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Homework 2 - Berkeley STAT 157

Handout 1/29/2019, due 2/5/2019 by 4pm in Git by committing to your repository.

Please generate pdf in pages rather the an one sheet.

In [1]:
```
from mxnet import nd, autograd, gluon
import numpy as np
from matplotlib import pyplot as plt
import string
```

1. Multinomial Sampling

Implement a sampler from a discrete distribution from scratch, mimicking the function `mxnet.ndarray.random.multinomial`. Its arguments should be a vector of probabilities p. You can assume that the probabilities are normalized, i.e. tha they sum up to 1. Make the call signature as follows:

```
samples = sampler(probs, shape)

probs   : An ndarray vector of size n of nonnegative numbers summing up to 1
shape   : A list of dimensions for the output
samples : Samples from probs with shape matching shape
```

Hints:

1. Use `mxnet.ndarray.random.uniform` to get a sample from $U[0,1]$.
2. You can simplify things for `probs` by computing the cumulative sum over `probs`.

In [168]:
```
def sampler(probs, shape):
    n = shape[0] * shape[1]
    sampled = nd.random.uniform(0, 1, n).reshape(shape)
    check = nd.zeros(shape)
    for prob in np.cumsum(probs):
        check += (sampled >= prob)
    return check

# a simple test
sampler(nd.array([0.2, 0.3, 0.5]), (2, 3))
```

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Out[168]:
```
[[2. 2. 1.]
 [1. 0. 1.]]
<NDArray 2x3 @cpu(0)>
```

2. Central Limit Theorem

Let's explore the Central Limit Theorem when applied to text processing.

- Download https://www.gutenberg.org/ebooks/84 from Project Gutenberg
- Remove punctuation, uppercase / lowercase, and split the text up into individual tokens (words).
- For the words `a`, `and`, `the`, `i`, `is` compute their respective counts as the book progresses, i.e.

$$n_{\mathrm{the}}[i] = \sum_{j=1}^{i} \{w_j = \mathrm{the}\}$$

- Plot the proportions $n_{\mathrm{word}}[i]/i$ over the document in one plot.
- Find an envelope of the shape $O(1/\sqrt{i})$ for each of these five words.
- Why can we **not** apply the Central Limit Theorem directly?
- How would we have to change the text for it to apply?
- Why does it still work quite well?

In [3]:
```
filename = gluon.utils.download('https://www.gutenberg.org/files/84/84-0.txt')
with open(filename) as f:
    book = f.read()
print(book[0:100])
```

```
Project Gutenberg's Frankenstein, by Mary Wollstonecraft (Godwin) Shelley

This eBook is for the u
```

In [4]:
```
# part 1
exclude = set(string.punctuation)

lowered_book = book.lower()
no_punctuation = "".join(ch for ch in lowered_book if ch not in exclude)
words = no_punctuation.split()
```

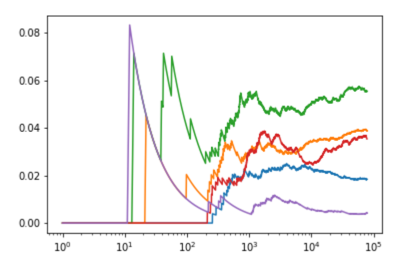
In [5]:
```
# part 2
a_counts = np.arange(len(words))
and_counts = np.arange(len(words))
the_counts = np.arange(len(words))
i_counts = np.arange(len(words))
is_counts = np.arange(len(words))

for i in range(len(words)):
    a_counts[i] = (words[i] == "a")
    and_counts[i] = (words[i] == "and")
    the_counts[i] = (words[i] == "the")
    i_counts[i] = (words[i] == "i")
    is_counts[i] = (words[i] == "is")
```

In [128]:
```
# part 3
range_n = np.arange(1, len(words) + 1)

prop_a = np.cumsum(a_counts)/range_n
prop_and = np.cumsum(and_counts)/range_n
prop_the = np.cumsum(the_counts)/range_n
prop_i = np.cumsum(i_counts)/range_n
prop_is = np.cumsum(is_counts)/range_n

# part 4
plt.semilogx(range_n, prop_a)
plt.semilogx(range_n, prop_and)
plt.semilogx(range_n, prop_the)
plt.semilogx(range_n, prop_i)
plt.semilogx(range_n, prop_is)
plt.show()
```



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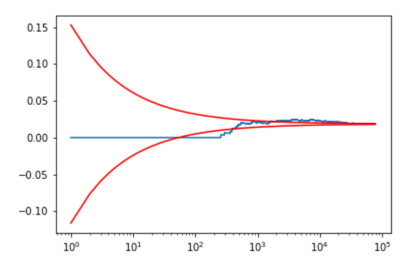
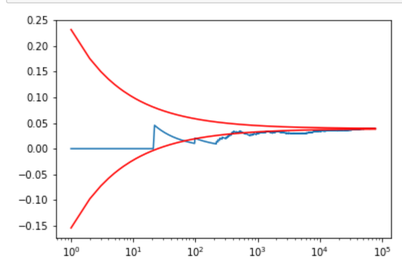
part 5

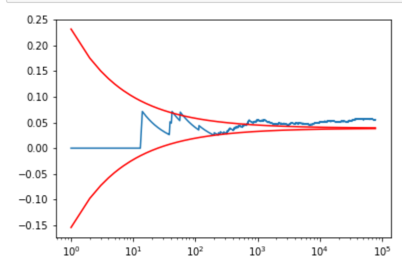
In [15]:
```
m_a = prop_a[-1]
var_a = m_a - m_a**2

y = np.arange(1, len(words)+1).reshape(len(words),1)
plt.semilogx(y, prop_a)

plt.semilogx(y,(var_a**0.5) * np.power(y,-0.5) + m_a,'r')
plt.semilogx(y,-(var_a**0.5) * np.power(y,-0.5) + m_a,'r')
plt.show()
```



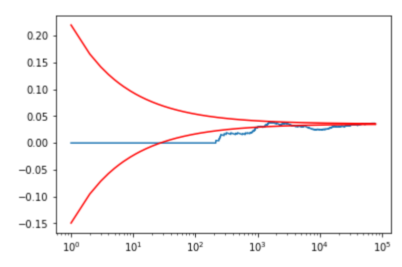
In [11]:
```
m_and = prop_and[-1]
var_and = m_and - m_and**2

plt.semilogx(y, prop_and)

plt.semilogx(y,(var_and**0.5) * np.power(y,-0.5) + m_and,'r')
plt.semilogx(y,-(var_and**0.5) * np.power(y,-0.5) + m_and,'r')
plt.show()
```



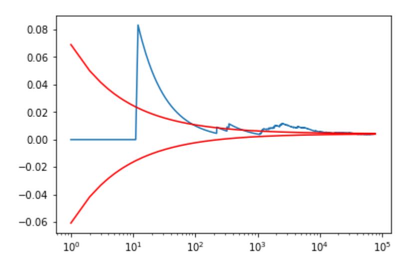
In [12]:
```
m_the = prop_and[-1]
var_the = m_the - m_the**2

plt.semilogx(y, prop_the)

plt.semilogx(y,(var_the**0.5) * np.power(y,-0.5) + m_the,'r')
plt.semilogx(y,-(var_the**0.5) * np.power(y,-0.5) + m_the,'r')
plt.show()
```



In [13]:
```
m_i = prop_i[-1]
var_i = m_i - m_i**2

plt.semilogx(y, prop_i)

plt.semilogx(y,(var_i**0.5) * np.power(y,-0.5) + m_i,'r')
plt.semilogx(y,-(var_i**0.5) * np.power(y,-0.5) + m_i,'r')
plt.show()
```



In [14]:
```
m_is = prop_is[-1]
var_is = m_is - m_is**2

plt.semilogx(y, prop_is)

plt.semilogx(y,(var_is**0.5) * np.power(y,-0.5) + m_is,'r')
plt.semilogx(y,-(var_is**0.5) * np.power(y,-0.5) + m_is,'r')
plt.show()
```



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part 6

we cannot apply the central limit theorem directly because the normal distribution is not discrete, its density is infinite and not representative of this specific multinomial distribution.

part 7

suggestion 1) use proportions of each word (which is equivalent to their mean/expected value), and let P(all other words) = 1 - sum(P(our 5 words)) and generate 80000 random uniform numbers from 0 to 1 like in the first problem. then, calculate their relative frequencies that way.

suggestion 2) we could randomize the order of the words, so that they are not dependent on each other.

suggestion 3) split each word up like how i did in part 5, so that each distribution revolves around one mean.

part 8

this distribution still works out fine because it still converges to the mean/expected value approximately due to n being pretty large **(~ 80000)**.

3. Denominator-layout notation

We used the numerator-layout notation for matrix calculus in class, now let's examine the denominator-layout notation.

Given $x, y \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$, we have

$$\frac{\partial y}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \\ \vdots \\ \frac{\partial y}{\partial x_n} \end{bmatrix}, \quad \frac{\partial \mathbf{y}}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x}, \frac{\partial y_2}{\partial x}, \ldots, \frac{\partial y_m}{\partial x} \end{bmatrix}$$

and

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial \mathbf{y}}{\partial x_1} \\ \frac{\partial \mathbf{y}}{\partial x_2} \\ \vdots \\ \frac{\partial \mathbf{y}}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1}, \frac{\partial y_2}{\partial x_1}, \ldots, \frac{\partial y_m}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2}, \frac{\partial y_2}{\partial x_2}, \ldots, \frac{\partial y_m}{\partial x_2} \\ \vdots \\ \frac{\partial y_1}{\partial x_n}, \frac{\partial y_2}{\partial x_n}, \ldots, \frac{\partial y_m}{\partial x_n} \end{bmatrix}$$

Questions:

1. Assume $\mathbf{y} = f(\mathbf{u})$ and $\mathbf{u} = g(\mathbf{x})$, write down the chain rule for $\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$
2. Given $\mathbf{X} \in \mathbb{R}^{m \times n}$, $\mathbf{w} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$, assume $z = \|\mathbf{X}\mathbf{w} - \mathbf{y}\|^2$, compute $\frac{\partial z}{\partial \mathbf{w}}$.

part 1

$$\frac{\partial y}{\partial \mathbf{x}} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial \mathbf{x}}$$
$$= \frac{\partial y}{\partial u}\begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{bmatrix}$$

part 2

decompose:

$\mathbf{a} = \mathbf{X}\mathbf{w}$

$\mathbf{b} = \mathbf{a} - \mathbf{y}$

$\mathbf{z} = \|\mathbf{b}\|^2$

compute:

$\frac{\partial z}{\partial \mathbf{w}} = \frac{\partial z}{\partial \mathbf{b}}\frac{\partial \mathbf{b}}{\partial \mathbf{a}}\frac{\partial \mathbf{a}}{\partial \mathbf{w}}$

$= \frac{\partial \|\mathbf{b}\|^2}{\partial \mathbf{b}}\frac{\partial \mathbf{a}-\mathbf{y}}{\partial \mathbf{a}}\frac{\partial \mathbf{X}\mathbf{w}}{\partial \mathbf{w}}$

$= 2\mathbf{b}^T \times \mathbf{I} \times \mathbf{X}$

$= 2(\mathbf{X}\mathbf{w} - \mathbf{y})^T\mathbf{X}^T$

2/3 dimension mismatch

4. Numerical Precision

Given scalars `x` and `y`, implement the following `log_exp` function such that it returns a numerically stable version of

$$-\log\left(\frac{e^x}{e^x + e^y}\right)$$

In [121]:
```
def log_exp(x, y):
    ## add your solution here
    inside = nd.exp(x)/(nd.exp(x) + nd.exp(y))
    #return inside
    return -nd.log(inside)
```

Test your codes with normal inputs:

In [122]:
```
x, y = nd.array([2]), nd.array([3])
z = log_exp(x, y)
z
```

Out[122]:
```
[1.3132617]
<NDArray 1 @cpu(0)>
```

Now implement a function to compute $\partial z/\partial x$ and $\partial z/\partial y$ with `autograd`

In [123]:
```
def grad(forward_func, x, y):
    ## Add your codes here

    x.attach_grad()
    y.attach_grad()

    with autograd.record():
        z = forward_func(x, y)
        z.backward()
    print('x.grad =', x.grad)
    print('y.grad =', y.grad)
```

Test your codes, it should print the results nicely.

In [124]:
```
grad(log_exp, x, y)
```

```
x.grad =
[-0.7310586]
<NDArray 1 @cpu(0)>
y.grad =
[0.7310586]
<NDArray 1 @cpu(0)>
```

But now let's try some "hard" inputs

In [126]:
```
x, y = nd.array([50]), nd.array([100])
grad(log_exp, x, y)
```

```
x.grad =
[nan]
<NDArray 1 @cpu(0)>
y.grad =
[nan]
<NDArray 1 @cpu(0)>
```

Does your code return correct results? If not, try to understand the reason. (Hint, evaluate `exp(100)`). Now develop a new function `stable_log_exp` that is identical to `log_exp` in math, but returns a more numerical stable result.

>> doesn't work because exp(100) is too small, so my original function rounds it to zero and cannot compute log(0)

>>i expanded x & y to have a sufficient amount of digits for their decimal approximations so they wouldn't be rounded down to zero.

2/3 see solutions

In [127]:
```
def stable_log_exp(x, y):
    ## Add your codes here
    #-x + log(e**x + e**y)
    x_large = nd.cast(x, dtype = 'float64')
    y_large = nd.cast(y, dtype = 'float64')
    return -nd.log(nd.exp(x_large)/(nd.exp(x_large) + nd.exp(y_large)))

grad(stable_log_exp, x, y)
```

```
x.grad =
[-1.]
<NDArray 1 @cpu(0)>
```