derived before.

Economic Interpretation

The condition $\lambda_{t-1} = (1+r)\lambda_t$ has a clear economic interpretation:

Marginal value of wealth must grow at rate $1+r$:

- λ_t = marginal utility value of having an extra unit of wealth in period t
- If you save \$1 in period $t-1$, you get \$$(1+r)$ in period t
- The value of \$1 in period $t-1$ must equal $(1+r)$ times the value of \$1 in period t
- This is a no-arbitrage condition: no benefit to shifting wealth between periods

Connection to single constraint: The value of \$1 in initial wealth must equal $(1+r)^t$ times the value of \$1 in period t

$$\lambda = (1+r)^t \lambda_t$$

Comparison: Single vs. Multiple Multipliers

Single Constraint	**Period-by-Period**
One multiplier λ	T multipliers $\{\lambda_t\}_{t=0}^{T-1}$
$\sum_{t=0}^{T-1} \frac{c_t}{(1+r)^t} = W_0$	$W_{t+1} = (1+r)(W_t - c_t)$ for all t
$\beta^t u'(c_t) = \frac{\lambda}{(1+r)^t}$	$\lambda_t = \beta^t u'(c_t)$
Direct Euler equation	Euler via $\lambda_{t-1} = (1+r)\lambda_t$
Simpler algebra	Clearer dynamics
Hides temporal structure	Shows sequential nature

Key insight: Both methods are equivalent, but the period-by-period approach better illustrates the dynamic structure.

Note: For more complex problems the Period-by-Period problem is typically easier to solve.

Limitations of Static Approach

- Becomes unwieldy with:
  - Long or infinite horizons
  - Uncertainty
  - State-dependent constraints
  - Complex state evolution
- Does not provide intuition about *sequential* decision-making
- Difficult to analyze policy functions and value functions
- Hard to extend to more complex environments

Solution: Dynamic programming approach

Bellman's Principle of Optimality

Principle: An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

Intuition:

- If you're on the optimal path, then from any point forward, the rest of the path must also be optimal
- This allows us to break down multi-period problems recursively
- Forms the foundation of dynamic programming

Value Functions

Define the **value function** $V_t(W_t)$ as the maximum utility attainable from period t onward, given wealth W_t:

$$V_t(W_t) = \max_{\{c_j, W_{j+1}\}_{j=t}^{T-1}} \sum_{j=t}^{T-1} \beta^{j-t} u(c_j)$$

subject to:

$$W_{j+1} = (1+r)(W_j - c_j) \quad \forall j = t, \ldots, T-1$$
$$W_T \geq 0$$

Boundary condition: $V_T(W_T) = 0$ (no utility from terminal assets)

Key insight: With no income, this is purely an asset decumulation problem - agent optimally spends down wealth over time.

Bellman Equation

By the principle of optimality:

$$V_t(W_t) = \max_{c_t, W_{t+1}} \{u(c_t) + \beta V_{t+1}(W_{t+1})\}$$

subject to: $W_{t+1} = (1 + r)(W_t - c_t)$

This is the **Bellman equation**. It states that the value function can be decomposed into:

- W_t is called a **state variable**
- c_t and W_{t+1} are called **choice variables**
- Current period utility: $u(c_t)$
- Discounted continuation value: $\beta V_{t+1}(W_{t+1})$

Economic interpretation: Choose consumption today vs. saving (with interest) for future consumption.

Policy Functions

The solution to the Bellman equation yields **policy functions**:

$$\begin{aligned} c_t^* &= g_t(W_t) \quad \text{(consumption policy)} \\ W_{t+1}^* &= h_t(W_t) \quad \text{(savings policy)} \end{aligned}$$

These functions tell us the optimal choice as a function of current wealth.

From the budget constraint: $W_{t+1}^* = (1+r)(W_t - g_t(W_t))$
So we only need to find the consumption policy function.

Economic insight: Policy functions show how consumption and saving decisions depend on current wealth level - richer agents may consume more but also save more in absolute terms.

Deriving the Euler Equation

Since $W_{t+1} = (1+r)(W_t - c_t)$, we can write:

$$V_t(W_{t+1}) = \max_{c_t} \{u(c_t) + \beta V_{t+1}((1+r)(W_t - c_t))\}$$

subject to: $0 \leq c_t \leq W_{t+1}$

The first-order condition for the Bellman equation:

$$\frac{\partial}{\partial c}[u(c_t) + \beta V_{t+1}((1+r)(W_t - c_t))] = 0$$

$$u'(c) - \beta V'_{t+1}((1+r)(W_t - c_t)) = 0$$

Therefore:

$$u'(c_t) = \beta V'_{t+1}(W_{t+1})$$

Interpretation: Marginal utility of consumption today equals discounted marginal value of wealth tomorrow.

Envelope Theorem

We need to find $V'_{t+t}(W_{t+1})$. For this we need to the Envelope Theorem.

Envelope Theorem: Suppose you have an optimization problem:

$$V(\alpha) = \max_{x} \; f(x, \alpha)$$

x is your choice variable and α is a parameter. Let $x^*(\alpha)$ be the optimal choice. The envelope theorem says:

$$\frac{dV}{d\alpha} = \frac{\partial f(x^*(\alpha), \alpha)}{\partial \alpha}$$

Envelope Theorem

Why it works: At the optimum, the first-order condition holds:

$$\frac{\partial f(x^*, \alpha)}{\partial x} = 0$$

So when you totally differentiate $V(\alpha)$:

$$\frac{dV}{d\alpha} = \frac{\partial f}{\partial x} \cdot \frac{dx^*}{d\alpha} + \frac{\partial f}{\partial \alpha}$$

The first term equals zero (by the FOC), leaving only the direct effect.

Deriving the Euler Equation

Apply the envelope theorem to $V_t(W)$:

$$V_t'(W_t) = \frac{\partial}{\partial W_t}\left[u(c_t^*(W_t)) + \beta V_{t+1}((1+r)(W_t - c_t^*(W_t)))\right]$$

Since $c_t^*(W_t)$ is optimal, the first-order condition holds, so:

$$V_t'(W_t) = \beta V_{t+1}'((1+r)(W_t - c_t^*(W_t))) \cdot (1+r)$$

$$V_t'(W_t) = \beta(1+r)V_{t+1}'(W_{t+1})$$

Economic interpretation: Marginal value of wealth today equals discounted marginal value of wealth tomorrow, adjusted for the gross interest rate.

Connection to Lagrangian: This is precisely the first order condition for wealth in the Lagrangian method $\lambda_t = (1+r)\lambda_{t+1}$.

Deriving the Euler Equation

From the first-order condition: $u'(c_t) = \beta V'_{t+1}(W_{t+1})$
From the envelope theorem: $V'_t(W_t) = \beta(1+r)V'_{t+1}(W_{t+1})$
Substituting back into the FOC:

$$u'(c_t) = \beta \cdot \frac{V'_t(W_t)}{\beta(1+r)} = \frac{V'_t(W_t)}{1+r}$$

For the next period: (Sometimes people call this the **Envelope condition**)

$$V'_{t+1}(W_{t+1}) = (1+r)u'(c_{t+1})$$

Combining with the first order condition:

$$u'(c_t) = \beta(1+r)u'(c_{t+1})$$

Result: Same Euler equation as Lagrangian method

Equivalence of Methods

Lagrangian Method	**Dynamic Programming**
Present value budget constraint	Sequential budget constraints
Single optimization problem	Sequence of optimization problems
Shadow price λ	Marginal value function $V'(W)$
FOC: $\beta^t u'(c_t) = \lambda/(1+r)^t$	FOC: $u'(c) = \beta V'_{t+1}(W')$
Euler equation directly	Euler equation via envelope theorem
Consumption path	Policy functions

Key insight: Both methods yield identical optimality conditions but provide different perspectives and tools.

When to Use Each Method

Lagrangian Method:

- Simpler for finite horizon problems
- Direct derivation of consumption paths
- Easier to handle with simple constraints
- Difficult with uncertainty
- Hard to extend to complex state evolution
- No policy functions

Dynamic Programming:

- Natural for sequential decision-making
- Easily extended to stochastic problems
- Provides value and policy functions
- Better for numerical computation
- More complex setup
- Requires envelope theorem

Matlab

You can get Matlab from the Uni here

- log in with your uni ID
- check terms and conditions then download
- you will need to create/sign in to Mathworks ID using your warwick email
- download latest version (you can also use the online version)

Download before next class.